

HOMOGENIZATION OF NONLINEAR STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS IN A GENERAL ERGODIC ENVIRONMENT

PAUL ANDRÉ RAZAFIMANDIMBY, MAMADOU SANGO, AND JEAN LOUIS WOUKENG

ABSTRACT. In this paper, we show that the concept of sigma-convergence associated to stochastic processes can tackle the homogenization of stochastic partial differential equations. In this regard, the homogenization of a stochastic nonlinear partial differential equation is addressed. Using some deep compactness results such as the Prokhorov and Skorokhod theorems, we prove that the sequence of solutions of this problem converges in probability towards the solution of an equation of the same type. To proceed with, we use the concept of *sigma-convergence for stochastic processes*, which takes into account both the deterministic and random behaviours of the solutions of the problem.

1. INTRODUCTION

Algebras with mean value have been highly efficient in deterministic homogenization theory. It is now a well known fact that given a partial differential equation (PDE) with oscillating coefficients, one can always, under some structural constraints on its coefficients, solve some homogenization problems related to this PDE.

Contrasted with deterministic homogenization, very few results are available as regards the homogenization of stochastic PDEs (SPDEs). We may cite [1, 14, 15, 25, 30, 31] in that context. In the just mentioned previous work, the homogenization of SPDEs is studied under the periodicity assumption on the coefficients of the equations considered. In addition, the convergence method used is either the G-convergence method [1, 14, 15] or the two-scale convergence method [30, 31]. Given the nature both random and deterministic of the solutions of these equations, it is more convenient to use an appropriate method taking into account both these two types of behaviour. As regards the SPDEs in a general ergodic environment, no results is available so far. The first attempt to generalize this to SPDEs beyond the periodic context is undertaken in [28] in which the authors consider the homogenization problem for a SPDE in an almost periodic setting. The present work is therefore the first one in which such a problem is considered.

To be more precise, we are concerned with the homogenization problem for the following non-linear stochastic partial differential equation

$$\begin{cases} du_\varepsilon = (\operatorname{div} a(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon}, u_\varepsilon, Du_\varepsilon) - a_0(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon}, u_\varepsilon)) dt + M(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, u_\varepsilon) dW & \text{in } Q_T \\ u_\varepsilon = 0 & \text{on } \partial Q \times (0, T) \\ u_\varepsilon(x, 0) = u^0(x) & \text{in } Q, \end{cases} \quad (1.1)$$

where $Q_T = Q \times (0, T)$, Q being a Lipschitz domain in \mathbb{R}^N with smooth boundary ∂Q , T is a positive real number and W is a cylindrical standard Wiener process defined on a given probability

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space $(\Omega, \mathcal{F}, \mathbb{P})$. Under a suitable assumption on the coefficients of (1.1) we prove that the sequence of solutions to (1.1) converges to the solution of an equation of the same type as (1.1). In view of the result obtained, one might be tempted to believe that the homogenization process of an SPDE is summarized in the homogenization of its deterministic part, added to the average of its stochastic part. This is not true in general. Indeed, one can obtain, passing to the limit, a homogenized equation of a type completely different from that of the initial problem; see e.g., [31].

The paper is presented as follows. In Section 2, we give some fundamentals of generalized Besicovitch spaces. Section 3 deals with the concept of sigma-convergence for stochastic processes. We state therein some compactness results that will be used in the sequel. In Section 4, we state the problem and prove some fundamental estimates. In Section 5 we collect some useful results necessary to the homogenization part, and we use them in Section 6 to study the homogenization of (1.1). We prove there the global homogenization result and we derive the homogenized problem. Section 7 deals with a corrector-type result. Finally in Section 8, we apply the result of Section 6 to some concrete physical situations.

Unless otherwise specified, vector spaces throughout are assumed to be real vector spaces, and scalar functions are assumed to take real values. We shall always assume that the numerical space \mathbb{R}^m (integer $m \geq 1$) and its open sets are each equipped with the Lebesgue measure $dx = dx_1 \dots dx_m$.

2. SOME PROPERTIES OF THE GENERALIZED BESICOVITCH SPACES

We begin this section by recalling some important properties of algebras with mean value [16, 8, 26, 35]. By an algebra with mean value (algebra wmv, in short) on \mathbb{R}^N we mean any closed subalgebra A of the C^* -algebra of bounded uniformly continuous functions $BUC(\mathbb{R}^N)$ which contains the constants, is translation invariant ($u(\cdot + a) \in A$ for any $u \in A$ and each $a \in \mathbb{R}^N$) and is such that each element possesses a mean value in the following sense:

(MV) For each $u \in A$, the sequence $(u^\varepsilon)_{\varepsilon > 0}$ (where $u^\varepsilon(x) = u(x/\varepsilon)$, $x \in \mathbb{R}^N$) weakly $*$ -converges in $L^\infty(\mathbb{R}^N)$ to some constant real-valued function $M(u)$ as $\varepsilon \rightarrow 0$.

It is known that A (endowed with the sup norm topology) is a commutative C^* -algebra with identity. We denote by $\Delta(A)$ the spectrum of A and by \mathcal{G} the Gelfand transformation on A . We recall that $\Delta(A)$ (a subset of the topological dual A' of A) is the set of all nonzero multiplicative linear functionals on A , and \mathcal{G} is the mapping of A into $\mathcal{C}(\Delta(A))$ such that $\mathcal{G}(u)(s) = \langle s, u \rangle$ ($s \in \Delta(A)$), where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between A' and A . We endow $\Delta(A)$ with the relative weak* topology on A' . Then using the well-known theorem of Stone (see e.g., [11, Theorem IV.6.18, p. 274]) one can easily show that the spectrum $\Delta(A)$ is a compact topological space, and the Gelfand transformation \mathcal{G} is an isometric isomorphism identifying A with $\mathcal{C}(\Delta(A))$ (the continuous functions on $\Delta(A)$) as C^* -algebras. Next, since each element of A possesses a mean value, this yields an application $u \mapsto M(u)$ (denoted by M and called the mean value) which is a nonnegative continuous linear functional on A with $M(1) = 1$, and so provides us with a linear nonnegative functional $\psi \mapsto M_1(\psi) = M(\mathcal{G}^{-1}(\psi))$ defined on $\mathcal{C}(\Delta(A)) = \mathcal{G}(A)$, which is clearly bounded. Therefore, by the Riesz-Markov theorem, $M_1(\psi)$ is representable by integration with respect to some Radon measure β (of total mass 1) in $\Delta(A)$, called the M -measure for A [19]. It is a fact that we have

$$M(u) = \int_{\Delta(A)} \mathcal{G}(u) d\beta \text{ for } u \in A.$$

Next, to any algebra with mean value A are associated the following subspaces: $A^m = \{\psi \in C^m(\mathbb{R}^N) : D_y^\alpha \psi \in A \text{ for every } \alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^N \text{ with } |\alpha| \leq m\}$ (where $D_y^\alpha \psi = \partial^{|\alpha|} \psi / \partial y_1^{\alpha_1} \dots$

$\partial y_N^{\alpha_N}$ and integer $m \geq 1$). Endowed with the norm $\|u\|_m = \sup_{|\alpha| \leq m} \|D_y^\alpha \psi\|_\infty$, A^m is a Banach space. We also define the space A^∞ as the space of $\psi \in C^\infty(\mathbb{R}^N)$ such that $D_y^\alpha \psi = \frac{\partial^{|\alpha|} \psi}{\partial y_1^{\alpha_1} \dots \partial y_N^{\alpha_N}} \in A$ for every $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^N$. Endowed with a suitable locally convex topology defined by the family of norms $\|\cdot\|_m$, A^∞ is a Fréchet space.

Now, the partial derivative of index i ($1 \leq i \leq N$) on $\Delta(A)$ is defined to be the mapping $\partial_i = \mathcal{G} \circ \partial / \partial y_i \circ \mathcal{G}^{-1}$ (usual composition) of $\mathcal{D}^1(\Delta(A)) = \{\varphi \in \mathcal{C}(\Delta(A)) : \mathcal{G}^{-1}(\varphi) \in A^1\}$ into $\mathcal{C}(\Delta(A))$. Higher order derivatives are defined analogously. At the present time, let $\mathcal{D}(\Delta(A)) = \{\varphi \in \mathcal{C}(\Delta(A)) : \mathcal{G}^{-1}(\varphi) \in A^\infty\}$. Endowed with a suitable locally convex topology $\mathcal{D}(\Delta(A))$ is a Fréchet space and further, \mathcal{G} viewed as defined on A^∞ is a topological isomorphism of A^∞ onto $\mathcal{D}(\Delta(A))$.

Analogously to the space $\mathcal{D}'(\mathbb{R}^N)$, we now define the space of distributions on $\Delta(A)$ to be the space of all continuous linear form on $\mathcal{D}(\Delta(A))$. We denote it by $\mathcal{D}'(\Delta(A))$ and we endow it with the strong dual topology. Since A^∞ is dense in A (see [33, Proposition 2.3]), it is easy to see that the space $L^p(\Delta(A))$ ($1 \leq p \leq \infty$) is a subspace of $\mathcal{D}'(\Delta(A))$ (with continuous embedding), so that one may define the Sobolev spaces on $\Delta(A)$ as follows.

$$W^{1,p}(\Delta(A)) = \{u \in L^p(\Delta(A)) : \partial_i u \in L^p(\Delta(A)) \ (1 \leq i \leq d)\} \ (1 \leq p < \infty)$$

where the derivative $\partial_i u$ is taken in the distribution sense on $\Delta(A)$. We equip $W^{1,p}(\Delta(A))$ with the norm

$$\|u\|_{W^{1,p}(\Delta(A))} = \left[\|u\|_{L^p(\Delta(A))}^p + \sum_{i=1}^N \|\partial_i u\|_{L^p(\Delta(A))}^p \right]^{\frac{1}{p}} \quad (u \in W^{1,p}(\Delta(A))),$$

$$1 \leq p < \infty,$$

which makes it a Banach space. To the above space is attached the space

$$W^{1,p}(\Delta(A))/\mathbb{R} = \{u \in W^{1,p}(\Delta(A)) : \int_{\Delta(A)} u d\beta = 0\}$$

equipped with the seminorm $u \mapsto (\sum_{i=1}^N \|\partial_i u\|_{L^p(\Delta(A))}^p)^{1/p}$, and its separated completion $W_{\#}^{1,p}(\Delta(A))$.

We will see in the sequel that $W_{\#}^{1,p}(\Delta(A))$ is in fact the completion of $W^{1,p}(\Delta(A))/\mathbb{R}$ since A will be taken to be an *ergodic* algebra; see the last part of this section.

The concept of a product algebra wmv will be useful in our study. Let A_y (resp. A_τ) be an algebra wmv on \mathbb{R}_y^N (resp. \mathbb{R}_τ). We define the product algebra wmv $A_y \odot A_\tau$ as the closure in $BUC(\mathbb{R}^{N+1})$ of the tensor product $A_y \otimes A_\tau = \{\sum_{\text{finite}} u_i \otimes v_i : u_i \in A_y \text{ and } v_i \in A_\tau\}$. This defines an algebra wmv on \mathbb{R}^{N+1} . A characterization of these products is given in the following result whose proof can be found in [20].

Theorem 1. *Let A_y, A_τ and A be as above. For $f \in BUC(\mathbb{R}_{y,\tau}^{N+1})$, we define $f_y \in BUC(\mathbb{R}_y^N)$ and $f^\tau \in BUC(\mathbb{R}_\tau^N)$ by*

$$f_y(\tau) = f^\tau(y) = f(y, \tau) \text{ for } (y, \tau) \in \mathbb{R}_y^N \times \mathbb{R}_\tau^N$$

and put

$$B_f = \{f^\tau : \tau \in \mathbb{R}\}, \quad C_f = \{f_y : y \in \mathbb{R}^N\}.$$

Then $B_f \subset A_y$ and $C_f \subset A_\tau$ for every $f \in A$. Also for $f \in A$ both B_f and C_f are relatively compact in A_y and in A_τ respectively (in the sup norm topology).

Let $AP(\mathbb{R}^N)$ denote the space of all Bohr almost periodic functions on \mathbb{R}^N [4, 5], that is the algebra of functions in $\mathcal{B}(\mathbb{R}^N)$ that are uniformly approximated by finite linear combinations of functions in the set $\{y \mapsto \cos(k \cdot y), y \mapsto \sin(k \cdot y) : k \in \mathbb{R}^N\}$. It is well-known that $AP(\mathbb{R}^N)$ is an algebra wmv on \mathbb{R}^N . As an example we have $AP(\mathbb{R}_y^N) \odot AP(\mathbb{R}_\tau) = AP(\mathbb{R}_y^N \times \mathbb{R}_\tau)$. We also have that $\mathcal{C}_{\text{per}}(Y) \odot \mathcal{C}_{\text{per}}(Z) = \mathcal{C}_{\text{per}}(Y \times Z)$ where $Y = (0, 1)^N$ and $Z = (0, 1)$. This follows from the identification $\mathcal{C}_{\text{per}}(Y) = \mathcal{C}(\mathbb{T}^N)$ where \mathbb{T}^N is the N -torus in \mathbb{R}^N . Similarly we have $\mathcal{C}_{\text{per}}(Z) \odot AP(\mathbb{R}_y^N) = \mathcal{C}_{\text{per}}(Z; AP(\mathbb{R}_y^N))$. Other examples of product algebras wmv can be given.

