Convergence for a splitting-up scheme for the 3D Stochastic Navier-Stokes-α model

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Abstract

We propose and analyze a splitting-up scheme for the numerical approximation of the 3D stochastic Navier-Stokes-α model. We prove the convergence of the scheme to the unique variational solution of the 3D stochastic Navier-Stokes-α model when the time step tends to zero.

Key words and phrases: splitting-up scheme, Navier-Stokes-α model, compactness, tightness
MSC(2010): 35R60, 35Q30, 60H15, 60H30, 60H35

1 Introduction

The Navier-Stokes-α model (also known as the Lagrangian averaged Navier-Stokes-α model or the viscous Camassa-Holm equations) was developed in an effort to provide an efficient numerical simulation of 3D turbulence. The mathematical analysis and the numerical study of the model have been intensively studied in [23], [19], [13], [18], [12], [9]-[11], [24]. In particular, the numerical study in [24] shows that this model captures most of the large scale features of a turbulent flow.

The study of the 3D stochastic Navier-Stokes-α model driven by a Wiener process was introduced and studied by Caraballo and his coworkers in [6]-[8]. In [6], they proved the existence and uniqueness of strong solution (in the probabilistic sense) under Lipschitz conditions on the external forces. The proof uses the Galerkin approximation, the properties of stopping time and some convergence principles from functional analysis. Moreover, they showed that the Galerkin approximation converges in mean square to the strong solution. In [7], they studied the asymptotic behavior of its solution when the time tends to infinity. Deugoué and Sango [14] extended the result in [6] to the case of non-Lipschitz assumptions on the coefficients. In particular, they proved the existence of a weak martingale solution for the 3D stochastic Navier-Stokes-α model. In [15], they studied the asymptotic behavior of a weak martingale solution when the parameter α approaches zero. Recently, they started the study of the model to the case where the driving noise is a Lévy noise [16]. While research on the analysis of the 3D stochastic Navier-Stokes-α model has been vigorously undertaken, the numerical analysis is completely missing.

This paper is concerned with the numerical approximation of the solution of the 3D stochastic Navier-Stokes-α model by a splitting method. The method was introduced in the context of stochastic partial differential equations in [4] and further developed in [3], [5], [17], [26], [20],[21]. The splitting method in [4] is an approximation method consisting of the construction of two sequences of equations with time discretization. The equations of the first sequence can be solved as deterministic equations. Those of the second sequence can be solved by the simulation of a stochastic integral.

In this paper, we propose and analyze a numerical scheme for the approximation of the solution of the 3D stochastic Navier-Stokes-α model. The approximation scheme is based on a splitting-up method. Here we construct a splitting approximation so that the equations of the first sequence are deterministic Navier-Stokes-α model with initial conditions, and the equations of the second sequence define a stochastic integral. The advantage of this method is that the first equations can be solved by deterministic methods for the Navier-Stokes-α problems and the second problems can be solved by the simulation of a stochastic integral. We prove the convergence of the sequence of approximation to the solution of the 3D stochastic Navier-Stokes-α model. The convergence holds in probability. Our proof of convergence relies on a compactness method and a lemma due

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to Gyöngy and Krylov [22]. This lemma is very useful and allows to get convergence in probability in the original probability space provided tightness of laws of the approximating sequence and uniqueness of solution of the continuous equation can be shown. Future work will deal with the order of convergence of the numerical scheme.

The paper is arranged as follows. In Section 2, we recall after some notations the result concerning the existence and uniqueness of the variational solution of the 3D stochastic Navier-Stokes-α model obtained in [6]. In Section 3, we introduce the splitting-up approximation scheme and state the main result i.e. the convergence of the sequence of the approximation (see Theorem 3). In Section 4, we establish a priori estimates for the solutions of the scheme and give the details of the proof of convergence of the scheme.

2 Variational and abstract formulation of the problem

2.1 The stochastic 3D Navier-Stokes-α model

Let D be a connected and bounded open subset of $\mathbb{R}^3$ with C^2 boundary ∂D and a final time $T > 0$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and let $\{\mathcal{F}_t\}_{t\in[0,T]}$ be an increasing and right continuous family of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets of $\mathcal{F}$. We denote by $\mathbb{E}$ the expectation with respect to $\mathbb{P}$. Let $W(t)$ be a standard m-dimensional Wiener process defined on this space and adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$.

For any separable Banach space X and $p \in [1, \infty]$, we will denote by $M^p_{\mathcal{F}_t}(0,T;X)$ the space of all processes $\varphi \in L^p(\Omega \times (0,T), d\mathbb{P} \times dt; X)$ that are $\mathcal{F}_t$- measurable. The space $M^p_{\mathcal{F}_t}(0,T;X)$ is a Banach subspace of $L^p(\Omega \times (0,T), d\mathbb{P} \times dt; X)$.

The stochastic 3D Navier-Stokes-α model reads as follows

$$\begin{cases} d(u-\alpha\Delta u) + [\nu(Au - \alpha\Delta(Au)) + (u.\nabla)(u-\alpha\Delta u) - \alpha(\nabla u)^*.\Delta u + \nabla p]\, dt \\ = F(t,u)dt + G(t,u)dW, \text{ in } (0,T)\times D, \\ \nabla.u = 0, \text{ in } (0,T)\times D, \\ u = 0, \ Au = 0, \text{ on } (0,T)\times \partial D, \\ u(0) = u_0, \text{ in } D, \end{cases} \tag{1}$$

where A is the Stokes operator. The constants $\nu > 0$ and $\alpha > 0$ are given and represent, respectively the kinematic viscosity of the fluid and the square of the spatial scale at which fluid motion is filtered. Here $u = (u_1, u_2, u_3)$ and p are unknown random fields in $D \times (0,T)$ representing respectively the large scale velocity and the pressure in each point of $D \times (0,T)$. Next the terms $F(t,u)$ and $G(t,u)$ are external forces depending on u; precise assumptions are given below. Finally u_0 is a given velocity field.

2.2 Notations and properties of the nonlinear term

Following [Caraballo1], we recall some properties regarding the nonlinear term $(u.\nabla)(u-\alpha\Delta u) - \alpha(\nabla u)^*.\Delta u$ appearing in (1).
We denote by $(.,.)$ and $|.|$, respectively, the scalar product and associated norm in $(L^2(D))^3$, and by $(\nabla u, \nabla v)$ the scalar product in $((L^2(D))^3)^3$ of the gradients of u and v. We consider the scalar product in $(H^1_0(D))^3$ defined by

$$((u,v)) = (u,v) + \alpha(\nabla u, \nabla v), \text{for } u, v \text{ in } (H^1_0(D))^3,$$

where its associated norm $\|.\|$ is, in fact, equivalent to the usual gradient norm. We denote by H the closure in $(L^2(D))^3$ of the set $\mathcal{V} = \{v \in (\mathcal{D}(D))^3 : \nabla v = 0 \text{ in } D\}$, and by V the closure of $\mathcal{V}$ in $(H^1_0(D))^3$. H is a Hilbert space equipped with the inner product of $(L^2(D))^3$, and V is a Hilbert subspace of $(H_0^1(D))^3$. Denote by A the Stokes operator, with domain $D(A) = (H^2(D))^3 \cap V$, defined by

$$Aw = -\mathcal{P}(\Delta w), w \in D(A),$$

where $\mathcal{P}$ is the projection operator from $(L^2(D))^3$ onto H. Recall that ∂D is C^2, $|Aw|$ defines in $D(A)$ a norm which is equivalent to the $(H^2(D))^3$-norm, i.e. there exists a constant $c_1(D) > 0$, depending only on D, such that

$$\|w\|_{(H^2(D))^3} \leq c_1(D)|Aw|, \forall w \in D(A).$$

So, $D(A)$ is a Hilbert space with respect to the scalar product

$$(u, w)_{D(A)} = (Av, Aw).$$

For $u \in D(A)$ and $v \in (L^2(D))^3$, we define $(u.\nabla)v$ as the element of $(H^{-1}(D))^3$ given by

$$\langle (u.\nabla)v, w\rangle_{-1} = \sum_{i,j=1}^{3} \langle \partial_i v_j, u_i w_j\rangle_{-1}, \tag{2}$$

for all $w \in (H_0^1(D))^3$. Here $\langle .,.\rangle_{-1}$ is the duality product between $(H^{-1}(D))^3$ and $(H_0^1(D))^3$ (respectively between $H^{-1}(D)$ and $H_0^1(D)$.
There exists a constant $c_2(D) > 0$ depending only on D, such that

$$|\langle (u.\nabla)v, w\rangle_{-1}| \leq c_2(D)|Au||v|\|w\|, \tag{3}$$

for all $(u, v, w) \in D(A) \times (L^2(D))^3 \times (H_0^1(D))^3$.
Now, if $u \in D(A)$, then $(\nabla u)^* \in (H^1(D))^{3\times 3} \subset (L^6(D))^{3\times 3}$, and consequently for $v \in (L^2(D))^3$, we have that $(\nabla u)^*.v \in (L^{\frac{3}{2}}(D))^3 \subset (H^{-1}(D))^3$, with

$$\langle (\nabla u)^*.v, w\rangle_{-1} = \sum_{i,j=1}^{3} \int_D (\partial_j u_i) v_i w_j \, dx, \text{for all } w \in (H_0^1(D))^3.$$

It follows that there exists a constant $c_3(D) > 0$, depending only on D, such that

$$|\langle (\nabla u)^*.v, w\rangle_{-1}| \leq c_3(D)|Au||v|\|w\|, \forall (u, v, w) \in D(A) \times (L^2(D))^3 \times (H_0^1(D))^3. \tag{4}$$

We have the following result (see [6])

Proposition 1. *For all $(u, w) \in D(A) \times D(A)$, and all $v \in (L^2(D))^3$, it follows that*

$$\langle (u.\nabla)v, w\rangle_{-1} = -\langle (\nabla w)^*.v, u\rangle_{-1}. \tag{5}$$

Consider now the bilinear form defined by

$$b^{\#}(u, v, w) = \langle (u.\nabla)v, w\rangle_{-1} + \langle (\nabla u)^*.v, w\rangle_{-1},$$

for $(u, v, w) \in D(A) \times (L^2(D))^3 \times (H_0^1(D))^3$.

