HEFAT2008
6th International Conference on Heat Transfer, Fluid Mechanics and Thermodynamics
30 June to 2 July 2008
Pretoria, South Africa
Paper number: K3

SUPER-TRANSPORT OF ENERGY IN ULTRA-SHORT PROCESSES: IMPLICATIONS TO HEAT TRANSFER, FLUID DYNAMICS AND QUANTUM MECHANICS

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ABSTRACT

The paradox of instantaneous propagation of energy is intrinsic to the classical models of energy transport. This paradox becomes well-pronounced during ultra-short processes. To remove the paradox, phase-lagged models of energy transport have been proposed. The analysis of the solutions to the phase-lagged energy equations suggests that when the characteristic time of the process is much less than the lag time, the wave mode of transport becomes the main mechanism of energy transfer.

In ultra-short heat transfer processes, the wave transport is manifested by an apparent increase (three-four orders of magnitude) of the thermal conductivity of the transporting medium. In viscous fluid flow, in addition, an apparent decrease of viscosity occurs, so that the liquid behaves as superfluid even at high temperatures.

It has been demonstrated that the lag time is inversely proportional to the temperature. Thus the life span of the super-transport phenomena should significantly increase at low temperatures, which is consistent with observing superfluids at lower temperatures for long periods of time.

From the quantum physics standpoint, the energy transporting waves can be viewed as narrow wave packets, whose wave functions describe the superfluid state of the liquid. This may provide an alternative explanation of the phenomenon of superfluidity: the Bose-Einstein condensation may not be needed for superfluidity to occur.

From the phase-lagged model of energy transport, it follows that the Schrödinger equation is but a zero-time-lag approximation. Hence, the phase-lagged Schrödinger equation must be used to describe ultra-short quantum interactions. It is suggested that the finiteness of Planck's constant and the finiteness of the speed of energy propagation are not independent. This circumstance may shed some light on the understanding of processes that took place in the beginning of our Universe.

INTRODUCTION

The assumption hidden behind the classical constitutive equations, such as Fick's or Fourier's laws, is the one of the energy transport speed being infinite. The application of the classical constitutive equations to diffusion problems leads to the appearance of the Poisson kernels in the solutions of these problems in infinite or semi-infinite domains, that is,

$$u(r,t)=\frac{1}{\left(2\sqrt{\pi Dt}\right)^{d}}\int_{\Omega}u(r,0)e^{-r^{2}/(4Dt)}dr \qquad (1)$$

where d is the dimension of the domain Ω and D is the diffusion coefficient of the domain in question. It is evident from (1) that even if the initial distribution $u(r, 0)$ is such that $u(r,0)\neq 0$ as $r\leq R$ and $u(r,0)=0$ as $r>R$, $u(r,t)>0$ for *any* values of r and t. In other words, however short the duration of the energy transport process is, the influence of the initial disturbance is sensed everywhere within the domain. Hence, the speed of energy transport has to be infinite, in order for the solution given by (1) to apply.

In practice, however, this paradox rarely causes a problem, because for a very small value of t, the value of the exponential function in (1) becomes practically indistinguishable from zero, so that even the most advanced means of measurement are not able to detect the difference. Yet, there are cases, in which the paradox of instantaneous propagation cannot be ignored. Thus, for instance, if $r\leq 2\sqrt{Dt}$, the classical assumptions become no longer valid and cannot be applied even in principle. Consequently, one may expect that when the characteristic size of the domain becomes small enough, the classical solutions can no longer be used and are to be substituted by other models.

To overcome this difficulty in heat transfer problems, a time lag is introduced between the onset of temperature

gradient and heat flux. This leads to hyperbolic models of heat transport, in which thermal energy is carried not only by means of diffusion but also by means of waves [1]. It has been demonstrated by many studies and experimentally confirmed that the hyperbolic energy equations work well at small scales [2-6]. This is likely to be due to the fact that the energy equations employed in those methods have direct analogy to the equation derived from the principles of statistical physics. Another strong argument in favor of the fact that the lag models are applicable to energy transport problems at very small scales can be provided based on the relationship established between the Navier-Stokes equations, diffusion equation and the Schrödinger equation [7]. Thus, physical quantities at all scales are related to the probability wave function found as the solution of the Schrödinger equation. In particular, the velocity field is the gradient of the phase of the wave function associated with the wave packet corresponding to the molecules of the fluid in question (see [7] for details). The Schrödinger equation, however, is valid at all scales.

Thermal energy is not the only form of energy to be transported during start-up processes. When a viscous fluid flow is to be considered, it is clear that momentum cannot propagate at an infinite speed. Hence, in the case of start-up flows or for flows in which the characteristic size of the flow is comparable with the molecular mean free path, a time lag between the velocity gradient and the resulting shear stress must be introduced to account for a finite speed of momentum transport.

GENERALIZED EQUATION OF ENERGY TRANSPORT

In this section, a generalized model of phase-lagged energy transport is presented.

A generalized energy transport equation with the convective term can be obtained from the conservation equation written in the differential form

$$\frac{\partial F(\mathbf{r},t)}{\partial t}+\nabla\cdot\boldsymbol{\varphi}(\mathbf{r},t)=S(\mathbf{r},t) \tag{2}$$

where F represents the transported property (such as, for instance, temperature, mass concentration, momentum, etc.), $\boldsymbol{\varphi}$ denotes the flux of F, $S(\mathbf{r}, t)$ is the source function, whereas $\mathbf{r}$ and t are the spatial and time independent variables, respectively. The conservation equation involves two unknown variables, F and $\boldsymbol{\varphi}$, and, hence, must be coupled with a constitutive equation that would relate these unknown quantities.

In general, constitutive equations are but assumptions and, unlike the conservation equation, cannot be derived from fundamental principles. Therefore, the final form of the equation of energy transport depends on the form of the constitutive equation used.

In the present model, the following constitutive equation is used:

$$\boldsymbol{\varphi}(\mathbf{r},t+\tau)=-D(\mathbf{r},t)\nabla F(\mathbf{r},t)+\mathbf{u}(\mathbf{r},t)F(\mathbf{r},t) \tag{3}$$

where D is the diffusion coefficient always measured in m^2/s (it can be, for example, mass diffusivity, kinematic viscosity, or the coefficient h/m that appears in the Schrödinger equation, where h is Planck's constant and m is the particle's mass) and $\mathbf{u}(\mathbf{r}, t)$ denotes the velocity vector – therefore, quantity F is transported by both diffusion (the first term in the right side of the constitutive equation) and convection (the second term in the right side). The parameter τ represents the time lag between the onset of the gradient of the transported quantity and the occurrence of the flux of that quantity. Hence, unlike Fick's or Fourier's constitutive equations, equation (3) accounts for a *finite speed* of the transport process and is more general than the latter.

The left and right sides of the constitutive relation are written for two different time moments. In order to overcome this difficulty, the left side of (3) is expanded into the Taylor series. The constitutive equation becomes

$$\sum_{n=0}^{\infty}\frac{\tau^n}{n!}\frac{\partial^n\boldsymbol{\varphi}(\mathbf{r},t)}{\partial t^n}=-D(\mathbf{r},t)\nabla F(\mathbf{r},t)+\mathbf{u}(\mathbf{r},t)F(\mathbf{r},t) \tag{4}$$

where $\dfrac{\partial^0\boldsymbol{\varphi}(\mathbf{r},t)}{\partial t^0}=\boldsymbol{\varphi}(\mathbf{r},t)$.

Upon applying the divergence operator to both parts of (4), the latter becomes

$$\sum_{n=0}^{\infty}\frac{\tau^n}{n!}\frac{\partial^n\left[\nabla\cdot\boldsymbol{\varphi}(\mathbf{r},t)\right]}{\partial t^n}= -\nabla\cdot\left[D(\mathbf{r},t)\nabla F(\mathbf{r},t)\right]+\nabla\cdot\left[\mathbf{u}(\mathbf{r},t)F(\mathbf{r},t)\right] \tag{5}$$

Now it follows from (2) that $\nabla\cdot\boldsymbol{\varphi}(\mathbf{r},t)=-\dfrac{\partial F(\mathbf{r},t)}{\partial t}+S(\mathbf{r},t)$. Upon substituting this into (5) and rearranging the terms, the equation becomes

$$\sum_{n=0}^{\infty}\frac{\tau^n}{n!}\frac{\partial^{n+1}F(\mathbf{r},t)}{\partial t^{n+1}}+\nabla\cdot\left[\mathbf{u}(\mathbf{r},t)F(\mathbf{r},t)\right]= \nabla\cdot\left[D(\mathbf{r},t)\nabla F(\mathbf{r},t)\right]+S(\mathbf{r},t)+\sum_{n=1}^{\infty}\frac{\tau^n}{n!}\frac{\partial^n S(\mathbf{r},t)}{\partial t^n} \tag{6}$$

Equation (6) is the generalized equation of phase-lagged energy transport. It reduces to the classical diffusion (heat) equation if $\tau = 0$ and $\mathbf{u}(\mathbf{r},t)=0$. If $\mathbf{u}(\mathbf{r},t)=0$ and $\tau<<1$ (all the terms whose order is larger than one can be

neglected in the series), the generalized equation of energy transport reduces to the classical wave equation, whereas $c = \sqrt{D/\tau}$ is the propagation speed of waves (for example, speed of light or sound). Note the presence of the apparent energy source $\sum_{n=1}^{\infty} \frac{\tau^n}{n!} \frac{\partial^n S(\mathbf{r},t)}{\partial t^n}$ in (6); it appears due to the finite time lag between the excitation and the response to it.