Next, let B_A^p ($1 \leq p < \infty$) denote the Besicovitch space associated to A , that is the closure of A with respect to the Besicovitch seminorm

$$\|u\|_p = \left(\limsup_{r \rightarrow +\infty} \frac{1}{|B_r|} \int_{B_r} |u(y)|^p dy \right)^{1/p}$$

where B_r is the open ball of \mathbb{R}^N of radius r . It is known that B_A^p is a complete seminormed vector space verifying $B_A^q \subset B_A^p$ for $1 \leq p \leq q < \infty$. From this last property one may naturally define the space B_A^∞ as follows:

$$B_A^\infty = \{f \in \cap_{1 \leq p < \infty} B_A^p : \sup_{1 \leq p < \infty} \|f\|_p < \infty\}.$$

We endow B_A^∞ with the seminorm $[f]_\infty = \sup_{1 \leq p < \infty} \|f\|_p$, which makes it a complete seminormed space. We recall that the spaces B_A^p ($1 \leq p \leq \infty$) are not in general Fréchet spaces since they are not separated in general. The following properties are worth noticing [20, 26]:

- (1) The Gelfand transformation $\mathcal{G} : A \rightarrow \mathcal{C}(\Delta(A))$ extends by continuity to a unique continuous linear mapping, still denoted by \mathcal{G} , of B_A^p into $L^p(\Delta(A))$, which in turn induces an isometric isomorphism \mathcal{G}_1 , of $B_A^p/\mathcal{N} = \mathcal{B}_A^p$ onto $L^p(\Delta(A))$ (where $\mathcal{N} = \{u \in B_A^p : \mathcal{G}(u) = 0\}$). Furthermore if $u \in B_A^p \cap L^\infty(\mathbb{R}^N)$ then $\mathcal{G}(u) \in L^\infty(\Delta(A))$ and $\|\mathcal{G}(u)\|_{L^\infty(\Delta(A))} \leq \|u\|_{L^\infty(\mathbb{R}^N)}$.
- (2) The mean value M viewed as defined on A , extends by continuity to a positive continuous linear form (still denoted by M) on B_A^p satisfying $M(u) = \int_{\Delta(A)} \mathcal{G}(u) d\beta$ ($u \in B_A^p$). Furthermore, $M(\tau_a u) = M(u)$ for each $u \in B_A^p$ and all $a \in \mathbb{R}^N$, where $\tau_a u(z) = u(z + a)$ for almost all $z \in \mathbb{R}^N$. Moreover for $u \in B_A^p$ we have $\|u\|_p = [M(|u|^p)]^{1/p}$.

In this work, we will deal with ergodic algebras (see [16, 35]). Let us recall the definition of an ergodic algebra.

Definition 1. An algebra wmv A on \mathbb{R}^N is *ergodic* if for every $u \in B_A^1$ such that $\|u - u(\cdot + a)\|_1 = 0$ for every $a \in \mathbb{R}^N$ we have $\|u - M(u)\|_1 = 0$.

The class of ergodic algebra plays a crucial role in homogenization theory as it will be seen in the following sections.

In order to simplify the text, we will henceforth use the same letter u (if there is no danger of confusion) to denote the equivalence class of an element $u \in B_A^p$. The symbol ϱ will denote the canonical mapping of B_A^p onto $\mathcal{B}_A^p = B_A^p/\mathcal{N}$. Our goal here is to define another space attached to \mathcal{B}_A^p . For that purpose, let us recall that the partial derivative of index $1 \leq i \leq N$ of a distribution $u \in \mathcal{D}'(\Delta(A))$, denoted by $\partial_i u$, is defined as follows:

$$\langle \partial_i u, \varphi \rangle = - \langle u, \partial_i \varphi \rangle \text{ for any } \varphi \in \mathcal{D}(\Delta(A)).$$

With this in mind, we define the formal derivative of index i , denoted by $\bar{\partial}/\partial y_i$, as follows:

$$\frac{\bar{\partial}}{\partial y_i} = \mathcal{G}_1^{-1} \circ \partial_i \circ \mathcal{G}_1.$$

Considered as defined from \mathcal{B}_A^p into itself, it is an unbounded operator with domain $\mathcal{D}_i = \{u \in \mathcal{B}_A^p : \bar{\partial}u/\partial y_i \in \mathcal{B}_A^p\}$. We set

$$\mathcal{B}_A^{1,p} = \cap_{1 \leq i \leq N} \mathcal{D}_i \equiv \left\{ u \in \mathcal{B}_A^p : \frac{\bar{\partial}u}{\partial y_i} \in \mathcal{B}_A^p \text{ for } 1 \leq i \leq N \right\}.$$

Since $\bar{\partial}/\partial y_i$ is closed, $\mathcal{B}_A^{1,p}$ is a Banach space under the norm

$$\|u\|_{\mathcal{B}_A^{1,p}} = \left[\|u\|_p^p + \sum_{i=1}^N \left\| \frac{\bar{\partial}u}{\partial y_i} \right\|_p^p \right]^{1/p} \quad (u \in \mathcal{B}_A^{1,p}).$$

Moreover, the restriction of \mathcal{G}_1 to $\mathcal{B}_A^{1,p}$ is an isometric isomorphism of $\mathcal{B}_A^{1,p}$ onto $W^{1,p}(\Delta(A))$. We assume for the remainder of this section that A is ergodic. Then according to Definition 1, the only elements of \mathcal{B}_A^1 that are $\|\cdot\|_1$ -invariant are constant functions. This infers that if $u \in \mathcal{B}_A^1$ satisfies $\bar{D}_y u = 0$, then u is constant. Indeed it can be shown that $u \in \mathcal{B}_A^1$ is $\|\cdot\|_1$ -invariant if and only if $\bar{D}_y u = 0$. So the mapping

$$\|\bar{D}_y \cdot\|_p : u \mapsto \|\bar{D}_y u\|_p := \left(\sum_{i=1}^N \left\| \frac{\bar{\partial}u}{\partial y_i} \right\|_p^p \right)^{1/p}$$

considered as defined on $\mathcal{B}_A^{1,p}$, is a norm on the subspace $\mathcal{B}_A^{1,p}/\mathbb{R}$ of $\mathcal{B}_A^{1,p}$ consisting of functions $u \in \mathcal{B}_A^{1,p}$ with $M(u) = 0$. Unfortunately, under this norm, $\mathcal{B}_A^{1,p}/\mathbb{R}$ is a normed vector space which is in general not complete. We denote by $\mathcal{B}_{\#A}^{1,p}$ the completion of $\mathcal{B}_A^{1,p}/\mathbb{R}$ with respect to that norm, and by J_1 the canonical embedding of $\mathcal{B}_A^{1,p}/\mathbb{R}$ into $\mathcal{B}_{\#A}^{1,p}$. By the theory of completion of uniform spaces [6, Chap. II], the mapping $\bar{\partial}/\partial y_i : \mathcal{B}_A^{1,p}/\mathbb{R} \rightarrow \mathcal{B}_A^p$ extends by continuity to a unique continuous linear mapping still denoted by $\bar{\partial}/\partial y_i : \mathcal{B}_{\#A}^{1,p} \rightarrow \mathcal{B}_A^p$ such that

$$\frac{\bar{\partial}}{\partial y_i} \circ J_1 = \frac{\bar{\partial}}{\partial y_i} \text{ and } \|u\|_{\mathcal{B}_{\#A}^{1,p}} = \|\bar{D}_y u\|_p \quad (u \in \mathcal{B}_{\#A}^{1,p}) \quad (2.1)$$

where $\bar{D}_y = (\bar{\partial}/\partial y_i)_{1 \leq i \leq N}$. Since \mathcal{G}_1 is an isometric isomorphism of $\mathcal{B}_A^{1,p}$ onto $W^{1,p}(\Delta(A))$ we have by the definition of $\bar{\partial}/\partial y_i$ that the restriction of \mathcal{G}_1 to $\mathcal{B}_A^{1,p}/\mathbb{R}$ sends isometrically and isomorphically $\mathcal{B}_A^{1,p}/\mathbb{R}$ onto $W^{1,p}(\Delta(A))/\mathbb{R}$. So by [6, Chap. II] there exists a unique isometric isomorphism $\bar{\mathcal{G}}_1 : \mathcal{B}_{\#A}^{1,p} \rightarrow W_{\#}^{1,p}(\Delta(A))$ such that

$$\bar{\mathcal{G}}_1 \circ J_1 = J \circ \mathcal{G}_1 \quad (2.2)$$

and

$$\partial_i \circ \bar{\mathcal{G}}_1 = \mathcal{G}_1 \circ \frac{\bar{\partial}}{\partial y_i} \quad (1 \leq i \leq N). \quad (2.3)$$

We recall that in this case (when A is ergodic), J is the canonical embedding of $W^{1,p}(\Delta(A))/\mathbb{R}$ into its completion $W_{\#}^{1,p}(\Delta(A))$ while J_1 is the canonical embedding of $\mathcal{B}_A^{1,p}/\mathbb{R}$ into $\mathcal{B}_{\#A}^{1,p}$. Furthermore, Since $\mathcal{B}_A^{1,p}/\mathbb{R}$ is dense in $\mathcal{B}_{\#A}^{1,p}$ (in fact by the embedding J_1 , $\mathcal{B}_A^{1,p}/\mathbb{R}$ is viewed as a subspace of $\mathcal{B}_{\#A}^{1,p}$,

and by the theory of completion, $J_1(\mathcal{B}_A^{1,p}/\mathbb{R})$ is dense in $\mathcal{B}_{\#A}^{1,p}$, it follows that, as A^∞ is dense in A , $\varrho(A^\infty/\mathbb{R})$ is dense in $\mathcal{B}_{\#A}^{1,p}$, where $A^\infty/\mathbb{R} = \{u \in A^\infty : M(u) = 0\}$.

Remark 1. For $u \in B_A^{1,p}$ (that is the space of $u \in B_A^p$ such that $D_y u \in (B_A^p)^N$) we have

$$\mathcal{G}_1 \left(\varrho \left(\frac{\partial u}{\partial y_i} \right) \right) = \mathcal{G} \left(\frac{\partial u}{\partial y_i} \right) = \partial_i \mathcal{G}(u) = \partial_i \mathcal{G}_1(\varrho(u)) = (\text{by definition}) \mathcal{G}_1 \left(\frac{\bar{\partial}}{\partial y_i}(\varrho(u)) \right),$$

hence

$$\varrho \left(\frac{\partial u}{\partial y_i} \right) = \frac{\bar{\partial}}{\partial y_i}(\varrho(u)),$$

or equivalently,

$$\varrho \circ \frac{\partial}{\partial y_i} = \frac{\bar{\partial}}{\partial y_i} \circ \varrho \text{ on } B_A^{1,p}. \quad (2.4)$$

Remark 2. The above remark shows that $\bar{\partial}/\partial y_i$, viewed as defined on \mathcal{B}_A^p , is in fact the infinitesimal generator of the group of transformations $T(y)$ defined on \mathcal{B}_A^p by

$$T(y)(u + \mathcal{N}) = u(\cdot + y) + \mathcal{N}.$$

This shows that all the above results can be obtained through the theory of strongly continuous groups as shown in [27] (see also [28]).

3. THE Σ -CONVERGENCE METHOD FOR STOCHASTIC PROCESSES

In this section we define an appropriate notion of the concept of Σ -convergence adapted to our situation. It is to be noted that it is built according to the original notion introduced by Nguetseng [19]. Here we adapt it to systems involving random behavior. In all that follows, Q is an open subset of \mathbb{R}^N (integer $N \geq 1$), T is a positive real number and $Q_T = Q \times (0, T)$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. The expectation on $(\Omega, \mathcal{F}, \mathbb{P})$ will throughout be denoted by \mathbb{E} . Let us first recall the definition of the Banach space of bounded \mathcal{F} -measurable functions. Denoting by $F(\Omega)$ the Banach space of all bounded functions $f : \Omega \rightarrow \mathbb{R}$ (with the sup norm), we define $B(\Omega)$ as the closure in $F(\Omega)$ of the vector space $H(\Omega)$ consisting of all finite linear combinations of the characteristic functions 1_X of sets $X \in \mathcal{F}$. Since \mathcal{F} is an σ -algebra, $B(\Omega)$ is the Banach space of all bounded \mathcal{F} -measurable functions. Likewise we define the space $B(\Omega; Z)$ of all bounded (\mathcal{F}, B_Z) -measurable functions $f : \Omega \rightarrow Z$, where Z is a Banach space endowed with the σ -algebra of Borelians B_Z . The tensor product $B(\Omega) \otimes Z$ is a dense subspace of $B(\Omega; Z)$: this follows from the obvious fact that $B(\Omega)$ can be viewed as a space of continuous functions over the *gamma-compactification* [36] of the measurable space (Ω, \mathcal{F}) , which is a compact topological space. Next, for X a Banach space, we denote by $L^p(\Omega, \mathcal{F}, \mathbb{P}; X)$ the space of X -valued random variables u such that $\|u\|_X$ is $L^p(\Omega, \mathcal{F}, \mathbb{P})$ -integrable.

This being so, let A_y and A_τ be two algebras wmv on \mathbb{R}_y^N and \mathbb{R}_τ respectively, and let $A = A_y \odot A_\tau$ be their product as defined in the preceding section. We know that A is the closure in $BUC(\mathbb{R}_{y,\tau}^{N+1})$ of the tensor product $A_y \otimes A_\tau$. We denote by $\Delta(A_y)$ (resp. $\Delta(A_\tau)$, $\Delta(A)$) the spectrum of A_y (resp. A_τ , A). The same letter \mathcal{G} will denote the Gelfand transformation on A_y , A_τ and A , as well. Points in $\Delta(A_y)$ (resp. $\Delta(A_\tau)$) are denoted by s (resp. s_0). The M -measure on the compact space $\Delta(A_y)$ (resp. $\Delta(A_\tau)$) is denoted by β_y (resp. β_τ). We have $\Delta(A) = \Delta(A_y) \times \Delta(A_\tau)$ (Cartesian product) and the M -measure on $\Delta(A)$ is precisely the product measure $\beta = \beta_y \otimes \beta_\tau$; the last equality follows in an obvious way by the density of $A_y \otimes A_\tau$ in A and by the Fubini's theorem. Points in Ω are as usual denoted by ω .

Unless otherwise stated, random variables will always be considered on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Finally, the letter E will throughout denote exclusively an ordinary sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ with $0 < \varepsilon_n \leq 1$ and $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$. In what follows, the notations are those of the preceding section.

Definition 2. A sequence of random variables $(u_\varepsilon)_{\varepsilon > 0} \subset L^p(\Omega, \mathcal{F}, \mathbb{P}; L^p(Q_T))$ ($1 \leq p < \infty$) is said to *weakly Σ -converge* in $L^p(Q_T \times \Omega)$ to some random variable $u_0 \in L^p(\Omega, \mathcal{F}, \mathbb{P}; L^p(Q_T; \mathcal{B}_A^p))$ if as $\varepsilon \rightarrow 0$, we have

$$\begin{aligned} & \int_{Q_T \times \Omega} u_\varepsilon(x, t, \omega) f\left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon}, \omega\right) dx dt d\mathbb{P} \\ & \rightarrow \iint_{Q_T \times \Omega \times \Delta(A)} \widehat{u}_0(x, t, s, s_0, \omega) \widehat{f}(x, t, s, s_0, \omega) dx dt d\mathbb{P} d\beta \end{aligned} \quad (3.1)$$

for every $f \in L^{p'}(\Omega, \mathcal{F}, \mathbb{P}; L^{p'}(Q_T; A))$ ($1/p' = 1 - 1/p$), where $\widehat{u}_0 = \mathcal{G}_1 \circ u_0$ and $\widehat{f} = \mathcal{G}_1 \circ (\varrho \circ f) = \mathcal{G} \circ f$. We express this by writing $u_\varepsilon \rightarrow u_0$ in $L^p(Q_T \times \Omega)$ -weak Σ .

