Approximation of the trajectory attractor of the 3D MHD System

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Abstract

We study the connection between the long-time dynamics of the 3D magnetohydrodynamic-α model and the exact 3D magnetohydrodynamic system. We prove that the trajectory attractor $\mathcal{U}_\alpha$ of the 3D magnetohydrodynamic-α model converges to the trajectory attractor $\mathcal{U}_0$ of the 3D magnetohydrodynamic system (in an appropriate topology) when α approaches zero.

Key words and phrases: magnetohydrodynamic-alpha model, trajectory attractor, magnetohydrodynamic equations, Navier-Stokes equations

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1 Introduction

In this paper, we study the relations between the global dynamics of the 3D magnetohydrodynamic-α model (MHD-α model) and the 3D magnetohydrodynamic system (MHD system) with periodic boundary conditions. The MHD-α model was introduced in [21] and was inspired by the Navier-Stokes-α model (also known as the viscous Camassa-Holm system or Lagrangien averaged Navier-Stokes-α equations) of turbulence (see [4, 5, 6, 7, 18, 19]). It was demonstrated analytically and computationally in many works that the Navier-Stokes-α model is a powerful tool in the study of turbulence (see [4, 5, 6, 7, 18, 19]). It was proved analytically in [21] that the 3D MHD-α model preserves some of the original properties of the 3D MHD system when alpha approaches zero. Direct numerical simulations were performed in [25] with periodic boundary conditions. This model is a regularized approximation of the 3D MHD system and depending on a small parameter α. For $\alpha = 0$, the model is reduced to the 3D MHD system.

It is well known that the uniqueness theorem for the solution of the boundary value problem still remains unproved for the 3D MHD system (see [28]). One cannot use the classical methods based on the analysis of the global attractor of the corresponding semigroup to discuss the behavior of solutions to this equation when the time approaches infinity.

The method of trajectory attractors for evolution partial differential equations was developed in ([8]-[11]). This approach is highly fruitful in the study of the long-time behavior of solutions to evolution equations for which the uniqueness theorem related to the corresponding initial-value problem is not proved yet (e.g. the 3D Navier-Stokes system, the 3D MHD system) or fails. For alternative approaches, the reader is referred to [2, 24, 20, 3, 30] and references therein.

In [21], the Cauchy problem for the 3D MHD-α model was studied, the global existence, uniqueness and regularity of weak solutions were established. It was proved that there exists a subsequence of solutions of the 3D MHD-α model that converges to one of the Leray-Hopf weak solutions of the 3D MHD system with periodic boundary conditions. Similar studies were investigated for the 3D Navier-Stokes-α model (see [19]). The stochastic version were also studied in [15].

In the present paper, we study the approximation of the trajectory attractor of the 3D MHD system by the trajectory attractor of the 3D MHD-α model. In the case of the 3D Navier-Stokes system, it was proved in [12] that the trajectory attractor of the 3D Navier-Stokes-α model converges to the trajectory attractor of the 3D Navier-Stokes system in an appropriate topology when alpha approaches zero. Similar results were established in [13, 14] for the Leray-α

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model of turbulence. Our object here is to generalize the results in [12] from the Navier-Stokes system to the MHD system. Our main result (see Theorem 6) states that bounded families of solutions $\{(u_\alpha(t), B_\alpha(t))\}$ of the 3D MHD-α model converge to the trajectory attractor $\mathcal{U}_0$ of the 3D MHD system when alpha approaches zero and t approaches infinity. In particular, the trajectory attractor $\mathcal{U}_\alpha$ of the 3D MHD-α model converges to the trajectory attractor $\mathcal{U}_0$ of the 3D MHD system. The proof is inspired by the work in [12]. Let us point out that one of the main difference between this work and that of [12] is the presence of magnetic field which makes the analysis of the problem studied in this article more involved. One of the main difficulty lies in obtaining needed a priori estimates in which the constants are independent of alpha and the passage to the limit which turns out to be rather complicated in view of the nature of the nonlinear terms involved in our 3D MHD-α model (see the proof of Theorem 5).

The article is structured as follows. In Section 2, we consider the 3D MHD system and we construct its trajectory attractor $\mathcal{U}_0$. For this purpose, we define spaces $\mathcal{F}_+^b$ and $\mathcal{F}_+^{loc}$ which contain weak solutions of the 3D MHD system. We then introduce the space of trajectory space $\mathcal{K}^+$ of Leray-Hopf weak solutions of the 3D MHD system on the semiaxis $0 < t < \infty$. The space $\mathcal{F}_+^{loc}$ is equipped with the weak topology $\mathcal{O}_+^{loc}$ generated by the weak convergence of sequences $\{(u_n(t), B_n(t))\} \subset \mathcal{F}_+^{loc}$. We prove that the trajectory space $\mathcal{K}^+$ is bounded in $\mathcal{F}_+^b$ and closed in the topology $\mathcal{O}_+^{loc}$. We consider the time translation semigroup $\{T(h)\} := \{T(h), h \geq 0\}$ acting on the trajectory space $\mathcal{K}^+$ by the formula

$$T(h)(u(t), B(t)) = (u(t+h), B(t+h)).$$

It follows from the definition of the trajectory space that $\mathcal{K}^+$ is invariant under $\{T(h)\}$. Using these facts and applying the theory of trajectory attractors, we prove that the translation semigroup $\{T(h)\}$ acting on $\mathcal{K}^+$ has a global attractor $\mathcal{U}_0$, which we call the trajectory attractor of the 3D MHD system. To describe the structure of the trajectory attractor $\mathcal{U}_0$, we define the kernel $\mathcal{K}_0$ of the 3D MHD system and prove that $\mathcal{U}_0 = \Pi_+\mathcal{K}_0$ where Π_+ is the restriction operator on the semiaxis $\mathbb{R}^+$. In Section 3, we consider the 3D MHD-α model. The corresponding initial value problem is well-posed and we construct the trajectory attractor $\mathcal{U}_\alpha$ for this system. In Section 4, we study the convergence of the solutions of the 3D MHD-α model as α approaches zero. For this, we study the system for which the couple $(w_\alpha(t), B_\alpha(t))$ is satisfied where $w_\alpha(t) = (1-\alpha^2\Delta)^{\frac{1}{2}} u_\alpha(t)$ and $(u_\alpha(t), B_\alpha(t))$ is the solution of the 3D MHD-α model. The main result of this section states that if a sequence of solutions $(w_{\alpha_n}(t), B_{\alpha_n}(t))$ of the mentioned above system converges to the limit $(w(t), B(t))$ in the space $\mathcal{O}_+^{loc}$ as α_n approaches zero and n tends to infinity, then $(w(t), B(t))$ is a Leray-Hopf weak solution of the 3D MHD system (see Theorem 5). In Section 5, using the result of Section 4, we prove the convergence of trajectory attractors $\mathcal{U}_\alpha$ to the trajectory attractor $\mathcal{U}_0$ in the space $\mathcal{O}_+^{loc}$ when alpha approaches zero.

2 Trajectory attractor of the 3D MHD system

2.1 The 3D MHD system

Let $\Omega = [0, L]^3$, where $L > 0$. We consider the autonomous 3D MHD system with periodic boundary conditions

$$\frac{\partial u}{\partial t} + (u.\nabla)u - \nu\Delta u + \nabla\pi + \frac{1}{2}\nabla|B|^2 = (B.\nabla)B + g(x), \tag{1}$$

$$\frac{\partial B}{\partial t} + (u.\nabla)B - (B.\nabla)u - \eta\Delta B = 0, \tag{2}$$

$$\nabla.u = 0, \ \nabla.B = 0, \tag{3}$$

$$\int_\Omega u(x,t)\,dx = 0, \ \int_\Omega B(x,t)\,dx = 0, \tag{4}$$

where $u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$, $B(x,t) = (B_1(x,t), B_2(x,t), B_3(x,t))$ and π are the unknown, representing respectively the velocity of the fluid, the magnetic field and the scalar pressure at each point of the fluid. In the system above, ν is the kinematic viscosity of the fluid , η the magnetic diffusivity and g is a given periodic field of external forces. We equip system (1)-(4) with the initial conditions

$$u(x,0) = u_0(x),\ B(x,0) = B_0(x). \tag{5}$$

We introduce some notations and background following the mathematical theory of Navier-Stokes equations (NSE) (see [27]). $L^p(\Omega)$ and $H^m(\Omega)$ denote the L^p-Lebesgue space and Sobolev space respectively. We denote by $|.|$ the L^2-norm, and by $(.,.)$ the L^2-inner product. Let X be a linear subspace of integrable functions defined on the domain Ω, we define

$$\dot{X} = \{\varphi \in X : \int_\Omega \varphi(x)\,dx = 0\},$$

and

$$\mathcal{V} = \{\varphi : \varphi \text{ is vector valued trigonometric polynomial defined in } \Omega, \nabla.\varphi = 0 \text{ and } \int_\Omega \varphi(x)\,dx = 0\}.$$

The spaces H and V are the closures of $\mathcal{V}$ in $L^2(\Omega)^3$ and $H^1(\Omega)^3$ respectively. Let $P : \dot{L}^2 \to H$ be the Helmholtz-Leray projection, and $A = -P\Delta$ be the Stokes operator with domain $D(A) = H^2(\Omega)^3 \cap V$. In the periodic boundary conditions $A = -\Delta|_{D(A)}$ is a self-adjoint positive operator with compact inverse. Hence the space H has an orthonormal basis $\{w_j\}_{j=1}^\infty$ of eigenfunctions of A, $Aw_j = \lambda_j w_j$, with $0 < \lambda_1 \leq \lambda_2 \leq ..., \leq \lambda_j \sim j^{\frac{2}{d}} L^{-2}$. On can show that $V = D(A^{\frac{1}{2}})$. We denote $((u,v)) = (A^{\frac{1}{2}}u, A^{\frac{1}{2}}v) = (\nabla u, \nabla v)$ and $\|u\| = |A^{\frac{1}{2}}u|$ the scalar product and the norm on V, respectively. For $f \in V'$, we denote by $\langle f, v\rangle$ the action of the functional $f \in V'$ on any $v \in V$. The operator A is an isomorphism from V to V' and $((u,v)) = \langle Au, v\rangle$ for $u, v \in V$. In the sequel, we identify H with its dual and we have the following inclusions

$$D(A) \subset V \subset H' = H \subset V' \subset D(A)', \tag{6}$$

where each space is densely and compactly embedded in the next one. Following the notation of the NSE, we denote

$$\mathbb{B}(u,v) = P[(u.\nabla)v] = P\sum_{j=1}^{3} u^j \partial_{x_j} v. \tag{7}$$

For u satisfying $\nabla.u = 0$, we have

$$\mathbb{B}(u,v) = P\sum_{j=1}^{3} \partial_{x_j}(u^j v). \tag{8}$$

It follows that

$$\langle \mathbb{B}(u,v), w\rangle = -\langle \mathbb{B}(u,w), v\rangle \text{ and } \langle \mathbb{B}(u,v), v\rangle = 0 \text{ for all } u, v, w \in V. \tag{9}$$

For all $w \in D(A)$ and $u, v \in V$, we have the estimate

$$|\langle \mathbb{B}(u,v), w\rangle| \leq C|u|\|v\|\|w\|_{L^\infty} \leq C_1 \lambda_1^{-\frac{1}{4}} |u|\|v\|\|w\|_{D(A)},$$

and therefore

$$\|\mathbb{B}(u,v)\|_{D(A)'} \leq C_1 \lambda_1^{-\frac{1}{4}} |u|\|v\|, \tag{10}$$

where $D(A)'$ is the dual space of $D(A)$.
The Pioncar$\acute{e}$ inequality implies that

$$\lambda_1|u|^2 \leq \|u\|^2 \text{ for } \text{ u}\in \text{V}, \quad (11)$$
$$\lambda_1\|u\|^2 \leq |Au|^2 \text{ for } \text{ u}\in \text{D(A)}. \quad (12)$$

