

## LEAST-SQUARES SPECTRAL ELEMENT METHODS FOR COMPRESSIBLE FLOWS

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### ABSTRACT

This paper describes the application of the least-squares spectral element method to compressible flow problems. The method is described and results are presented for subsonic, transonic and supersonic flow problems over a bump.

### INTRODUCTION

#### Compressible flows

The least-squares spectral element method has been applied successfully to compressible and incompressible viscous flows, [32, 30, 31, 29, 36, 33, 34, 35], but the method has not been applied to compressible inviscid flows. The numerical simulation of inviscid, compressible flow problems has been an active area of research over the last decades. Transonic and supersonic flows admit discontinuous solutions and a proper numerical setting is required to predict the correct shock location and shock strength. In addition, many compressible flow problems are not well-posed in the sense that they do not possess a unique so-

lution. The physical solution – the entropy solution<sup>1</sup> – is the solution obtained by taking the limit of the (unique) viscous problem for the viscosity tending to zero, [26].

An important feature in compressible flows is the use of a conservative scheme which means that the conservation laws (mass, momentum and energy) are satisfied at the discrete level. The Rankine-Hugoniot relations which relate discontinuities before and after a shock are essentially a restatement of these conservation laws in the vicinity of the shock. So by employing a conservative scheme many of the continuous relations also hold true at the discrete level. This is the main reason why finite volume methods are so popular in compressible flow dynamics, see for instance [21, 27, 42].

The converse is however not true: conservative schemes are not necessary to converge to the exact solution. Least-squares formulations are known to suffer from lack of conservation, [3, 11, 35, 20], and it is therefore rather challeng-

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<sup>1</sup>The entropy condition is a restatement of the vanishing viscosity limit

ing to apply this weak formulation to problems which contain discontinuous solutions (shocks, contact discontinuities).

The aim of this paper is to show that LSQSEM is capable of approximating the discontinuous solutions of non-linear hyperbolic equations. In combination with adaptive strategies such as described in [13], this might be a viable alternative to finite volume methods.

### Higher order/spectral

Very little work has been done on inviscid, compressible flow problems in the framework of higher order/spectral methods. The main reason why so little work has been done in this field using spectral methods is mainly due to the appearance of shocks and contact discontinuities. Spectral methods work best when the coefficients of the higher orthogonal basis functions in the solution decay sufficiently fast, [23], in which case exponential convergence to the exact solution may result. The smoothness of the solution dictates the decay rate of the coefficients of the global spectral basis functions, see for example Gottlieb and Hesthaven, [16]. In case of discontinuous solutions the coefficients of the higher order modes decay very slowly and the approximate solution tends to oscillate in the vicinity of large gradients. These wiggles are prone to pollute the entire computational domain. Damping or filtering of these unwanted oscillations is therefore required. The application of spectral methods to non-linear hyperbolic equations has been mainly restricted to one dimensional model problems, such as the Burgers equation, see for instance [17, 28, 39] and references therein.

### Least-squares formulation

The least-squares formulation is gaining renewed interest due to some favorable properties. The least-squares formulation often shows optimal convergence, in contrast to the conventional Galerkin approximation which generally yields sub-optimal convergence rates. Furthermore, the least-squares formulation is inherently stable and

does not require artificial dissipation/viscosity to stabilize the scheme. In addition, a well-posed least-squares approximation always leads to symmetric, positive definite (SPD) systems which is very convenient from a computational point of view since only half of the system matrix needs to be computed and SPD systems are highly amenable to well-established iterative solvers such as the preconditioned conjugate gradient algorithm.

Despite these attractive features, very little work has been done in the field of linear and non-linear hyperbolic equations. In the least-squares finite element (LSFEM) framework work has been done by Jiang, [24]. De Sterck et al., [9, 10], showed that the use of the conservative formulation employed by Jiang is necessary for a proper description of non-linear hyperbolic equations. For the Burgers equation it is shown that the solution is not in  $H^1$ , but the velocity-flux pair  $(u, f)$  is a member of  $H(\text{div})$ . This analysis has been used by De Maerschalck and Heinrichs in the least-squares spectral element context, [6, 8, 4, 7, 19, 18].

Taghaddosi et al., [40, 41], applied the least-squares finite element formulation to the two-dimensional Euler equations in combination with grid adaptation.

For a general overview of the least-squares formulation, the reader is referred to [1, 24] and references therein.

## COMPRESSIBLE FLOWS

Compressible flows in the absence of dissipative terms are governed by the Euler equations. There are several ways in which the Euler equations in differential form can be written, but only the conservative form in terms of conservation quantities will be presented.

