A SUZUKI TYPE FIXED POINT THEOREM FOR A GENERALIZED MULTIVALUED MAPPING ON PARTIAL HAUSDORFF METRIC SPACES

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ABSTRACT. In this paper, we obtain a Suzuki type fixed point theorem for a generalized multivalued mapping on a partial Hausdorff metric space. As a consequence of the presented results, we discuss the existence and uniqueness of the bounded solution of a functional equation arising in dynamic programming.

1. Introduction and Preliminaries

In 1937, Von Neumann [34] initiated the fixed point theory for multivalued mappings in the study of game theory. Indeed, the fixed point theorems for multivalued mappings are quite useful in control theory and have been frequently used in solving many problems of economics and game theory. Successively, Nadler [24] initiated the development of the geometric fixed point theory for multivalued mappings. He used the concept of the Hausdorff metric to establish the multivalued contraction principle containing the Banach contraction principle as a special case.

Here, we recall that a Hausdorff metric H induced by a metric d on a set X is given by

$$H(A,B) = \max\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\}$$

for every $A, B \in CB(X)$, where $d(x,B) = \inf\{d(x,y) : y \in B\}$ and $CB(X)$ is the collection of the closed and bounded subsets of X.

Theorem 1.1. [24] *Let (X,d) be a complete metric space and $T : X \to CB(X)$ be a multivalued mapping satisfying*

$$H(Tx,Ty) \leq kd(x,y)$$

for all $x, y \in X$ and $k \in (0,1)$. Then T has a fixed point, that is, there exists a point $z \in X$ such that $z \in Tz$.

In the last decades, a number of fixed point results (see, for example, [10, 11, 13, 14, 20, 21, 22, 27, 31]) have been obtained in attempts to generalize the Theorem 1.1. One of the most significant fixed point theorems for multivalued contractions appeared in [18, Theorem 2.1]. To state this theorem, we need the following notation and notion.

Let (X,d) be a metric space and $T : X \to CB(X)$ be a multivalued mapping. Define

$$M_d(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\} \tag{1}$$

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for all $x, y \in X$.
Also, let $\psi : [0,1) \to (0,1]$ be the non-increasing function defined by

$$\psi(r) = \begin{cases} 1 & \text{if } 0 \leq r < \dfrac{1}{2} \\ 1 - r & \text{if } \frac{1}{2} \leq r < 1 \end{cases}. \tag{2}$$

Then, we have the following result:

Theorem 1.2. [18] *Let (X, d) be a complete metric space and $T : X \to CB(X)$. If there exists $0 \leq r < 1$ such that T satisfies the condition*

$$\psi(r)d(x, Tx) \leq d(x, y) \Longrightarrow H(Tx, Ty) \leq rM_d(x, y)$$

for all $x, y \in X$, where ψ is given by (2), then T has a fixed point, that is, there exists a point $z \in X$ such that $z \in Tz$.

The other basic notion for the development of our work is the concept of the partial metric space, that was introduced by Matthews [23] as a part of the study of denotational semantics of dataflow networks. He presented a modified version of the Banach contraction principle, more suitable in this context, see also [15, 25]. In fact, the (complete) partial metric spaces constitute a suitable framework to model several distinguished examples of the theory of computation and also to model metric spaces via domain theory, see [12, 16, 17, 19, 23, 28, 30, 33]. In this direction, Aydi et al. [4] introduced the concept of a partial Hausdorff metric and extended Nadler's fixed point theorem in the setting of partial metric spaces.

In view of the above considerations, the aim of this paper is to obtain a Kikkawa-Suzuki type fixed point theorem for multivalued mappings in a partial Hausdorff metric space. The presented results extend and unify some recently obtained comparable results for multivalued mappings (see [18] and the references therein). Moreover, using our results, we will prove the existence and uniqueness of the bounded solution of a functional equation arising in dynamic programming.

Now on the letters $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$ will denote the set of all real numbers, the set of all non-negative real numbers and the set of all positive integers, respectively. Consistent with [2, 3, 4, 23], the following definitions and results will be needed in the sequel.

Definition 1.3. [23] *Let X be any non-empty set. A function $p : X \times X \to \mathbb{R}^+$ is said to be a partial metric if and only if for all $x, y, z \in X$ the following conditions are satisfied:*

- **(P1):** $p(x, x) = p(y, y) = p(x, y)$ *if and only if* $x = y$;
- **(P2):** $p(x, x) \leq p(x, y)$;
- **(P3):** $p(x, y) = p(y, x)$;
- **(P4):** $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

The pair (X, p) is called a partial metric space. If $p(x, y) = 0$, then (P1) and (P2) imply that $x = y$ but the converse does not hold in general. A trivial example of a partial metric space is the pair $(\mathbb{R}^+, p)$, where $p : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is defined as $p(x, y) = \max\{x, y\}$, see also [1].

