DISTRIBUTION-FREE EXCEEDANCE CUSUM CONTROL CHARTS FOR LOCATION

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Abstract

Distribution-free (nonparametric) control charts can be useful to the quality practitioner when the underlying distribution is not known. A Phase II nonparametric CUSUM chart based on the exceedance statistics, called the exceedance CUSUM chart, is proposed here for detecting a shift in the unknown location parameter of a continuous distribution. The exceedance statistics can be more efficient than rank-based methods when the underlying distribution is heavy-tailed and/or right-skewed, which may be the case in some applications, particularly with certain lifetime data. Moreover, exceedance statistics can save testing time and resources as they can be applied as soon as a certain order statistic of the reference sample is available. Guidelines and recommendations are provided for the chart's design parameters along with an illustrative example. The in- and out-of-control performance of the chart are studied through extensive simulations on the basis of the average run-length (*ARL*), the standard deviation of run-length (*SDRL*), the median run-length (*MDRL*) and some percentiles of run-length. Further, a comparison with a number of existing control charts, including the parametric CUSUM $\overline{X}$ chart and a recent nonparametric CUSUM chart based on the Wilcoxon rank-sum statistic, called the rank-sum CUSUM chart, is made. It is seen that the exceedance CUSUM chart performs well in many cases and thus can be a useful alternative chart in practice. A summary and some concluding remarks are given.

Keywords: Binomial; CUSUM chart; Exceedance statistic; Markov chain; Nonparametric; Precedence statistic; Quality control; Robust; Simulation.

Subject Classifications: 62G99, 62P30.

1. INTRODUCTION

Woodall and Montgomery (1999, page 380) pointed out the increasing role for nonparametric methods in Statistical Quality Control (SQC) or Statistical Process Control (SPC) problems, particularly as data availability increases. Since that time, there has been a lot of interest in and a significant amount of work on nonparametric control charts. Chakraborti et al. (2001), Chakraborti and Graham (2007) and most recently Chakraborti et al. (2011) provide thorough overviews of the area. If the in-control (IC) run-length distribution of a control chart is the same for every continuous distribution, the chart is called nonparametric or distribution-free. Chakraborti et al. (2001) summarized the advantages of nonparametric control charts as follows: (i) simplicity, (ii) no need to assume a particular parametric distribution for the underlying process, (iii) the IC run-length distribution is the same for all continuous distributions (the same is true for the false alarm rate (*FAR*) and the IC average run-length (ARL_0); and thus different nonparametric charts can be compared more easily), (iv) more robust and outlier resistant and (v) more efficiency in detecting changes when the true distribution is markedly non-normal, particularly with heavier tails.

Several nonparametric charts currently exist for monitoring the location parameter (mean, median, etc.) of a continuous distribution. In a process control setting, the location parameter of interest can be specified and hence known, say because of the existing standards or specifications, or it may be unknown as in a start-up situation. In this paper we consider nonparametric Phase II CUSUM control charts for monitoring an unknown location parameter based on the exceedance statistic.

Precedence/Exceedance Charts

Phase II nonparametric charts are constructed by adapting suitable two-sample nonparametric tests. Among these, the Wilcoxon-Mann-Whitney (WMW) test, also known as the rank-sum test (see Gibbons and Chakraborti (2003)), is perhaps the most popular. Chakraborti and Van de Wiel (2008) considered a class of Shewhart-type charts based on the WMW statistics. Following this line of research, Li et al. (2010) considered a class of cumulative sum (CUSUM) and exponentially weighted moving average (EWMA) control charts based on the WMW statistics. Another useful two-sample nonparametric test is the precedence (or the exceedance) test. The precedence test was developed in Nelson (1963) to meet the "demand for a statistical test that could give an early indication of the difference between two samples placed simultaneously on life test," (Nelson, 1993, page 140). While the rank-sum statistic is based on the sum of ranks of the observations from one sample in the combined sample, the precedence statistic is based on the number of observations from one of the samples that precede a specified (say the r^{th}) order statistic of the other sample. The precedence statistic is linearly related to the exceedance statistic, which is the number of observations from one of the samples that exceed the r^{th} order statistic of the other sample, so that precedence and exceedance tests are equivalent. Exceedance tests have been found to be useful in a number of applications including quality control and reliability studies with lifetime data where the exceedance probability can be associated with the so-called 'warranty time' of a product (see, for example, Chakraborti and Van der Laan (2000)). The reader is referred to Balakrishnan and Ng (2006) for the vast literature on precedence/exceedance tests. In particular, they noted that (page 51) "Wilcoxon's rank-sum test performs better than the precedence tests if the underlying distributions are close to symmetry, such as the normal distribution, gamma distribution with large values of shape parameter, and lognormal distribution with small values of shape parameter. However, under some right-skewed distributions such as the exponential distribution, gamma distribution with shape parameter 2.0, and lognormal distribution with shape parameter 0.5, the precedence tests have higher power values than the Wilcoxon's rank-sum test for small values of r. It is evident that the more right-skewed the underlying distribution is, the more powerful the precedence test is."

Chakraborti et al. (2004) studied a class of nonparametric Phase II Shewhart-type charts based on the precedence statistics, called the Shewhart precedence charts. This paper has been the starting point for a number of papers in this area. They showed that the precedence charts perform admirably compared to their normal theory counterparts and hence can be legitimate competitors in practice. However, while the Shewhart charts are most widely known and used in practice, it is well known that the CUSUM charts are useful for detecting smaller and persistent shifts more quickly. Moreover, the CUSUM charts can be more natural in a process control environment because of the sequential nature of data collection. The reader is referred to

Hawkins and Olwell (1998) for a general overview on CUSUM charts. Motivated by these facts, we consider a class of Phase II CUSUM control charts based on exceedance statistics, called exceedance CUSUM charts, in this paper. Recently, Graham et al. (2012) considered a Phase II EWMA control chart based on exceedance statistics which performs better than Li et al. (2010)'s EWMA rank-sum chart; particularly for distributions that are heavier-tailed and more peaked than the normal distribution. Subsequently, in this paper we show that the exceedance CUSUM chart outperforms Li et al. (2010)'s CUSUM rank-sum chart for most of the heavy-tailed distributions under consideration irrespective of the peakedness of the distribution.

This article is organized as follows: In Section 2 the nonparametric exceedance CUSUM chart is introduced. In particular we consider exceedances over the median and call the resulting chart the exceedance CUSUM median chart. In Section 3 the implementation of the chart is discussed in terms of the chart design parameters. In Section 4 the run-length distribution is studied. The IC and out-of-control (OOC) chart performance are studied in Section 5 and compared with a number of existing Phase II charts including the CUSUM $\overline{X}$ chart and the nonparametric rank-sum CUSUM chart based on the Wilcoxon rank-sum statistic considered by Li et al. (2010). In Section 6 we discuss an example showing application of the proposed chart. We conclude with a summary and some recommendations for future research in Section 7.

2. EXCEEDANCE CUSUM CHART

Assume that a (Phase I) reference sample $X_1, X_2, \ldots, X_m$ is available from an IC process with an unknown continuous cumulative distribution function (cdf) $F(x)$. Let $Y_{j1}, Y_{j2}, \ldots, Y_{jn_j}$, $j = 1,2, \ldots$, denote the j^{th} test (Phase II) sample of size n_j. Let $G(y)$ denote the cdf of the distribution of the j^{th} Phase II sample. Both F and G are unknown continuous distribution functions and the process is IC when $F = G$. For detecting a change in the location, we use the location model $G_Y(x) = F(x - \theta)$ where $\theta \epsilon [0, \infty)$ is the unknown location parameter. It is often the case that the Phase II samples (subgroups) are all of the same size, n, so that the subscript j in n_j can be suppressed.

Let $U_{j,r}$ denote the number of exceedances, that is, the number of Y observations in the j^{th} Phase II sample that exceeds $X_{(r)}$, the r^{th} ordered observation in the reference sample. The statistic $U_{j,r}$ is called an exceedance statistic and the probability $p_r = P[Y > X_{(r)} | X_{(r)}]$ is called an exceedance probability. The number of Y observations in the j^{th} Phase II sample that precede $X_{(r)}$ is called a precedence statistic and has been used by Chakraborti et al. (2004) to study a nonparametric Shewhart-type chart called the Shewhart precedence chart. In this paper we work with the exceedances as they seem more natural to consider while detecting a shift to the right (increase) in the location parameter θ. For inference purposes, the exceedance and precedence tests are equivalent in the sense that they are linearly related and provide the same amount of information about θ.

Construction of an exceedance CUSUM chart is straightforward. Given $X_{(r)} = x_{(r)}$, it can be shown (see Result A.1 in Appendix A) that the $U_{j,r}$ follows a binomial distribution with parameters (n, p_r) and thus, conditionally on $X_{(r)}$, we can use a binomial-type CUSUM chart based on the $U_{j,r}$'s to monitor the process location (via the exceedance probabilities). Based on this idea, an upper one-sided exceedance CUSUM chart, to detect an increase in the location, may be based on

$$\max\left[0, C_{j-1} + \left(U_{j,r} - \mu_U\right) - k\right],\ j = 1,2,3, \ldots,$$

where the starting value $C_0 = 0$, $\mu_U = E\left(U_{j,r}\middle|X_{(r)}\right) = np_r$ and $k \geq 0$ is the so-called reference value. Since the conditional probability p_r is unknown, we replace it by its unconditional (averaging over the distribution of $X_{(r)}$) IC value $d = \frac{(m-r+1)}{(m+1)}$ (see Result A.4 in Appendix A). Hence the proposed upper one-sided exceedance CUSUM plotting statistic is defined as

$$C_j = \max[0, C_{j-1} + \left(U_{j,r} - nd\right) - k], \text{ say, for } j = 1,2,3, \ldots \quad (1)$$

with starting value $C_0 = 0$.

It can be shown that (see Result A.3 in Appendix A) the unconditional joint distribution of the exceedance statistics is distribution-free and hence the proposed exceedance CUSUM chart is unconditionally distribution-free. However, note that unconditionally, the proposed chart is not a binomial CUSUM chart.

The chart signals a possible OOC situation for the first j such that $C_j > H$, where $H > 0$, may be looked upon as the Upper Control Limit (UCL), and at that point a search for assignable causes is started; the Lower Control Limit (LCL) is 0 by default. Otherwise, the process is considered IC and process monitoring continues without interruption. We study the upper one-sided chart here but a lower one-sided as well as a two-sided exceedance CUSUM chart can be studied along similar lines. Note that the CUSUM statistic in (1) actually gives a class of control charts for various choice of *r*. From a practical point of view, and as used in Chakraborti et al. (2004), we take θ to be the median and $X_{(r)}$ to be the median of the reference sample. In this case, d is taken to be equal to 0.5. The reasons for focusing on the median are clear; it is robust and a better representative of the central reference value. However, in general, the precedence/exceedance chart can be used to monitor other parameters, for example, the 1st quartile or the 70th percentile.

3. IMPLEMENTATION OF THE CHART: CHOICE OF DESIGN PARAMETERS

Implementation of the proposed nonparametric exceedance CUSUM chart requires specifying the following quantities: (i) *m*: the size of the IC Phase I reference sample, (ii) *n*: the size of each Phase II test sample (the subgroup size), (iii) *t*: the desired ARL_0, (iv) k: the reference value, essentially a rational number and (v) H: the decision interval/UCL depending on *m*, *n*, *r*, *t* and *k*. It is up to the experimenter to specify the parameters *m*, *n* and *t* in a given situation. The choice of the IC Phase I reference sample size can be profound and is discussed in detail in Appendix B. The design parameters (*k, H*) are chosen so that the chart has a specified nominal $ARL_0 = t$, say, and is capable of detecting a shift, specially a small shift, as soon as possible. The first step is to choose *k*. Under the normal distribution, the choice of *k* has been discussed, for example, in Montgomery (2009). Let us consider the traditional CUSUM chart for monitoring of normal mean with individual data (n = 1) with no reference sample. To examine the impact of k, we examine the OOC ARL (denoted ARL_δ, where δ represents the shift in the mean) for the normal distribution in Figure 1, taking the IC mean $\mu_0 = 0$ and standard deviation $\sigma = 1$ (without loss of generality) and setting $t = 500$, for μ_1 = 0.1, 0.25, 0.5 and 1.0. Note that μ_1 represents the increased value of μ to be detected "quickly" from $\mu_0 = 0$; hence μ_1 represents the true shift in the mean, that is, $\delta = \mu_1$.

< Figure 1 >

From Figure 1 several interesting observations can be made. When the shift is small (see panels A and B of Figure 1) and a larger value of k is chosen, the ARL_δ values become unacceptably high. On the other hand, if the shift is large (see panels C and D of Figure 1) and a smaller value of k is chosen, the ARL_δ values are also high, but not as high as in the latter case. This suggests that when there is little or no a-priori information regarding the size of the shift, a smaller value of k is the safest choice (to protect against any unnecessary delays in detection). Later we shall see that similar conclusions can be drawn about the proposed chart and that can be seen from the Tables 4.A to 4.E while comparing the ARL_δ values under $k = 0$ with $k > 0$. Note that although we are considering an unknown shift, we are primarily interested in detecting a smaller and moderate shift with a CUSUM chart. Therefore, we recommend using $k = 0$ (or letting δ tend to 0). Hence the upper one-sided exceedance CUSUM median chart based on the reference sample median, is given by the plotting statistic

$$C_j = \max[0, C_{j-1} + (U_{j,r} - n/2)], \quad \text{for } j = 1,2,3,\ldots \qquad (2)$$

with a starting value $C_0 = 0$.

The next step is to choose H, in conjunction with the chosen k, so that a desired nominal ARL_0 is attained. Tables 1.A and 1.B in Section 5 lists different values of H with $k = 0$ for the industry standard ARL_0 values of 370 and 500 and for m = 1000, 500 and n = 5, 11 and 25, respectively. These tables should be useful for implementing the exceedance CUSUM chart for location in practice. It is seen that the proposed chart can attain ARL_0 values of 370 and 500 almost exactly. To this end, however, we discuss the run-length distribution and the performance of the chart first.

4. RUN-LENGTH DISTRIBUTION

There are two main approaches to studying the run-length distribution of a CUSUM chart. For continuous observations, Page (1954) described an integral equation approach. An alternative method based on Markov chain theory was developed by Brook and Evans (1972). Since the proposed chart is a binomial CUSUM chart conditionally on $X_{(r)}$, we can use the results of Gan (1993) to derive the conditional run-length distribution. Then the unconditional run-length distribution is obtained by simply averaging over the distribution of $X_{(r)}$.

In order to implement the Markov chain approach, we introduce some new notations. Write $k = nk^*$ such that $(nd + k) = n(d + k^*) = nd^*$, say, so that $k = n(d^* - d)$. Note that when $k = 0$, $d^* = d$. Thus the plotting statistic in (1) can be expressed as

$$C_j = \max[0, C_{j-1} + (U_{j,r} - nd^*)], \quad \text{for } j = 1,2,3,\ldots$$

with $C_0 = 0$.