Remark 3. The above weak Σ -convergence in $L^p(Q_T \times \Omega)$ implies the weak convergence in $L^p(Q_T \times \Omega)$. One can show as in the usual setting of Σ -convergence method [19] that each $f \in L^p(\Omega, \mathcal{F}, \mathbb{P}; L^p(Q_T; A))$ weakly Σ -converges to $\varrho \circ f$.

In order to simplify the notation, we will henceforth denote $L^p(\Omega, \mathcal{F}, \mathbb{P}; X)$ merely by $L^p(\Omega; X)$ if it is understood from the context and there is no danger of confusion. Definition 2 can be formally motivated by the following fact. Assume $p = 2$; then using the chaos decomposition (see [7, 32]) of u_ε and f we get $u_\varepsilon(x, t, \omega) = \sum_{j=1}^{\infty} u_{\varepsilon,j}(x, t) \Phi_j(\omega)$ and $f(x, t, y, \tau, \omega) = \sum_{k=1}^{\infty} f_k(x, t, y, \tau) \Phi_k(\omega)$ where $u_{\varepsilon,j} \in L^2(Q_T)$ and $f_k \in L^2(Q_T; A)$, so that

$$\int_{Q_T \times \Omega} u_\varepsilon(x, t, \omega) f\left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon}, \omega\right) dx dt d\mathbb{P}$$

can be formally written as

$$\sum_{j,k} \int_{\Omega} \Phi_j(\omega) \Phi_k(\omega) d\mathbb{P} \int_{Q_T} u_{\varepsilon,j}(x, t) f_k\left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right) dx dt,$$

and by the usual Σ -convergence method (see [26, 19]), as $\varepsilon \rightarrow 0$,

$$\int_{Q_T} u_{\varepsilon,j}(x, t) f_k\left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right) dx dt \rightarrow \iint_{Q_T \times \Delta(A)} \widehat{u}_{0,j}(x, t, s, s_0) \widehat{f}_k(x, t, s, s_0) dx dt d\beta.$$

Hence, by setting

$$\widehat{u}_0(x, t, s, s_0, \omega) = \sum_{j=1}^{\infty} \widehat{u}_{0,j}(x, t, s, s_0) \Phi_j(\omega); \quad \widehat{f}(x, t, s, s_0, \omega) = \sum_{k=1}^{\infty} \widehat{f}_k(x, t, s, s_0) \Phi_k(\omega)$$

we get (3.1). We can also see that (3.1) is a straight generalization of the usual concept of Σ -convergence.

The following result holds.

Theorem 2. Let $1 < p < \infty$. Let $(u_\varepsilon)_{\varepsilon \in E} \subset L^p(\Omega; L^p(Q_T))$ be a sequence of random variables verifying the following boundedness condition:

$$\sup_{\varepsilon \in E} \mathbb{E} \|u_\varepsilon\|_{L^p(Q_T)}^p < \infty.$$

Then there exists a subsequence E' from E such that the sequence $(u_\varepsilon)_{\varepsilon \in E'}$ is weakly Σ -convergent in $L^p(Q_T \times \Omega)$.

Proof. Let us set $Y = L^{p'}(Q_T \times \Omega \times \Delta(A))$ and $X = L^{p'}(\Omega; L^{p'}(Q_T; \mathcal{C}(\Delta(A)))) = \mathcal{G}(L^{p'}(\Omega; L^{p'}(Q_T; A)))$. Applying [20, Theorem 3.1] with Y and X we are led at once to the result. ■

The following result will be very useful in the homogenization process.

Theorem 3. *Let $1 < p < \infty$. Let $A = A_y \odot A_\tau$ be an algebra wmv on $\mathbb{R}_y^N \times \mathbb{R}_\tau$ with the further property that A_y is ergodic. Finally let $(u_\varepsilon)_{\varepsilon \in E} \subset L^p(\Omega; L^p(0, T; W_0^{1,p}(Q)))$ be a sequence of random variables which satisfies the following estimate:*

$$\sup_{\varepsilon \in E} \mathbb{E} \|u_\varepsilon\|_{L^p(0, T; W_0^{1,p}(Q))}^p < \infty.$$

Then there exist a subsequence E' of E and a couple of random variables (u_0, u_1) with $u_0 \in L^p(\Omega; L^p(0, T; W_0^{1,p}(Q)))$ and $u_1 \in L^p(\Omega; L^p(Q_T; \mathcal{B}_{A_\tau}^p(\mathbb{R}_\tau; \mathcal{B}_{\#A_y}^{1,p})))$ such that, as $E' \ni \varepsilon \rightarrow 0$,

$$u_\varepsilon \rightarrow u_0 \text{ in } L^p(Q_T \times \Omega)\text{-weak}; \quad (3.2)$$

$$\frac{\partial u_\varepsilon}{\partial x_i} \rightarrow \frac{\partial u_0}{\partial x_i} + \frac{\bar{\partial} u_1}{\partial y_i} \text{ in } L^p(Q_T \times \Omega)\text{-weak } \Sigma, 1 \leq i \leq N. \quad (3.3)$$

Proof. The proof of the above theorem follows exactly the same lines of reasoning as the one of [26, Theorem 3.6]. ■

In practice, we will mostly deal with the following modified version of the above theorem.

Theorem 4. *Assume that the hypotheses of Theorem 3 are satisfied. Assume further that $p \geq 2$ and that there exist a subsequence E' from E and a random variable $u_0 \in L^p(\Omega; L^p(0, T; W_0^{1,p}(Q)))$ such that, as $E' \ni \varepsilon \rightarrow 0$,*

$$u_\varepsilon \rightarrow u_0 \text{ in } L^2(Q_T \times \Omega). \quad (3.4)$$

Then there exist a subsequence of E' (not relabeled) and a $\mathcal{B}_{A_\tau}^p(\mathbb{R}_\tau; \mathcal{B}_{\#A_y}^{1,p})$ -valued stochastic process $u_1 \in L^p(\Omega; L^p(Q_T; \mathcal{B}_{A_\tau}^p(\mathbb{R}_\tau; \mathcal{B}_{\#A_y}^{1,p})))$ such that (3.3) holds when $E' \ni \varepsilon \rightarrow 0$.

Proof. Since $(u_\varepsilon)_{\varepsilon \in E'}$ is bounded in $L^p(\Omega; L^p(0, T; W_0^{1,p}(Q)))$, there exist a subsequence of E' not relabeled and $v_0 \in L^p(\Omega; L^p(0, T; W_0^{1,p}(Q)))$ such that $u_\varepsilon \rightarrow v_0$ in $L^p(Q_T \times \Omega)$ -weak (and hence in $L^2(Q_T \times \Omega)$ -weak since $p \geq 2$) as $E' \ni \varepsilon \rightarrow 0$. From (3.4) and owing to the uniqueness of the weak-limit, we infer that $u_0 = v_0$, so that (3.2) holds true with u_0 as in (3.4). The remainder of the proof follows exactly the same lines of reasoning as in the proof of [26, Theorem 3.6]. ■

4. STATEMENT OF THE PROBLEM: A PRIORI ESTIMATES AND TIGHTNESS PROPERTY

4.1. Problem setting. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which is defined an infinite sequence of independent standard 1-d Brownian motion $(W_k)_{k \geq 1}$. We equip the probability space by the natural filtration, denoted by \mathcal{F}^t , of W_k . Now let \mathcal{U} be a fixed Hilbert space with orthonormal basis $\{e_k : k \geq 1\}$. We may define a cylindrical Wiener process W by setting $W = \sum_{k=1}^{\infty} W_k e_k$ (see [9]). By $L_2(\mathcal{U}, X)$ we denote the space of Hilbert-Schmidt operators from \mathcal{U} to the Hilbert space X :

$$L_2(\mathcal{U}, X) = \left\{ R \in L(\mathcal{U}, X) : \sum_{k=1}^{\infty} |Re_k|_X^2 < \infty \right\}.$$

We can define another Hilbert space $\mathcal{U}_0 \subset \mathcal{U}$ by setting

$$\mathcal{U}_0 = \left\{ v = \sum_{k=1}^{\infty} \alpha_k e_k : \sum_{k=1}^{\infty} \alpha_k^2 k^{-2} < \infty \right\}.$$

Note that the embedding $\mathcal{U}_0 \subset \mathcal{U}$ is Hilbert-Schmidt. We endow \mathcal{U}_0 with the norm $|v|_{\mathcal{U}_0}^2 = \sum_{k=1}^{\infty} \alpha_k^2 k^{-2}$. It is a well known fact that there exists $\Omega' \in \mathcal{F}$ with $\mathbb{P}(\Omega') = 1$ such that $W(\omega) \in \mathcal{C}(0, T; \mathcal{U}_0)$ for any $\omega \in \Omega'$ (see, for example, [9]).

For any given $G \in L^2(\Omega; L^2(0, T; L_2(\mathcal{U}, X)))$ such that $G(t)$ is \mathcal{F}^t -adapted we may define the stochastic integral

$$\int_0^t G dW = \sum_{k=1}^{\infty} \int_0^t G e_k dW_k,$$

as an element of the space of X -valued square integrable martingale. Moreover we have

$$\mathbb{E} \sup_{t \in [0, T]} \left| \int_0^t G dW \right|^r \leq C \mathbb{E} \left(\int_0^T |G|_{L_2(\mathcal{U}, X)}^2 \right)^{\frac{r}{2}},$$

for any $r \geq 1$. For the two results and more details on stochastic calculus in infinite dimension we refer to [9]. From now we will set $|G|_{L_2} = |G|_{L_2(\mathcal{U}, X)}$ for any Hilbert space X and for any $G \in L_2(\mathcal{U}, X)$.

Let $Q \subset \mathbb{R}^N$ be an open and bounded domain with smooth boundary. Throughout we will set $H = L^2(Q)$, $V = W_0^{1,p}(Q)$ and denote by $|u|, u \in H$, $\|v\|, v \in V$ their respective norms. We will also denote by $|\nu|, \nu \in \mathbb{R}^N$ the Euclidian norm on \mathbb{R}^N . The symbol V' will denote the dual of V and $\langle u, v \rangle$ denotes the duality pairing between $u \in V'$ and $v \in V$. The inner product in H is denoted by (u, v) for any $u, v \in H$. In this work we are interested in the asymptotic behaviour as $\varepsilon \rightarrow 0$ of the solution of (1.1) which is defined on the stochastic system $(\Omega, \mathcal{F}, \mathbb{P}), \mathcal{F}^t, W$. We assume that all the coefficients in (1.1) are measurable with respect to each of their arguments. Furthermore, for a.e $(x, t) \in Q \times (0, T)$, $(y, \tau) \in \mathbb{R}^N \times \mathbb{R}$ and for all $\mu \in \mathbb{R}$ and $\lambda \in \mathbb{R}^N$, we assume that

- A1.** $a(x, t, y, \tau, \mu, 0) = 0$,
- A2.** $(a(x, t, y, \tau, \mu, \lambda) - a(x, t, y, \tau, \mu, \lambda')) \cdot (\lambda - \lambda') \geq c_1 |\lambda - \lambda'|^p$,
- A3.** $|a(x, t, y, \tau, \mu, \lambda)| \leq c_2(1 + |\mu|^{p-1} + |\lambda|^{p-1})$,
- A4.** $|a_0(x, t, y, \tau, \mu)| \leq c_3(1 + |\mu|)$,
- A5.** $|a_0(x, t, y, \tau, \mu) - a_0(x, t, y, \tau, \mu')| \leq c_4 |\mu - \mu'|$,
- A6. (a)** $|a_0(x, t, y, \tau, u) - a_0(x', t', y, \tau, u')| \leq m(|x - x'| + |t - t'| + |u - u'|)(1 + |u| + |u'|)$,
- A6. (b)** $|a(x, t, y, \tau, u, \mathbf{v}) - a(x', t', y, \tau, u', \mathbf{v}')| \leq m(|x - x'| + |t - t'| + |u - u'|^{p-1} + |u'|^{p-1} + |\mathbf{v}|^{p-1} + |\mathbf{v}'|^{p-1}) + C(1 + |u| + |u'| + |\mathbf{v}| + |\mathbf{v}'|)^{p-2} |\mathbf{v} - \mathbf{v}'|$,

where m is a continuity modulus (i.e., a nondecreasing continuous function on $[0, +\infty)$ such that $m(0) = 0, m(r) > 0$ if $r > 0$, and $m(r) = 1$ if $r > 1$).

As far as the operator M is concerned, we will suppose that

A7. it is a measurable mapping from $\mathbb{R}^N \times \mathbb{R} \times H$ into $L_2(\mathcal{U}, H)$ such that

- (a) $|M(y, \tau, u) - M(y, \tau, v)|_{L_2} \leq c_6 |u - v|$,
- (b) $|M(y, \tau, u)|_{L_2} \leq c_7(1 + |u|)$.

We note that an example of nontrivial functions a, a_0 and M satisfying **A1.-A7.** are the functions $a(x, t, y, \tau, \mu, \lambda) = g(x, t, y, \tau) |\lambda|^{p-2} \lambda$, $a_0(x, t, y, \tau, \mu) = g_0(x, t, y, \tau) h(\mu)$, $M(y, \tau, u) = (M_k(y, \tau, u))_{k \geq 1}$ with $M_k(y, \tau, u) = g_1(y, \tau) \lambda_k u \geq 0$, where $\sum_{k=1}^{\infty} |\lambda_k|^2 < \infty$, $g, g_0 \in \mathcal{C}(\overline{Q}_T; \mathcal{B}(\mathbb{R}_{y, \tau}^{N+1}))$, $g_1 \in \mathcal{B}(\mathbb{R}_{y, \tau}^{N+1})$ and h is a continuous Lipschitz function on \mathbb{R} . We recall that $\mathcal{B}(\mathbb{R}_{y, \tau}^{N+1})$ is the space of bounded continuous real-valued functions defined on $\mathbb{R}_{y, \tau}^{N+1}$.

