

Design and Implementation of CUSUM Exceedance Control Charts for Unknown Location

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Abstract

Nonparametric control charts provide a robust alternative in practice when the form of the underlying distribution is unknown. Nonparametric CUSUM (NPCUSUM) charts blend the advantages of a CUSUM with that of a nonparametric chart in detecting small to moderate shifts. In this paper, we examine efficient design and implementation of Phase II NPCUSUM charts based on exceedance (EX) statistics, called the NPCUSUM-EX chart. We investigate the choice of the order statistic from the reference (Phase I) sample that defines the exceedance statistic. We see that choices other than the median, such as the 75th percentile, can yield improved performance of the chart in certain situations. Furthermore, observing certain shortcomings of the average run-length (*ARL*), we use the median run-length (*MRL*) as the performance metric. The NPCUSUM-EX chart is compared with the NPCUSUM-Rank chart proposed by Li et al. (2010) based on the popular Wilcoxon rank-sum statistic. We also study the choice of the reference value, k , of the CUSUM charts. An illustration with real data is provided.

Keywords: Nonparametric; Robustness; Precedence; Average run-length (*ARL*); Median run-length (*MRL*); Order Statistics.

1. Introduction

Nonparametric control charts provide a robust alternative when there is lack of knowledge about the underlying distribution. A key advantage of nonparametric charts is that their in-control (IC) run-length distribution remains invariant for all continuous process distributions. This is not true for parametric control charts in general and consequently, their IC robustness can be a legitimate concern. Moreover, nonparametric charts are often more robust and efficient under some heavy-tailed symmetric and skewed distributions. For more details on the nonparametric control charts literature see Chakraborti and Graham (2007) and Chakraborti et al. (2011).

The Shewhart-type charts are the most widely known charts in practice because of their simplicity. Nevertheless, the cumulative sum (CUSUM) charts are useful specially for detecting small and persistent shifts. With the availability of commercial software, CUSUM charts have now moved far beyond manufacturing into biology, medicine, athletics, finance and other areas. The advantages of CUSUM charts have been well documented and the reader is referred to Khoo and Teh

(2009) and the references therein. For a detailed discussion on CUSUM charts the reader is referred to Hawkins and Olwell (1998). More recently, Goel (2011) has summarized the basic principles and characteristics of CUSUM charts.

In this paper we study a class of Phase II nonparametric CUSUM (denoted by NPCUSUM) charts for monitoring the unknown location parameter based on the exceedance statistic which is constructed using a pre-defined order statistic from a reference sample. Although the main focus of our study is to design the most efficient NPCUSUM-EX chart using the appropriate order statistic from the Phase I reference sample, some other key questions that we address are as follows. A NPCUSUM chart using the reference sample median was first introduced and studied by Mukherjee et al. (2013). Unlike the traditional exceedance/precedence charts, instead of the reference sample median, we first investigate the efficient choice of the order statistic from an IC reference sample. More details about the exceedance/precedence statistic are given in Section 2. Second, in order to apply the NPCUSUM-EX chart, the reference value and the decision interval need to be specified. Mukherjee et al. (2013) did not provide much detail about the choice of these parameters in relation with the performance of the chart. This issue is addressed here. Finally, the performance of the NPCUSUM-EX chart is compared with a competing rank-based nonparametric chart, the NPCUSUM chart based on the Wilcoxon rank-sum statistic, proposed by Li et al. (2010) and denoted NPCUSUM-Rank chart. More details about the NPCUSUM-Rank chart are given in Appendix A.1.

Here we consider two-sided charts for detecting shifts in both directions (increasing or decreasing) unlike Mukherjee et al. (2013), who studied an upper one-sided CUSUM chart. Moreover we use the median run-length (*MRL*), motivated by Khoo et al. (2011) and others, as the performance metric in implementing the charts. The *MRL* has certain practical advantages over the traditional practice of using the *ARL*. More information about the benefits of *MRL* is provided later.

The remainder of this paper is organized as follows: In Section 2, the NPCUSUM-EX chart is presented. In Section 3, the implementation of the charts is discussed and the out-of-control (OOC) chart performance of the NPCUSUM-EX charts is studied and compared to that of the NPCUSUM-Rank chart. An illustrative example is given in Section 4. We conclude with a summary and some recommendations.

2. Statistical Framework and Preliminaries of CUSUM Charts

Host of researchers have addressed nonparametric control charts for process monitoring with the specified location (Case K). Here, we consider the problem of monitoring the unknown location (Case U) of a process: (i) McDonald (1990) considered a CUSUM procedure for individual observations based on statistics called ‘sequential ranks’ (ii) Jones et al. (2004) discussed the run-length distribution of the CUSUM with estimated parameters and provided a method for

approximating the distribution and moments; (iii) Chatterjee and Qiu (2009) proposed NPCUSUM charts using bootstrap control limits; (iv) Liu et al. (2013) proposed an adaptive NPCUSUM scheme for detecting unknown shifts in location and (v) Yang and Cheng (2011) proposed a NPCUSUM chart for variables data to monitor the process mean.

We consider a reference sample X_1, X_2, \dots, X_m from an IC process with an unknown continuous cdf $F(x)$. Let $Y_{j1}, Y_{j2}, \dots, Y_{jn}$, $j = 1, 2, \dots$, denote the j^{th} test Phase II sample of size n with cdf $G(y)$. Both F and G are unknown continuous cdf's and the process is IC when $F = G$. For detecting a change in the location, we use the location model $G_Y(x) = F(x - \delta)$ where $\delta \in (-\infty, \infty)$ is the unknown location parameter so the process is IC when $\delta = 0$.

NPCUSUM-EX Chart

Let $U_{j,r}$ denote the the number of Y observations in the j^{th} Phase II sample that exceeds $X_{(r)}$, the r^{th} ordered observation in the reference sample. The statistic $U_{j,r}$ is called an exceedance statistic and the probability $p_r = P[Y > X_{(r)} | X_{(r)}]$ is called an exceedance probability. The number of Y observations in the j^{th} Phase II sample that precede $X_{(r)}$ is called a precedence statistic and has been used by Chakraborti et al. (2004) to study the nonparametric Shewhart precedence chart. As in the Mukherjee et al. (2013), here we consider exceedance which may be advantageous in some cases, although for inference purposes; the exceedance and precedence tests are equivalent in the sense that the two statistics are linearly related.

Given $X_{(r)} = x_{(r)}$, it can be shown that the $U_{j,r}$ follows a binomial distribution with parameters (n, p_r) and thus, conditionally on $X_{(r)}$, we can use a binomial-type CUSUM chart based on the $U_{j,r}$'s to monitor the process location (via the exceedance probabilities). Noting that $E(U_{j,r} | X_{(r)}) = np_r$ and the conditional probability p_r is unknown, we may replace it by its unconditional IC value denoted by d . The latter is obtained by observing that when the process is IC, p_r follows a beta distribution with parameters $m - r + 1$ and r . Thus $d = E(p_r | \text{IC}) = P(Y > X_{(r)} | \text{IC}) = 1 - \frac{r}{m+1} = \frac{m-r+1}{m+1}$. Hence the two-sided NPCUSUM-EX chart has plotting statistics

$$C_j^+ = \max[0, C_{j-1}^+ + (U_{j,r} - nd) - k] \quad (1)$$

and

$$C_j^- = \min[0, C_{j-1}^- + (U_{j,r} - nd) + k] \quad (2)$$

for $j = 1, 2, 3, \dots$, with starting values $C_0^+ = C_0^- = 0$ and $k \geq 0$ is the reference value. The chart signals a possible OOC situation for the first j at which either $C_j^- < -H$ or $C_j^+ > H$, where $H > 0$ is the decision interval. Otherwise, the process is considered IC and process monitoring continues without interruption.

The traditional NPCUSUM-EX chart is based on the median of the reference sample as the median is a robust and good representative of the reference data and one of the popular percentiles in practice. However, the question of which reference sample order statistic should be chosen to create the chart, has never been examined. To this end, we investigate the performance of the NPCUSUM-EX chart systematically, based on the 25th, 40th, 50th, 60th and 75th percentiles, respectively. Note that for brevity, a shorthand notation is used to describe the charts. For example, the NPCUSUM-EX chart based on the 50th percentile is denoted by EX(50). We also study the effect of the choice of the reference value, k .

3. Implementation and Performance

The two key quantities necessary to implement the CUSUM chart are the reference value, k , and the decision interval, H . These are typically based on the magnitude of the shift to be detected and some nominal value of an IC chart performance characteristic. The most widely used chart performance metric is the *ARL* and determining the charting constants typically involves specifying a nominal IC *ARL* value, such as 370. However, since the run-length distribution is significantly right-skewed, researchers have often recommended other measures for the assessment of chart performance. More specifically, the *MRL* provides additional and more meaningful information about the in-control and out-of-control performances of control charts compared to the *ARL*. The idea of looking at percentiles in SPC goes back to Barnard (1959). Recently Khoo et al. (2011) have advocated the use of percentiles, such as the median, for assessment of chart performance. The disadvantages of using the *ARL* are summarized in Appendix A.2. Further, the use of the *MRL* will ensure a guaranteed control over the *FAR* in the sense that no more than 50% of the false alarms will be realized before the MRL_0 . Moreover, there is practically no disadvantage of using the *MRL* except possibly one. For the *MRL*, finding the exact standard error is not straightforward unless one uses bootstrapping or other sophisticated techniques..

When comparing the performance of charts the MRL_0 is fixed at an acceptably ‘high’ level and the chart with the smaller MRL_δ is preferred. To this end, note that the run-length distribution of a Shewhart chart follows a Geometric distribution with success probability $p = \frac{1}{ARL_0}$. For industry standard values for ARL_0 such as 500, $= \frac{1}{500} = 0.002$. Consequently, it follows that the MRL_0 equals $\frac{-1}{\log_2(1-0.002)} = 346.2$. Therefore, keeping parity with the traditional Shewhart-type chart and for simplicity, we recommend 350 as the target nominal IC MRL_0 .

3.1. Implementation: Chart Design Parameters

Practical implementation of the NPCUSUM-EX charts requires specifying the following quantities: (i) m : the size of the IC Phase I reference sample^{*}, (ii) r : the order of the reference sample order statistic, (iii) n : the size of each Phase II test sample, (iv) t : the desired MRL_0 , (v) k : the reference value and (vi) H : the decision interval depending on m, n, r, t and k . The experimenter need to specify the parameters m, n, r and t in a given situation. The choice of r will be discussed later.

The design parameters k and H are chosen so that the chart has a specified nominal MRL_0 and is capable of detecting a specified amount of shift as soon as possible. The first step in this direction is to choose k . For the parametric CUSUM chart the choice of k has been discussed by many authors where, typically, $k = \frac{1}{2}\delta$ with δ representing the size of the shift in the mean expressed in standard deviation units. Hawkins and Olwell (1998) investigated the sensitivity of the ARL considering four values of k , namely $k = 0.25, 0.5, 1$ and 2 and concluded that the CUSUM with $k = 0.25$ is the best of the four for all $\Delta < 0.73$. The CUSUM with $k = 0.50$ is the best in the range $0.73 < \Delta < 1.46$. The CUSUM with $k = 1$ leads in the range $1.46 < \Delta < 2.87$, whereas the CUSUM with $k = 2$ is the best of these four for $\Delta > 2.87$. In their conclusion, Δ denotes the size of the actual mean shift[†]. Kim et al. (2007) considered two values of the reference value k , namely, $k = 0$ and $k = 0.5$ and found that the CUSUM chart with $k = 0$ is more effective in detecting shifts of size $0.25\sigma_Y$, where σ_Y is the process standard deviation, whereas the chart with $k = 0.5$ detects any shift exceeding $0.25\sigma_Y$ much faster. Recently, Mukherjee et al. (2013) examined the impact of k , for the parametric CUSUM chart, and concluded that when there is little or no a-priori information regarding the size of the shift a smaller value of k is the safest choice to protect against any unnecessary delays in detection. They came to a similar conclusion for their proposed NPCUSUM-EX chart and recommended using a small value of k or even $k = 0$. Here, in order to investigate the impact of the reference value on the performance of the chart more thoroughly, we consider a number of reference values, $k_l = \tau_l STDEV(U_{j,r})$ for $l = 1, 2, 3, 4$, respectively, where $\tau_l = c_l \delta$, c_l is a positive constant and δ is the amount of the shift to be detected in terms of standard deviation units, $\delta = 0.00(0.25)1.00, 1.50, 2.00$ and 3.00 . The constant c_l ranged from small to medium to large, namely, $0.25, 0.5, 1$ and 2 . A similar approach is used for the NPCUSUM-Rank chart and thus, in summary, for the NPCUSUM-EX and the NPCUSUM-Rank charts we set k_l equal to $\tau_l STDEV(U_{j,r})$ and $\tau_l STDEV(W_j)$ for $l = 1, 2, 3, 4$, respectively. Note that

^{*} The reference sample size affects the performance of the Phase II chart; see Appendix B of Mukherjee et al. (2013) for details.

[†] Relationship between δ and Δ : Let μ_0 denote the target IC mean and let μ_1 denote the OOC value of the mean. Then δ is the size of the shift in the mean expressed in standard deviation units, i.e. $\mu_1 = \mu_0 + \delta\sigma$ so that $\delta = |\mu_1 - \mu_0|/\sigma$. In Hawkins and Olwell (1998) Δ denotes the size of the actual mean shift, i.e. $\Delta = \delta\sigma$.

although shifts as large as $\delta = 3.00$ were considered in this study, the largest value of δ reported in the paper is $\delta = 1.00$, since, for larger shifts, (i) the run-length characteristics of the charts tend to converge to some small values, i.e., if the shift is large enough the any of the charts will signal (almost) immediately and (ii) in Section 3.2 it is shown that for moderate to large shifts the practitioner should use the EX(75) chart which signals very quickly for all k under consideration.

The next step is to find the decision interval H , in conjunction with the chosen k , so that a desired nominal MRL_0 is attained. Using an appropriate search algorithm (see Appendix A.3) H is found so that the attained MRL_0 is less than or equal to the desired nominal MRL_0 .

3.2. Out-of-control Chart Performance Comparisons

Our performance study includes a collection of non-normal distributions that are heavy-tailed, symmetric and skewed. Specifically, the distributions considered in the study are: the standard normal distribution, $N(0,1)$, the exponential distribution, $EXP(1)$, which is positively skewed, the Student's t -distribution, $t(3)$, the Laplace (or double exponential) distribution, $DE(0,1/\sqrt{2})$, which is standard normal like, but has heavier tails, the Log-Normal(0,1) distribution, denoted $LogN(0,1)$, the Symmetric Mixture Normal distribution $[0.6N(\mu_1 = 0, \sigma_1 = 0.25) + 0.4N(\mu_2 = 0, \sigma_2 = 4)]$, denoted $SymmMixN$, two Asymmetric Mixture Normal distributions $[0.6N(\mu_1 = 1, \sigma_1 = 1) + 0.4N(\mu_2 = 0, \sigma_2 = 4)]$ and $[0.6N(\mu_1 = -1, \sigma_1 = 1) + 0.4N(\mu_2 = 0, \sigma_2 = 4)]$, denoted $AsymmMixN1$ and $AsymmMixN2$, respectively. Graham et al. (2012) considered these mixture normal distributions being relatively heavier tailed than the normal and with higher kurtosis. All distributions in the study have been standardised to mean/median equals 0 and the standard deviation equals 1 so that the results are easily comparable across the distributions.

The first row of each cell in the tables shows the MRL and the corresponding interquartile range (IQR) in parentheses, whereas the second row shows the values of the 5th, 25th, 75th and 95th percentiles (in this order) of the run-length distribution. For each row, i.e. for each distribution, the chart that is performing the best is indicated by use of grey shading. If any two or more blocks (in the same row) are shaded, it means that the performance of the charts is similar.