Proposition 2. *The trilinear form $b^{\#}$ satisfies*

$$b^{\#}(u, v, w) = -b^{\#}(w, v, u), \forall (u, v, w) \in D(A) \times (L^2(D))^3 \times D(A). \tag{6}$$

and consequently

$$b^{\#}(u, v, u) = 0, \forall (u, v) \in D(A) \times (L^2(D))^3. \tag{7}$$

Moreover, there exists a constant $c(D) > 0$, depending only on D such that

$$|b^{\#}(u, v, w)| \leq c(D)|Au||v|\|w\|, \text{ for all } (u, v, w) \in D(A) \times (L^2(D))^3 \times (H_0^1(D))^3, \tag{8}$$

$$|b^{\#}(u, v, w)| \leq c(D)\|u\||v||Au|, \text{ for all } (u, v, w) \in D(A) \times (L^2(D))^3 \times D(A). \tag{9}$$

In particular, $b^{\#}$ is continuous on $D(A) \times (L^2(D))^3 \times (H_0^1(D))^3$.

2.3 Existence and uniqueness of the variational solution

We recall the result concerning the existence and uniqueness of the variational solution of the 3D stochastic Navier-Stokes-α model. We assume that F and G are measurable Lipschitz mappings from $(0,T)\times V$ into $(H^{-1}(D))^3$ and from $(0,T)\times V$ into $\left[(L^2(D))^3\right]^m$ respectively. More precisely, suppose that for all $u, v \in V$,

$$\|F(t,u) - F(t,v)\|_{(H^{-1}(D))^3} \leq L_F\|u - v\|, \ dt - a.e., \tag{10}$$

$$\|G(t,u) - G(t,v)\|_{[(L^2(D))^3]^m} \leq L_G\|u - v\|, \ dt - a.e., \tag{11}$$

where L_F and L_G are positive constants.
We also suppose that

$$F(t,0) = 0, \ dt - a.e., \tag{12}$$

$$G(t,0) = 0, \ dt - a.e., \tag{13}$$

$$\|G(t,u) - G(s,u)\|^2_{[(L^2(D))^3]^m} \leq o(|t-s|)(1 + \|u\|^2) \tag{14}$$

for all $t, s \in [0,T]$ and $u \in V$, with $o(h)$ monotone increasing and $o(h) \to 0$ as $h \to 0$. Finally we assume that $u_0 \in L^4(\Omega, V)$.
We recall from [6] the definition of a variational solution to problem (1).

Definition 1. *A variational solution to problem (1) is a stochastic process $u \in M^2_{\mathcal{F}_t}(0,T;D(A)) \cap L^2(\Omega; L^\infty(0,T;V))$ weakly continuous with values in V, such that for all $w \in D(A)$ and $t \in [0,T]$,*

$$\begin{aligned}((u(t),w)) &+ \nu\int_0^t (u(s) + \alpha Au(s), Aw)\,ds + \int_0^t b^\#(u(s), u(s) - \alpha\Delta u(s), w)\,ds \\ &= ((u_0,w)) + \int_0^t \langle F(s,u(s)), w\rangle_{-1}\,ds + \left(\int_0^t G(s,u(s))dW(s), w\right).\end{aligned} \tag{15}$$

Remark 1. *Observe that (15) follows from (1) by multiplying the first equation in (1) by $w \in D(A)$, taking into account of the scalar product $((.,.))$, the definition of $b^\#$ and the equality (5). Now, as a consequence of Theorem 3.3 in [6], we have the following result (for the existence of the pressure see [6]) concerning the existence and uniqueness of a variational solution of problem (1).*

Theorem 1. *Under the hypotheses (10)-(13) and $u_0 \in L^4(\Omega; V)$, there exists a unique variational solution of problem (1). Moreover $u \in L^4(\Omega; C([0,T]; V)) \cap L^4(\Omega; L^2(0,T; D(A)))$. In fact there exists a constant $C > 0$ depending only on α, ν, T, L_F and L_G such that*

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\|u(t)\|^4 + \left(\int_0^T \|u(s)\|^2_{D(A)}\,ds\right)^2\right] \leq C\left(\mathbb{E}(\|u_0\|^4) + 1\right). \tag{16}$$

Remark 2. *The hypotheses (10)-(13) are sufficient to prove the existence and uniqueness of a variational solution of problem (1). The assumption (14) is used to prove the convergence of the scheme (20)-(23), see (105).*

2.4 Formulation of problem (1) as an abstract problem

In this section, we rewrite problem (1) as an abstract problem. We identify V with its topological dual V' and we have the Gelfand triple $D(A) \subset V \subset D(A)'$.
We denote by $\langle .,. \rangle$ the duality product between $D(A)'$ and $D(A)$.
We define

$$\langle \widetilde{A}u, v\rangle = \nu(Au, v) + \nu\alpha(Au, Av),$$

for $u, v \in D(A)$.
It is clear that for all $v \in D(A)$

$$2\langle \widetilde{A}v, v\rangle \geq 2\nu\alpha|Av|^2,$$

and, if we denote by μ_k and $w_k, k \geq 1$, the eigenvalues and their corresponding eigenvectors associated to A, then

$$\langle \widetilde{A}w_k, v\rangle = \nu\mu_k((w_k, v)).$$

Taking $\widetilde{\alpha} = 2\nu\alpha$, we have

(a) $\widetilde{A}$ is a linear continuous operator $\widetilde{A} \in \mathcal{L}(D(A), D(A)^{'})$ such that

  $(a1)$ $\widetilde{A}$ is self-adjoint,

  $(a2)$ there is a constant $\widetilde{\alpha} > 0$, such that $2\langle \widetilde{A}v, v\rangle \geq \widetilde{\alpha}\|v\|^2_{D(A)}$, for all $v \in D(A)$.

There exists a Hilbert basis $\{v_k, k \geq 1\} \subset D(A)$ of V and an increasing sequence $\{\lambda_k; k \geq 1\} \subset (0, \infty)$ such that

$$\widetilde{A}v_k = \lambda_k v_k \tag{17}$$

where $\lambda_k = \nu\mu_k$ and $v_k = \frac{w_k}{\sqrt{1+\alpha\mu_k}}$.
On the other hand, denote

$$\langle \widetilde{B}(u, v), w\rangle = b^{\#}(u, v - \alpha\Delta v, w), \text{ for } (u, v, w) \text{ in } (D(A))^3,$$

$$((\widetilde{F}(t, u), w)) = \langle F(t, u), w\rangle_{-1}, \text{ for } (u, w) \in V \times D(A).$$

Then it is straightforward to check that if we take

$$c_1 = (1 + \alpha)c_1(D)c(D) \text{ and } L_{\widetilde{F}} = L_F,$$

then we obtain

(b) $\widetilde{B} : D(A) \times D(A) \to D(A)^{'}$ is a bilinear mapping such that

  $(b1)$ $\langle \widetilde{B}(u, v), u\rangle = 0$, for all $u, v \in D(A)$,

  $(b2)$ $\|\widetilde{B}(u, v)\|_{D(A)'} \leq c_1\|u\|\|v\|_{D(A)}$, for all $(u, v) \in D(A) \times D(A)$,

  $(b3)$ $|\langle \widetilde{B}(u, v), w\rangle| \leq c_1\|u\|_{D(A)}\|v\|_{D(A)}\|w\|$, for all $u, v, w \in D(A)$.

(c) $\widetilde{F} : [0, T] \times V \to V$ is a mapping such that

  $(c1)$ $\widetilde{F}(t, 0) = 0$, $dt - a.e.$,

  $(c2)$ $\|\widetilde{F}(t, u) - \widetilde{F}(t, v)\| \leq L_{\widetilde{F}}\|u - v\|$, for all $u, v \in V$.

We denote by $V^{\otimes m}$ the product of m copies of V. Let I denote the identity operator in H, and define $\widetilde{G}(t, u)$ as

$$\widetilde{G}(t, u) = (I + \alpha A)^{-1} \circ \mathcal{P} \circ G(t, u),$$

for $u \in V$.
We have

$$\|(I + \alpha A)^{-1}f\|^2 \leq \frac{1}{1 + \alpha\mu_1}|f|^2$$

for all $f \in H$. Also observe that

$$\|u\| \leq c_4\|u\|_{D(A)}, \ \forall u \in D(A) \tag{18}$$

where $c_4 = \frac{\sqrt{1+\alpha\mu_1}}{\mu_1}$. See the proof in [6].
Consequently, taking

$$L_{\widetilde{G}} = \frac{L_G}{\sqrt{1+\alpha\mu_1}},\ C_3 = \frac{1}{1+\alpha\mu_1},$$

we obtain

(d) $\widetilde{G} : [0,T] \times V \to V^{\otimes m}$ is a mapping such that

$(d1)$ $\widetilde{G}(t,0) = 0, dt - a.e.,$

$(d2)$ $\|\widetilde{G}(t,u) - \widetilde{G}(t,v)\|_{V^{\otimes m}} \leq L_{\widetilde{G}}\|u-v\|,$ for all $u, v \in V$,

$(d3)$ $\|\widetilde{G}(t,u) - \widetilde{G}(t,u)\|^2_{V^{\otimes m}} \leq C_3 o(|t-s|)(1+\|u\|^2),$ for all $t, s \in [0,T]$ *and* $u \in V$.

Next, for each $j \in [1,m]$ and all $(t,u,w) \in (0,T) \times V \times D(A)$, we have

$$(G(t,u),w) = ((I+\alpha A)\widetilde{G}(t,u),w) = ((\widetilde{G}(t,u),w)),$$

and for all $u \in L^2(\Omega,\mathcal{F},\mathbb{P};L^\infty(0,T;V)), (t,w) \in (0,T) \times D(A)$, it follows that

$$\begin{aligned}\left(\int_0^t G(s,u(s))dW(s),w\right) &= \sum_{j=1}^m \int_0^t (G_j(s,u(s)),w)dW_j(s) \\ &= \sum_{j=1}^m \int_0^t ((\widetilde{G}_j(s,u(s)),w))dW_j(s) \\ &= \left(\left(\int_0^t \widetilde{G}(s,u(s))dW(s),w\right)\right).\end{aligned}$$

Consequently a variational solution of problem (1) is equivalently a stochastic process $u \in M^2_{\mathcal{F}_t}(0,T;D(A)) \cap L^2(\Omega;L^\infty(0,T;V))$ such that the equation

$$\begin{aligned}&u(t) + \int_0^t \widetilde{A}u(s)ds + \int_0^t \widetilde{B}(u(s),u(s))ds \\ &= u_0 + \int_0^t \widetilde{F}(s,u(s))ds + \int_0^t \widetilde{G}(s,u(s))dW(s),\end{aligned} \tag{19}$$

is satisfied in $D(A)'$, a.s. for all $t \in [0,T]$.