We recall the following inequality in three dimension

$$\|f\|_{L^4(\Omega)} \leq C\|f\|^{\frac{1}{4}}_{L^2(\Omega)}\|\nabla f\|^{\frac{3}{4}}_{L^2(\Omega)}, \quad (13)$$

where $f \in V$ and C is a constant independent of Ω (see [27], [23]). It follows from inequality (13) that

$$|u^i v|_{L^2(\Omega)} \leq C_1|u|^{\frac{1}{4}}\|u\|^{\frac{3}{4}}|v|^{\frac{1}{4}}\|v\|^{\frac{3}{4}}, \quad (14)$$
$$|u^i u|_{L^2(\Omega)} \leq C_1|u|^{\frac{1}{2}}\|u\|^{\frac{3}{2}} \quad (15)$$

for $u, v \in V$. By virtue of (8), we also have

$$\|\mathbb{B}(u)\|_{V'} \leq C_1|u|^{\frac{1}{2}}\|u\|^{\frac{3}{2}}, \quad (16)$$
$$\|\mathbb{B}(u,v)\|_{V'} \leq C_1|u|^{\frac{1}{4}}\|u\|^{\frac{3}{4}}|v|^{\frac{1}{4}}\|v\|^{\frac{3}{4}}, \quad (17)$$

for $u, v \in V$.
We assume $g \in H$. Using the above notations, we apply the Leray-Helmholtz projector P to the system (1)-(4) to obtain as for the case of the NSE, the equivalent system of equations

$$\partial_t u + \nu Au + \mathbb{B}(u,u) - \mathbb{B}(B,B) = g, \quad (18)$$
$$\partial_t B + \eta AB + \mathbb{B}(u,B) - \mathbb{B}(B,u) = 0. \quad (19)$$

For a given $M > 0$, and for any couple of functions $(u(.), B(.)) \in L^2(0,M; V\times V) \cap L^\infty(0,M; H\times H)$, it follows that

$$Au \in L^2(0,M;V'), \quad (20)$$
$$AB \in L^2(0,M;V'), \quad (21)$$

and due to (10), we also have

$$\mathbb{B}(u(.),u(.)) \in L^2(0,M;D(A)'), \quad (22)$$
$$\mathbb{B}(u(.),B(.)) \in L^2(0,M;D(A)'), \quad (23)$$
$$\mathbb{B}(B(.),u(.)) \in L^2(0,M;D(A)'), \quad (24)$$
$$\mathbb{B}(B(.),B(.)) \in L^2(0,M;D(A)'). \quad (25)$$

We can now find a couple of functions (u, B) of equations (18),(19) in the space of distributions $\mathcal{D}'(0,M;D(A)')$ such that

$$\partial_t u \in L^2(0,M;D(A)') \text{ and } \partial_t B \in L^2(0,M;D(A)').$$

A couple of vector functions $(u(.), B(.))$ is said to be a weak solution of system (18)-(19) if for every $M > 0$,

$$(u,B) \in L^2(0,M;V\times V) \cap L^\infty(0,M;H\times H), \quad (26)$$

the function u satisfies equation (18) in the distribution sense of the space $\mathcal{D}'(0,M;D(A)')$ and the function B satisfies equation (19) in the space of distribution $\mathcal{D}'(0,M;D(A)')$.

Since a weak solution (u, B) belongs to $L^\infty(0, M; H \times H)$, then using the well-known Lions-Magenes lemma (see [22]), we have

$$u(.) \in C_w([0, M]; H) \text{ and } B(.) \in C_w([0, M]; H),$$

where $C_w([0, M]; H)$ denotes the space of weakly continuous function from $[0, M]$ to H. Consequently for every $t \geq 0$, the values $u(t)$ and $B(t)$ make sense in the space H and, in particular, the initial conditions

$$u(0) = u_0(x) \in H, \tag{27}$$
$$B(0) = B_0(x) \in H, \tag{28}$$

are meaningful.

We now formulate the classical theorem on the existence of a weak solution of the Cauchy problem for the 3D MHD system in the form we need in the sequel (see [17],[26]).

Theorem 1. *Let $g \in H$ and $(u_0, B_0) \in H \times H$. Then for every $M > 0$, there exists a weak solution (u, B) of system (18)-(19) from the space $L^2(0, M; V \times V) \cap L^\infty(0, M; H \times H)$ such that $u(0) = u_0, B(0) = B_0$ and (u, B) satisfies the energy inequality*

$$\frac{1}{2}\frac{d}{dt}(|u(t)|^2 + |B(t)|^2) + \nu|u(t)\|^2 + \eta\|B(t)\|^2 \leq \langle g, u(t)\rangle, \text{ for almost all } t \in [0.M]. \tag{29}$$

Remark 1. *Inequality (29) means that for any $\psi \in C_0^\infty(]0, M[), \psi(t) \geq 0$,*

$$-\frac{1}{2}\int_0^M (|u(t)|^2 + |B(t)|^2)\psi'(t)dt + \int_0^M (\nu\|u(t)\|^2 + \eta\|B(t)\|^2)\psi(t)dt \leq \int_0^M \langle g, u(t)\rangle\psi(t)dt. \tag{30}$$

The proof of Theorem 1 uses the Galerkin approximation method. For every $m \in \mathbb{N}$, we construct the Galerkin approximation $(u_m(x,t), B_m(x,t)) \in C^1([0, M]; H^2 \cap V)^2$ of order m, that is a solution of the corresponding system of ordinary differential equations, and prove the existence of a subsequence $\{m_j\} \subset \{m\}$ such that $(u_{m_j}(x,t), B_{m_j}(x,t))$ converges in a weak sense to a weak solution $(u(x,t), B(x,t))$ of problem (18)-(19),(27)-(28). The Galerkin approximation $(u_m(x,t), B_m(x,t))$ satisfies

$$\frac{1}{2}\frac{d}{dt}|u_m(t)|^2 + \nu\|u_m(t)\|^2 - \langle \mathbb{B}(B_m(t), B_m(t)), u_m(t)\rangle = \langle g, u_m(t)\rangle, \tag{31}$$
$$\frac{1}{2}\frac{d}{dt}|B_m(t)|^2 + \eta\|B_m(t)\|^2 - \langle \mathbb{B}(B_m(t), u_m(t)), B_m(t)\rangle = 0 \tag{32}$$

Summing up (31) and (32) and taking into account (9), we obtain the following energy equality

$$\frac{1}{2}\frac{d}{dt}(|u_m(t)|^2 + |B_m(t)|^2) + \nu\|u_m(t)\|^2 + \eta\|B_m(t)\|^2 = \langle g, u_m(t)\rangle, t \in [0, M]. \tag{33}$$

Passing to a limit in (33) in a weak sense as $m_j \to \infty$, we obtain (29) in the form (30)(see [17],[26]).

Remark 2. *For the 3D MHD system, the question of the uniqueness of a weak solution of problem (18)-(19),(27)-(28) remains open. It is also unknown, whether every weak solution satisfies the energy inequality (29). Nevertheless, it is known that every weak solution resulting from the Galerkin approximation method satisfies the energy inequality (29). The class of weak solutions which satisfy the energy inequality (29) is called Leray-Hopf weak solutions.*

Now, we establish some estimates for a weak solution of the 3D MHD system.

Proposition 1. ***(A Priori estimates)***
For any weak solution $(u(t), B(t))$ of problem (18)-(19),(27)-(28), the following inequalities hold:

$$i)\quad |u(t)|^2 + |B(t)|^2 \leq (|u(0)|^2 + |B(0)|^2)e^{-\mu\lambda_1 t} + \frac{|g|^2}{\lambda_1^2\mu^2}, \tag{34}$$

$$ii)\quad \mu\int_t^{t+1}(\|u(s)\|^2 + \|B(s)\|^2)ds \leq (|u(0)|^2 + |B(0)|^2)e^{-\mu\lambda_1 t} + \frac{|g|^2}{\lambda_1^2\mu^2} + \frac{|g|^2}{\mu\lambda_1}, \tag{35}$$

where $\mu = min(\nu, \eta)$.

Proof. The proof uses the energy inequality (29), the Poincaré inequality and follows the same line as in a case of the Navier-Stokes system (see [11]). □

2.2 Construction of the trajectory attractor of 3D MHD system

We now construct the trajectory attractor of the 3D MHD system. At first, we define the trajectory space $\mathcal{K}^+$ of system (18),(19).

Definition 1. *The trajectory space $\mathcal{K}^+$ is the set of all Leray-Hopf weak solutions (u, B) of system (18),(19) in the space $L_2^{loc}(\mathbb{R}_+; V \times V) \cap L_\infty^{loc}(\mathbb{R}_+; H \times H)$ that satisfy the energy inequality (29) for $t \geq 0$, that is*

$$-\frac{1}{2}\int_0^\infty (|u(t)|^2 + |B(t)|^2)\psi'(t)dt + \int_0^\infty (\nu\|u(t)\|^2 + \eta\|B(t)\|^2)\psi(t)dt \leq \int_0^\infty < g, u(t) > \psi(t)dt,$$

for all $\psi \in C_0^\infty(\mathbb{R}_+), \psi \geq 0$.

We note that by Theorem 1, the trajectory space $\mathcal{K}^+$ is nonempty that is for any $(u_0, B_0) \in H \times H$, there is a trajectory (u, B) such that $u(0) = u_0$ and $B(0) = B_0$.
Let us now define the spaces $\mathcal{F}_+^{loc}$, $\mathcal{F}_+^b$ and the topology Θ_+^{loc}. Set

$$\mathcal{F}_+^{loc} = \{z = (u, B) \;/\; (u, B)(.) \in L_2^{loc}(\mathbb{R}_+; V \times V) \cap L_\infty^{loc}(\mathbb{R}_+; H \times H),$$
$$\partial_t u(.) \in L_2^{loc}(\mathbb{R}_+; D(A)'), \partial_t B(.) \in L_2^{loc}(\mathbb{R}_+; D(A)')\}.$$

In the space $\mathcal{F}_+^{loc}$, we define the following local weak convergence topology. By definition, a sequence of functions $\{(u_n(.), B_n(.)), n \in \mathbb{N}\} \subset \mathcal{F}_+^{loc}$ converges to $(u(.), B(.)) \in \mathcal{F}_+^{loc}$ in Θ_+^{loc} as $n \to \infty$ if, for each $M > 0$, the following limit relations hold:

$$\begin{aligned}
&u_n \rightharpoonup u \text{ weakly star in } L^\infty(0, M; H) \textit{ and weakly in } L^2(0, M; V),\\
&\partial_t u_n \rightharpoonup \partial_t u \text{ weakly in } L^2(0, M; D(A)'),\\
&B_n \rightharpoonup B \text{ weakly star in } L^\infty(0, M; H) \text{ and weakly in } L^2(0, M; V),\\
&\partial_t B_n \rightharpoonup \partial_t B \textit{ in } L^2(0, M; D(A)').
\end{aligned}$$

The space $\mathcal{F}_+^{loc}$ equipped with the topology Θ_+^{loc} is a Hausdorff Fréchet-Uryhson topological vector space with a countable base (see [11]).
We consider a linear subspace $\mathcal{F}_+^b \subset \mathcal{F}_+^{loc}$ consisting of vector functions $(u, B) \in \mathcal{F}_+^{loc}$ with finite norm

$$\begin{aligned}\|(u, B)\|_{\mathcal{F}_+^b} :=& \|(u, B)\|_{L_2^b(\mathbb{R}_+; V\times V)} + \|(u, B)\|_{L^\infty(\mathbb{R}_+; H\times H)} +\\
&\|\partial_t u\|_{L_2^b(\mathbb{R}_+; D(A)')} + \|\partial_t B\|_{L_2^b(\mathbb{R}_+; D(A)')},\end{aligned} \tag{36}$$

where

$$\|(u,B)\|^2_{L^b_2(\mathbb{R}_+;V\times V)} = \sup_{t\geq 0}\int_t^{t+1}\|u(s)\|^2\,ds + \sup_{t\geq 0}\int_t^{t+1}\|B(s)\|^2\,ds,$$
$$\|(u,B)\|_{L^\infty(\mathbb{R}_+;H\times H)} = ess\sup_{t\geq 0}|u(t)| + ess\sup_{t\geq 0}|B(t)|,$$
$$\|\partial_t u\|^2_{L^b_2(\mathbb{R}_+;D(A)')} = \sup_{t\geq 0}\int_t^{t+1}\|\partial_t u(s)\|^2_{D(A)'}\,ds,$$
$$\|\partial_t B\|^2_{L^b_2(\mathbb{R}_+;D(A)')} = \sup_{t\geq 0}\int_t^{t+1}\|\partial_t B(s)\|^2_{D(A)'}\,ds.$$

Recall that the norm of a function ϕ in the space $L^b_p(\mathbb{R}_+;X)$ where X is Banach space and $p\geq 1$, is defined by the formula $\|\phi\|^p_{L^b_p(\mathbb{R}_+;X)} := \sup_{t\geq 0}\int_t^{t+1}\|\phi(s)\|^p_X\,ds$.
The space $\mathcal{F}^b_+$ with the norm (36) is a Banach space.