Several observations can be made from an examination of Tables 1 to 4. The decision interval, H , for the NPCUSUM-EX chart is the same for the 25th and 75th percentiles and for the 40th and 60th percentiles, respectively. Note that, since the $STDEV(U_{j,r})$ is the same for the pair of percentiles $(M - \Delta_1, M + \Delta_1)$ where M denotes the median and Δ_1 is an integer between 1 and 49, the reference value k would be the same (since k is a function of $STDEV(U_{j,r})$) and, consequently,

the decision interval H would be the same (the reader is referred to Section 3.1 where the choice of the design parameters k and H are explained in detail).

As expected, all the run-length distribution percentiles decrease sharply with increasing shift, which is expected, indicating that NPCUSUM charts are reasonably effective in detecting shifts in location. However, the effectiveness (speed of detection) of the charts varies depending on the magnitude of the shift, the underlying process distribution and the type of the reference percentile being considered in forming the exceedance statistic.

Table 1. OOC run-length characteristics (target $MRL_0 = 350$, $m = 100$, $n = 5$, and $\delta = 0.25$)

| | | CUSUM-Rank | CUSUM-EX | | | | |
|--------------|--------------------------|----------------------------|------------------------------------|------------------------------------|------------------------------------|------------------------------------|------------------------------------|
| | | | $r = 25^{\text{th}}$ percentile | $r = 40^{\text{th}}$ percentile | $r = 50^{\text{th}}$ percentile | $r = 60^{\text{th}}$ percentile | $r = 75^{\text{th}}$ percentile |
| $c_1 = 2$ | Dist / H | $H=386$ | $H=5.71$ | $H=6.5$ | $H=6.55$ | $H=6.5$ | $H=5.71$ |
| | $N(0,1)$ | 34 (67) 8, 17, 84, 489 | 139 (433) 18, 49, 482, 2133 | 76 (207) 12, 30, 237, 1286 | 60 (160) 11, 25, 185, 1043 | 53 (137) 9, 22, 159, 902 | 47 (123) 8, 19, 142, 847 |
| | $EXP(1)$ | 16 (18) 6, 10, 28, 125 | 9 (7) 8, 8, 15, 44 | 18 (27) 6, 11, 38, 240 | 26 (55) 8, 13, 68, 507 | 38 (106) 8, 17, 123, 881 | 66 (215) 8, 23, 238, 1370 |
| | $t(3)$ | 18 (21) 6, 11, 32, 112 | 71 (157) 15, 32, 189, 977 | 30 (44) 9, 17, 61, 248 | 23 (31) 8, 13, 44, 168 | 20 (28) 7, 12, 40, 40, 160 | 21 (38) 5, 11, 49, 266 |
| | $DE(0, 1/\sqrt{2})$ | 22 (31) 7, 13, 44, 196 | 134 (363) 19, 50, 413, 1867 | 40 (67) 11, 21, 88, 344 | 24 (32) 8, 14, 46, 156 | 20 (29) 7, 11, 40, 214 | 30 (77) 6, 13, 90, 638 |
| | $GAM(0.5,1)$ | 9 (5) 5, 7, 12, 24 | 8 (0) 8, 8, 8, 8 | 5 (0) 5, 5, 5, 14 | 9 (12) 4, 6, 18, 105 | 21 (47) 6, 11, 58, 505 | 60 (202) 8, 21, 223, 1368 |
| | $LogN(0,1)$ | 8 (5) 5, 6, 11, 19 | 8 (0) 8, 8, 8, 8 | 6 (3) 5, 6, 9, 15 | 8 (6) 4, 6, 12, 32 | 11 (14) 4, 8, 22, 96 | 27 (77) 6, 12, 89, 761 |
| | $SymmMixN$ | 6 (3) 4, 5, 8, 11 | 127 (177) 24, 61, 238, 512 | 11 (5) 6, 9, 14, 20 | 7 (2) 4, 6, 8, 11 | 5 (2) 4, 4, 6, 7 | 4 (1) 3, 3, 4, 6 |
| | $AsymmMixN1$ | 16 (18) 6, 10, 28, 89 | 267 (723) 29, 94, 817, 4123 | 31 (41) 11, 19, 60, 203 | 20 (21) 8, 13, 34, 99 | 15 (16) 6, 10, 26, 75 | 14 (18) 5, 8, 26, 108 |
| | $AsymmMixN2$ | 16 (18) 6, 10, 28, 91 | 50 (82) 15, 28, 110, 473 | 21 (23) 8, 13, 36, 105 | 17 (18) 7, 11, 29, 84 | 15 (17) 6, 10, 27, 93 | 22 (50) 5, 11, 61, 659 |
| $c_1 = 1$ | Dist / H | $H=800$ | $H=11.7$ | $H=13.5$ | $H=14$ | $H=13.5$ | $H=11.7$ |
| | $N(0,1)$ | 37 (42) 15, 24, 66, 369 | 75 (163) 23, 40, 203, 1802 | 60 (114) 20, 34, 148, 1347 | 56 (104) 18, 32, 136, 1373 | 52 (96) 17, 30, 126, 1172 | 52 (115) 15, 28, 143, 1386 |
| | $EXP(1)$ | 20 (14) 11, 15, 29, 69 | 14 (7) 12, 12, 19, 35 | 23 (20) 11, 17, 37, 139 | 32 (39) 13, 20, 59, 433 | 41 (77) 14, 24, 101, 1153 | 67 (229) 15, 31, 260, 2525 |
| | $t(3)$ | 23 (18) 12, 17, 35, 78 | 49 (54) 21, 32, 86, 368 | 32 (27) 15, 23, 50, 140 | 28 (23) 14, 21, 44, 118 | 27 (23) 12, 19, 42, 125 | 28 (34) 11, 18, 52, 298 |
| | $DE(0, 1/\sqrt{2})$ | 28 (25) 12, 19, 44, 132 | 73 (127) 24, 41, 168, 1302 | 38 (35) 17, 27, 62, 182 | 29 (22) 15, 22, 44, 107 | 26 (24) 12, 18, 42, 173 | 37 (65) 12, 21, 86, 968 |
| | $GAM(0.5,1)$ | 13 (5) 9, 11, 16, 25 | 12 (0) 12, 12, 12, 12 | 8 (0) 8, 8, 8, 19 | 15 (15) 7, 10, 25, 81 | 27 (39) 10, 17, 56, 585 | 63 (215) 14, 29, 244, 2320 |
| | $LogN(0,1)$ | 12 (5) 8, 10, 15, 23 | 12 (0) 12, 12, 12, 13 | 10 (4) 8, 9, 13, 19 | 13 (6) 8, 11, 17, 34 | 17 (15) 8, 12, 27, 82 | 33 (62) 11, 19, 81, 941 |
| | $SymmMixN$ | 10 (4) 7, 8, 12, 15 | 58 (36) 27, 42, 78, 118 | 16 (4) 12, 14, 18, 23 | 11 (2) 9, 10, 12, 15 | 8 (2) 7, 7, 9, 10 | 6 (1) 4, 5, 6, 8 |
| | $AsymmMixN1$ | 22 (16) 11, 16, 32, 65 | 84 (114) 30, 53, 167, 968 | 32 (22) 17, 24, 46, 98 | 25 (17) 14, 19, 36, 70 | 22 (14) 11, 16, 30, 62 | 20 (18) 9, 14, 32, 101 |
| | $AsymmMixN2$ | 22 (15) 11, 16, 31, 64 | 40 (33) 20, 28, 61, 148 | 25 (16) 14, 19, 35, 68 | 22 (15) 12, 17, 32, 66 | 22 (16) 11, 16, 32, 71 | 28 (40) 11, 17, 57, 643 |
| $c_1 = 0.5$ | Dist / H | $H=1650$ | $H=24.5$ | $H=28$ | $H=28$ | $H=28$ | $H=24.5$ |
| | $N(0,1)$ | 59 (50) 29, 42, 92, 326 | 100 (126) 44, 66, 192, 2191 | 85 (100) 37, 56, 156, 1388 | 79 (99) 34, 52, 151, 1351 | 79 (103) 33, 51, 154, 1681 | 83 (125) 30, 51, 176, 2138 |
| | $EXP(1)$ | 35 (19) 21, 28, 47, 88 | 27 (12) 22, 22, 34, 57 | 40 (28) 23, 30, 58, 148 | 50 (48) 25, 35, 83, 378 | 66 (88) 27, 42, 130, 1352 | 105 (249) 32, 57, 306, 5141 |
| | $t(3)$ | 40 (23) 23, 31, 54, 102 | 73 (52) 40, 55, 107, 276 | 53 (31) 30, 41, 72, 145 | 46 (28) 27, 35, 63, 125 | 45 (30) 25, 34, 64, 145 | 47 (44) 22, 33, 77, 343 |
| | $DE(0, 1/\sqrt{2})$ | 46 (30) 25, 35, 65, 140 | 95 (99) 46, 66, 165, 1058 | 61 (39) 34, 47, 86, 173 | 48 (28) 28, 37, 65, 124 | 44 (31) 25, 33, 64, 197 | 61 (88) 24, 39, 127, 1538 |
| $GAM(0.5,1)$ | 23 (9) 16, 20, 29, 41 | 22 (0) 22, 22, 22, 22 | 16 (0) 16, 16, 16, 33 | 25 (21) 13, 18, 39, 104 | 46 (50) 20, 51, 81, 557 | 93 (216) 29, 51, 267, 4501 | |

| | | | | | | | |
|--------------|--------------------|-----------------------------|--------------------------------|-------------------------------|-------------------------------|-------------------------------|--------------------------------|
| $c_1 = 0.25$ | <i>LogN(0,1)</i> | 22 (7) 16, 19, 26, 37 | 22 (0) 22, 22, 22, 23 | 20 (6) 16, 17, 23, 33 | 23 (11) 15, 19, 30, 52 | 31 (23) 17, 23, 46, 108 | 57 (86) 22, 35, 121, 1625 |
| | <i>SymmMixN</i> | 18 (5) 14, 16, 21, 25 | 78 (26) 52, 66, 92, 119 | 29 (4) 23, 27, 31, 36 | 20 (3) 17, 19, 22, 24 | 15 (2) 13, 14, 16, 19 | 11 (2) 9, 10, 12, 15 |
| | <i>AsymmMixN1</i> | 37 (19) 22, 30, 49, 83 | 103 (80) 54, 75, 155, 423 | 52 (26) 33, 42, 68, 113 | 41 (19) 26, 34, 53, 91 | 37 (20) 23, 29, 49, 85 | 36 (25) 20, 27, 52, 122 |
| | <i>AsymmMixN2</i> | 38 (19) 22, 30, 49, 82 | 62 (33) 39, 49, 82, 144 | 43 (20) 28, 35, 55, 91 | 38 (18) 23, 31, 49, 85 | 37 (21) 22, 29, 50, 97 | 48 (53) 22, 33, 86, 537 |
| | <i>Dist / H</i> | H=2640 | H=38 | H=43 | H=45 | H=43 | H=38 |
| | <i>N(0,1)</i> | 83 (60) 45, 62, 122, 317 | 128 (130) 64, 90, 220, 1383 | 111 (107) 55, 78, 185, 989 | 108 (105) 53, 76, 181, 962 | 103 (105) 48, 71, 176, 860 | 108 (133) 46, 71, 204, 1377 |
| | <i>EXP(1)</i> | 52 (25) 33, 42, 67, 111 | 39 (15) 33, 33, 48, 77 | 56 (35) 33, 43, 78, 171 | 72 (57) 38, 53, 110, 354 | 89 (93) 41, 60, 153, 808 | 133 (230) 47, 79, 309, 2877 |
| | <i>t(3)</i> | 58 (29) 36, 47, 76, 127 | 97 (56) 58, 76, 132, 286 | 72 (37) 45, 58, 95, 173 | 67 (35) 41, 53, 88, 161 | 63 (36) 37, 49, 85, 163 | 67 (56) 34, 48, 104, 322 |
| | <i>DE(0, 1/√2)</i> | 67 (38) 39, 52, 90, 175 | 123 (98) 65, 91, 189, 647 | 83 (46) 50, 66, 112, 199 | 69 (31) 44, 56, 87, 148 | 60 (37) 37, 48, 85, 208 | 84 (97) 36, 56, 152, 1071 |
| | <i>GAM(0.5,1)</i> | 35 (12) 25, 30, 42, 57 | 33 (0) 33, 33, 33, 33 | 23 (0) 23, 23, 23, 48 | 39 (31) 19, 27, 58, 140 | 64 (62) 29, 45, 107, 478 | 125 (227) 43, 73, 300, 2779 |
| $c_1 = 1$ | <i>LogN(0,1)</i> | 33 (10) 24, 29, 39, 53 | 33 (0) 33, 33, 33, 35 | 28 (8) 23, 25, 33, 47 | 35 (16) 23, 28, 44, 72 | 44 (28) 26, 34, 62, 130 | 79 (97) 33, 51, 148, 991 |
| | <i>SymmMixN</i> | 27 (6) 21, 25, 31, 37 | 103 (25) 74, 91, 116, 142 | 41 (5) 35, 39, 44, 49 | 30 (4) 26, 28, 32, 35 | 23 (2) 20, 22, 24, 26 | 16 (3) 14, 15, 18, 23 |
| | <i>AsymmMixN1</i> | 55 (25) 35, 45, 70, 114 | 125 (73) 77, 102, 175, 355 | 72 (29) 48, 60, 89, 138 | 60 (25) 40, 50, 75, 114 | 53 (24) 34, 43, 67, 108 | 52 (33) 30, 40, 73, 153 |
| | <i>AsymmMixN2</i> | 56 (26) 35, 45, 71, 114 | 85 (36) 57, 70, 106, 171 | 60 (24) 41, 50, 74, 114 | 55 (24) 37, 46, 70, 110 | 53 (26) 33, 43, 69, 120 | 67 (62) 33, 48, 110, 470 |