Example 1.4. [23] *Let $X = \{[a, b] : a, b \in R, a \leq b\}$. It is easy to show that the function $p : X \times X \to \mathbb{R}^+$ given by*

$$p([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\}$$

defines a partial metric on X.

For further examples we refer to [2, 9, 28, 29, 30]. Note that each partial metric p on X generates a T_0 topology τ_p on X which has as a base, the family of the open balls (p-balls) $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$ where

$$B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$$

for all $x \in X$ and $\varepsilon > 0$. A sequence $\{x_n\}_{n\in N}$ in a partial metric space (X,p) is called convergent to a point $x \in X$, with respect to τ_p, if and only if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$, see [23] for details. If p is a partial metric on X, then the function $p^S(x,y) = 2p(x,y) - p(x,x) - p(y,y)$ defines a metric on X. Further a sequence $\{x_n\}_{n\in N}$ converges in the metric space (X,p^S) to a point $x \in X$ if and only if

$$p(x,x) = \lim_{n\to+\infty} p(x,x_n) = \lim_{n,m\to+\infty} p(x_n,x_m).$$

Definition 1.5. [23] *Let (X,p) be a partial metric space. Then*

- **(a):** *A sequence $\{x_n\}_{n\in N}$ in X is called Cauchy if and only if $\lim_{n,m\to+\infty} p(x_n,x_m)$ exists and is finite.*
- **(b):** *A partial metric space (X,p) is said to be complete if every Cauchy sequence $\{x_n\}_{n\in N}$ in X converges with respect to τ_p to a point $x \in X$ such that $p(x,x) = \lim_{n\to+\infty} p(x,x_n)$.*

Lemma 1.6. [2, 23] *Let (X,p) be a partial metric space. Then*

- **(a):** *A sequence $\{x_n\}_{n\in N}$ in X is Cauchy in (X,p) if and only if it is Cauchy in (X,p^S).*
- **(b):** *A partial metric space (X,p) is complete if and only if the metric space (X,p^S) is complete.*

Consistent with [4], let (X,p) be a partial metric space and let $CB^p(X)$ be the family of all non-empty, closed and bounded subsets of the partial metric space (X,p), induced by the partial metric p. Note that the closedness is taken from (X,τ_p) (τ_p is the topology induced by p) and the boundedness is given as follows: A is a bounded subset in (X,p) if there exist $x_0 \in X$ and $M \geq 0$ such that for all $a \in A$, we have $a \in B_p(x_0,M)$, that is, $p(x_0,a) < p(x_0,x_0) + M$. For $A,B \in CB^p(X)$, $x \in X$, $\delta_p : CB^p(X) \times CB^p(X) \to \mathbb{R}^+$ define

$$\begin{aligned} p(x,A) &= \inf\{p(x,a) : a \in A\}, \\ p(A,B) &= \inf\{p(x,y) : x \in A, y \in B\}, \\ \delta_p(A,B) &= \sup\{p(a,B) : a \in A\}, \\ \delta_p(B,A) &= \sup\{p(b,A) : b \in B\}, \\ H_p(A,B) &= \max\{\delta_p(A,B), \delta_p(B,A)\}. \end{aligned}$$

It is easy to show that $p(x,A) = 0$ implies that $p^S(x,A) = 0$, where

$$p^S(x,A) = \inf\{p^S(x,a) : a \in A\}.$$

Lemma 1.7. [2] *Let (X,p) be a partial metric space and A be any non-empty subset of X, then $a \in \overline{A}$ if and only if $p(a,A) = p(a,a)$.*

Proposition 1.8. [4] *Let (X,p) be a partial metric space. For any $A,B,C \in CB^p(X)$, we have the following:*

(i): $\delta_p(A, A) = \sup\{p(a, a) : a \in A\}$;
(ii): $\delta_p(A, A) \leq \delta_p(A, B)$;
(iii): $\delta_p(A, B) = 0 \Longrightarrow A \subseteq B$;
(iv): $\delta_p(A, B) \leq \delta_p(A, C) + \delta_p(C, B) - \inf_{c \in C} p(c, c)$.

Proposition 1.9. [4] *Let (X, p) be a partial metric space. For any $A, B, C \in CB^p(X)$, we have the following:*

(h1): $H_p(A, A) \leq H_p(A, B)$;
(h2): $H_p(A, B) = H_p(B, A)$;
(h3): $H_p(A, B) \leq H_p(A, C) + H_p(C, B) - \inf_{c \in C} p(c, c)$;
(h4): $H_p(A, B) = 0 \Longrightarrow A = B$.

The mapping $H_p : CB^p(X) \times CB^p(X) \to \mathbb{R}^+$ is called the partial Hausdorff metric induced by p. Every Hausdorff metric is a partial Hausdorff metric but the converse is not true, see Example 2.6 in [4].