Remark 3. *Any ball $B_r = \{(u,B)\in\mathcal{F}^b_+ \ /\ \|(u,B)\|_{\mathcal{F}^b_+}\leq r\}$ in the space $\mathcal{F}^b_+$ is compact in the topology Θ^{loc}_+. Moreover the corresponding topological subspace $B_r|_{\Theta^{loc}_+}$ is metrisable (see [11]). Note that the space $\mathcal{F}^{loc}_+|_{\Theta^{loc}_+}$ is not metrisable.*

Consider the translation semigroup $\{T(h)\} := \{T(h), h\geq 0\}$ acting on $\mathcal{F}^{loc}_+$ by the formula

$$T(h)(u(t),B(t)) = (u(t+h),B(t+h)), t\geq 0. \tag{37}$$

The semigroup $\{T(h)\}$ takes $\mathcal{K}^+$ to itself that is $T(h):\mathcal{K}^+\to\mathcal{K}^+$ for all $h\geq 0$.
We are going to construct the global attractor of the translation semigroup $\{T(h)\}$ on $\mathcal{K}^+$. We call this attractor the trajectory attractor. The following key proposition is crucial for the proof.

Proposition 2. *Let $g\in H$, then*

1. *The space $\mathcal{K}^+\subset\mathcal{F}^b_+$.*
2. *For any couple of functions $(u,B)\in\mathcal{K}^+$*

$$\|T(h)(u,B)\|_{\mathcal{F}^b_+}\leq C_0\|(u,B)(.)\|^2_{L^\infty(0,1;H\times H)}e^{-\mu\lambda_1 h} + R_0^2, \tag{38}$$

where the constant C_0 depends on μ,λ_1 and R_0 depends on $\mu,\lambda_1,|g|$.

For the proof of Proposition 2, we will need some additional estimates on weak solution of the 3D MHD system.

Proposition 3. *If $(u,B)\in\mathcal{K}^+$, then*

1.
$$\left(\int_t^{t+1}\|\partial_t u(s)\|^2_{D(A)'}\,ds\right)^{\frac{1}{2}}\leq C_2\left(|u(0)|^2+|B(0)|^2\right)e^{-\mu\lambda_1 t} + R_1^2, \tag{39}$$

where C_2 depends on λ_1,μ; R_1 depends on λ_1,μ and $|g|$.

2.
$$\left(\int_t^{t+1}\|\partial_t B(s)\|^2_{D(A)'}\,ds\right)^{\frac{1}{2}}\leq C_3\left(|u(0)|^2+|B(0)|^2\right)e^{-\mu\lambda_1 t} + R_2^2, \tag{40}$$

where C_3 depends on λ_1,μ; R_2 depends on λ_1,μ and $|g|$.

Proof. For $(u,B)\in\mathcal{K}^+$, we have

$$\partial_t u = -\nu Au - \mathbb{B}(u,u) + \mathbb{B}(B,B) + g, \tag{41}$$
$$\partial_t B = -\eta AB - \mathbb{B}(u,B) + \mathbb{B}(B,u). \tag{42}$$

Using (41) and applying the Minkowski inequality, we have

$$\begin{aligned}\left(\int_t^{t+1}\|\partial_t u(s)\|^2_{D(A)'}\right)^{\frac{1}{2}} \le \nu\left(\int_t^{t+1}\|Au(s)\|^2_{D(A)'}\,ds\right)^{\frac{1}{2}} + \left(\int_t^{t+1}\|\mathbb{B}(u(s),u(s))\|^2_{D(A)'}\,ds\right)^{\frac{1}{2}}\\ + \left(\int_t^{t+1}\|\mathbb{B}(B(s),B(s))\|^2_{D(A)'}\,ds\right)^{\frac{1}{2}} + \|g\|_{D(A)'}.\end{aligned} \tag{43}$$

We now estimate each term on the right.

$$\begin{aligned}\int_t^{t+1}\|Au(s)\|^2_{D(A)'} &= \int_t^{t+1}|u(s)|^2\,ds\\ &\le (|u(0)|^2+|B(0)|^2)e^{-\mu\lambda_1 t} + \frac{|g|^2}{\mu^2\lambda_1^2}.\end{aligned} \tag{44}$$

From the estimates (10), (34), (35), we have

$$\begin{aligned}&\int_t^{t+1}\|\mathbb{B}(u(s),u(s)\|^2_{D(A)'}\,ds\\ &\le C_4\int_t^{t+1}\|u(s)\|^2|u(s)|^2\,ds\\ &\le C_4 ess\sup_{t\le s\le t+1}|u(s)|^2\int_t^{t+1}\|u(s)\|^2\,ds\\ &\le C_4\mu^{-1}\left((|u(0)|^2+|B(0)|^2)e^{-\mu\lambda_1 t} + \frac{|g|^2}{\lambda_1^2\mu^2} + \frac{|g|^2}{\lambda_1\mu}\right)^2.\end{aligned}$$

We then deduce that

$$\left(\int_t^{t+1}\|\mathbb{B}(u(s),u(s))\|^2_{D(A)'}\,ds\right)^{\frac{1}{2}} \le C_5\left(|u(0)|^2+|B(0)|^2\right)e^{-\mu\lambda_1 t} + R_2^2, \tag{45}$$

where $C_5 = C_1\lambda_1^{-\frac{1}{4}}\mu^{-\frac{1}{2}}$ and $R_2^2 = \frac{|g|^2}{\lambda_1^2\mu^2} + \frac{|g|^2}{\lambda_1\mu}$.
We also have from the estimates (10), (34), (35),

$$\left(\int_t^{t+1}\|\mathbb{B}(B(s),B(s))\|^2_{D(A)'}\,ds\right)^{\frac{1}{2}} \le C_5\left(|u(0)|^2+|B(0)|^2\right)e^{-\mu\lambda_1 t} + R_2^2. \tag{46}$$

Conbining (43), (44), (45) and (46), we arrive at

$$\begin{aligned}&\left(\int_t^{t+1}\|\partial_t u(s)\|^2_{D(A)'}\right)^{\frac{1}{2}}\\ &\le \nu\left((|u(0)|^2+|B(0)|^2)e^{-\mu\lambda_1 t} + R_3^2\right)^{\frac{1}{2}} + 2C_5\left((|u(0)|^2+|B(0)|^2)e^{-\nu\lambda_1 t} + R_2^2\right) + \lambda_1^{-1}|g|\\ &\le C_2\left(|u(0)|^2+|B(0)|^2\right)e^{-\nu\lambda_1 t} + R_1^2,\end{aligned}$$

where $C_2 = \nu + 2C_5$, $R_1^2 = \nu(R_3^2+1) + 2C_5R_2^2 + \lambda_1^{-1}|g|$, $R_3^2 = \frac{|g|^2}{\mu^2\lambda_1^2}$ and $R_2^2 = \frac{|g|^2}{\lambda_1^2\mu^2} + \frac{|g|^2}{\lambda_1\mu}$.
This completes the proof of (39). Using (42), we can also prove the estimate (40) in a similar way. □

Proof. (Proof of Proposition 2)

The proof of (38) follows from Proposition 1 and Proposition 3. The proof of the inclusion $\mathcal{K}^+ \subset \mathcal{F}_+^b$ follows from (38) by setting $h = 0$.

□

It follows from the definition of the topology Θ_+^{loc} that the translation semigroup $\{T(h)\}$ is continuous in Θ_+^{loc}. Hence also on $\mathcal{K}^+$ as well. The following assertion proves that the trajectory space $\mathcal{K}^+$ is closed in the space Θ_+^{loc}.

Proposition 4. *The space $\mathcal{K}^+$ is closed in Θ_+^{loc}.*

Proof. Consider an arbitrary sequence $(u_n(t), B_n(t)) \in \mathcal{K}^+$ which converges as $n \to \infty$ *in* Θ_+^{loc} to an element $(u(t), B(t)) \in \mathcal{F}_+^{loc}$. We prove that $(u(t), B(t)) \in \mathcal{K}^+$. By the definition of the topology Θ_+^{loc}, for every segment $[0, M]$, the following convergence hold as $n \to \infty$

$$u_n \rightharpoonup u \textit{ weakly star in } L^\infty(0, M; H) \textit{ and weakly} \text{ in } L^2(0, M; V), \tag{47}$$

$$\partial_t u_n \rightharpoonup \partial_t u \textit{ weakly} \text{ in } L^2(0, M; D(A)'), \tag{48}$$

$$B_n \rightharpoonup B \textit{ weakly star} \text{ in } L^\infty(0, M; H) \textit{ and weakly} \text{ in } L^2(0, M; V), \tag{49}$$

$$\partial_t B_n \rightharpoonup \partial_t B \text{ in } L^2(0, M; D(A)'). \tag{50}$$

In particular the sequence (u_n) is bounded in $L^\infty(0, M; H)$ and $L^2(0, M; V)$, the sequence (B_n) is bounded in $L^\infty(0, M; H)$ and $L^2(0, M; V)$, whereas the sequences $(\partial_t u_n)$ and $(\partial_t B_n)$ are bounded in $L^2(0, M; D(A)')$. Hence , due to inequalities $(16) - (17)$, the sequences $(\mathbb{B}(u_n, u_n))$, $(\mathbb{B}(u_n, B_n))$, $(\mathbb{B}(B_n, B_n))$, $(\mathbb{B}(B_n, u_n))$ are bounded in the space $L^{\frac{4}{3}}(0, M; V')$. Then, passing to a subsequence $\{n'\} \subset \{n\}$ and keeping the notation $\{n\}$, we can assume that

$$\mathbb{B}(u_n, u_n) \rightharpoonup B_1(.) \;\text{ weakly }\; \text{in } L^{\frac{4}{3}}(0, M; V'), \tag{51}$$

$$\mathbb{B}(u_n, B_n) \rightharpoonup B_2(.) \;\text{ weakly }\; \text{in } L^{\frac{4}{3}}(0, M; V'), \tag{52}$$

$$\mathbb{B}(B_n, B_n) \rightharpoonup B_3(.) \;\text{ weakly }\; \text{in } L^{\frac{4}{3}}(0, M; V'), \tag{53}$$

$$\mathbb{B}(B_n, u_n) \rightharpoonup B_4(.) \text{ weakly }\; \text{in } \; L^{\frac{4}{3}}(0, M; V'), \tag{54}$$

where $B_i = B_i(x, t), i = 1, 2, 3, 4$ are some elements of the space $L^{\frac{4}{3}}(0, M; V')$.
Since (u_n, B_n) is a weak solution of system (18), (19), we have

$$\partial_t u_n + \nu A u_n + \mathbb{B}(u_n, u_n) - \mathbb{B}(B_n, B_n) = g(x),$$
$$\partial_t B_n + \eta A B_n + \mathbb{B}(u_n, B_n) - \mathbb{B}(B_n, u_n) = 0.$$

Using (47)-(50) and (51)-(54), we conclude that the couple (u, B) satisfies

$$\partial_t u + \nu A u + B_1(x, t) - B_3(x, t) = g(x), t \geq 0,$$
$$\partial_t B + \eta A B + B_2(x, t) - B_4(x, t) = 0,$$

in the distribution sense. By the Aubin compactness theorem (see [23],[1],[16]), we have

$$u_n \to u \text{ strongly }\; \text{in } L^2(0, M; H), \tag{55}$$

$$B_n \to B \text{ strongly }\; \text{in } L^2(0, M; H). \tag{56}$$

Passing to a subsequence gives

$$u_n \to \text{ u for a.e. (x,t) } \in \Omega \times]0, M[,$$
$$B_n \to \text{ u for a.e. (x,t) } \in \Omega \times]0, M[.$$

Applying the known Lions lemma concerning the weak convergence (see [23],Chap.1,Lemma 1.3), we have the following limit relations as $n \to \infty$:

$$\begin{aligned}
&\mathbb{B}(u_n, u_n) \rightharpoonup \mathbb{B}(u, u) \text{ weakly in } L^{\frac{4}{3}}(0, M; V'),\\
&\mathbb{B}(u_n, B_n) \rightharpoonup \mathbb{B}(\text{u,B}) \text{ weakly in } L^{\frac{4}{3}}(0, M; V'),\\
&\mathbb{B}(B_n, u_n) \rightharpoonup \mathbb{B}(B, u) \text{ weakly in } L^{\frac{4}{3}}(0, M; V'),\\
&\mathbb{B}(B_n, B_n) \rightharpoonup \mathbb{B}(B, B) \text{ weakly in } L^{\frac{4}{3}}(0, M; V').
\end{aligned} \tag{57}$$