Note that **A1.-A3.** imply that $A(x, t, y, \tau, u, Du) \equiv -\operatorname{div} a(x, t, y, \tau, u, Du)$ satisfies

C1.

$$\begin{aligned} & \langle A(x, t, y, \tau, u, Du) - A(x, t, y, \tau, v, Dv), u - v \rangle \\ & \geq \int_Q (a(x, t, y, \tau, u, Du) - a(x, t, y, \tau, v, Dv)) \cdot (u - v) dx, \end{aligned}$$

C2. $\langle A(x, t, y, \tau, u, Du), u \rangle \geq c_1 |Du|^p$,

C3. $\|A(x, t, y, \tau, u, Du)\|_{W^{-1, p'}(Q)}^{p'} \leq c'_2(1 + |Du|^p)$ for some positive constant c'_2 depending only on Q_T and on c_2 ,

C4. the mapping $\theta \rightarrow \langle A(x, t, y, \tau, u + \theta v, D(u + \theta v)), w \rangle : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function for any $u, v, w \in V$.

To simplify the notations we will set throughout

$$\begin{aligned} a^\varepsilon(\cdot, u_\varepsilon, Du_\varepsilon)(x, t) &= a\left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon}, u_\varepsilon, Du_\varepsilon\right), \\ a_0^\varepsilon(\cdot, u_\varepsilon)(x, t) &= a_0\left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon}, u_\varepsilon\right), \\ M^\varepsilon(\cdot, u_\varepsilon)(x, t) &= M\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, u_\varepsilon\right), \end{aligned}$$

and

$$A^\varepsilon(\cdot, u_\varepsilon, Du_\varepsilon)(x, t) = A\left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon}, u_\varepsilon, Du_\varepsilon\right).$$

It is to be noted that the just defined functions make sense as trace functions; see e.g. [26, 34] for the justification. By a strong probabilistic solution of (1.1) we mean an \mathcal{F}^t -adapted stochastic process u_ε such that:

$$u_\varepsilon \in L^p(\Omega, \mathcal{F}, \mathbb{P}; L^p(0, T; V)) \cap L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathcal{C}(0, T; H)),$$

and for all $\phi \in V$ and for almost every $(\omega, t) \in \Omega \times [0, T]$ the following holds true

$$\begin{aligned} (u_\varepsilon(t), \phi) + \int_0^t (a^\varepsilon(\cdot, u_\varepsilon(s), Du_\varepsilon(s)), D\phi) ds &= (u^0, \phi) - \int_0^t (a_0^\varepsilon(\cdot, u_\varepsilon(s)), \phi) ds \\ &+ \sum_{k=1}^{\infty} \int_0^t (M_k^\varepsilon(\cdot, u_\varepsilon(s)), \phi) dW_k, \end{aligned}$$

where $M_k^\varepsilon(\cdot, u_\varepsilon(s)) = M^\varepsilon(\cdot, u_\varepsilon(s))e_k$. Under the above conditions, it is easily seen that if u_ε and v_ε are two solutions to (1.1) on the same stochastic system $(\Omega, \mathcal{F}, \mathbb{P}), \mathcal{F}^t, W$ with the same initial condition u^0 , then $u_\varepsilon(t) = v_\varepsilon(t)$ in H almost surely for any t . Thanks to this fact together with the Yamada-Watanabe's Theorem (see [24]) and the existence result of martingale solutions in [2], we see that (1.1) has a unique strong probabilistic solution.

4.2. The a priori estimates. Throughout C will denote a generic constant independent of ε . We have the following result whose proof can be obtained in a standard way.

Lemma 1. *The solution u_ε of (1.1) satisfies the following inequalities*

$$\mathbb{E} \sup_{0 \leq t \leq T} |u_\varepsilon(t)|^4 \leq C, \quad (4.1)$$

$$\mathbb{E} \int_0^T |Du_\varepsilon(t)|^p dt \leq C. \quad (4.2)$$

The following result is very crucial for the proof of the tightness property of u_ε .

Lemma 2. *There exists a constant $C > 0$ such that*

$$\mathbb{E} \sup_{|\theta| \leq \delta} \int_0^T |u_\varepsilon(t + \theta) - u_\varepsilon(t)|_{V'}^{p'} dt \leq C \delta^{\frac{1}{p-1}},$$

for any ε , and $\delta \in (0, 1)$. Here we assume that $u_\varepsilon(t)$ has zero extension outside the interval $[0, T]$.

Proof. Let us assume $\theta \geq 0$ (as we will see in what follows the same argument will apply for $\theta < 0$). We will denote by p' the Hölder conjugate of p (i.e., $\frac{1}{p} + \frac{1}{p'} = 1$). It is clear that

$$\begin{aligned} |u_\varepsilon(t + \theta) - u_\varepsilon(t)|_{V'}^{p'} &\leq C \left| \int_t^{t+\theta} A^\varepsilon(\cdot, u_\varepsilon(s), Du_\varepsilon(s)) ds \right|_{V'}^{p'} + C \left| \int_t^{t+\theta} a_0^\varepsilon(\cdot, u_\varepsilon(s)) ds \right|_{V'}^{p'} \\ &\quad + C \left| \int_0^t M^\varepsilon(\cdot, u_\varepsilon(s)) dW \right|_{V'}^{p'}. \\ &\leq I_1(t, \theta) + I_2(t, \theta) + I_3(t, \theta). \end{aligned} \quad (4.3)$$

It is not difficult to show that

$$I_1(t, \theta) \leq \theta^{\frac{p'}{p}} \int_t^{t+\theta} |A^\varepsilon(\cdot, u_\varepsilon(s), Du_\varepsilon(s))|^{p'} ds.$$

Therefore

$$\mathbb{E} \sup_{\theta \leq \delta} \int_0^T I_1(t, \theta) dt \leq C \delta^{\frac{p'}{p}} \mathbb{E} \int_0^T \int_t^{t+\delta} |Du_\varepsilon|^p ds.$$

Thanks to (4.2) we have that

$$\mathbb{E} \sup_{\theta \leq \delta} \int_0^T I_1(t, \theta) dt \leq C \delta^{\frac{p'}{p}}. \quad (4.4)$$

Thanks to **A4.**, we have that

$$I_2(t, \theta) \leq C \left(\int_t^{t+\theta} (1 + |u_\varepsilon(s)|) ds \right)^{p'},$$

which implies that

$$\mathbb{E} \sup_{\theta \leq \delta} \int_0^T I_2(t, \theta) dt \leq \int_0^T \mathbb{E} \left(\delta + \int_t^{t+\theta} |u_\varepsilon(s)| ds \right)^{p'} dt.$$

We invoke from this that

$$\mathbb{E} \sup_{\theta \leq \delta} \int_0^T I_2(t, \theta) dt \leq C \left(\int_0^T \mathbb{E} \left(\delta + \int_t^{t+\delta} |u_\varepsilon(s)| ds \right)^2 dt \right)^{\frac{p'}{2}}.$$

Thanks to (4.1) we deduce from this that

$$\mathbb{E} \sup_{\theta \leq \delta} \int_0^T I_2(t, \theta) dt \leq C \delta^{p'}. \quad (4.5)$$

Next, by using Burkholder-Davis-Gundy's inequality we see that

$$\begin{aligned} \mathbb{E} \sup_{0 \leq \theta \leq \delta} \int_0^T I_3(t, \theta) dt &\leq C \int_0^T \mathbb{E} \left(\int_t^{t+\delta} |M^\varepsilon(\cdot, u_\varepsilon(s))|^2 ds \right)^{\frac{p'}{2}} dt \\ &\leq C \left[\int_0^T \mathbb{E} \left(\int_t^{t+\delta} |M^\varepsilon(\cdot, u_\varepsilon(s))|^2 ds \right)^2 dt \right]^{\frac{p'}{4}} \end{aligned}$$

By condition **A7.**,

$$\mathbb{E} \sup_{0 \leq \theta \leq \delta} \int_0^T I_3(t, \theta) dt \leq C \left[\int_0^T \delta^2 + \mathbb{E} \sup_{s \in [0, T]} |u_\varepsilon(s)|^4 dt \right]^{\frac{p'}{4}},$$

which clearly implies that

$$\mathbb{E} \sup_{0 \leq \theta \leq \delta} \int_0^T I_3(t, \theta) dt \leq C \delta^{\frac{p'}{2}}. \quad (4.6)$$

Combining (4.4), (4.5) and (4.6) we infer from (4.3) that

$$\mathbb{E} \sup_{0 \leq \theta \leq \delta} \int_0^T |u_\varepsilon(t + \theta) - u_\varepsilon(t)|^{p'} dt \leq C \delta^{\frac{p'}{p}},$$

since $\frac{p'}{p} \leq 1$. A same inequality holds for $\theta < 0$. This ends the proof of the lemma. \blacksquare

4.3. Tightness property. To prove the tightness of the law of (u_ε, W) we will mainly follow the idea in [2] and in [10]. Let us consider the mappings:

$$\begin{aligned} \psi_1^\varepsilon : \omega \in \Omega &\mapsto u_\varepsilon(\omega) \in L^p(0, T; H) \\ \psi_2^\varepsilon : \omega \in \Omega &\mapsto W(\omega) \in \mathcal{C}(0, T; \mathcal{U}_0). \end{aligned}$$

We denote by $\mathfrak{S}_1 = L^p(0, T, H)$ (resp., $\mathfrak{S}_2 = \mathcal{C}(0, T; \mathcal{U}_0)$) and $\mathcal{B}(\mathfrak{S}_1)$ (resp., $\mathcal{B}(\mathfrak{S}_2)$) its Borel σ -algebra. The mappings

$$\begin{aligned} \Pi_1^\varepsilon(A) &= \mathbb{P} \circ \psi_1^\varepsilon(A) \equiv \mathbb{P}((\psi_1^\varepsilon)^{-1}(A)), \quad A \in \mathcal{B}(\mathfrak{S}_1), \\ \Pi_2^\varepsilon(A) &= \mathbb{P} \circ \psi_2^\varepsilon(A) \equiv \mathbb{P}((\psi_2^\varepsilon)^{-1}(A)), \quad A \in \mathcal{B}(\mathfrak{S}_2), \end{aligned}$$

and

$$\Pi^\varepsilon = \Pi_1^\varepsilon \times \Pi_2^\varepsilon$$

define families of probability measures on $(\mathfrak{S}_1, \mathcal{B}(\mathfrak{S}_1))$, $(\mathfrak{S}_2, \mathcal{B}(\mathfrak{S}_2))$ and $(\mathfrak{S} = \mathfrak{S}_1 \times \mathfrak{S}_2, \mathcal{B}(\mathfrak{S}_1 \times \mathfrak{S}_2))$, respectively.

Lemma 3. *Let μ_n, ν_n be sequences of positive numbers such that $\mu_n, \nu_n \rightarrow 0$ as $n \rightarrow \infty$. The set*

$$Z = \left\{ z : \int_0^T \|Dz\|^p dt \leq L, |z(t)|^2 \leq K \text{ a.e. } t, \sup_{|\theta| \leq \mu_n} \int_0^T |z(t + \theta) - z(t)|_{V'}^{p'} \leq \nu_n M \right\}$$

is a compact subset of $L^p(0, T; H)$.

Proof. The proof is the same as in [2, Proposition 3.1]. \blacksquare

The following result is of great importance for the rest of the work.

Lemma 4. *The family Π^ε is tight on \mathfrak{S} .*

Proof. Let $\delta > 0$ and let $L_\delta, K_\delta, M_\delta$ positive constants depending only on δ to be fixed later. It follows from Lemma 3 that

$$Z_\delta = \left\{ z : \int_0^T |Dz|^p dt \leq L_\delta, |z(t)|^2 \leq K_\delta \text{ a.e. } t, \sup_{|\theta| \leq \mu_n} \int_0^T |z(t+\theta) - z(t)|_{V'}^{p'} \leq \nu_n M_\delta \right\}$$

is a compact subset of $L^p(0, T; H)$ for any $\delta > 0$. Here we choose the sequence μ_n, ν_n so that $\sum \frac{1}{\nu_n} (\mu_n)^{\frac{p'}{p}} < \infty$. We have that

$$\begin{aligned} \mathbb{P}(u_\varepsilon \notin Z_\delta) &\leq \mathbb{P}\left(\int_0^T |Du_\varepsilon(s)|^p ds \geq L_\delta\right) + \mathbb{P}\left(\sup_{s \in [0, T]} |u_\varepsilon(s)|^2 \geq K_\delta\right) \\ &\quad + \mathbb{P}\left(\sup_{|\theta| \leq \mu_n} \int_0^T |u_\varepsilon(t+\theta) - u_\varepsilon(t)|^{p'} dt \geq \nu_n M_\delta\right). \end{aligned}$$

Thanks to Tchebychev's inequality we have

$$\begin{aligned} \mathbb{P}(u_\varepsilon \notin Z_\delta) &\leq \frac{1}{L_\delta} \mathbb{E} \int_0^T |Du_\varepsilon(s)|^p ds + \frac{1}{K_\delta} \mathbb{E} \sup_{s \in [0, T]} |u_\varepsilon(s)|^2 \\ &\quad + \sum \frac{1}{\nu_n M_\delta} \mathbb{E} \sup_{|\theta| \leq \mu_n} \int_0^T |u_\varepsilon(t+\theta) - u_\varepsilon(t)|^{p'} dt. \end{aligned}$$

From Lemmata 1 and 2 it follows that

$$\mathbb{P}(u_\varepsilon \notin Z_\delta) \leq \frac{C}{L_\delta} + \frac{C}{K_\delta} + \frac{C}{M_\delta} \sum \frac{1}{\nu_n} (\mu_n)^{\frac{p'}{p}}.$$

By Choosing

$$K_\delta = L_\delta = \frac{6C}{\delta} \text{ and } M_\delta = \frac{6C \left(\sum \frac{1}{\nu_n} (\mu_n)^{\frac{p'}{p}} \right)}{\delta},$$

we have that

$$\mathbb{P}(u_\varepsilon \notin Z_\delta) \leq \frac{\delta}{2}. \quad (4.7)$$

The sequence of probability measure $\Pi_2^\varepsilon = \mathbb{P} \circ \psi_2^\varepsilon(A) = \mathbb{P}(W \in A)$ for any $A \in \mathcal{B}(\mathfrak{S}_2)$ is constantly consisting of one element so it is weakly compact. As $\mathcal{C}(0, T; \mathcal{U}_0)$ is a Polish space we have that a sequence of probability measures which is weakly compact is tight. Therefore for any $\delta > 0$ there exists a compact $\mathcal{K}_\delta \subset \mathfrak{S}_2$ such that $\mathbb{P}(W \in \mathcal{K}_\delta) \geq 1 - \frac{\delta}{2}$. It follows from this and (4.7) that

$$\mathbb{P}((u_\varepsilon, W) \in Z_\delta \times \mathcal{K}_\delta) \geq 1 - \delta.$$