The two-dimensional Euler equations in conservation form are given by

$$\frac{\partial}{\partial t} \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{bmatrix} + \frac{\partial}{\partial x} \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho uH \end{bmatrix} + \frac{\partial}{\partial y} \begin{bmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ \rho vH \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

These equations express conservation of mass,

conservation of momentum in the  $x$ - and  $y$ -direction and conservation of energy, respectively. Here  $\rho$  is the local density,  $p$  is the pressure and  $(u, v)$  denotes the fluid velocity. The total energy per unit mass is denoted by  $E$ . The total energy can be decomposed into internal energy  $e$  and the kinetic energy per unit mass

$$\rho E = \rho e + \frac{\rho}{2} (u^2 + v^2) = \frac{p}{\gamma - 1} + \frac{\rho}{2} (u^2 + v^2) ,$$

where in the last equality we assume a calorically ideal, perfect gas. The total enthalpy,  $H$ , is defined as

$$H = E + \frac{p}{\rho} .$$

For steady flows the enthalpy is constant along the streamlines.

If the spatial fluxes depend continuously on the conserved quantities  $\mathbf{u} = (\rho, \rho u, \rho v, \rho E)^T$  we can write the governing equation in non-conservative form as

$$\mathbf{u}_t + A(\mathbf{u})\mathbf{u}_x + B(\mathbf{u})\mathbf{u}_y = \mathbf{0} , \quad (1)$$

where  $A(\mathbf{u})$  and  $B(\mathbf{u})$  are the Jacobian matrices.

## LEAST-SQUARES FORMULATION

The least-squares formulation is based on the equivalence of the residual in a certain norm and the error in an associated norm. If this equivalence is established, one aims to minimize the residual norm which then provides an upper bound for the error in the associated norm.

In order to explain the method, consider the abstract differential equation given by

$$\mathcal{L}u = f , \quad x \in \Omega , \quad (2)$$

with

$$\mathcal{R}u = g , \quad x \in \Gamma \subset \partial\Omega . \quad (3)$$

Here  $\mathcal{L}$  denotes a linear (or linearized) partial differential operator, which for the linearized Euler

equations is given in [14, 43].  $\mathcal{R}$  denotes a linear trace operator by which Dirichlet boundary conditions are prescribed. The data  $f$  and  $g$  are known vectors. Without loss of generality we can set  $g = 0$ .

If the problem is well-posed, the operator  $(\mathcal{L}, \mathcal{R})$  will be a continuous mapping from the underlying Hilbert space  $X = X(\Omega)$  onto the Hilbert spaces  $Y = Y(\Omega)$  and  $Z = Z(\Gamma)$ , with a continuous inverse. Here  $\Gamma \subset \partial\Omega$  is the part of the boundary where boundary conditions are prescribed. This can be expressed by the following inequalities

$$C_1 \|u\|_X \leq \|\mathcal{L}u\|_Y + \|\mathcal{R}u\|_Z \leq C_2 \|u\|_X , \quad \forall u \in X . \quad (4)$$

These inequalities establish norm equivalence between the residuals and the error. We assume that the exact solution  $u_{ex} \in X$ , then by the linearity of  $\mathcal{L}$  and  $\mathcal{R}$  we have

$$\begin{aligned} C_1 \|u - u_{ex}\|_X &\leq \|\mathcal{L}u - f\|_Y + \|\mathcal{R}u - g\|_Z \\ &\leq C_2 \|u - u_{ex}\|_X , \quad \forall u \in X . \end{aligned} \quad (5)$$

These inequalities state that if the residuals of the differential equation measured in the  $Y$ -norm and the traces measured in the  $Z$ -norm go to zero, the exact solution is approximated in the  $X$ -norm. Based on this observation, we introduce the least-squares functional

$$\mathcal{I}(u) = \frac{1}{2} \left( \|\mathcal{L}u - f\|_Y^2 + \|\mathcal{R}u - g\|_Z^2 \right) , \quad \forall u \in X . \quad (6)$$

Minimization of this functional with respect to  $u$  gives the weak formulation

$$\begin{aligned} (\mathcal{L}u, \mathcal{L}v)_Y + W(\mathcal{R}u, \mathcal{R}v)_Z &= \\ (f, \mathcal{L}v)_Y + W(g, \mathcal{R}v)_Z , \quad \forall v \in X . \end{aligned} \quad (7)$$

Generally, the least-squares method is applied to over-determined systems where one has more equations than unknowns, see for instance [22]. The least-squares solution is the solution which minimizes the residual in the  $L^2$ -norm. By

adding weights larger than one to some of the equations, one can force the solution to reduce the residual for that particular equation. By taking a weight smaller than one, one allows the residual of these particular equations to become larger. Here a weighing factor  $W$  is inserted for the boundary terms which allows one to increase or decrease the contribution of the boundary residuals to the overall residual norm. In case the trial solution satisfies the condition  $\mathcal{R}u = 0$  the boundary terms vanish from the weak formulation.