Hence, due to (51)-(54), we conclude that $B_1(x,t) = \mathbb{B}(u,u), B_2(x,t) = \mathbb{B}(u,B), B_3(x,t) = \mathbb{B}(B,B), B_4(x,t) = \mathbb{B}(B,u)$ for a.e. $\Omega \times (0,M)$. That is the couple (u,B) is a weak solution of system (18),(19). It remains to prove that (u,B) satisfies the energy inequality (30):

$$-\frac{1}{2}\int_0^M (|u(t)|^2 + |B(t)|^2)\psi'(t)dt + \int_0^M (\nu\|u(t)\|^2 + \eta\|B(t)\|^2)\psi(t)dt \le \int_0^M \langle g, u(t)\rangle\psi(t)dt. \tag{58}$$

for all $\psi \in C_0^\infty(\mathbb{R}_+), \psi \ge 0$.
The couple (u_n, B_n) satisfies the energy inequality (30) that is

$$-\frac{1}{2}\int_0^M (|u_n(t)|^2 + |B_n(t)|^2)\psi'(t)dt + \int_0^M (\nu\|u_n(t)\|^2 + \eta\|B_n(t)\|^2)\psi(t)dt \le \int_0^M \langle g, u_n(t)\rangle\psi(t)dt. \tag{59}$$

for all $\psi \in C_0^\infty(\mathbb{R}_+), \psi \ge 0$.
From $(55) - (56)$ and the Lebesgue dominant convergence theorem, we have

$$\int_0^M |u_n(t)|^2\psi'(t)\,dt \to \int_0^M |u(t)|^2\psi'(t)\,dt \ as\ n \ \to \infty, \tag{60}$$

$$\int_0^M |B_n(t)|^2\psi'(t)\,dt \to \int_0^M |B(t)|^2\psi'(t)\,dt \ as\ n \ \to \infty. \tag{61}$$

We note that $u_n\sqrt{\psi(t)} \to u\sqrt{\psi(t)}$ weakly in $L^2(0,M;V)$ and $B_n\sqrt{\psi(t)} \to B\sqrt{\psi(t)}$ weakly in $L^2(0,M;V)$. Consequently

$$\int_0^M \|u(t)\|^2\psi(t)\,dt \le \lim_{n\to\infty}\inf \int_0^M \|u_n(t)\|^2\psi(t)\,dt, \tag{62}$$

$$\int_0^M \|B(t)\|^2\psi(t)\,dt \le \lim_{n\to\infty}\inf \int_0^M \|B_n(t)\|^2\psi(t)\,dt. \tag{63}$$

From (55), we also have

$$\int_0^M \langle g, u_n(t)\rangle\psi(t)\,dt \to \int_0^M \langle g, u(t)\rangle\psi(t)\,dt. \tag{64}$$

Using (60)-(64), and passing to the limit in (59), we obtain (58).
Thus we have proved that the limit (u,B) is a weak solution of the 3D MHD system and satisfies the energy inequality (30), that is $(u,B) \in \mathcal{K}^+$. This completes the proof of Proposition 4.

□

We have defined the trajectory space $\mathcal{K}^+$ of system (18)-(19) on $\mathbb{R}_+$. We now extend this definition on $\mathbb{R}$. The kernel $\mathcal{K}_0$ of system (18)-(19) is the set of all weak solutions $(u(t),B(t)), t \in \mathbb{R}$ bounded in the space

$$\begin{aligned}
\mathcal{F}^b = \{z = (u,B)\ /\ &(u,B)(.) \in L_2^b(\mathbb{R}; V\times V) \cap L^\infty(\mathbb{R}; H\times H),\\
&\partial_t u(.) \in L_2^b(\mathbb{R}; D(A)'), \partial_t B(.) \in L_2^b(\mathbb{R}; D(A)')\},
\end{aligned}$$

that satisfies the following inequality:

$$-\frac{1}{2}\int_{-\infty}^{+\infty}(|u(t)|^2+|B(t)|^2)\psi'(t)dt+\int_{-\infty}^{+\infty}(\nu\|u(t)\|^2+\eta\|B(t)\|^2)\psi(t)dt\le\int_{-\infty}^{+\infty}\langle g,u(t)\rangle\psi(t)dt, \tag{65}$$

for all $\psi\in C_0^\infty(\mathbb{R}),\psi\ge 0$.

The norm in $\mathcal{F}^b$ is defined in a similar way that the norm in $\mathcal{F}_+^b$ replacing $\mathbb{R}_+$ by $\mathbb{R}$. The same definition also holds for $\mathcal{F}^{loc}$ with the topology Θ^{loc} where the intervals $(0,M)$ are replaced by $(-M,M)$. We denote by Π_+ the restriction operator onto $\mathbb{R}_+$. This operator takes a function $\{\phi(t),t\in\mathbb{R}\}$ to the function $\{\Pi_+\phi(t),t\ge 0\}$, where $\Pi_+\phi(t)=\phi(t)$ for all $t\ge 0$.

Let us now study the translation semigroup $\{T(h)\}$ acting on the trajectory $\mathcal{K}^+$. We start with the main definitions

Definition 2. *A set $P\subset\mathcal{K}^+$ is said to be absorbing for the semigroup $\{T(h)\}$if for every bounded set $B\subset\mathcal{K}^+$ in $\mathcal{F}_+^b$, there is a $h_1=h_1(B)$ such that*

$$T(h)B\subseteq P \ \textit{for} \ \ \textit{all}\ h\ge h_1.$$

Definition 3. *A set $P\subseteq\mathcal{K}^+$ is said to be attracting for the semigroup $\{T(h)\}$ if any neighborhood $O(P)$ of the set P in the topology Θ_+^{loc} is an absorbing set for $\{T(h)\}$, i.e., for every bounded set $B\subset\mathcal{K}^+$ in $\mathcal{F}_+^b$, there is a $h_1=h_1(B,O)\ge 0$ such that $T(h)B\subseteq O(P)$ for all $h\ge h_1$.*

Definition 4. *A set $\mathcal{U}\subset\mathcal{K}^+$ is called a trajectory attractor for the semigroup $\{T(h)\}$ on $\mathcal{K}^+$ if $\mathcal{U}$ is bounded in $\mathcal{F}_+^b$, compact with respect to Θ_+^{loc}, strictly invariant with respect to $\{T(h)\}$, i.e. $T(h)\mathcal{U}=\mathcal{U}$, $\forall h\ge 0$, and $\mathcal{U}$ is an attracting set for $\{T(h)\}$.*

Let us now construct a trajectory attractor for $\{T(h)\}$ on $\mathcal{K}^+$ and describe its structure by using the kernel of system (18)-(19). It is the main result of this section

Theorem 2. *If $g\in H$, then the translation semigroup $\{T(h)\}$ acting on $\mathcal{K}^+$ has a trajectory attractor $\mathcal{U}_0$. The set $\mathcal{U}_0$ is bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc}. Moreover*

$$\mathcal{U}_0=\Pi_+\mathcal{K}_0,$$

the set $\mathcal{K}_0$ is bounded in $\mathcal{F}^b$ and compact in Θ^{loc}.

Proof. It is clear that $T(t)\mathcal{K}^+\subseteq\mathcal{K}^+,\forall t\ge 0$. Thanks to Proposition 2, the set $P=\{(u,B)\in\mathcal{F}_+^b/\|(u,B)(.)\|_{\mathcal{F}_+^b}\le 2R_0^2\}$ is an absorbing set for $\mathcal{K}^+$. The ball P is compact in Θ_+^{loc} and bounded in $\mathcal{F}_+^b$. Thus the conditions of Theorem XII.2.1 and Theorem XII.2.2 in [11] are valid and Theorem 2 is proved. □

Remark 4. *The trajectory attractor for the 3D Navier-Stokes system has been constructed [11, 29]. As far as we know , Theorem 2 is the first result dealing with the trajectory attractor for the 3D MHD system.*

3 The 3D MHD-α model and its trajectory attractor

3.1 The 3D MHD-α model and some properties

We consider the 3D MHD-α model with periodic boundary conditions. The model reads as follows:

$$\frac{\partial v}{\partial t} + (u.\nabla)v + \sum_{j=1}^{3} v_j \nabla u_j - \nu \Delta v + \nabla \pi' + \frac{1}{2}\nabla |B|^2 = (B.\nabla)B + g, \tag{66}$$

$$\frac{\partial B}{\partial t} + (u.\nabla)B - (B.\nabla)u - \eta \Delta B = 0, \tag{67}$$

$$v = (1 - \alpha^2 \Delta)u, \tag{68}$$

$$\nabla.u = \nabla.v = \nabla.B = 0, \tag{69}$$

$$\int_\Omega u(x,t)\, dx = 0, \tag{70}$$

$$\int_\Omega B(x,t)\, dx = 0. \tag{71}$$

This is an approximation of the 3D MHD system (1)-(4) discussed in the previous section. The unknown are u, B and π' defined on $\Omega \times [0, M]$, representing respectively the "filtered" fluid velocity , the magnetic field and the pressure at each point of the fluid. We assume that the functions $u(x,t), B(x,t)$ and the known external force g(x) are periodic in $x \in \Omega$ and have zero spatial mean i.e. $\int_\Omega g(x)\, dx = 0$. We observe that for $\alpha = 0$, the function $v = u$ and we formally obtain the system $(1)-(4)$. Recall that α is a fixed positive parameter called "the sub-grid (filter) length scale" of the model (see the motivations in [21] and references therein).

Following the notation of the NSE, we denote

$$\widetilde{B}(u,v) = -P\left((\nabla \times v) \times u\right) \text{for any u,v} \in \mathcal{V},$$

the bilinear operator. We have

$$(\widetilde{B}(u,v), w) = (\mathbb{B}(u,v), w) - (\mathbb{B}(w,v), u), \tag{72}$$

for any $u, v, w \in \mathcal{V}$. In fact the equality (72) follows from the identity

$$(b.\nabla)a + \sum_{j=1}^{3} a_j \nabla b_j = -b \times (\nabla \times a) + \nabla(a.b), \tag{73}$$

for $a, b \in \mathbb{R}^3$. The symbol $\times$ represents the vector product in $\mathbb{R}^3$. We recall that

$$\widetilde{B}(u,u) = \mathbb{B}(u,u), \tag{74}$$

where $\mathbb{B}(u,v) = P(u.\nabla)v$ (see (7)).
We now rewrite the system $(66) - (71)$ in the short form

$$\partial_t v + \nu A v + \widetilde{B}(u,v) - \mathbb{B}(B,B) = g, \tag{75}$$

$$\partial_t B + \eta A B + \mathbb{B}(u,B) - \mathbb{B}(B,u) = 0, \tag{76}$$

$$v = u + \alpha^2 A u. \tag{77}$$

For $\alpha = 0$, system $(75) - (77)$ coincides with the 3D MHD system $(18) - (19)$.
We supplement $(75) - (77)$ with initial conditions :

$$u(0) = u_0 \in V, \tag{78}$$

$$B(0) = B_0 \in H. \tag{79}$$

We now formulate some properties of the bilinear operator $\widetilde{B}$ that are analogous to the properties of the operator B. The operator $\widetilde{B}$ maps $V \times V$ *to* V' and the following inequalities hold (see [21, 19] for the proof)

$$|\langle \widetilde{B}(u,v), w\rangle| \leq C|u|^{\frac{1}{4}}\|u\|^{\frac{3}{4}}\|v\||w|^{\frac{1}{4}}\|w\|^{\frac{3}{4}},$$
$$|\langle \widetilde{B}(u,v), w\rangle| \leq C|u|^{\frac{1}{2}}\|u\|^{\frac{1}{2}}\|v\|\|w\|,$$

for all $u, v \in V$.
We also have

$$\langle \widetilde{B}(u,v), w\rangle = -\langle \widetilde{B}(w,v), u)\rangle, \tag{80}$$
$$\langle \widetilde{B}(u,v), u\rangle = 0, \tag{81}$$

for all $u, v \in V$. We also need the following inequality proved in [21, 19]:

$$|\langle \widetilde{B}(u,v), w\rangle| \leq C\|u\||v||Aw|, \tag{82}$$

for all $u \in V$, $v \in H$ and $w \in D(A)$.
This means that $\widetilde{B}$ maps $V \times H$ into $D(A)'$ and

$$\|\widetilde{B}(u,v)\|_{D(A)'} \leq C\|u\||v|, \tag{83}$$

for all $u \in V$, $v \in H$.

3.2 The Cauchy problem for the 3D MHD-α model

We recall from [21] the definition of weak solution of the 3D MHD-α model.

Definition 5. *Let $M > 0$, $(u_0, B_0) \in V \times H$ and $g \in H$. A couple of functions (u, B) is weak solution of system $(75) - (77), (78) - (79)$ on $[0, M]$ if:*

i) u and B satisfy

$$\begin{aligned}
&u \in C([0,M]; V) \cap L^2(0,M; D(A)),\\
&\frac{du}{dt} \in L^2(0,M; H),\\
&B \in C([0,M]; H) \cap L^2(0,M; V),\\
&\frac{dB}{dt} \in L^2(0,M; V').
\end{aligned}$$

ii) (u,B) satisfies the system $(75) - (77)$ in the sense of distributions, i.e.,

$$\begin{aligned}
&\langle \frac{dv}{dt}, w\rangle_{D(A)'} + \langle \widetilde{B}(u,v), w\rangle_{D(A)'} + \nu(v, Aw) = \langle \mathbb{B}(B,B), w\rangle_{V'} + \langle g, w\rangle_{V'},\\
&\langle \frac{dB}{dt}, \xi\rangle_{V'} + (\mathbb{B}(u,B), \xi) - (\mathbb{B}(B,u), \xi) + \eta((B,\xi)) = 0,
\end{aligned} \tag{84}$$

for every $w \in D(A), \xi \in V$ and for almost every $t \in (0, M)$.

iii) $u(0) = u_0$ and $B(0) = B_0$.