So we have found that for any $\delta > 0$ there is a compact $Z_\delta \times \mathcal{K}_\delta \subset \mathfrak{S}$ such that

$$\Pi^\varepsilon(Z_\delta \times \mathcal{K}_\delta) \geq 1 - \delta.$$

This prove that the family Π^ε is tight on $\mathfrak{S} = L^p(0, T; H) \times \mathcal{C}(0, T; \mathcal{U}_0)$. ■

It follows from Lemma 4 and Prokhorov's theorem that there exists a subsequence Π^{ε_j} of Π^ε converging weakly (in the sense of measure) to a probability measure Π . It emerges from Skorokhod's theorem that we can find a new probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ and random variables $(u_{\varepsilon_j}, W^{\varepsilon_j}), (u_0, \bar{W})$ defined on this new probability space and taking values in \mathfrak{S} such that:

$$\text{The probability law of } (W^{\varepsilon_j}, u_{\varepsilon_j}) \text{ is } \Pi^{\varepsilon_j}, \quad (4.8)$$

$$\text{The probability law of } (\bar{W}, u_0) \text{ is } \Pi, \quad (4.9)$$

$$W^{\varepsilon_j} \rightarrow \bar{W} \text{ in } \mathcal{C}(0, T; \mathcal{U}_0) \bar{\mathbb{P}}\text{-a.s.}, \quad (4.10)$$

$$u_{\varepsilon_j} \rightarrow u_0 \text{ in } L^p(0, T; H) \bar{\mathbb{P}}\text{-a.s.} \quad (4.11)$$

We can see that $\{W^{\varepsilon_j}\}$ is a sequence of cylindrical Brownian Motions evolving on \mathcal{U} . We let $\bar{\mathcal{F}}^t$ be the σ -algebra generated by $(\bar{W}(s), u_0(s)), 0 \leq s \leq t$ and the null sets of $\bar{\mathcal{F}}$. We can show by arguing as in [2] that \bar{W} is an $\bar{\mathcal{F}}^t$ -adapted cylindrical Wiener process evolving in \mathcal{U} . By the same argument as in [3] we can show that for all $\phi \in V$ and for almost every $(\omega, t) \in \bar{\Omega} \times [0, T]$ the following holds true

$$\begin{aligned} (u_{\varepsilon_j}(t), \phi) + \int_0^t a^{\varepsilon_j}(\cdot, u_{\varepsilon_j}(s), Du_{\varepsilon_j}) \cdot D\phi ds &= (u^0, \phi) - \int_0^t a_0^{\varepsilon_j}(\cdot, u_{\varepsilon_j}(s)) \phi ds \\ &+ \sum_{k=1}^{\infty} \int_0^t (M_k^{\varepsilon_j}(\cdot, u_{\varepsilon_j}(s)), \phi) dW_k^{\varepsilon_j}. \end{aligned} \quad (4.12)$$

5. PRELIMINARY RESULTS

In this section we collect some useful results that will be necessary in the homogenization process. The notation is that of the preceding sections. Before we can go further, let us however observe that property (3.1) (in Definition 2) still holds true for f in $B(\Omega; \mathcal{C}(\bar{Q}_T; B_A^{p', \infty}))$ where $B_A^{p', \infty} = B_A^{p'} \cap L^\infty(\mathbb{R}_{y, \tau}^{N+1})$ and $p' = p/(p-1)$.

With this in mind, the following assumption will be fundamental in the rest of the paper:

$$\begin{aligned} a_i(x, t, \cdot, \cdot, \mu, \lambda) &\in B_A^{p'} \text{ and } a_0(x, t, \cdot, \cdot, \mu), M_k(\cdot, \cdot, \mu) \in B_A^2 \\ \text{for any } (x, t) &\in \bar{Q}_T \text{ and each } (\mu, \lambda) \in \mathbb{R}^{N+1}, 1 \leq i \leq N, k \geq 1 \end{aligned} \quad (5.1)$$

where $p' = p/(p-1)$ with $2 \leq p < \infty$.

Arguing exactly as in [26, Proposition 4.5] we have the following result.

Proposition 1. *Let $1 \leq i \leq N$. Assume (5.1) holds true. Then for every $(\psi_0, \Psi) \in (A)^{N+1}$ and every $(x, t) \in \bar{Q}_T$, the functions $(y, \tau) \mapsto a_i(x, t, y, \tau, \psi_0(y, \tau), \Psi(y, \tau))$, $(y, \tau) \mapsto M_k(y, \tau, \psi_0(y, \tau))$ and $(y, \tau) \mapsto a_0(x, t, y, \tau, \psi_0(y, \tau))$ denoted respectively by $a_i(x, t, \cdot, \cdot, \psi_0, \Psi)$, $M_k(\cdot, \cdot, \psi_0)$ and $a_0(x, t, \cdot, \cdot, \psi_0)$, lie respectively in $B_A^{p'}$, B_A^2 and B_A^2 .*

Now, let $(\psi_0, \Psi) \in B(\Omega; \mathcal{C}(\bar{Q}_T; (A)^{N+1}))$. Assuming (5.1), it can be easily shown that the function $(x, t, y, \tau, \omega) \mapsto a_i(x, t, y, \tau, \psi_0(x, t, y, \tau, \omega), \Psi(x, t, y, \tau, \omega))$, denoted by $a_i(\cdot, \psi_0, \Psi)$, lies in $B(\Omega; \mathcal{C}(\bar{Q}_T; B_A^{p', \infty}))$ (use also Proposition 1). We can then define its trace

$$(x, t, \omega) \mapsto a_i(x, t, x/\varepsilon, t/\varepsilon, \psi_0(x, t, x/\varepsilon, t/\varepsilon, \omega), \Psi(x, t, x/\varepsilon, t/\varepsilon, \omega)),$$

from $Q_T \times \Omega$ into \mathbb{R} , as an element of $L^\infty(Q_T \times \Omega)$, which we denote by $a_i^\varepsilon(\cdot, \psi_0^\varepsilon, \Psi^\varepsilon)$. Likewise we can define the functions $a_0(\cdot, \psi_0)$ and $a_0^\varepsilon(\cdot, \psi_0^\varepsilon)$, $M_k(\cdot, \psi_0)$ and $M_k^\varepsilon(\cdot, \psi_0^\varepsilon)$.

The next result allows us to rigorously set the homogenized problem.

Proposition 2. *Let $2 \leq p < \infty$ and let $1 \leq i \leq N$. Assume (5.1) holds. For any $(\psi_0, \Psi) \in B(\Omega; \mathcal{C}(\bar{Q}_T; (A)^{N+1}))$ we have*

$$a_i^\varepsilon(\cdot, \psi_0^\varepsilon, \Psi^\varepsilon) \rightarrow a_i(\cdot, \psi_0, \Psi) \text{ in } L^{p'}(Q_T \times \Omega)\text{-weak } \Sigma \text{ as } \varepsilon \rightarrow 0. \quad (5.2)$$

Let $a(\cdot, \psi_0, \Psi) = (a_i(\cdot, \psi_0, \Psi))_{1 \leq i \leq N}$. The mapping $(\psi_0, \Psi) \mapsto a(\cdot, \psi_0, \Psi)$ of $B(\Omega; \mathcal{C}(\bar{Q}_T; (A)^{N+1}))$ into $L^{p'}(Q_T \times \Omega; B_A^{p'})^N$ extends by continuity to a unique mapping still denoted by a , of $L^p(Q_T \times$

$\Omega; (B_A^p)^{N+1}$) into $L^{p'}(Q_T \times \Omega; B_A^{p'})^N$ such that

$$\begin{aligned}
& (a(\cdot, u, \mathbf{v}) - a(\cdot, u, \mathbf{w})) \cdot (\mathbf{v} - \mathbf{w}) \geq c_1 |\mathbf{v} - \mathbf{w}|^p \quad \text{a.e. in } Q_T \times \Omega \times \mathbb{R}_y^N \times \mathbb{R}_\tau \\
& \|a_i(\cdot, u, \mathbf{v})\|_{L^{p'}(Q_T \times \Omega; B_A^{p'})} \leq c_2'' \left(1 + \|u\|_{L^p(Q_T \times \Omega; B_A^p)}^{p-1} + \|\mathbf{v}\|_{L^p(Q_T \times \Omega; (B_A^p)^N)}^{p-1} \right) \\
& \|a_i(\cdot, u, \mathbf{v}) - a_i(\cdot, u, \mathbf{w})\|_{L^{p'}(Q_T \times \Omega; B_A^{p'})} \\
& \quad \leq c_0 \|1 + |u| + |\mathbf{v}| + |\mathbf{w}|\|_{L^p(Q_T \times \Omega; B_A^p)}^{p-2} \|\mathbf{v} - \mathbf{w}\|_{L^p(Q_T \times \Omega; (B_A^p)^N)} \\
& |a_i(x, t, y, \tau, u, \mathbf{w}) - a_i(x', t', y, \tau, v, \mathbf{w})| \leq \\
& \quad \leq m(|x - x'| + |t - t'| + |u - v|) \left(1 + |u|^{p-1} + |v|^{p-1} + |\mathbf{w}|^{p-1} \right) \\
& \quad \text{a.e. in } Q_T \times \Omega \times \mathbb{R}_y^N \times \mathbb{R}_\tau
\end{aligned} \tag{5.3}$$

for all $u, v \in L^p(Q_T \times \Omega; B_A^p)$, $\mathbf{v}, \mathbf{w} \in L^p(Q_T \times \Omega; (B_A^p)^N)$ and all $(x, t), (x', t') \in Q_T$, where the constant c_2'' depends only on c_2 and on Q_T .

Proof. As discussed above before the statement of the proposition, we know that the function $a_i(\cdot, \psi_0, \Psi)$ lies in $B(\Omega; \mathcal{C}(\overline{Q_T}; B_A^{p', \infty}))$. Since Property (3.1) (in Definition 2) still holds for $f \in B(\Omega; \mathcal{C}(\overline{Q_T}; B_A^{p', \infty}))$ the convergence result (5.2) follows at once. Besides, it is immediate from the definition of the function $a_i(\cdot, \psi_0, \Psi)$ (for $(\psi_0, \Psi) \in B(\Omega; \mathcal{C}(\overline{Q_T}; (A)^{N+1}))$) and from some obvious arguments that the remainder of the proposition follows from [34, Proposition 3.1]. ■

It emerges from the preceding proposition, the following important corollary.

Corollary 1. (1) Let $(u_\varepsilon)_{\varepsilon \in E}$ be a sequence in $L^2(Q_T \times \Omega)$ such that $u_\varepsilon \rightarrow u_0$ in $L^2(Q_T \times \Omega)$ as $E \ni \varepsilon \rightarrow 0$, where $u_0 \in L^p(Q_T \times \Omega)$. Let $\Psi \in B(\Omega; \mathcal{C}(\overline{Q_T}; (A)^N))$, and finally let $1 \leq i \leq N$. Then, as $E \ni \varepsilon \rightarrow 0$,

$$a_i^\varepsilon(\cdot, u_\varepsilon, \Psi^\varepsilon) \rightarrow a_i(\cdot, u_0, \Psi) \text{ in } L^{p'}(Q_T \times \Omega)\text{-weak } \Sigma.$$

(2) Let $\psi_0 \in B(\Omega) \otimes \mathcal{C}_0^\infty(Q_T)$ and $\psi_1 \in B(\Omega) \otimes \mathcal{C}_0^\infty(Q_T) \otimes A^\infty$. For $\varepsilon > 0$, let

$$\Phi_\varepsilon = \psi_0 + \varepsilon \psi_1, \tag{5.4}$$

i.e., $\Phi_\varepsilon(x, t, \omega) = \psi_0(x, t, \omega) + \varepsilon \psi_1(x, t, x/\varepsilon, t/\varepsilon, \omega)$ for $(x, t, \omega) \in Q_T \times \Omega$. Let $(u_\varepsilon)_{\varepsilon \in E}$ be a sequence in $L^2(Q_T \times \Omega)$ such that $u_\varepsilon \rightarrow u_0$ in $L^2(Q_T \times \Omega)$ as $E \ni \varepsilon \rightarrow 0$ where $u_0 \in L^2(Q_T \times \Omega)$. Then, as $E \ni \varepsilon \rightarrow 0$, one has

$$(i) \ a_i^\varepsilon(\cdot, u_\varepsilon, D\Phi_\varepsilon) \rightarrow a_i(\cdot, u_0, D\psi_0 + D_y\psi_1) \text{ in } L^{p'}(Q_T \times \Omega)\text{-weak } \Sigma.$$

Moreover, if $(v_\varepsilon)_{\varepsilon \in E}$ is a sequence in $L^p(Q_T \times \Omega)$ such that $v_\varepsilon \rightarrow v_0$ in $L^p(Q_T \times \Omega)$ -weak Σ as $E \ni \varepsilon \rightarrow 0$ where $v_0 \in L^p(Q_T \times \Omega; \mathcal{B}_A^p)$, then, as $E \ni \varepsilon \rightarrow 0$,

$$(ii) \ \int_{Q_T \times \Omega} a_i^\varepsilon(\cdot, u_\varepsilon, D\Phi_\varepsilon) v_\varepsilon dx dt d\mathbb{P} \rightarrow \iint_{Q_T \times \Omega \times \Delta(A)} \widehat{a}_i(\cdot, u_0, D\psi_0 + \partial \widehat{\psi}_1) \widehat{v}_0 dx dt d\mathbb{P} d\beta.$$