Now, as in Gan (1993), in general, suppose that, $d^* = a/nb$ and $H = c/nb$ where a, b and c are all positive integers. Then, again as in Gan (1993), it is easy to see that when the process is declared to be IC the possible values of C_j are given by $\{0, 1/nb, 2/nb, \ldots, c/nb\}$; these are the transient states of the Markov chain. If $C_j > c/nb$, then the process is declared to be OOC and C_j is said to be in the absorbing state. Using the simplified notation structure of Gan (1993) by labeling the transient states as $\{1, 2, \ldots, c + 1\}$ corresponding to $\left\{C_j = \frac{i}{nb};\ i = 0,1,2,\ldots,c\right\}$, respectively, and by denoting the $(c+2)^{\text{th}}$ state as the absorbing state, we can write the one-step transition probability matrix in a partitioned form

$$M = \begin{pmatrix} T & \tilde{P} \\ \tilde{O} & 1 \end{pmatrix} = \begin{pmatrix} p_{11} & p_{12} & \cdots\cdots & p_{1,c+1} & p_{1,c+2} \\ p_{21} & p_{22} & \cdots\cdots & p_{2,c+1} & p_{2,c+2} \\ & \cdots & & \cdots & \cdots \\ & \cdots & & \cdots & \cdots \\ p_{c+1,1} & p_{c+1,0} & \cdots & p_{c+1,c+1} & p_{c+1,c+2} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

where p_{ij} denotes the one-step transition probability from state *i* to state *j*; the essential transition probability sub-matrix T contains all the probabilities of going from one transient state to another; the column vector $\tilde{P}$ contains all the probabilities of going from each transient state to the absorbing state; $\tilde{O}$ a row vector of zeros which contains all the probabilities of going from the absorbing state to each transient state and the scalar value 1 is the probability of going from the absorbing state to the absorbing state. The elements of the essential transition probability sub-matrix *T* may be calculated from the conditional distribution of *Y* given $X_{(r)} = x_{(r)}$. It is easy to see that, for $i = 1, 2, \ldots, c + 1$,

$$p_{i1} = P\left(C_j = 0 \left| C_{j-1} = \frac{i-1}{b};\ X_{(r)} = x_{(r)}\right.\right)$$

$$= \begin{cases} P\left(U_{j,r} \le \dfrac{a - i + 1}{b} \middle| X_{(r)} = x_{(r)}\right) & \text{if } \dfrac{a - i + 1}{b} \ge 0 \\ 0 & \text{otherwise} \end{cases}.$$

Similarly for $i = 1, 2, \ldots, c + 1,$ and for $j = 2, 3, \ldots, c + 1$, we have that

$$p_{ij} = P\left(C_j = \frac{j-1}{b} \middle| C_{k-1} = \frac{i-1}{b};\ X_{(r)} = x_{(r)}\right)$$

$$= \begin{cases} P\left(U_{j,r} \le \dfrac{a - i + j}{b} \middle| X_{(r)} = x_{(r)}\right) & \text{if } \dfrac{a - i + j}{b} = 0,1,\ldots,n \\ 0 & \text{otherwise} \end{cases}.$$

Note that the conditional probabilities can be calculated directly using result A.1, that is, given $X_{(r)} = x_{(r)}$, $U_{j,r}$ follows a binomial distribution with parameters (n, p_r) where $p_r = P[Y > X_{(r)} | X_{(r)}]$. For the proposed exceedance CUSUM median chart based on the reference sample median, we may substitute *n* and 2 for a and b, respectively (so that $d = 0.5$). Note that when the process is IC, $p_r = 1 - V_{(r)}$, where $V_{(r)} = F(X_{(r)})$ follows a beta distribution with parameters *r* and *m* – *r* + 1 whatever the continuous *F* may be. Now defining N_i as the run-length variable with a starting value equal to $(i-1)/b$, i.e. $C_0 = \frac{i-1}{nb}$ and $\mu_i = E(N_i|\, X_{(r)})$ as the conditional average run-length for $i = 1, 2, \ldots, c+1$, we have, from the properties of Markov chains,

$$\tilde{\mu} = (\mu_1, \mu_2, \ldots, \mu_{c+1})' = (I - T)^{-1}\tilde{1}. \quad (3)$$

The unconditional *ARL* is given by averaging this over the probability distribution of $X_{(r)}$. Thus, the unconditional average run-length is given by

$$\mu_1^* = EE\,(N_1|\, X_{(r)}) = \int E\,(N_1|\, X_{(r)})\, dF(X_{(r)}) = \int \mu_1\, dF(X_{(r)}) = \int \mu_1\, dV_{(r)}. \quad (4)$$

Expressions for the conditional and unconditional run-length distributions can be obtained similarly using properties and results of Markov chains from which other run-length distribution characteristics such as the standard deviation and the percentiles can be found.

Note that using Eq. (4), one can approximate the unconditional ARL_0, replacing the integral in Eq. (4) by a sum, which yields

$$ARL_0 = \mu_1^*(\text{IC}) = \lim_{l\to 0} \sum_{\xi} (\mu_1|\, p_r = \xi, \text{IC}) \int_{\xi-\frac{l}{2}}^{\xi+\frac{l}{2}} \frac{1}{B(r, m-r+1)} y^{r-1}(1-y)^{m-r} dy, \quad (5)$$

where ξ ranges from $a - l/2$ to $b + l/2$ in steps of l; l is a small positive proper fraction; a and b are such that $0 < a < b < 1$, satisfying

$$\int_0^a \mu_1\, dV_{(r)} \cong 0 \;\; and \;\; \int_b^1 \mu_1\, dV_{(r)} \cong 0.$$

The IC conditional mean, $(\mu_1|p_r = \xi, \text{IC})$, can be calculated by using the Markov chain formula in (3). Let us consider an example. Suppose m = 1000, n = 5, H = 15 and we consider exceedance over the median. For even m, the quantity r is not unique but approximately take r = 500.5. Hence the IC distribution of p_r is (approximately) a $Beta$ (500.5,500.5). Take a value of p_r, say 0.35. In the IC case, $P[p_r < 0.35|\text{IC}] = 3.963355 \times 10^{-22}$. Further, using (3), we find $(\mu_1|p_r = 0.35, \text{IC}) = 9.39 \times 10^9$. Since $P[p_r < 0.35|\text{IC}] \approx 0$, the contribution of $(\mu_1|p_r = 0.35, \text{IC})\, P\big[p_r \in \Delta_{0.35,\varepsilon}\big]$ is of the order 10^{-13} and is therefore negligible, where $\Delta_{0.35,\varepsilon}$ is an ε-neighbourhood (ε close to zero) containing 0.35.

Similarly $P[Z > 0.65|\, Beta\ (500.5,500.5)] \approx 0$. The main point is that for calculating the unconditional ARL_0 using (5), it is sufficient to consider values for ξ in the interval $(0.3, 0.7)$ as other values of ξ do not contribute any significant amounts in the sum. This interval, however, will vary with m as well as r and has to be determined with care. For example, for m = 1000, n = 5, H = 15 we may set $a = 0.3, l = 0.0001$ and $b = 0.7$. Thus we find from (5)

$$\mu_1^*(IC) = \sum_{\xi=0.29995(0.0001)0.70005} (\mu_1|\, p_r = \xi) \int_{\xi-0.00005}^{\xi+0.00005} f(t)dt,$$

where $f(t)$ is the pdf of the $Beta$ (550.5,500.5) distribution. This yields $\mu_1^* = 352.359$ (the ARL_0). If on the other hand, we set $l = 0.00005$, that is, if we use a smaller partition of the interval, we get a slightly better approximation $\mu_1^* = 352.3584$. However, these two results are pretty close for all practical proposes.

Therefore, we use l = 0.0001 and calculate ARL_0 for m = 1000, n = 5, for H = 15.5, 16, 16.5 and 17, respectively, m = 1000, n = 5, H = 15 employing the above technique. The results obtained are 388.7368, 429.1888, 474.3201 and 524.8474, respectively. Note that these findings are very similar to the results in Table 1.A obtained via Monte-Carlo simulation, in course of finding H under a nominal ARL_0 of 370 and 500, for m = 1000 and n = 5. Moreover, matrix inversion is often troublesome when ξ is close to 0 or 1 and hence this process is not very efficient if m is small. Thus in this paper we use Monte-Carlo simulations to evaluate the run-length distribution instead of the above method, which requires extreme care and large m, to work efficiently. The free software R.2.11.0 is used and the results are verified using SAS® v 9.1.3. Although the values of the run-length percentiles are found to be very stable under the two methods (the reader is referred to the tables in Section 5 for the values of the percentiles under discussion), slight sampling fluctuations were observed. We used 100,000 Monte-Carlo simulations to achieve reasonably small standard error of the estimated values.

5. PERFORMANCE OF THE CHART

Performance of a control chart is examined based on the run-length distribution and some associated characteristics such as the first two moments and some percentiles.

IC and OOC Performance

In this section, we examine the performance of the proposed exceedance CUSUM median chart in both the IC and the OOC cases. We also present a comparison between the proposed chart with some of its competitors based on the *ARL, SDRL,* the 5th, the 25th, the median, the 75th and the 95th percentiles. The competitors include the parametric CUSUM $\overline{X}$ chart and the rank-sum CUSUM chart considered in Li et al. (2010). The IC performance of a chart is typically used to assess its robustness (as it relates to the *FAR*) to different distributional assumptions whereas the OOC performance of the chart is examined to assess its efficacy in detecting a shift in the underlying process. Along with normal distribution, our study includes a collection of non-normal distributions and considers heavy-tailed, symmetric and asymmetric distributions. Specifically, the distributions considered in the study are: (a) the standard normal distribution, $N(0,1)$, (b) the Student's t-distribution, $t(v)$, with degrees of freedom (d.f.) $v = 3$, which is symmetric about 0 but with heavier tails than the $N(0,1)$, (c) the gamma distribution, $GAM(\alpha,\beta)$, with parameters $(\alpha, \beta) = (3,1)$ which is right-skewed but bell-shaped, (d) the exponential distribution with mean 1, which is $GAM(1,1)$ and (e) the Laplace (or double exponential $DE(0,1)$) distribution with mean 0 and variance 2 which is also symmetric but highly leptokurtic and has heavier tails.

IC Robustness

Because the exceedance CUSUM chart is nonparametric, the IC run-length distribution and the associated characteristics should remain the same for all continuous distributions. However, this is not true for the parametric charts. The values of the ARL_0, the $SDRL_0$ and the IC 5th, 25th, 50th, 75th and 95th percentiles (in this order) for the proposed chart are shown in Tables 1.A and 1.B for m = 1000 and 500, respectively, for different values of n (n = 5, 11 and 25).

<Tables 1.A and 1.B>

Implementation

Tables 1A and 1B provide useful information for implementing the exceedance CUSUM median chart. First, it is seen that the IC run-length characteristics of the proposed chart are approximately the same for all continuous distributions for fixed m and n which confirms its nonparametric characteristics. Secondly, the proposed chart can attain the industry standard ARL_0 values of 370 and 500 almost exactly. Note however that the ARL_0 values in Tables 1.A and 1.B were obtained for relatively large reference samples m (= 1000 and 500, respectively) and using k = 0. While these m values seem large, note that m = 500 means 100 samples each of size 5 from Phase I, which is reasonable. Several authors have discussed and made recommendations about the size of the reference sample and the consensus seems to be around 300 to 500. For smaller reference sample sizes, the calculations are rather difficult and need special care. A modified approach based on winsorization is discussed in Appendix C for this case. For small fixed m and n, as H increases, the Monte-Carlo simulations start producing some extreme runs and, consequently, estimating ARL_0 values accurately becomes difficult after a certain stage. Thus while for larger values of m one can easily reach a nominal ARL_0 of 500, it is not so easy for small m, especially when $k = 0$ (so that $d = d^*$) and in such cases, higher values of d^* may be preferable. However, when d^* is taken to be marginally higher, say 0.51 (in place of 0.50), much higher ARL_0 values may not be attainable. Nevertheless, if d^* is further increased to 0.52 or 0.53, the ARL_0 increases.

Keeping in mind the needs in practice, in Table 2, we compute some ARL_0 values for different values of d^* when m is not too large. Note that specifying d^* is equivalent to specifying k as they are linearly related through the relationship $k = n(d^* - d)$. We consider m to be small to moderate and an odd number, equal to 49, 99 and 149, respectively, so that the reference sample median is a unique order statistic. The subgroup size n is taken to be 5 and 11, respectively. Table 2 shows the ARL_0 values for $d^* = 0.50$, 0.51, 0.52 and 0.53 and for various values of H. This is helpful to implement the exceedance CUSUM median chart for small to moderate reference sample sizes. For example, when m = 49, n = 5, $d = 0.5$ and $d^* = 0.53$, the upper one-sided exceedance CUSUM median plotting statistic is given by

$$C_j = \max[0, C_{j-1} + (U_{j,50} - 2.65)], \text{ for } \; j = 1,2,3, \ldots; \; C_0 = 0,$$

Note that, $k = 5(0.53 - 0.5) = 0.15$, so this is just an equivalent form given by (1) as $nd = 2.5$. Now from Table 2, using H = 4.5, the ARL_0 is found to be equal to 394.7. It is easy to see from Table 2 that higher ARL_0 values may be obtained by either increasing H to a desirable level or by increasing d^* to a certain extent.

< Table 2 >

OOC Performance

While the IC robustness is a key factor in a performance comparison, it is also important to examine the OOC performance for a more complete comparison. In five Tables 3.A to 3.E, we show the OOC characteristics of the run-length distribution for different values of n with m = 1000 and d = 0.5 for the normal, exponential, two-parameter gamma, t with d.f. = 3 and Laplace distributions, respectively.

< Tables 3.A, 3.B, 3.C, 3.D and 3.E >

< Figures 2.A to 2.E >

Note that the ARL_0 values of all the charts under comparison are fixed at or close to the industry standard values of 370 and 500, respectively. Denote, the IC mean and the standard deviation by μ_0 and σ_0 respectively, and the OOC mean by $\mu_0 + \delta$. The ARL_δ values are calculated for a range of positive shifts in the mean (δ) where $\delta = \gamma\sigma_0/\sqrt{n}$ to facilitate appropriate comparisons among different distributions and different test sample sizes (n), for $\gamma = 0.25(0.25)1$, 1.50, 2.0 and 3.0. For an efficient control chart the ARL_0 should be large and the ARL_δ should be small. From Figures 2.A to 2.E we see that the shift detection capability of the chart increases as n increases for all distributions under consideration, which is to be expected. From Tables 3.A to 3.E we see that the chart performance is very similar for the $DE(0,1)$ and $t(3)$ distributions where the chart has the best shift detection capability. Following this, the chart performs much better under the $EXP(1)$ distribution than the $GAM(3,1)$ distribution. This is expected as the $EXP(1)$ (which is also $GAM(1,1)$) is more highly skewed than the $GAM(3,1)$ distribution (the skewness of the $GAM(\alpha, \beta)$ distribution increases as α decreases). The proposed chart is not so efficient under the $N(0,1)$ distribution, but this is very common with exceedance statistics. Similar results for the above five distributions for m =100 with n = 5 and d = 0.5 as well as d > 0.5 are presented in Tables 4.A to 4.E along with other existing charts which we shall discuss in the next subsection.