Table 2. OOC run-length characteristics (target $MRL_0 = 350$, $m = 100$, $n = 5$, and $\delta = 0.50$)

| | <i>Dist / H</i> | CUSUM-Rank | CUSUM-EX | | | | |
|-----------|--------------------|-------------------------|--|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| | | | $r = 25^{\text{th}}$ percentile | $r = 40^{\text{th}}$ percentile | $r = 50^{\text{th}}$ percentile | $r = 60^{\text{th}}$ percentile | $r = 75^{\text{th}}$ percentile |
| $c_1 = 2$ | <i>Dist / H</i> | H=175 | H=2.71 | H=2.809 | H=2.900 | H=2.730 | H=2.71 |
| | <i>N(0,1)</i> | 11 (14) 3, 6, 20, 59 | 923 (2487) 46, 273, 2760, 9163 | 40 (79) 7, 17, 96, 364 | 20 (34) 4, 10, 44, 143 | 16 (26) 4, 8, 34, 112 | 12 (19) 3, 6, 25, 79 |
| | <i>EXP(1)</i> | 6 (5) 3, 4, 9, 22 | 11 (0) 11, 11, 11, 11 | 4 (1) 4, 4, 5, 14 | 5 (6) 3, 3, 9, 32 | 8 (12) 3, 5, 17, 70 | 14 (30) 3, 6, 36, 154 |
| | <i>t(3)</i> | 5 (5) 2, 4, 9, 18 | 353 (792) 30, 130, 922, 3490 | 14 (18) 5, 8, 26, 63 | 8 (8) 3, 5, 13, 28 | 6 (6) 3, 4, 10, 21 | 5 (6) 2, 3, 9, 22 |
| | <i>DE(0, 1/√2)</i> | 7 (8) 2, 4, 12, 28 | 1151 (2677) 63, 379, 3056, 9218 | 23 (32) 6, 12, 44, 113 | 10 (11) 3, 6, 17, 36 | 7 (7) 3, 4, 11, 24 | 6 (8) 2, 4, 12, 40 |
| | <i>GAM(0.5,1)</i> | 4 (2) 2, 3, 5, 8 | 11 (0) 11, 11, 11, 11 | 4 (0) 4, 4, 4, 4 | 3 (0) 3, 3, 3, 3 | 2 (3) 2, 2, 5, 19 | 11 (26) 2, 5, 31, 146 |
| | <i>LogN(0,1)</i> | 3 (2) 2, 2, 4, 5 | 11 (0) 11, 11, 11, 11 | 4 (0) 4, 4, 4, 4 | 3 (0) 3, 3, 3, 3 | 2 (1) 2, 2, 3, 6 | 5 (6) 1, 3, 9, 40 |
| | <i>SymmMixN</i> | 3 (2) 2, 2, 4, 7 | 10531 (17141) 759, 4158, 21298, 49458 | 12 (11) 5, 8, 19, 36 | 5 (3) 3, 3, 6, 10 | 3 (1) 2, 3, 4, 5 | 2 (1) 1, 1, 2, 3 |
| | <i>AsymmMixN1</i> | 5 (5) 2, 3, 8, 16 | 10644 (24253) 576, 3552, 27805, 96918 | 24 (32) 6, 13, 45, 101 | 9 (7) 3, 6, 13, 27 | 6 (4) 3, 4, 8, 15 | 4 (3) 1, 3, 6, 11 |
| | <i>AsymmMixN2</i> | 5 (4) 2, 4, 8, 17 | 322 (586) 32, 131, 717, 2234 | 10 (10) 5, 7, 17, 34 | 6 (5) 3, 4, 9, 17 | 5 (4) 2, 3, 7, 13 | 4 (5) 1, 3, 8, 23 |
| $c_1 = 1$ | <i>Dist / H</i> | H=390 | H=5.75 | H=6.5 | H=6.6 | H=6.5 | H=5.75 |
| | <i>N(0,1)</i> | 10 (8) 5, 7, 15, 30 | 27 (32) 11, 17, 49, 170 | 18 (19) 7, 11, 30, 81 | 15 (14) 7, 10, 24, 65 | 13 (13) 5, 9, 22, 57 | 12 (13) 4, 8, 21, 61 |
| | <i>EXP(1)</i> | 6 (3) 4, 5, 8, 12 | 8 (0) 8, 8, 8, 8 | 5 (1) 5, 5, 6, 11 | 7 (4) 4, 5, 9, 19 | 9 (8) 4, 6, 14, 39 | 15 (23) 4, 9, 32, 167 |
| | <i>t(3)</i> | 7 (4) 4, 5, 9, 13 | 20 (14) 10, 15, 29, 57 | 11 (5) 6, 9, 14, 24 | 9 (4) 4, 7, 11, 18 | 7 (4) 4, 6, 10, 16 | 6 (4) 4, 5, 9, 18 |
| | <i>DE(0, 1/√2)</i> | 8 (5) 4, 6, 11, 18 | 28 (29) 12, 19, 48, 131 | 14 (9) 6, 10, 19, 33 | 10 (5) 5, 8, 13, 21 | 8 (4) 4, 6, 10, 16 | 8 (7) 4, 5, 12, 32 |
| | <i>GAM(0.5,1)</i> | 5 (2) 4, 4, 6, 7 | 8 (0) 8, 8, 8, 8 | 5 (0) 5, 5, 5, 5 | 4 (0) 4, 4, 4, 4 | 3 (3) 3, 3, 6, 15 | 12 (20) 4, 7, 27, 158 |
| | <i>LogN(0,1)</i> | 4 (1) 3, 4, 5, 6 | 8 (0) 8, 8, 8, 8 | 5 (0) 5, 5, 5, 5 | 4 (0) 4, 4, 4, 4 | 4 (1) 3, 3, 4, 7 | 6 (6) 3, 4, 10, 30 |

| | | | | | | | |
|-------------------|--------------------------|---------------------------|----------------------------|---------------------------|---------------------------|---------------------------|----------------------------|
| $c_1 = 0.5$ | <i>SymmMixN</i> | 5 (2) 3, 4, 6, 7 | 46 (45) 16, 29, 74, 138 | 9 (4) 6, 8, 12, 16 | 6 (1) 4, 6, 7, 9 | 4 (1) 3, 4, 5, 6 | 3 (0) 2, 3, 3, 4 |
| | <i>AsymmMixN1</i> | 6 (3) 4, 5, 8, 13 | 47 (57) 16, 28, 85, 206 | 12 (6) 7, 10, 16, 25 | 9 (4) 5, 7, 11, 15 | 7 (4) 4, 5, 9, 12 | 5 (3) 3, 4, 7, 11 |
| | <i>AsymmMixN2</i> | 6 (3) 4, 5, 8, 13 | 19 (11) 11, 14, 25, 42 | 9 (4) 6, 7, 11, 16 | 7 (3) 4, 6, 9, 13 | 6 (3) 4, 5, 8, 11 | 6 (5) 3, 4, 9, 18 |
| | Dist / H | H=800 | H=12 | H=13.5 | H=13.5 | H=13.5 | H=12 |
| | <i>N(0,1)</i> | 15 (8) 9, 12, 20, 32 | 30 (18) 18, 24, 42, 80 | 23 (13) 13, 18, 31, 60 | 20 (13) 11, 15, 28, 52 | 20 (13) 10, 14, 27, 52 | 19 (14) 9, 13, 27, 57 |
| | <i>EXP(1)</i> | 10 (3) 7, 9, 12, 16 | 13 (0) 13, 13, 13, 13 | 8 (2) 8, 8, 10, 15 | 10 (6) 7, 8, 14, 23 | 14 (10) 7, 10, 20, 40 | 21 (22) 9, 14, 36, 157 |
| | <i>t(3)</i> | 11 (4) 7, 9, 13, 17 | 25 (10) 16, 21, 31, 45 | 15 (6) 11, 13, 19, 26 | 13 (5) 9, 11, 16, 22 | 11 (4) 8, 10, 14, 20 | 11 (5) 7, 9, 14, 23 |
| | <i>DE(0, 1/√2)</i> | 12 (5) 8, 10, 15, 22 | 31 (18) 18, 24, 42, 71 | 18 (8) 12, 15, 23, 32 | 14 (5) 9, 12, 17, 24 | 12 (5) 8, 10, 15, 21 | 12 (9) 7, 9, 18, 37 |
| | <i>GAM(0.5,1)</i> | 8 (2) 6, 7, 9, 11 | 13 (0) 13, 13, 13, 13 | 8 (0) 8, 8, 8, 8 | 7 (0) 7, 7, 7, 7 | 6 (4) 6, 6, 10, 20 | 18 (20) 7, 12, 32, 155 |
| | <i>LogN(0,1)</i> | 7 (2) 6, 6, 8, 10 | 13 (0) 13, 13, 13, 13 | 8 (0) 8, 8, 8, 8 | 7 (0) 7, 7, 7, 7 | 6 (1) 6, 6, 7, 10 | 10 (8) 5, 7, 15, 37 |
| $c_1 = 0.25$ | <i>SymmMixN</i> | 8 (2) 6, 7, 9, 11 | 39 (17) 23, 31, 48, 66 | 14 (3) 11, 13, 16, 19 | 10 (2) 8, 9, 11, 13 | 7 (1) 6, 7, 8, 9 | 5 (1) 6, 7, 8, 9 |
| | <i>AsymmMixN1</i> | 10 (3) 7, 9, 12, 17 | 40 (21) 23, 31, 52, 79 | 18 (6) 12, 15, 21, 27 | 13 (4) 9, 11, 15, 19 | 11 (4) 7, 9, 13, 16 | 9 (4) 6, 7, 11, 15 |
| | <i>AsymmMixN2</i> | 10 (3) 7, 9, 12, 17 | 24 (8) 16, 20, 28, 37 | 14 (4) 10, 12, 16, 20 | 11 (4) 8, 9, 13, 17 | 10 (3) 7, 8, 11, 16 | 10 (5) 6, 8, 13, 23 |
| | Dist / H | H=1610 | H=24 | H=27.5 | H=28.5 | H=27.5 | H=24 |
| | <i>N(0,1)</i> | 27 (11) 18, 22, 33, 47 | 46 (21) 31, 38, 59, 95 | 38 (16) 25, 32, 48, 78 | 36 (17) 23, 29, 46, 73 | 33 (17) 20, 27, 44, 73 | 31 (19) 17, 24, 43, 76 |
| | <i>EXP(1)</i> | 18 (5) 14, 16, 21, 27 | 22 (0) 22, 22, 22, 22 | 15 (4) 15, 15, 19, 26 | 20 (8) 14, 17, 25, 39 | 25 (14) 15, 19, 33, 58 | 36 (31) 17, 25, 56, 164 |
| | <i>t(3)</i> | 19 (5) 14, 17, 22, 29 | 39 (10) 29, 35, 45, 60 | 28 (7) 21, 25, 32, 40 | 24 (7) 18, 21, 28, 35 | 21 (7) 16, 18, 25, 32 | 19 (8) 12, 16, 24, 35 |
| | <i>DE(0, 1/√2)</i> | 22 (7) 15, 19, 26, 34 | 47 (19) 32, 39, 58, 86 | 32 (10) 24, 28, 38, 48 | 26 (7) 20, 23, 30, 38 | 22 (6) 16, 19, 25, 34 | 22 (12) 13, 17, 29, 54 |
| | <i>GAM(0.5,1)</i> | 15 (3) 12, 13, 16, 20 | 22 (0) 22, 22, 22, 22 | 15 (0) 15, 15, 15, 15 | 13 (0) 13, 13, 13, 13 | 10 (8) 10, 10, 18, 35 | 31 (29) 14, 21, 50, 155 |
| | <i>LogN(0,1)</i> | 13 (2) 11, 12, 14, 17 | 22 (0) 22, 22, 22, 22 | 15 (0) 15, 15, 15, 15 | 13 (0) 13, 13, 13, 14 | 11 (3) 10, 10, 13, 18 | 18 (12) 10, 14, 26, 53 |
| <i>SymmMixN</i> | 14 (3) 11, 13, 16, 19 | 56 (15) 40, 49, 64, 79 | 25 (3) 21, 24, 27, 31 | 18 (3) 16, 17, 20, 22 | 14 (2) 12, 13, 15, 16 | 9 (1) 8, 9, 10, 11 | |
| <i>AsymmMixN1</i> | 19 (5) 14, 16, 21, 27 | 57 (21) 39, 48, 69, 90 | 31 (7) 24, 27, 34, 41 | 23 (5) 19, 21, 26, 32 | 19 (5) 15, 17, 22, 26 | 16 (4) 11, 14, 18, 24 | |
| <i>AsymmMixN2</i> | 19 (5) 14, 16, 21, 27 | 37 (9) 29, 34, 43, 52 | 25 (5) 20, 22, 27, 32 | 20 (4) 17, 19, 23, 28 | 18 (4) 14, 16, 20, 26 | 17 (9) 12, 14, 23, 37 | |

Table 3. OOC run-length characteristics (target $MRL_0 = 350$, $m = 100$, $n = 5$, and $\delta = 0.75$)

| | CUSUM-Rank | CUSUM-EX | | | | | |
|-----------|--------------------|------------------------------------|------------------------------------|------------------------------------|------------------------------------|------------------------------------|------------------------|
| | | $r = 25^{\text{th}}$ percentile | $r = 40^{\text{th}}$ percentile | $r = 50^{\text{th}}$ percentile | $r = 60^{\text{th}}$ percentile | $r = 75^{\text{th}}$ percentile | |
| $c_1 = 2$ | Dist / H | H=98 | H=1.575 | H=1.350 | H=1.530 | H=1.350 | H=1.575 |
| | <i>N(0,1)</i> | 6 (8) 2, 3, 11, 25 | *** | 79 (158) 8, 31, 189, 675 | 11 (18) 2, 5, 23, 59 | 7 (10) 2, 4, 14, 33 | 6 (9) 1, 3, 12, 31 |
| | <i>EXP(1)</i> | 3 (3) 2, 2, 3, 13 | *** | 5 (0) 5, 5, 5, 5 | 2 (0) 2, 2, 2, 5 | 2 (2) 2, 2, 4, 12 | 6 (10) 1, 3, 13, 48 |
| | <i>t(3)</i> | 3 (2) 1, 2, 4, 9 | *** | 24 (33) 5, 12, 45, 105 | 5 (5) 2, 3, 8, 17 | 3 (3) 2, 2, 5, 9 | 3 (2) 1, 2, 4, 8 |
| | <i>DE(0, 1/√2)</i> | 4 (4) 1, 2, 6, 13 | *** | 48 (76) 7, 22, 98, 243 | 7 (8) 2, 4, 12, 24 | 4 (4) 2, 2, 6, 12 | 3 (3) 1, 2, 5, 11 |
| | <i>GAM(0.5,1)</i> | 2 (1) 2, 2, 3, 5 | *** | 5 (0) 5, 5, 5, 5 | 2 (0) 2, 2, 2, 2 | 2 (0) 2, 2, 2, 2 | 3 (6) 1, 2, 8, 36 |
| | <i>LogN(0,1)</i> | 2 (0) 1, 2, 2, 3 | *** | 5 (0) 5, 5, 5, 5 | 2 (0) 2, 2, 2, 2 | 2 (0) 2, 2, 2, 2 | 1 (2) 1, 1, 3, 6 |
| | <i>SymmMixN</i> | 2 (2) 1, 2, 4, 6 | *** | 46 (68) 8, 22, 90, 186 | 5 (5) 2, 3, 8, 15 | 2 (1) 2, 2, 3, 5 | 2 (1) 1, 1, 2, 3 |
| | <i>AsymmMixN1</i> | 3 (3) | *** | 85 (133) | 7 (9) | 3 (3) | 2 (1) |