If the diffusion coefficient and velocity are both constant, that is $D(\mathbf{r},t) = D$ and $\mathbf{u}(\mathbf{r},t) = u$, equation (6) becomes

$$\sum_{n=0}^{\infty} \frac{\tau^n}{n!} \frac{\partial^{n+1} F(\mathbf{r},t)}{\partial t^{n+1}} + u\nabla F(\mathbf{r},t) = D\nabla^2 F(\mathbf{r},t) + S(\mathbf{r},t) + \sum_{n=1}^{\infty} \frac{\tau^n}{n!} \frac{\partial^n S(\mathbf{r},t)}{\partial t^n} \tag{7}$$

PROBLEM FORMULATION

Consider an energy transport process that occurs in a homogeneous (no preferred direction), semi-infinite domain whose boundary moves with a constant speed *u* and whose diffusion coefficient depends neither on spatial variable nor on time. In this case, the generalized equation of energy transport becomes

$$\sum_{n=0}^{\infty} \frac{\tau^n}{n!} \frac{\partial^{n+1} F(r,t)}{\partial t^{n+1}} + u\frac{\partial F(r,t)}{\partial r} = D\left[\frac{\partial^2 F(r,t)}{\partial r^2} + \frac{2\gamma}{r}\frac{\partial F(r,t)}{\partial r}\right] + S(r,t) + \sum_{n=1}^{\infty} \frac{\tau^n}{n!} \frac{\partial^n S(r,t)}{\partial t^n} \tag{8}$$

The parameter γ characterizes the domain geometry. Thus, $\gamma = 0$ corresponds to the domain with the flat boundary (no curvature); $\gamma = \pm 1$ represents the spherical case with the convex and concave boundary, respectively; whereas $\gamma = \pm 1/2$ describes the cylinder whose boundary is either convex ($\gamma = 1/2$) or concave ($\gamma = -1/2$).

The spatial variable $r = x \pm R$, where *x* is the actual distance from the origin and *R* is the initial radius of curvature. Note that the sign of *R* must be the same as the sign of γ.

Initially, at *t* = 0, the domain is in equilibrium with a constant value of the transported quantity, F_0, throughout the domain, $0 \le r < +\infty$.

As the energy transport process goes on, the condition $\lim_{r\to\infty} F(r,t) = F_0$ must be imposed in order to comply with the principle of energy conservation.

At this point, the second boundary condition is deliberately not imposed. This issue will be clarified in the following section.

SOLUTION PROCEDURE

In order to treat (8), we now apply the technique that was first discussed by Oldham and Spanier [8]. Whilst this equation can be solved by using Laplace transforms, the technique adopted allows for obtaining integral equations that relate local values of energy density (e.g. temperature, mass concentration, velocity, etc.) and the corresponding local values of energy flux (e,g. heat flux, mass flux, shear stress, etc.). The same method was successfully used in numerous applications [9-11]. In recent works, the method was extended and applied to problems involving combustion [12], hyperbolic heat transfer [13-15], turbulent flows [16], and such problems in biomedical engineering as modeling of the neural response to an external stimulus [17] and the alveolar gas exchange [18]. Some other developments of the method are presented in [19-23].

Upon introducing the new variable, $\rho = r/\sqrt{D}$, and the excess of the transported quantity $\hat{F} = F - F_0$, the transport equation becomes

$$\sum_{n=0}^{\infty} \frac{\tau^n}{n!} \frac{\partial^{n+1} \hat{F}(\rho,t)}{\partial t^{n+1}} = \frac{\partial^2 \hat{F}(\rho,t)}{\partial \rho^2} + 2\left(\frac{\gamma}{\rho} - \omega\right)\frac{\partial \hat{F}(\rho,t)}{\partial \rho} + \sum_{n=0}^{\infty} \frac{\tau^n}{n!} \frac{\partial^n S(\rho,t)}{\partial t^n} \tag{9}$$

where $\omega = \frac{u}{2\sqrt{D}}$ and $\frac{\partial^0 S(\rho,t)}{\partial t^0} = S(\rho,t)$.

The initial condition becomes $\hat{F}(\rho,0) = 0$, and the boundary condition is now $\lim_{\rho\to\infty} \hat{F}(\rho,t) = 0$.

Upon taking the Laplace transform of equation (9), the latter becomes

$$\frac{d^2\Phi(\rho;s)}{d\rho^2} + 2\left(\frac{\gamma}{\rho} - \omega\right)\frac{d\Phi(\rho;s)}{d\rho} - se^{s\tau}\Phi(\rho;s) = -e^{s\tau}Q(\rho,s) \tag{10}$$

where Φ is the Laplace transform of the excess of the transported quantity $\hat{F}$, *s* is the Laplace transform variable, and $Q(\rho,s)$ represents the Laplace transform of the source function *S*, provided that this Laplace transform exists.

The general solution of Eqn. (10) is

$$\Phi(\rho;s) = \left\{C_1(s)M_{\kappa,\mu}\left[2\rho f(s)\right] + C_2(s)W_{\kappa,\mu}\left[2\rho f(s)\right]\right\}\frac{e^{\omega\rho}}{\rho^{\gamma}} + P(\rho,s) \tag{11}$$

where $f(s)=\sqrt{se^{s\tau}+\omega^2}$, $P(\rho,s)$ is a particular solution of (10), $C_1(s)$ and $C_2(s)$ are two arbitrary functions of the Laplace transform variable *s*, $M_{\kappa,\mu}(z)$ and $W_{\kappa,\mu}(z)$ are Whittaker's functions, defined as

$$M_{\kappa,\mu}(z) = e^{-z/2}z^{\mu+1/2}M(1/2+\mu-\kappa, 1+2\mu, z) \tag{12a}$$

and

$$W_{\kappa,\mu}(z) = e^{-z/2}z^{\mu+1/2}U(1/2+\mu-\kappa, 1+2\mu, z) \tag{12b}$$

where $z=2\rho f(s)$, $\kappa=\gamma\omega/f(s)$, and $\mu=\gamma-1/2$. Note that the value of *z* increases as ρ increases. Functions *M* and *U* in (12) are Kummer's confluent hyper-geometric functions, defined as

$$U(a,b,z) = \frac{\pi}{\sin(\pi b)}\left[\frac{M(a,b,z)}{\Gamma(1+a-b)\Gamma(b)} - z^{1-b}\frac{M(1+a-b,2-b,z)}{\Gamma(a)\Gamma(2-b)}\right] \tag{13a}$$

$$M(a,b,z) = 1+\frac{az}{b}+\frac{(a)_2 z^2}{(b)_2 2!}+\ldots+\frac{(a)_n z^n}{(b)_n n!}+\ldots \tag{13b}$$

where $(a)_n = a(a+1)(a+2)\ldots(a+n-1)$, $(b)_n = b(b+1)(b+2)\ldots(b+n-1)$, and $(a)_0=(b)_0=1$ (see [24, p. 504]).

Written in terms of Kummer's functions, equation (11) becomes

$$\Phi(z;s) = \left[C_1(s)M(a,b,z)+C_2(s)U(a,b,z)\right]\left[2f(s)\right]^{\gamma}e^{-z(1+\omega/f(s))/2} + P(z,s) \tag{14}$$

where $a=\gamma\left(1-\omega/f(s)\right)$ and $b=2\gamma$.

Now, since $\lim_{|z|\to\infty} M(a,b,z) = \frac{\Gamma(b)}{\Gamma(a)}e^{z}z^{a-b}$ (see [24, p. 504]), the first term in (14) becomes unbounded for large values of *z*. This, however, contradicts the boundary condition $\lim_{z\to\infty}\Phi(z;s)=0$. Hence, for the solution to be bounded, the arbitrary function $C_1(s)$ must be identically zero. On the other hand, $\lim_{|z|\to\infty} U(a,b,z) = z^{-a}$ (see [24, p. 504]). Therefore, provided that $s > 0$ (this is really the case, because the Laplace transform variable corresponds to time and is always positive), the second term in equation (14) decreases as *z* increases and vanishes as *z* becomes infinitely large. Consequently, the solution becomes

$$\Phi(\rho;s) = C(s)U\left[a,b,2\rho f(s)\right]\left[2f(s)\right]^{\gamma}e^{\rho[\omega-f(s)]} + P(\rho,s) \tag{15}$$

Upon differentiating (15) with respect to ρ, the equation becomes

$$\frac{d\Xi}{d\rho} = C(s)\left[2f(s)\right]^{\gamma}\left\{\begin{matrix}-2af(s)U\left[a+1,b+1,2\rho f(s)\right]\\ +U\left[a,b,2\rho f(s)\right]\left[\omega-f(s)\right]\end{matrix}\right\}e^{\rho[\omega-f(s)]} \tag{16}$$

where $\Xi(\rho;s)=\Phi(\rho;s)-P(\rho;s)$. Note that

$$\frac{dU\left[a,b,2\rho f(s)\right]}{d\rho} = -2af(s)U\left[a+1,b+1,2\rho f(s)\right]$$

Furthermore, it follows from (15) that

$$C(s) = \frac{1}{U\left[a,b,2\rho f(s)\right]}\Xi(\rho;s)\left[2f(s)\right]^{-\gamma}e^{-\rho[\omega-f(s)]} \tag{17}$$

Upon substituting this into (16), the latter reduces to

$$\frac{d\Xi}{d\rho} = \left\{\left[\omega-f(s)\right]+2\gamma\left[\omega-f(s)\right]\frac{U\left[a+1,b+1,2\rho f(s)\right]}{U\left[a,b,2\rho f(s)\right]}\right\}\Xi(\rho;s) \tag{18}$$

Dividing (18) by $f(s)$,

$$-\frac{1}{f(s)}\frac{d\Xi}{d\rho}= \left\{1-\frac{\omega}{f(s)}+ 2\gamma\left[1-\frac{\omega}{f(s)}\right]\frac{U\left[\gamma\left(1-\omega/f(s)\right)+1,\,2\gamma+1,\,2\rho f(s)\right]}{U\left[\gamma\left(1-\omega/f(s)\right),\,2\gamma,\,2\rho f(s)\right]}\right\} \times\Xi(\rho;s) \tag{19}$$

Note the minus sign in the left side.