Comparison with other charts

Next we study the performance of the exceedance CUSUM median chart relative to a number of existing CUSUM charts, both parametric and nonparametric. These charts include the parametric CUSUM $\overline{X}$ chart and the rank-sum CUSUM chart. These charts are candidates to monitor small shifts in the location. The comparisons are based on the ARL_δ with a given ARL_0. These are shown in Tables 4.A to 4.E and also graphically represented in Figures 3.A to 3.E and Figures 4.A to 4.E for m = 100, n = 5 and $k = 0$ and $k > 0$, for each distribution under consideration. Note that as above, (i) the ARL_0 values of all the charts under comparison are fixed at or close to 500 and (ii) the ARL_δ values are calculated for a range of positive values of the parameter $\gamma = 0.25(0.25)1$, 1.50, 2.0 and 3.0; the chart with the smaller ARL_δ value is preferred. For the parametric CUSUM $\overline{X}$ chart the standards (parameters) are estimated from a Phase I reference sample duly taking care of the issues related to estimation.

< Tables 4.A to 4.E >

< Figures 3.A to 3.E >

< Figures 4.A to 4.E >

From Figures 3.A and 4.A we see that for all k and when the underlying process distribution is $N(0,1)$, the CUSUM $\overline{X}$ chart outperforms the other charts, which is not surprising, since it is natural for parametric methods to outperform their nonparametric counterparts when all assumptions are satisfied. We also find that the proposed chart outperforms the rank-sum CUSUM chart for larger ($\gamma > 1.5$) magnitutes of shifts.

From Figures 3.B and 4.B we see that when the underlying process distribution is *EXP*(1), although the proposed chart and the CUSUM $\overline{X}$ chart have a very similar performance for $k = 0$, the proposed chart outperforms the CUSUM $\overline{X}$ chart for $k > 0$. In addition, for all k and for shifts of moderately larger ($\gamma > 0.75$) magnitudes, the proposed chart outperforms the rank-sum CUSUM chart.

From Figures 3.C and 4.C we see that for all k and when the underlying process distribution is GAM(3,1), the proposed chart and the CUSUM $\overline{X}$ chart have a very similar performance. In addition, for all k and for shifts of moderately larger ($\gamma > 0.75$) magnitudes, the proposed chart outperforms the rank-sum CUSUM chart.

From Figures 3.D and 4.D we see that when the underlying process distribution is *t*(3), the proposed chart outperforms the competing charts for shifts of moderate ($0.25 < \gamma < 2.00$) magnitudes for $k = 0$ and the superiority is even more visible for $k > 0$. Our proposed chart is the best for the *DE*(0,1) distribution. It outperforms the competing charts for all k and for shifts all magnitudes for the *DE*(0,1) distribution and this can be observed from Figures 3.E and 4.E.

In summary, it is seen that in comparison with the CUSUM $\overline{X}$ chart, the proposed exceedance CUSUM median chart is outperformed only when the underlying distribution is normal. In all other cases the performances of the two charts are either similar or the exceedance CUSUM median chart has superior performance. Finally, the proposed exceedance CUSUM median chart outperforms the rank-sum CUSUM chart in all instances.

6. EXAMPLE

We illustrate the exceedance CUSUM median chart using a well-known dataset from Montgomery (2001; Tables 5.1 and 5.2) on the inside diameters of piston rings manufactured by a forging process. The data given in Table 5.1 contains twenty-five retrospective or Phase I samples, each of size five, that were collected when the process was thought to be IC, i.e. m = 125. These data are considered to be the Phase I reference data for which a goodness of fit test for normality is not rejected. The reference sample has a median equal to 74.001, i.e. $X_{(r)} = 74.001$.

Table 5.2 of Montgomery (2001) contains fifteen prospective (Phase II) samples each of five observations (n = 5). For the exceedance CUSUM median chart, we use k = 0 (i.e. $d = d^*$) and set H = 7.5 for an $ARL_0 \approx$ 370. The values of the exceedance and the exceedance CUSUM statistics are shown for illustration in Table 5. For the parametric CUSUM $\overline{X}$ chart we also use $k = 0$ and set H = 18 for an $ARL_0 \approx$ 370. It should be noted that this value of H was found using a search algorithm. The CUSUM $\overline{X}$ and the exceedance CUSUM median charts are shown in panels (a) and (b), respectively, of Figure 5.

< Table 5 >

< Figure 5 >

From Figure 5 we can see that the performances of the two charts are very similar. The exceedance CUSUM median chart signals at sample 13, whereas the CUSUM $\overline{X}$ chart signals at the very next sample, 14. However, recall that the nonparametric exceedance CUSUM chart doesn't require normality or any distributional assumption other than continuity to guarantee the $ARL_0 \approx 370$ but the same couldn't be said about the CUSUM $\overline{X}$ chart unless the underlying distribution was normal or close to it. In fact, the actual ARL_0 of the CUSUM $\overline{X}$ chart is unknown and most likely not the nominal 370.

As we have mentioned before, in practice the normality assumption can be in doubt or can't be justified for lack of enough information or data and a nonparametric method may be more desirable.

7. SUMMARY AND CONCLUDING REMARKS

CUSUM charts are popular control charts used in practice; they take advantage of the sequential accumulation of data arising in a typical SQC/SPC environment and are known to be more efficient than Shewhart charts in detecting smaller and persistent shifts. However, the traditional (parametric) CUSUM charts can lack in-control robustness and as such the possibility of varying and unknown false alarm rates is a practical concern with their applications. Nonparametric CUSUM charts offer an attractive alternative as they combine the inherent advantages of nonparametric charts (in-control robustness; same, fixed, false alarm rate for all continuous distributions) with the better small shift detection capability of CUSUM-type charts. We propose a nonparametric Phase II CUSUM chart based on the well-known exceedance statistics for detecting an increasing shift in the location parameter of a continuous distribution and is referred as the exceedance CUSUM chart. A performance comparison of the proposed chart with existing parametric and nonparametric CUSUM charts show that the exceedance CUSUM chart performs better than its competitors in detecting shifts under various contexts. Moreover, in certain situations where the data become available in a natural time order, the exceedance charts can be advantageous as they can be applied early, leading to savings in time and resources. Thus, on the basis of practicality, minimal assumptions, robustness of the in-control run-length distribution and out-of-control performance, the exceedance CUSUM chart is a strong contender in practical SPC applications. In terms of further research, exceedance/precedence statistics can be considered in a EWMA framework. This is currently being investigated and will be reported in a separate paper.

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APPENDIX A. SOME MATHEMATICAL RESULTS.

Result A.1. Given (conditionally on) $X_{(r)}$, the $U_{j,r}$'s are independently binomially distributed with parameters (n, p_r) for any $j = 1,2,3, \dots$.

Proof. Since $U_{j,r}$ is the number of Y-observations in the j^{th} Phase II sample that exceeds $X_{(r)}$, given $X_{(r)}$, the random variable $U_{j,r}$ follows a binomial distribution with parameters (n, p_r) where $p_r = P[Y > X_{(r)} | X_{(r)}]$.

Result A.2. The (unconditional) IC distribution of $U_{j,r}$ is distribution-free and is given by the pmf

$$P(U_{j,r} = u) = \frac{\binom{u+m-r}{u}\binom{n-u+r-1}{n-u}}{\binom{m+n}{n}} \text{ with } u = 0,1,2, \dots, n.$$

Proof. Using Result A.1 and averaging over the distribution of $X_{(r)}$, we find the unconditional distribution of $U_{j,r}$. Thus

$$P(U_{j,r} = u) = E[P(U_{j,r} = u|X_{(r)}] = \binom{n}{u}\int p_r^u(1-p_r)^{n-u}\, dF(X_{(r)})$$
$$= \binom{n}{u}\int [1-G(t)]^u[G(t)]^{n-u}\, dF_{X_{(r)}}(t).$$

When the process is IC we have $F = G$ and the above integral can be shown to simplify, via a beta function, to the given result. Hence, the statistics $U_{j,r}$ are unconditionally distribution-free when the process is IC. The same result can be obtained by combinatorial arguments; details are skipped. Note that the pmf of $U_{j,r}$ shown above is known as the negative hypergeometric distribution.

Next we extend Result A.2 and show that the (unconditional) joint distribution of $U_{j,r}$ for $j = 1,2, \dots, v$ is distribution-free when the process is IC. This establishes that the proposed chart is distribution-free.

Result A.3. The unconditional IC distribution of $U_{j,r}$ for $j = 1,2, \dots, v$ where $v > 1$ is a positive integer, is distribution-free.

Proof. Noting that by Result A.1, given $X_{(r)}$ the $U_{j,r}, j = 1,2, \dots, v$ are independent binomial (n, p_r) variables, the joint distribution of $(U_{1,r}, U_{2,r}, \dots, U_{v,r})$, when the process is IC, is given by

$$P[U_{1,r} = u_1, U_{2,r} = u_2, \dots, U_{v,r} = u_v]$$
$$= E[P[U_{1,r} = u_1\, U_{2,r} = u_2, \dots, U_{v,r} = u_v] \,|\, X_{(r)}]$$

$$= \binom{n}{u_1}\binom{n}{u_2}\cdots\binom{n}{u_v}\int (1-p_r)^{nv}\left[\frac{p_r}{1-p_r}\right]^{\sum_{j=1}^{v} u_j} dF(X_{(r)})$$

$$= \binom{n}{u_1}\binom{n}{u_2}\cdots\binom{n}{u_v}\frac{B(\sum_{j=1}^{v} u_j+m-r+1,\ nv+r-\sum_{j=1}^{v} u_j)}{B(r,\ m-r+1)}.$$

The last result follows again by using the fact that $F = G$ when the process is IC and simplifying the integral via a beta function. Hence the unconditional IC joint distribution of exceedance statistics from any number of independent Phase II samples is distribution-free. This proves that the proposed chart is distribution-free.

Result A.4. The unconditional exceedance probability $P[Y > X_{(r)}]$ equals $(m - r + 1) / (m + 1)$ when the process is IC.

Proof. Note that $P[Y > X_{(r)}] = E[P(Y > X_{(r)}|X_{(r)})] = E(p_r)$. When the process is IC, $F = G$ and then $p_r = 1 - F(X_{(r)}) = 1 - V_r$ (say). Since $X_{(r)}$ is the r^{th} order statistic in a random sample of size *m* from F, using the probability integral transform, it can be shown (see, for example, Gibbons and Chakraborti, 2003) that V_r follows the distribution of the r^{th} order statistic in a random sample of size *m* from the uniform (0, 1) distribution as long as F is continuous. This latter distribution is known to be a beta distribution with parameters r and $m - r + 1$, respectively. Moreover, when the process is IC, p_r follows a beta distribution with parameters $m - r + 1$ and r. Thus $E(p_r) = P[Y > X_{(r)}] = 1 - \frac{r}{m+1} = \frac{m-r+1}{m+1}$, when the process is IC using the expectation formula for a beta distribution.

APPENDIX B. EFFECT OF THE REFERENCE SAMPLE (ESTIMATION OF PARAMETERS)

The effect of the reference sample data on the performance of the Phase II chart can be profound. To investigate this question for our nonparametric chart, let us consider the IC situation and suppose that the IC underlying true process distribution is standard normal and we obtain 100 observations from it, as done in a Phase I study. The characteristics of this IC sample can and will vary and that will impact the chart. For example, if the sample median turns out to be the same as true median (0) we have $\hat{p}_r = 1 - \Phi(0) = 1 - 0.5 = 0.5$. In that case the performance of the chart will be satisfactory. In practice, however, it is hard to realise such a perfect situation; the sample median may be less than 0, that is, may have a downward bias or it may be greater than 0 with an upward bias. Let us consider these two situations separately. The bias in the Phase I sample (reference sample) median will introduce bias in $\hat{p}_r$, which, in turn, will affect the IC performance of the control chart.

To examine the effect, suppose that the sample median has 1% downward bias and is found to be -0.01. Given this sample median, we have $\hat{p}_r = 1 - \Phi(-0.01) = 1 - 0.496 = 0.504$. Then given $\hat{p}_r = 0.504$, using the Markov chain approach as in Section 4, we find that for $H = 5.5$ and $d = 0.5$, the exact IC conditional average run-length is 505.72. Now, the approximate distribution of the sample median based on a sample of size $m = 100$ is normal with mean 0 and standard deviation $\frac{1}{\sqrt{8\pi m}} = 0.0199$. Thus there is a 30.8% chance that the downward bias may be more than 1% since $\Phi(-0.01/0.0199) = 0.308$. Next suppose that the sample median has a 5% downward bias and thus equal 0.05. Then $\hat{p}_r = 1 - \Phi(-0.05) = 1 - 0.480 = 0.520$ and given $\hat{p}_r = 0.520$ and using the Markov chain approach, we find that for $H = 5.5, d = 0.5$, the exact IC conditional *ARL* drops down to 228.27. Further, since $\Phi(-0.05/0.0199) = 0.0061$, there is a 0.6% chance that the downward bias in the median may be more than 5%. Thus the downward bias in the reference sample median does not seem to produce very low conditional IC *ARL*'s which provides fair protection against the possibility of an early false alarm.

On the other hand, suppose that the sample median has a 1% upward bias and thus equals 0.01. Then $\hat{p}_r = 1 - \Phi(+0.01) = 1 - 0.504 = 0.496$ and given $\hat{p}_r = 0.496$, using the Markov chain approach, we find that for $H = 5.5$ and $d = 0.5$, the exact IC conditional *ARL* is 941.04. Since, $1 - \Phi(+0.01/0.0199) = 0.308$, we have about 30.8% chance that the upward bias may be more than 1%. On the other hand if the sample median has a 5% upward bias, we have $\hat{p}_r = 1 - \Phi(+0.05) = 1 - 0.520 = 0.480$ and given $\hat{p}_r = 0.480$, using the Markov chain approach, we find that for $H = 5.5$ and $d = 0.5$, the exact IC conditional *ARL* jumps up to 6147.45. Thus the upward bias in the sample median can be of some concern as a much larger than nominal ARL_0 can unnecessarily defer the detection of a shift even if it has occurred. If there is an upward bias of over 2.5%, the IC conditional *ARL* exceeds 1712.15, more than three times the nominal ARL_0. This will produce nearly 1-2.5% of the extreme values in the unconditional run-length distribution and will make estimation of the unconditional ARL_0 very unreliable and most likely result in a large IC *SDRL*.