| | | | | | | | | |
|--------------------|--------------------|-------------------------|-----------------------------|--------------------------|--------------------------|--------------------------|--------------------------|-------------------------|
| | | 1, 2, 5, 9 | | 10, 38, 171, 394 | 2, 4, 13, 27 | 2, 2, 5, 10 | 1, 2, 3, 5 | |
| | <i>AsymmMixN2</i> | 3 (3) 1, 2, 5, 9 | *** | 19 (25) 5, 10, 35, 74 | 4 (5) 2, 2, 7, 12 | 2 (2) 2, 2, 4, 7 | 2 (1) 1, 2, 3, 6 | |
| $c_l = 1$ | Dist / H | H=250 | H=3.5 | H=4.1 | H=4.2 | H=4.1 | H=3.5 | |
| | <i>N(0,1)</i> | 5 (3) 3, 4, 7, 12 | 20 (20) 8, 14, 34, 86 | 10 (7) 5, 8, 15, 30 | 8 (5) 4, 6, 11, 23 | 7 (6) 3, 4, 10, 20 | 5 (4) 2, 4, 8, 17 | |
| | <i>EXP(1)</i> | 4 (1) 3, 3, 4, 6 | 8 (0) 8, 8, 8, 8 | 4 (0) 4, 4, 4, 4 | 3 (0) 3, 3, 3, 5 | 3 (3) 2, 2, 5, 9 | 5 (6) 2, 3, 9, 26 | |
| | <i>t(3)</i> | 3 (1) 2, 3, 4, 6 | 16 (12) 8, 12, 24, 44 | 7 (3) 4, 5, 8, 13 | 5 (2) 3, 4, 6, 9 | 4 (2) 2, 3, 5, 8 | 3 (2) 2, 2, 4, 6 | |
| | <i>DE(0, 1/√2)</i> | 4 (2) 2, 3, 5, 8 | 25 (26) 10, 16, 42, 98 | 8 (4) 4, 7, 11, 19 | 6 (3) 3, 4, 7, 11 | 4 (3) 2, 3, 6, 9 | 3 (2) 2, 2, 4, 8 | |
| | <i>GAM(0.5,1)</i> | 3 (0) 2, 3, 3, 4 | 8 (0) 8, 8, 8, 8 | 4 (0) 4, 4, 4, 4 | 3 (0) 3, 3, 3, 3 | 2 (0) 2, 2, 2, 2 | 3 (4) 2, 2, 6, 21 | |
| | <i>LogN(0,1)</i> | 3 (1) 2, 2, 3, 3 | 8 (0) 8, 8, 8, 8 | 4 (0) 4, 4, 4, 4 | 3 (0) 3, 3, 3, 3 | 2 (0) 2, 2, 2, 2 | 2 (1) 2, 2, 3, 5 | |
| | <i>SymmMixN</i> | 3 (1) 2, 3, 4, 5 | 73 (98) 16, 37, 135, 282 | 8 (4) 4, 6, 10, 15 | 4 (2) 3, 4, 6, 7 | 3 (1) 2, 3, 4, 5 | 2 (0) 2, 2, 2, 3 | |
| | <i>AsymmMixN1</i> | 3 (1) 2, 3, 4, 6 | 65 (96) 14, 32, 128, 314 | 9 (5) 5, 7, 12, 19 | 6 (3) 4, 4, 7, 10 | 4 (2) 2, 3, 5, 7 | 3 (1) 2, 2, 3, 5 | |
| | <i>AsymmMixN2</i> | 3 (1) 2, 3, 4, 6 | 18 (13) 8, 13, 26, 46 | 7 (3) 4, 5, 8, 11 | 4 (2) 3, 4, 6, 7 | 3 (1) 2, 3, 4, 6 | 3 (1) 2, 2, 3, 5 | |
| | $c_l = 0.5$ | Dist / H | H=530 | H=7.8 | H=8.9 | H=8.9 | H=8.9 | H=7.8 |
| | | <i>N(0,1)</i> | 7 (3) 5, 6, 9, 13 | 16 (7) 11, 13, 20, 31 | 11 (6) 7, 9, 15, 22 | 10 (5) 6, 8, 13, 20 | 9 (5) 6, 7, 12, 18 | 8 (5) 4, 6, 11, 18 |
| <i>EXP(1)</i> | | 5 (1) 4, 5, 6, 8 | 9 (0) 9, 9, 9, 9 | 6 (0) 6, 6, 6, 6 | 5 (0) 5, 5, 5, 7 | 5 (3) 4, 4, 7, 11 | 8 (6) 4, 6, 12, 26 | |
| <i>t(3)</i> | | 5 (1) 4, 5, 6, 8 | 14 (5) 10, 12, 17, 23 | 9 (4) 7, 7, 11, 14 | 7 (2) 5, 6, 8, 11 | 6 (2) 4, 5, 7, 9 | 5 (2) 3, 4, 6, 8 | |
| <i>DE(0, 1/√2)</i> | | 6 (2) 4, 5, 7, 10 | 17 (8) 11, 14, 22, 32 | 10 (4) 7, 9, 13, 17 | 8 (2) 5, 7, 9, 12 | 6 (2) 4, 6, 8, 10 | 6 (3) 3, 4, 7, 10 | |
| <i>GAM(0.5,1)</i> | | 5 (1) 4, 4, 5, 6 | 9 (0) 9, 9, 9, 9 | 6 (0) 6, 6, 6, 6 | 5 (0) 5, 5, 5, 5 | 4 (0) 4, 4, 4, 4 | 6 (5) 3, 4, 9, 22 | |
| <i>LogN(0,1)</i> | | 4 (0) 4, 4, 4, 5 | 9 (0) 9, 9, 9, 9 | 6 (0) 6, 6, 6, 6 | 5 (0) 5, 5, 5, 5 | 4 (0) 4, 4, 4, 4 | 3 (1) 3, 3, 4, 8 | |
| <i>SymmMixN</i> | | 5 (2) 4, 4, 6, 7 | 24 (13) 14, 19, 32, 47 | 9 (3) 7, 8, 11, 14 | 7 (2) 5, 6, 8, 9 | 5 (1) 4, 5, 6, 7 | 3 (1) 3, 3, 4, 4 | |
| <i>AsymmMixN1</i> | | 5 (1) 4, 5, 6, 8 | 24 (12) 14, 19, 31, 47 | 11 (4) 7, 9, 13, 16 | 8 (2) 6, 7, 9, 11 | 6 (1) 4, 6, 7, 9 | 4 (1) 3, 4, 5, 6 | |
| <i>AsymmMixN2</i> | | 5 (1) 4, 5, 6, 8 | 15 (5) 11, 13, 18, 23 | 8 (2) 6, 7, 9, 12 | 6 (1) 5, 6, 7, 9 | 6 (1) 4, 5, 6, 8 | 4 (2) 3, 4, 6, 8 | |
| $c_l = 0.25$ | | Dist / H | H=1090 | H=16.2 | H=18.5 | H=18.5 | H=18.5 | H=16.2 |
| | | <i>N(0,1)</i> | 13 (4) 9, 11, 15, 19 | 24 (6) 19, 22, 28, 39 | 19 (6) 14, 17, 23, 30 | 17 (6) 12, 14, 20, 27 | 16 (6) 10, 13, 19, 26 | 14 (7) 9, 11, 18, 26 |
| | <i>EXP(1)</i> | 10 (2) 8, 9, 11, 13 | 33 (0) 33, 33, 33, 33 | 11 (0) 11, 11, 11, 11 | 9 (0) 9, 9, 9, 11 | 10 (4) 7, 8, 12, 18 | 14 (8) 8, 11, 19, 36 | |
| | <i>t(3)</i> | 10 (2) 7, 9, 11, 13 | 23 (4) 18, 21, 25, 31 | 15 (3) 13, 14, 17, 20 | 12 (3) 10, 11, 14, 16 | 10 (2) 9, 10, 12, 14 | 9 (2) 7, 8, 10, 13 | |
| | <i>DE(0, 1/√2)</i> | 11 (2) 8, 10, 12, 15 | 26 (7) 20, 23, 30, 40 | 17 (5) 13, 15, 20, 24 | 14 (3) 10, 12, 15, 19 | 12 (3) 9, 10, 13, 16 | 10 (3) 7, 8, 11, 16 | |
| | <i>GAM(0.5,1)</i> | 8 (1) 7, 8, 9, 10 | 16 (0) 16, 16, 16, 16 | 11 (0) 11, 11, 11, 11 | 9 (0) 9, 9, 9, 9 | 7 (0) 7, 7, 7, 7 | 10 (8) 5, 7, 15, 21 | |
| | <i>LogN(0,1)</i> | 7 (1) 6, 7, 8, 9 | 16 (0) 16, 16, 16, 16 | 11 (0) 11, 11, 11, 11 | 9 (0) 9, 9, 9, 9 | 7 (0) 7, 7, 7, 7 | 6 (3) 5, 5, 8, 13 | |
| | <i>SymmMixN</i> | 9 (2) 7, 8, 10, 12 | 34 (11) 23, 29, 40, 49 | 17 (3) 13, 15, 18, 21 | 12 (2) 10, 11, 13, 14 | 9 (1) 8, 9, 10, 11 | 6 (1) 5, 6, 7, 7 | |
| | <i>AsymmMixN1</i> | 10 (2) 7, 9, 11, 13 | 33 (12) 23, 28, 40, 50 | 18 (4) 14, 16, 20, 23 | 14 (2) 11, 12, 14, 17 | 10 (2) 9, 10, 12, 13 | 8 (2) 6, 7, 9, 10 | |
| | <i>AsymmMixN2</i> | 10 (2) 8, 9, 11, 13 | 23 (4) 19, 21, 25, 30 | 14 (2) 12, 13, 15, 18 | 11 (2) 10, 10, 12, 14 | 10 (1) 8, 9, 10, 12 | 8 (2) 6, 7, 9, 12 | |

*** Values could not be computed within a reasonable time. This indicates that the values are extremely large and the corresponding chart is performing poorly.

Table 4. OOC run-length characteristics (target $MRL_0 = 350$, $m = 100$, $n = 5$, and $\delta = 1.00$)

| | | CUSUM-Rank | CUSUM-EX | | | | |
|--------------|---------------------|--------------------------|------------------------------------|------------------------------------|---------------------------------|------------------------------------|------------------------------------|
| | | | $r = 25^{\text{th}}$ percentile | $r = 40^{\text{th}}$ percentile | $r = 50^{\text{th}}$ percentile | $r = 60^{\text{th}}$ percentile | $r = 75^{\text{th}}$ percentile |
| $c_I = 2$ | Dist / H | H=57 | H=0.75 | H=0.700 | H=0.390 | H=0.700 | H=0.75 |
| | N(0,1) | 4 (5) 1, 2, 7, 15 | *** | *** | 6 (8) 2, 3, 11, 25 | 3 (4) 1, 1, 5, 11 | 4 (6) 1, 2, 8, 18 |
| | EXP(1) | 3 (2) 1, 2, 4, 11 | *** | *** | 2 (0) 2, 2, 2, 2 | 1 (0) 1, 1, 1, 1 | 3 (4) 1, 2, 6, 19 |
| | t(3) | 2 (2) 1, 1, 3, 6 | *** | *** | 3 (3) 2, 2, 5, 10 | 1 (1) 1, 1, 2, 5 | 2 (2) 1, 1, 3, 5 |
| | DE(0, 1/√2) | 3 (2) 1, 2, 4, 9 | *** | *** | 4 (5) 2, 2, 7, 14 | 2 (2) 1, 1, 3, 6 | 2 (1) 1, 2, 3, 7 |
| | GAM(0.5,1) | 2 (0) 1, 2, 2, 5 | *** | *** | 2 (0) 2, 2, 2, 2 | 1 (0) 1, 1, 1, 1 | 1 (1) 1, 1, 2, 8 |
| | LogN(0,1) | 1 (1) 1, 1, 2, 2 | *** | *** | 2 (0) 2, 2, 2, 2 | 1 (0) 1, 1, 1, 1 | 1 (0) 1, 1, 1, 2 |
| | SymmMixN | 2 (2) 1, 1, 3, 6 | *** | *** | 4 (5) 2, 2, 7, 13 | 1 (1) 1, 1, 2, 4 | 1 (1) 1, 1, 2, 3 |
| | AsymmMixN1 | 2 (2) 1, 1, 3, 7 | *** | *** | 5 (6) 2, 3, 9, 19 | 2 (2) 1, 1, 3, 6 | 2 (2) 1, 1, 3, 4 |
| | AsymmMixN2 | 2 (2) 1, 1, 3, 7 | *** | *** | 3 (3) 2, 2, 5, 9 | 1 (1) 1, 1, 2, 4 | 2 (1) 1, 1, 2, 4 |
| $c_I = 1$ | Dist / H | H=175 | H=2.715 | H=2.701 | H=2.925 | H=2.701 | H=2.715 |
| | N(0,1) | 3 (2) 2, 2, 4, 7 | 30 (37) 11, 18, 55, 149 | 7 (5) 4, 5, 10, 20 | 5 (4) 3, 3, 7, 13 | 4 (4) 2, 2, 6, 10 | 3 (3) 1, 2, 5, 9 |
| | EXP(1) | 2 (1) 2, 2, 3, 4 | 11 (0) 11, 11, 11, 11 | 4 (0) 4, 4, 4, 4 | 3 (0) 3, 3, 3, 3 | 2 (0) 2, 2, 2, 2 | 3 (2) 1, 2, 4, 10 |
| | t(3) | 2 (1) 2, 2, 3, 4 | 26 (26) 11, 17, 43, 88 | 6 (2) 4, 5, 7, 10 | 3 (1) 3, 3, 4, 6 | 2 (1) 2, 2, 3, 5 | 2 (2) 1, 1, 3, 4 |
| | DE(0, 1/√2) | 3 (1) 2, 2, 3, 5 | 48 (69) 13, 26, 95, 242 | 7 (4) 4, 5, 9, 15 | 4 (3) 3, 3, 6, 8 | 2 (2) 2, 2, 4, 6 | 2 (1) 1, 2, 3, 4 |
| | GAM(0.5,1) | 2 (0) 2, 2, 2, 3 | 11 (0) 11, 11, 11, 11 | 4 (0) 4, 4, 4, 4 | 3 (0) 3, 3, 3, 3 | 2 (0) 2, 2, 2, 2 | 1 (1) 1, 1, 2, 6 |
| | LogN(0,1) | 2 (0) 2, 2, 2, 2 | 11 (0) 11, 11, 11, 11 | 4 (0) 4, 4, 4, 4 | 3 (0) 3, 3, 3, 3 | 2 (0) 2, 2, 2, 2 | 1 (0) 1, 1, 1, 2 |
| | SymmMixN | 2 (1) 2, 2, 3, 4 | 405 (606) 45, 173, 779, 1610 | 7 (3) 4, 6, 9, 15 | 3 (2) 3, 3, 5, 7 | 2 (1) 2, 2, 3, 4 | 2 (1) 1, 1, 2, 3 |
| | AsymmMixN1 | 2 (1) 2, 2, 3, 4 | 278 (484) 32, 127, 611, 1438 | 8 (6) 4, 6, 12, 20 | 4 (3) 3, 3, 6, 9 | 2 (2) 2, 2, 4, 5 | 2 (2) 1, 1, 3, 3 |
| | AsymmMixN2 | 2 (1) 2, 2, 3, 4 | 35 (39) 12, 21, 60, 118 | 5 (2) 4, 5, 7, 9 | 3 (1) 3, 3, 4, 6 | 2 (1) 2, 2, 3, 4 | 2 (1) 1, 1, 2, 3 |
| $c_I = 0.5$ | Dist / H | H=390 | H=5.8 | H=6.5 | H=6.5 | H=6.5 | H=5.8 |
| | N(0,1) | 4 (1) 3, 4, 5, 7 | 11 (4) 8, 10, 14, 20 | 7 (3) 5, 6, 9, 13 | 6 (3) 4, 5, 8, 11 | 5 (3) 4, 4, 7, 10 | 4 (2) 3, 4, 6, 9 |
| | EXP(1) | 4 (1) 3, 3, 4, 5 | 8 (0) 8, 8, 8, 8 | 5 (0) 5, 5, 5, 5 | 4 (0) 4, 4, 4, 4 | 3 (0) 3, 3, 3, 4 | 4 (2) 2, 3, 5, 10 |
| | t(3) | 3 (1) 3, 3, 4, 5 | 11 (3) 8, 10, 13, 17 | 6 (1) 5, 6, 7, 9 | 5 (1) 4, 4, 5, 7 | 4 (0) 3, 4, 4, 6 | 3 (1) 2, 3, 4, 5 |
| | DE(0, 1/√2) | 4 (2) 3, 3, 5, 6 | 12 (5) 8, 11, 16, 23 | 7 (3) 5, 6, 9, 11 | 5 (2) 4, 4, 6, 8 | 4 (1) 3, 4, 5, 7 | 4 (1) 2, 3, 4, 5 |
| | GAM(0.5,1) | 3 (1) 3, 3, 4, 4 | 8 (0) 8, 8, 8, 8 | 5 (0) 5, 5, 5, 5 | 4 (0) 4, 4, 4, 4 | 3 (0) 3, 3, 3, 3 | 2 (1) 2, 2, 3, 7 |
| | LogN(0,1) | 3 (0) 3, 3, 3, 4 | 8 (0) 8, 8, 8, 8 | 5 (0) 5, 5, 5, 5 | 4 (0) 4, 4, 4, 4 | 3 (0) 3, 3, 3, 3 | 2 (0) 2, 2, 2, 3 |
| | SymmMixN | 4 (1) 3, 3, 4, 5 | 19 (10) 11, 15, 25, 37 | 7 (3) 5, 6, 9, 11 | 5 (2) 4, 4, 6, 7 | 4 (0) 3, 4, 4, 5 | 3 (1) 2, 2, 3, 4 |
| | AsymmMixN1 | 4 (1) 3, 3, 4, 5 | 18 (10) 10, 14, 24, 37 | 8 (3) 6, 6, 9, 12 | 5 (2) 4, 5, 7, 8 | 4 (1) 3, 4, 5, 6 | 3 (1) 2, 3, 4, 4 |
| | AsymmMixN2 | 4 (1) 3, 3, 4, 5 | 12 (4) 8, 10, 14, 19 | 6 (1) 5, 6, 7, 9 | 4 (1) 4, 4, 5, 7 | 4 (0) 3, 4, 4, 5 | 3 (0) 2, 3, 3, 4 |
| $c_I = 0.25$ | Dist / H | H=797 | H=11.75 | H=13.511 | H=13.905 | H=13.511 | H=11.75 |
| | N(0,1) | 7 (2) 6, 7, 9, 11 | 15 (3) 12, 14, 17, 22 | 12 (3) 9, 11, 14, 17 | 11 (3) 8, 9, 12, 15 | 9 (3) 7, 8, 11, 14 | 8 (3) 5, 6, 9, 13 |
| EXP(1) | 6 (1) 5, 6, 7, 8 | 12 (0) 12, 12, 12, 12 | 8 (0) 8, 8, 8, 8 | 7 (0) 7, 7, 7, 7 | 6 (0) 6, 6, 6, 7 | 6 (4) 4, 5, 9, 14 | |