Proof. We just sketch the proof since it is very similar to the one of [26, Corollaries 4.7-4.8]. For part (1), let $f \in L^p(Q_T \times \Omega; A)$, and let $(\psi_j)_j$ be a sequence in $B(\Omega) \otimes \mathcal{C}_0^\infty(Q_T)$ such that $\psi_j \rightarrow u_0$

in $L^p(Q_T \times \Omega)$ as $j \rightarrow \infty$. We have

$$\begin{aligned} & \int_{Q_T \times \Omega} a_i^\varepsilon(\cdot, u_\varepsilon, \Psi^\varepsilon) f^\varepsilon dx dt d\mathbb{P} - \iint_{Q_T \times \Omega \times \Delta(A)} \widehat{a}_i(\cdot, u_0, \widehat{\Psi}) \widehat{f} dx dt d\mathbb{P} d\beta \\ &= \int_{Q_T \times \Omega} [a_i^\varepsilon(\cdot, u_\varepsilon, \Psi^\varepsilon) - a_i^\varepsilon(\cdot, u_0, \Psi^\varepsilon)] f^\varepsilon dx dt d\mathbb{P} \\ & \quad + \int_{Q_T \times \Omega} [a_i^\varepsilon(\cdot, u_0, \Psi^\varepsilon) - a_i^\varepsilon(\cdot, \psi_j, \Psi^\varepsilon)] f^\varepsilon dx dt d\mathbb{P} \\ & \quad + \int_{Q_T \times \Omega} a_i^\varepsilon(\cdot, \psi_j, \Psi^\varepsilon) f^\varepsilon dx dt - \iint_{Q_T \times \Omega \times \Delta(A)} \widehat{a}_i(\cdot, u_0, \widehat{\Psi}) \widehat{f} dx dt d\mathbb{P} d\beta \\ &= A_\varepsilon + B_{\varepsilon, j} + C_{\varepsilon, j} \end{aligned}$$

where

$$\begin{aligned} A_\varepsilon &= \int_{Q_T \times \Omega} [a_i^\varepsilon(\cdot, u_\varepsilon, \Psi^\varepsilon) - a_i^\varepsilon(\cdot, u_0, \Psi^\varepsilon)] f^\varepsilon dx dt d\mathbb{P}, \\ B_{\varepsilon, j} &= \int_{Q_T \times \Omega} [a_i^\varepsilon(\cdot, u_0, \Psi^\varepsilon) - a_i^\varepsilon(\cdot, \psi_j, \Psi^\varepsilon)] f^\varepsilon dx dt d\mathbb{P}, \\ C_{\varepsilon, j} &= \int_{Q_T \times \Omega} a_i^\varepsilon(\cdot, \psi_j, \Psi^\varepsilon) f^\varepsilon dx dt - \iint_{Q_T \times \Omega \times \Delta(A)} \widehat{a}_i(\cdot, u_0, \widehat{\Psi}) \widehat{f} dx dt d\mathbb{P} d\beta. \end{aligned}$$

As far as A_ε is concerned, we have

$$|A_\varepsilon| \leq \int_{Q_T \times \Omega} m(|u_\varepsilon - u_0|) \left(1 + |u_\varepsilon|^{p-1} + |u_0|^{p-1} + |\Psi^\varepsilon|^{p-1}\right) |f^\varepsilon| dx dt d\mathbb{P}.$$

From the convergence result $u_\varepsilon \rightarrow u_0$ in $L^2(Q_T \times \Omega)$, we infer that $m(|u_\varepsilon - u_0|) \rightarrow 0$ a.e. in $Q_T \times \Omega$ as $E \ni \varepsilon \rightarrow 0$, so that, by Egorov's theorem, $A_\varepsilon \rightarrow 0$ as $E \ni \varepsilon \rightarrow 0$. As for $C_{\varepsilon, j}$, we see that the function $(x, t, \omega) \mapsto a_i(x, t, \cdot, \cdot, \psi_j(x, t, \omega), \Psi(x, t, \cdot, \cdot, \omega))$ belongs to $B(\Omega; \mathcal{C}(\overline{Q}_T; B_A^{p', \infty}))$, in such a way that we use the convergence result (5.2) to get

$$C_{\varepsilon, j} \rightarrow \iint_{Q_T \times \Omega \times \Delta(A)} \left(\widehat{a}_i(\cdot, \psi_j, \widehat{\Psi}) - \widehat{a}_i(\cdot, u_0, \widehat{\Psi})\right) \widehat{f} dx dt d\mathbb{P} d\beta \equiv \widehat{C}_j \text{ as } E \ni \varepsilon \rightarrow 0.$$

But as

$$|\widehat{C}_j| \leq \iint_{Q_T \times \Omega \times \Delta(A)} m(|\psi_j - u_0|) \left(1 + |\psi_j|^{p-1} + |u_0|^{p-1} + |\widehat{\Psi}|^{p-1}\right) |\widehat{f}| dx dt d\mathbb{P} d\beta,$$

arguing as before we get $\widehat{C}_j \rightarrow 0$ as $j \rightarrow \infty$. We also have $\lim_{E \ni \varepsilon \rightarrow 0} \lim_{j \rightarrow \infty} B_{\varepsilon, j} = 0$, so that part (1) follows from the equality

$$\begin{aligned} & \lim_{E \ni \varepsilon \rightarrow 0} \left(\int_{Q_T \times \Omega} a_i^\varepsilon(\cdot, u_\varepsilon, \Psi^\varepsilon) f^\varepsilon dx dt d\mathbb{P} - \iint_{Q_T \times \Omega \times \Delta(A)} \widehat{a}_i(\cdot, u_0, \widehat{\Psi}) \widehat{f} dx dt d\mathbb{P} d\beta \right) \\ &= \lim_{E \ni \varepsilon \rightarrow 0} A_\varepsilon + \lim_{E \ni \varepsilon \rightarrow 0} \lim_{j \rightarrow \infty} B_{\varepsilon, j} + \lim_{E \ni \varepsilon \rightarrow 0} \lim_{j \rightarrow \infty} C_{\varepsilon, j} = 0. \end{aligned}$$

Part (2) is a mere consequence of part (1). ■

Another important result which will be needed is the

Lemma 5. *Let $(u_\varepsilon)_\varepsilon$ be a sequence in $L^2(Q_T \times \Omega)$ such that $u_\varepsilon \rightarrow u_0$ in $L^2(Q_T \times \Omega)$ as $\varepsilon \rightarrow 0$ where $u_0 \in L^2(Q_T \times \Omega)$. Then for each positive integer k we have ,*

$$M_k^\varepsilon(\cdot, u_\varepsilon) \rightarrow M_k(\cdot, u_0) \text{ in } L^2(Q_T \times \Omega)\text{-weak } \Sigma \text{ as } \varepsilon \rightarrow 0.$$

Proof. First of all, let $u \in B(\Omega; \mathcal{C}(\overline{Q}_T))$; then the function $(x, t, y, \tau, \omega) \mapsto M_k(y, \tau, u(x, t, \omega))$ lies in $B(\Omega; \mathcal{C}(\overline{Q}_T; B_A^{2, \infty}))$, so that we have $M_k^\varepsilon(\cdot, u) \rightarrow M_k(\cdot, u)$ in $L^2(Q_T \times \Omega)$ -weak Σ as $\varepsilon \rightarrow 0$. Next, since $B(\Omega; \mathcal{C}(\overline{Q}_T))$ is dense in $L^2(Q_T \times \Omega)$, it can be easily shown that

$$M_k^\varepsilon(\cdot, u_0) \rightarrow M_k(\cdot, u_0) \text{ in } L^2(Q_T \times \Omega)\text{-weak } \Sigma \text{ as } \varepsilon \rightarrow 0. \quad (5.5)$$

Now, let $f \in L^2(\Omega; L^2(Q_T; A))$; then

$$\begin{aligned} & \int_{Q_T \times \Omega} M_k^\varepsilon(\cdot, u_\varepsilon) f^\varepsilon dx dt d\mathbb{P} - \iint_{Q_T \times \Omega \times \Delta(A)} \widehat{M}_k(\cdot, u_0) \widehat{f} dx dt d\mathbb{P} d\beta \\ &= \int_{Q_T \times \Omega} (M_k^\varepsilon(\cdot, u_\varepsilon) - M_k^\varepsilon(\cdot, u_0)) f^\varepsilon dx dt d\mathbb{P} + \int_{Q_T \times \Omega} M_k^\varepsilon(\cdot, u_0) f^\varepsilon dx dt d\mathbb{P} \\ & \quad - \iint_{Q_T \times \Omega \times \Delta(A)} \widehat{M}_k(\cdot, u_0) \widehat{f} dx dt d\mathbb{P} d\beta. \end{aligned}$$

Using the inequality

$$\left| \int_{Q_T \times \Omega} (M_k^\varepsilon(\cdot, u_\varepsilon) - M_k^\varepsilon(\cdot, u_0)) f^\varepsilon dx dt d\mathbb{P} \right| \leq C \|u_\varepsilon - u_0\|_{L^2(Q_T \times \Omega)} \|f^\varepsilon\|_{L^2(Q_T \times \Omega)}$$

in conjunction with (5.5) leads at once to the result. \blacksquare

Remark 4. From the Lipschitz property of the function M_k we may get more information on the limit of the sequence $M_k^\varepsilon(\cdot, u_\varepsilon)$. Indeed, since $|M_k^\varepsilon(\cdot, u_\varepsilon) - M_k^\varepsilon(\cdot, u_0)| \leq C |u_\varepsilon - u_0|$, we deduce the following convergence result:

$$M_k^\varepsilon(\cdot, u_\varepsilon) \rightarrow \widetilde{M}_k(u_0) \text{ in } L^2(Q_T \times \Omega) \text{ as } \varepsilon \rightarrow 0$$

where $\widetilde{M}_k(u_0)(x, t, \omega) = \int_{\Delta(A)} \widehat{M}_k(s, s_0, u_0(x, t, \omega)) d\beta$, so that we can derive the existence of a subsequence of $M_k^\varepsilon(\cdot, u_\varepsilon)$ that converges a.e. in $Q_T \times \Omega$ to $\widetilde{M}_k(u_0)$. For the next sections we will need the following function: $\widetilde{M}(u_0) = (\widetilde{M}_k(u_0))_{k \geq 1}$.

We end this section with some useful spaces. Let

$$\mathbb{F}_0^{1,p} = L^p(\bar{\Omega} \times (0, T); W_0^{1,p}(Q)) \times L^p(Q_T \times \bar{\Omega}; \mathcal{V})$$

and

$$\mathcal{F}_0^\infty = [B(\bar{\Omega}) \otimes \mathcal{C}_0^\infty(Q_T)] \times [B(\bar{\Omega}) \otimes \mathcal{C}_0^\infty(Q_T) \otimes \mathcal{E}]$$

where $\mathcal{V} = \mathcal{B}_{A_\tau}^p(\mathbb{R}_\tau; \mathcal{B}_{\#A_y}^{1,p})$ and $\mathcal{E} = \varrho_\tau(A_\tau^\infty) \otimes [\varrho_y(A_y^\infty / \mathbb{R})]$, and ϱ_τ (resp. ϱ_y) denotes the canonical mapping of $B_{A_\tau}^p$ (resp. $B_{A_y}^p$) onto $\mathcal{B}_{A_\tau}^p$ (resp. $\mathcal{B}_{A_y}^p$). $\mathbb{F}_0^{1,p}$ is a Banach space under the norm

$$\|(u_0, u_1)\|_{\mathbb{F}_0^{1,p}} = \|u_0\|_{L^p(\bar{\Omega} \times (0, T); W_0^{1,p}(Q))} + \|u_1\|_{L^p(Q_T \times \bar{\Omega}; \mathcal{V})}.$$

Moreover, since $B(\bar{\Omega})$ is dense in $L^p(\bar{\Omega})$, it is an easy matter to check that \mathcal{F}_0^∞ is dense in $\mathbb{F}_0^{1,p}$.

6. HOMOGENIZATION RESULTS

Let (u_{ε_j}) be the sequence determined in Section 4 and satisfying Eq. (4.12). It therefore satisfies the a priori estimates (4.1)-(4.2), so that, by the diagonal process, one can find a subsequence of $(u_{\varepsilon_j})_j$ not relabeled, which weakly converges in $L^p(\bar{\Omega}; L^p(0, T; W_0^{1,p}(Q)))$ to u_0 determined by the Skorokhod's theorem and satisfying (4.11). Next, due to the estimate (4.1) (which yields the uniform integrability of the sequence $(u_{\varepsilon_j})_j$ with respect to ω) and the Vitali's theorem, we deduce from (4.11) that, as $j \rightarrow \infty$,

$$u_{\varepsilon_j} \rightarrow u_0 \text{ in } L^2(Q_T \times \bar{\Omega}). \quad (6.1)$$

Then, from Theorem 4, we infer the existence of a function $u_1 \in L^p(\bar{\Omega}; L^p(Q_T; \mathcal{B}_{A_\tau}^p(\mathbb{R}_\tau; \mathcal{B}_{\#A_y}^{1,p})))$ such that

$$\frac{\partial u_{\varepsilon_j}}{\partial x_i} \rightarrow \frac{\partial u_0}{\partial x_i} + \frac{\bar{\partial} u_1}{\partial y_i} \text{ in } L^p(Q_T \times \bar{\Omega})\text{-weak } \Sigma \text{ (} 1 \leq i \leq N \text{)} \quad (6.2)$$

hold when $\varepsilon_j \rightarrow 0$.

With this in mind, the following *global* homogenization result holds.