Lemma 1.10. [4] *Let (X, p) be a partial metric space, $A, B \in CB^p(X)$ and $h > 1$, then for any $a \in A$, there exists $b(a) \in B$ such that $p(a, b(a)) \leq hH_p(A, B)$.*

Theorem 1.11. [4] *Let (X, p) be a partial metric space. If $T : X \to CB^p(X)$ is a multivalued mapping such that for all $x, y \in X$, we have $H_p(Tx, Ty) \leq kp(x, y)$, where $k \in (0, 1)$, then T has a fixed point, that is, there exists a point $z \in X$ such that $z \in Tz$.*

2. Fixed Point Results in Partial Hausdorff Metric Spaces

Let $p : X \times X \to \mathbb{R}^+$ be a partial metric and $T : X \to CB^p(X)$ be a multivalued mapping. We define

$$M_p(x, y) = \max\{p(x, y), p(x, Tx), p(y, Ty), \frac{p(x, Ty) + p(y, Tx)}{2}\}.$$

Now we state and prove our main result.

Theorem 2.1. *Let (X, p) be a complete partial metric space, $T : X \to CB^p(X)$ be a multivalued mapping and $\psi : [0, 1) \to (0, 1]$ be the non-increasing function defined by (2). If there exists $0 \leq r < 1$ such that T satisfies the condition*

$$\psi(r)p(x, Tx) \leq p(x, y) \Longrightarrow H_p(Tx, Ty) \leq rM_p(x, y) \tag{3}$$

for all $x, y \in X$ whenever $x \neq y$, then T has a fixed point, that is, there exists a point $z \in X$ such that $z \in Tz$.

Proof. Let r_1 be a real number such that $0 \leq r < r_1 < 1$ and $u_1 \in X$. As Tu_1 is non-empty, we can choose $u_2 \in Tu_1$. Clearly, if $u_2 = u_1$ the proof is finished and so we assume $u_2 \neq u_1$. From Lemma 1.10, with $h = \dfrac{1}{\sqrt{r_1}}$, there exists $u_3 \in Tu_2$ such that $p(u_2, u_3) \leq \frac{1}{\sqrt{r_1}} H_p(Tu_1, Tu_2)$. Again, if $u_3 = u_2$ the proof is finished and so we assume $u_3 \neq u_2$. Since $\psi(r) \leq 1$, we have $\psi(r)p(u_1, Tu_1) \leq p(u_1, Tu_1) \leq p(u_1, u_2)$.

Now from (3), we obtain

$$\begin{aligned}
p(u_2,u_3) &\le \frac{1}{\sqrt{r_1}} H_p(Tu_1,Tu_2)\\
&\le \sqrt{r_1}\max\{p(u_1,u_2),p(u_1,Tu_1),p(u_2,Tu_2),\\
&\quad \frac{p(u_1,Tu_2)+p(u_2,Tu_1)}{2}\}\\
&\le \sqrt{r_1}\max\{p(u_1,u_2),p(u_2,u_3),\frac{p(u_1,u_3)+p(u_2,u_2)}{2}\}\\
&\le \sqrt{r_1}\max\{p(u_1,u_2),p(u_2,u_3),\frac{p(u_1,u_2)+p(u_2,u_3)}{2}\}.
\end{aligned}$$

Thus

$$p(u_2,u_3) \le \sqrt{r_1}\max\{p(u_1,u_2),p(u_2,u_3)\}.$$

If $\max\{p(u_1,u_2),p(u_2,u_3)\}=p(u_2,u_3)$, then $p(u_2,u_3)\le \sqrt{r_1}p(u_2,u_3)$ implies that $p(u_2,u_3)=0$ and we obtain $p^S(u_2,u_3)\le 2p(u_2,u_3)=0$ which further implies that $p^S(u_2,u_3)=0$. Hence $u_2=u_3\in Tu_2$ and the proof is finished. On the contrary if $\max\{p(u_1,u_2),p(u_2,u_3)\}=p(u_1,u_2)$, then we have

$$p(u_2,u_3)\le \sqrt{r_1}p(u_1,u_2).$$

Continuing this process, we can obtain a sequence $\{u_n\}$ in X such that $u_{n+1}\in Tu_n$, $u_{n+1}\ne u_n$ and

$$p(u_n,u_{n+1}) \le (\sqrt{r_1})^{n-1}p(u_1,u_2) \tag{4}$$

for every $n>1$. This shows that $\lim\limits_{n\to+\infty} p(u_n,u_{n+1})=0$. Since

$$p(u_n,u_n)+p(u_{n+1},u_{n+1}) \le 2p(u_n,u_{n+1}),$$

then we get

$$\lim_{n\to+\infty} p(u_n,u_n)=0 \text{ and } \lim_{n\to+\infty} p(u_{n+1},u_{n+1})=0. \tag{5}$$