| | | | | | | |
|---------------------|---------------------|--------------------------|--------------------------|------------------------|----------------------|----------------------|
| $t(3)$ | 6 (2) 5, 5, 7, 8 | 15 (3) 12, 14, 17, 20 | 10 (1) 8, 10, 11, 13 | 8 (1) 7, 8, 9, 11 | 7 (2) 6, 6, 8, 9 | 5 (1) 4, 5, 6, 7 |
| $DE(0, 1/\sqrt{2})$ | 7 (1) 5, 6, 7, 9 | 17 (4) 13, 15, 19, 24 | 11 (3) 9, 10, 13, 15 | 9 (3) 7, 8, 11, 12 | 7 (1) 6, 7, 8, 10 | 6 (1) 4, 5, 6, 8 |
| $GAM(0.5,1)$ | 6 (1) 5, 5, 6, 7 | 12 (0) 12, 12, 12, 12 | 8 (0) 8, 8, 8, 8 | 7 (0) 7, 7, 7, 7 | 6 (0) 6, 6, 6, 6 | 4 (1) 4, 4, 5, 11 |
| $LogN(0,1)$ | 5 (0) 5, 5, 5, 6 | 12 (0) 12, 12, 12, 12 | 8 (0) 8, 8, 8, 8 | 7 (0) 7, 7, 7, 7 | 6 (0) 6, 6, 6, 6 | 4 (0) 4, 4, 4, 4 |
| $SymmMixN$ | 6 (2) 5, 5, 7, 8 | 22 (7) 15, 19, 26, 34 | 12 (2) 10, 11, 13, 15 | 9 (2) 7, 8, 10, 11 | 7 (1) 6, 6, 7, 8 | 4 (1) 4, 4, 5, 5 |
| $AsymmMixN1$ | 6 (1) 5, 6, 7, 8 | 21 (7) 15, 18, 25, 33 | 13 (3) 10, 11, 14, 16 | 10 (2) 8, 9, 11, 12 | 7 (1) 6, 7, 8, 9 | 5 (1) 4, 5, 6, 6 |
| $AsymmMixN2$ | 6 (1) 5, 6, 7, 8 | 16 (4) 13, 14, 18, 21 | 10 (1) 8, 10, 11, 13 | 8 (1) 7, 8, 9, 10 | 7 (1) 6, 6, 7, 8 | 5 (1) 4, 4, 5, 6 |

*** Values could not be computed within a reasonable time. This indicates that the values are extremely large and the corresponding chart is performing poorly.

The observations from Tables 1 to 4 are summarized in Table 5 along with some recommendations. Note that if two charts perform similarly, for example, if the NPCUSUM-EX chart based on the 50th and 60th percentiles perform similarly, the notation EX(50,60) is used.

Table 5. Summary of the efficacy of different reference sample percentiles for the NPCUSUM-EX chart

| Dist | Size of Location Shift | | | |
|-------------------------|--|--|---|---|
| | Small ($\delta \leq 0.25$) | Small to moderate ($0.25 < \delta \leq 0.50$) | Moderate to large ($0.50 < \delta \leq 0.75$) | Large ($0.75 < \delta \leq 1.00$) |
| Symmetric distributions | | | | |
| $N(0,1)$ | The Rank chart performs the best for all c_l | The Rank chart performs the best for all c_l | The Rank chart performs the best for $c_l = 0.25$ and 0.5 . For larger values of $c_l = 1$ and 2 the Rank and EX(75) charts perform similarly | The Rank chart and the higher order percentile based charts, EX(75) and EX(60), perform best. More specifically, for each c_l , the chart performing the best is: $c_l = 2$: EX(60) $c_l = 1$: Rank, EX(75) $c_l = 0.5$: Rank, EX(75) $c_l = 0.25$: Rank |
| $t(3)$ | The Rank chart performs the best for all c_l | The Rank and the EX(60,75) charts perform almost similarly and the best. More specifically, for each c_l , the chart performing the best is: $c_l = 2$: Rank, EX(75) $c_l = 1$: EX(75) $c_l = 0.5$: Rank, EX(60,75) $c_l = 0.25$: Rank, EX(75) | The Rank and the EX(75) charts perform similarly for all c_l except for $c_l = 0.25$ where the EX(75) chart performs best. | The Rank chart and the higher order percentile based charts, EX(75) and EX(60), perform best. More specifically, for each c_l , the chart performing the best is: $c_l = 2$: EX(60) $c_l = 1$: Rank, EX(60,75) $c_l = 0.5$: Rank, EX(75) $c_l = 0.25$: EX(75) |

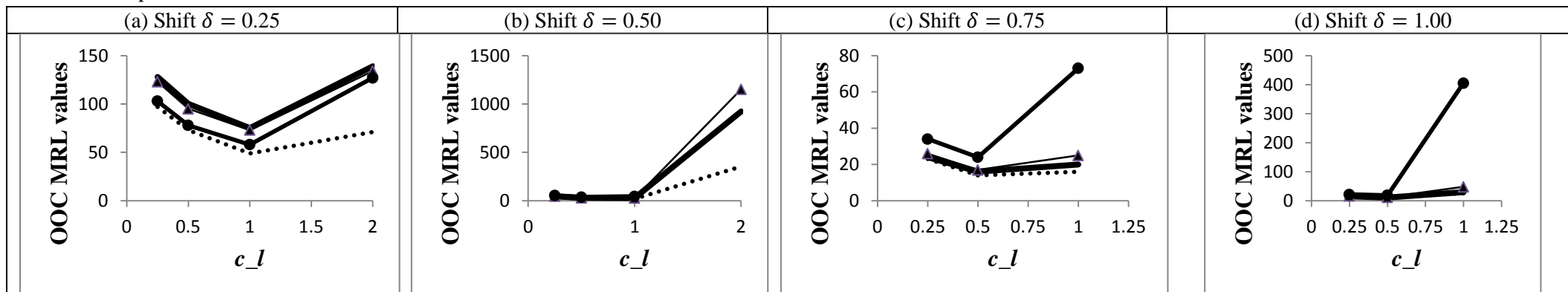
| | | | | |
|--------------------------|--|--|--|--|
| $DE(0,1/\sqrt{2})$ | The EX(60) chart performs the best for all c_l | The Rank and the EX(60,75) charts perform similarly and the best for all c_l | The EX(75) chart performs the best for most c_l , however, for $c_l = 0.5$, the Rank and the EX(60,75) charts perform similarly and the best | The Rank chart and the higher order percentile based charts, EX(75) and EX(60), perform best. More specifically, for each c_l , the chart performing the best is: $c_l = 2$: EX(75) $c_l = 1$: EX(75) $c_l = 0.5$: Rank, EX(60,75) $c_l = 0.25$: EX(75) |
| <i>SymmMixN</i> | The EX(75) chart performs the best for all c_l | The EX(75) chart performs the best for all c_l | The EX(75) chart performs the best for all c_l | The EX(75) chart performs the best for all c_l , except for $c_l = 1$ where the Rank and EX(60,75) charts perform similarly and $c_l = 2$ where the EX(60,75) charts perform similarly |
| Asymmetric distributions | | | | |
| $EXP(1)$ | The EX(25) chart performs the best for all c_l | The EX(40) chart performs the best for all c_l | The EX(50) chart has the overall advantage, though the EX(60) chart is equally good for higher values of c_l . The Rank chart can also be used for $c_l = 0.5$. | The Rank chart and the higher order percentile based charts, EX(75) and EX(60), perform best. More specifically, for each c_l , the chart performing the best is: $c_l = 2$: EX(60) $c_l = 1$: Rank, EX(60) $c_l = 0.5$: EX(60) $c_l = 0.25$: Rank, EX(60) |
| $GAM(0.5,1)$ | The EX(40) chart performs the best for all c_l | The EX(60) chart performs the best for all c_l | The EX(60) chart performs the best for all c_l , except for $c_l = 2$ where EX(50) and EX(60) perform similarly | The EX(75) chart performs the best for all c_l , except for $c_l = 2$ where EX(60) performs best |
| $LogN(0,1)$ | The EX(40) chart performs the best for all c_l | The EX(60) chart performs the best for most c_l , however, for $c_l = 1$, the Rank and the EX(50,60) charts perform similarly and the best | The EX(75) chart performs the best for most c_l , however, for $c_l = 1$, the EX(60,75) charts perform similarly and the best | The EX(75) chart performs the best for all c_l , except for $c_l = 2$ where the EX(75) and EX(60) charts perform similarly |
| 2 <i>AsymmMixN</i> | The EX(75) chart performs the best for <i>AsymmMixN1</i> , whereas the EX(60) chart is preferable for <i>AsymmMixN2</i> . Note that the Rank and EX(50) charts can also be used for <i>AsymmMixN2</i> when $c_l = 1$. | The Rank chart and the EX(60,75) charts perform similarly and the best for <i>AsymmMixN2</i> for all c_l except for $c_l = 0.25$ where EX(75) performs best. For <i>AsymmMixN1</i> the EX(75) chart always performs best for all c_l . | The EX(75) chart performs the best for all $c_l = 0.25$ and 0.5 . For larger values of $c_l = 1$ and 2 then Rank and EX(60) charts can also be implemented. | The EX(75) chart performs the best for all c_l , except for $c_l = 1$ and 2 where the Rank and EX(60,75) charts perform similarly |

From Table 5, we find that a NPCUSUM-EX chart using the reference sample percentiles other than the median can provide a useful nonparametric chart in practice. In fact, it appears that the EX(75) chart is a good overall chart for detecting a larger location shift for a wide class of distributions. This finding is an important contribution to the literature on exceedance/precedence tests and charts. Other reference sample percentiles, such as the 25th or the 40th, can also be used when a small shift is expected. It is worth mentioning that we recommend avoiding the choice of percentiles lower than the 25th or higher than the 75th percentile as there may be some problems with extreme values and higher variances, particularly for heavy-tailed distributions. Therefore, our recommendation is that the NPCUSUM-EX chart should be constructed choosing the appropriate order statistic from the Phase I sample that lies within the 25th and 75th percentile, both inclusive. Moreover, when we extend the comparison of the NPCUSUM-EX chart with the NPCUSUM-Rank chart, we find that the former performs better than the latter for some choices of the reference sample percentiles, for various well-known process distributions.

Choice of Reference Value

The reference value k is an important parameter of a CUSUM chart and is generally chosen based on the amount of the expected shift. In the parametric context, given the population distribution, one can choose the optimal k from the notion of a sequential probability ratio test; see Hawkins and Olwell (1998). Traditionally, for the parametric CUSUM chart it is recommended that smaller (larger) values of the reference value are preferable for detecting smaller (larger) shifts. In the nonparametric setting, however, the choice of such k is more complicated and we investigate this issue more thoroughly, starting with revisiting the parametric CUSUM chart. Note that when there is an expected shift, the goal is to choose that value of k which gives the shortest MRL_δ . Our simulation study suggests that for both parametric and nonparametric charts, when small shifts are under consideration, the MRL_δ values initially decrease, as the reference value k increases from 0, reaches a minimum and then sharply increases. Thus, roughly, the MRL_δ is a U-shaped function as a function of k . Since the MRL_0 is fixed at the same value, the k (or equivalently the c_l value) with the smallest or lowest MRL_δ is selected to be the winner. For the parametric CUSUM chart the U-shape (with ARL_δ on the vertical axis against k on the horizontal axis) is illustrated graphically in Figure 1 of Mukherjee et al. (2013). For the NPCUSUM-EX chart we illustrate the shape of the MRL_δ profile graphically in Figures 1 to 8 for different selected percentiles of the Phase I reference sample. In order to make the graphs more visually appealing, we separate the symmetric (see Figures 1 to 5) and asymmetric (see Figures 6 to 8) distributions. To summarize, in the nonparametric setting, the choice of k is not straightforward for an exceedance chart, since the choice depends on the reference sample

Figure 1. Out-of-control MRL_{δ} values for different symmetric distributions and under different choices of c_l for the NPCUSUM-EX chart based on the 25th percentile of the reference sample



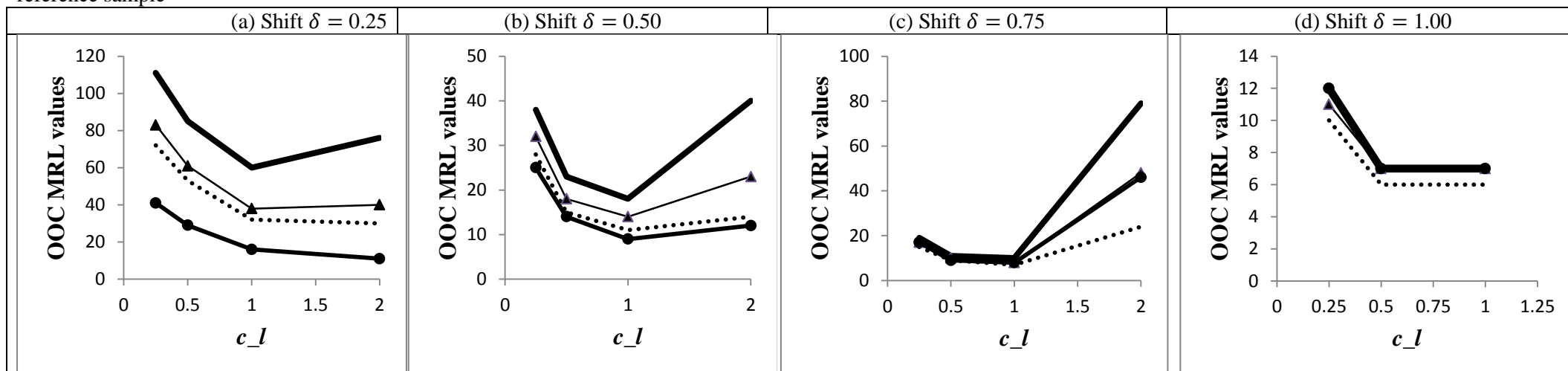
Notes on Figures:

Figure 1b. There is no marker for the *SymmMixN* distribution at $c_l = 2$, since the run-length characteristics could not be computed within a reasonable time, indicating that the run-length characteristics are extremely large and, consequently, $c_l = 2$ would not be chosen

Figures 1 c and d. There are no markers for $c_l = 2$, at any of the distributions. Reasoning follows along the lines as the note above.