In the work [21], the following theorem on the existence and uniqueness of weak solution for the 3D MHD-α model was proved.

Theorem 3. *Let $g \in H$ and $(u_0, B_0) \in V \times H$. For every $M > 0$, the Cauchy problem (75) − (77), (78) − (79) has a unique weak solution $(u(t), B(t))$ in the sense of Definition 5. Moreover (u, B) satisfies the following energy equality :*

$$\frac{1}{2}\frac{d}{dt}\{|u(t)|^2 + \alpha^2\|u(t)\|^2 + |B(t)|^2\} + \{\nu\left(\|u(t)\|^2 + \alpha^2|Au(t)|^2\right) + \eta\|B(t)\|^2\} = \langle g, u(t)\rangle, \quad (85)$$

for almost every $t \in (0, M)$.

The energy equality (85) implies the main a priori estimates of problem (75) − (77), (78) − (79).

Proposition 5. *Let $(u(t), B(t))$ be a weak solution of system (75) − (77), (78) − (79). Then the following inequalities hold:*

$$i)\ \ |u(t)|^2 + \alpha^2\|u(t)\|^2 + |B(t)|^2 \le \left(|u(0)|^2 + \alpha^2\|u(0)\|^2 + |B(0)|^2\right)e^{-\mu\lambda_1 t} + \frac{|g|^2}{\lambda_1^2\mu^2}, \quad (86)$$

$$ii)\ \ \mu\int_t^{t+1}\left(\|u(t)\|^2 + \alpha^2|Au(s)|^2 + \|B(s)\|^2\right)ds \le \left(|u(0)|^2 + \alpha^2\|u(0)\|^2 + |B(0)|^2\right)e^{-\mu\lambda_1 t} + \frac{|g|^2}{\lambda_1^2\mu^2} + \frac{|g|^2}{\mu\lambda_1}, \quad (87)$$

$$iii)\ \ \left(\int_t^{t+1}\|\partial_t v(s)\|^2_{D(A)'}\,ds\right)^{\frac{1}{2}} \le C_6\left(|u(0)|^2 + \alpha^2\|u(0)\|^2 + |B(0)|^2\right)e^{-\mu\lambda_1 t} + R_6^2, \quad (88)$$

$$iv)\ \ \left(\int_t^{t+1}\|\partial_t u(s)\|^2_{D(A)'}\,ds\right)^{\frac{1}{2}} \le C_6\left(|u(0)|^2 + \alpha^2\|u(0)\|^2 + |B(0)|^2\right)e^{-\mu\lambda_1 t} + R_6^2, \quad (89)$$

$$v)\ \ \left(\int_t^{t+1}\|\partial_t B(s)\|^2_{D(A)'}\,ds\right)^{\frac{1}{2}} \le C_{10}\left(|u(0)|^2 + \alpha^2\|u(0)\|^2 + |B(0)|^2\right)e^{-\mu\lambda_1 t} + R_7^2,, \quad (90)$$

where the constants C_6, C_{10} depend on λ_1 , μ. R_6 and R_7 depend on λ_1, μ and $|g|$.

Proof. i) The proof uses the energy equality (85) and the Poincaré inequality.

ii) The proof follows from *i*).

iii) The function v satisfies (75) that is

$$\partial_t v + \nu Av + \widetilde{B}(u, v) - B(B, B) = g. \quad (91)$$

We apply to (91) the Minkowski inequality and obtain

$$\left(\int_t^{t+1}\|\partial_t v(s)\|^2_{D(A)'}\,ds\right)^{\frac{1}{2}} \le \nu\left(\int_t^{t+1}\|Av(s)\|^2_{D(A)'}\,ds\right)^{\frac{1}{2}} + \left(\int_t^{t+1}\|\widetilde{B}(u(s), v(s))\|^2_{D(A)'}\,ds\right)^{\frac{1}{2}}$$
$$\left(\int_t^{t+1}\|\mathbb{B}(B(s), B(s))\|^2_{D(A)'}\,ds\right)^{\frac{1}{2}} + \|g\|_{D(A)'}. \quad (92)$$

We now estimate each of the terms on the right.
From the estimate $\|Av\|_{D(A)'} = |v| \le |u| + \alpha^2|Au|$, we have

$$\int_t^{t+1}\|Av(s)\|^2_{D(A)'}\,ds \le 2\left(\int_t^{t+1}|u(s)|^2\,ds + \alpha^2\int_t^{t+1}\alpha^2|Au(s)|^2\,ds\right) \quad (93)$$
$$\le 2\int_t^{t+1}(|u(s)|^2 + \alpha^2|Au(s)|^2)\,ds, \text{ since } \alpha^2 \le 1. \quad (94)$$

Using the Poincaré inequality and (87), we obtain

$$\int_t^{t+1}\|Av(s)\|^2_{D(A)'}\,ds \le C_7\left(|u(0)|^2 + \alpha^2\|u(0)\|^2 + |B(0)|^2\right)e^{-\mu\lambda_1 t} + R_2^2, \quad (95)$$

where the constant C_7 depends on λ_1 and μ,and $R_2^2 = \frac{|g|^2}{\lambda_1^2\mu^2} + \frac{|g|^2}{\lambda_1\mu}$.
From the inequality (83), we have

$$
\begin{aligned}
\|\widetilde{B}(u(t), v(t))\|_{D(A)'} \leq C\|u(t)\||v(t)| &\leq C\|u(t)\| \left(|u(t)| + \alpha^2|Au(t)|\right) \\
&\leq C\left(|u(t)|\|u(t)\| + \alpha\|u(t)\|\alpha|Au(t)|\right) \\
&\leq C\left(|u(t)|^2 + \alpha^2\|u(t)\|^2\right)^{\frac{1}{2}} \left(\|u(t)\|^2 + \alpha^2|Au(t)|^2\right)^{\frac{1}{2}}.
\end{aligned}
$$

where we have used the Cauchy inequality. Applying inequality (86), we have

$$
\|\widetilde{B}(u(t), v(t))\|_{D(A)'}^2 \leq C^2 \left(\left(|u(0)|^2 + \alpha^2\|u(0)\|^2 + |B(0)|^2\right) e^{-\mu\lambda_1 t} + \frac{|g|^2}{\lambda_1^2\mu^2} \right) \left(\|u(t)\|^2 + \alpha^2|Au(t)|^2\right).
$$

Integrating this inequality over $[t, t+1]$, we find

$$
\begin{aligned}
&\int_t^{t+1} \|\widetilde{B}(u(s), v(s))\|_{D(A)'}^2 \, ds \\
&\leq C^2 \left(\left(|u(0)|^2 + \alpha^2\|u(0)\|^2 + |B(0)|^2\right) e^{-\mu\lambda_1 t} + \frac{|g|^2}{\lambda_1^2\mu^2} \right) \int_t^{t+1} \left(\|u(s)\|^2 + \alpha^2|Au(s)|^2\right) ds.
\end{aligned}
$$

We now use (87) and obtain

$$
\left(\int_t^{t+1} \|\widetilde{B}(u(s), v(s))\|_{D(A)'}^2 \, ds\right)^{\frac{1}{2}} \leq C_8 \left(|u(0)|^2 + \alpha^2\|u(0)\|^2 + |B(0)|^2\right) e^{-\mu\lambda_1 t} + R_2^2, \quad (96)
$$

where $R_2^2 = \frac{|g|^2}{\lambda_1^2\mu^2} + \frac{|g|^2}{\lambda_1\mu}$ and C_8 depends on λ_1 and μ.
Using inequality (10), we have

$$
\begin{aligned}
\int_t^{t+1} \|\mathbb{B}(B(s), B(s))\|_{D(A)'}^2 \, ds &\leq C^2 \int_t^{t+1} \|B(s)\|^2 |B(s)|^2 \, ds \\
&\leq C^2 ess \sup_{0 \leq s \leq M} |B(s)|^2 \int_t^{t+1} \|B(s)\|^2 \, ds.
\end{aligned}
$$

Taking into account of $(86) - (87)$, on obtain

$$
\int_t^{t+1} \|\mathbb{B}(B(s), B(s))\|_{D(A)'}^2 \, ds \quad \leq C^2\mu^{-1} \left(\left(|u(0)|^2 + \alpha^2\|u(0)\|^2 + |B(0)|^2\right) e^{-\mu\lambda_1 t} + \frac{|g|^2}{\lambda_1^2\mu^2} + \frac{|g|^2}{\lambda_1\mu} \right)^2.
$$

This inequality implies that

$$
\left(\int_t^{t+1} \|\mathbb{B}(B(s), B(s))\|_{D(A)'}^2 \, ds\right)^{\frac{1}{2}} \leq C_9 \left(|u(0)|^2 + \alpha^2\|u(0)\|^2 + |B(0)|^2\right) e^{-\mu\lambda_1 t} + R_2^2, \quad (97)
$$

where C_9 depends on λ_1 and μ.
Putting $(95), (96), (97)$ and (92) together , it follows that

$$
\begin{aligned}
\left(\int_t^{t+1} \|\partial_t v(s)\|_{D(A)'}^2 \, ds\right)^{\frac{1}{2}} &\leq \nu \left(C_7 \left(|u(0)|^2 + \alpha^2\|u(0)\|^2 + |B(0)|^2\right) e^{-\mu\lambda_1 t} + 2R_2^2 + 1 \right) \\
&\quad + C_8 \left(|u(0)|^2 + \alpha^2\|u(0)\|^2 + |B(0)|^2\right) e^{-\mu\lambda_1 t} + |g|\lambda_1^{-1}, \\
&\leq C_6 \left(|u(0)|^2 + \alpha^2\|u(0)\|^2 + |B(0)|^2\right) e^{-\mu\lambda_1 t} + R_6^2,
\end{aligned}
$$

where $C_6 = \nu C_7 + C_8 + C_9$ and $R_6^2 = \nu(R_2^2 + 1) + 2R_2^2 + |g|\lambda_1^{-1}$. This completes the proof of iii).

iv) The proof of $iv)$ follows from $iii)$ since $\|\partial_t u\|_{D(A)'} \leq \|\partial_t v\|_{D(A)'}$.

v) From (76), we have

$$\left(\int_t^{t+1} \|\partial_t B(s)\|^2_{D(A)'}\, ds\right)^{\frac{1}{2}} \leq \eta \left(\int_t^{t+1} \|AB(s)\|^2_{D(A)'}\, ds\right)^{\frac{1}{2}} + \left(\int_t^{t+1} \|\mathbb{B}(u(s), B(s))\|^2_{D(A)'}\, ds\right)^{\frac{1}{2}} + \left(\int_t^{t+1} \|\mathbb{B}(B(s), u(s))\|^2_{D(A)'}\|^2_{D(A)'}\, ds\right)^{\frac{1}{2}}. \tag{98}$$

We will bound each of the terms on the right.

$$\int_t^{t+1} \|AB(s)\|^2_{D(A)'} = \int_t^{t+1} |B(s)|^2\, ds \leq \left(|u(0)|^2 + \alpha^2\|u(0)\|^2 + |B(0)|^2\right) e^{-\mu\lambda_1 t} + \frac{|g|^2}{\lambda_1^2\mu^2}. \tag{99}$$

In view of (10), (86) − (87), we have

$$\left(\int_t^{t+1} \|\mathbb{B}(u(s), B(s))\|^2_{D(A)'}\right)^{\frac{1}{2}} \leq C\mu^{-\frac{1}{2}} \left(\left(|u(0)|^2 + \alpha^2\|u(0)\|^2 + |B(0)|^2\right) e^{-\mu\lambda_1 t} + \frac{|g|^2}{\lambda_1^2\mu^2} + \frac{|g|^2}{\lambda_1\mu}\right). \tag{100}$$

We also have the estimate

$$\left(\int_t^{t+1} \|\mathbb{B}(B(s), u(s))\|^2_{D(A)'}\right)^{\frac{1}{2}} \leq C\mu^{-\frac{1}{2}} \left(\left(|u(0)|^2 + \alpha^2\|u(0)\|^2 + |B(0)|^2\right) e^{-\mu\lambda_1 t} + \frac{|g|^2}{\lambda_1^2\mu^2} + \frac{|g|^2}{\lambda_1\mu}\right). \tag{101}$$

Substituting (99)-(101) into (98), we obtain

$$\left(\int_t^{t+1} \|\partial_t B(s)\|^2_{D(A)'}\, ds\right)^{\frac{1}{2}} \leq C_{10} \left(|u(0)|^2 + \alpha^2\|u(0)\|^2 + |B(0)|^2\right) e^{-\mu\lambda_1 t} + R_7^2,$$

where $C_{10} = (\eta + 2C\mu^{-\frac{1}{2}})$ and $R_7^2 = \eta(\frac{|g|^2}{\lambda_1^2\mu^2} + 1) + 2C\mu^{-\frac{1}{2}} R_2^2$. This completes the proof of Proposition 5.