Now, upon noticing that

$$U(a+1,b+1,z)/U(a,b,z)=1/z=1/\left[2\rho f(s)\right],$$

equation (19) simplifies into

$$-\frac{1}{f(s)}\frac{d\Xi}{d\rho}= \left[1+\left(\frac{\gamma}{\rho}-\omega\right)\frac{1}{f(s)}-\frac{\gamma\omega}{\rho}\frac{1}{f^2(s)}\right]\Xi(\rho;s) \tag{20}$$

In most practical applications, values of the time lag τ are very small. In fact, $\tau=D/c^2$, where c is the speed of the energy transporting waves (it is usually very large, while the diffusion coefficient D is small). Hence, in this case, $f(s)=\sqrt{se^{s\tau}+\omega^2}$ can be written as $f(s)=\sqrt{s(s\tau+1)+\omega^2}$. The inverse Laplace transform of $1/f(s)$ is [24, p. 1025]

$$L^{-1}\left[\frac{1}{\sqrt{s(s\tau+1)+\omega^2}}\right]= \frac{1}{\sqrt{\tau}}L^{-1}\left[\frac{1}{\sqrt{\left(s+\frac{1}{2\tau}\right)^2-\left(\frac{1}{4\tau^2}-\frac{\omega^2}{\tau}\right)}}\right]= \frac{e^{-t/(2\tau)}}{\sqrt{\tau}}I_0\left(\frac{t}{2\tau}\sqrt{1-4\tau\omega^2}\right)$$

where I_0 is the modified Bessel function. The inverse Laplace transform of $1/f^2(s)$ is [24, p. 1022]

$$L^{-1}\left[\frac{1}{s(s\tau+1)+\omega^2}\right]= \frac{1}{\tau}L^{-1}\left[\frac{1}{\left(s+\frac{1}{2\tau}\right)^2-\left(\frac{1}{4\tau^2}-\frac{\omega^2}{\tau}\right)}\right]= \frac{2e^{-t/(2\tau)}}{\sqrt{1-4\tau\omega^2}}\sinh\left(\frac{t}{2\tau}\sqrt{1-4\tau\omega^2}\right).$$

Upon taking the inverse Laplace transform of (20) and restoring the original variables, the solution becomes

$$\begin{aligned}F(r,t)=F_0 &- c\int_0^t e^{-(t-\zeta)/(2\tau)}I_0\left(\frac{t-\zeta}{2\tau}\sqrt{1-\frac{u^2}{c^2}}\right)\frac{\partial\left[F(r,\zeta)-p(r,\zeta)\right]}{\partial r}d\zeta \\ &+\left(\frac{u}{2\pi c}-\frac{\gamma c}{r}\right)\times \int_0^t e^{-(t-\zeta)/(2\tau)}I_0\left(\frac{t-\zeta}{2\tau}\sqrt{1-\frac{u^2}{c^2}}\right)\left[F(r,\zeta)-F_0-p(r,\zeta)\right]d\zeta \\ &+\frac{\gamma u}{r\sqrt{1-\frac{u^2}{c^2}}}\times \int_0^t e^{-(t-\zeta)/(2\tau)}\sinh\left(\frac{t-\zeta}{2\tau}\sqrt{1-\frac{u^2}{c^2}}\right)\left[F(r,\zeta)-F_0-p(r,\zeta)\right]d\zeta \\ &+p(r,t)\end{aligned} \tag{21}$$

where $p(r, t)$ is the inverse Laplace transform of the particular solution $P(r;s)$.

Finally, upon substituting the constitutive equation relating the transported quantity F with its flux ϕ, namely

$$-\frac{\partial F(r,t)}{\partial r}=\frac{1}{D}\left[\phi(r,t)+\tau\frac{\partial\phi(r,t)}{\partial t}-uF(r,t)\right],$$

the solution becomes

$$
\begin{aligned}
F(x,t) &= F_0 + \\
&\frac{1}{\ell}\int_0^t e^{-(t-\zeta)/(2\tau)} I_0\left(\frac{t-\zeta}{2\tau}\sqrt{1-\frac{u^2}{c^2}}\right) \\
&\times\left[\phi(x,t)+\tau\frac{\partial\phi(x,\zeta)}{\partial t}-uF(x,\zeta)+\beta(x,\zeta)\right]d\zeta \\
&+\left(\frac{u}{2\ell}-\frac{\gamma c}{x\pm R}\right)\times \\
&\int_0^t e^{-(t-\zeta)/(2\tau)} I_0\left(\frac{t-\zeta}{2\tau}\sqrt{1-\frac{u^2}{c^2}}\right)\left[F(x,\zeta)-F_0-p(x,\zeta)\right]d\zeta \\
&+\frac{\gamma u}{(x\pm R)\sqrt{1-\frac{u^2}{c^2}}}\times \\
&\int_0^t e^{-(t-\zeta)/(2\tau)} \sinh\left(\frac{t-\zeta}{2\tau}\sqrt{1-\frac{u^2}{c^2}}\right)\left[F(x,\zeta)-F_0-p(x,\zeta)\right]d\zeta \\
&+ p(x,t)
\end{aligned}
\tag{22}
$$

where $\beta(x,t) = -D\partial p(x,t)/\partial x$ denotes the effective flux due to the presence of the source function in the original equation, R represents the radius of curvature of the surface. The sign of R must be the same as the sign of the parameter γ. The parameter $\ell = c\tau$ is the "space quantum".

Equation (22) is an integral solution of the generalized phase-lagged problem of energy transport modeled by equation (8) in the case of small values of the phase lag τ. It provides a relationship between the *local* values of the transported quantity F and its flux ϕ and is *valid everywhere within the domain, including the domain boundary*.

The solution given by equation (22) provides some important cues of how energy transport processes take place in general. Thus, for instance, it follows that a certain maximal speed of energy transport should exist such that no process of energy transport may occur with the speed larger than that maximum speed (the parameter c and the term $\sqrt{1-u^2/c^2}$ in the solution). Furthermore, although allowed being very small, the time lag, τ, in the solution is finite. This time lag may be viewed as the "time quantum" of the process in question. Moreover, the solution contains the "space quantum", ℓ, defined as $\ell = c\tau$.

Curiously enough, in the case of a non-zero value of the velocity u, the solution given by equation (22) becomes a mapping of the form $F_{n+1} = \Im(F_n)$, where $\Im$ denotes the integral operator in (22). Therefore, the solution of the generalized equation of energy transport allows of chaotic or even biotic (self-organized) solutions. This may become the topic of future studies.

In the following sections, the implications of the proposed model to fluid dynamics, heat transfer and quantum mechanics will be discussed.

PHASE-LAGGED NAVIER-STOKES EQUATION

If a viscous fluid flow is considered, the generalized approach to energy transport leads to an extended version of the classical Navier-Stokes equation, which, unlike the latter, is valid at *all* scales. For the sake of simplicity, but without the loss of generality, the derivation is performed in the case of an incompressible Newtonian fluid with constant physical properties.

Newton's second law of motion, written for a fluid particle, becomes

$$
\frac{\partial \mathbf{v}(\mathbf{x},t)}{\partial t} + \left[\mathbf{v}(\mathbf{x},t)\cdot\nabla\right]\mathbf{v}(\mathbf{x},t) = \frac{1}{\rho}\nabla\cdot\boldsymbol{\varphi}(\mathbf{x},t) + \mathbf{g}(\mathbf{x},t) \tag{23}
$$

where $\mathbf{v}$ is the velocity field, ρ represents density of the fluid, $\boldsymbol{\varphi}$ is the stress tensor, and $\mathbf{g}$ denotes body forces. The Navier-Stokes equation is derived from equation (23) by coupling it with an additional relation between the stress tensor and velocity field. Thus, for instance, such a relation (otherwise called the constitutive equation) in the case of a Newtonian fluid is

$$
\boldsymbol{\varphi}(\mathbf{x},t) = -p(\mathbf{x},t) + \mu\nabla\mathbf{v}(\mathbf{x},t) + \lambda\nabla\cdot\mathbf{v}(\mathbf{x},t) \tag{24}
$$

where p is pressure, μ denotes the dynamic viscosity, and λ is the second viscosity coefficient. Equation (24) establishes direct proportionality between the stress value and velocity gradient. If the fluid is assumed to be of constant density, i.e., $\nabla\cdot\mathbf{v} = 0$, the last term in the right hand side of (24) vanishes and equations simplifies into

$$
\boldsymbol{\varphi}(\mathbf{x},t) = -p(\mathbf{x},t) + \mu\nabla\mathbf{v}(\mathbf{x},t) \tag{25}
$$

Hence, the Navier-Stokes equation becomes in this case

$$
\begin{aligned}
&\frac{\partial \mathbf{v}(\mathbf{x},t)}{\partial t} + \left[\mathbf{v}(\mathbf{x},t)\cdot\nabla\right]\mathbf{v}(\mathbf{x},t) = \\
&\qquad -\frac{1}{\rho}\nabla p(\mathbf{x},t) + \nu\nabla^2\mathbf{v}(\mathbf{x},t) + \mathbf{g}(\mathbf{x},t)
\end{aligned}
\tag{26}
$$

This equation has been derived from Eqn. (23) by using the assumptions given by (26). This constitutive relation, however, assumes that the onset of the stress happens simultaneously with the onset of the velocity gradient, which

is equivalent to saying that the momentum transport happens with an infinite speed. As has been already mentioned, this may be a good assumption for relatively large scales, but it obviously fails at small scales.