Legend. — $N(0,1)$ $t(3)$ —▲— *DoubleExp* —●— *SymmMixN*

Figure 2. Out-of-control MRL_{δ} values for different symmetric distributions under different choices of c_l for the NPCUSUM-EX chart based on the 40th percentile of the reference sample



Note on Figure 2d. There is no marker for $c_l = 2$, since the run-length characteristics could not be computed within a reasonable time, indicating that the run-length characteristics are extremely large and, consequently, $c_l = 2$ would not be chosen.

Legend. — $N(0,1)$ $t(3)$ —▲— *DoubleExp* —●— *SymmMixN*

Figure 3. Out-of-control MRL_{δ} values for different symmetric distributions under different choices of c_l for the NPCUSUM-EX chart based median of the reference sample

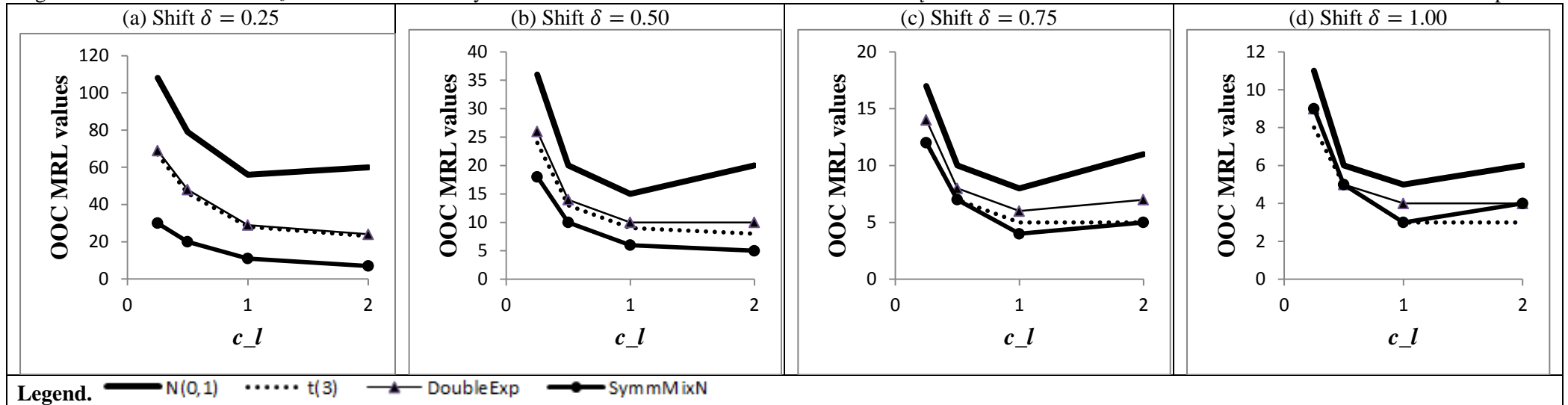


Figure 4. Out-of-control MRL_{δ} values for different symmetric distributions under different choices of c_l for the NPCUSUM-EX chart based on the 60th percentile of the reference sample

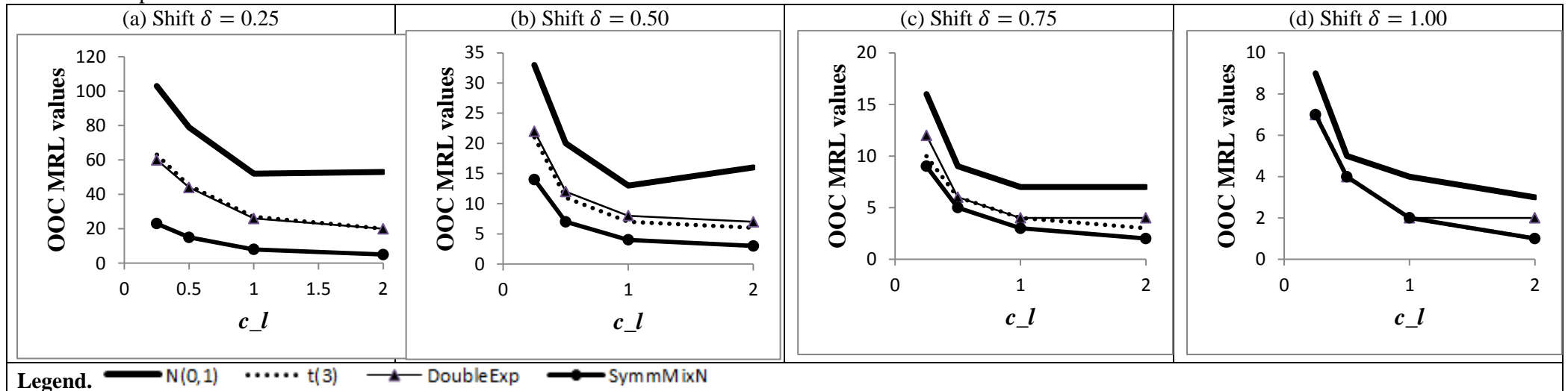


Figure 5. Out-of-control MRL_{δ} values for different symmetric distributions under different choices of c_l for the NPCUSUM-EX chart based on the 75th percentile of the reference sample

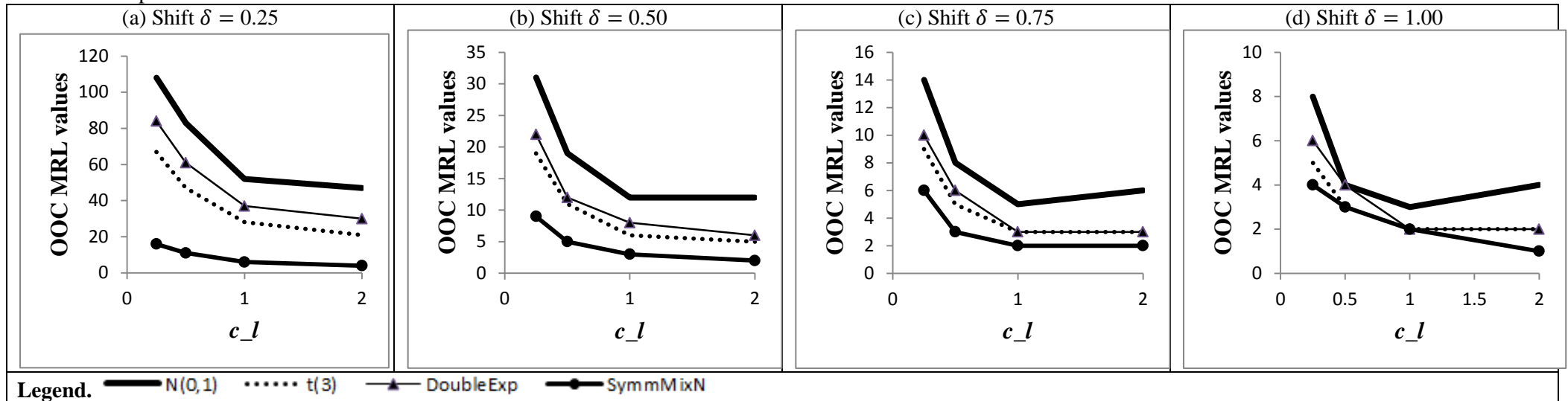
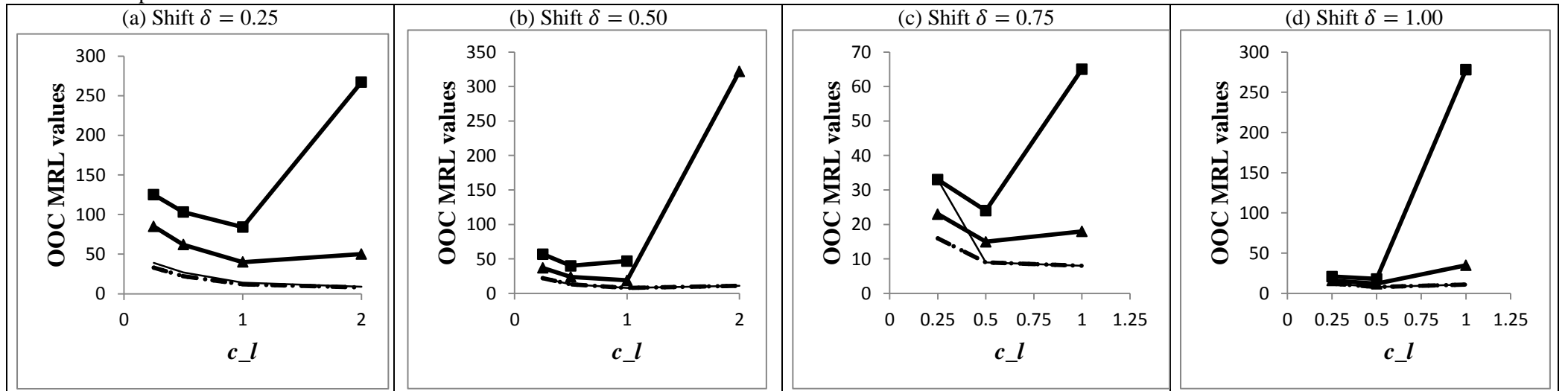


Figure 6. Out-of-control MRL_{δ} values for different asymmetric distributions and under different choices of c_l for the NPCUSUM-EX chart based on the 25th percentile of the reference sample



Notes on Figures:

Figure 6b. There is no marker for the $AsymmMixN1$ distribution at $c_l = 2$, since the run-length characteristics could not be computed within a reasonable time, indicating that the run-length characteristics are extremely large and, consequently, $c_l = 2$ would not be chosen

Figures 6 c and d. There are no markers for $c_l = 2$, at any of the distributions. Reasoning following the same lines as the note above.

Legend. — $EXP(1)$ $GAM(0.5,1)$ - - - $LogN$ —■— $AsymmMixN1$ —▲— $AsymmMixN2$

order statistic, the magnitude of shift that one wants to detect and the shape of the underlying process distribution. A detailed discussion is given below.

If we consider the MRL_δ values for various symmetric distributions under different choices of c_l , for the EX(25) chart we usually see that the MRL_δ values initially decline as c_l increases and reaches a minimum around $c_l = 1$ for small to moderate shifts ($\delta = 0.25$ and 0.50) (see Figures 1a,b) and around $c_l = 0.5$ for moderate to large shifts ($\delta = 0.75$ and 1.00) (see Figures 1c,d). After the minimum is reached the MRL_δ values begin to increase sharply. Thus, for symmetric distributions, the NPCUSUM charts have a U-shape function when the MRL_δ is plotted against the reference value k when the 25th percentile of the reference sample is used. However, for the *SymmMixN* distribution, the U-shape only holds for small to moderate shifts ($\delta = 0.25$ and 0.50) (see Figures 1a,b). That being said, the minimum MRL_δ values, for the *SymmMixN* distribution, is at the same c_l values as the other distributions for all shifts under consideration.

Subsequently, the graphs for the EX(40), EX(50), EX(60) and EX(75) charts are shown in Figures 2-5 for the symmetric distributions.

From Figure 2 we see that a U-shape is observed in all cases with the minimum at $c_l = 1$, except for the non-normal distributions when the desired magnitude of shift to be detected is small ($\delta = 0.25$) (see Figure 2a). From Figure 3 it can be seen that a U-shape isn't as prominent as with the NPCUSUM-EX chart based on smaller percentiles, but it is still seen in all cases for the normal distribution and it's very prominent for moderate to large shifts ($\delta = 0.75$ and 1.00) for the other distributions (see Figures 3c,d). For the EX(60) chart the U-shape isn't as prominent as with the NPCUSUM-EX chart based on smaller percentiles, as it is only clearly seen for the normal distribution when $\delta = 0.50$ (see Figure 4b). In all other cases it seems that larger values of k (where $c_l \geq 1$ and recall that $k = c_l \delta STDEV(U_{j,r})$) are to be preferred regardless of the size of this shift. Finally, for the EX(75) chart the U-shape isn't as prominent as with the NPCUSUM-EX chart based on smaller percentiles, as it is only clearly seen for the normal distribution when the shift is moderate to large ($\delta = 0.75$ and 1.00) (see Figures 5c,d). In all other cases it seems that larger values of k are preferred regardless of the size of this shift.

Next we consider the asymmetric distributions (see Figures 6, 7 and 8). It should be noted that the OOC performance of gamma and lognormal distributions are very similarly and, consequently, the plots of MRL_δ values for of these two distributions almost overlap on each other in Figures 6 to 8. We first consider the graphs for the EX(25) chart.

If we consider the MRL_δ values for various asymmetric distributions under different choices of c_l , for the EX(25) chart we usually see the U-shape pattern for c_l vs MRL_δ plot for moderate to

Figure 7. Out-of-control MRL_{δ} values for different asymmetric distributions under different choices of c_l for the NPCUSUM-EX chart based on the 40th percentile of the reference sample

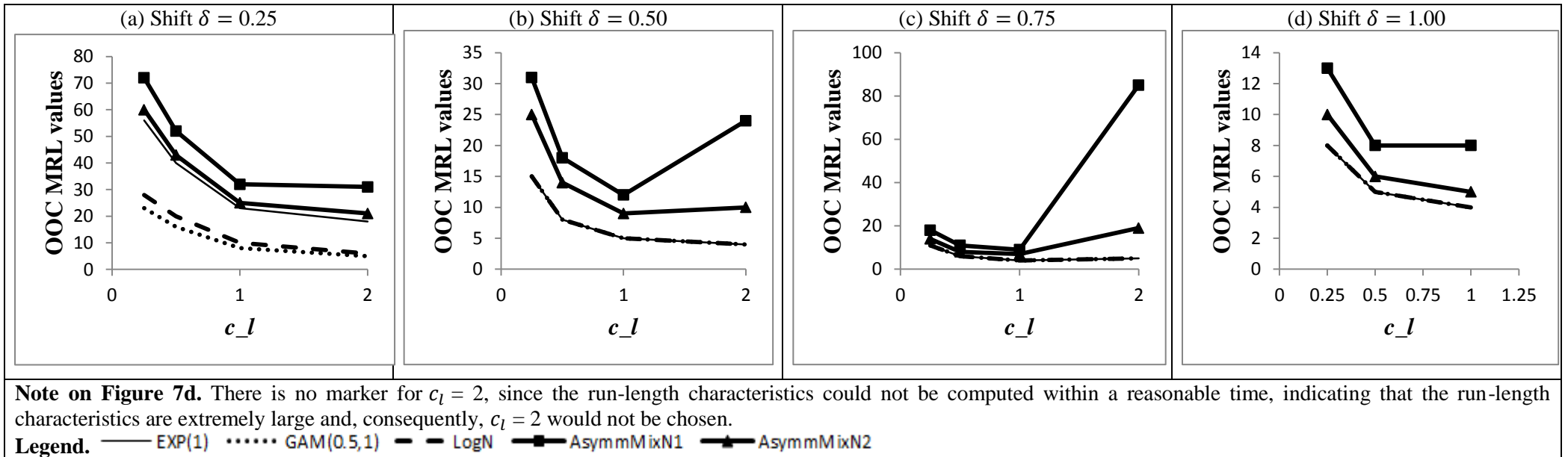
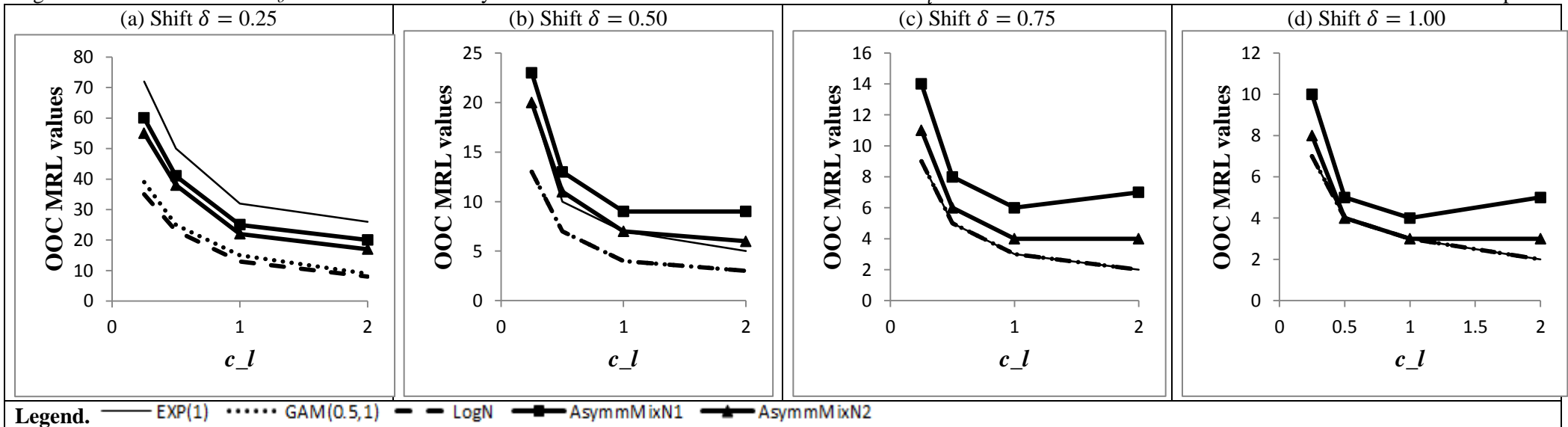