As the reference sample size *m* increases however, the chances of a large bias in the Phase I estimate gradually decreases and the picture improves. With m = 100, n = 5, from the above calculations we see in only 68.6% (1-30.8-0.6)% cases the conditional ARL_0 lies within the interval (228, 942). The same interval contains 71%, 72.9%, 74.6%, 76.1%, 78.6%,86.9% and 94.4% of the conditional ARL_0 values, respectively, for m =125, 150, 175, 200, 250, 500 and 1000 with n = 5. For academic interest, we can study with fixed *H* and *d* and see how the interval (228, 942) works for n = 3 and 7 for various values of *m*. With n = 3, the interval can handle (that is, if the upward bias is up to 3.6% conditional *ARL* will be within (228, 942). If upward bias is more the conditional ARL_0will be more than 942) up to 3.6% upward bias compared to only 1% with n = 5. On the other hand it can handle (that is if the downward bias is up to 3.8% conditional *ARL* will be within (228, 942). If downward bias is more the conditional ARL_0will be less than 228) up to 3.8% downward bias (compared to 5% with n = 5). With n = 3, the same interval contains 93.6%, 96.2%, 97.7%, 98.6%, 99.1%, and 99.6% conditional ARL_0, respectively, for m = 100,125, 150, 175, 200 and 250 and 500; it contains almost 100% of the conditional ARL_0 if *m* is 500 or more. On the other hand, when n = 7, it can address up to 5.4% downward bias (compared to 5% for n = 5) but unfortunately, the interval can tolerate only about 0.01% upward bias. Intuitively, if one already has an upward bias in the Phase I sample, one cannot expect a quick detection of an upward sift or a quick boundary hitting of the Markov chain. It appears that a smaller test sample size *n* is preferable when *m* is not very high, as it gives more a stable run-length distribution.

Thus, based on our observations, when m = 100, *n* should be 3 or 5, if m = 1000, *n* should be less than or equal 11 and if n = 25 has to adopt then *m* should be even higher.

It may be noted that the effect of the reference sample is important to understand not only for our proposed chart but in all Phase II charts, including the normal theory CUSUM charts when parameters are estimated in Phase I. For some related details, see Hawkins and Olwell (1998). In the present scenario, we see that if *m* is relatively small, a choice of smaller *n* is better in the sense that it produces less extreme runs given a Phase I sample. Larger values of *n* tend to produce more and more extreme runs for a given *H*. Now if *H* is reduced, high extreme values may be avoided but instead some low values of conditional run-length (lower tail extremes) will arise. One possibility to avoid the hazard caused from extreme runs is to use, instead of the ARL_0, the IC median run length (*MDRL*) as suggested by Gan (1993) or some percentile of the run-length distribution as advocated in Chakraborti and Van de Wiel (2008). Nevertheless, in the industrial set up, the *ARL* is still more commonly used. Therefore, we propose a systematic approach of using Monte-Carlo simulation along with windsorization to estimate the *ARL*. More details are presented in Appendix C.

APPENDIX C. MONTE-CARLO SIMULATION UNDER SMALL SAMPLES: WINSORIZATION

When m < 200, we recommend to set a-priori the maximum allowable length of monitoring at a certain high level, say, S. This will eliminate the possibilities of high extreme runs by induced termination at *S*, which may be 10 to 15 times the nominal ARL_0.Therefore, in course of estimation of the run-length via Monte-Carlo studies; a particular replicate will at most inspect S test samples (each of size *n*). In other words, if we don’t observe a value of the run-length variable (that is the chart does not signal) less than or equal to S, we shall enforce termination of the monitoring process (simulation of data) and set the run-length value equal to S. As a result, we obtain a winsorized *ARL* with winsorization at the upper tail of the run-length distribution. While Table 4.A-4.E are based on a Monte-Carlo with termination enforced after S = 5,000 simulations, in Tables 6.A and 6.B, we present a case study when a termination is enforced after S = 2,000 simulations. In all those tables, we record the percentage of simulation replicates that naturally terminate before S and refer it as Winsorization level (WL). Tables 6.A and 6.B shows the effect of choice of lower value of S. For brevity, only the normal distribution and target ARL_0 =500 (for Table 6.A) and Target ARL_0 =370 (for Table 6.B) are considered. We see from Table 6.A and 6.B that the control limits are naturally overestimated and as a consequence the OOC run-lengths also increase a little when the shift is small. The following points are essential to note while working with winsorisation.

A. The winsorized ARL with winsorization at the upper tail of the run-length distribution slightly underestimates the true ARL.
B. Winsorization at upper tail of the run-length distribution stabilizes the variance and, consequently, increases the efficiency of the estimate of the ARL_0.
C. H* determined on the basis of winsorized ARL_0 overestimates true H.
D. If the calculated ARL_δ based on H* with ARL_0 = A of a chart is found to be lower than the ARL_δ of any other charts with ARL_0 = A, it guarantees that the former chart is better provided the probability that a conditional ARL_δ exceeds the winsorization point, is practically nil.

<Tables 6.A and 6.B>

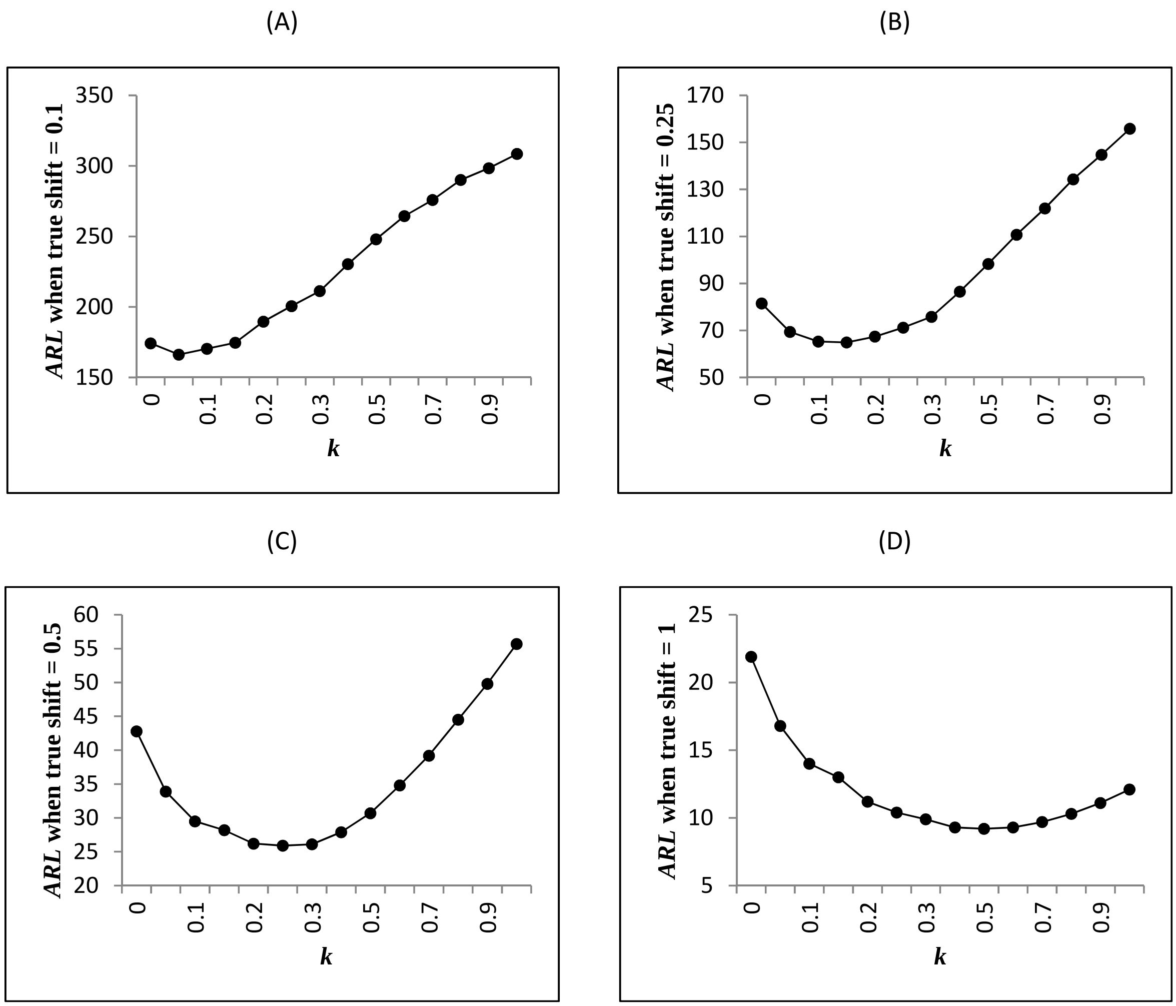


Figure 1. ARL_δ values of the exceedance CUSUM median chart with t = 500 for different values of k and μ_1 = 0.1, 0.25, 0.5 and 1.0.

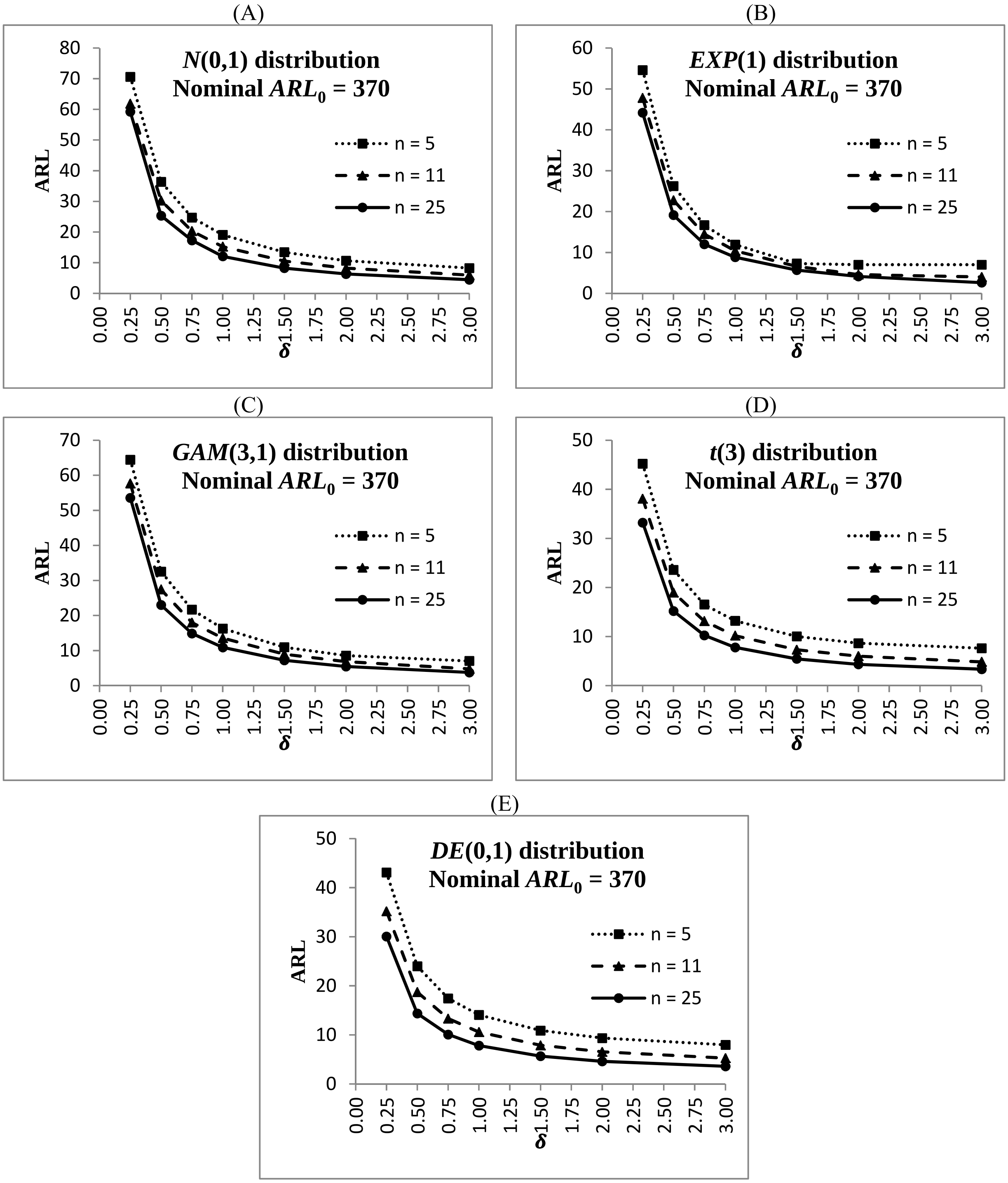


Figure 2. OOC performance comparison of the exceedance CUSUM median chart for different values of *n* and various distributions for $m = 1000$, $d = 0.5$ or $k = 0$.