3 The splitting-up approximation scheme and the main result

In this section, we propose an approximation scheme for the variational solution of problem (1) by splitting it into a sequence of deterministic Navier-Stokes-α equations and a sequence of Itô integral. We construct the scheme in the following way.
Let N be an integer and $k = \frac{T}{N+1}$. We split the interval $[0,T]$ into subintervals $[rk,(r+1)k[$ where $r = 0,1,...,N$. For each $B(.) = (B_1(.),...,B_m(.)) \in C(0,T;\mathbb{R}^m)$, we shall define a process $z_k(t)$ depending on k and B. Consider an interval $[rk,(r+1)k[, r = 0,1,2,...,N$; then z_k is defined on this interval by the relation

$$\begin{cases}\frac{dz_k}{dt} + \widetilde{A}z_k + \widetilde{B}(z_k,z_k) = 0,\ t \in]rk,(r+1)k[, \\ z_k(rk) = z_k^r,\end{cases} \tag{20}$$

where

$$\begin{aligned}z_k^{r+1} &= z_k((r+1)k-0) + \int_{rk}^{(r+1)k} \widetilde{F}(t,z_k(t))dt \\ &+ \widetilde{G}(rk,\bar{z}_k^r).\left(B((r+1)k) - B(rk)\right),\end{aligned} \tag{21}$$

$$z_k^0 = u_0, \tag{22}$$

$$\bar{z}_k^r = \frac{1}{k}\int_{rk}^{(r+1)k} z_k(t)dt, \tag{23}$$

where $v(q-0)$ stands for the limit of v from the left at q.
The existence and uniqueness of $z_k(.)$ on $[0,T[$ is given by the following classical result (see [25], [6], [13]).

Theorem 2. *Let $y_0 \in V$ and $f \in L^2(0,T;D(A)')$. Then there exists one and only one solution $y \in L^2(0,T;D(A))$ with $y' \in L^2(0,T;D(A)')$ such that*

$$\begin{cases} y' + \widetilde{A}y + \widetilde{B}(y,y) = f, \\ y(0) = y_0. \end{cases}$$

Remark 3. *The proof of this theorem uses the Galerkin approximation, the properties (a_2) of $\widetilde{A}$, (b) of $\widetilde{B}$ and the Aubin-Lions compactness theorem (see [25]).*

We set for completeness $z_k(T) = z_k^{N+1}$ and we note that z_k is discontinuous at points $k, ..., (N+1)k$ and has left limits.
We can define a map $\Psi_k : C(0,T;\mathbb{R}^m) \to L^2(0,T;D(A))$ by $\psi_k(B) = z_k =$ the solution of (20)-(23) corresponding to $B \in C(0,T;R^m)$. It is easy to see that Ψ_k is a continuous map from $C(0,T;\mathbb{R}^m)$ to $L^2(0,T;D(A))$, where $C(0,T;\mathbb{R}^m)$ is equipped with the uniform topology and $L^2(0,T;D(A))$ with the strong topology.
Put $z_k = \Psi_k(W)$. Now, we are ready to state the main result of this paper concerning the convergence of the scheme (20)-(23).

Theorem 3. *Assume that the assumptions (10)-(14) hold and $u_0 \in V$. The sequence (z_k) converges to the variational solution u of problem (1). The convergence of the sequence (z_k) holds in probability in $L^2(0,T;V)$.*

Remark 4. *The proof of Theorem 3 is the object of the next section. A compactness method will be used. The uniqueness of the variational solution of problem (1) is necessary to derive the convergence in probability of the sequence (z_k). Indeed in order to obtain the convergence of the sequence (z_k), we will make use of the following powerful lemma which was first used by Gyöngy and Krylov in [22].*

Lemma 1. *Let (Z_n) be a sequence of random elements in a Polish space E equipped with the Borel σ-algebra. Then Z_n converges in probability to an E-valued random element if and only if for every pair of subsequence $(Z_{\Phi(n)}, Z_{\Psi(n)})$, there is a subsequence of $(Z_{\Phi(n)}, Z_{\Psi(n)})$ which converges in law to a random element supported on the diagonal $\{(x,y) \in E \times E, x = y\}$.*

4 Proof of Theorem 3

The proof of Theorem 3 will be divided into three steps. In the first step, we establish some a priori estimates for the solution of the scheme in suitable functions spaces. In the second step, we prove the tightness for the approximating solutions. In the last step, we proceed with the passage to the limit in the equation and the conclusion of the proof of Theorem 3.

4.1 A priori estimates of the scheme

We shall prove the following a priori estimates of the solution of the scheme.

Lemma 2. *Under the conditions of Theorem 1, there is constant $C > 0$ independent of k such that*

$$\sup_{0\le t\le T} \mathbb{E}\|z_k(t)\|^2 \le C, \tag{24}$$

$$\mathbb{E}\int_0^T \|z_k(t)\|^2_{D(A)} \le C, \tag{25}$$

$$\sup_{0\le t\le T} \mathbb{E}\|z_k(t)\|^4 \le C, \tag{26}$$

$$\mathbb{E}\sup_{0\le t\le T} \|z_k(t)\|^2 \le C, \tag{27}$$

$$\mathbb{E}\left(\int_0^T \|z_k(t)\|^2_{D(A)}\right)^2 \le C, \tag{28}$$

$$\mathbb{E}\sup_{0\le t\le T} \|z_k(t)\|^4 \le C. \tag{29}$$

Remark 5. *Hereafter we denote by C a constant independent of k and $r = 0, 1, , , , , N$.*

Proof. Proof of Lemma 2.
Proof of the estimate (24).
From the energy equality related to (10) and the property $(b1)$ of $\widetilde{B}$, we get

$$\|z_k(t)\|^2 + 2\int_{rk}^t \langle \widetilde{A}(z_k(s)), z_k(s)\rangle\, ds = \|z_k^r\|^2 \text{ for } t \in [rk, (r+1)k[. \tag{30}$$

Using the property $(a2)$ of $\widetilde{A}$, we deduce

$$\|z_k(t)\|^2 + \widetilde{\alpha}\int_{rk}^t \|z_k(s)\|^2_{D(A)}\, ds = \|z_k^r\|^2 \text{ for } t \in [rk, (r+1)k[. \tag{31}$$

It follows in particular

$$\|z_k(t)\| \le \|z_k^r\| \text{ for } t \in [rk, (r+1)k[. \tag{32}$$

Therefore

$$\|\bar{z}_k^r\| \le \|z_k^r\|. \tag{33}$$

Consider next the process

$$\bar{z}_k(t) = z_k((r+1)k - 0) + + \int_{rk}^t \widetilde{F}(s, z_k(s))\, ds + \widetilde{G}(rk, \bar{z}_k^r).(W(t) - W(rk)) \tag{34}$$

for $t \in [rk, (r+1)k[$. We have $z_k^{r+1} = \bar{z}_k((r+1)k)$.
Applying $\mathrm{It}\hat{o}'s$ formula to (34) we have

$$d\|\bar{z}_k(t)\|^2 = 2((\widetilde{F}(t, z_k(t)), \bar{z}_k(t)))dt + 2((\widetilde{G}(rk, \bar{z}_k^r), \bar{z}_k(t)))dW(t) + \|\widetilde{G}(rk, \bar{z}_k^r)\|^2 dt, \tag{35}$$

for $t \in [rk, (r+1)k[$ and $r = 0, 1, 2, ..., N$.
We then deduce that

$$\begin{aligned}
\mathbb{E}\|\bar{z}_k(t)\|^2 &= \mathbb{E}\|z_k((r+1)k - 0)\|^2 + 2\mathbb{E}\left[\int_{rk}^t ((\widetilde{F}(s, z_k(s)), \bar{z}_k(s)))\, ds\right] + \mathbb{E}\left[\int_{rk}^t \|\widetilde{G}(rk, \bar{z}_k^r)\|^2\, ds\right] \\
&\le \mathbb{E}\|z_k^r\|^2 + c(1 + \mathbb{E}\|z_k^r\|^2)(t - rk) + \mathbb{E}\int_{rk}^t \|\bar{z}_k(s)\|^2\, ds \\
&\le \mathbb{E}\|z_k^r\|^2 + c(1 + \mathbb{E}\|z_k^r\|^2)k + \mathbb{E}\int_{rk}^t \|\bar{z}_k(s)\|^2\, ds \\
&\le (1 + Ck)\mathbb{E}\|z_k^r\|^2 + ck + \mathbb{E}\int_{rk}^t \|\bar{z}_k(s)\|^2\, ds,
\end{aligned} \tag{36}$$

for $t \in [rk,(r+1)k[$ and $r = 0,1,2,...,N$.
So Gronwall's inequality derives

$$\begin{aligned}\mathbb{E}\|z_k^{r+1}\|^2 &\le \left[(1+ck)\mathbb{E}\|z_k^r\|^2 + ck\right] e^k \\ &\le (1+Ck)\mathbb{E}\|z_k^r\|^2 + ck \text{ since } e^k \le e^T,\end{aligned}$$

which yields

$$\begin{aligned}\mathbb{E}\|z_k^r\|^2 &\le (1+ck)^r(\mathbb{E}\|u_0\|^2+1) \\ &\le \left(1 + C\frac{T}{N+1}\right)^{N+1}(\mathbb{E}\|u_0\|^2+1)\end{aligned}$$

for any $r = 0,1,2,...,N$. Hence we obtain

$$\mathbb{E}\|z_k^r\|^2 \le C, \ \forall k, \ \forall r = 0,1,...,N. \tag{37}$$

Combining (32) and (37), the inequality (24) follows. This completes the proof of (24).