Figure 8. Out-of-control MRL_{δ} values for different asymmetric distributions under different choices of c_l for the NPCUSUM-EX chart based median of the reference sample



large shifts ($\delta = 0.75$ and 1.00) and MRL_δ reaches a minimum around $c_l = 0.5$ (see Figures 6c,d). However, for a small shift ($\delta = 0.25$) the U-shape isn't as prominent for the exponential, gamma and lognormal distributions, having very long right tails and, consequently, larger values of c_l are preferred. For a moderate shift of $\delta = 0.50$ the minimum seems to be at $c_l = 1.00$.

Similarly, for the EX(40) chart all the distributions have very long right tails and, consequently, larger values of c_l are preferred when a small shift ($\delta = 0.25$) is in question. For moderate ($\delta = 0.50$ and 0.75) all distributions have the U-shape, except for the exponential, gamma and lognormal, again showing very long right tails and, consequently, larger values of c_l are preferred[‡]. For a large shift ($\delta = 1.00$) the U-shape is prominent for all distributions and it reaches a minimum at $c_l = 1.00$ which, of course, is then the desired value to use.

From Figure 8 it is clear that for the EX(50) chart the U-shape isn't as prominent as with the NPCUSUM-EX based on smaller percentiles. All the distributions are showing very long right tails and, consequently, larger values of c_l are preferred. This also holds for the EX(60) and EX(75) charts and, consequently, these figures are omitted to conserve space.

For the NPCUSUM-Rank chart we illustrate the shape of the MRL_δ profile graphically for the symmetric (see Figure 9) and asymmetric (see Figure 10) distributions. For the NPCUSUM-Rank chart the U-shape pattern for c_l vs MRL_δ plot isn't as prominent as with the NPCUSUM-EX chart based on smaller percentiles, as it is only clearly seen for the normal distribution when the shift is moderate to large ($\delta = 0.75$ and 1.00) (see Figures 9c,d). In all other cases it seems that larger values of k are to be preferred regardless of the size of this shift.

Overall, it is seen that the EX(25) chart performs exceptionally well in detecting smaller shifts when the underlying process distribution is exponential. The EX(40) chart is appropriate for detection of smaller shifts when the underlying process distribution is lognormal or gamma and moderate shifts when the underlying process distribution is exponential. The EX(75) chart is the best for detecting larger shifts in almost all major univariate uni-modal bell-shaped distributions. Figures 1 to 8 and Tables 1 to 5 should be helpful for the practitioner in implementing the NPCUSUM-EX charts. To further assist the practitioner in implementing the chart and, more specifically, which value of c_l to choose, a summary of the discussions above are given in Table 6. The choice of k values was also investigated for the NPCUSUM-Rank chart and it seems that, in most cases, larger values of k are preferred regardless of the size of this shift for all distributions that are non-normal.

[‡] Recall that $k = c_l \delta STDEV(U_{j,r})$.

Figure 9. Out-of-control MRL_{δ} values for different symmetric distributions and under different choices of c_l for the NPCUSUM-Rank chart

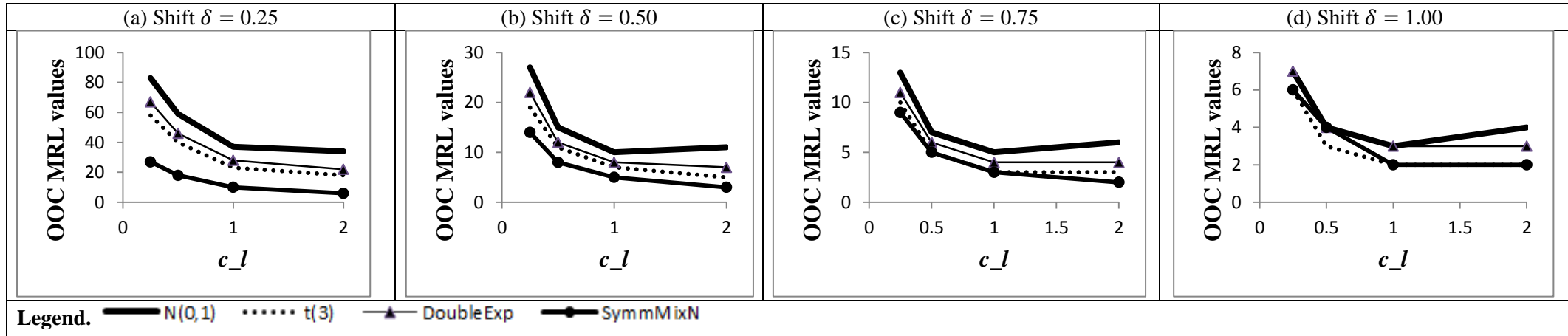


Figure 10. Out-of-control MRL_{δ} values for different asymmetric distributions and under different choices of c_l for the NPCUSUM-Rank chart

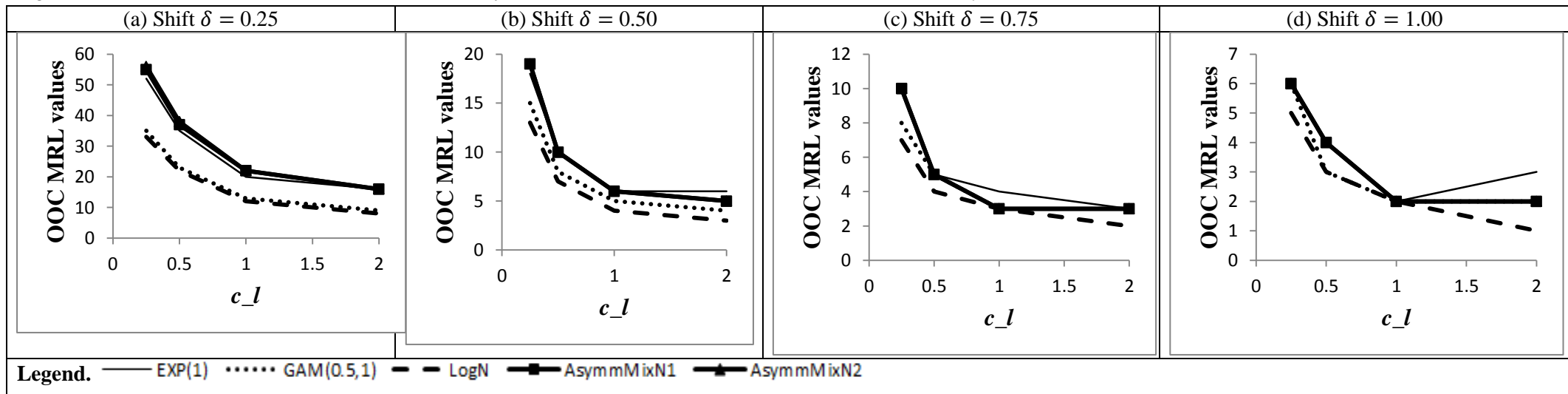


Table 6. Summary of the different reference sample percentiles and the choices of c_l

| If the practitioner has reason to suspect that the underlying process distribution is symmetric | | | | | |
|---|---|---|--|--|--|
| Desired shift / Chart | EX(25) | EX(40) | EX(50) | EX(60) | EX(75) |
| 0.25 | $c_l = 1.00$ | $c_l = 1.00$ | Larger values of c_l are preferred, preferably $c_l \geq 1$ | Larger values of c_l are preferred regardless of the size of this shift, preferably $c_l \geq 1$ | Larger values of c_l are preferred regardless of the size of this shift, preferably $c_l \geq 1$ |
| 0.50 | | | | | |
| 0.75 | $c_l = 0.50$ | $c_l = 1.00$ | $c_l = 1.00$ | | |
| 1.00 | | $c_l = 0.50$ or 1.00 | | | |
| If the practitioner has reason to suspect that the underlying process distribution is asymmetric | | | | | |
| 0.25 | Larger values of c_l are preferred, preferably $c_l \geq 1$ | Larger values of c_l are preferred, preferably $c_l \geq 1$ | Larger values of c_l are preferred regardless of the size of this shift, preferably $c_l \geq 1$ | Larger values of c_l are preferred regardless of the size of this shift, preferably $c_l \geq 1$ | Larger values of c_l are preferred regardless of the size of this shift, preferably $c_l \geq 1$ |
| 0.50 | $c_l = 1.00$ | | | | |
| 0.75 | $c_l = 0.50$ | $c_l = 1.00$ | | | |
| 1.00 | | | | | |

4. Example

We illustrate our proposals first using a well-known dataset from Montgomery (2001; Tables 5.1 and 5.2) on the inside diameters of piston rings manufactured by a forging process. The data given in Table 5.1 contains twenty-five retrospective or Phase I samples, each of size five, that were collected when the process was thought to be IC, i.e. $m = 125$. An analysis in Montgomery (2001) showed that these data are from an IC process and thus can be considered to be Phase I reference data. Note also that for these data, a goodness of fit test for normality is not rejected. This does not guarantee that the normality assumption for a parametric CUSUM chart is valid but often the practical implication is as such. For the EX(25), EX(40), EX(50), EX(60) and EX(75) charts we use $c_l = 1.00$ so that $k = \delta STDEV(U_{j,r})$. For comparison purposes we include the NPCUSUM-Rank chart with $c_l = 1.00$ so that $k = \delta STDEV(W_j)$. For all the charts we use a search algorithm to find the value of H such that $MRL_0 \approx 350$. Note that, the values of the respective reference sample percentiles are as follows: 25th percentile = 73.995, 40th percentile = 73.998, median = 74.001, 60th percentile = 74.004 and 75th percentile = 74.008. All of the measurements are in mm.

In order to calculate the Phase II exceedance control charts, we use the data in Table 5.2 of Montgomery (2001) that contains fifteen prospective (Phase II) samples each of five observations ($n = 5$). The desired shift to be detected was taken to be small $\delta = 0.25$ (see Figures 11 and 14 for the exceedance and rank charts, respectively), moderate $\delta = 0.50$ (see Figures 12 and 15 for the exceedance and rank charts, respectively) and large $\delta = 1.00$ (see Figures 13 and 16 for the exceedance and rank charts, respectively). Note that for Figures 11 to 13 the EX(25) and EX(40) charts weren't drawn, since they performed the worst overall. This also simplified the presentation in the figures (instead of 10 curves each figure now only has 8 curves). Also note that Figures 11 to 13 only show the exceedance charts and that the NPCUSUM-Rank charts are shown in Figures 14 to 16. The reason is that the vertical axes of the NPCUSUM-EX chart and that of the NPCUSUM-Rank chart differ dramatically and, accordingly, cannot be drawn on the same figure.

When the underlying process distribution is normal and we have a small shift ($\delta = 0.25$), Table 1 suggests that the NPCUSUM-Rank chart performs best followed by the EX(75) chart. From Figures 11 and 14 we can see that the EX(50) chart performs the worst and doesn't signal at all. The EX(60) chart almost signals, whereas the EX(75) chart actually does signal at sample number 14 with a counter of $N^+ = 6$, meaning that the shift most likely occurred at sample number 8. The same is true for the NPCUSUM-Rank chart, i.e. it signals at sample number 14 with a counter of $N^+ = 6$, meaning that the shift most likely occurred at sample number 8. In conclusion, both the EX(75) chart and the NPCUSUM-Rank chart perform similarly and are the best.

Figure 11. NPCUSUM-EX chart for Montgomery (2001) data with $\delta = 0.25$

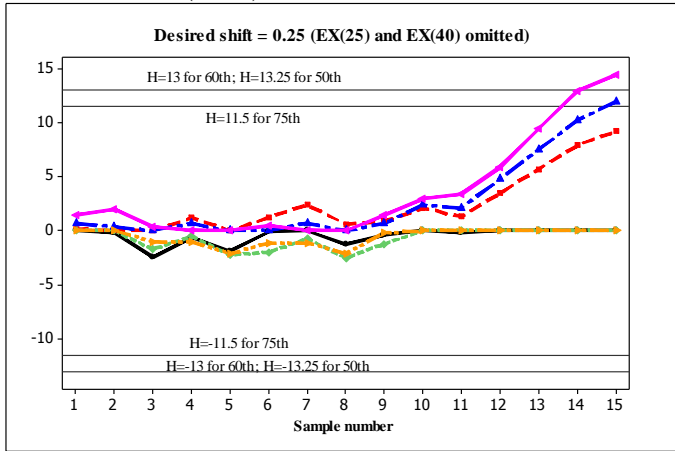


Figure 12. NPCUSUM-EX chart for Montgomery (2001) data with $\delta = 0.50$

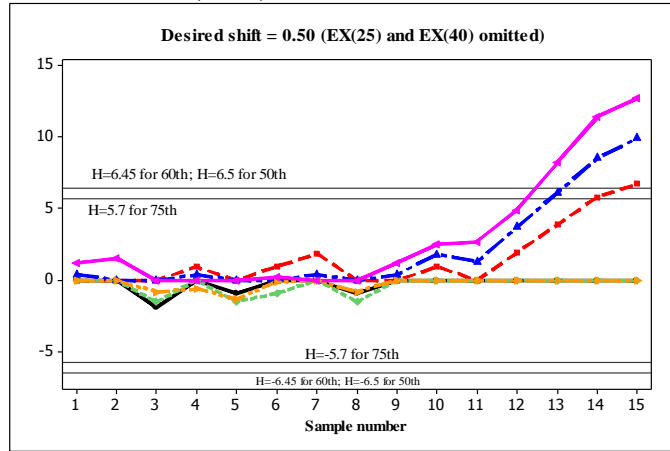
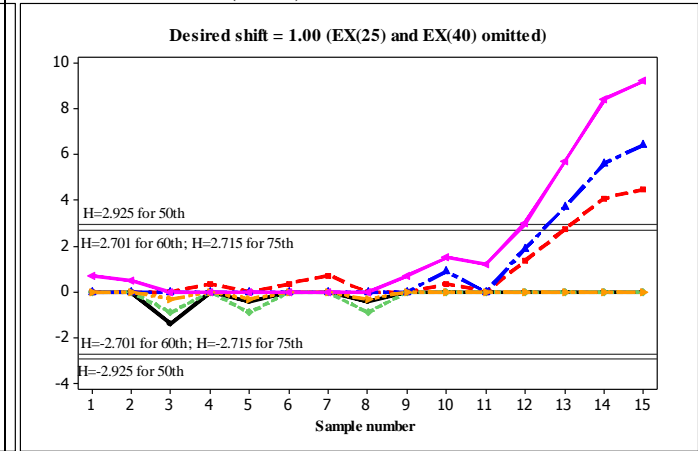