Let $\epsilon>0$ and pick $N\in\mathbb{N}$ large enough that for $n\ge N$ we have

$$2(\sqrt{r_1})^{n-1}\frac{1}{1-\sqrt{r_1}}p(u_1,u_2)<\epsilon.$$

Then, for every positive integer $k>n\ge N$ there is a $m\in\mathbb{N}$ such that $k=n+m$ and we have

$$\begin{aligned}
p^S(u_n,u_{n+m}) &\le 2p(u_n,u_{n+m})\\
&\le 2[p(u_n,u_{n+1})+\cdots+p(u_{n+m-1},u_{n+m})]\\
&\le 2[(\sqrt{r_1})^{n-1}p(u_1,u_2)+\cdots+(\sqrt{r_1})^{n+m-2}p(u_1,u_2)]\\
&\le 2(\sqrt{r_1})^{n-1}\frac{1}{1-\sqrt{r_1}}p(u_1,u_2)<\epsilon.
\end{aligned}$$

It is immediate to deduce that $\{u_n\}$ is a Cauchy sequence in (X,p^S) and by Lemma 1.6 $\{u_n\}$ is Cauchy in (X,p). Moreover, since (X,p) is complete, again by Lemma 1.6 we have the completeness of (X,p^S). It follows that there exists $z\in X$ such that $\lim\limits_{n\to+\infty} u_n=z$ in (X,p^S). Therefore $\lim\limits_{n\to+\infty} p^S(u_n,z)=0$ implies

$$p(z,z)=\lim_{n\to+\infty} p(u_n,z)=\lim_{m,n\to+\infty} p(u_n,u_m).$$

Now, since $\{u_n\}$ is a Cauchy sequence in (X, p^S), then we have

$$\lim_{m,n\to+\infty} p^S(u_n, u_m) = 0$$

and so

$$\lim_{m,n\to+\infty} 2p(u_n, u_m) - \lim_{m\to+\infty} p(u_m, u_m) - \lim_{n\to+\infty} p(u_n, u_n) = 0.$$

It follows from (5) that

$$\lim_{m\to+\infty} p(u_m, u_m) = \lim_{n\to+\infty} p(u_n, u_n) = 0$$

which further implies that

$$\lim_{m,n\to+\infty} p(u_n, u_m) = 0$$

and

$$p(z, z) = \lim_{n\to\infty} p(u_n, z) = \lim_{n\to\infty} p(u_n, u_n) = 0.$$

Now, the inequalities

$$p(z, Tx) \leq p(z, u_{n+1}) + p(u_{n+1}, Tx) - p(u_{n+1}, u_{n+1})$$

and

$$p(u_{n+1}, Tx) \leq p(u_{n+1}, u_n) + p(u_n, z) + p(z, Tx) - p(u_n, u_n) - p(z, z)$$

give us

$$\lim_{n\to+\infty} p(u_{n+1}, Tx) = \lim_{n\to+\infty} p(u_n, Tx) = p(z, Tx).$$

We claim that

$$p(z, Tx) \leq r \max\{p(z, x), p(x, Tx)\}$$

for all $x \neq z$. Since $p(z, z) = \lim_{n\to+\infty} p(u_n, z) = 0$, then there exists a positive integer $n_0 \in \mathbb{N}$ such that

$$p(z, u_n) \leq \frac{1}{3} p(z, x)$$

for all $n \geq n_0$ and this implies $u_n \neq x$. Since $u_{n+1} \in Tu_n$, then we have

$$\begin{aligned}
\psi(r)p(u_n, Tu_n) &\leq p(u_n, Tu_n) \leq p(u_n, u_{n+1}) \leq p(u_n, z) + p(z, u_{n+1}) \\
&\leq \frac{2}{3} p(z, x) = p(z, x) - \frac{1}{3} p(z, x) \leq p(z, x) - p(u_n, z) \\
&\leq p(u_n, x) - p(u_n, u_n) \leq p(u_n, x).
\end{aligned}$$

Hence, for any $n \geq n_0$ we get $\psi(r)p(u_n, Tu_n) \leq p(u_n, x)$.
Now from (3), we obtain

$$\begin{aligned}
p(u_{n+1}, Tx) &\leq H_p(Tu_n, Tx) \\
&\leq r \max\{p(u_n, x), p(u_n, Tu_n), p(x, Tx), \frac{p(u_n, Tx) + p(x, Tu_n)}{2}\} \\
&\leq r \max\{p(u_n, x), p(u_n, u_{n+1}), p(x, Tx), \frac{p(u_n, Tx) + p(x, u_{n+1})}{2}\} \\
&\leq r \max\{p(u_n, z) + p(z, x) - p(z, z), p(u_n, u_{n+1}), p(x, Tx), \\
&\quad \frac{p(u_n, z) + p(z, Tx) - p(z, z) + p(x, z) + p(z, u_{n+1}) - p(z, z)}{2}\} \\
&\leq r \max\{p(u_n, z) + p(z, x), p(u_n, u_{n+1}), p(x, Tx), \\
&\quad \frac{p(u_n, z) + p(z, Tx) + p(x, z) + p(z, u_{n+1})}{2}\}.
\end{aligned}$$