Remark 6. *Gronwall's inequality together with the estimates (36) and (37) give*

$$\sup_{0\le t\le T} \mathbb{E}\|\bar{z}_k(t)\|^2 \le C, \ \forall k = 1,2,.... \tag{38}$$

Proof of the estimate (25).
From (35), we have

$$\mathbb{E}\|z_k^{r+1}\|^2 = 2\mathbb{E}\int_{rk}^{(r+1)k}((\widetilde{F}(s,z_k(s)),\bar{z}_k(s)))\,ds + k\mathbb{E}\|\widetilde{G}(rk,\bar{z}_k^r)\|^2 + \mathbb{E}\|z_k((r+1)k-0)\|^2. \tag{39}$$

Also from (30), we get

$$\|z_k((r+1)k-0)\|^2 + 2\int_{rk}^{(r+1)k}\langle\widetilde{A}(z_k(s)),z_k(s)\rangle\,ds = \|z_k^r\|^2. \tag{40}$$

Combining (39) and (40) we get

$$\begin{aligned}&\mathbb{E}\|z_k^{r+1}\|^2 + 2\mathbb{E}\int_{rk}^{(r+1)k}\langle\widetilde{A}(z_k(s)),z_k(s)\rangle\,ds \\ &= 2\mathbb{E}\int_{rk}^{(r+1)k}((\widetilde{F}(s,z_k(s)),\bar{z}_k(s)))\,ds + k\mathbb{E}\|\widetilde{G}(rk,\bar{z}_k^r)\|^2 + \mathbb{E}\|z_k^r\|^2\end{aligned}$$

for $r = 0,1,2,...,N$.
Addition of these relations gives

$$\begin{aligned}&\mathbb{E}\|z_k^{N+1}\|^2 + 2\mathbb{E}\int_0^T\langle\widetilde{A}(z_k(s),z_k(s)\rangle\,ds \\ &= \|u_0\|^2 + 2\mathbb{E}\int_0^T((\widetilde{F}(s,z_k(s)),\bar{z}_k(s)))\,ds + k\sum_{r=0}^N\mathbb{E}\|\widetilde{G}(rk,\bar{z}_k^r)\|^2.\end{aligned}$$

Using the properties $(a2)$ of $\widetilde{A}$, $(c2)$ of $\widetilde{F}$ and the estimate (33), we arrive at

$$\begin{aligned}\widetilde{\alpha}\mathbb{E}\int_0^T\|z_k(t)\|_{D(A)}^2\,dt &\le \mathbb{E}\int_0^T\left(\|\widetilde{F}(s,z_k(s))\|^2 + \|\bar{z}_k(s)\|^2\right)ds + k\mathbb{E}\sum_{r=0}^N c(1+\|\bar{z}_k^r\|^2) \\ &\le \mathbb{E}\int_0^T\left(c(1+\|z_k(s)\|^2) + \|\bar{z}_k(s)\|^2\right)\,ds + TC.\end{aligned} \tag{41}$$

Using the estimates (24), (38) and (41), the estimate (25) follows.
Proof of the estimate (26).
From (32), we have

$$\|z_k(t)\|^4 \le \|z_k^r\|^4 \text{ for } t \in [rk,(r+1)k[. \tag{42}$$

Applying It$\hat{o}'$s formula to (35) we get

$$\begin{aligned} d\|\bar{z}_k(t)\|^4 &= 2\|\bar{z}_k(t)\|^2 \left(2((\widetilde{F}(t,z_k(t)),\bar{z}_k(t))) + \|\widetilde{G}(rk,\bar{z}_k^r)\|^2\right) dt \\ &\quad + 4\|\bar{z}_k(t)\|^2((\widetilde{G}(rk,\bar{z}_k^r),\bar{z}_k(t)))dW + 4((\widetilde{G}(rk,\bar{z}_k^r),\bar{z}_k(t)))^2 dt \end{aligned} \tag{43}$$

for $t \in [rk,(r+1)k[$ and $r = 0,1,2,...,N$. Taking the expectation in (43), we have

$$\begin{aligned} \mathbb{E}\|\bar{z}_k(t)\|^4 &= \mathbb{E}\|z_k((r+1)k-0)\|^2 + 4\mathbb{E}\int_{rk}^t \|\bar{z}_k(s)\|^2((\widetilde{F}(s,z_k(s)),\bar{z}_k(s)))ds \\ &+ 2\mathbb{E}\int_{rk}^t \|\bar{z}_k(s)\|^2\|\widetilde{G}(rk,\bar{z}_k^r)\|^2 ds + 4\mathbb{E}\int_{rk}^t ((\widetilde{G}(rk,\bar{z}_k^r),\bar{z}_k(s)))^2 ds. \end{aligned} \tag{44}$$

for $t \in [rk,(r+1)k[$ and $r = 0,1,2,...,N$.
We are going to estimate the terms on the right hand side of (44).

$$\begin{aligned} 4\|\bar{z}_k(s)\|^2\|\widetilde{F}(s,z_k(s))\|\|\bar{z}_k(s)\| &\le 2\|\bar{z}_k(s)\|^2\left(\|\widetilde{F}(s,z_k(s))\|^2 + \|\bar{z}_k(s)\|^2\right) \\ &\le 2\|\bar{z}_k(s)\|^2\|\widetilde{F}(s,z_k(s))\|^2 + 2\|\bar{z}_k(s)\|^4 \\ &\le 3\|z_k(s)\|^4 + C(1+\|z_k(s)\|^4) \\ &\le 3\|z_k(s)\|^4 + C(1+\|z_k^r\|^4). \end{aligned} \tag{45}$$

$$\begin{aligned} 2\|\bar{z}(t)\|^2\|\widetilde{G}(rk,\bar{z}_k^r)\|^2 &\le \|\bar{z}_k(t)\|^4 + \|\widetilde{G}(rk,\bar{z}_k^r)\|^4 \\ &\le \|\bar{z}_k(t)\|^4 + C(1+\|z_k^r\|^4). \end{aligned} \tag{46}$$

Combining (45), (46) and (44), we arrive at

$$\mathbb{E}\|\bar{z}_k(t)\|^4 \le \mathbb{E}\|z_k^r\|^4 + 4\int_{rk}^t \mathbb{E}\|\bar{z}_k(s)\|^4 ds + C(1+\mathbb{E}\|z_k^r\|^4)k. \tag{47}$$

So Gronwall's lemma yields

$$\begin{aligned} \mathbb{E}\|\bar{z}_k(t)\|^4 &\le (Ck + (1+Ck)\mathbb{E}\|z_k^r\|^4)e^{4k} \\ &\le (1+Ck)\mathbb{E}\|z_k^r\|^4 + Ck. \end{aligned}$$

And this implies that

$$\mathbb{E}\|z_k^{r+1}\|^4 \le (1+Ck)\mathbb{E}\|z_k^r\|^4 + Ck, \tag{48}$$

which yields

$$\begin{aligned} \mathbb{E}\|z_k^r\|^4 &\le (1+Ck)^r(\mathbb{E}\|u_0\|^4 + 1) \\ &\le (1+C\frac{T}{N+1})^{N+1}(\mathbb{E}\|u_0\|^4 + 1), \end{aligned}$$

for any $r = 0,1,...,N$. Hence we obtain

$$\mathbb{E}\|z_k^r\|^4 \le C, \forall k \text{ and } r = 0,1,...,N. \tag{49}$$

Using the estimate (42), the estimate (26) follows.

Remark 7. *Combining the estimates (47) and (49), we have*

$$\mathbb{E}\|\bar{z}_k(t)\|^4 \le C, \tag{50}$$

for all $t\in [0,T]$.

Proof the estimate (27).
From (32), we have

$$\sup_{t\in[0,T]} \|z_k(t)\|^2 \le \max_{r=0,1,\dots,N} \|z_k^r\|^2. \tag{51}$$

Also from (30) and (35), we have

$$\begin{aligned}&\|z_k^{r+1}\|^2 + 2\int_{rk}^{(r+1)k} \langle \widetilde{A}(z_k(s)), z_k(s)\rangle\, ds \le \|z_k^r\|^2 + 2\int_{rk}^{(r+1)k} ((\widetilde{F}(s, z_k(s)), \bar{z}_k(s)))\, ds\\ &+ 2\int_{rk}^{(r+1)k} ((\widetilde{G}(rk, \bar{z}_k^r), \bar{z}_k(s)))dW + \int_{rk}^{(r+1)k} \|\widetilde{G}(rk, \bar{z}_k^r)\|^2\, ds,\end{aligned} \tag{52}$$

for $r = 0, 1, 2, ..., N$. Adding up these relations, we arrive at

$$\begin{aligned}&2\int_0^{(N+1)k} \langle \widetilde{A}(z_k(s)), z_k(s)\rangle\, ds + \|z_k^r\|^2\\ &\le \|u_0\|^2 + 2\int_0^{rk} ((\widetilde{F}(s, z_k(s)), \bar{z}_k(s)))\, ds + 2\int_0^{rk} ((\widetilde{G}(rk, \bar{z}_k^r), \bar{z}_k(s)))dW + k\sum_{r=0}^{N} \|\widetilde{G}(rk, \bar{z}_k^r)\|^2\\ &\le \|u_0\|^2 + \int_0^T \|\widetilde{F}(s, z_k(s))\|^2\, ds + \int_0^T \|\bar{z}_k(s)\|^2\, ds + 2\int_0^{rk} ((\widetilde{G}(rk, \bar{z}_k^r), \bar{z}_k(s)))dW\\ &+ k\sum_{r=0}^{N} \|\widetilde{G}(rk, \bar{z}_k^r)\|^2.\end{aligned} \tag{53}$$

Taking the expectation, we then deduce

$$\begin{aligned}\mathbb{E} \max_{r=0,1,\dots,N} \|z_k^r\|^2 &\le \mathbb{E}\|u_0\|^2 + cT + \mathbb{E}\int_0^T \|z_k(s)\|^2\, ds + \mathbb{E}\int_0^T \|\bar{z}_k(s)\|^2\, ds\\ &+ k\sum_{r=0}^{N} \mathbb{E}\|\widetilde{G}(rk, \bar{z}_k^r)\|^2 + 2\mathbb{E} \sup_{t\in[0,T]} \left|\int_0^t ((\widetilde{G}(rk, \bar{z}_k^r), \bar{z}_k(s)))dW\right|.\end{aligned}$$