Proposition 3. *The couple $(u_0, u_1) \in \mathbb{F}_0^{1,p}$ determined above solves the following variational problem*

$$\begin{cases} - \int_{Q_T \times \bar{\Omega}} u_0 \psi'_0 dx dt d\bar{\mathbb{P}} + \iint_{Q_T \times \bar{\Omega} \times \Delta(A)} \widehat{a}_0(\cdot, u_0) \psi_0 dx dt d\bar{\mathbb{P}} d\beta \\ + \iint_{Q_T \times \bar{\Omega} \times \Delta(A)} \widehat{a}(\cdot, u_0, Du_0 + \partial \widehat{u}_1) \cdot (D\psi_0 + \partial \widehat{\psi}_1) dx dt d\bar{\mathbb{P}} d\beta \\ = \int_{\bar{\Omega}} \int_0^T \left(\widetilde{M}(u_0, \psi_0) \right) d\bar{W} d\bar{\mathbb{P}} \text{ for all } (\psi_0, \psi_1) \in \mathcal{F}_0^\infty. \end{cases} \quad (6.3)$$

Proof. In what follows, we omit the index j from the sequence ε_j . So we will merely write ε for ε_j . With this in mind, let $\Phi = (\psi_0, \varrho \circ \psi_1) \in \mathcal{F}_0^\infty$ with $\psi_0 \in B(\bar{\Omega}) \otimes \mathcal{C}_0^\infty(Q_T)$, $\psi_1 \in B(\bar{\Omega}) \otimes \mathcal{C}_0^\infty(Q_T) \otimes [A_\tau^\infty \otimes (A_y^\infty/\mathbb{R})]$. Define Φ_ε as in (5.4) (see Corollary 1). Then, $\Phi_\varepsilon \in B(\bar{\Omega}) \otimes \mathcal{C}_0^\infty(Q_T)$ and, using Φ_ε as a test function in the variational formulation of (4.12) we get

$$\begin{aligned} \int_{\bar{\Omega}} (u_\varepsilon(T), \Phi_\varepsilon(T)) d\bar{\mathbb{P}} &= \int_{\bar{\Omega}} (u^0, \Phi_\varepsilon(0)) d\bar{\mathbb{P}} + \int_{Q_T \times \bar{\Omega}} u_\varepsilon \frac{\partial \Phi_\varepsilon}{\partial t} dx dt d\bar{\mathbb{P}} \\ &\quad - \int_{Q_T \times \bar{\Omega}} a^\varepsilon(\cdot, u_\varepsilon, Du_\varepsilon) \cdot D\Phi_\varepsilon dx dt d\bar{\mathbb{P}} \\ &\quad - \int_{Q_T \times \bar{\Omega}} a_0^\varepsilon(\cdot, u_\varepsilon) \Phi_\varepsilon dx dt d\bar{\mathbb{P}} \\ &\quad + \int_{\bar{\Omega}} \int_0^T (M^\varepsilon(\cdot, u_\varepsilon), \Phi_\varepsilon) dW^\varepsilon d\bar{\mathbb{P}}, \end{aligned}$$

or equivalently, taking into account the fact that $\Phi_\varepsilon(0) = \Phi_\varepsilon(T) = 0$,

$$\begin{aligned} - \int_{Q_T \times \bar{\Omega}} u_\varepsilon \frac{\partial \Phi_\varepsilon}{\partial t} dx dt d\bar{\mathbb{P}} + \int_{Q_T \times \bar{\Omega}} a^\varepsilon(\cdot, u_\varepsilon, Du_\varepsilon) \cdot D\Phi_\varepsilon dx dt d\bar{\mathbb{P}} \\ + \int_{Q_T \times \bar{\Omega}} a_0^\varepsilon(\cdot, u_\varepsilon) \Phi_\varepsilon dx dt d\bar{\mathbb{P}} = \int_0^T \int_{\bar{\Omega}} (M^\varepsilon(\cdot, u_\varepsilon), \Phi_\varepsilon) dW^\varepsilon d\bar{\mathbb{P}}. \end{aligned} \quad (6.4)$$

We consider the terms in (6.4) respectively.

Firstly we have

$$\begin{aligned} \int_{Q_T \times \bar{\Omega}} u_\varepsilon \frac{\partial \Phi_\varepsilon}{\partial t} dx dt d\bar{\mathbb{P}} &= \int_{Q_T \times \bar{\Omega}} u_\varepsilon \frac{\partial \psi_0}{\partial t} dx dt d\bar{\mathbb{P}} + \varepsilon \int_{Q_T \times \bar{\Omega}} u_\varepsilon \left(\frac{\partial \psi_1}{\partial t} \right)^\varepsilon dx dt d\bar{\mathbb{P}} \\ &\quad + \int_{Q_T \times \bar{\Omega}} u_\varepsilon \left(\frac{\partial \psi_1}{\partial \tau} \right)^\varepsilon dx dt d\bar{\mathbb{P}}. \end{aligned}$$

But in view of (4.11) we have that $u_\varepsilon \rightarrow u_0$ in $L^2(Q_T \times \bar{\Omega})$ (strong). Moreover, since $(\partial \psi_1 / \partial \tau)^\varepsilon \rightarrow M(\partial \psi_1 / \partial \tau) = 0$ in $L^2(Q_T \times \bar{\Omega})$ -weak, we deduce from the preceding strong convergence result that

$$\int_{Q_T \times \bar{\Omega}} u_\varepsilon \frac{\partial \Phi_\varepsilon}{\partial t} dx dt d\bar{\mathbb{P}} \rightarrow \int_{Q_T \times \bar{\Omega}} u_0 \frac{\partial \psi_0}{\partial t} dx dt d\bar{\mathbb{P}}.$$

Next, from Corollary 1, it follows that $a_0^\varepsilon(\cdot, u_\varepsilon) \rightarrow a_0(\cdot, u_0)$ in $L^2(Q_T \times \bar{\Omega})$ -weak Σ , so that

$$\int_{Q_T \times \bar{\Omega}} a_0^\varepsilon(\cdot, u_\varepsilon) \Phi_\varepsilon dx dt d\bar{\mathbb{P}} \rightarrow \iint_{Q_T \times \bar{\Omega} \times \Delta(A)} \widehat{a}_0(\cdot, u_0) \psi_0 dx dt d\bar{\mathbb{P}} d\beta.$$

As far as the term $\int_0^T \int_{\bar{\Omega}} (M^\varepsilon(\cdot, u_\varepsilon), \Phi_\varepsilon) dW^\varepsilon d\bar{\mathbb{P}}$ is concerned, thanks to Remark 4 we get at once

$$\int_{\bar{\Omega}} \int_0^T (M^\varepsilon(\cdot, u_\varepsilon), \Phi_\varepsilon) dW^\varepsilon d\bar{\mathbb{P}} \rightarrow \int_{\bar{\Omega}} \int_0^T (\widetilde{M}(u_0), \psi_0) d\bar{W} d\bar{\mathbb{P}}.$$

The last term is more involved. Indeed, by the monotonicity argument, it emerges that

$$\int_{Q_T \times \bar{\Omega}} (a^\varepsilon(\cdot, u_\varepsilon, Du_\varepsilon) - a^\varepsilon(\cdot, u_\varepsilon, D\Phi_\varepsilon)) \cdot (Du_\varepsilon - D\Phi_\varepsilon) dx dt d\bar{\mathbb{P}} \geq 0. \quad (6.5)$$

Owing to the estimate (4.2) (denoting by $\bar{\mathbb{E}}$ the mathematical expectation on $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$) we infer that

$$\sup_{\varepsilon > 0} \bar{\mathbb{E}} \|a^\varepsilon(\cdot, u_\varepsilon, Du_\varepsilon)\|_{L^{p'}(Q_T)^N}^{p'} < \infty,$$

so that, from Theorem 2, there exist a function $\chi \in L^{p'}(Q_T \times \bar{\Omega}; \mathcal{B}_A^{p'})^N$ and a subsequence of ε not relabeled, such that $a^\varepsilon(\cdot, u_\varepsilon, Du_\varepsilon) \rightarrow \chi$ in $L^{p'}(Q_T \times \bar{\Omega})^N$ -weak Σ as $\varepsilon \rightarrow 0$. We therefore pass to the limit in (6.5) (as $\varepsilon \rightarrow 0$) using Corollary 1 to get

$$\iint_{Q_T \times \bar{\Omega} \times \Delta(A)} (\widehat{\chi} - \widehat{a}(\cdot, u_0, \mathbb{D}\Phi)) \cdot (\mathbb{D}\mathbf{u} - \mathbb{D}\Phi) dx dt d\bar{\mathbb{P}} d\beta \geq 0 \quad (6.6)$$

for any $\Phi \in \mathcal{F}_0^\infty$ where $\mathbb{D}\mathbf{u} = Du_0 + \partial\widehat{u}_1$ ($\mathbf{u} = (u_0, u_1)$) and $\mathbb{D}\Phi = D\psi_0 + \partial\widehat{\psi}_1$. By a density and continuity arguments (6.6) still holds for $\Phi \in \mathbb{F}_0^{1,p}$. Hence by taking $\Phi = \mathbf{u} + \lambda\mathbf{v}$ for $\mathbf{v} = (v_0, v_1) \in \mathbb{F}_0^{1,p}$ and $\lambda > 0$ arbitrarily fixed, we get

$$\lambda \iint_{Q_T \times \bar{\Omega} \times \Delta(A)} (\widehat{\chi} - \widehat{a}(\cdot, u_0, \mathbb{D}\mathbf{u} + \lambda\mathbb{D}\mathbf{v})) \cdot \mathbb{D}\mathbf{v} dx dt d\bar{\mathbb{P}} d\beta \geq 0 \quad \forall \mathbf{v} \in \mathbb{F}_0^{1,p}.$$

Therefore by a mere routine, we deduce that $\chi = a(\cdot, u_0, Du_0 + \bar{D}_y u_1)$. Putting all the above facts together we are led to (6.3), and the proof is completed. \blacksquare

The problem (6.3) is equivalent to the following system:

$$\iint_{Q_T \times \bar{\Omega} \times \Delta(A)} \widehat{a}(\cdot, u_0, \mathbb{D}\mathbf{u}) \cdot \partial\widehat{\psi}_1 dx dt d\bar{\mathbb{P}} d\beta = 0 \quad \text{for all } \psi_1 \in B(\bar{\Omega}) \otimes \mathcal{C}_0^\infty(Q_T) \otimes \mathcal{E} \quad (6.7)$$

and

$$\begin{cases} - \int_{Q_T \times \bar{\Omega}} u_0 \psi_0' dx dt d\bar{\mathbb{P}} + \iint_{Q_T \times \bar{\Omega} \times \Delta(A)} \widehat{a}(\cdot, u_0, \mathbb{D}\mathbf{u}) \cdot D\psi_0 dx dt d\bar{\mathbb{P}} d\beta \\ + \iint_{Q_T \times \bar{\Omega} \times \Delta(A)} \widehat{a}_0(\cdot, u_0) \psi_0 dx dt d\bar{\mathbb{P}} d\beta = \int_{\bar{\Omega}} \int_0^T (\widetilde{M}(u_0), \psi_0) d\bar{W} d\bar{\mathbb{P}} \\ \text{for all } \psi_0 \in B(\bar{\Omega}) \otimes \mathcal{C}_0^\infty(Q_T). \end{cases} \quad (6.8)$$

As far as (6.7) is concerned, let $(x, t) \in Q_T$ and let $(r, \xi) \in \mathbb{R} \times \mathbb{R}^N$ be freely fixed. Let $\pi(x, t, r, \xi)$ be defined by the cell problem

$$\begin{aligned} \pi(x, t, r, \xi) &\in \mathcal{V} = \mathcal{B}_{A_\tau}^p(\mathbb{R}_\tau; \mathcal{B}_{\#A_y}^{1,p}) : \\ \int_{\Delta(A)} \widehat{a}(\cdot, r, \xi + \partial\widehat{\pi}(x, t, r, \xi)) \cdot \partial\widehat{w} d\beta &= 0 \quad \text{for all } w \in \mathcal{V}. \end{aligned} \quad (6.9)$$

Then from the properties of the function a , it follows by [18, Chap. 2] that Eq. (6.9) admits at least a solution. Now if $\pi_1 \equiv \pi_1(x, t, r, \xi)$ and $\pi_2 \equiv \pi_2(x, t, r, \xi)$ are two solutions of (6.9), then we must have

$$\int_{\Delta(A)} (\widehat{a}(\cdot, r, \xi + \partial\widehat{\pi}_1) - \widehat{a}(\cdot, r, \xi + \partial\widehat{\pi}_2)) \cdot (\partial\widehat{\pi}_1 - \partial\widehat{\pi}_2) d\beta = 0,$$

and so, by assumption A2., $\partial\widehat{\pi}_1 = \partial\widehat{\pi}_2$, so that $\frac{\partial\pi_1}{\partial y_i} = \frac{\partial\pi_2}{\partial y_i}$ ($1 \leq i \leq N$) since $\partial_i\widehat{\pi}_j = \mathcal{G}_1\left(\frac{\partial\pi_j}{\partial y_i}\right)$ for $j = 1, 2$. Hence $\pi_1 = \pi_2$ since they belong to \mathcal{V} . Next, taking in particular $r = u_0(x, t, \omega)$ and $\xi = Du_0(x, t, \omega)$ with (x, t, ω) arbitrarily chosen in $Q_T \times \bar{\Omega}$, and then choosing in (6.7) the particular test functions $\psi_1(x, t, \omega) = \phi(\omega)\varphi(x, t)w$ ($(x, t, \omega) \in Q_T \times \bar{\Omega}$) with $\varphi \in \mathcal{C}_0^\infty(Q_T)$, $\phi \in B(\bar{\Omega})$ and $w \in \mathcal{E}$, and finally comparing the resulting equation with (6.9) (note that \mathcal{E} is dense in \mathcal{V}), the uniqueness of the solution to (6.7) tells us that $u_1 = \pi(\cdot, u_0, Du_0)$, where the right-hand side of the preceding equality stands for the function $(x, t, \omega) \mapsto \pi(x, t, u_0(x, t, \omega), Du_0(x, t, \omega))$ from $Q_T \times \bar{\Omega}$ into \mathcal{V} .

We have just proved the

Proposition 4. *The solution of the variational problem (6.7) is unique.*

Let us now deal with the variational problem (6.8). For that, set

$$q(x, t, r, \xi) = \int_{\Delta(A)} \widehat{a}(\cdot, r, \xi + \partial\widehat{\pi}(x, t, r, \xi))d\beta$$

and

$$q_0(x, t, r) = \int_{\Delta(A)} \widehat{a}_0(\cdot, r)d\beta; \quad \widetilde{M}(r) = \int_{\Delta(A)} \widetilde{M}(\cdot, r)d\beta$$

for $(x, t) \in Q_T$ and $(r, \xi) \in \mathbb{R} \times \mathbb{R}^N$ arbitrarily fixed. Substituting $u_1 = \pi(\cdot, u_0, Du_0)$ in (6.8) and choosing there the particular test functions $\psi_0(x, t, \omega) = \varphi(x, t)\phi(\omega)$ for $\varphi \in \mathcal{C}_0^\infty(Q_T)$ and $\phi \in B(\bar{\Omega})$ we get by Itô's formula, the macroscopic homogenized problem, viz.

$$\begin{cases} du_0 = (\operatorname{div} q(\cdot, \cdot, u_0, Du_0) - q_0(\cdot, \cdot, u_0)) dt + \widetilde{M}(u_0)d\bar{W} \text{ in } Q_T \\ u_0 = 0 \text{ on } \partial Q \times (0, T) \\ u_0(x, 0) = u^0(x) \text{ in } Q. \end{cases} \quad (6.10)$$

In view of (6.3), (6.10) admits at least a solution. Moreover the following uniqueness result holds.