In order to account for a finite value of the time lag between the onset of the velocity gradient and the stress caused by it, a new constitutive equation is assumed. Such an equation is

$$\boldsymbol{\varphi}(\mathbf{x},t+\tau) = -p(\mathbf{x},t) + \mu\nabla\mathbf{v}(\mathbf{x},t) \qquad (27)$$

where τ is called *the relaxation time* or *the time lag*.

The same procedure as was used in [25] is applied from this point onwards.

The left and right sides of the constitutive relation are written for two different time moments. In order to overcome this difficulty, the left side of (27) is expanded into the Taylor series. The constitutive equation becomes

$$\sum_{n=0}^{\infty}\frac{\tau^n}{n!}\frac{\partial^n\boldsymbol{\varphi}(\mathbf{x},t)}{\partial t^n} = -p(\mathbf{x},t) + \mu\nabla\mathbf{v}(\mathbf{x},t) \qquad (28)$$

where $\frac{\partial^0\boldsymbol{\varphi}(\mathbf{x},t)}{\partial t^0} = \boldsymbol{\varphi}(\mathbf{x},t)$.

Upon applying the divergence operator to both parts of (28), the latter becomes,

$$\sum_{n=0}^{\infty}\frac{\tau^n}{n!}\frac{\partial^n\left[\nabla\cdot\boldsymbol{\varphi}(\mathbf{x},t)\right]}{\partial t^n} = -\nabla p(\mathbf{x},t) + \mu\nabla^2\mathbf{v}(\mathbf{x},t) \qquad (29)$$

It follows from (23) that

$$\nabla\cdot\boldsymbol{\varphi}(\mathbf{x},t) = \rho\left\{\frac{\partial\mathbf{v}(\mathbf{x},t)}{\partial t} + \left[\mathbf{v}(\mathbf{x},t)\cdot\nabla\right]\mathbf{v}(\mathbf{x},t) - \mathbf{g}(\mathbf{x},t)\right\} \qquad (30)$$

Upon substituting this into (29) and rearranging the terms, it yields

$$\sum_{n=0}^{\infty}\frac{\tau^n}{n!}\left\{\begin{array}{l}\dfrac{\partial^{n+1}\mathbf{v}(\mathbf{x},t)}{\partial t^{n+1}} + \left[\mathbf{v}(\mathbf{x},t)\cdot\nabla\right]\dfrac{\partial^n\mathbf{v}(\mathbf{x},t)}{\partial t^n} \\ + \left(\dfrac{\partial^n\mathbf{v}(\mathbf{x},t)}{\partial t^n}\cdot\nabla\right)\mathbf{v}(\mathbf{x},t)\end{array}\right\} =$$

$$-\frac{\nabla p(\mathbf{x},t)}{\rho} + \nu\nabla^2\mathbf{v}(\mathbf{x},t) + \mathbf{g}(\mathbf{x},t) + \sum_{n=1}^{\infty}\frac{\tau^n}{n!}\frac{\partial^n\mathbf{g}(\mathbf{x},t)}{\partial t^n} \qquad (31)$$

where ν is the kinematic viscosity of the fluid.

Equation (31) is the phase-lagged version of the Navier-Stokes equation in the case of a finite time lag, τ, between the onset of the velocity gradient and shear stress. This equation is valid at all scales, provided that the fluid can be viewed as incompressible with constant physical properties. Note the presence of the apparent energy source due to the body force $\sum_{n=1}^{\infty}\frac{\tau^n}{n!}\frac{\partial^n\mathbf{g}(\mathbf{x},t)}{\partial t^n}$ in (31); it appears due to the finite time lag between the excitation and the response to it.

Now due to the smallness of the time lag, equation (31) can be written in the truncated form as

$$\tau\left[\frac{\partial^2\mathbf{v}}{\partial t^2} + (\mathbf{v}\cdot\nabla)\frac{\partial\mathbf{v}}{\partial t} + \left(\frac{\partial\mathbf{v}}{\partial t}\cdot\nabla\right)\mathbf{v}\right] + \frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{\nabla p}{\rho} + \nu\nabla^2\mathbf{v} + \mathbf{g} + \tau\frac{\partial\mathbf{g}}{\partial t} \qquad (32)$$

so that all the terms in (31) for which $n \geq 2$ are neglected. The significance of these terms will be explained in subsequent sections.

AN ILLUSTRATIVE EXAMPLE: START-UP FLOW IN A FLAT NANO-CHANNEL

In order to provide an illustration of how the developed model can be applied to problems in fluid mechanics, a very simple problem is considered: the start-up Couette flow in a flat nano-channel. It will be shown that, although the problem is very simple, the conclusions that can be drawn from the application of the present model to it are far reaching.

Consider the Couette flow between two parallel plates the distance between which is small, $h \sim 10^{-9}$ m. The fluid between the plates is initially at rest, $u(z,0) = 0$. The flow is created as one of the plates begins to move with a constant velocity, $u(h,t) = u_0 = const$. The other plate remains at rest at all times. In this case, equation (32) simplifies into

$$\tau\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial z^2} \qquad (33)$$

where z denotes the spatial variable, $0 \leq z \leq h$. Mathematically equation (33) is equivalent to the so-called telegraph equation that accounts for the damping of transporting waves due to friction (viscous dissipation, in the case of this study). The second initial condition is obviously $\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$.

The same equation was treated in the case of hyperbolic heat conduction in a semi-infinite domain previously [13].

The solution is obtained by the method of Kulish [23], the one that was used to treat (7):

$$\int_0^t e^{-(t-t')/(2\tau)}\left[I_0\left(\frac{t-t'}{2\tau}\right)-I_0\left(\frac{\sqrt{(t-t')^2-4\tau h^2/\nu}}{2\tau}\right)H\left[t-t'-2h\sqrt{\frac{\tau}{\nu}}\right]\right]u(z,t')dt' = u_0\int_0^t e^{-(t-t')/(2\tau)}\left[\begin{array}{l} I_0\left(\frac{\sqrt{(t-t')^2-\tau(h-z)^2/\nu}}{2\tau}\right)H\left[t-t'-(h-z)\sqrt{\frac{\tau}{\nu}}\right] \\ -I_0\left(\frac{\sqrt{(t-t')^2-\tau(h+z)^2/\nu}}{2\tau}\right)H\left[t-t'-(h+z)\sqrt{\frac{\tau}{\nu}}\right]\end{array}\right]dt' \quad (34)$$

where I_0 is the modified Bessel function and *H* is the unit step function [24].

SOLUTION APPLICABILITY

To establish the range of the solution applicability, a simple scale analysis of (33) is performed in this section.

Equation (33) is now written in terms of characteristic scales of the process, that is,

$$\tau\frac{u}{t_W^2}+\frac{u}{t_D}\sim\nu\frac{u}{h^2} \quad (35)$$

If both terms on the left side are of the same order of magnitude, one obtains expressions for the wave and diffusion time scales, namely, t_W and t_D, using the fact that each of these terms have to be of the same order as the term on the right side, that is,

$$t_W \propto \frac{h}{C} \quad (36)$$

and

$$t_D \propto \frac{h^2}{\nu} \quad (37)$$

where *h* is the scale of linear dimension and $C=(\nu/\tau)^{1/2}$ is the speed of momentum waves.

In fact, the wave component of the momentum transport dominates if the term u/t_D, responsible for the change due to diffusion, is much smaller than the term $\tau u/t_W^2$, responsible for the wave transport. In other words, the wave transport dominates if $\frac{t_W^2}{t_D\tau}<<1$.

Therefore, when the intrinsic length of the momentum diffusion

$$\lambda_D=(\nu t)^{1/2} \quad (38)$$

is significantly smaller than the intrinsic length scale of the momentum wave

$$\lambda_W = Ct \quad (39)$$

where C is the speed of momentum waves, the effect of wave transport can be neglected. Otherwise, this effect must be taken into account.

The lag time, τ, is such a time moment when $\lambda_D=\lambda_W$. Hence, it follows from (38) and (39) that $\tau=\nu/C^2$.

It is clear from the above that if the characteristic time of momentum transport is smaller than the lag time, i.e., $t<<\tau$, the transport by means of diffusion can be neglected, so that the transport by means of wave can be viewed as the only mechanism of energy transfer. On the contrary, when $t>>\tau$, the diffusion can be seen as the only mechanism of energy transport. However, if $t\sim\tau$, both the mechanisms of energy transport – waves and diffusion – must be taken into account.

Based on the scale analysis of (33), one can now introduce a dimensionless criterion to distinguish between the possible types of transport processes. Such a criterion is the relaxation number $N_r=\frac{t}{\tau}=\frac{C^2t}{\nu}$. If one now notices that the length scale of the process is the distance traveled by the wave, $d_W=Ct$, one obtains

$$N_r=\frac{t}{\tau}=\frac{d_W C}{\nu} = \begin{cases} <<1 & \text{transport by means of waves} \\ \sim 1 & \text{transport by means of both} \\ & \quad \text{waves and diffusion (transition)} \\ >>1 & \text{transport by means of diffusion} \end{cases} \quad (40)$$

It is amazing that this criterion is the Reynolds number based on the speed of momentum waves within the domain. This circumstance, however, is by no means coincidental. As has been shown [26], a similar scale analysis performed for buckling of streams in a fluid flow leads to the definition of the Reynolds number, seen as the factor of competition between the buckling waves and viscous diffusion.