On taking the limit as $n \to +\infty$, we get

$$p(z,Tx) \le r \max\{p(z,x), p(x,Tx), \frac{p(z,Tx)+p(x,z)}{2}\}.$$

If $\max\{p(z,x), p(x,Tx), \frac{p(z,Tx)+p(x,z)}{2}\} = \frac{p(z,Tx)+p(x,z)}{2}$, then we have

$$p(z,Tx) \le r\frac{p(z,Tx)+p(x,z)}{2}.$$

Since $r < 1$, it follows that

$$p(z,Tx) \le \frac{r}{2-r}p(x,z) < rp(x,z) \le r \max\{p(z,x), p(x,Tx)\}$$

and hence

$$p(z,Tx) \le r \max\{p(z,x), p(x,Tx)\} \tag{6}$$

for all $x \ne z$. On the other hand, if $\max\{p(z,x), p(x,Tx), \frac{p(z,Tx)+p(x,z)}{2}\}$ is equal to $p(x,z)$ or $p(z,Tx)$, we can deduce easily that (6) holds.

Now we shall prove that $z \in Tz$. First, consider the case when $0 \le r < 1/2$ and suppose, by contradiction, that $z \notin Tz = \overline{Tz}$, as Tz is closed. Hence by Lemma 1.7, $p(z,Tz) \ne p(z,z) = 0$. Then we can choose $a \in Tz$ such that $2rp(a,z) < p(z,Tz)$. Since $a \in Tz$ and $z \notin Tz$ imply $a \ne z$, then from (6) with $x = a$, we have

$$p(z,Ta) \le r \max\{p(z,a), p(a,Ta)\}. \tag{7}$$

Since $a \in Tz$ and $\psi(r)p(z,Tz) \le p(z,Tz) \le p(z,a)$, therefore from (3) we have

$$\begin{aligned} H_p(Tz,Ta) &\le r \max\{p(z,a), p(z,Tz), p(a,Ta), \frac{p(z,Ta)+p(a,Tz)}{2}\} \\ &\le r \max\{p(z,a), p(a,Ta), \frac{p(z,Ta)+p(a,a)}{2}\} \\ &\le r \max\{p(z,a), p(a,Ta), \frac{p(z,a)+p(a,Ta)}{2}\}, \end{aligned}$$

that is,

$$H_p(Tz,Ta) \le r \max\left\{p(z,a), p(a,Ta)\right\}. \tag{8}$$

Since $a \in Tz$, then $p(a,Ta) \le H_p(Tz,Ta)$. Therefore from (8) we obtain

$$H_p(Tz,Ta) \le r \max\left\{p(z,a), H_p(Tz,Ta)\right\}.$$

Since $r < 1$, it follows that

$$H_p(Tz,Ta) \le rp(z,a). \tag{9}$$

From (9), we obtain

$$p(a,Ta) \le rp(z,a).$$

Starting from (P4), using the fact that $p(Ta,Tz) = \inf\{p(x,y) : x \in Ta, y \in Tz\} \le \inf\{p(x,a) : x \in Ta\} = p(Ta,a)$ since $a \in Tz$ and by (6), (8) and (9) we have

$$\begin{aligned} p(z,Tz) &\le p(z,Ta) + p(Ta,Tz) \\ &\le p(z,Ta) + H_p(Tz,Ta) \\ &\le r \max\{p(z,a), p(a,Ta)\} + rp(z,a) \\ &= rp(z,a) + rp(z,a) = 2rp(z,a) \\ &< p(z,Tz) \end{aligned}$$

that is a contradiction and hence $z \in Tz$.

Now consider the case when $\frac{1}{2} \le r < 1$. First we prove that

$$H_p(Tx, Tz) \le r \max\{p(x,z), p(x,Tx), p(z,Tz), \frac{p(x,Tz)+p(z,Tx)}{2}\} \tag{10}$$

for all $x \in X$ such that $x \neq z$. For each $n \in \mathbb{N}$, there exists $y_n \in Tx$ such that $p(z, y_n) < p(z, Tx) + \frac{1}{n} p(x, z)$ and consequently we have

$$\begin{aligned} p(x,Tx) &\le p(x,y_n) \\ &\le p(x,z) + p(z,y_n) - p(z,z) \\ &< p(x,z) + p(z,Tx) + \frac{1}{n} p(x,z). \end{aligned}$$

Hence by (6) we get

$$p(x,Tx) < p(x,z) + r\max\{p(z,x), p(x,Tx)\} + \frac{1}{n} p(x,z). \tag{11}$$

If $\max\{p(z,x), p(x,Tx)\} = p(x,z)$, then from (11) we obtain

$$\begin{aligned} p(x,Tx) &< p(x,z) + rp(z,x) + \frac{1}{n} p(x,z) \\ &< [(1+r) + \frac{1}{n}] p(x,z). \end{aligned}$$

This implies that $\frac{1}{(1+r)} p(x,Tx) < [1 + \frac{1}{(1+r)n}] p(x,z)$ and since $\psi(r) = 1 - r$, it follows that

$$\begin{aligned} \psi(r)p(x,Tx) &= (1-r)p(x,Tx) \\ &\le \frac{1}{(1+r)} p(x,Tx) < [1 + \frac{1}{(1+r)n}] p(x,z). \end{aligned}$$