□

Remark 5. *We note that the constants on the right of estimates* (86), (87), (88), (89), (90) *are independent of α. This fact plays the crucial role in the proof of convergence of solutions of the 3D MHD-α model to the solutions of the 3D MHD system as α approaches* 0.

3.3 Existence of the trajectory attractor of the 3D MHD-α model

To construct the trajectory attractor for the system (75)-(77), we have to pass to new function variable w that occupies an intermediate position between the function u and v.
Following [12], we set $w = (1 + \alpha^2 A)^{\frac{1}{2}} u$. We have the following identities:

$$v = (1 + \alpha^2 A)u = (1 + \alpha^2 A)^{\frac{1}{2}} w,$$

$$|w|^2 = |u|^2 + \alpha^2\|u\|^2, \tag{102}$$

$$\|w\|^2 = \|u\|^2 + \alpha^2|Au|^2. \tag{103}$$

The couple of functions (w, B) satisfies the following system:

$$\partial_t w + \nu Aw + (1 + \alpha^2 A)^{-\frac{1}{2}} \widetilde{B}\left((1 + \alpha^2 A)^{-\frac{1}{2}} w, (1 + \alpha^2 A)^{\frac{1}{2}} w\right) - (1 + \alpha^2 A)^{-\frac{1}{2}} \mathbb{B}(B, B) = (1 + \alpha^2 A)^{-\frac{1}{2}} g, \tag{104}$$

$$\partial_t B + \eta AB + \mathbb{B}\left((1 + \alpha^2 A)^{-\frac{1}{2}} w, B\right) - \mathbb{B}\left(B, (1 + \alpha^2 A)^{-\frac{1}{2}} w\right) = 0. \tag{105}$$

Using the function w, we rewrite inequalities (86), (87), (88), (90).

Corollary 1.

$$i)\ \ |w(t)|^2 + |B(t)|^2 \le \left(|w(0)|^2 + |B(0)|^2\right) e^{-\mu\lambda_1 t} + \frac{|g|^2}{\lambda_1^2\mu^2}. \tag{106}$$

$$ii)\ \ \mu \int_t^{t+1} \left(\|w(t)\|^2 + \|B(s)\|^2\right) ds \le \left(|w(0)|^2 + |B(0)|^2\right) e^{-\mu\lambda_1 t} + \frac{|g|^2}{\lambda_1^2\mu^2} + \frac{|g|^2}{\mu\lambda_1}. \tag{107}$$

$$iii)\ \ \left(\int_t^{t+1} \|\partial_t w(s)\|^2_{D(A)'}\, ds\right)^{\frac{1}{2}} \le C_6 \left(|w(0)|^2 + |B(0)|^2\right) e^{-\mu\lambda_1 t} + R_6^2. \tag{108}$$

$$iv)\ \ \left(\int_t^{t+1} \|\partial_t B(s)\|^2_{D(A)'}\, ds\right)^{\frac{1}{2}} \le C_{10} \left(|w(0)|^2 + |B(0)|^2\right) e^{-\mu\lambda_1 t} + R_7^2. \tag{109}$$

Proof. The proof of *i)* and *ii)* follow from (86) – (87) and (102) – (103).
From the inequality

$$\int_t^{t+1} \|\partial_t w(s)\|^2_{D(A)'}\, ds \le \int_t^{t+1} \|\partial_t v(s)\|^2_{D(A)'}\, ds,$$

we obtain *iii)*.
iv) also follows from (90) and (102). □

Consider the Banach space $\mathcal{F}_+^b$ in Section 2. Recall that

$$\mathcal{F}_+^b = \{z = (w, B)\ /\ (w, B)(.) \in L_2^b(\mathbb{R}_+; V \times V) \cap L^\infty(\mathbb{R}_+; H \times H),$$
$$\partial_t w(.) \in L_2^b(\mathbb{R}_+; D(A)'),\ \partial_t B(.) \in L_2^b(\mathbb{R}_+; D(A)')\}.$$

Inequalities (106) – (109) of Corollary 1 provide the following

Proposition 6. *If $g \in H$, for any solution (u, B) of problem (75) – (79), the corresponding couple $(w(t), B(t))$ being a solution of system (104)-(105) satisfies the inequality*

$$\|T(h)(w, B)(.)\|_{\mathcal{F}_+^b} \le C_{11} \left(|w(0)|^2 + |B(0)|^2\right) e^{-\mu\lambda_1 h} + R_8^2, \tag{110}$$

where the constant C_{11} depends on μ, λ_1 and R_8 depends on $\mu, \lambda_1, |g|$.

Remark 6. *We note that the constants C_{11} and R_8 are independent of α.*

Let us now construct the trajectory attractor for the 3D MHD-α model. The trajectory space K_α^+ for system (75) – (77) is defined as follows

Definition 6. *The trajectory space K_α^+ is the union of all couple $(w(t), B(t))$ where $(u(t), B(t))$ is a solution of system (75) – (77) with arbitrary $(u_0, B_0) \in V \times H$.*

Using Theorem 3, we prove that the trajectory space $\mathcal{K}_\alpha^+$ is nonempty.
Proposition 6 implies that $\mathcal{K}_\alpha^+ \subset \mathcal{F}_+^b$ for all $\alpha > 0$. We also consider the topological space Θ_+^{loc} introduced in Section 2 in connection with the 3D MHD system. Recall that $\mathcal{F}_+^b \subset \Theta_+^{loc}$.
We consider the topology Θ_+^{loc} on $\mathcal{K}_\alpha^+$. We prove that the space $\mathcal{K}_\alpha^+$ is closed in Θ_+^{loc}.

Proposition 7. *The space $\mathcal{K}_\alpha^+$ is closed in Θ_+^{loc}.*

Proof. The proof follows the similar argument as in the proof of Proposition 4. □

The translation semigroup $\{T(h)\}$ acts on $\mathcal{K}_\alpha^+$ by the formula:

$$T(h)\left(w_\alpha(t), B_\alpha(t)\right) = \left(w_\alpha(t+h), B_\alpha(t+h)\right),$$

for $h \ge 0$.
From the definition of $\mathcal{K}_\alpha^+$, it follows that $T(h)\mathcal{K}_\alpha^+ \subseteq \mathcal{K}_\alpha^+$ for all $h \ge 0$.
Our main result in this section is the following

Theorem 4. ***(Existence of the trajectory attractor of the 3D MHD-α model)***
If $g \in H$, then the translation semigroup $\{T(h)\}$ acting on $\mathcal{K}_\alpha^+$ has a trajectory attractor $\mathcal{U}_\alpha$. The set $\mathcal{U}_\alpha$ is bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc}. Moreover

$$\mathcal{U}_\alpha = \Pi_+ \mathcal{K}_\alpha,$$

where $\mathcal{K}_\alpha$ is the kernel of system $(75)-(77)$.

Proof. We have $T(t)\mathcal{K}_\alpha^+ \subseteq \mathcal{K}_\alpha^+,\ for\ all\ t \geq 0$. The set $P' = \{(w,B) \in \mathcal{F}_+^b / \|(w,B)(.)\|_{\mathcal{F}_+^b} \leq 2R_8^2\}$ is an absorbing set for $\mathcal{K}_\alpha^+$ (see Proposition 6). The ball P' is compact in Θ_+^{loc} and bounded in $\mathcal{F}_+^b$. This absorbing set does not depend on α since the constants C_{11} and R_8 in (110) are independent of α. Thus the conditions of Theorems XII.2.1 and XII.2.2 in [11] are valid. Thus there exists a trajectory attractor $\mathcal{U}_\alpha \subset \mathcal{K}_\alpha^+$ such that $\mathcal{U}_\alpha$ is bounded in $\mathcal{F}_+^b$, compact in Θ_+^{loc}. □

Remark 7. *Since $\mathcal{U}_\alpha \subsetneq P'$, then the trajectory attractors $\mathcal{U}_\alpha$ are uniformly (with respect to $\alpha \in]0,1]$) bounded in $\mathcal{F}_+^b$, that is*

$$\|\mathcal{U}_\alpha\|_{\mathcal{F}_+^b} \leq\ R\ , \forall \alpha \in]0,1], \tag{111}$$

where R is a constant independent on α.

4 Convergence of the solutions of the 3D MHD-α model

We formulate and prove the main result of this section concerning the behavior of the solutions of the 3D MHD-α model when α approaches 0.

Theorem 5. *Let a sequence $\{(w_n(t), B_n(t))\} \subset \mathcal{K}_{\alpha_n}^+$ be given such that*

$$\begin{aligned}
&1.\ \ \{(w_n(t), B_n(t)), n \in \mathbb{N}\}\ \ is\ \ bounded\ \ in\ \mathcal{F}_+^b,\\
&2.\ \ \alpha_n \to 0^+\ \ as\ n \to \infty,\\
&3.\ \ (w_n(t), B_n(t)) \to (w(t), B(t))\ \ in\ \Theta_+^{loc}\ \ as\ n \to \infty.
\end{aligned} \tag{112}$$

Then $(w(t), B(t))$ is a weak solution of the 3D MHD system such that (w, B) satisfies the energy inequality

$$-\frac{1}{2}\int_0^M (|w(t)|^2 + |B(t)|^2)\psi'(t)dt + \int_0^M (\nu\|w(t)\|^2 + \eta\|B(t)\|^2)\psi(t)dt \leq \int_0^M \langle g, w(t)\rangle \psi(t)dt, \tag{113}$$

for all $\psi \in C_0^\infty(0, M), \psi \geq 0$, that is $(w, B) \in \mathcal{K}^+$, where $\mathcal{K}^+$ is the trajectory space of the 3D MHD system.

For the proof of Theorem 5, we will need the following lemma

Lemma 1. *Let two sequences $(u_n(t), B_n(t)) \in \mathcal{F}_+^b$ and $\{\alpha_n\} \subset]0,1]$ be given such that $\alpha_n \to 0^+$ as $n \to \infty$. We denote $w_n = (1 + \alpha_n^2 A)^{\frac{1}{2}} u_n$ for $n \in \mathbb{N}$. We assume that the sequence $(w_n(t), B_n(t))$ is bounded in $\mathcal{F}_+^b$ and $(w_n(t), B_n(t)) \to (w(t), B(t))$ in Θ_+^{loc} as $n \to \infty$. Then the sequence $(u_n(t), B_n(t))$ is bounded in $\mathcal{F}_+^b$ and $(u_n(t), B_n(t)) \to (w(t), B(t))$ in Θ_+^{loc} as $n \to \infty$.*

Proof. (Proof of Lemma 1)
The proof follows the one given in [12]. For the reader's convenience, we will the details of the proof. From the inequalities

$$\begin{aligned}
&|u_n|^2 \leq |u_n|^2 + \alpha^2\|u_n\|^2 = |w_n|^2,\\
&\|u_n\|^2 \leq \|u_n\|^2 + \alpha^2|Au_n|^2 = \|w_n\|^2,\\
&\int_t^{t+1} \|\partial_t u_n(s)\|_{D(A)'}^2\, ds, \leq \int_t^{t+1} \|\partial_t w_n(s)\|_{D(A)'}^2\, ds,
\end{aligned}$$

we have

$$\|(u_n, B_n)\|_{\mathcal{F}_+^b} \leq \|(w_n, B_n)\|_{\mathcal{F}_+^b}, \tag{114}$$

for all n $\in \mathbb{N}$.
From (114), we conclude that $\{(u_n(t), B_n(t))\}$ is bounded in $\mathcal{F}_+^b$. Since a ball in $\mathcal{F}_+^b$ is weakly compact set in Θ_+^{loc}, we can extract from $(u_n(t), B_n(t))$ a convergent subsequence and we denote the limit of this subsequence by $(u(t), B(t))$. For simplicity, we denote this subsequence by $(u_n(t), B_n(t))$. We also keep the corresponding subsequence of $(w_n(t), B_n(t))$. Then we have

$$(u_n(t), B_n(t)) \to (u(t), B(t)) \text{ in } \Theta_+^{loc} \text{ as } n \to \infty$$

$$(w_n(t), B_n(t)) \to (w(t), B(t)) \text{ in } \Theta_+^{loc} \text{ as } n \to \infty.$$