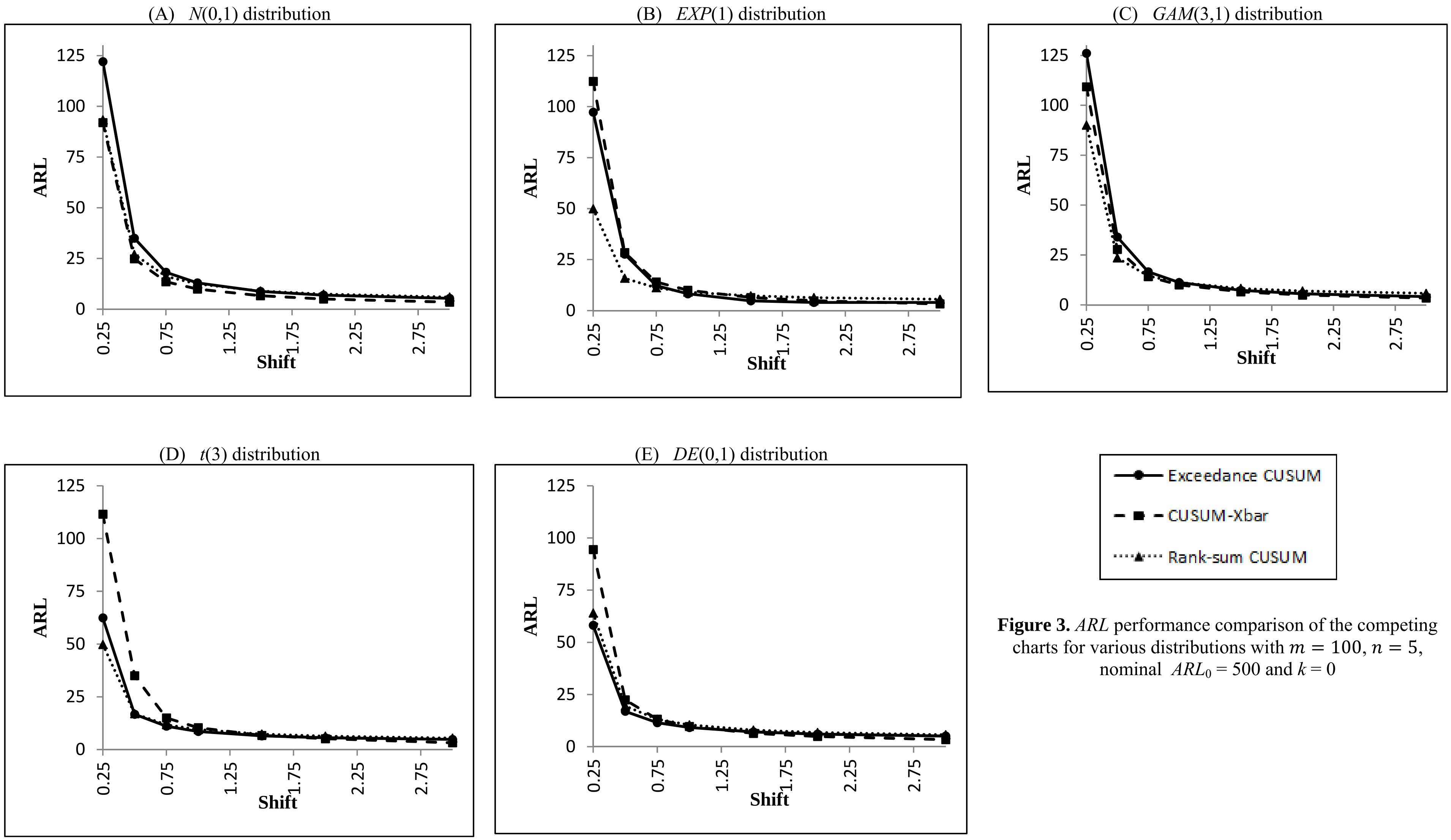
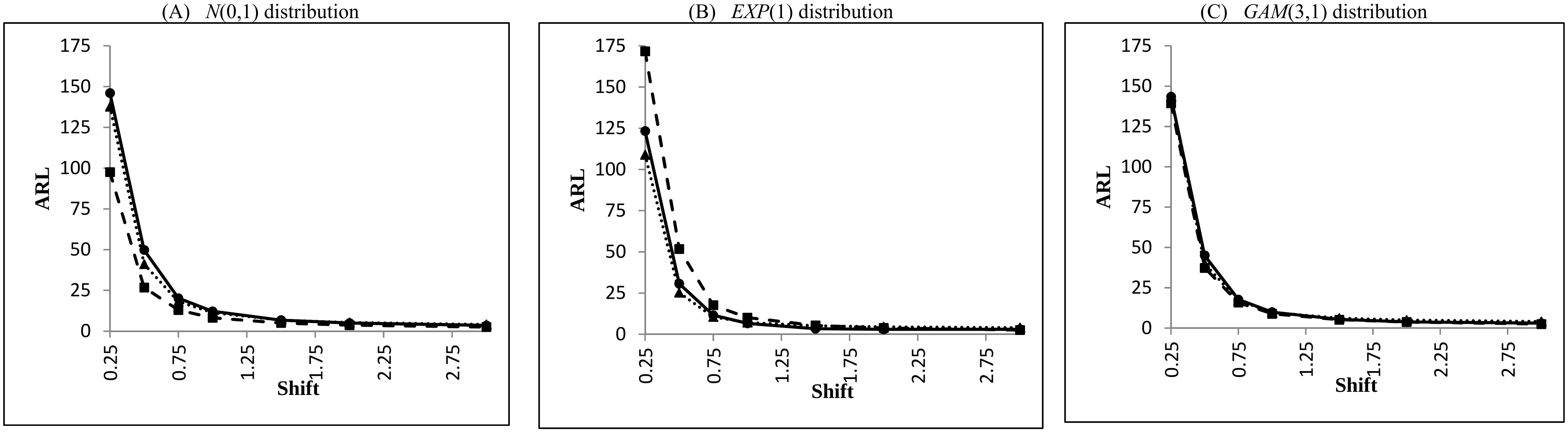
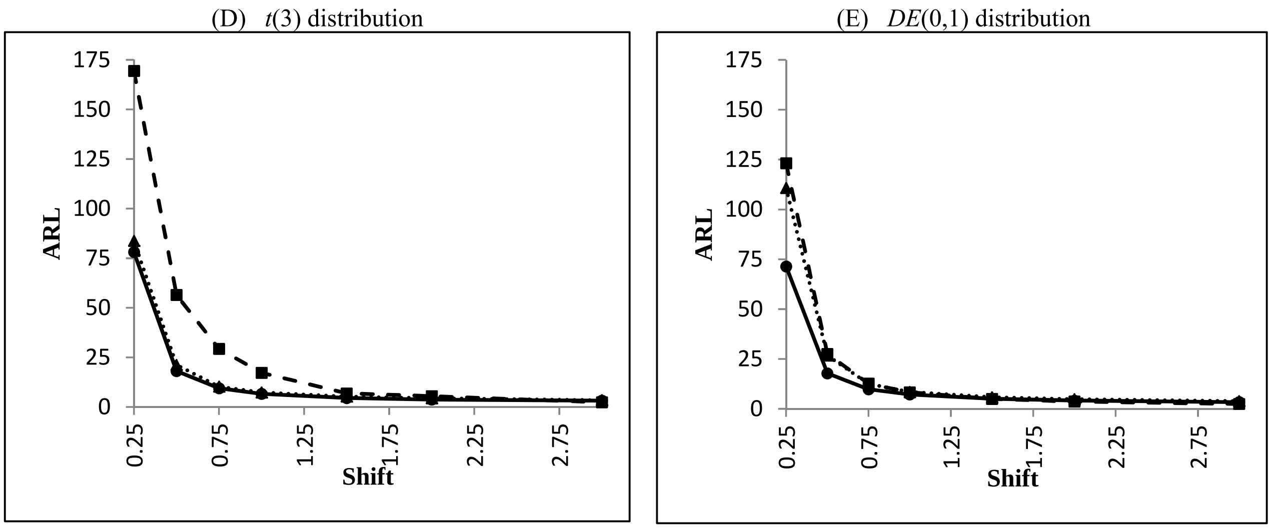


Figure 3. *ARL* performance comparison of the competing charts for various distributions with $m = 100$, $n = 5$, nominal $ARL_0 = 500$ and $k = 0$





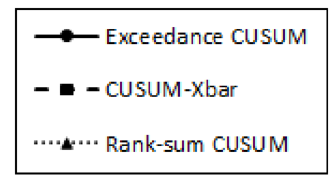


Figure 4. *ARL* performance comparison of the competing charts for various distributions with $m = 100$, $n = 5$, nominal $ARL_0 = 500$ and $k > 0$.

Table 1A.

The IC characteristics of the run-length distribution of the exceedance CUSUM median chart for different n with m =1000, d= 0.5 (or k = 0)

	Nominal ARL_0 = 370							Nominal ARL_0 = 500						
Distribution Type	**Mean**	**Standard deviation**	**5%**	**25%**	**Median**	**75%**	**95%**	**Mean**	**Standard deviation**	**5%**	**25%**	**Median**	**75%**	**95%**
	***n* = 5, *H* = 15.5**							***n* = 5, *H* = 16.5**						
***N*(0,1)**	394.68	911.58	42	91	173	366	1361	487.10	1548.10	46	102	198	439	1690
***EXP*(1)**	384.42	972.02	42	89	173	372	1291	474.69	1668.50	46	99	195	426	1585
***GAM*(3,1)**	383.33	1048.65	42	89	172	370	1273	470.56	1395.70	46	99	194	427	1571
***t*(3)**	389.89	1084.26	42	89	174	373	1289	470.19	1277.29	46	99	195	428	1586
***DE*(0,1)**	385.70	918.64	42	90	173	371	1300	475.51	1628.32	47	99	194	422	1590
	***n* = 11, *H* = 18.5**							***n* = 11, *H* = 20.0**						
***N*(0,1)**	371.10	1810.85	27	58	116	272	1204	516.43	3079.99	30	65	133	326	1631
***EXP*(1)**	355.46	1428.46	26	56	114	260	1174	505.77	3121.91	30	66	135	331	1591
***GAM*(3,1)**	381.93	2315.71	27	58	117	273	1203	552.18	5437.10	30	66	133	328	1613
***t*(3)**	372.83	2008.43	27	58	116	273	1202	536.60	4091.31	31	66	134	325	1602
***DE*(0,1)**	368.17	2172.19	27	58	116	272	1190	528.42	4412.10	31	66	133	322	1611
	***n* = 25, *H* = 21.5**							***n* = 25, *H* = 22.5**						
***N*(0,1)**	369.96	2451.44	16	34	73	190	1104	486.27	11749.40	17	36	75	203	1409
***EXP*(1)**	370.80	2219.84	16	36	76	192	1211	493.66	7470.19	17	38	79	218	1504
***GAM*(3,1)**	416.70	4703.45	16	35	72	188	1136	555.21	10451.15	17	37	78	213	1405
***t*(3)**	442.81	6440.04	16	35	72	188	1153	576.02	9028.16	17	37	79	212	1416
***DE*(0,1)**	454.72	6960.93	16	35	72	190	1156	539.45	5942.58	17	37	79	214	1428

Table 1B.

The IC characteristics of the run-length distribution of the exceedance CUSUM median chart for different *n* with *m* =1000, *d*= 0.5 (or *k* = 0)

	Nominal ARL_0 = 370							Nominal ARL_0 = 500						
Distribution Type	**Mean**	**Standard deviation**	**5%**	**25%**	**Median**	**75%**	**95%**	**Mean**	**Standard deviation**	**5%**	**25%**	**Median**	**75%**	**95%**
	***n* = 5, *H* = 13.0**							***n* = 5, *H* = 13.5**						
***N*(0,1)**	394.06	2279.96	29	63	126	294	1266	480.12	3542.63	31	67	134	321	1494
***EXP*(1)**	406.99	2707.59	29	63	125	293	1278	467.90	2669.46	31	67	135	319	1483
***GAM*(3,1)**	392.48	1828.10	29	63	126	295	1295	453.43	2129.83	31	67	136	324	1489
***t*(3)**	409.22	3368.77	29	63	126	294	1290	488.18	5288.29	31	67	134	320	1496
***DE*(0,1)**	404.24	3566.87	29	62	124	291	1280	486.71	3590.93	31	67	135	322	1501
	***n* = 11, *H* = 14.5**							***n* = 11, *H* = 15.5**						
***N*(0,1)**	361.20	3476.82	17	37	76	191	1050	544.91	8365.88	19	40	85	223	1387
***EXP*(1)**	365.10	3824.06	17	36	75	189	1047	538.40	6505.87	19	40	84	222	1407
***GAM*(3,1)**	370.10	5205.87	17	36	75	190	1060	520.38	8939.23	19	40	84	222	1411
***t*(3)**	370.71	4236.95	17	37	75	191	1071	558.72	6986.16	19	40	85	223	1403
***DE*(0,1)**	347.59	2205.70	17	36	75	192	1081	533.52	8816.49	19	41	85	226	1378
	***n* = 25, *H* = 15.5**							***n* = 25, *H* = 16.5**						
***N*(0,1)**	335.38	4094.52	9	19	41	115	815	555.66	11960.87	10	21	46	138	1059
***EXP*(1)**	319.67	4547.30	9	19	41	114	801	555.31	13151.20	10	21	46	134	1066
***GAM*(3,1)**	384.44	10013.27	9	20	42	114	799	647.13	32693.61	10	21	46	131	1053
***t*(3)**	350.98	4977.05	9	20	42	114	827	590.27	21776.30	10	21	45	130	1047
***DE*(0,1)**	341.56	4254.38	9	20	41	115	809	555.81	15907.33	10	21	46	133	1068

Table 2. Some ARL_0 values of the exceedance CUSUM median chart for given m, n, H and d^*

m	H	$d^* = 0.50$		$d^* = 0.51$		$d^* = 0.52$		$d^* = 0.53$	
		$n = 5$	$n = 11$	$n = 5$	$n = 11$	$n = 5$	$n = 11$	$n = 5$	$n = 11$
49	3	45.3	26.8	39.4	30.5	44.1	36.4	54.2	38.5
	3.5	55.1	50.5	63.2	52.1	74.3	54.3	89.9	94.3
	4	112.2	116.2	118.9	137.6	119.1	133.1	145.6	194.8
	4.5	204.0	175.7	184.3	232.9	279.9	238.9	394.7	418.1
	5	328.4	265.0	324.5	342.2	452.7	>500	>500	>>500
99	3	22.5	15.3	23.8	15.7	26.8	18.8	27.8	21.2
	3.5	31.0	21.2	31.2	22.5	36.8	24.7	48.9	33.3
	4	43.7	31.5	45.0	30.3	55.1	39.3	63.6	54.9
	4.5	59.6	41.1	64.7	47.2	77.1	56.5	111.1	75.3
	5	81.5	57.3	91.5	68.5	122.7	100.5	163.0	145.5
	5.5	113.5	85.8	120.6	89.8	168.5	136.6	312.6	204.2
	6	164.65	125.6	188.6	168.2	292.9	234.7	374.2	346.5
149	3	20.4	12.9	20.5	13.3	23.1	15.1	25.2	17.6
	3.5	25.8	16.4	27.2	17.2	31.0	20.1	39.6	26.1
	4	34.3	21.6	35.8	24.0	41.2	29.4	50.1	36.3
	4.5	44.2	26.5	44.9	29.7	59.3	36.1	78.9	48.0
	5	54.3	34.0	65.1	43.0	79.0	60.3	105.0	84.9
	5.5	71.0	45.1	81.9	55.5	120.7	73.8	165.4	103.1
	6	96.7	64.5	101.9	74.6	144.5	108.9	250.4	162.1
	6.5	127.8	77.1	149.9	103.1	208.5	178.4	328.5	212.9
	7	160.3	112.6	195.1	170.2	291.6	229.1	446.1	379.2

Table 3.A. The OOC characteristics of the run-length distribution for different n with $m = 1000$ and $d = 0.5$ (or $k = 0$) for the $N(0,1)$* distribution

	Nominal $ARL_0 = 370$							Nominal $ARL_0 = 500$						
γ	ARL_δ	Standard deviation	5%	25%	Median	75%	95%	ARL_δ	Standard deviation	5%	25%	Median	75%	95%
	$n = 5$, $H = 15.5$							$n = 5$, $H = 16.5$						
0.25	70.60	53.31	24	39	56	84	164	74.57	55.70	26	42	60	89	173
0.50	36.38	16.08	18	25	33	44	67	38.52	16.78	19	27	35	46	70
0.75	24.72	8.53	14	19	23	29	41	26.35	8.82	15	20	25	31	43
1.00	19.07	5.51	12	15	18	22	29	20.72	5.60	13	16	19	23	30
1.50	13.40	2.94	10	11	13	15	19	14.13	2.96	10	12	16	18	20
2.00	10.64	1.84	8	9	10	12	14	11.30	1.92	8	10	11	12	15
3.00	8.24	0.90	7	8	8	9	10	8.71	0.95	8	8	8	9	10
	$n = 11$, $H = 18.5$							$n = 11$, $H = 20.0$						
γ														
0.25	61.83	79.05	17	29	43	69	158	67.94	80.43	19	31	47	75	181
0.50	30.25	17.84	13	19	26	36	62	32.41	19.15	14	21	28	38	65
0.75	20.30	8.48	10	14	18	24	36	21.59	8.56	11	16	20	26	38
1.00	15.22	5.13	9	12	14	18	25	16.53	5.47	10	13	15	19	27
1.50	10.54	2.74	7	9	10	12	16	11.33	2.84	7	9	11	13	17
2.00	8.20	1.76	6	7	8	9	12	8.82	1.84	6	7	9	10	12
3.00	5.95	0.94	5	5	6	6	8	6.37	1.00	5	6	6	7	8
	$n = 25$, $H = 21.5$							$n = 25$, $H = 22.5$						
γ														
0.25	59.22	120.60	11	20	32	57	174	61.74	123.92	12	21	33	58	183
0.50	25.32	25.05	9	14	19	29	59	26.58	26.16	9	14	20	30	62
0.75	17.27	9.24	7	10	14	19	32	16.83	9.76	8	11	15	20	33
1.00	12.08	5.16	6	9	11	14	22	12.67	5.38	6	9	12	15	23
1.50	8.23	2.60	5	6	8	10	13	8.56	2.66	5	7	8	10	14
2.00	6.31	1.65	4	5	6	7	9	6.55	1.66	4	5	6	8	10
3.00	4.44	0.89	3	4	4	5	6	4.60	0.90	3	4	4	5	6