From (40) one can observe that for fluids with $\nu\sim10^{-5}$ m^2/s and $C\sim10^3$ m/s, the transport by momentum

waves must be taken into account as $d_W \sim 10^{-9}$ m and below. It is equivalent to the time scale of $t \sim 10^{-11}$ s or smaller.

SUPERFLUID BEHAVIOR OF START-UP FLOWS

The scale analysis performed in the preceding section suggests that during fluid flow one part of momentum is transported by waves, while another part is transported by viscous diffusion. The part of momentum transported by viscous diffusion is determined by the relaxation number, N_r. Hence, the flow behaves as if the fluid consisted of two interpenetrating liquids: a wave component and a viscous component.

This picture of flow is strikingly reminiscent of the two-fluid model proposed by Landau [27] to explain behavior of liquid helium at low temperatures – the phenomenon known as superfluidity. In the start-up flow, the amount of momentum carried by waves can be identified with the amount of momentum carried by the superfluid component in Landau's model, that is, the part of momentum transported not by viscous diffusion.

The analogy becomes even more striking, if one notices that in the flow considered here, a certain part of energy will be transported by means of a temperature wave, whose lag time is determined as

$$\tau_T = \frac{\alpha}{C^2} \tag{41}$$

where α is the thermal diffusivity of the fluid in question. This phenomenon is known as the second sound in superfluids [27]. In fact, it can be noted that the ratio of τ to τ_T is the Prandtl number of the fluid.

From 1938, when superfluidity was discovered [28], helium-4 was the only known liquid that exhibits the superfluid behavior. In 1972, the phenomenon was seen, at much lower temperatures, in helium-3 [29]. Despite many searches for superfluidity in other systems, it remained an uncommon phenomenon until 1995, when Bose-Einstein condensation was achieved in rubidium vapor, adding to the list of superfluid systems [30]. This list now includes other gases, such as spin-polarized hydrogen gas, and, most recently, molecular gases of paired fermions [31].

It follows from the model presented in this paper that superfluidity is a common phenomenon that could be observed when particles constituting the medium transport their energy by both wave and diffusion. Hence, the superfluid component of flow is the part of momentum transported by means of waves, while the normal (viscous) component corresponds to the transport by means of diffusion. This provides a clear and very simple physical explanation of the superfluidity phenomenon and supports the intuitive two-liquid model suggested by Landau.

Yet the superfluidity of start-up flows is a very short-living occurrence. As the transient time of the process increases, the normal component of the flow (viscous diffusion) becomes more pronounced, while the superfluid component (the wave transport of momentum) decreases, so that it becomes negligible as $N_r >> 1$, when the flow can be considered as completely normal, so that all the superfluid effects vanish.

It is necessary to point out here that if the complete, non-truncated version of the generalized equation of fluid flow is considered [see Eqn. (31)], it follows that start-up flows have to exhibit additional modes of transport. Interestingly, the phenomena of the third, fourth and even the fifth sound have been recently reported in literature, for the nano-channel flows of liquid helium at very low temperatures [32].

It follows from the presented model that the phenomenon of superfluidity, albeit it has a very short life span within start-up flows, is a common phenomenon and is not confined to near absolute zero temperatures; it can be observed in all start-up flows. The start-up superfluidity becomes fully pronounced in flows, in which characteristic size of the channel is comparable with the distance traveled by momentum waves, that is, $d_W \sim 10^{-9}$ m. Otherwise, the phenomenon can only be observed in a thin nano-layer adjacent to the moving boundary and completely degenerates as the transient time of the flow becomes much larger than the relaxation time, $t >> \tau$.

SOME ILLUSTRATIVE SOLUTIONS

In order to demonstrate the wave effects (superfluidity) throughout the entire channel, the air flow through a nano-channel where $h = 3\times10^{-9}\,\text{m}$, $\nu = 1.55\times10^{-5}\,\text{m}^2/\text{s}$ and $C = 200.14\,\text{m/s}$ was selected. Note that the speed of momentum waves is defined as $C = C_s/\sqrt{3}$, where $C_s = 346.65\,\text{m/s}$ is the speed of sound in air.

Based on these values, $\tau = 3.87\times10^{-10}$ s. At this time moment, the contributions from wave transport and diffusion are equal.

This means that as the transient time is shorter than the relaxation time, τ, the wave effect has to be taken into consideration since it is the dominant term. In particular, it means that *any* start-up flow is to be described by the phase-lagged Navier-Stokes equations (31).

For a given constant velocity value of the moving wall, integration of equation (34) is straightforward, and is carried out by explicit time advancement using trapezoidal rule. In order to handle singularity, analytical integration is performed in the vicinity of the upper limit. The time step is chosen in such a way that the relative error of the solution does not exceed $\varepsilon = 10^{-6}$. Upon achieving this criterion, solution also becomes independent of further reduction in time step.

Figure 1 shows the velocity profiles versus time for different locations within the channel. It is observed that for the first one pico-second, the velocity in the channel is zero, i.e., there is a time lag before the effects of the upper plate are

propagated to the fluid in the channel. At slightly less than 2 pico-seconds, at a location of z/h = 0.9, the fluid velocity increases rapidly to a maximum before reaching its equilibrium value at time t = 30 pico-seconds. For smaller values of z/h, the wave takes a longer time to propagate from the upper plate to the lower plate, e.g. at z/h = 0.1, it takes almost 14 pico-seconds for the waves to reach that location. The velocity then increases rapidly to a maximum before decreasing to its equilibrium value at a time of 30 pico-seconds. This trend is also observed for other locations in the channel. It is noted that for the first 30 pico-seconds, wave transport is the main mechanism in the propagation of the velocity. In fact, for all the positions within the channel, the equilibrium velocity is reached after 30 pico-seconds.

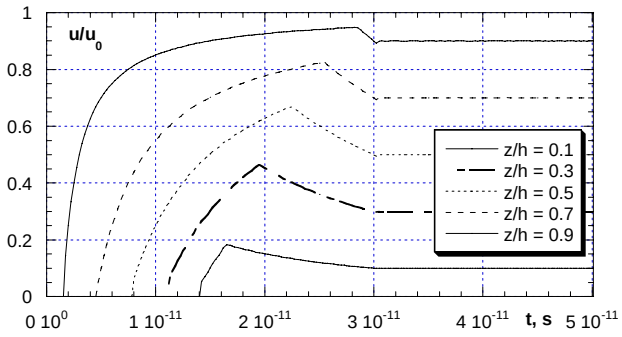

Figure 1 Transient velocity profiles at different locations within the channel

This can be explained by the following analysis:

Because $C = C_s/\sqrt{3}$ where $C_s = 346.65$ m/s is the speed of sound in air.

Then,

$$t_{relax} = 2h/C = 2\times 3\times 10^{-9}\ \text{m}/200.14\ \text{m/s}$$
$$= 29.97\times 10^{-12}\ \text{s} \approx 30\ \text{ps}$$

which is in excellent accord with the numerical solution.

This means that the speed of momentum wave is exactly equal to the velocity of the density disturbances in the phonon gas, defined as $c_x^2 + c_y^2 + c_z^2 = 3c_{phonon}^2 = c_{sound}^2$ in the kinetic theory (molecular physics). The relaxation time of 30 pico-seconds is the time necessary for the momentum wave, generated by the moving wall, to reach the opposite wall and then, being reflected by that wall, to travel back across the channel. No diffusion has to be accounted at those timescales. Hence the flow behaves as if it consisted of its superfluid component only.

The velocity profiles across the channel at different moments of time are shown in Figure 2. This figure is the most telling for visualizing the momentum wave that travels across the channel. In Fig. 2 (above), the position of the wave front within the channel corresponds to the point of zero velocity value, beyond which the velocity profile has been formed. As the transient time is 15 ps, the wave front reaches the opposite wall. At this particular moment, the velocity profile is indistinguishable from the steady Couette profile. At the same moment, the momentum wave, being reflected by the resting wall, begins to travel back through the channel, towards the moving wall. This motion of the reflected wave can be seen in Fig. 2 (below) as the kink in the velocity profile. The reflected wave moves from the resting wall towards the moving wall, so that the reflected wave reaches that wall as the transient time is 30 ps. As soon as the reflected momentum wave reaches the moving wall, it is absorbed by the latter, so that the velocity profile becomes the classical linear profile of the Couette flow – the velocity profile remains steady beyond this point. However, the superfluid component remains dominant, for the transient time of the process is still much less than $\tau = 387$ ps. Yet, as the transient time of the process increases, more and more momentum is transported by diffusion, because the probability of collisions among air molecules within the channel increases with time. Consequently, the content of the normal (viscous) component in the flow increases, while the superfluid effects become less pronounced. The flow becomes normal – with no superfluid component present – when the transient time of the process exceeds the relaxation time of 387 ps.