We prove that $u = w$. In fact consider an arbitrary interval $[0, M]$. By the assumption $w_n(t) \to w(t)$ weakly in $L^2(0, M; V)$ and $\partial_t w_n(t) \to \partial_t w(t)$ weakly in $L^2(0, M; D(A)')$. Then by the Aubin compactness theorem (see [23, 1]), we obtain that $w_n(t) \to w(t)$ strongly in $L^2(0, M; H)$. Arguing similarly, we also have $u_n(t) \to u(t)$ strongly in $L^2(0, M; H)$.
We note that $\|(1 + \alpha_n^2 A)^{-\frac{1}{2}}\|_{\mathcal{L}(H,H)} \leq 1$ and therefore

$$\begin{aligned}&\|(1 + \alpha_n^2 A)^{-\frac{1}{2}} w_n - (1 + \alpha_n^2 A)^{-\frac{1}{2}} w\|_{L^2(0,M;H)} \\ &\leq \|w_n - w\|_{L^2(0,M;H)} \to \ 0 \text{ as } n \to \infty.\end{aligned} \tag{115}$$

On the other hand, Lemma 3.2 in [12] implies that

$$\|(1 + \alpha_n^2 A)^{-\frac{1}{2}} w - w\|_{L^2(0,M;H)} \to 0 \ as \ n \to \infty. \tag{116}$$

Combining (115) and (116), we obtain

$$\begin{aligned}&\|u_n - w\|_{L^2(0,M;H)} \\ &= \|(1 + \alpha_n^2 A)^{-\frac{1}{2}} w_n - w\|_{L^2(0,M;H)} \\ &\leq \|(1 + \alpha_n^2 A)^{-\frac{1}{2}} w_n - (1 + \alpha_n^2 A)^{-\frac{1}{2}}\|_{L^2(0,M;H)} + \quad \|(1 + \alpha_n^2 A)^{-\frac{1}{2}} w - w\|_{L^2(0,M;H)} \to 0 \ as \ n \to \infty.\end{aligned}$$

Therefore $u_n(t) \to w(t)$ strongly in $L^2(0, M; H)$. Consequently $u(t) = w(t)$ and Lemma 1 is proved. □

Proof. (Proof of Theorem 5)
The proof follows the one given in [12] but the presence here of the magnetic field makes the analysis more involved.
Since

$$\|(w_n, B_n)\|_{\mathcal{F}_+^b} \leq C, \ \forall n \in \mathbb{N}, \tag{117}$$

and $(w_n, B_n) \to (w(t), B(t))$ in Θ_+^{loc} as $n \to \infty$, then we have $\|(w, B)\|_{\mathcal{F}_+^b} \leq C$. We set $u_n = (1 + \alpha_n^2 A)^{-\frac{1}{2}} w_n$. The couple (u_n, B_n) is a solution of the original problem (75) – (77). The estimates (117),(102) and (103) imply that

$$ess \sup_{t \geq 0} \left(|u_n(t)|^2 + \alpha_n^2 \|u_n(t)\|^2 + |B_n(t)|^2\right) \leq C^2, \tag{118}$$

$$\sup_{t \geq 0} \int_t^{t+1} \left(\|u_n(s)\|^2 + \alpha_n^2 |Au_n(s)|^2 + \|B_n(s)\|^2\right) ds \leq C^2, \tag{119}$$

$$\sup_{t \geq 0} \int_t^{t+1} \|\partial_t u_n(s)\|_{D(A)'}^2 ds \leq \sup_{t \geq 0} \int_t^{t+1} \|\partial_t w_n(s)\|_{D(A)'}^2 ds \leq C^2, \tag{120}$$

$$\sup_{t \geq 0} \int_t^{t+1} \|\partial_t B_n(s)\|_{D(A)'}^2 ds \leq C^2. \tag{121}$$

We now prove that $(w(t), B(t))$ is a weak solution of the 3D MHD system on any interval $(0, M)$. The couple $(w_n(t), B_n(t))$ satisfies the system

$$\partial_t w_n + \nu A w_n + (1+\alpha_n^2 A)^{-\frac{1}{2}} \widetilde{B}(u_n, v_n) - (1+\alpha_n^2 A)^{-\frac{1}{2}} \mathbb{B}(B_n, B_n) = (1+\alpha_n^2 A)^{-\frac{1}{2}} g, \tag{122}$$

$$\partial_t B_n + \eta A B_n + \mathbb{B}(u_n, B_n) - \mathbb{B}(B_n, u_n) = 0, \tag{123}$$

in the sense of distributions. Here $v_n = u_n + \alpha_n^2 A u_n$.
From the assumption (112), we have

$$w_n(t) \rightharpoonup w(t) \text{ weakly in } L^2(0, M; V), \text{ weakly star in } L^\infty(0, M; H), \tag{124}$$

$$\partial_t w_n(t) \rightharpoonup \partial_t \mathrm{w(t)} \text{ weakly in } L^2(0, M; D(A)'), \tag{125}$$

$$B_n(t) \rightharpoonup B(t) \text{ weakly in } L^2(0, M; V), \text{ weakly star in } L^\infty(0, M; H), \tag{126}$$

$$\partial_t B_n(t) \rightharpoonup \partial_t B(t) \text{ weakly in } L^2(0, M; D(A)'). \tag{127}$$

It follows from $(124), (126)$ that

$$A w_n(t) \rightharpoonup A w(t) \text{ weakly in } L^2(0, M; V'), \tag{128}$$

$$A B_n(t) \rightharpoonup A B(t) \text{ weakly in } L^2(0, M; V'), \tag{129}$$

and hence in the topology $\mathcal{D}'(0, M; D(A)')$ as well.
Besides combining Lemma 1,(126),(127) and Aubin compactness theorem, we also have

$$u_n(t) \to w(t) \text{ strongly in } L^2(0, M; H), \tag{130}$$

$$B_n(t) \to B(t) \text{ strongly in } L^2(0, M; H). \tag{131}$$

Arguing as in the proof of Proposition 4 (see (57)), we also have

$$\mathbb{B}(u_n.B_n) \rightharpoonup \mathbb{B}(u, B) \text{ weakly in } L^{\frac{4}{3}}(0, M; V'), \tag{132}$$

$$\mathbb{B}(B_n, u_n) \rightharpoonup \mathbb{B}(B, u) \text{ weakly in } L^{\frac{4}{3}}(0, M; V'), \tag{133}$$

$$\mathbb{B}(B_n, B_n) \rightharpoonup \mathbb{B}(B, B) \text{ weakly in } L^{\frac{4}{3}}(0, M; V'), \tag{134}$$

and therefore in $\mathcal{D}'(0, M; D(A)')$.
From Lemma 3.3 in [12] and (134), we deduce that

$$(1+\alpha_n^2 A)^{-\frac{1}{2}} \mathbb{B}(B_n, B_n) \rightharpoonup \mathbb{B}(B, B) \text{ weakly in } L^{\frac{4}{3}}(0, M; D(A)'). \tag{135}$$

Applying Lemma 3.2 in [12] , we have

$$(1+\alpha_n^2 A)^{-\frac{1}{2}} g \to \mathrm{g} \text{ strongly in } L^2(0, M; H). \tag{136}$$

Therefore having $(124), (130), (131), (132), (133), (135), (136)$, to prove that (w, B) satisfies the system

$$\partial_t w + \nu A w + \mathbb{B}(w, w) - \mathbb{B}(B, B) = g, \tag{137}$$

$$\partial_t B + \eta A B + \mathbb{B}(u, B) - \mathbb{B}(B, u) = 0, \tag{138}$$

we must establish that

$$(1+\alpha_n^2 A)^{-\frac{1}{2}} \widetilde{B}(u_n, v_n) \rightharpoonup \mathbb{B}(w, w) \text{ weakly in } L^{\frac{4}{3}}(0, M; V'). \tag{139}$$

Following [12], we first prove that

$$\widetilde{B}(u_n, v_n) \rightharpoonup \mathbb{B}(w, w) \text{ weakly in the space } L^q(0, M; D(A)') \tag{140}$$

for some q, $1<q<2$.
We rewrite $\widetilde{B}(u_n, v_n)$ as follows:

$$\begin{aligned}\widetilde{B}(u_n, v_n) &= \widetilde{B}(u_n, u_n + \alpha_n^2 A u_n)\\ &= \widetilde{B}(u_n, u_n) + \alpha_n^2 \widetilde{B}(u_n, Au_n)\\ &= \mathbb{B}(u_n, u_n) + \alpha_n^2 \widetilde{B}(u_n, Au_n),\end{aligned} \tag{141}$$

where we have used the identity $\widetilde{B}(u,u) = \mathbb{B}(u,u)$ (see (74)). Consider both terms of (141) separately. We start with the second. By (82), we have

$$\|\alpha_n^2 \widetilde{B}(u_n, Au_n\|_{D(A)'} \leq C\alpha_n^2 \|u_n\||Au_n|. \tag{142}$$

Fixing an arbitrary β, $1<\beta<2$, we obtain the following chain of inequalities

$$\begin{aligned}&\int_0^M \|\alpha_n^2 \widetilde{B}(u_n(t), Au_n(t))\|_{D(A)'}^{\beta}\, dt\\ &\leq C^\beta \alpha_n^{2\beta} \int_0^M \|u_n(t)\|^\beta |Au_n(t)|^\beta\, dt\\ &\leq C^\beta \alpha_n^{2\beta} \left(\sup_{t\in[0,M]} \|u_n(t)\|^\gamma\right) \int_0^M \|u_n(t)\|^{\beta-\gamma} |Au_n(t)|^\beta\, dt\\ &\leq C^\beta \alpha_n^{2\beta} \left(\sup_{t\in[0,M]} \|u_n(t)\|^\gamma\right) \left(\int_0^M \|u_n(t)\|^{q(\beta-\gamma)}\, dt\right)^{\frac{1}{q}} \left(\int_0^M |Au_n(t)|^{p\beta}\, dt\right)^{\frac{1}{p}},\end{aligned} \tag{143}$$

where γ is an arbitrary number such that $0<\gamma<\beta$, and in (143) we have applied the Hölder inequality with $\frac{1}{p}+\frac{1}{q}=1$ (these numbers will be determined later on). Combining the chain of inequalities after (143), we have

$$\begin{aligned}&\int_0^M \|\alpha_n^2 \widetilde{B}(u_n(t), Au_n(t)\|_{D(A)'}^{\beta}\, dt\\ &\leq C^\beta \alpha_n^{2\beta} \left(\sup_{t\in[0,M]} \|u_n(t)\|^2\right)^{\frac{\gamma}{2}} \left(\int_0^M \|u_n(t)\|^{q(\beta-\gamma)}\, dt\right)^{\frac{1}{q}} \left(\int_0^M |Au_n(t)|^{p\beta}\, dt\right)^{\frac{1}{p}}.\end{aligned} \tag{144}$$

We now set $p=\frac{2}{\beta}, q=\frac{2}{2-\beta}$, and find the number γ from the equation $q(\beta-\gamma)=2$, that is $\gamma=2(\beta-1)$. We see that such γ satisfies the inequality $\gamma<\beta \iff \beta<2$. Replacing such p,q and γ into (144), we obtain the following estimate

$$\begin{aligned}&\int_0^M \|\alpha_n^2 \widetilde{B}(u_n(t), Au_n(t)\|_{D(A)'}^{\beta}\, dt\\ &\leq C^\beta \alpha_n^{2-\beta} \left(\sup_{t\in[0,M]} \alpha_n^2\|u_n(t)\|^2\right)^{\beta-1} \left(\int_0^M \|u_n(t)\|^2\, dt\right)^{\frac{2-\beta}{2}} \left(\int_0^M \alpha_n^2|Au_n(t)|^2\, dt\right)^{\frac{\beta}{2}}.\end{aligned} \tag{145}$$

We now use estimates (118) – (119) and find that the right hand side of (145) is less than or equal than $C_1\alpha_n^{2-\beta}$:

$$\int_0^M \|\alpha_n^2 \widetilde{B}(u_n(t), Au_n(t)\|_{D(A)'}^{\beta}\, dt \leq C_1\alpha_n^{2-\beta}, 1<\beta<2. \tag{146}$$

Therefore, the term

$$\alpha_n^2 \widetilde{B}(u_n, Au_n) \to\ 0 \quad \text{strongly}\ \ \text{in}\ L^\beta(0,M;D(A)') \tag{147}$$

for any β, $1 < \beta < 2$.
On the other hand, arguing as in the proof of Proposition 4(see (57)), we also have

$$\mathbb{B}(u_n, u_n) \rightharpoonup \mathbb{B}(w, w) \text{ weakly in } L^{\frac{4}{3}}(0, M; V'). \tag{148}$$

Now combining (141), (147) *and* (148), we find that

$$\widetilde{B}(u_n, v_n) \rightharpoonup \mathbb{B}(w, w) \text{ weakly in } L^{\frac{4}{3}}(0, M; D(A)'). \tag{149}$$