On taking the limit as $n \to +\infty$ we obtain $\psi(r)p(x,Tx) \le p(x,z)$. Then from (3) with $y = z$ we get (10). If $p(x,z) < p(x,Tx)$, then from (11) we obtain

$$p(x,Tx) \le p(x,z) + rp(x,Tx) + \frac{1}{n} p(x,z)$$

and hence $(1-r)p(x,Tx) \le (1 + \frac{1}{n}) p(x,z)$.
On taking again the limit as $n \to +\infty$, we get $(1-r)p(x,Tx) \le p(x,z)$, that is, $\psi(r)p(x,Tx) \le p(x,z)$. Then from (3) with $y = z$ we get (10).
Since $u_{n+1} \neq u_n$ for each $n \in \mathbb{N}$, we have $u_{n+1} \neq z$ or $u_n \neq z$ and so the set $J = \{n : u_n \neq z\}$ is infinite. From (10) with $x = u_n$, $n \in J$, we have

$$\begin{aligned} p(u_{n+1}, Tz) &\le H_p(Tu_n, Tz) \\ &\le r\max\{p(u_n,z), p(u_n,Tu_n), p(z,Tz), \frac{p(u_n,Tz)+p(z,Tu_n)}{2}\} \\ &\le r\max\{p(u_n,z),\ p(u_n,u_{n+1}), p(z,Tz), \frac{p(u_n,Tz)+p(z,u_{n+1})}{2}\}. \end{aligned}$$

On taking the limit as $n \to +\infty$, $n \in J$, we obtain

$$p(z,Tz) \le rp(z,Tz).$$

Since $r < 1$, therefore $p(z,Tz) = 0 = p(z,z)$ and hence by Lemma 1.7 we have $z \in Tz$, that is, z is a fixed point of T. $\square$

From the Theorem 2.1 we deduce the following corollaries.

Corollary 2.2. *Let (X,p) be a complete partial metric space, $T : X \to CB^p(X)$ be a multivalued mapping and $\psi : [0,1) \to (0,1]$ be the non-increasing function defined by (2). If there exists $0 \leq r < 1$ such that T satisfies the condition*

$$\psi(r)p(x,Tx) \leq p(x,y) \Longrightarrow H_p(Tx,Ty) \leq r\max\{p(x,y), p(x,Tx), p(y,Ty)\} \quad (12)$$

for all $x,y \in X$ whenever $x \neq y$, then T has a fixed point, that is, there exists a point $z \in X$ such that $z \in Tz$.

Corollary 2.3. *Let (X,p) be a complete partial metric space, $T : X \to CB^p(X)$ be a multivalued mapping and $\psi : [0,1) \to (0,1]$ be the non-increasing function defined by (2). If there exists $0 \leq r < 1$ such that T satisfies the condition*

$$\psi(r)p(x,Tx) \leq p(x,y) \Longrightarrow H_p(Tx,Ty) \leq \frac{r}{3}[p(x,y) + p(x,Tx) + p(y,Ty)] \quad (13)$$

for all $x,y \in X$ whenever $x \neq y$, then T has a fixed point, that is, there exists a point $z \in X$ such that $z \in Tz$.

In the case of single-valued mappings, the Theorem 2.1 reduces to the following important corollary that will be used in the last section to prove the existence and uniqueness of the bounded solution of a functional equation arising in dynamic programming.

Corollary 2.4. *Let (X,p) be a complete partial metric space, $T : X \to X$ be a single-valued mapping and $\psi : [0,1) \to (0,1]$ be the non-increasing function defined by (2). If there exists $0 \leq r < 1$ such that T satisfies*

$$\psi(r)p(x,Tx) \leq p(x,y) \Longrightarrow p(Tx,Ty) \leq rM_p(x,y)$$

for all $x,y \in X$ whenever $x \neq y$, then T has a unique fixed point, that is, there exists a unique point $z \in X$ such that $z = Tz$.

Proof. The existence of the fixed point follows from the Theorem 2.1. To prove the uniqueness, assume that there exist $z_1, z_2 \in X$ with $z_1 \neq z_2$ such that $z_1 = Tz_1$ and $z_2 = Tz_2$. Then

$$\psi(r)p(z_1,Tz_1) \leq p(z_1,Tz_1) = p(z_1,z_1) \leq p(z_1,z_2)$$

that implies

$$\begin{aligned} p(z_1,z_2) &= p(Tz_1,Tz_2) \\ &\leq r\max\{p(z_1,z_2), p(z_1,Tz_1), p(z_2,Tz_2), \frac{p(z_2,Tz_1)+p(z_1,Tz_2)}{2}\} \\ &\leq r\max\{p(z_1,z_2), p(z_1,z_1), p(z_2,z_2)\} \leq rp(z_1,z_2). \end{aligned}$$