Observe that using (24), (38), (37) we get

$$\begin{aligned}\mathbb{E} \max_{r=0,1,\dots,N} \|z_k^r\|^2 &\le \mathbb{E}\|u_0\|^2 + C + 2\mathbb{E} \sup_{t\in[0,T]} \left|\int_0^t ((\widetilde{G}(rk, \bar{z}_k^r), \bar{z}_k(s)))dW\right|\\ &\le C + 2\mathbb{E}\left[\sup_{t\in[0,T]} \left|\int_0^t ((\widetilde{G}(rk, \bar{z}_k^r), \bar{z}_k(s)))dW\right|\right].\end{aligned} \tag{54}$$

Using the Burkholder-Gundy's inequality, we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t((\widetilde{G}(rk,\bar{z}_k^r),\bar{z}_k(s)))dW(s)\right|\right]\\
&\le c\mathbb{E}\left[\left|\int_0^T((\widetilde{G}(rk,\bar{z}_k^r),\bar{z}_k(s)))^2\,ds\right|^{\frac{1}{2}}\right]\\
&\le c\left(\mathbb{E}\left[\int_0^T\|\bar{z}_k(t)\|^4\,dt\right]+\mathbb{E}\int_0^T\|\widetilde{G}(rk,\bar{z}_k^r)\|^4\,ds\right)^{\frac{1}{2}}\\
&\le c\left(\mathbb{E}\left[\int_0^T\|\bar{z}_k(t)\|^4\,dt\right]+\mathbb{E}\sum_{r=0}^N\int_{rk}^{(r+1)k}\|\widetilde{G}(rk,\bar{z}_k^r)\|^4\,ds\right)^{\frac{1}{2}}\\
&\le c\left(\mathbb{E}\int_0^T\|\bar{z}_k(s)\|^4\,ds+k\sum_0^N(1+\mathbb{E}\|\bar{z}_k^r\|^4)\right)^{\frac{1}{2}}\\
&\le C,
\end{aligned}$$

where we have used the estimates (50), (49) and (42). We then deduce from (54) that

$$\mathbb{E}\max_{r=0,1,\dots,N}\|z_k^r\|^2\le C, \tag{55}$$

and from the estimate (51), we have

$$\mathbb{E}\sup_{t\in[0,T]}\|z_k(t)\|^2\le C. \tag{56}$$

This proves (27).
Proof of the estimate (28).
Relation (53) implies that

$$\begin{aligned}
2\int_0^T\|z_k(s)\|_{D(A)}^2\,ds&\le\|u_0\|^2+2\int_0^{rk}((\widetilde{F}(s,z_k(s)),\bar{z}_k(s)))\,ds\\
&\quad+2\int_0^{rk}((\widetilde{G}(rk,\bar{z}_k^r),\bar{z}_k(s)))dW+k\sum_{r=0}^N\|\widetilde{G}(rk,\bar{z}_k^r)\|^2.
\end{aligned}\tag{57}$$

Squaring both side of this inequality and taking the expectation, we obtain

$$\begin{aligned}
\mathbb{E}\left(\int_0^T\|z_k(s)\|_{D(A)}^2\,ds\right)^2&\le c\mathbb{E}\|u_0\|^4+4c\mathbb{E}\left(\int_0^T((\widetilde{F}(s,z_k(s)),\bar{z}_k(s)))\,ds\right)^2+k^2\mathbb{E}\sum_{r=0}^N\|\widetilde{G}(rk,\bar{z}_k^r)\|^4\\
&\quad+c\mathbb{E}\sup_{t\in[0,T]}\left|((\widetilde{G}(rk,\bar{z}_k^r),\bar{z}_k(s)))dW\right|^2.
\end{aligned}\tag{58}$$

Let us estimate the integrals of the right-hand side . We have

$$\begin{aligned}
\mathbb{E}\left(\int_0^T((\widetilde{F}(s,z_k(s)),\bar{z}(s)))ds\right)^2&\le C+\mathbb{E}\int_0^T\|z_k(s)\|^4\,ds+\mathbb{E}\int_0^T\|\bar{z}_k(s)\|^4\,ds\\
&\le C,
\end{aligned}\tag{59}$$

where we have used the estimates (50) and (26).

$$\begin{aligned}
\mathbb{E}k^2\sum_{r=0}^{N}\|\widetilde{G}(rk,\bar{z}_k^r)\|^4 &\le k^2\mathbb{E}\sum_{r=0}^{N}C(1+\|\bar{z}_k^r\|^4)\\
&\le k^2(N+1)C+k^2(N+1)C\mathbb{E}\|\bar{z}_k^r\|^4\\
&\le CkT+CkT\mathbb{E}\|\bar{z}_k^r\|^4\\
&\le C.
\end{aligned}\tag{60}$$

Using the martingale inequality, we get

$$\begin{aligned}
\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t((\widetilde{G}(rk,\bar{z}_k^r),\bar{z}_k(s)))dW\right|^2 &\le \mathbb{E}\int_0^T((\widetilde{G}(rk,\bar{z}_k^r),\bar{z}_k(s)))^2\,ds\\
&\le \mathbb{E}\int_0^T\|\widetilde{G}(rk,\bar{z}_k^r)\|^2\|\bar{z}_k(s)\|^2\,ds\\
&\le C\mathbb{E}\int_0^T\|\widetilde{G}(rk,\bar{z}_k^r)\|^4\,ds+C\mathbb{E}\int_0^T\|\bar{z}_k(s)\|^4\,ds\\
&\le C\mathbb{E}\sum_{r=0}^{N}\int_{rk}^{(r+1)k}\|\widetilde{G}(rk,\bar{z}_k^r)\|^4\,ds+C\mathbb{E}\int_0^T\|\bar{z}_k(s)\|^4\,ds\\
&\le C\mathbb{E}\sum_{r=0}^{N}k(1+\|\bar{z}_k^r\|^4)+C\mathbb{E}\int_0^T\|\bar{z}_k(s)\|^4\,ds\\
&\le C(N+1)k+ck(N+1)\mathbb{E}\|\bar{z}_k^r\|^4+C\mathbb{E}\int_0^T\|\bar{z}_k(s)\|^4\,ds\\
&\le C,
\end{aligned}\tag{61}$$

since $\mathbb{E}\|\bar{z}_k^r\|^4$ and $\mathbb{E}\|\bar{z}_k(t)\|^4$ is bounded uniformly on k. Combining (59), (60),(61) with (58), we get $\mathbb{E}\left(\int_0^T\|z_k(s)\|_{D(A)}^2\,ds\right)^2$ is bounded uniformly on k. This ends the proof of (28).

Proof of the estimate (29).

Using (42), we have

$$\sup_{t\in[0,T]}\|z_k(t)\|^4\le\max_{r=0,1,\dots,N}\|z_k^r\|^4.\tag{62}$$

Also relation (53) implies that

$$\begin{aligned}
\max_{r=0,1,\dots,N}\|z_k^r\|^2 \le \|u_0\|^2+cT+\int_0^T\|z_k(s)\|^2\,ds+\int_0^T\|\bar{z}_k(s)\|^2\,ds\\
+k\sum_{r=0}^{N}\|\widetilde{G}(rk,\bar{z}_k^r)\|^2+2\sup_{t\in[0,T]}\left|\int_0^t((\widetilde{G}(rk,\bar{z}_k^r),\bar{z}_k(s)))dW\right|.
\end{aligned}$$

Squaring both side on this inequality and taking the expectation, we arrive at

$$\begin{aligned}
\mathbb{E}\max_{r=0,1,\dots,N}\|z_k^r\|^4 \le \mathbb{E}\|u_0\|^4+\mathbb{E}\int_0^T\|z_k(s)\|^4\,ds+\mathbb{E}\int_0^T\|\bar{z}_k(s)\|^4\,ds\\
+k^2\sum_{r=0}^{N}\mathbb{E}\|\widetilde{G}(rk,\bar{z}_k^r)\|^4+4\sup_{0\le t\le T}\left|\int_0^t((\widetilde{G}(rk,\bar{z}_k^r),\bar{z}_k(s)))dW\right|^2.
\end{aligned}\tag{63}$$

Arguing as in the proof of the estimate (28) (see (58)), we have

$$\mathbb{E}\max_{r=0,1,\dots,N}\|z_k^r\|^4$$

is bounded uniformly on k. We then deduce that

$$\mathbb{E}\sup_{t\in[0,T]}\|z_k(t)\|^4 \le C.$$

This proves the estimate (29) and completes the proof of Lemma 2. $\square$

We next proof an important estimate.

Lemma 3. *Under the conditions of Theorem 1, we have*

$$\mathbb{E}\left[\sup_{|\theta|\le\delta}\int_0^T \|z_k(t+\theta)-z_k(t)\|^2_{D(A)'}\,dt\right] \le C\delta, \tag{64}$$

for any $0<\delta\le 1$*, where* z_k *is extended by* 0 *outside* $[0,T]$.