Finally using Lemma 3.3 in [12], we deduce that

$$(1 + \alpha_n^2 A)^{-\frac{1}{2}} \widetilde{B}(u_n, v_n) \rightharpoonup \mathbb{B}(w, w) \text{ weakly in } L^{\frac{4}{3}}(0, M; D(A)'). \tag{150}$$

We have then established that the couple $(w(t), B(t))$ satisfies the system (137)-(138).
It is left to prove that $(w(t), B(t))$ satisfies the energy inequality (113). The proof is similar to the case of (58) since the couple $(w_n(t), B_n(t))$ satisfies the energy inequality

$$-\frac{1}{2}\int_0^M (|w_n(t)|^2 + |B_n(t)|^2)\psi'(t)dt + \int_0^M (\nu\|w_n(t)\|^2 + \eta\|B_n(t)\|^2)\psi(t)dt \leq \int_0^M \langle g, w_n(t)\rangle\psi(t)dt, \tag{151}$$

for all $\psi \in C_0^\infty(0, M), \psi \geq 0$. This completes the proof of Theorem 5. □

5 Convergence of the trajectory attractor of the 3D MHD-α model as α approaches zero

In Section 2, we have constructed the trajectory attractor $\mathcal{U}_0$ of the 3D MHD system:

$$\partial_t v + \nu A v + \mathbb{B}(v, v) - \mathbb{B}(B, B) = g, \ t \geq 0, \tag{152}$$
$$\partial_t B + \eta A B + \mathbb{B}(v, B) - \mathbb{B}(B, v) = 0. \tag{153}$$

Recall that the set $\mathcal{U}_0$ is bounded in $\mathcal{F}_+^b$, compact in Θ_+^{loc} and $\mathcal{U}_0 \subset \mathcal{K}^+$. We have also proved that

$$\mathcal{U}_0 = \Pi_+ \mathcal{K}_0, \tag{154}$$

where $\mathcal{K}_0$ is the kernel of system (152) – (153). $\mathcal{K}_0$ is the union of all bounded (in the norm $\mathcal{F}^b$) complete weak solutions $(v(t), B(t))$, t$\in \mathbb{R}$ of the 3D MHD system that satisfy the energy inequality (65).
We denote by

$$\mathcal{B}_\alpha = \{(w_\alpha(t), B_\alpha(t)), t \geq 0\}, 0 < \alpha \leq 1,$$

a family of couple $(w_\alpha(t), B_\alpha(t))$ where $w_\alpha(t) = (1 + \alpha^2 A)^{\frac{1}{2}} u_\alpha(t)$ and $(u_\alpha(t), B_\alpha(t))$ is a solution of system (75) – (77). The norm of $(w_\alpha(t), B_\alpha(t))$ in $\mathcal{F}_+^b$ are uniformly bounded with respect to α, that is

$$\|(w_\alpha, B_\alpha)\|_{\mathcal{F}_+^b} \leq R$$

where R is an arbitrary number independent of α (see (110)) and

$$\|(w_\alpha, B_\alpha)\|_{\mathcal{F}_+^b} = \|(w_\alpha, B_\alpha)\|_{L_2^b(\mathbb{R}_+; V\times V)} + \|(w_\alpha, B_\alpha\|_{L^\infty(\mathbb{R}_+; H\times H)} + \\ \|\partial_t w_\alpha\|_{L_2^b(\mathbb{R}_+; D(A)')} + \|\partial_t B_\alpha\|_{L_2^b(\mathbb{R}_+; D(A)')}. \tag{155}$$

Recall that (w_α, B_α) satisfies the system

$$\partial_t w_\alpha + \nu A w_\alpha + (1+\alpha^2 A)^{-\frac{1}{2}} \widetilde{B}(u_\alpha, v_\alpha) - (1+\alpha^2 A)^{-\frac{1}{2}} \mathbb{B}(B_\alpha, B_\alpha) = (1+\alpha^2 A)^{-\frac{1}{2}} g, \tag{156}$$

$$\partial_t B_\alpha + \eta A B_\alpha + \mathbb{B}(u_\alpha, B_\alpha) - \mathbb{B}(B_\alpha, u_\alpha) = 0, \tag{157}$$

where $v_\alpha = (1+\alpha^2)^{\frac{1}{2}} w_\alpha(t)$ and $u_\alpha = (1+\alpha^2 A)^{-\frac{1}{2}} w_\alpha(t)$. We also recall that

$$T(h)(w_\alpha(t), B_\alpha(t)) = (w_\alpha(t+h), B_\alpha(t+h)). \tag{158}$$

The main result of this paper is the following theorem

Theorem 6. *1) The trajectory attractor $\mathcal{U}_\alpha$ of the system* $(75)-(77)$ *converges in the topology Θ_+^{loc} as $\alpha \to 0^+$ to the trajectory attractor $\mathcal{U}_0$ of the 3D MHD system* $(152)-(153)$ *:*

$$\mathcal{U}_\alpha \to \mathcal{U}_0 \text{ in } \Theta_+^{loc} \text{ as } \alpha \to 0^+. \tag{159}$$

2) Let $\mathcal{B}_\alpha = \{(w_\alpha(t,x), B_\alpha(t,x)), t \geq 0\}$, $0 < \alpha \leq 1$, be bounded sets of solutions of system $(156)-(157)$ *that satisfy the inequality*

$$\|(w_\alpha, B_\alpha)\|_{\mathcal{F}_+^b} \leq R, \ \forall \alpha, \ 0 < \alpha \leq 1. \tag{160}$$

Then the sets of shifted solutions $\{T(h)\mathcal{B}_\alpha\}$ converge to the trajectory attractor $\mathcal{U}_0$ of the 3D MHD system $(152)-(153)$ *in the topology Θ_+^{loc} as $h \to \infty$ and $\alpha \to 0^+$:*

$$T(h)\mathcal{B}_\alpha \to \mathcal{U}_0 \text{ in } \Theta_+^{loc} \text{ as } \alpha \to 0^+, \ h \to \infty. \tag{161}$$

Proof. it suffices to prove 2) which implies (159) if we take $\mathcal{B}_\alpha = \mathcal{U}_\alpha = T(h)\mathcal{U}_\alpha \ \forall h \geq 0$.
Assume that relation (161) fails to hold. Then there is a neighborhood $\Theta(\mathcal{U}_0)$ of $\mathcal{U}_0$ in Θ_+^{loc} and two sequences $\alpha_n \to 0^+$, $h_n \to \infty$ as $n \to \infty$ such that

$$T(h_n)\mathcal{B}_{\alpha_n} \not\subseteq \Theta(\mathcal{U}_0).$$

Hence, there are couple $(w_{\alpha_n}(.), B_{\alpha_n}(.)) \in \mathcal{B}_{\alpha_n}$, such that $X_{\alpha_n}(.) := (w_{\alpha_n}(.), B_{\alpha_n}(.)) \in \mathcal{B}_{\alpha_n}$, and the functions

$$\begin{aligned} W_{\alpha_n}(t) &:= T(h_n) X_{\alpha_n}(t) \\ &= (w_{\alpha_n}(t+h_n), B_{\alpha_n}(t+h_n)), t \geq 0 \end{aligned}$$

do not belong to $\Theta(\mathcal{U}_0)$, that is

$$W_{\alpha_n}(t) \notin \Theta(\mathcal{U}_0). \tag{162}$$

The couple $W_{\alpha_n}(t) = (U_{\alpha_n}(t), V_{\alpha_n}(t))$ is a solution of system $(156)-(157)$ on the interval $[-h_n, +\infty[$ with $\alpha = \alpha_n$, since $(w_{\alpha_n}(t+h_n), B_{\alpha_n}(t+h_n))$ is a solution of the system for $t+h_n \geq 0$ and the system is autonomous. Moreover, it follows from (160) that

$$\left(\sup_{t \geq -h_n} \int_t^{t+1} \|U_{\alpha_n}(s)\|^2 \, ds + \sup_{t \geq -h_n} \int_t^{t+1} \|V_{\alpha_n}(s)\|^2 \, ds \right)^{\frac{1}{2}} + ess \sup_{t \geq -h_n} |U_{\alpha_n}(t)| + \\ ess \sup_{t \geq -h_n} |V_{\alpha_n}(t)| + \left(\sup_{t \geq -h_n} \int_t^{t+1} \|\partial_t U_{\alpha_n}(s)\|_{D(A)'}^2 \, ds \right)^{\frac{1}{2}} + \left(\sup_{t \geq -h_n} \int_t^{t+1} \|\partial_t V_{\alpha_n}(s)\|_{D(A)'}^2 \, ds \right)^{\frac{1}{2}} \leq R. \tag{163}$$

This inequality implies that the sequence $\{(U_{\alpha_n}(.), V_{\alpha_n}(.)), n \in \mathbb{N}\}$ is weakly compact in the space

$$\begin{aligned} &\Theta_{-M,M} = L^2(-M, M; V \times V) \cap L^\infty(-M, M; H \times H) \cap \\ &\{(v, b) / \partial_t v \in L^2(-M, M; D(A)'), \partial_t b \in L^2(-M, M; D(A)')\} \end{aligned}$$

for every M, if we consider α_n with indices n such that $h_n \geq M$. Therefore, for every fixed $M > 0$, we can choose a subsequence $\{\alpha_{n'}\} \subset \{\alpha_n\}$ such that $\{W_{\alpha_n}(.) = (U_{\alpha_n}(.), V_{\alpha_n}(.)), n \in \mathbb{N}\}$ converges in $\Theta_{-M,M'}$. Thus using the well-known Cantor diagonal procedure, we can construct a couple of functions $W(.) = (U(t), V(t)), t \in \mathbb{R}$ and a subsequence $\{\alpha_{n"}\} \subset \{\alpha_n\}$ such that

$$W_{\alpha_{n"}} = (U_{\alpha_{n"}}, V_{\alpha_{n"}}) \to \text{ W=(U,V) weakly in } \Theta_{-M,M}, \tag{164}$$

as $n" \to \infty$ for every $M > 0$.
From (163), we obtain the inequality for the limit function $W(t) = (U(t), V(t)), t \in \mathbb{R}$:

$$\begin{aligned}&\left(\sup_{t\in\mathbb{R}} \int_t^{t+1} \|U(s)\|^2\, ds + \sup_{t\in\mathbb{R}} \int_t^{t+1} \|V(s)\|^2\, ds\right)^{\frac{1}{2}} + ess\sup_{t\in\mathbb{R}} |U(t)| + \\ &ess\sup_{t\in\mathbb{R}} |V(t)| + \left(\sup_{t\in\mathbb{R}} \int_t^{t+1} \|\partial_t U(s)\|^2_{D(A)'}\, ds\right)^{\frac{1}{2}} + \left(\sup_{t\in\mathbb{R}} \int_t^{t+1} \|\partial_t V(s)\|^2_{D(A)'}\, ds\right)^{\frac{1}{2}} \leq R.\end{aligned} \tag{165}$$

In particular, we have

$$\begin{aligned}W(t) = (U(t), V(t)) \in \mathcal{F}^b = L_2^b(\mathbb{R}; V \times V) \cap L^\infty(\mathbb{R}; H \times H)& \\ \cap \{(u,v)/\partial_t u \in L_2^b(\mathbb{R}; D(A)'), \partial_t v \in L_2^b(\mathbb{R}; D(A)')\}.&\end{aligned} \tag{166}$$

We now apply Theorem 5, where we can assume that all the functions $U_{\alpha_{n"}}, V_{\alpha_{n"}}$ are defined in the semiaxis $[-M; +\infty[$ instead of $[0, +\infty[$ (equations are autonomous). Then from (164) and (165), we conclude that $W(t) = (U(t), V(t))$ is a weak solution of the 3D MHD system for all $t \in \mathbb{R}$ and $W(t) = (U(t), V(t))$ satisfies the energy inequality. Therefore $W(t) = (U(t), V(t)) \in \mathcal{K}_0$ where $\mathcal{K}_0$ is the kernel of the system $(152) - (153)$.
Since $\Pi_+\mathcal{K}_0 = \mathcal{U}_0$ and $W(t) \in \mathcal{K}_0$, we have $\Pi_+ W \in \mathcal{U}_0$. On the other hand, we have established from (164) that

$$\Pi_+ W_{\alpha_{n"}} \to \Pi_+ W \text{ in } \Theta_+^{loc} \text{ as } \text{ n"} \to \infty. \tag{167}$$

In particular for a large $n"$,

$$\Pi_+ W_{\alpha_{n"}} \in \Theta(\Pi_+ W) \subseteq \Theta(\mathcal{U}_0). \tag{168}$$

This contradicts (162). Therefore (161) is true . This completes the proof of Theorem 6. □

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