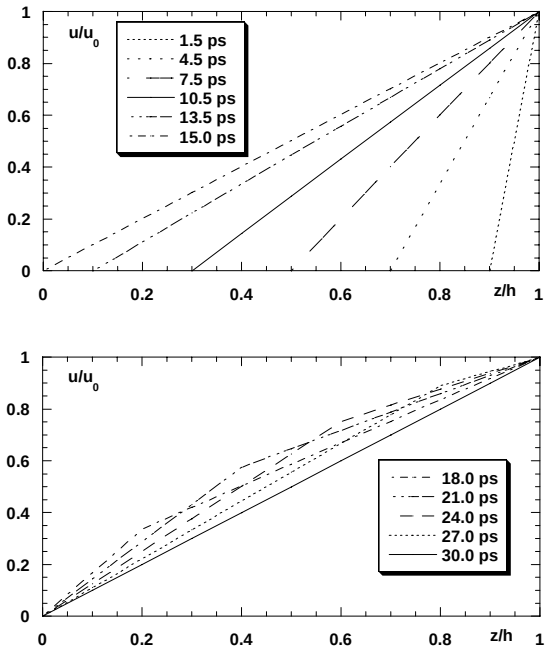

Figure 2 Evolution of velocity profiles across the channel – between 0 and 15 ps (above) and between 15 and 30 ps (below)

The evolution of the shear stress at different locations within the channel is shown in Figure 3. The first peak corresponds to the increase in velocity due to the momentum

wave coming from the moving plate. The wave then hits the resting plate and is reflected back to the moving plate. This explains the second peak in the shear stress value. It is seen from the figure that the equilibrium value of the shear stress at all the locations becomes the same, namely 6127 Pa, which corresponds to the constant value of the shear stress for the classical Couette flow. It is noted that the maximum shear stress throughout the channel occurs at the moving plate and has a value of nearly 120,000 Pa. This is almost 20 times the equilibrium shear stress value. Such a huge value is solely due to the instantaneous jerk of the moving wall and should be much smaller if the wall velocity were a smooth function of time.

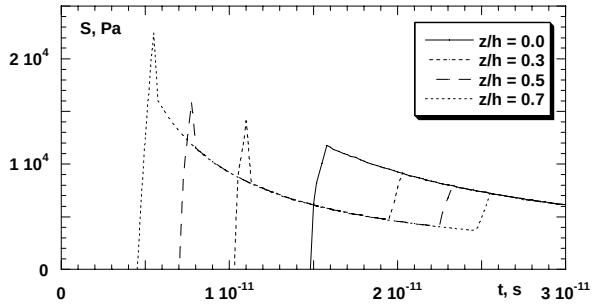

Figure 3 Transient shear stress profiles at different locations within the channel

Figure 4 shows the shear stress profiles across the channel at different time moments. In the first 1.5 ps, the momentum wave has propagated to h/H =0.9, this explains the increase in shear stress. As the transient time increases, the shear stress in the channel increases (until the transient time becomes 15 ps). At this point, the wave hits the resting plate, and the shear stress becomes constant throughout the channel. This figure is consistent with Figure 2 where the velocity profile throughout the channel has a Couette-type profile. Subsequently, a reflected wave, which is produced by the resting plate, moves towards the moving plate and a jump in the shear stress value is noticed. As the reflected momentum wave reaches the moving plate, the shear stress becomes constant across the channel and its value is equal to the shear stress in the steady Couette flow.

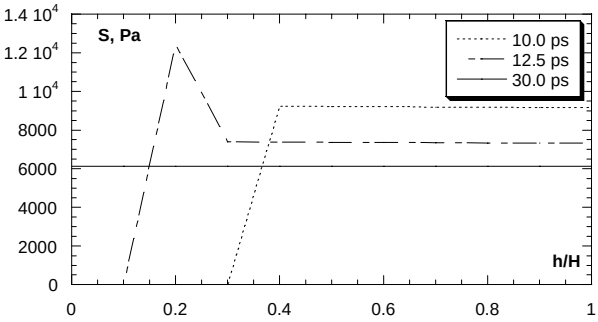

Figure 4 Evolution of shear stress profiles across the channel

AN APPARENT INCREASE OF THERMAL DIFFUSIVITY DURING SUPER-TRANSPORT PROCESSES

It is well-known that the phenomenon of superfluidity is always accompanied by a tremendous increase of the thermal conductivity of the superfluid liquid [27]. Thus, for instance, the thermal conductivity of superfluid helium exceeds that of copper by about three orders of magnitude. The purpose of this section is, therefore, to show that an apparent increase of the thermal diffusivity occurs during ultra-short processes as well and derive an expression that will allow us to estimate this increase.

In the wave model of heat conduction, the temperature evolution is given by

$$T(x,t) = T_0 - \sqrt{\frac{\alpha}{\tau}} \int_0^t \frac{\partial T}{\partial x} K_w\left(\frac{t-\zeta}{2\tau}\right) d\zeta \qquad (42)$$

where α denotes the thermal diffusivity, T_0 is the initial temperature, τ is the lag between the onset of the temperature gradient and the corresponding heat flux [13]. The kernel, K_w, is

$$K_w(z) = I_0(z) e^{-z} \qquad (43)$$

where $I_0(z)$ is the modified Bessel function.

If, however, the classical model is to be applied instead, then a different value of the thermal diffusivity has to be assumed, α_{eff}, in order to match the experimental results [9-10]. Thus,

$$T(x,t) = T_0 - \sqrt{\alpha_{eff}} \int_0^t \frac{\partial T}{\partial x} K(t-\zeta) d\zeta \qquad (44)$$

where the kernel is given by

$$K(z) = \frac{1}{\sqrt{\pi z}} \qquad (45)$$

Combining (42) and (44),

$$\int_0^t \frac{\partial T}{\partial x}\left[\sqrt{\frac{\alpha}{\tau}} K_w\left(\frac{t-\zeta}{2\tau}\right) - \sqrt{\alpha_{eff}} K(t-\zeta)\right] d\zeta = 0 \qquad (46)$$

Because (46) is to be valid for any $t > 0$, it yields

$$\sqrt{\frac{\alpha}{\tau}}K_w\left(\frac{t}{2\tau}\right)-\sqrt{\alpha_{eff}}K(t)=0 \qquad (47)$$

Consequently,

$$\alpha_{eff}=\frac{\alpha}{\tau}\left[\frac{K_w\left(\frac{t}{2\tau}\right)}{K(t)}\right]^2 \qquad (48)$$

However,

$$I_0(z)\approx\frac{e^z}{\sqrt{2\pi z}}\left[1+\frac{1}{8z}+\frac{9}{2!(8z)^2}+\frac{225}{3!(8z)^3}+...\right]$$

[24]. Using this and substituting (43) and (45) into (48),

$$\alpha_{eff}\approx\alpha\left[1+\frac{\tau}{4t}+\frac{9\tau^2}{2!(4t)^2}+...\right]^2 \qquad (49)$$

Hence, an apparent increase of thermal diffusivity during wave transport takes place. Thus, for instance, for $t=0.01\tau$, the apparent thermal diffusivity may exceed the real thermal diffusivity by almost three orders of magnitude.

SUPERFLUIDITY FROM THE POINT OF VIEW OF QUANTUM MECHANICS

It is usually believed that the phenomenon of superfluidity can be explained by the phenomenon nowadays known as the Bose-Einstein condensation. A gas of non-interacting atoms, obeying the Bose statistics, below a characteristic temperature, which depends on the mass and density, should manifest a peculiar behavior: a finite fraction of all the atoms (and at zero temperature, all of them) should occupy a *single* one-particle state [33]. This state is thought to correspond to the superfluid phase of a medium.

However, in the case of superfluid 3-He [29], since the 3-He atom obeys Fermi rather than Bose statistics, the mechanism of superfluidity cannot be explained by simple Bose-Einstein condensation [33]. This is consistent with the phase-lag model of fluid flow that predicts the existence of the phenomenon of superfluidity under some conditions, in all fluids [34].

Nevertheless, in general, the phenomenon of superfluidity has to be explained from the point of view of quantum mechanics.

It seems that the phase-lag model, although it explains the mechanism of superfluidity and relates the phenomena observed at low temperatures and the phenomena that should exist during ultra-short processes, is not related with any explanation made from the point of view of quantum mechanics. This, however, may be misleading.

As has been pointed out, the wave energy transport, competing with viscous diffusion, is the reason for superfluidity of a fraction of fluid. The fraction of particles participating in this wave transport, in a sense, behaves as a "single" particle, united by the wave motion. As a matter of fact, the particles constituting the transporting wave can be viewed as a narrow wave packet. It has been shown [7] that the group velocity of such a wave packet is the same as the flow velocity of the particles constituting the wave packet. Moreover, it has been demonstrated that this velocity is proportional to the gradient of the phase of the wave function that represents the quantum state of the packet. In other words,

$$\mathbf{v}=\frac{\hbar}{m}\nabla\theta \qquad (50)$$

where $\hbar$ is Planck's constant, m is the mass of the fluid element and θ denotes the phase of the corresponding wave function, $\psi=Ae^{i\theta}$. Therefore, the velocity of the wave packet satisfies $\nabla\times\mathbf{v}=0$ and, hence, part of the flow that transports momentum by means of waves behaves as inviscid, which in terms of the phase-lag model formulation is the same as being superfluid.

Furthermore, written in terms of the wave function, equation (50) becomes

$$\mathbf{v}=-\frac{i\hbar}{m}\frac{\nabla\psi}{\psi} \qquad (51)$$

which is known as the Cole-Hopf transformation. Using (51), the Navier-Stokes equations reduce to the Einstein-Kolmogorov equation for the wave function

$$\frac{\partial\psi}{\partial t}-\nu\nabla^2\psi=\frac{\Delta p}{2\mu}\psi \qquad (52)$$

where μ and ν denote dynamic and kinematic viscosities, respectively, and Δp is the pressure difference between the local pressure and a certain reference pressure (for details see [7]).

It has been shown, therefore, that no Bose-Einstein condensation may be needed for the phenomenon of superfluidity to occur. It is sufficient that narrow wave packets were present in the medium, which is equivalent to saying that super-transport of energy is carried by waves and not by diffusion.