Figure 13. NPCUSUM-EX chart for Montgomery (2001) with $\delta = 1.00$



Legend: \bullet — $C_{j_minus\ 50th}$ \blacksquare — $C_{j_plus\ 50th}$ \blacklozenge — $C_{j_minus\ 60th}$ \blacktriangle — $C_{j_plus\ 60th}$ \blacktriangleright — $C_{j_minus\ 75th}$ \blacktriangleleft — $C_{j_plus\ 75th}$

Figure 14. NPCUSUM-Rank chart for Montgomery (2001) data with $\delta = 0.25$

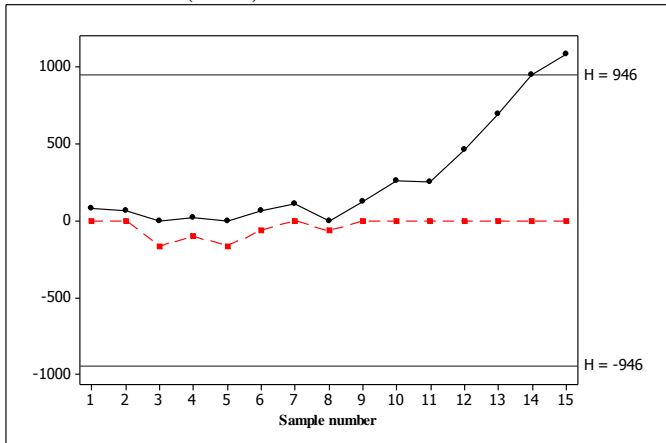


Figure 15. NPCUSUM-Rank chart for Montgomery (2001) data with $\delta = 0.50$

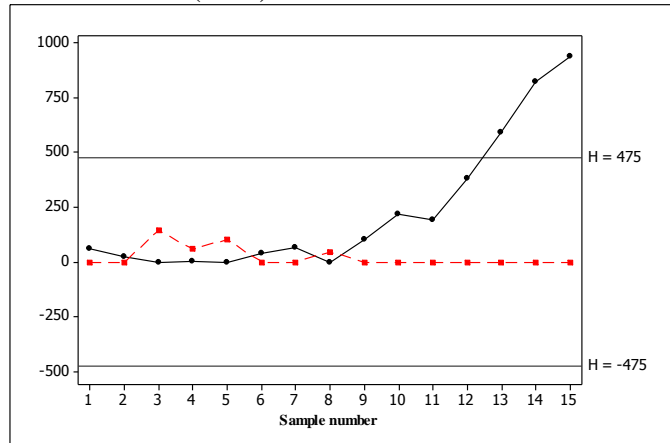
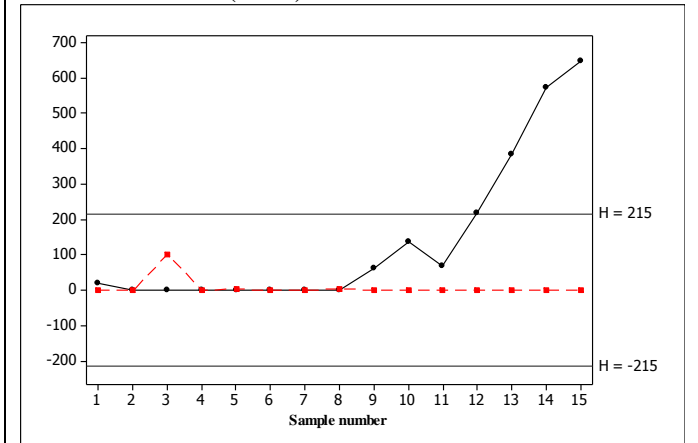


Figure 16. NPCUSUM-Rank chart for Montgomery (2001) with $\delta = 1.00$



Legend: Solid line = C_j^+ and Dotted line = C_j^-

Again, for a normally distributed process and for a moderate shift ($\delta = 0.50$), Table 5 suggests that NPCUSUM-Rank performs best. From Figures 12 and 15 we can see that the EX(50) chart signals at sample number 15 with a counter of $N^+ = 4$, indicating that the shift most likely occurred at sample number 11. The EX(60) chart signals at sample number 14 with a counter of $N^+ = 5$, indicating that the shift most likely occurred at sample number 9. The EX(75) chart and the NPCUSUM-Rank chart signal at the earliest, at sample number 13, both with a counter of $N^+ = 5$, indicating that the shift most likely occurred at sample number 8. Thus, we find that the EX(75) chart and the NPCUSUM-Rank chart performs similarly and outperforms their competitors when tuned for a moderate shift ($\delta = 0.50$).

Further, if the process distribution is normal and we have a large shift ($\delta = 1.00$), Table 5 suggests that the NPCUSUM-Rank chart and the EX(75) chart perform similarly and the best. From Figures 13 and 16 we can see that the EX(50) chart signals on sample number 14 with a counter of $N^+ = 3$, indicating that the shift most likely occurred at sample number 11. The EX(60) chart signals a little quicker, at sample number 13 with a counter of $N^+ = 2$, also indicating that the shift most likely occurred at sample number 11. The EX(75) chart signals at sample number 12 with a counter of $N^+ = 3$, indicating that the shift most likely occurred at sample number 9. The NPCUSUM-Rank chart also signals at sample number 12, however the counter is $N^+ = 4$, indicating that the the shift most likely occurred at sample number 8.

The findings above are very interesting, because the small shift ($\delta = 0.25$) goes nearly undetected, except by the NPCUSUM-EX chart based on the 75th percentile and the NPCUSUM-Rank chart which signal, but only at the second last sample. Also, for the same data, the parametric CUSUM- \bar{X} chart, with design parameters equal to $h = 4$ and $k = 0.5$ (a popular choice for the design parameters in the parametric setting providing a good IC performance) signals on sample number 12 (the reader is referred to the overview paper by Chakraborti et al. (2011) for details). This is the earliest signal, except for the NPCUSUM-Rank chart and the EX(75) chart designed for a large shift ($\delta = 1.00$) which also signal at sample number 12. Thus, overall, the parametric CUSUM- \bar{X} chart is performing best, which isn't surprising, as it is expected that parametric methods will outperform their parametric counterparts when all assumptions are met.

Summary and Concluding Remarks

The NPCUSUM charts offer an attractive alternative in practice as they combine the inherent advantages of nonparametric charts with the better small shift detection capability of CUSUM-type charts. We examine a class of NPCUSUM charts based on the exceedance statistic by investigating

Figure 17. NPCUSUM-EX chart for simulated data with $\delta = 0.25$

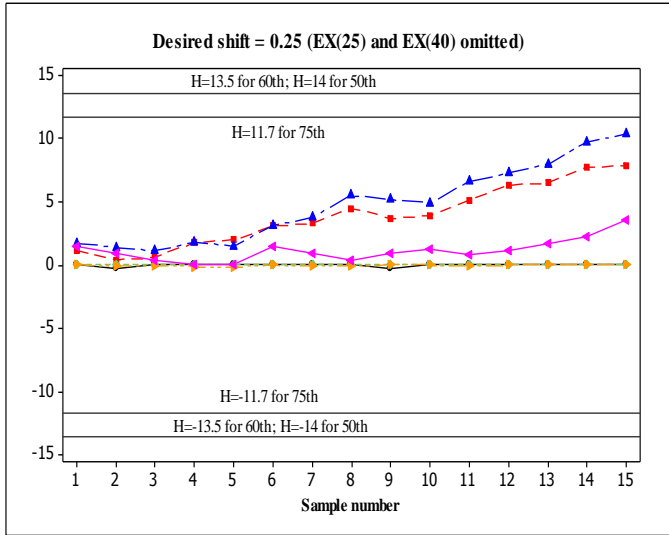


Figure 18. NPCUSUM-EX chart for simulated data with $\delta = 0.50$

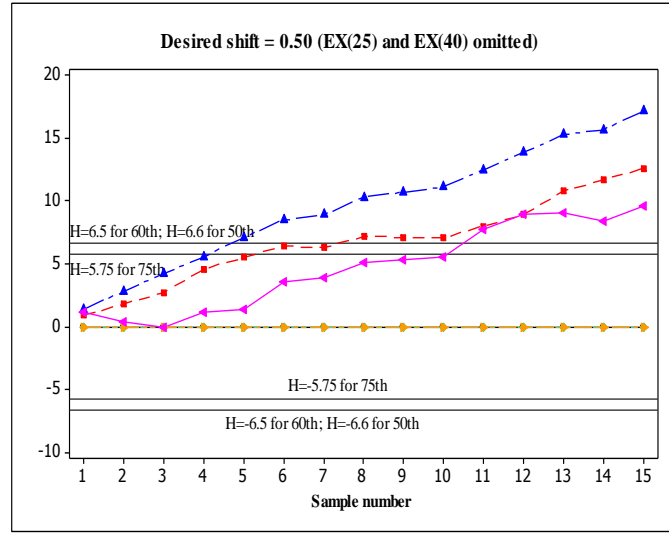
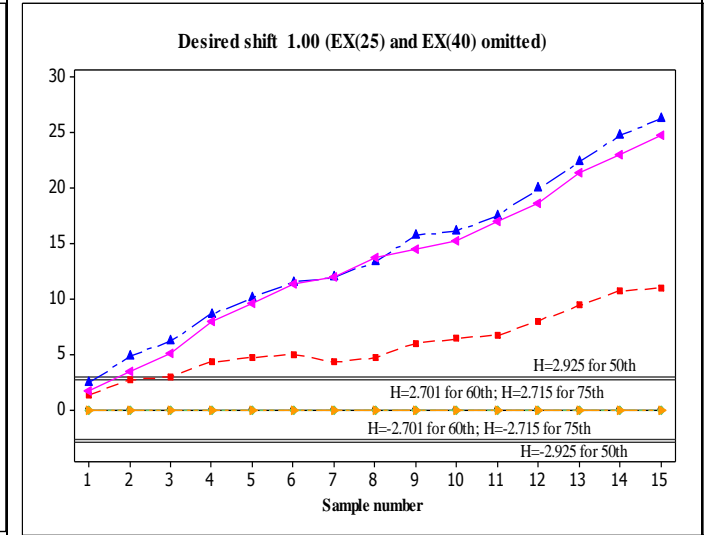


Figure 19. NPCUSUM-EX chart for simulated data with $\delta = 1.00$



Legend: —●— Cj_minus 50th - - - ■ - - Cj_plus 50th - - - ◆ - - Cj_minus 60th - - - ▲ - - Cj_plus 60th - - - ● - - Cj_minus 75th - - - ▼ - - Cj_plus 75th

Figure 20. NPCUSUM-Rank chart for simulated data with $\delta = 0.25$

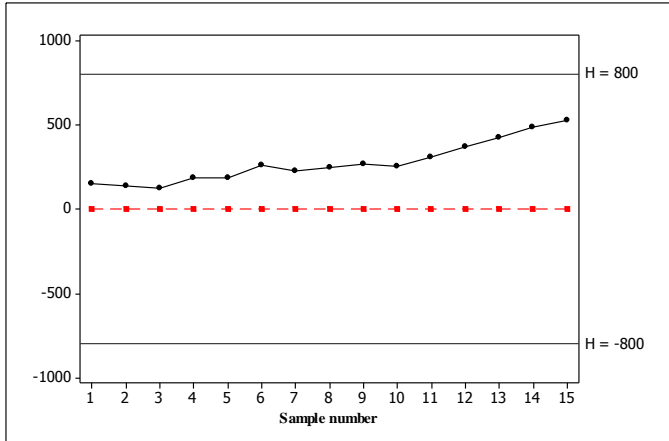


Figure 21. NPCUSUM-Rank chart for simulated data with $\delta = 0.50$

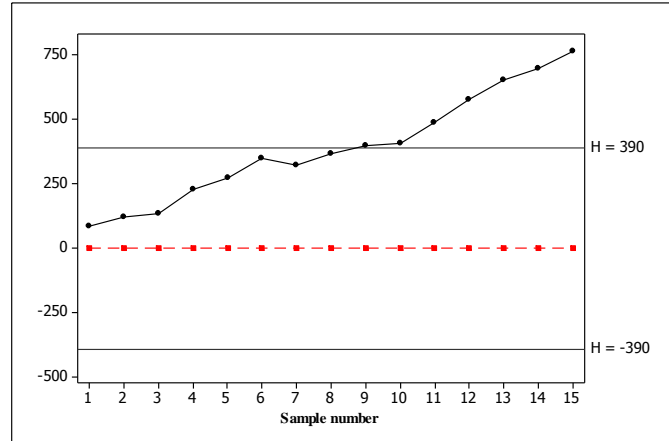
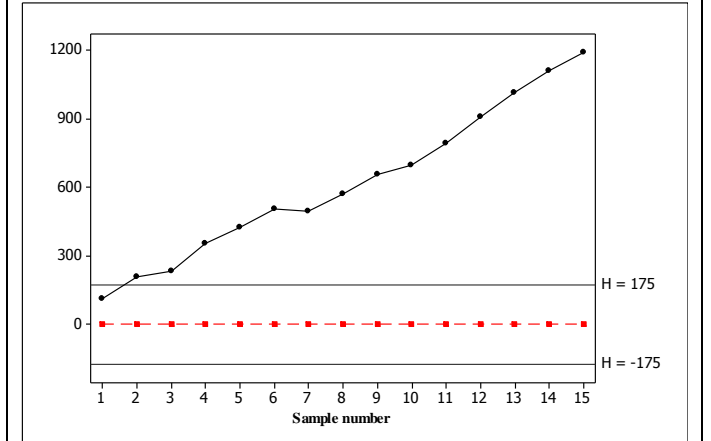


Figure 22. NPCUSUM-Rank chart for simulated data with $\delta = 1.00$



Legend: Solid line = C_j^+ and Dotted line = C_j^-

which order statistic (percentile), from the reference sample, should be used for good overall performance. The chart based on exceedance statistics is compared with the Wilcoxon rank-sum statistic based nonparametric chart. We see that in general, the exceedance CUSUM chart based on higher percentiles performs better in many cases for a number of common distributions. We also examine the impact of the reference value, k , on the performance of the chart and conclude that the NPCUSUM-EX chart, using the 75th percentile of the reference sample, is a good overall chart for detecting a larger location shift. Other reference sample percentiles, such as the 25th or the 40th, can also be used when a smaller shift in location is expected. More specifically, for moderate to large shifts there is little doubt that the practitioner should use the exceedance chart based on the 75th percentile which signals quickly for all reference values under consideration. This is an interesting result in the literature on nonparametric exceedance/precedence tests and control charts. Note that our metric of comparison is the MRL , which we recommend over the ARL . In terms of future research, it would be interesting to compare the NPCUSUM-EX chart with some self-starting charts. The authors also wish to extend this work to the nonparametric exponentially weighted moving average control chart based on the exceedance statistic. This is currently being investigated and will be presented in a separate paper. Other possibilities for future research include studying the economic design of the NPCUSUM-EX chart and investing implementation procedure that will provide guarantee over unconditional Phase II run-length.

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