Proof. Assume $\theta>0$. A similar calculation is done whenever $\theta<0$. We write

$$I=\mathbb{E}\left[\sup_{0\le\theta\le\delta}\int_0^T \|z_k(t+\theta)-z_k(t)\|^2_{D(A)'}\,dt\right] \le I_1+I_2,$$

where

$$I_1=\mathbb{E}\left[\sup_{0\le\theta\le\delta}\int_0^{T-\delta} \|z_k(t+\theta)-z_k(t)\|^2_{D(A)'}dt\right],$$

and

$$I_2=\mathbb{E}\left[\sup_{0\le\theta\le\delta}\int_{T-\delta}^T \|z_k(t+\theta)-z_k(t)\|^2_{D(A)'}\,dt\right].$$

From (25), we obtain $I_2\le c\delta$.
Now we deal with I_1. Using (20) and (34), we have

$$\begin{aligned} &z_k(t+\theta)-z_k(t)+\int_t^{t+\theta}\widetilde{A}(z_k(s))\,ds+\int_t^{t+\theta}\widetilde{B}(z_k(s),z_k(s))\,ds\\ &=\int_{k[\frac{t}{k}]}^{k[\frac{(t+\theta)}{k}]}\widetilde{F}(s,z_k(s))\,ds+\int_{k[\frac{t}{k}]}^{[\frac{(t+\theta)}{k}]k}\widetilde{G}(rk,\bar{z}_k^r)dW(s), \end{aligned} \tag{65}$$

for $t\in[0,T-\delta]$ and $0\le\theta\delta$, where $[t]$ denotes the integer part of t.
Next we have

$$\begin{aligned} \left\|\int_t^{t+\theta}\widetilde{A}(z_k(s))\,ds\right\|_{D(A)'} &\le \int_t^{t+\theta}\|\widetilde{A}(z_k(s))\|_{D(A)'}\,ds\\ &\le \theta^{\frac12}\left[\int_t^{t+\theta}\|\widetilde{A}(z_k(s))\|^2_{D(A)'}\,ds\right]^{\frac12}. \end{aligned}$$

Using the estimate (25) of Lemma 65, we get

$$\begin{aligned} \mathbb{E}\left[\sup_{0\le\theta\le\delta}\int_0^{T-\delta}\left\|\int_t^{t+\theta}\widetilde{A}(z_k(s))ds\right\|^2_{D(A)'}\,dt\right] &\le \delta\mathbb{E}\left[\int_0^{T-\delta}\int_0^T\|z_k(s)\|^2dsdt\right]\\ &\le C\delta. \end{aligned} \tag{66}$$

From the property $(b2)$ of $\widetilde{B}$, the Hölder's inequality and the estimates (28), (29), we obtain

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{0\le\theta\le\delta}\int_0^{T-\delta}\left\|\int_t^{t+\theta}\widetilde{B}(z_k(s),z_k(s))\,ds\right\|_{D(A)'}^2 dt\right]\\
&\le c\delta\int_0^{T-\delta}\left(\mathbb{E}\sup_{s\in[0,T]}\|z_k(s)\|^2\int_t^{t+\delta}\|z_k(s)\|_{D(A)}^2\,ds\right)dt\\
c\delta&\int_0^{T-\delta}\left(\mathbb{E}\sup_{s\in[0,T]}\|z_k(s)\|^4\right)^{\frac{1}{2}}\left[\mathbb{E}\left(\int_t^{t+\delta}\|z_k(s)\|_{D(A)}^2\,ds\right)^2\right]^{\frac{1}{2}}dt\\
&\le c\delta\mathbb{E}\sup_{s\in[0,T]}\|z_k(s)\|^4\left[\mathbb{E}\left(\int_0^{T}\|z_k(s)\|_{D(A)}^2\,ds\right)^2\right]^{\frac{1}{2}}\\
&\le C\delta.
\end{aligned}\tag{67}
$$

For the first term on the right-hand side of (65), we have

$$
\sup_{0\le\theta\le\delta}\left\|\int_{k[\frac{t}{k}]}^{k[\frac{(t+\theta)}{k}]}\widetilde{F}(s,z_k(s))\,ds\right\|_{D(A)'}^2\le C\left(\int_{k[\frac{t}{k}]}^{k[\frac{(t+\delta)}{k}]}(1+\|z_k(s)\|)\,ds\right)^2. \tag{68}
$$

Using (27), we get

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{0\le\theta\le\delta}\int_0^{T-\delta}\left\|\int_{k[\frac{t}{k}]}^{k[\frac{(t+\theta)}{k}]}\widetilde{F}(s,z_k(s))\,ds\right\|_{D(A)'}^2 dt\right]\\
&\le\mathbb{E}\left[\sup_{0\le s\le T}(1+\|z_k(s)\|^2)\int_0^{T-\delta}\left([\frac{t+\delta}{k}]k-[\frac{t}{k}]k\right)^2 dt\right]\\
&\le\int_0^{T-\delta}\left([\frac{t+\delta}{k}]k-[\frac{t}{k}]k\right)^2 dt\\
&\le C\delta.
\end{aligned}\tag{69}
$$

Finally using the Burkholder-Gundy inequality, we have

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{0\le\theta\le\delta}\int_0^{T-\delta}\left\|\int_{k[\frac{t}{k}]}^{\left[\frac{(t+\theta)}{k}\right]k}\widetilde{G}(rk,\bar{z}_k^r)dW\right\|_{D(A)'}^2 dt\right]\\
&\le\int_0^{T-\delta}\mathbb{E}\left[\int_{k[\frac{t}{k}]}^{[\frac{(t+\delta)}{k}]k}\|\widetilde{G}(rk,\bar{z}_k^r)\|^2\,ds\right]dt\\
&\le C\int_0^{T-\delta}\left(\left[\frac{t+\delta}{k}\right]k-\left[\frac{t}{k}\right]k\right)dt\\
&\le C\delta.
\end{aligned}\tag{70}
$$

Combining (66)-(70) and (65), we obtain $I_1\le c\delta$ and this completes the proof of Lemma 3. $\square$

4.2 Tightness for the approximating solutions

We introduce the space $\mathcal{U}_{\mu_n,\nu_n}$ of functions $v=v(w,t,x)$ defined on $\Omega\times[0,T]\times D$ and such that

(1) v is measurable with respect to (w,t,x) and for each $t\in[0,T]$, v is $\mathcal{F}_t$-measurable,

(2) v satisfies

$$\mathbb{E}\sup_{0\le t\le T}\|v(t)\|^2\le C;\quad \mathbb{E}\int_0^T\|v(s)\|^2_{D(A)}\,ds\le C,$$
$$\mathbb{E}\sup_n \nu_n^{-1}\sup_{0\le\theta\le\mu_n}\int_0^T\|v(t+\theta)-v(t)\|^2_{D(A)'}\,dt\le C,$$

where the sequences $\{\nu_n\}$ and $\{\mu_n\}$ are positives sequences converging to zero as $n\to\infty$.

We endow $\mathcal{U}_{\mu_n,\nu_n}$ with the norm

$$\begin{aligned}\|v\|_{\mathcal{U}_{\mu_n,\nu_n}} &= \left(\mathbb{E}ess\sup_{0\le t\le T}\|v(s)\|^2\right)^{\frac{1}{2}}+\left(\mathbb{E}\int_0^T\|v(s)\|^2_{D(A)}\,ds\right)^{\frac{1}{2}}\\ &\quad+\mathbb{E}\sup_n\nu_n^{-1}\sup_{0\le\theta\le\mu_n}\left(\int_0^T\|v(t+\theta)-v(t)\|^2_{D(A)'}\,dt\right)^{\frac{1}{2}}.\end{aligned}$$

We summarize our findings so far in the following theorem.

Theorem 4. *For any μ_n, ν_n such that the serie $\sum_n\nu_n^{-1}\sqrt{\mu_n}$ converges, the sequence $\{z_k\}$ is bounded in $\mathcal{U}_{\mu_n,\nu_n}$.*

Now, we consider the set

$$S=C(0,T;\mathbb{R}^m)\times L^2(0,T;V)$$

equipped with its Borel σ-algebra $\mathcal{B}(S)$. For each k, let Ψ_k be the map

$$\begin{aligned}\Phi_k :&\Omega\to S\\ &w\mapsto (W(w,.),z_k(w,.)).\end{aligned}$$

For each k, we introduce a measure Π_k on $(S,\mathcal{B}(S))$ given by

$$\Pi_k(A)=\mathbb{P}(\Phi_k(A))$$

for all $A\in\mathcal{B}(S)$.
The main result of this section is the following theorem.

Theorem 5. *The family of measures $\{\Pi_k\}$ is tight uniformly in S.*

For the proof of Theorem 5, we will use the following compactness result from [Bensoussan].

Lemma 4. *For any sequences of positives real numbers μ_n,ν_n which tend to zero as $n\to\infty$, the injection of*

$$Z=\left\{q/q\in L^2(0,T;D(A))\cap L^\infty(0,T;V);\quad \sup_n\nu_n^{-1}\sup_{|\theta|\le\mu_n}\left(\int_0^T\|q(t+\theta)-q(t)\|^2_{D(A)'}\,dt\right)^{\frac{1}{2}}<\infty\right\}$$

in $L^2(0,T;V)$ is compact.

Remark 8. *Endowed Z with the following norm*

$$\begin{aligned}\|v\|_Z &= ess\sup_{0\le s\le T}\|v(s)\|^2+\left(\int_0^T\|v(s)\|^2_{D(A)}\,ds\right)^{\frac{1}{2}}\\ &\quad+\sup_n\nu_n^{-1}\sup_{|\theta|\le\mu_n}\left(\int_0^T\|q(t+\theta)-q(t)\|^2_{D(A)'}\,dt\right)^{\frac{1}{2}},\end{aligned}$$

Z is a Banach space.

Proof. Proof of Theorem 5.
For any $\epsilon > 0$, we should find the compacts subsets $H_\epsilon \subset C(0,T;\mathbb{R}^m)$, $K_\epsilon \subset L^2(0,T;V)$ such that

$$\mathbb{P}\{w;\ W(w,.) \notin H_\epsilon\} \leq \frac{\epsilon}{2} \tag{71}$$

and

$$\mathbb{P}\{w;\ z_k(w,.) \notin K_\epsilon\} \leq \frac{\epsilon}{2}. \tag{72}$$

The choice of H_ϵ is made by taking account of some facts about the Wiener process, such as the formula

$$\mathbb{E}|W(t_2) - W(t_1)|^{2j} = (2j-1)!(t_2-t_1)^j, j = 1,2,...$$

For a constant L_ϵ depending on ϵ to be chosen later, we consider the set H_ϵ of all $C(.) \in C(0,T;\mathbb{R}^m)$ such that

$$sup\left\{n|C(t_2) - C(t_1)|; t_1, t_2 \in [0,T], |t_2 - t_1| < n^{-6}\right\} \leq L_\epsilon$$

which is compact in $C(0,T;\mathbb{R}^m)$, thanks to Arzelá-Ascoli's theorem.
Making use of Markov's inequality, we get

$$\begin{aligned}
\mathbb{P}\left\{w : W(w,.) \notin H_\epsilon\right\} &\leq \mathbb{P}\left[\bigcup_n \left\{w : \sup_{t_1,t_2:|t_2-t_1|<n^{-6}} |W(t_2) - W(t_1)| > \frac{L_\epsilon}{n}\right\}\right] \\
&\leq \sum_{n=1}^{\infty}\sum_{i=0}^{n^6-1}\left(\frac{n}{L_\epsilon}\right)^4 \mathbb{E} \sup_{iTn^{-6}\leq t\leq (i+1)Tn^{-6}} |W(t) - W(iTn^{-6})|^4 \\
&\leq C\sum_{n=1}^{\infty}\left(\frac{n}{L_\epsilon}\right)^4 (Tn^{-6})^2 n^6 = \frac{C}{L_\epsilon^4}\sum_{n=1}^{\infty}\frac{1}{n^2}.
\end{aligned}$$