* IC set up: mean $\mu_0 = 0$ and standard deviation $\sigma_0 = 1$; OOC set up: mean $= \mu_0 + \frac{\gamma\sigma_0}{\sqrt{n}}$ and standard deviation $= 1$

Table 3.B. The OOC characteristics of the run-length distribution for different n with $m = 1000$ and $d = 0.5$ (or $k = 0$) for the $EXP(1)$* distribution

	Nominal ARL_0 = 370							Nominal ARL_0 = 500						
γ	ARL_δ	Standard deviation	5%	25%	Median	75%	95%	ARL_δ	Standard deviation	5%	25%	Median	75%	95%
	$n = 5$, $H = 15.5$							$n = 5$, $H = 16.5$						
0.25	54.58	36.53	21	33	45	65	118	58.15	38.03	23	35	49	70	124
0.50	26.24	9.98	14	20	24	31	45	27.60	10.23	15	20	26	32	46
0.75	16.67	4.54	11	14	16	19	25	17.70	4.72	12	14	17	20	26
1.00	11.92	2.53	8	10	12	13	16	12.60	2.62	9	11	12	14	18
1.50	7.30	0.56	7	7	7	8	8	7.56	0.70	7	7	7	8	9
2.00	7	0	7	7	7	7	7	7	0	7	7	7	7	7
3.00	7	0	7	7	7	7	7	7	0	7	7	7	7	7
	$n = 11$, $H = 18.5$							$n = 11$, $H = 20.0$						
γ														
0.25	47.78	47.76	16	25	36	55	114	52.88	56.47	17	27	39	60	129
0.50	22.75	11.01	11	16	20	27	43	24.17	11.35	12	17	22	29	45
0.75	14.49	5.04	8	11	14	17	24	15.59	5.29	9	12	15	18	25
1.00	10.41	2.88	7	8	10	12	16	11.29	3.07	7	9	11	13	17
1.50	6.60	1.27	5	6	6	7	9	7.08	1.34	5	6	7	8	9
2.00	4.59	0.66	4	4	4	5	6	5.01	0.60	4	5	5	5	6
3.00	4	0	4	4	4	4	4	4	0	4	4	4	4	4
	$n = 25$, $H = 21.5$							$n = 25$, $H = 22.5$						
γ														
0.25	44.17	85.33	10	17	26	44	122	47.50	96.67	11	18	28	46	126
0.50	19.13	20.13	8	11	16	22	40	19.92	16.39	8	12	16	23	42
0.75	12.01	5.43	6	8	11	14	22	12.61	5.85	6	9	11	15	23
1.00	8.85	3.30	5	7	8	10	15	9.18	3.28	5	7	8	11	15
1.50	5.69	1.48	4	5	6	6	8	5.91	1.52	4	5	6	7	9
2.00	4.14	0.83	3	4	4	5	6	4.30	0.86	3	4	4	5	6
3.00	2.62	0.50	2	2	3	3	3	2.79	0.44	2	3	3	3	3

* IC set up: mean $\mu_0 = 1$ and standard deviation $\sigma_0 = 1$; OOC set up: mean $= \mu_0 + \frac{\gamma\sigma_0}{\sqrt{n}}$ and standard deviation $= 1$

Table 3.C. The OOC characteristics of the run-length distribution for different n with $m = 1000$ and $d = 0.5$ (or $k = 0$) for the $GAM(3,1)$* distribution

	Nominal ARL_0 = 370							**Nominal ARL_0 = 500**						
γ	ARL_δ	Standard deviation	5%	25%	Median	75%	95%	ARL_δ	Standard deviation	5%	25%	Median	75%	95%
	$n = 5$, $H = 15.5$							$n = 5$, $H = 16.5$						
0.25	64.43	46.09	24	37	52	77	147	70.35	51.39	25	40	57	84	159
0.50	32.53	13.68	16	23	30	39	59	34.01	14.21	18	24	32	41	61
0.75	21.68	6.96	13	17	20	25	34	22.98	7.20	14	18	22	27	36
1.00	16.28	4.34	10	13	16	18	24	17.20	4.35	12	14	16	20	25
1.50	10.96	2.01	8	10	11	12	14	11.62	2.09	9	10	11	13	16
2.00	8.56	1.07	7	8	8	9	10	9.04	1.12	8	8	9	10	11
3.00	7.02	0.14	7	7	7	7	7	7.19	0.39	7	7	7	7	8
	$n = 11$, $H = 18.5$							$n = 11$, $H = 20.0$						
γ														
0.25	57.66	66.31	17	28	41	65	142	63.10	75.50	19	30	45	71	159
0.50	27.42	15.10	12	18	24	33	55	29.74	16.19	13	19	26	35	69
0.75	17.98	7.04	10	13	16	21	31	19.43	7.58	11	14	18	23	33
1.00	13.56	4.35	8	10	13	16	22	14.51	4.53	9	11	14	17	23
1.50	8.98	2.15	6	8	9	10	13	9.68	2.22	7	8	9	11	14
2.00	6.83	1.28	5	6	7	8	9	7.30	1.31	5	6	7	8	10
3.00	4.77	0.64	4	4	5	5	6	5.16	0.55	4	5	5	5	6
	$n = 25$, $H = 21.5$							$n = 25$, $H = 22.5$						
γ														
0.25	53.57	107.16	11	19	30	52	154	58.41	110.50	12	20	32	56	177
0.50	23.00	19.17	8	13	18	26	53	24.25	23.68	9	14	19	28	54
0.75	14.87	8.03	7	10	13	18	29	15.43	8.20	7	10	14	18	30
1.00	10.90	4.45	6	8	10	13	19	11.45	4.80	6	8	10	14	20
1.50	7.22	2.14	4	6	7	8	11	7.54	2.23	5	6	7	9	12
2.00	5.44	1.32	4	4	5	6	8	5.69	1.34	4	5	6	6	8
3.00	3.75	0.66	3	3	4	4	5	3.88	0.67	3	3	4	4	5

* IC set up: mean $\mu_0 = 3$ and standard deviation $\sigma_0 = \sqrt{3}$; OOC set up: mean $= \mu_0 + \frac{\gamma\sigma_0}{\sqrt{n}}$ and standard deviation $= \sqrt{3}$

Table 3.D. The OOC characteristics of the run-length distribution for different n with $m = 1000$ and $d = 0.5$ (or $k = 0$) for the $t(3)$* distribution

	Nominal ARL_0 = 370							Nominal ARL_0 = 500						
γ	ARL_δ	Standard deviation	5%	25%	Median	75%	95%	ARL_δ	Standard deviation	5%	25%	Median	75%	95%
	$n = 5$, $H = 15.5$							$n = 5$, $H = 16.5$						
0.25	45.20	23.68	20	30	40	54	89	47.84	24.83	21	31	42	57	94
0.50	23.59	7.81	14	18	22	28	38	25.13	8.23	14	19	24	29	40
0.75	16.52	4.24	11	14	16	19	24	17.64	4.42	12	14	17	20	26
1.00	13.20	2.75	9	11	13	15	18	13.96	2.88	10	12	14	16	19
1.50	10.02	1.58	8	9	10	11	13	10.58	1.61	8	10	10	12	14
2.00	8.64	1.06	7	8	8	9	10	9.14	1.12	8	8	9	10	11
3.00	7.61	0.62	7	7	8	8	9	8.02	0.62	7	8	8	8	9
	$n = 11$, $H = 18.5$							$n = 11$, $H = 20.0$						
γ														
0.25	38.08	29.33	14	22	31	45	84	41.26	29.24	16	24	34	49	90
0.50	18.97	7.55	10	14	18	22	33	20.52	7.94	11	15	19	24	35
0.75	13.13	3.97	8	10	12	15	20	14.16	4.21	9	11	13	16	22
1.00	10.17	2.56	7	8	10	12	15	10.93	2.70	7	9	11	12	16
1.50	7.31	1.40	5	6	7	8	10	7.84	1.46	6	7	8	9	11
2.00	6.00	0.93	5	5	6	6	8	6.46	1.02	5	6	6	7	8
3.00	4.84	0.64	4	4	5	5	6	5.22	0.54	5	5	5	5	6
	$n = 25$, $H = 21.5$							$n = 25$, $H = 22.5$						
γ														
0.25	33.20	47.53	10	16	23	36	81	35.23	48.64	10	16	24	38	90
0.50	15.19	7.99	7	10	13	18	29	16.04	8.44	8	11	14	19	31
0.75	10.24	3.85	6	8	9	12	18	10.75	4.01	6	8	10	13	18
1.00	7.79	2.35	5	6	7	9	12	8.15	2.43	5	6	8	9	13
1.50	5.45	1.28	4	5	5	6	8	5.70	1.33	4	5	6	6	8
2.00	4.34	0.84	3	4	4	5	6	4.53	0.86	3	4	4	5	6
3.00	3.34	0.51	3	3	3	4	4	3.45	0.54	3	3	3	4	4

* IC set up: mean $\mu_0 = 0$ and standard deviation $\sigma_0 = \sqrt{3}$; OOC set up: mean $= \mu_0 + \frac{\gamma\sigma_0}{\sqrt{n}}$ and standard deviation $= \sqrt{3}$

Table 3.E. The OOC characteristics of the run-length distribution for different n with m = 1000 and d = 0.5 (or $k = 0$) for the DE(0,1)* distribution

	Nominal ARL_0 = 370							Nominal ARL_0 = 500						
γ	ARL_δ	Standard deviation	5%	25%	Median	75%	95%	ARL_δ	Standard deviation	5%	25%	Median	75%	95%
	$n = 5$, $H = 15.5$							$n = 5$, $H = 16.5$						
0.25	43.10	21.54	19	28	38	52	83	46.10	21.9	21	31	41	55	86
0.50	23.99	7.89	14	18	23	28	39	25.50	8.27	15	20	24	30	41
0.75	17.45	4.56	11	14	17	20	26	18.45	4.77	12	15	18	21	27
1.00	14.07	3.11	10	12	14	16	20	15.05	3.26	10	13	15	17	21
1.50	10.90	1.90	8	10	11	12	14	11.59	1.96	9	10	11	13	15
2.00	9.35	1.36	8	8	9	10	12	9.89	1.38	8	9	10	11	12
3.00	7.98	0.77	7	8	8	8	9	8.42	0.81	7	8	8	9	10
	$n = 11$, $H = 18.5$							$n = 11$, $H = 20.0$						
γ														
0.25	35.20	23.2	14	21	29	42	76	38.60	25.82	15	23	32	46	82
0.50	18.73	7.23	10	14	17	22	32	20.30	7.55	11	15	19	24	34
0.75	13.32	3.98	8	10	13	16	21	14.26	4.14	9	11	14	17	22
1.00	10.58	2.72	7	9	10	12	16	11.33	2.77	7	9	11	13	16
1.50	7.89	1.60	6	7	8	9	11	8.45	1.65	6	7	8	9	11
2.00	6.55	1.10	5	6	6	7	8	7.00	1.15	5	6	7	8	9
3.00	5.27	0.73	4	5	5	6	6	5.62	0.73	5	5	6	6	7
	$n = 25$, $H = 21.5$							$n = 25$, $H = 22.5$						
γ														
0.25	30.07	42.88	9	15	22	34	71	32.05	42.75	10	16	23	55	78
0.50	14.37	6.93	7	10	13	17	26	15.06	6.90	7	10	14	18	28
0.75	10.09	3.58	6	8	10	12	17	10.52	3.66	6	8	10	12	17
1.00	7.83	2.31	5	6	8	9	12	8.18	2.40	5	6	8	10	13
1.50	5.67	1.33	4	5	6	6	8	5.92	1.37	4	5	6	7	8
2.00	4.61	0.92	3	4	4	5	6	4.81	0.95	4	4	5	5	6
3.00	3.60	0.60	3	3	4	4	4	3.73	0.60	3	3	4	4	5

* IC set up: mean $\mu_0 = 3$ and standard deviation $\sigma_0 = \sqrt{2}$; OOC set up: mean $= \mu_0 + \frac{\gamma\sigma_0}{\sqrt{n}}$ and standard deviation $= \sqrt{2}$

Table 4.A. The IC and OOC characteristics of the run-length distribution for $m = 100$ and $n = 5$ for the *N*(0,1) distribution* with nominal $ARL_0 = 500$ and winsorization at the 5000th step**

	Chart Type					
	Parametric CUSUM $\overline{X}$ chart with Parameters estimated from a Phase I sample		Rank-sum CUSUM chart		Exceedance CUSUM median chart	
Winsorization level	WL = 95.4	WL = 96.6	WL = 95.6	WL = 97.7	WL = 95.9	WL = 97.3
Control limits	$H = 8.50$	$H = 5.17$	$H = 563.0$	$H = 225$	$H = 9.55$	$H = 5.18$
k	0	$0.5\sigma/\sqrt{n}$	0	$0.5\sqrt{mn(m+n+1)/12}$	0	$k = n(d^* - d)$
d^* / γ	NA	NA	NA	NA	0.50	$0.5\sqrt{\frac{n(m+n+1)}{4(m+2)}} \approx 0.57$
0.00	507.44 (1177.56) 14, 31, 72, 256, 4449	493.51 (1070.55) 11, 32, 99, 337, 3139	503.52 (1171.43) 16,33, 74, 264, 4236	505.53 (972.27) 12, 48, 144, 444, 2499	503.24 (1137.31) 14, 31, 70, 252, 3880	502.27 (1023.94) 10, 37, 113, 404, 2844
0.25	92.00 (380.94) 10, 17, 27, 51, 240	97.55 (307.37) 7, 14, 29, 68, 346	93.33 (343.36) 12, 19, 29, 56, 255	137.73 (351.71) 8, 19, 44, 114, 521	122.89 (464.30) 11, 19, 32, 65, 380	146.06 (430.42) 8, 17, 39, 105, 547
0.50	24.81 (61.19) 7, 11, 16, 25, 56	26.74 (69.05) 5, 9, 14, 25, 76	26.98 (72.05) 10, 14, 19, 27, 59	41.09 (114.10) 6, 11, 19, 40, 129	35.97 (143.89) 9, 13, 20, 31, 83	49.78 (180.86) 6, 11, 19, 40, 153
0.75	13.52 (8.15) 6, 9, 12, 16, 27	12.93 (16.50) 4, 6, 10, 15, 31	16.15 (9.80) 8, 11, 14,18, 30	18.11 (26.46) 5, 8, 12, 19, 48	18.21 (22.74) 7, 10, 14, 20, 39	20.22 (30.90) 5, 8, 12, 21, 58
1.00	9.93 (4.28) 5, 7, 9, 12, 18	8.22 (5.12) 3, 5, 7, 10, 17	12.43 (4.73) 7, 9, 11, 14, 21	11.35 (13.77) 5, 6, 9, 13, 26	12.97 (7.39) 6, 9, 11, 15, 24	12.17 (11.72) 4, 6, 9, 14, 29
1.50	6.62 (2.08) 4, 5, 6, 8, 10	5.00 (2.01) 3, 4, 5, 6, 9	8.96 (2.08) 6, 8, 9,10, 13	6.77 (2.67) 4, 5, 6, 8, 12	8.76 (2.82) 6, 7, 8, 10, 14	6.77 (3.37) 3, 4, 6, 8, 13
2.00	5.01 (1.29) 3, 4, 5, 6, 7	3.67 (1.19) 2, 3, 3, 4, 6	7.41 (1.23) 6, 7, 7, 8, 10	5.31 (1.38) 4, 5, 6, 8, 20	6.95 (1.69) 5, 6, 6, 8, 10	4.99 (1.87) 3, 4, 5, 6, 8
3.00	3.46 (0.71) 3, 3, 3, 4, 5	2.49 (0.63) 2, 2, 2, 3, 4	5.99 (0.67) 5, 6, 6, 6, 7	4.10 (0.98) 1, 4, 4, 5, 5	5.33 (0.86) 4, 5, 5, 6, 7	3.54 (0.78) 3, 3, 3, 4, 5

* IC set up: mean $\mu_0 = 0$ and standard deviation $\sigma_0 = 1$; OOC set up: mean $= \mu_0 + \frac{\gamma\sigma_0}{\sqrt{n}}$ and standard deviation $= 1$

** Note that, the first row of each of the cells shows the *ARL* and *SDRL* values whereas the second row shows the 5th, 25th, 50th, 75th and 95th percentiles (in this order).