It is worth noting that the two fluids model by Landau does not include the idea of quantum condensation either (it

seems Landau has been opposed to such an idea, regarding a non-interacting gas as pathology) [33].

THE PARADOX OF INSTANTANEOUS PROPAGATION IN THE SCHRÖDINGER EQUATION

Like other classical transport equations, the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2\psi+\Phi(\mathbf{r},t)\psi \qquad (53)$$

where $\hbar$ is Planck's constant, m denotes the mass of the system and $\Phi(\mathbf{r},t)$ represents the potential field influencing the system, allows for instantaneous propagation of energy. It yields the same kernels as the diffusion equation would yield; although the presence of the imaginary unity in the Schrödinger equation allows for periodic solutions, while the diffusion equation allows for time-decaying solutions only.

Hence, it may be tempting to assume that the Schrödinger equation, in the same way as the diffusion equation, leads to the paradox of instantaneous propagation and that a modified version of the Schrödinger equation is required if problems in which $r\le\sqrt{2\hbar t/m}$ are to be considered.

Indeed, the solution for the wave function, which is similar to the solution given by (1), is

$$\psi(\mathbf{r},t)=\left(\sqrt{\frac{m}{2\pi\hbar ti}}\right)^n\int_\Omega\psi(\mathbf{r},0)e^{ir^2m/(2\hbar t)}d\mathbf{r} \qquad (54)$$

The latter equation implies that the influence from the domain Ω is *instantaneously* felt everywhere in space. In other words, energy can propagate with an infinite speed. In most general applications, the error incurred is negligible but it must be noted that this is factually wrong since in reality energy cannot propagate at an infinite speed.

Therefore, the underlying assumption in the derivation of the Schrödinger equation needs to be re-examined. It is evident from the analysis presented in this section that this contradiction becomes manifest as the characteristic size of the domain and transient time of the process become small enough.

HEURISTIC DERIVATION OF THE SCHRÖDINGER EQUATION

In this section, a heuristic derivation of the Schrödinger equation is provided.

It is necessary to emphasize at this point that, historically, the Schrödinger equation was deduced, but *not derived*, from the reasoning that if the wave function $\psi(\mathbf{r},t)$ uniquely defines the state of a system, then, according to the causality principle, the value of the wave function at any moment of time, and hence, the state of the system, can be found from its initial value $\psi(\mathbf{r},t_0)$.

Consider the moment of time t after the initial moment t_0, such that $\Delta t=t-t_0<<1$. Then,

$$\psi(\mathbf{r},t)=\psi(\mathbf{r},t_0)+\left.\frac{\partial\psi}{\partial t}\right|_{t=t_0}\Delta t+\ldots \qquad (55)$$

According to what is said previously, the value of $\left.\frac{\partial\psi}{\partial t}\right|_{t=t_0}$ can be found from $\psi(\mathbf{r},t_0)$, that is,

$$\left.\frac{\partial\psi}{\partial t}\right|_{t=t_0}=\hat{T}(\mathbf{r},t_0)\psi(\mathbf{r},t_0) \qquad (56)$$

where $\hat{T}$, usually called *the time translation operator* [35], is a certain operation, to which $\psi(\mathbf{r},t_0)$ is to be subjected, in order to obtain $\left.\frac{\partial\psi}{\partial t}\right|_{t=t_0}$.

Since the initial time moment has been arbitrarily chosen, it follows that

$$\frac{\partial\psi}{\partial t}=\hat{T}(\mathbf{r},t)\psi(\mathbf{r},t) \qquad (57)$$

It follows from the superposition of states principle that the operator $\hat{T}$ is linear. Furthermore, $\hat{T}$ is independent of derivatives and integrals of ψ with respect to time. Indeed, if $\hat{T}$ depends on such derivatives, it means that it would be necessary to know their initial values to determine the states of the system following its initial state. In this case, however, the wave function $\psi(\mathbf{r},t)$ does not uniquely define the state of the system, which contradicts the initial assumption. The presence of an integral in (57) would mean that the state of the system depends on a piece of the past history of this system. Hence, $\hat{T}$ can only contain t as a parameter.

Unfortunately, the form of the time translation operator cannot be found from the above argument. A correct choice of $\hat{T}$ is hinted by considering a free motion with the value of momentum **p**. The wave function in this case is the de Broglie wave, given by

$$\psi(r,t)=Ne^{-\frac{i}{\hbar}(Et-\mathbf{pr})} \qquad (58)$$

where $E = p^2/(2m)$ is the energy of the particle of mass *m*. The wave function given by (58) satisfies

$$\frac{\partial \psi}{\partial t} = \frac{i\hbar}{2m}\nabla^2\psi \tag{59}$$

The latter equation can be written as

$$i\hbar\frac{\partial \psi}{\partial t} = \hat{T}\psi \tag{60}$$

It follows that, in the case of *free motion*, the time translation operator

$$\hat{T} = -\frac{\hbar^2}{2m}\nabla^2 \tag{61}$$

In quantum mechanics, one generalizes this particular result: it is *assumed* that the time translation operator is *always* given by (61).

THE PHASE-LAGGED SCHRÖDINGER EQUATION

In this section, the phase-lagged Schrödinger equation is derived, assuming that a finite time lag exists between the onset of the energy density gradient and the corresponding energy flux. It is shown that the classical Schrödinger equation arises in the case of the zero lag.

It is now claimed in many courses of quantum mechanics that the Schrödinger equation cannot be derived [36, 37]. Yet, it has been shown that if a partial differential equation is to adequately describe physical processes, such an equation can be derived from the energy conservation equation [38]. Usually, an auxiliary assumption is needed, because the conservation equation contains two unknown functions – energy density and energy flux. Thus, the final form of a partial differential equation depends not only on the conservation equation but also on the chosen form of the auxiliary equation, usually called the constitutive equation, of which examples are Fick's and Fourier's laws, assuming a direct proportionality between the energy flux and the energy density gradient. This is exactly why there are so many different partial differential equations. Each of them is valid only in those cases in which the corresponding constitutive equation is valid.

The Schrödinger equation is not an exception. A link between it and the conservation equation can be established only if an auxiliary equation of the Fickean type is to be employed. This circumstance makes the Schrödinger equation inexact and valid only at relatively large scales. Consider a domain Ω, whose total energy content is initially E_0. Upon introducing the dimensionless energy $w = E/E_0$, the energy conservation equation can be written as

$$\frac{\partial w}{\partial t} + \nabla\cdot\mathbf{j} = S(\mathbf{r},t) \tag{62}$$

where **j** is the flux and $S(\mathbf{r}, t)$ denotes the source function.

Note that the dimensionless variable *w* is more convenient as *E*, because it can be normalized and treated as a distribution

$$\int_\Omega w d\mathbf{r} = 1 \tag{63}$$

In this sense, it is possible to speak of the probability to find energy at a certain location within the domain at a certain time.

To overcome the paradox of instantaneous propagation, it is assumed that there is a finite time lag, τ, between the onset of the energy density gradient and the corresponding flux, that is,

$$\mathbf{j}(\mathbf{r}, t+\tau) = -D\nabla w(\mathbf{r},t) \tag{64}$$

where *D* is a proportionality coefficient that can in general depend on **r** and *t*. However, for the sake of clarity but without loss in generality, *D* is assumed constant in this study. Equation (64) in turn can be written as

$$\sum_{k=0}^{\infty}\frac{\tau^k}{k!}\frac{\partial^k \mathbf{j}(\mathbf{r},t)}{\partial t^k} = -D\nabla w(\mathbf{r},t) \tag{65}$$

where $\frac{\partial^0 \mathbf{j}(\mathbf{r},t)}{\partial t^0} = \mathbf{j}(\mathbf{r},t)$.

Upon applying the divergence operator to (65), the latter becomes

$$\sum_{k=0}^{\infty}\frac{\tau^k}{k!}\frac{\partial^k \left[\nabla\cdot\mathbf{j}(\mathbf{r},t)\right]}{\partial t^k} = -D\nabla^2 w(\mathbf{r},t) \tag{66}$$

Now, upon combining (66) and (62), one obtains

$$\frac{\partial w(\mathbf{r},t)}{\partial t} + \sum_{k=1}^{\infty}\frac{\tau^k}{k!}\frac{\partial^{k+1} w(\mathbf{r},t)}{\partial t^{k+1}} = D\nabla^2 w(\mathbf{r},t) + S(\mathbf{r},t) + \sum_{k=1}^{\infty}\frac{\tau^k}{k!}\frac{\partial^k S(\mathbf{r},t)}{\partial t^k} \tag{67}$$

which is the phase-lagged energy transport equation without the convective term [25]. Note the presence of the apparent energy source in the right side of (67). This apparent energy source is responsible for the resonance states within the energy field. Its physical meaning is still to be analyzed.