For numerical calculations we need to restrict the infinite dimensional space  $X$  to a finite dimensional subspace, denoted by  $X^h \subset X$ . Here  $h$  denotes a generic discretization parameter which in this paper will refer to the mesh size or the polynomial degree used in the approximation.

## SPECTRAL ELEMENTS

Instead of seeking the minimizer over the infinite dimensional space  $X$  we restrict our search to a conforming subspace  $X^h \subset X$  by performing a domain decomposition where the solution within each sub-domain is expanded with respect to a polynomial basis. The domain  $\Omega$  is sub-divided into  $K$  non-overlapping quadrilateral sub-domains  $\Omega^k$ :

$$\Omega = \bigcup_{k=1}^K \Omega^k, \quad \overset{\circ}{\Omega}^k \cap \overset{\circ}{\Omega}^l = \emptyset, \quad k \neq l. \quad (8)$$

Each sub-domain is mapped onto the unit cube  $[-1, 1]^d$ , where  $d = \dim(\Omega)$ . Within this unit cube the unknown function is approximated by polynomials. In this paper a spectral element method based on *Legendre* polynomials,  $L_k(x)$  over the interval  $[-1, 1]$ , is employed, [2, 12, 25]. We define the Gauss-Lobatto-Legendre (GLL) nodes by the zeroes of the polynomial

$$(1 - x^2) L'_N(x), \quad (9)$$

and the Lagrange polynomials,  $h_i(x)$ , through

these GLL-points,  $x_i$ , by

$$h_i(x) = \frac{1}{N(N+1)} \frac{(x^2 - 1)L'_N(x)}{L_N(x_i)(x - x_i)} \quad \text{for } i = 0, \dots, N, \quad (10)$$

where  $L'_N(x)$  denotes the derivative of the  $N$ th Legendre polynomial. For multi-dimensional problems tensor products of the one-dimensional basis functions are employed in the expansion of the approximate solution. We can therefore expand the approximate solution in each sub-domain in terms of a truncated series of these Lagrangian basis functions, which for  $d = 2$  yields

$$u^N(x, y) = \sum_{i=0}^N \sum_{j=0}^N \hat{u}_{ij} h_i(x) h_j(y), \quad (11)$$

where the  $\hat{u}_{ij}$ 's are to be determined by the least-squares method. Since we have converted a general higher order PDE to an equivalent first order system,  $C^0$ -continuity suffices to patch the solutions on the individual subdomains together. The integrals appearing in the least squares formulation, (7), are approximated by Gauss-Lobatto quadrature

$$\int_{-1}^1 f(x) dx \approx \sum_{i=0}^P f(x_i) w_i, \quad (12)$$

where  $w_i$  are the GL weights given by

$$w_i = \frac{2}{P(P+1)} \frac{1}{L_P^2(x_i)}, \quad i = 0, \dots, P \geq N. \quad (13)$$

It has been shown in [5] that it is beneficial for non-linear equations possessing large gradients to choose the integration order  $P$  higher than the approximation of the solution,  $N$ .

## RESULTS

In this section results are given for the flow over a circular bump in a 2D channel. Results will be given for subsonic flow,  $M_\infty = 0.5$ , transonic flow,  $M_\infty = 0.85$  and supersonic flow,  $M_\infty = 1.4$ .

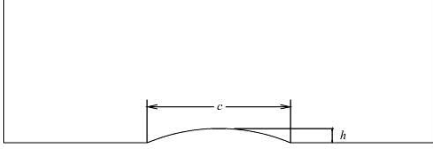


Figure 1: The general geometry of the 2D channel with a circular bump.

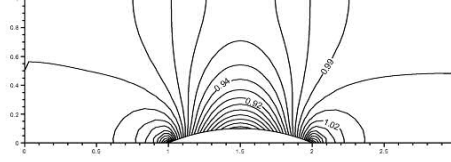


Figure 3: Pressure contours for the subsonic flow with boundary weight  $W = 1$  and  $\Delta t = 0.1$ .

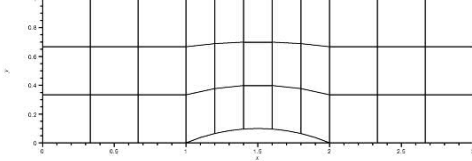


Figure 2: The mesh used for the subsonic test case. The height of the bump is 10% of the chord length and 33 elements are used.