We choose

$$L_\epsilon^4 = \frac{1}{2c\epsilon}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^{-1}$$

to get (71).
Next we choose K_ϵ as a closed ball of radius M_ϵ in Z centered at zero and with μ_n, ν_n independent of ϵ, converging to zero and such that the serie $\sum_n \nu_n^{-1}\sqrt{\mu_n}$ converges. Lemma 4 implies that K_ϵ is a compact subset of $L^2(0,T;V)$. We have further

$$\begin{aligned}
\mathbb{P}\left\{w :\quad z_k \in K_\epsilon\right\} &\leq \mathbb{P}\left\{w :\quad \|z_k\|_Z > M_\epsilon\right\} \\
&\leq \frac{1}{M_\epsilon}\mathbb{E}\|z_k\|_Z \\
&\leq \frac{1}{M_\epsilon}\|z_k\|_{\mathcal{U}_{\nu_n,\mu_n}} \\
&\leq \frac{C}{M_\epsilon},
\end{aligned}$$

where in the last inequality we make use of Theorem 4. Choosing $M_\epsilon = c\epsilon^{-1}$, we get (72). This ends the proof of Theorem 5. ☐

5 Passage to the limit and conclusion

Now, we are in position to use the Skorokhod's theorem, the uniqueness of the variational solution of problem (1) and Lemma 1 to prove that the whole sequence $\{z_k\}$ converges to the variational solution u of problem (1) in probability in $L^2(0,T;V)$.

Indeed since the sequence $\{z_k\}$ is tight in $L^2(0,T;V)$ uniformly in k then by Skorokhod's theorem for a given pair of subsequences z_{k_j} and z_{l_j}, there exist subsequences which are denoted by the same symbol $\{k_j\}$ and $\{l_j\}$ and a sequence of random elements $(\tilde{z}_{k_j}, \bar{z}_{l_j}, B_j)$ in $X = L^2(0,T;V) \times L^2(0,T;V) \times C(0,T;\mathbb{R}^m)$ carried by some probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ such that

$$(\tilde{z}_{k_j}, \bar{z}_{l_j}, B_j)_{j\geq 1} \text{ converges almost surely in X to a random element } (\tilde{z}, \bar{z}, \widetilde{W}). \tag{73}$$

Moreover the corresponding joint laws are equal, that is

$$\mathcal{L}(\tilde{z}_{k_j}, \bar{z}_{l_j}, B_j) = \mathcal{L}(z_{k_j}, z_{l_j}, W), \tag{74}$$

for all $j \geq 1$.
We set $\bar{\mathcal{F}}^t = \sigma\{\tilde{z}(s), \bar{z}(s), \widetilde{W}(s) : 0 \leq s \leq t\}$ and $\bar{\mathcal{F}}^t_j = \sigma\{\tilde{z}_{k_j}(s), \bar{z}_{l_j}(s), B_j(s) : 0 \leq s \leq t\}$.
Arguing as in [5],[26] we can prove that $\widetilde{W}$ is an $\mathbb{R}^m-$valued $\bar{\mathcal{F}}^t$- Wiener process on the probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$. Also B_j is an $\mathbb{R}^m$ − valued $\bar{\mathcal{F}}^t_j$-Wiener process in the same probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$.
We are going to prove that

$$\tilde{z}_{k_j} = \Psi_{k_j}\left(B_j(.,w)\right), \hat{\mathbb{P}} - a.s.. \tag{75}$$

We denote by $\hat{\mathbb{E}}$ the expectation with respect to $\hat{\mathbb{P}}$. We recall that Ψ_k is continuous from $C(0,T;\mathbb{R}^m)$ to $L^2(0,T;V)$.
Consider Θ_k: S$\rightarrow \mathbb{R}$ defined by

$$\Theta_k(b(.), z(.)) = \frac{\int_0^T \|z(t) - \Psi_k(b(.))(t)\|^2\, dt}{1 + \int_0^T \|z(t) - \Psi_k(b(.))(t)\|^2\, dt},$$

which is continuous and bounded.
Since $\mathcal{L}(\tilde{z}_{k_j}, B_j)$=$\mathcal{L}(z_{k_j}, W)$, we have

$$\begin{aligned}
\hat{\mathbb{E}}\Theta_{k_j}(B_j(.), \tilde{z}_{k_j}(.)) &= \int \Theta_{k_j}(b(.), z(.))d\mathcal{L}(B_j, \tilde{z}_{k_j})\\
&= \mathbb{E}\Theta_{k_j}(W, z_{k_j})\\
&= \mathbb{E}\Theta_{k_j}(W, \Psi_{k_j}(W)) = 0.
\end{aligned}$$

This implies that $\tilde{z}_{k_j} = \Psi_{k_j}\left(B_j(.,w)\right), \hat{\mathbb{P}} - a.s.$, and we then have (75).
Therefore we may write from the definition of Ψ_k, the relations

$$\begin{cases}
\frac{d\tilde{z}_{k_j}}{dt} + \widetilde{A}\tilde{z}_{k_j} + \widetilde{B}(\tilde{z}_{k_j}, \tilde{z}_{k_j}) = 0, \;\; t \in]rk_j, (r+1)k_j[,\\
\tilde{z}_{k_j}(rk_j) = \tilde{z}^r_{k_j},\\
\tilde{z}^{r+1}_{k_j} = \tilde{z}_{k_j}((r+1)k_j - 0) + \int_{rk_j}^{(r+1)k_j} \widetilde{F}(t, \tilde{z}_{k_j}(t))dt + \widetilde{G}(rk_j, \tilde{\bar{z}}^r_{k_j}).\left(B_j((r+1)k_j) - B_j(rk_j)\right),\\
\tilde{z}^0_{k_j} = u_0,\\
\tilde{z}_{k_j}(T) = \tilde{z}^{N_j+1}_{k_j},
\end{cases} \tag{76}$$

where $r = 0, 1, ..., N_j$ and $N_j + 1 = \frac{T}{k_j}$.
From the system (76) and Lemmas 2, 3, it is also clear that the following estimates are valid:

$$\sup_{0\leq t\leq T} \hat{\mathbb{E}}\|\tilde{z}_{k_j}(t)\|^2 \leq C, \tag{77}$$

$$\hat{\mathbb{E}}\int_0^T \|\tilde{z}_{k_j}(t)\|^2_{D(A)} \leq C, \tag{78}$$

$$\sup_{0\leq t\leq T} \hat{\mathbb{E}}\|\tilde{z}_{k_j}(t)\|^4 \leq C, \tag{79}$$

$$\hat{\mathbb{E}} \sup_{0\leq t\leq T} \|\tilde{z}_{k_j}(t)\|^2 \leq C, \tag{80}$$

$$\hat{\mathbb{E}}\left(\int_0^T \|\tilde{z}_{k_j}(t)\|^2_{D(A)}\right)^2 \leq C, \tag{81}$$

$$\hat{\mathbb{E}} \sup_{0\leq t\leq T} \|\tilde{z}_{k_j}(t)\|^4 \leq C. \tag{82}$$

$$\hat{\mathbb{E}}\left[\sup_{|\theta|\leq\delta}\int_0^T \|\tilde{z}_{k_j}(t+\theta) - \tilde{z}_{k_j}(t)\|^2_{D(A)'}\, dt\right] \leq C\delta, \tag{83}$$

Therefore we may also assume, by extracting a new subsequence still denoted $\tilde{z}_{k_j}$ to save the notation, that:

$$\tilde{z}_{k_j} \rightharpoonup \tilde{z} \text{ in } L^2(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}; L^2(0,T; D(A)) \text{ weakly}, \tag{84}$$

$$\tilde{z}_{k_j} \rightharpoonup \tilde{z} \text{ in } L^4(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}; L^2(0,T; D(A)) \text{ weakly}, \tag{85}$$

$$\tilde{z}_{k_j} \rightharpoonup \tilde{z} \text{ in } L^\infty(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}; L^4(0,T; V) \text{ weakly star}. \tag{86}$$

From this and the previous estimates (77)-(83), we can state that

$$\sup_{0\leq t\leq T} \hat{\mathbb{E}}\|\tilde{z}(t)\|^2 \leq C, \tag{87}$$

$$\hat{\mathbb{E}}\int_0^T \|\tilde{z}(t)\|^2_{D(A)} \leq C, \tag{88}$$

$$\sup_{0\leq t\leq T} \hat{\mathbb{E}}\|\tilde{z}(t)\|^4 \leq C, \tag{89}$$

$$\hat{\mathbb{E}} \sup_{0\leq t\leq T} \|\tilde{z}(t)\|^2 \leq C, \tag{90}$$

$$\hat{\mathbb{E}}\left(\int_0^T \|\tilde{z}(t)\|^2_{D(A)}\right)^2 \leq C, \tag{91}$$

$$\hat{\mathbb{E}} \sup_{0\leq t\leq T} \|\tilde{z}(t)\|^4 \leq C. \tag{92}$$

$$\hat{\mathbb{E}}\left[\sup_{|\theta|\leq\delta}\int_0^T \|\tilde{z}(t+\theta) - \tilde{z}(t)\|^2_{D(A)'}\, dt\right] \leq C\delta. \tag{93}$$

Note that by (73) and the estimate (79), we have

$$\tilde{z}_{k_j} \to \tilde{z} \text{ in } L^2(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}; L^2(0,T; V)) \text{ strongly}, \tag{94}$$

and thus extracting a new subsequence still denoted by $\tilde{z}_{k_j}$ to save notation, we can also assert that

$$\tilde{z}_{k_j} \to \tilde{z} \text{ in } V, \tag{95}$$

for almost w, t with respect to the measure d$\hat{\mathbb{P}}\otimes$ dt.
According to the Lipschitz conditions on $\widetilde{F}$ and $\widetilde{G}$ combined with (94), we obtain

$$\widetilde{F}(., \tilde{z}_{k_j}(.)) \to \widetilde{F}(., \tilde{z}(.)) \text{ in } L^2(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}; L^2(0,T; V)), \tag{96}$$

$$\widetilde{G}(.,\tilde{z}_{k_j}(.)) \to \widetilde{G}(.,\tilde{z}(.)) \text{ in } L^2(\hat{\Omega},\hat{\mathcal{F}},\hat{\mathbb{P}};L^2(0,T;V)). \tag{97}$$