Table 4.B. The IC and OOC characteristics of the run-length distribution for $m = 100$ and $n = 5$ for the *EXP*(1) distribution* with nominal $ARL_0 = 500$ and winsorization at the 5000th step**

	Chart Type					
	Parametric CUSUM $\overline{X}$ chart with Parameters estimated from Phase I sample		Rank-sum CUSUM chart		Exceedance CUSUM median chart	
Winsorization level	WL = 95.7	WL = 96.6	WL = 95.6	WL = 97.9	WL = 95.5	WL = 97.2
Control limits	$H = 8.00$	$H = 5.25$	$H = 563.0$	$H = 225.0$	$H = 9.55$	$H = 5.18$
k	0	$0.5\sigma/\sqrt{n}$	0	$0.5\sqrt{mn(m+n+1)/1}$	0	$k = n(d^* - d)$
d^* / γ	NA	NA	NA	NA	0.50	$0.5\sqrt{\frac{n(m+n+1)}{4(m+2)}} \approx 0.57$
0.00	497.99 (1151.67) 11, 28, 69, 262, 3990	494.40 (1080.71) 9, 29, 89, 332, 3220	502.40 (1155.50) 16, 33, 74, 257, 4067	500.02 (953.54) 12, 47, 141, 448, 2464	501.01 (1158.61) 14, 31, 72, 264, 4223	491.60 (1021.17) 10, 37, 110, 383, 2763
0.25	112.39 (475.83) 8, 15, 25, 51, 329	171.68 (584.54) 6, 14, 30, 87, 636	49.89 (216.36) 11, 15, 22, 34, 107	108.92 (338.02) 7, 14, 30, 75, 401	97.27 (400.71) 10, 16, 26, 51, 248	123.30 (399.82) 7, 14, 30, 81, 440
0.50	28.47 (91.94) 6, 10, 16, 26, 68	51.78 (247.44) 4, 9, 15, 30, 149	15.84 (16.01) 9, 11, 13, 17, 29	25.24 (73.72) 6, 8, 12, 21, 73	27.69 (142.54) 7, 10, 15, 23, 58	30.71 (110.29) 4, 8, 13, 24, 92
0.75	14.06 (15.56) 5, 8, 11, 16, 30	17.64 (56.61) 4, 6, 10, 17, 44	11.19 (3.48) 7, 9, 10, 13, 17	10.52 (17.21) 5, 6, 8, 11, 21	12.26 (12.33) 6, 8, 10, 14, 25	11.54 (22.89) 3, 5, 8, 12, 29
1.00	9.99 (6.25) 4, 6, 9, 12, 19	10.12 (31.63) 3, 5, 7, 11, 22	9.05 (1.95) 7, 8, 9, 10, 13	7.14 (3.10) 5, 5, 6, 8, 12	8.25 (3.95) 5, 6, 7, 9, 15	6.60 (7.87) 3, 4, 5, 8, 15
1.50	6.45 (2.44) 3, 5, 6, 8, 11	5.32 (2.62) 2, 4, 5, 6, 10	7.24 (1.00) 6, 7, 7, 8, 9	5.18 (0.98) 4, 5, 5, 6, 7	4.79 (1.28) 4, 4, 4, 5, 7	3.39 (1.01) 3, 3, 3, 3, 5
2.00	4.82 (1.48) 3, 4, 5, 6, 7	3.82 (1.45) 2, 3, 4, 4, 6	6.37 (0.61) 6, 6, 6, 7, 7	4.50 (0.62) 4, 4, 4, 5, 5	4.02 (0.20) 4, 4, 4, 4, 4	3.01 (0.08) 3, 3, 3, 3, 3
3.00	3.28 (0.82) 2, 3, 3, 4, 5	2.54 (0.72) 2, 2, 2, 3, 4	5.55 (0.50) 5, 5, 6, 6, 6	3.97 (0.43) 4, 4, 4, 4, 4	4.00 (0.00) 4, 4, 4, 4, 4	3.00 (0.00) 3, 3, 3, 3, 3

* IC set up: mean $\mu_0 = 1$ and standard deviation $\sigma_0 = 1$; OOC set up: mean $= \mu_0 + \frac{\gamma\sigma_0}{\sqrt{n}}$ and standard deviation $= 1$

** Note that, the first row of each of the cells shows the *ARL* and *SDRL* values whereas the second row shows the 5th, 25th, 50th, 75th and 95th percentiles (in this order).

Table 4.C. The IC and OOC characteristics of the run-length distribution for $m = 100$ and $n = 5$ for the *GAM*(3,1) distribution* with nominal $ARL_0 = 500$ and winsorization at the 5000th step**

	Chart Type					
	Parametric CUSUM $\overline{X}$ chart with Parameters estimated from Phase I sample		Rank-sum CUSUM chart		Exceedance CUSUM median chart	
Windsorization level	WL = 95.6	WL = 96.5	WL = 95.7	WL = 97.9	WL = 95.3	WL = 97.2
Control limits	$H = 8.50$	$H = 5.20$	$H = 563.0$	$H = 225.0$	$H = 9.55$	$H = 5.18$
k	0	$0.5\sigma/\sqrt{n}$	0	$0.5\sqrt{mn(m+n+1)/12}$	0	$k = n(d^* - d)$
d^* / γ	NA	NA	NA	NA	0.50	$0.5\sqrt{\frac{n(m+n+1)}{4(m+2)}} \approx 0.57$
0.00	494.98 (1152.81) 12, 29, 70, 255, 4117	504.68 (1085.77) 10, 30, 90, 352, 3226	496.82 (1138.20) 16, 33, 74, 262, 3752	494.35 (945.50) 11, 45, 140, 450, 2393	509.83 (1186.24) 14, 31, 74, 269, 4546	493.35 (1013.53) 11, 38, 108, 364, 2739
0.25	109.26 (443.83) 9, 16, 27, 55, 303	139.34 (477.73) 6, 14, 29, 79, 505	90.09 (371.13) 12, 18, 27, 49, 222	143.62 (391.30) 8, 18, 41, 108, 555	126.09 (494.92) 10, 18, 29, 61, 356	143.49 (435.83) 7, 16, 36, 100, 534
0.50	27.76 (101.83) 7, 11, 16, 25, 61	37.35 (137.94) 5, 9, 15, 29, 109	23.54 (39.17) 10, 13, 17, 24, 49	39.89 (101.63) 6, 10, 18, 35, 131	34.02 (142.31) 8, 12, 18, 28, 76	45.05 (142.86) 5, 9, 17, 36, 145
0.75	14.13 (17.18) 6, 8, 11, 16, 30	15.82 (72.91) 4, 6, 10, 16, 38	14.45 (7.44) 8, 10, 13, 16, 26	17.15 (55.18) 5, 8, 11, 17, 43	16.60 (53.03) 7, 9, 12, 18, 34	17.70 (36.70) 4, 7, 10, 18, 49
1.00	10.03 (5.87) 5, 7, 9, 12, 19	8.88 (8.02) 3, 5, 7, 10, 20	11.25 (3.51) 7, 9, 10, 13, 17	9.65 (6.38) 5, 6, 8, 11, 20	11.23 (6.82) 6, 8, 10, 13, 21	9.91 (14.59) 3, 5, 8, 11, 23
1.50	6.54 (2.27) 4, 5, 6, 8, 11	5.19 (2.38) 3, 4, 5, 6, 10	8.24 (1.54) 6, 7, 8, 9, 11	6.14 (1.88) 4, 5, 6, 7, 9	7.29 (2.18) 5, 6, 7, 8, 11	5.34 (2.52) 3, 4, 5, 6, 10
2.00	4.92 (1.40) 3, 4, 5, 6, 7	3.75 (1.31) 2, 3, 4, 4, 6	6.97 (0.94) 6, 6, 7, 7, 9	4.97 (0.95) 4, 4, 5, 5, 7	5.60 (1.16) 4, 5, 5, 6, 8	3.79 (1.16) 3, 3, 3, 4, 6
3.00	3.40 (0.78) 2, 3, 3, 4, 5	2.51 (0.66) 2, 2, 2, 3, 4	5.80 (0.51) 5, 5, 6, 6, 6	4.04 (0.58) 4, 4, 4, 4, 5	4.17 (0.40) 4, 4, 4, 4, 5	3.02 (0.14) 3, 3, 3, 3, 3

* IC set up: mean $\mu_0 = 3$ and standard deviation $\sigma_0 = \sqrt{3}$; OOC set up: mean $= \mu_0 + \frac{\gamma\sigma_0}{\sqrt{n}}$ and standard deviation $= \sqrt{3}$

** Note that, the first row of each of the cells shows the *ARL* and *SDRL* values whereas the second row shows the 5th, 25th, 50th, 75th and 95th percentiles (in this order).

Table 4.D. The IC and OOC characteristics of the run-length distribution for $m = 100$ and $n = 5$ for the *t*(3) distribution* with nominal $ARL_0 =$ 500 and winsorization at the 5000th step**

	Chart Type					
	Parametric CUSUM $\overline{X}$ chart with Parameters estimated from Phase I sample		Rank-sum CUSUM chart		Exceedance CUSUM median chart	
Winsorization level	WL = 95.2	WL = 96.6	WL = 95.6	WL = 98.0	WL = 95.7	WL = 97.4
Control limits	$H = 8.02$	$H = 5.05$	$H = 563.0$	$H = 225.0$	$H = 9.55$	$H = 5.18$
k	0	$0.5\sigma/\sqrt{n}$	0	$0.5\sqrt{mn(m+n+1)/12}$	0	$k = n(d^* - d)$
d^* / γ	NA	NA	NA	NA	0.50	$0.5\sqrt{\frac{n(m+n+1)}{4(m+2)}} \approx 0.57$
0.00	496.93 (1179.16) 13, 28, 64, 230, 4539	501.25 (1104.27) 9, 31, 90, 319, 3331	501.88 (1142.80) 16, 32, 71, 256, 3931	492.11 (944.01) 12, 47, 138, 435, 2434	498.96 (1139.02) 14, 31, 71, 258, 3933	494.62 (1010.80) 10, 37, 111, 375, 2748
0.25	111.54 (514.50) 8, 15, 24, 45, 237	169.39 (655.76) 6, 13, 26, 66, 519	49.81 (189.48) 11, 16, 24, 39, 114	83.73 (221.83) 7, 15, 30, 71, 303	62.42 (267.48) 10, 15, 24, 40, 148	78.14 (242.04) 6, 13, 25, 60, 273
0.50	34.96 (247.14) 6, 10, 15, 22, 52	56.46 (366.38) 5, 8, 13, 24, 89	16.97 (10.60) 9, 11, 15,19, 33	21.22 (37.98) 5, 8, 13, 22, 60	16.69 (12.84) 7, 10, 14, 19, 34	18.09 (29.15) 4, 8, 12, 20, 49
0.75	15.00 (95.56) 5, 8, 10, 14, 26	29.29 (293.97) 4, 6, 9, 13, 31	11.78 (4.01) 7, 9, 11, 13, 19	10.24 (7.67) 5, 6, 8,12, 22	11.05 (4.55) 6, 8, 10, 13, 19	9.46 (6.84) 4, 6, 8, 11, 21
1.00	10.37 (75.34) 4, 6, 8, 10, 17	17.23 (202.69) 3, 5, 6, 9, 17	9.50 (2.37) 7, 8, 9, 11, 14	7.33 (3.13) 4, 5, 6, 8, 13	8.57 (2.68) 6, 7, 8, 10, 14	6.60 (3.08) 3, 4, 6, 8, 12
1.50	6.71 (53.64) 3, 5, 6, 7, 10	6.88 (100.17) 2, 3, 4, 5, 9	7.31 (1.22) 6, 6, 7, 8, 10	5.23 (1.28) 4, 4, 5, 6, 8	6.48 (1.36) 5, 6, 6, 7, 9	4.53 (1.47) 3, 3, 4, 5, 8
2.00	5.13 (50.05) 3, 4, 4, 5, 7	5.47 (99.93) 2, 3, 3, 4, 6	6.31 (0.79) 5, 6, 6, 7, 8	4.42 (0.92) 4, 4, 4, 5, 6	5.56 (0.92) 4, 5, 6, 6, 7	3.73 (0.91) 3, 3, 4, 4, 5
3.00	3.20 (1.14) 2, 3, 3, 4, 5	2.36 (0.84) 2, 2, 2, 3, 4	5.39 (0.52) 5, 5, 5, 6, 6	3.43 (1.28) 1,4, 4, 4, 5	4.79 (0.68) 4, 4, 5, 5, 6	3.19 (0.45) 3, 3, 3, 3, 4

* IC set up: mean $\mu_0 = 0$ and standard deviation $\sigma_0 = \sqrt{3}$; OOC set up: mean $= \mu_0 + \frac{\gamma\sigma_0}{\sqrt{n}}$ and standard deviation $= \sqrt{3}$

** Note that, the first row of each of the cells shows the *ARL* and *SDRL* values whereas the second row shows the 5th, 25th, 50th, 75th and 95th percentiles (in this order).