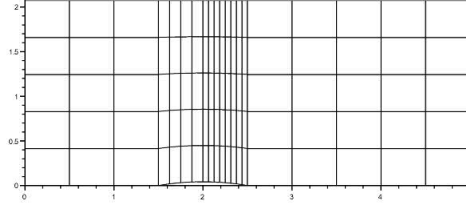


Figure 4: The mesh used for the transonic test case. The height of the bump is 4.2% of the chord length and 100 elements are used.

This is a difficult test problem over the entire Mach range for spectral methods due to the presence of stagnation points at the leading and trailing edge of the bump.

The general geometry for the channel flow with a circular bump is shown in Fig. 1. The bump is modeled by curved elements using the transfinite mapping by Gordon and Hall, [15].

The entropy variation  $s$  in the domain is calculated with the freestream entropy as a reference:

$$s = \frac{\hat{s} - \hat{s}_\infty}{\hat{s}_\infty}, \quad \text{where } \hat{s} = p\rho^{-\gamma}. \quad (14)$$

### Results for subsonic flow

To test a subsonic flow problem the chord length of the bump is set at  $c = 1$ . The length of the channel is three times the chord length whereas the height is set equal to the chord length. The height of bump is 10% of the chord length. The mesh contains 33 elements (Fig. 2) and the polynomial degree chosen is  $N = 6$ . An integration order  $P = 8$  is chosen, see (12).

At the outflow boundary the exit pressure is set at  $p = 1$ . At the inflow boundary the density

is prescribed and set at  $\rho = 1.4$ ; the velocity components are fixed at  $u = 0.5$  and  $v = 0$ . Pressure contours are given in Fig. 3.

### Results for transonic flow

To investigate the transonic flow over a bump the geometry is the same as that for the transonic flow problems described by Spekrijse, [38], and Rizzi and Viviand, [37].

As in the subsonic case, the chord length of the bump is  $c = 1$ . The length of the channel however, is 5 times the chord length and the height is set at 2.073 times the chord length. The height of the bump is 4.2% of the chord length. The mesh used for this test case is shown in Fig. 4. The polynomial degree is  $N = 5$  whereas the integration order is  $P = 6$ .

In Fig. 5 the Mach contours at an inflow Mach number of  $M = 0.85$  and a time step of  $\Delta t = 0.075$  are compared to the finite volume results produced by Spekrijse, [38].

The shock is positioned at approximately 86% of

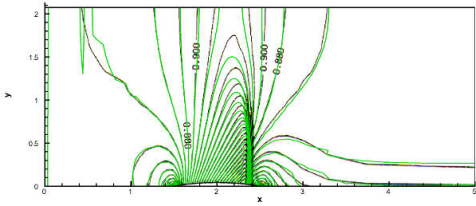


Figure 5: Comparison of the iso-Mach lines for transonic flow,  $M = 0.85$  obtained by LSQSEM (green) and Finite Volume Method by Spekreijse [38], (black).

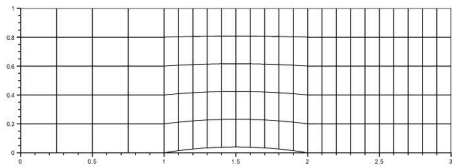


Figure 6: The mesh used for the supersonic test case. The height of the bump is 4% of the chord length and 120 elements are used.

the bump with the Mach number just upstream of the shock being  $M \approx 1.32$ . These results are quantitatively in agreement with the finite volume results obtained by Spekreijse.

### Results for supersonic flow

The geometry used for the supersonic test case is similar to that considered for the subsonic flow test case. The only difference is the height of the bump which is 4% of the chord length for this test case. The mesh has a total of 120 elements as can be seen in Fig. 6.

At inflow the density is set to  $\rho = 1.4$  and the pressure to  $p = 1$ . The Mach number of the flow at the inlet boundary is  $M = 1.4$ .

The iso-Mach contours for this test case are shown in Fig. 7 together with the results obtained by Spekreijse, [38]. This figure reveals that the shock structure over the bump agrees.

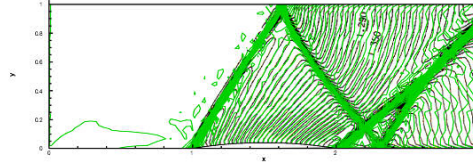


Figure 7: Iso-Mach lines and shock structure obtained by LSQSEM (green) and Finite Volume Method by Spekreijse [38]

## CONCLUSIONS

This paper described the least-squares spectral element formulation in which time stepping was used to reach steady state solutions.

In the transonic and supersonic test case over a circular bump shocks develop. Direct comparison with results from literature demonstrates that the LSQSEM method is capable of solving these type of flow problems.

Despite the fact that the least-squares formulation is not conservative and the fact that high order polynomials tend to oscillate in the vicinity of shocks, the results presented in this paper demonstrate that the least-squares formulation is able to cover the whole Mach range.

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