We deduce that $p(z_1,z_2) = 0$, which further implies that $p^S(z_1,z_2) \leq 2p(z_1,z_2) = 0$ and hence $z_1 = z_2$. □

Example 2.5. *Let* $X = \{0,1,2\}$ *and* $p : X \times X \to \mathbb{R}^+$ *be defined by*

$$\begin{aligned} p(0,0) &= p(1,1) = 0 \text{ and } p(2,2) = \frac{1}{3}, \\ p(0,1) &= p(1,0) = \frac{1}{4}, \\ p(0,2) &= p(2,0) = \frac{2}{5}, \\ p(1,2) &= p(2,1) = \frac{11}{15}. \end{aligned}$$

Then p *is a partial metric on* X. *Let* $\psi(r)$ *be given by* (2) *and define* $T : X \to CB^p(X)$ *by*

$$Tx = \begin{cases} \{0\} & \text{if } x \neq 2 \\ \{0,1\} & \text{if } x = 2 \end{cases}.$$

Therefore, we get

$$\max\{p(x,Tx) : x \in X\} = \frac{2}{5} \text{ and } \min\{p(x,y) : x,y \in X \text{ and } x \neq y\} = \frac{1}{3}.$$

Note that for any $r \geq \frac{1}{6}$, *we have* $\psi(r) \leq \frac{5}{6}$ *and then*

$$\psi(r)p(x,Tx) \leq p(x,y)$$

holds for all $x, y \in X$ *with* $x \neq y$. *Put* $r = \frac{5}{6}$ *and so* $\psi(r) = \frac{1}{6}$. *Consequently we have*

$$\begin{aligned} H_p(T0,T1) &= p(0,0) = 0 \leq rM_p(0,1), \\ H_p(T0,T2) &= p(0,1) = \frac{1}{4} < \frac{1}{3} = rp(0,2) \leq rM_p(0,2), \\ H_p(T1,T2) &= p(0,1) = \frac{1}{4} < \frac{11}{18} = rp(1,2) \leq rM_p(1,2) \end{aligned}$$

and similarly $H_p(Tx,ty) \leq rM_p(x,y)$ *also holds true for* $x = y$. *Hence, for all* $x, y \in X$, *we get*

$$\psi(r)p(x,Tx) \leq p(x,y) \Longrightarrow H_p(Tx,Ty) \leq rM_p(x,y).$$

Thus all the conditions of Theorem 2.1 are satisfied and $x = 0$ *is the only fixed point of* T. *On the other hand, the metric* p^S *induced by the partial metric* p *is given by*

$$\begin{aligned} p^S(0,0) &= p^S(1,1) = p^S(2,2) = 0, \\ p^S(0,1) &= p^S(1,0) = \frac{1}{2}, \\ p^S(1,2) &= p^S(2,1) = \frac{17}{15}, \\ p^S(0,2) &= p^S(2,0) = \frac{7}{15}. \end{aligned}$$

Now it is easy to show that Theorem 2.1 is not applicable in this case. Since

$$\psi(r)p^S(0,T0) = \psi(r)p^S(0,0) = 0 \leq p^S(0,y)$$

is satisfied for any $0 \leq r < 1$ *and* $y \in X$, *therefore for* $y = 2$ *we must have*

$$H_{p^S}(T0,T2) \leq rM_{p^S}(0,2).$$

Now, we get

$$H_{p^S}(T0,T2)=H_{p^S}(\{0\},\{0,1\})=\frac{1}{2}$$

and

$$\begin{aligned} M_{p^S}(0,2) &= \max\left\{p^S(0,2),p^S(0,T0),p^S(2,T2),\frac{p^S(0,T2)+p^S(2,T0)}{2}\right\} \\ &= \max\left\{\frac{7}{15},0,\frac{7}{15},\frac{0+\frac{7}{15}}{2}\right\}=\frac{7}{15}<\frac{1}{2}. \end{aligned}$$

Thus, for any $0\le r<1$ we have

$$H_{p^S}(T0,T2)\nleq rM_{p^S}(0,2).$$

3. An application

Generally, a dynamical process consists of a state space and a decision space. The state space is the set of the initial state, actions and transition model of the process; the decision space is the set of possible actions that are allowed for the process.

In this section we assume that U and V are Banach spaces, $W\subseteq U$ is a state space and $D\subseteq V$ is a decision space. It is well known that the dynamic programming provides useful tools for mathematical optimization and computer programming as well. In particular, the problem of dynamic programming related to multistage process reduces to the problem of solving the functional equation

$$q(x)=\sup_{y\in D}\{f(x,y)+G(x,y,q(\tau(x,y)))\},\qquad x\in W,$$

which further can be reformulated as

$$q(x)=\sup_{y\in D}\{g(x,y)+G(x,y,q(\tau(x,y)))\}-b,\qquad x\in W, \tag{14}$$

where $\tau:W\times D\to W$, $f,g:W\times D\to\mathbb{R}$, $G:W\times D\times\mathbb{R}\to\mathbb{R}$ and $b>0$. However, for the detailed background of the problem, the reader can refer to [5, 6, 7, 8, 26, 32]. Here, we study the existence and uniqueness of the bounded solution of the functional equation (14).