Put

$$\bar{\tilde{z}}_k(t) = \frac{1}{k}\int_{rk}^{(r+1)k} \tilde{z}_k(s)\,ds,\ \ t \in]rk,(r+1)k[,\ \ r=0,1,...,N, \tag{98}$$

$$\text{and}\ \ \bar{\tilde{z}}^k(t) = \frac{1}{k}\int_{rk}^{(r+1)k} \tilde{z}(s)\,ds,\ \ t \in [rk,(r+1)k[,\ \ r=0,1,...,N. \tag{99}$$

Then by (94), we have

$$\hat{\mathbb{E}}\left[\int_0^T \|\bar{\tilde{z}}_k(t) - \bar{\tilde{z}}^k(t)\|^2\,dt\right] \le \hat{\mathbb{E}}\left[\int_0^T \|\tilde{z}_k(s) - \tilde{z}(s)\|^2\,ds\right] \to 0 \text{ as } k \to 0 \tag{100}$$

On the other hand, by Lebesgue's theorem

$$\bar{\tilde{z}}^k(t) \to \tilde{z}(t) \text{ in } L^2(\hat{\Omega};V)\text{for almost all t.} \tag{101}$$

From this and (79), it follows that

$$\bar{\tilde{z}}^k(t) \to \tilde{z}(t) \text{ in } L^2(\hat{\Omega},\hat{\mathcal{F}},\hat{\mathbb{P}};L^2(0,T;V)), \tag{102}$$

which yields

$$\bar{\tilde{z}}_k(t) \to \tilde{z}(t) \text{ in } L^2(\hat{\Omega},\hat{\mathcal{F}},\hat{\mathbb{P}};L^2(0,T;V)). \tag{103}$$

Let $\widetilde{G}_k^r(u) = \widetilde{G}(rk;u)$ and

$$X_{k_j}(t) = \widetilde{G}_{k_j}^r(\bar{\tilde{z}}_{k_j}^r) = \widetilde{G}(rk_j;\bar{\tilde{z}}_{k_j}^r) \text{ for } t \in [rk_j,(r+1)k_j[. \tag{104}$$

By assumption $(d3)$ of $\widetilde{G}$, the convergences (103), (97) it follows that

$$X_{k_j}(.) \to \widetilde{G}(.,\tilde{z}(.)) \text{ in } L^2(\hat{\Omega},\hat{\mathcal{F}},\hat{\mathbb{P}};L^2(0,T;V^m)) \text{ strongly.} \tag{105}$$

Let us write (76) in the following more convenient way

$$\begin{aligned}
&\tilde{z}_{k_j}(t) + \int_0^t \widetilde{A}(\tilde{z}_{k_j}(s))\,ds + \int_0^t \widetilde{B}(\tilde{z}_{k_j}(s),\tilde{z}_{k_j}(s))\,ds \\
&= u_0 + \int_0^{k_j[\frac{t}{k_j}]} \widetilde{F}(s,\tilde{z}_{k_j}(s))\,ds + \int_0^{k_j[\frac{t}{k_j}]} X_{k_j}(s)dB_j(s),
\end{aligned} \tag{106}$$

for $t \in [0,T]$.

Since $\widetilde{A}$ is a linear bounded operator, then the convergence (84) implies that

$$\int_0^t \widetilde{A}(\tilde{z}_{k_j}(s)\,ds \rightharpoonup \int_0^t \widetilde{A}(\tilde{z}(s))\,ds \text{ weakly } \text{ in } L^2(\hat{\Omega},\hat{\mathcal{F}},\hat{\mathbb{P}};L^2(0,T;D(A)')). \tag{107}$$

We also have

$$\int_0^t \widetilde{B}(\tilde{z}_{k_j}(s),\tilde{z}_{k_j}(s))\,ds \rightharpoonup \int_0^t \widetilde{B}(\tilde{z}(s),\tilde{z}(s))\,ds \text{ weakly } \text{ in } L^2(\hat{\Omega},\hat{\mathcal{F}},\hat{\mathbb{P}};L^2(0,T;D(A)')). \tag{108}$$

In fact, since $L^\infty(\hat{\Omega}\times[0,T],d\hat{\mathbb{P}}\times dt;D(A))$ is dense in $L^2(\hat{\Omega},\hat{\mathcal{F}},\hat{\mathbb{P}};L^2(0,T;D(A)))$ and $\widetilde{B}(\tilde{z}_{k_j}(.),\tilde{z}_{k_j}(.))$ is bounded in $L^2(\hat{\Omega},\hat{\mathcal{F}},\hat{\mathbb{P}};L^2(0,T;D(A)'))$(by using the estimate $(b2)$ of $\widetilde{B}$ and the estimates (82), (81)), it suffices to prove that for all $\varphi \in L^\infty(\hat{\Omega}\times[0,T],d\hat{\mathbb{P}}\times dt;D(A))$,

$$\hat{\mathbb{E}}\int_0^T \langle \widetilde{B}(\tilde{z}_{k_j}(s),\tilde{z}_{k_j}(s)),\varphi(s)\rangle\,ds \to \hat{\mathbb{E}}\int_0^T \langle \widetilde{B}(\tilde{z}(s),\tilde{z}(s)),\varphi(s)\rangle\,ds. \tag{109}$$

To prove (109), we write

$$\begin{aligned}
&\hat{\mathbb{E}}\int_0^T \langle \widetilde{B}(\tilde{z}_{k_j}(s), \tilde{z}_{k_j}(s)) - \widetilde{B}(\tilde{z}(s), \tilde{z}(s)), \varphi(s)\rangle\, ds \\
&= \hat{\mathbb{E}}\int_0^T \langle \widetilde{B}(\tilde{z}_{k_j}(s) - \tilde{z}(s), \tilde{z}_{k_j}(s)), \varphi(s)\rangle\, ds \\
&+ \hat{\mathbb{E}}\int_0^T \langle \widetilde{B}(\tilde{z}(s), \tilde{z}_{k_j}(s) - \tilde{z}(s)), \varphi(s)\rangle\, ds \\
&= I_{1j} + I_{2j},
\end{aligned} \tag{110}$$

where

$$I_{1j} = \hat{\mathbb{E}}\int_0^T \langle \widetilde{B}(\tilde{z}_{k_j}(s) - \tilde{z}(s), \tilde{z}_{k_j}(s)), \varphi(s)\rangle\, ds.$$

By the property $(b2)$ of $\widetilde{B}$, we have

$$I_{1j} \leq C\hat{\mathbb{E}}\int_0^T \|\tilde{z}_{k_j}(s) - \tilde{z}(s)\| \|\tilde{z}_{k_j}(s)\|_{D(A)} |A\varphi(s)|\, ds$$

Applying Cauchy-Schwarz's inequality, we get

$$I_{1j} \leq C_\varphi \left(\hat{\mathbb{E}}\int_0^T \|\tilde{z}_{k_j}(s) - \tilde{z}(s)\|^2\, ds\right)^{\frac{1}{2}} \left(\hat{\mathbb{E}}\int_0^T \|\tilde{z}_{k_j}(s)\|^2_{D(A)}\, ds\right)^{\frac{1}{2}}.$$

Using the strong convergence (94) and the boundedness of $\tilde{z}_{k_j}$ in $L^2(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}; L^2(0,T;D(A)))$, we infer that $I_{1j} \to 0$ as $j \to \infty$.
For the second term

$$I_{2j} = \hat{\mathbb{E}}\int_0^T \langle \widetilde{B}(\tilde{z}(s), \tilde{z}_{k_j}(s) - \tilde{z}(s)), \varphi(s)\rangle\, ds$$

we have $I_{2j} \to 0\ as\ j \to \infty$. In fact, the proof uses the property $(b2)$ of $\widetilde{B}$ and the weak convergence (84) since any strongly continuous linear operator is weakly continuous. This completes the proof of (109).
Next using the convergence (96), we also have

$$\int_0^{k_j[\frac{t}{k_j}]} \widetilde{F}(s, \tilde{z}_{k_j}(s))\, ds \to \int_0^t \widetilde{F}(s, \tilde{z}(s))\, ds \text{ in } L^2(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}; L^2(0,T;V)). \tag{111}$$

Finally by the same argument as in [5], [26] we can deduce from (73) and (105) that

$$\int_0^{k_j[\frac{t}{k_j}]} X_{k_j}(s) dB_j(s) \rightharpoonup \int_0^t \widetilde{G}(s, \tilde{z}(s)) d\widetilde{W}(s) \quad \text{weakly} \quad \text{in } L^2(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}; V). \tag{112}$$

Using the convergences (107)-(112), we can pass to the limit in (106) and obtain

$$\begin{aligned}
&\tilde{z}(t) + \int_0^t \widetilde{A}(\tilde{z}(s))\, ds + \int_0^t \widetilde{B}(\tilde{z}(s), \tilde{z}(s))\, ds \\
&= u_0 + \int_0^t \widetilde{F}(s, \tilde{z}(s))\, ds + \int_0^t \widetilde{G}(s, \tilde{z}(s)) d\widetilde{W}(s),
\end{aligned} \tag{113}$$

$\hat{\mathbb{P}}$-a.s. and for all $t \in [0,T]$.
In the same way $\bar{z}$ also satisfies (113). By Theorem 1, we know that the solution of problem (113) is unique. Therefore $\tilde{z} = \bar{z}$. Hence (z_{k_j}, z_{l_j}) tends to (u,u) in distribution, u given by Theorem 1. Then Lemma 1 implies that the whole sequence $\{z_k\}$ converges in $L^2(0,T;V)$ in probability to some random element u. Taking now the limit when $k \to 0$, we obtain that u is the variational solution of problem (1) given by Theorem 1. The proof of Theorem 3 is complete.

Acknowledgments

This work is supported by the Claude Leon Foundation Postdoctoral Fellowship and the University of Pretoria.

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