Table 4.E. The IC and OOC characteristics of the run-length distribution for $m = 100$ and $n = 5$ for the *DE*(0,1) distribution* with nominal $ARL_0 = 500$ and winsorization at the 5000th step**

	Chart Type					
	Parametric CUSUM $\overline{X}$ chart with Parameters estimated from Phase I sample		Rank-sum CUSUM chart		Exceedance CUSUM median chart	
Winsorization level	WL = 95.5	WL = 96.7	WL = 95.5	WL = 97.6	WL = 95.7	WL = 97.0
Control limits	$H = 8.25$	$H = 5.15$	$H = 563.0$	$H = 225$	$H = 9.55$	$H = 5.18$
k	0	$0.5\sigma/\sqrt{n}$	0	$0.5\sqrt{mn(m+n+1)/12}$	0	NA
d^* / γ	NA	NA	NA	NA	0.50	$0.5\sqrt{\frac{n(m+n+1)}{4(m+2)}} \approx 0.57$
0.00	492.36 (1164.91) 13, 29, 68, 241, 4367	505.43 (1085.83) 10, 32, 93, 341, 3169	508.74 (1160.94) 16, 33, 73, 269, 4103	507.43 (987.93) 12, 46, 143, 450, 2661	493.02 (1138.46) 14, 31, 72, 270, 3953	507.55 (1051.32) 10, 37, 112, 384, 3043
0.25	94.44 (388.64) 9, 16, 26, 51, 247	123.21 (442.64) 7, 14, 29, 72, 383	64.02 (259.88) 11, 17, 26, 43, 144	110.76 (313.85) 7, 16, 35, 87, 390	58.15 (249.72) 10, 15, 23, 38, 132	71.41 (238.15) 6, 13, 25, 54, 221
0.50	22.44 (58.34) 7, 11, 16, 23, 52	27.55 (72.73) 5, 9, 14, 25, 77	19.48 (17.22) 9, 12, 16, 22, 39	26.73 (54.03) 6, 9, 14, 26, 77	16.94 (17.05) 8, 10, 14, 19, 33	17.82 (27.62) 5, 8, 12, 20, 46
0.75	13.17 (8.19) 6, 8, 11, 15, 26	12.64 (14.38) 4, 6, 9, 14, 31	13.16 (5.32) 8, 10, 12, 15, 22	12.55 (11.21) 5, 7, 9, 14, 30	11.48 (4.76) 6, 8, 10, 13, 20	9.83 (6.80) 4, 6, 8, 11, 22
1.00	9.68 (4.42) 5, 7, 9, 11, 18	8.31 (6.16) 3, 5, 7, 10, 17	10.44 (3.06) 7, 8, 10, 12, 16	8.46 (4.62) 4, 6, 7, 10, 16	9.17 (2.88) 6, 7, 9, 10, 14	7.22 (3.51) 3, 5, 7, 8, 14
1.50	6.40 (2.06) 4, 5, 6, 7, 10	4.95 (2.07) 3, 4, 5, 6, 9	7.95 (1.54) 6, 7, 8, 9, 11	5.81 (1.67) 4, 5, 5, 6, 9	7.06 (1.63) 5, 6, 7, 8, 10	5.08 (1.81) 3, 4, 5, 6, 8
2.00	4.88 (1.32) 3, 4, 5, 6, 7	3.66 (1.24) 2, 3, 3, 4, 6	6.78 (0.99) 6, 6, 7, 7, 9	4.82 (1.04) 4, 4, 5, 5, 7	6.03 (1.12) 4, 5, 6, 6, 8	4.16 (1.19) 3, 3, 4, 5, 6
3.00	3.36 (0.75) 2, 3, 3, 4, 5	2.46 (0.65) 2, 2, 2, 3, 4	5.71 (0.63) 5, 5, 6, 6, 7	3.88 (1.04) 1, 4, 4, 4, 5	5.10 (0.77) 4, 5, 5, 6, 6	3.38 (0.62) 3, 3, 3, 4, 5

* IC set up: mean $\mu_0 = 0$ and standard deviation $\sigma_0 = \sqrt{2}$; OOC set up: mean $= \mu_0 + \frac{\gamma\sigma_0}{\sqrt{n}}$ and standard deviation $= \sqrt{2}$

** Note that, the first row of each of the cells shows the *ARL* and *SDRL* values whereas the second row shows the 5th, 25th, 50th, 75th and 95th percentiles (in this order).

Table 5. The exceedance and the exceedance CUSUM median statistics

j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$U_{j,r}$	3	2	0	4	1	4	4	1	3	4	2	5	5	5	4
C_j	0.5	0	0	1.5	0	1.5	3	1.5	2	3.5	3	5.5	8	10.5	12

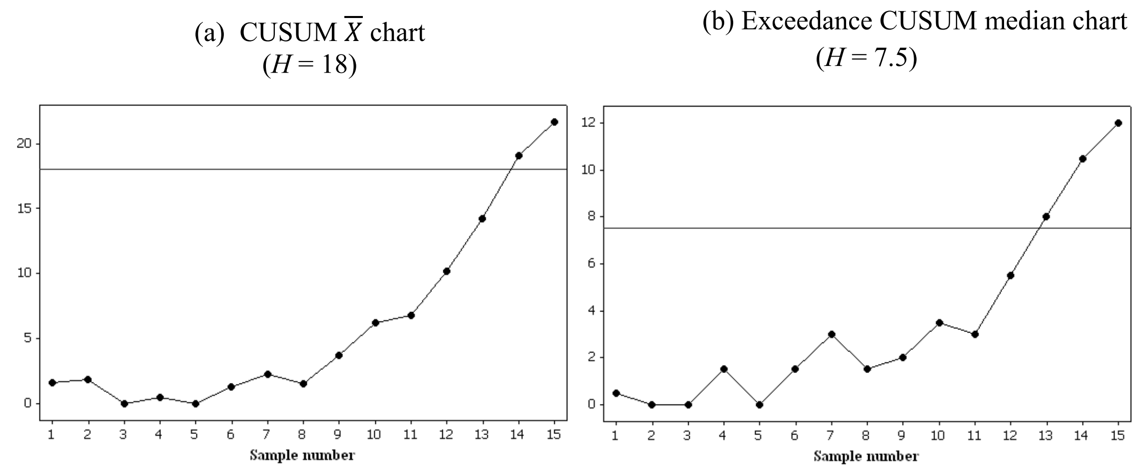


Figure 5. The CUSUM $\overline{X}$ and the exceedance CUSUM charts for the Montgomery (2001) piston ring data

Table 6.A. The IC and OOC characteristics of the run-length distribution for $m = 100$ and $n = 5$ for the Normal distribution with nominal $ARL_0 = 500$ and winsorization at the 2000th step*

	Chart Type					
	Parametric CUSUM $\overline{X}$ chart with Parameters estimated from Phase I sample		Rank-sum CUSUM chart		Exceedance CUSUM median chart	
Winsorization level (WL)	WL = 85.1	WL = 86.7	WL = 84.7	WL = 88.9	WL = 85.2	WL = 89.1
Control limits	$H = 11.05$	$H = 5.95$	$H = 725.0$	$H = 246$	$H = 12.10$	$H = 5.85$
k	0	$0.5\sigma/\sqrt{n}$	0	$0.5\sqrt{mn(m+n+1)/12}$	0	$k = n(d^* - d) = 5 \times 0.07 = 0.35$
d^* / γ	NA	NA	NA	NA	0.50	$0.5\sqrt{\frac{n(m+n+1)}{4(m+2)}} \approx 0.57$
0.00	498.49 (712.37) 20, 45, 111, 586, 2000	505.28 (687.54) 14, 45, 146, 636, 2000	505.34 (715.71) 20, 44, 118, 606, 2000	503.9843 (656.38) 10,57, 189, 660, 2000	489.23 (709.53) 19, 43, 109, 553, 2000	502.26 (676.35) 14, 46, 15, 650, 2000
0.25	125.02 (323.68) 13, 23, 36, 73, 488	134.21 (320.59) 8, 18, 35, 90, 605	123.61 (314.23) 13, 23, 36, 73, 490	168.01 (340.48) 5, 19, 51, 146, 762	141.68 (341.30) 6, 14, 24, 40, 88, 636	178.81 (360.39) 9, 22, 51, 149, 871
0.50	32.09 (69.69) 10, 15, 21, 32, 71	34.71 (88.19) 6, 11, 17, 31, 103	33.01 (80.32) 10, 15, 21, 32, 72	45.76 (108.06) 3, 10, 19, 44, 163	49.18 (130.78) 11, 17, 25, 40, 126	61.53 (152.20) 6, 13, 23, 51, 220
0.75	17.66 (12.21) 8, 11, 15, 20, 35	15.00 (20.13) 5, 7, 11, 17, 36	17.92 (11.75) 8, 12, 15, 21, 35	18.14 (32.02) 3, 6, 11, 20, 54	23.20 (27.99) 9, 14, 18, 25, 47	25.09 (56.04) 6, 9, 14, 24, 70
1.00	12.67 (5.28) 7, 9, 12, 15, 22	9.36 (5.58) 4, 6, 8, 11, 19	12.90 (5.28) 7, 9, 12, 15, 23	9.71 (10.00) 2, 5, 7, 12, 24	16.19 (9.79) 9, 11, 14, 19, 30	13.78 (16.79) 5, 7, 10, 15, 34
1.50	8.38 (2.41) 5, 7, 8, 10, 13	5.65 (2.23) 3, 4, 5, 7, 10	8.79 (2.57) 5, 7, 8, 10, 13	5.18 (2.88) 2, 3, 5, 6, 10	10.92 (3.46) 7, 9, 10, 13, 17	7.58 (3.77) 4, 6, 6, 9, 14
2.00	6.32 (1.50) 4, 5, 6, 7, 9	4.15 (1.28) 3, 3, 4, 5, 6	6.77 (1.57) 5, 6, 7, 8, 10	3.60 (1.55) 2, 2, 3, 4, 6	8.56 (1.96) 6, 7, 8, 9, 12	5.63 (1.81) 3, 4, 5, 6, 9
3.00	4.35 (0.85) 3, 4, 4, 5, 6	2.78 (0.70) 2, 2, 3, 3, 4	4.96 (0.83) 4, 4, 5, 5, 6	2.51 (0.71) 2, 2, 2, 3, 4	6.55 (0.92) 5, 6, 7, 7, 8	4.12 (0.91) 3, 4, 4, 5, 6

* IC set up: mean $\mu_0 = 0$ and standard deviation $\sigma_0 = 1$; OOC set up: mean $= \mu_0 + \frac{\gamma\sigma_0}{\sqrt{n}}$ and standard deviation $= 1$

* Note that, the first row of each of the cells shows the *ARL* and *SDRL* values whereas the second row shows the 5th, 25th, 50th, 75th and 95th percentiles (in this order).

Table 6.B. The IC and OOC characteristics of the run-length distribution for $m = 100$ and $n = 5$ for the Normal distribution with nominal $ARL_0 = 370$ and winsorization at the 2000th step*

	Chart Type					
	Parametric CUSUM $\overline{X}$ chart with Parameters estimated from Phase I sample		Rank-sum CUSUM chart		Exceedance CUSUM median chart	
Winsorization level (WL)	WL = 90.6	WL = 92.6	WL = 90.6	WL = 220	WL = 90.7	WL = 92.9
Control limits	$H = 9.00$	$H = 5.20$	$H = 595.0$	$H = 94.1$	$H = 10.35$	$H = 5.15$
k	0	$0.5\sigma/\sqrt{n}$	0	$0.5\sqrt{mn(m+n+1)/12}$	0	$k = n(d^* - d) = 5 \times 0.07 = 0.35$
d^* / γ	NA	NA	NA	NA	0.50	$0.5\sqrt{\frac{n(m+n+1)}{4(m+2)}} \approx 0.57$
0.00	367.44 (612.93) 15, 34, 78, 303, 2000	365.84 (576.38) 11, 34, 101, 359, 2000	369.49 (612.77) 15, 33, 80, 309, 2000	368.60 (543.36) 7, 39, 129, 412, 2000	366.25 (611.08) 15, 34, 80, 304, 2000	371.06 (568.33) 11, 37, 112, 387, 2000
0.25	81.82 (224.01) 10, 18, 29, 55, 258	98.61 (247.75) 7, 15, 29, 72, 386	84.66 (237.11) 10, 17, 28, 55, 270	109.51 (232.28) 4, 14, 38, 101, 440	110.48 (285.48) 11, 20, 33, 70, 427	133.13 (284.64) 8, 18, 42, 104, 547
0.50	25.60 (50.31) 8, 12, 17, 26, 59	26.99 (62.22) 5, 9, 14, 25, 78	26.19 (58.50) 8, 12, 17, 26, 61	36.57 (81.63) 3, 8, 16, 36, 124	37.97 (102.56) 9, 14, 21, 33, 105	45.90 (114.99) 6, 11, 19, 39, 158
0.75	14.37 (9.07) 7, 9, 12, 17, 29	12.71 (14.03) 4, 6, 9, 15, 31	14.75 (9.76) 6, 9, 12, 17, 30	15.52 (24.71) 2, 5, 9, 17, 46	18.93 (17.13) 8, 11, 15, 21, 41	20.89 (36.14) 5, 8, 13, 22, 58
1.00	10.51 (4.81) 5, 8, 10, 12, 19	8.26 (5.23) 3, 5, 7, 10, 17	10.76 (4.94) 5, 8, 10, 13, 19	8.55 (8.27) 2, 4, 7, 10, 22	13.62 (7.38) 7, 9, 12, 16, 25	12.13 (11.61) 4, 7, 9, 14, 30
1.50	6.98 (2.16) 4, 5, 7, 8, 11	5.07 (2.08) 3, 4, 5, 6, 9	7.31 (2.31) 4, 6, 7, 9, 11	4.63 (2.61) 2, 3, 4, 6, 9	9.19 (2.95) 6, 7, 9, 11, 15	6.97 (3.60) 3, 5, 6, 8, 14
2.00	5.28 (1.35) 3, 4, 5, 6, 8	3.70 (1.23) 2, 3, 3, 4, 6	5.70 (1.43) 4, 5, 5, 6, 8	3.34 (1.45) 2, 2, 3, 4, 6	7.22 (1.83) 5, 6, 7, 8, 11	5.10 (2.04) 3, 4, 5, 6, 8
3.00	3.63 (0.74) 3, 3, 4, 4, 5	2.49 (0.63) 2, 2, 2, 3, 4	4.17 (0.78) 3, 4, 4, 5, 6	2.36 (0.62) 2, 2, 2, 3, 4	5.53 (0.78) 5, 5, 5, 6, 7	3.57 (0.82) 3, 3, 3, 4, 5

* IC set up: mean $\mu_0 = 0$ and standard deviation $\sigma_0 = 1$; OOC set up: mean $= \mu_0 + \frac{\gamma\sigma_0}{\sqrt{n}}$ and standard deviation $= 1$

* Note that, the first row of each of the cells shows the *ARL* and *SDRL* values whereas the second row shows the 5th, 25th, 50th, 75th and 95th percentiles (in this order).