Let $B(W)$ denote the set of all bounded real-valued functions on W and, for an arbitrary $h\in B(W)$, define $\|h\|=\sup_{x\in W}|h(x)|$. Clearly, $(B(W),\|\cdot\|)$ endowed with the metric d defined by

$$d(h,k)=\sup_{x\in W}|h(x)-k(x)|$$

for all $h,k\in B(W)$, is a Banach space. Indeed, the convergence in the space $B(W)$ with respect to $\|\cdot\|$ is uniform. Thus, if we consider a Cauchy sequence $\{h_n\}$ in $B(W)$, then $\{h_n\}$ converges uniformly to a function, say h^*, that is bounded and so $h^*\in B(W)$.

Moreover, we can define a partial metric p_B by

$$p_B(h,k)=d(h,k)+b \tag{15}$$

for all $h,k\in B(W)$ and $b>0$.

We also define $T : B(W) \to B(W)$ by

$$T(h)(x) = \sup_{y \in D}\{g(x,y) + G(x,y,h(\tau(x,y)))\} - b, \tag{16}$$

for all $h \in B(W)$ and $x \in W$. Obviously, if the functions g and G are bounded then T is well-defined.
Finally, let

$$M_{p_B}(h,k) = \max\{p_B(h,k), p_B(h,T(h)), p_B(k,T(k)), \frac{p_B(h,T(k)) + p_B(k,T(h))}{2}\}.$$

We will prove the following theorem.

Theorem 3.1. *Assume that there exists* $0 \le r < 1$ *such that the following condition holds:*

$$|G(x,y,h(x)) - G(x,y,k(x))| \le rM_{p_B}(h,k)$$

where $x \in W$, $y \in D$, $T : B(W) \to B(W)$ *is given by* (16)*, the functions* $G : W \times D \times \mathbb{R} \to \mathbb{R}$ *and* $g : W \times D \to \mathbb{R}$ *are bounded. Then the functional equation* (14) *has a unique bounded solution.*

Proof. Since $(B(W), d)$ is complete and $p^S_B(h,k) = 2d(h,k)$ for all $h, k \in B(W)$ and $x \in W$, by Lemma 1.6 we deduce that $(B(W), p_B)$ is a complete partial metric space. Let λ be an arbitrary positive number, $x \in W$ and $h_1, h_2 \in B(W)$, then there exist $y_1, y_2 \in D$ such that

$$\begin{aligned}
T(h_1)(x) &< g(x,y_1) + G(x,y_1,h_1(\tau(x,y_1))) - b + \lambda, &(17)\\
T(h_2)(x) &< g(x,y_2) + G(x,y_2,h_2(\tau(x,y_2))) - b + \lambda, &(18)\\
T(h_1)(x) &\ge g(x,y_2) + G(x,y_2,h_1(\tau(x,y_2))), &(19)\\
T(h_2)(x) &\ge g(x,y_1) + G(x,y_1,h_2(\tau(x,y_1))). &(20)
\end{aligned}$$

Then from (17) and (20), it follows easily that

$$\begin{aligned}
T(h_1)(x) - T(h_2)(x) &< G(x,y_1,h_1(\tau(x,y_1))) - G(x,y_1,h_2(\tau(x,y_1))) - b + \lambda\\
&\le |G(x,y_1,h_1(\tau(x,y_1))) - G(x,y_1,h_2(\tau(x,y_1)))| - b + \lambda\\
&\le rM_{p_B}(h_1,h_2) - b + \lambda.
\end{aligned}$$

Hence we get

$$T(h_1)(x) - T(h_2)(x) < rM_{p_B}(h_1,h_2) - b + \lambda. \tag{21}$$

Similarly, from (18) and (19) we obtain

$$T(h_2)(x) - T(h_1)(x) < rM_{p_B}(h_1,h_2) - b + \lambda. \tag{22}$$

Therefore, from (21) and (22) we have

$$|T(h_1)(x) - T(h_2)(x)| < rM_{p_B}(h_1,h_2) - b + \lambda, \tag{23}$$

that is,

$$p_B(T(h_1), T(h_2)) < rM_{p_B}(h_1,h_2) + \lambda.$$

Since the above inequality does not depend on $x \in W$ and $\lambda > 0$ is taken arbitrary, then we conclude immediately that

$$p_B(T(h_1), T(h_2)) \le rM_{p_B}(h_1,h_2).$$

In particular, the last inequality holds for any $x \in W$ such that $\psi(r)p_B(h,T(h)) \le p_B(h,k)$ and so the Corollary 2.4 is applicable in this case. Consequently, the mapping T has a unique fixed point, that is, the functional equation (14) has a unique bounded solution. $\square$

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