If now one denotes $w(\mathbf{r},t)=\psi\psi^*$ and $\mathbf{j}(\mathbf{r},t)=D\left(\psi\nabla\psi^*-\psi^*\nabla\psi\right)$, where ψ and ψ^* are complex conjugate functions of $\mathbf{r}$ and t, then, upon applying the Leibniz rule,

$$\frac{\partial^k(\psi\psi^*)}{\partial t^k}=\sum_{n=0}^{k}C_k^n\frac{\partial^n\psi}{\partial t^n}\frac{\partial^{k-n}\psi^*}{\partial t^{k-n}} \tag{68}$$

where $C_k^n=\dfrac{k!}{n!(k-n)!}$,

$$\begin{aligned}&\psi^*\frac{\partial\psi}{\partial t}+\psi\frac{\partial\psi^*}{\partial t}+\\&\sum_{k=1}^{\infty}\tau^k\left[\sum_{n=0}^{k+1}\frac{k+1}{n!(k+1-n)!}\frac{\partial^n\psi}{\partial t^n}\frac{\partial^{k+1-n}\psi^*}{\partial t^{k+1-n}}\right]=\\&D\left[\psi^*\nabla^2\psi-\psi\nabla^2\psi^*\right]+\\&S(\mathbf{r},t)+\sum_{k=1}^{\infty}\frac{\tau^k}{k!}\frac{\partial^k S(\mathbf{r},t)}{\partial t^k}\end{aligned} \tag{69}$$

where the identity

$$\nabla\cdot\left(\psi\nabla\psi^*-\psi\nabla\psi^*\right)=\psi\nabla^2\psi^*-\psi^*\nabla^2\psi$$

has been used to write the first two terms in the right side.

Equation (69) is the phase-lagged Schrödinger equation. It accounts for the presence of a finite time lag between the onset of the energy gradient and the corresponding energy flux. As such, equation (69) eliminates the paradox of instantaneous propagation hidden behind the classical Schrödinger equation. Therefore, the phase-lagged version of the Schrödinger equation presented here – unlike the classical Schrödinger equation – does *not* contradict the fact that energy propagation takes place at a finite speed.

Note also that, if $\tau=0$ in (69), the latter reduces to the two classical Schrödinger equations, written for ψ and ψ^*, respectively, that is,

$$i\frac{\partial\psi}{\partial t}=-\frac{\hbar}{2m}\nabla^2\psi+\Phi(\mathbf{r},t)\psi \tag{70a}$$

and

$$-i\frac{\partial\psi^*}{\partial t}=-\frac{\hbar}{2m}\nabla^2\psi^*+\Phi(\mathbf{r},t)\psi^* \tag{70b}$$

provided that one assumes $D=i\hbar/(2m)$ and $S(\mathbf{r},t)=0$, where m is the mass of the system in question and $\Phi(\mathbf{r},t)$ denotes the field acting on this system (see [39] for details). Hence, the classical Schrödinger equation is but a limit case of (69).

TRUNCATED VERSION OF THE PHASE-LAGGED SCHRÖDINGER EQUATION

In this section a truncated version of the phase-lagged Schrödinger equation is considered. It is evident that, due to the presence of the non-linear term in the equation, it cannot be separated into two equations: one for the wave function and the other for its conjugate. The importance of this circumstance is discussed.

Due to the smallness of the time lag in (69), one can neglect higher order terms and write (69) in its truncated form, that is,

$$\begin{aligned}&\psi^*\frac{\partial\psi}{\partial t}+\psi\frac{\partial\psi^*}{\partial t}+\\&\tau\left[\psi^*\frac{\partial^2\psi}{\partial t^2}+2\frac{\partial\psi}{\partial t}\frac{\partial\psi^*}{\partial t}+\psi\frac{\partial^2\psi^*}{\partial t^2}\right]=\\&\frac{i\hbar}{2m}\left[\psi^*\nabla^2\psi-\psi\nabla^2\psi^*\right]+S(\mathbf{r},t)+\tau\frac{\partial S(\mathbf{r},t)}{\partial t}\end{aligned} \tag{71}$$

It is evident from (71) that, unlike the case of the zero time lag, the equation cannot be split into two equations: one for the wave function and the other for its complex conjugate. This implies that duality – the co-existence of the two complementary features in any physical process – is fundamental. These complementary tendencies of the process are essential for the process to take place and, if separated, lead to solutions, behind which paradoxes, such as, for instance, the paradox of instantaneous propagation, are hidden.

Moreover, since the time lag is proportional to Planck's constant $\hbar$, whose value is finite, the finiteness of the time lag is fundamental and is related to the quantum structure of the universe.

The finiteness of the time lag in all the physical systems furthermore implies that the wave-corpuscular duality of matter is to be manifested at small scales. Indeed, the finite time lag leads to the appearance of both the wave term and the diffusion term in the equations of energy transport [25]. Yet, the finiteness of the time lag further implies that higher modes of energy transport are to be present, for the complete version of the transport equation contains higher derivatives with respect to time [see (67)].

SOME PHILOSOPHICAL IMPLICATIONS OF THE PHASE-LAGGED SCHRÖDINGER EQUATION

The phase-lagged Schrödinger equation given by (69) is written for the wave function and its complex conjugate and cannot be split into two equations, each for one of the specified functions only. Thus, from observing (69), it becomes evident that ***duality is a fundamental symmetry of matter***. That is, natural processes are results arising from the interaction of two complementary (complex conjugate) tendencies, described by ψ and ψ^*, respectively. Moreover, this interaction is non-linear (all the terms in (69) that involve partial derivatives of ψ and ψ^* are non-linear).

The phase-lagged Schrödinger equation suggests that higher than wave and corpuscular (diffusion) modes of energy transport are possible: this is due to the presence of higher order time derivatives in (69).

The finiteness of the time lag τ implies both the finiteness of the energy propagation speed and the finiteness of Planck's constant. In fact, the time lag can be defined as $\tau = \hbar/(2mc^2)$, where c represents the speed of energy propagation. Hence, in order for the time lag to be finite, both $\hbar$ and c are to be finite.

TEMPERATURE DEPENDENCE OF THE TIME LAG

In this section, it will be demonstrated that the phase-lagged model not only explains super-transport phenomena – such as superfluidity and super-thermal conductivity – during ultra-short processes at higher temperatures, but that, as a matter of fact, these phenomena are of the same nature as the well-known phenomenon of superfluidity observable at ultra-low temperatures.

Consider an ideal gas of phonons. The temperature gradient, in the situation depicted in Fig. 5, can be approximated as

$$\nabla T \approx \frac{T_2 - T_1}{\lambda} \tag{72}$$

where λ is the mean free path.

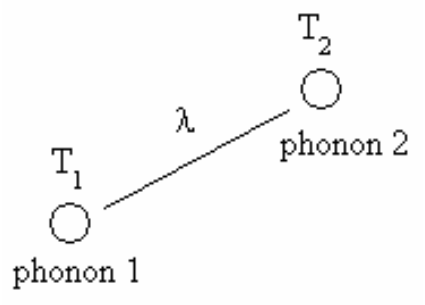

Figure 5 A single collision of two phonons

It is obvious that, although the temperature gradient is not zero, the energy flux will begin only upon a collision between the phonons. Hence, the time lag between the onset of the temperature gradient and the corresponding heat flux is

$$\tau_T = \frac{\lambda}{\langle v \rangle} \tag{73}$$

where $\langle v \rangle$ stays for the average speed of phonons in the ideal gas.

Now, from the kinetic theory,

$$\lambda = \frac{3N_A \kappa}{nC_V \langle v \rangle} \text{ and } \langle v \rangle = \sqrt{\frac{8kT}{\pi m}} \tag{74}$$

where N_A is the Avogadro number, κ denotes the thermal conductivity of the phonon gas, n is the amount of phonons per volume, C_V is the specific heat of the phonon gas at constant volume, k stays for the Boltzmann constant, and m is the phonon mass.

Combining equations (73) and (74) and using $C_V = \frac{3}{2} k N_A$ for the ideal gas of phonons,

$$\tau_T = \frac{\pi m \kappa}{4nk^2 T} \tag{75}$$

Therefore, $\lim_{T \to 0} \tau_T(T) = \infty$ and, hence, the life span of super-transport phenomena should increase as temperature decreases.

For the superfluidity effect, because $\tau_{visc}/\tau_T = \nu/\alpha$, where α denotes the thermal diffusivity and ν is the kinematic viscosity of the fluid in question,

$$\tau_{visc} = \frac{\pi m \rho C_P \nu}{4nk^2 T} \tag{76}$$

where ρ denotes density and C_P is the specific heat at constant pressure. In other words, the life span of the superfluid phenomena increases as T decreases.

Note that Kittel (1986) obtained a similar expression for the relaxation times for phonon-electron collisions in solids

$$\tau_{ph-e} = \frac{3m_e \kappa_e}{(\pi k)^2 n_e T} \tag{77}$$

where the subscript *e* refers to electrons. The coefficients in (77) are different from those in (75) and (76), because, unlike the present case – in which the classical distributions were used – the Fermi and Bose statistics were employed to derive equation (77).

Note also that the only purpose of equations (75)-(77) is to illustrate that the time lag increases as temperature decreases *in principle*. Thus, for instance, if a well-known relationship from statistical physics is employed, $\kappa \sim T^{1/2}$, it will lead to a slower increase of the time lag with temperature, $\tau \sim T^{-1/2}$ but not $\tau \sim T^{-1}$ as follows from (75)-(77). It will imply that lower temperatures – than predicted by (75)-(77) – are needed in order to achieve significant time lags. Interestingly, this is consistent with what is observed in liquid 3-He and some alkali gases: they achieve the superfluid state at much lower temperatures than 4-He, for which the theory predicts the transition temperature $T_\lambda = 2.17\ \mathrm{K}$ [27, 33, 41].

CONCLUSIONS

It seems that the phase-lagged model of energy transport may provide a deeper understanding of the physics of ultra-short processes. The models, discussed in this paper, may pave the way towards harnessing the state of energy super-transport, leading to the development of means for its practical use.

ACKNOWLEDGEMENTS

The author wishes to express his gratitude to Professor Sir Anthony Leggett (University of Illinois at Urbana-Champaign) and Professor Douglas Osheroff (Stanford University) for their valuable suggestions relevant to the subject of this paper.

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