Proposition 5. *Let u_0 and $u_0^\#$ be two solutions of (6.10) on the same probabilistic system $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$, $\bar{W}, \bar{\mathcal{F}}^t$ with the same initial condition u^0 . We have that $u_0 = u_0^\#$ $\bar{\mathbb{P}}$ -almost surely.*

Proof. From the definition of q_0 and \widetilde{M} , it is not difficult to see that they are Lipschitz continuous with respect to the variable u_0 . It also follows from the definition of the operator q that it satisfies properties similar to A1.-A3.. Now the proof is quite standard but we give the detail for sake of completeness. The functions u_0 and $u_0^\#$ given in the proposition satisfy

$$\begin{aligned} dw_0 = & \left(\operatorname{div} q(x, t, u_0, Du_0) - \operatorname{div} q(x, t, u_0^\#, Du_0^\#) \right) dt - \left(q_0(x, t, u_0) - q_0(x, t, u_0^\#) \right) dt \\ & + (\widetilde{M}(u_0) - \widetilde{M}(u_0^\#))d\bar{W}, \end{aligned}$$

where $w_0 = u_0 - u_0^\#$. For sake of simplicity we will omit the dependence on the variables x, t in the following computations. Thanks to Itô's formula we have

$$\begin{aligned} d|w_0|^2 = & 2\langle \operatorname{div}(q(u_0, Du_0) - q(u_0^\#, Du_0^\#)), w_0 \rangle dt - 2(q_0(u_0) - q_0(u_0^\#), w_0)dt \\ & + |\widetilde{M}(u_0) - \widetilde{M}(u_0^\#)|_{L^2}^2 dt + 2(\widetilde{M}(u_0) - \widetilde{M}(u_0^\#), w_0)d\bar{W}. \end{aligned}$$

Due to the monotonicity of $\operatorname{div}(q(u, Du))$, the Lipschitz continuity of $q_0(\cdot)$ and \widetilde{M} we have that

$$d|w_0|^2 + C|Dw_0|^p \leq C|w_0|^2 dt + 2(\widetilde{M}(u_0) - \widetilde{M}(u_0^\#), w_0)d\bar{W}.$$

Note that we also used the Cauchy-Schwartz' inequality to get the above estimate. Integrating over $[0, t]$ and taking the mathematical expectation to both sides of the latter equations yield

$$\bar{\mathbb{E}}|w_0(t)|^2 \leq C\bar{\mathbb{E}} \int_0^t |w_0(s)|^2 ds.$$

Now we can conclude the proof of the proposition by invoking the Gronwall's lemma. ■

Remark 5. The pathwise uniqueness result in Proposition 5 and Yamada-Watanabe's Theorem (see, for instance, [24]) implies the existence of a unique strong probabilistic solution of (6.10) on a prescribed probabilistic system $(\Omega, \mathcal{F}, \mathbb{P}), \mathcal{F}^t, W$.

We are now in a position to formulate the main homogenization result.

Theorem 5. *Assume that A1.-A7. hold. Suppose moreover that (5.1) holds true. Let $2 \leq p < \infty$. For each $\varepsilon > 0$ let u_ε be the unique solution of (1.1) on a given stochastic system $(\Omega, \mathcal{F}, \mathbb{P}), \mathcal{F}^t, W$ defined as in Section 4. Then as $\varepsilon \rightarrow 0$, the whole sequence u_ε converges in probability to u_0 in $L^2(Q_T)$ (i.e., $\|u_\varepsilon - u_0\|_{L^2(Q_T)}$ converges to zero in probability) where u_0 is the unique strong probabilistic solution of (6.10).*

The main ingredients for the proof of this theorem are the pathwise uniqueness for (6.10) and the following criteria for convergence in probability whose proof can be found in [13].

Lemma 6. *Let X be a Polish space. A sequence of a X -valued random variables $\{x_n; n \geq 0\}$ converges in probability if and only if for every subsequence of joint probability laws, $\{\nu_{n_k, m_k}; k \geq 0\}$, there exists a further subsequence which converges weakly to a probability measure ν such that*

$$\nu(\{(x, y) \in X \times X; x = y\}) = 1.$$

Let us set $L^p = L^p(0, T, H)$, $L^{p,2} = L^p(0, T, H) \times L^p(0, T, H)$, $\mathfrak{S}^W = \mathcal{C}(0, T; \mathcal{U}_0)$, and finally $\mathfrak{S} = L^p \times L^p \times \mathfrak{S}^W$. For any $S \in \mathcal{B}(L^p)$ we set $\Pi^\varepsilon(S) = \mathbb{P}(u_\varepsilon \in S)$ and $\Pi^W = \mathbb{P}(W \in S)$ for any $S \in \mathcal{B}(\mathfrak{S}^W)$. Next we define the joint probability laws :

$$\begin{aligned} \Pi^{\varepsilon, \varepsilon'} &= \Pi^\varepsilon \times \Pi^{\varepsilon'} \\ \nu^{\varepsilon, \varepsilon'} &= \Pi^\varepsilon \times \Pi^{\varepsilon'} \times \Pi^W. \end{aligned}$$

The following tightness property holds.

Lemma 7. *The collection $\{\nu^{\varepsilon, \varepsilon'}; \varepsilon, \varepsilon' \in E\}$ (and hence any subsequence $\{\nu^{\varepsilon_j, \varepsilon'_j}; \varepsilon_j, \varepsilon'_j \in E'\}$) is tight on \mathfrak{S} .*

Proof. The proof is very similar to Lemma 4. For any $\delta > 0$ we choose the sets Z_δ and \mathcal{K}_δ exactly as in the proof of Lemma 4 with appropriate modification on the constants $K_\delta, L_\delta, M_\delta$ so that $\Pi^\varepsilon(Z_\delta) \geq 1 - \frac{\delta}{4}$ and $\Pi^W(\mathcal{K}_\delta) \geq 1 - \frac{\delta}{2}$ for every $\varepsilon \in E$. Now let us take $K_\delta = Z_\delta \times Z_\delta \times \mathcal{K}_\delta$ which is compact in \mathfrak{S} ; it is not difficult to see that $\{\nu^{\varepsilon, \varepsilon'}(K_\delta) \geq (1 - \frac{\delta}{4})^2(1 - \frac{\delta}{2}) \geq 1 - \delta$ for all $\varepsilon, \varepsilon'$. This completes the proof of the lemma. ■

Proof of Theorem 5. To prove Theorem 5 we will mainly use the idea in [10]. Lemma 7 implies that there exists a subsequence from $\{\nu^{\varepsilon_j, \varepsilon'_j}\}$ still denoted by $\{\nu^{\varepsilon_j, \varepsilon'_j}\}$ which converges to a probability measure ν . By Skorokhod's theorem there exists a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ on which a sequence

$(u_{\varepsilon_j}, u_{\varepsilon'_j}, W^j)$ is defined and converges almost surely in $L^{p,2} \times \mathfrak{S}^W$ to a couple of random variables (u_0, v_0, \bar{W}) . Furthermore, we have

$$\begin{aligned} Law(u_{\varepsilon_j}, u_{\varepsilon'_j}, W^j) &= \nu^{\varepsilon_j, \varepsilon'_j}, \\ Law(u_0, v_0, \bar{W}) &= \nu. \end{aligned}$$

Now let $Z_j^{u_\varepsilon} = (u_{\varepsilon_j}, W^j)$, $Z_j^{u_{\varepsilon'}} = (u_{\varepsilon'_j}, W^j)$, $Z^{u_0} = (u_0, \bar{W})$ and $Z^{v_0} = (v_0, \bar{W})$. We can infer from the above argument that $(\Pi^{\varepsilon_j, \varepsilon'_j})$ converges to a measure Π such that

$$\Pi(\cdot) = \bar{\mathbb{P}}((u_0, v_0) \in \cdot).$$

As above we can show that $Z_j^{u_\varepsilon}$ and $Z_j^{u_{\varepsilon'}}$ satisfy (4.12) and that Z^u and Z^v satisfy (6.10) on the same stochastic system $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$, $\bar{\mathcal{F}}^t, \bar{W}$, where $\bar{\mathcal{F}}^t$ is the filtration generated by the couple (u_0, v_0, \bar{W}) . Since we have the uniqueness result above, then we see that $u^0 = v^0$ almost surely and $u_0 = v_0$ in $L^p(0, T; H)$. Therefore

$$\Pi(\{(x, y) \in L^{p,2}; x = y\}) = \bar{\mathbb{P}}(u_0 = v_0 \text{ in } L^p(0, T; H)) = 1.$$

This fact together with Lemma 6 imply that the original sequence (u_ε) defined on the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$, \mathcal{F}^t, W converges in probability to an element u_0 in the topology of $L^p(0, T; H)$. By a passage to the limit's argument as in the previous subsection it is not difficult to show that u_0 is the unique solution of (6.10) (on the original probability system $(\Omega, \mathcal{F}, \mathbb{P})$, \mathcal{F}^t, W). This ends the proof of Theorem 5. ■

7. A CORRECTOR-TYPE RESULT

Our aim in this section is to prove some general corrector-type results.

Here and henceforth, we set, for a function $\mathbf{v} = (v_0, v_1) \in \mathbb{F}_0^{1,p}$, $\mathbb{D}_y \mathbf{v} = Dv_0 + \bar{D}_y v_1$ and $\mathbb{D}\mathbf{v} = Dv_0 + \partial \hat{v}_1 = \mathcal{G}_1^N(\mathbb{D}_y \mathbf{v})$. The following result holds.

Theorem 6. *Let the hypotheses be those of Theorem 5. There exists a continuous nondecreasing function $\bar{\nu} : [0, \infty) \rightarrow [0, \infty)$ with $\bar{\nu}(0) = 0$ such that for all $\Phi = (\psi_0, \varrho(\psi_1))$ with $\psi_0 \in L^p(\Omega \times (0, T); W_0^{1,p}(Q))$ and $\psi_1 \in L^p(\Omega \times (0, T); W_0^{1,p}(Q)) \otimes [A_\tau \otimes (A_y^1/\mathbb{R})]$, if we define Φ_ε as in (5.4) (see Corollary 1), then*

$$\limsup_{\varepsilon \rightarrow 0} \|Du_\varepsilon - D\Phi_\varepsilon\|_{L^p(Q_T \times \Omega)^N} \leq \bar{\nu} \left(\|\mathbb{D}_y \mathbf{u} - \mathbb{D}_y \Phi\|_{L^p(Q_T \times \Omega; \mathcal{B}_A^p)^N} \right). \quad (7.1)$$

Proof. The proof follows closely the one of its homologue in [26]. We repeat it here for reader's convenience. Let F_0^1 be the vector space of all Φ as in the statement of Theorem 6. Endowed with an obvious topology, F_0^1 has \mathcal{F}_0^∞ as a dense subspace (this is straightforward). Thus, we first establish (7.1) for Φ in \mathcal{F}_0^∞ . Owing to A2., for $\Phi \in \mathcal{F}_0^\infty$,

$$\begin{aligned} & c_1 \|Du_\varepsilon - D\Phi_\varepsilon\|_{L^p(Q_T \times \Omega)^N}^p \\ & \leq \int_{Q_T \times \Omega} (a^\varepsilon(\cdot, u_\varepsilon, Du_\varepsilon) - a^\varepsilon(\cdot, u_\varepsilon, D\Phi_\varepsilon)) \cdot D(u_\varepsilon - \Phi_\varepsilon) dx dt d\mathbb{P} \equiv B_\varepsilon. \end{aligned}$$

As shown in the proof of Proposition 3, we see that, as $\varepsilon \rightarrow 0$,

$$B_\varepsilon \rightarrow \iint_{Q_T \times \Omega \times \Delta(A)} (\hat{a}(\cdot, u_0, \mathbb{D}\mathbf{u}) - \hat{a}(\cdot, u_0, \mathbb{D}\Phi)) \cdot \mathbb{D}(\mathbf{u} - \Phi) dx dt d\mathbb{P} d\beta \equiv B,$$

where $\mathbf{u} = (u_0, u_1)$ is as in Proposition 3. It follows that

$$\limsup_{\varepsilon \rightarrow 0} \|Du_\varepsilon - D\Phi_\varepsilon\|_{L^p(Q_T \times \Omega)^N} \leq \left(\frac{B}{c_1}\right)^{\frac{1}{p}}.$$

But using Hölder's inequality together with the properties of the function a (see especially assumption A6. in Section 4), we get

$$B \leq c_0 \|1 + |u_0| + |\mathbb{D}\mathbf{u}| + |\mathbb{D}\Phi|\|_{L^p(Q_T \times \Omega \times \Delta(A))}^{p-2} \|\mathbb{D}(\mathbf{u} - \Phi)\|_{L^p(Q_T \times \Omega \times \Delta(A))}^2,$$

and by the obvious inequality $|\mathbb{D}\Phi| \leq |\mathbb{D}\mathbf{u} - \mathbb{D}\Phi| + |\mathbb{D}\mathbf{u}|$,

$$B \leq c_0 \left(\|1 + |u_0| + 2\|Du_0\|_{L^p(Q_T \times \Omega \times \Delta(A))} + \|\mathbb{D}(\mathbf{u} - \Phi)\|_{L^p(Q_T \times \Omega \times \Delta(A))} \right)^{p-2} \times \|\mathbb{D}(\mathbf{u} - \Phi)\|_{L^p(Q_T \times \Omega \times \Delta(A))}^2.$$

Now, set $\alpha = \|1 + |u_0| + 2\|Du_0\|_{L^p(Q_T \times \Omega \times \Delta(A))}$ and

$$\bar{\nu}(r) = \frac{c_0}{c_1^{\frac{1}{p}}} r^{\frac{2}{p}} (\alpha + r)^{1 - \frac{2}{p}} \text{ for } r \geq 0.$$

Then the function $\bar{\nu}$ is independent of Φ and satisfies hypotheses stated in Theorem 6 (this is straightforward by observing that $\|\mathbb{D}(\mathbf{u} - \Phi)\|_{L^p(Q_T \times \Omega \times \Delta(A))}^N = \|\mathbb{D}_y \mathbf{u} - \mathbb{D}_y \Phi\|_{L^p(Q_T \times \Omega; \mathcal{B}_A^p)^N}$). Whence (7.1) is shown for Φ in \mathcal{F}_0^∞ .

Now, let $\Phi \in F_0^1$. Let $(\Psi_j)_j$ be a sequence in \mathcal{F}_0^∞ such that $\Psi_j \rightarrow \Phi$ in F_0^1 as $j \rightarrow \infty$. Set

$$\Psi_j = (\varphi_{0j}, \varrho(\varphi_{1j})) \text{ and } \Phi = (\psi_0, \varrho(\psi_1)),$$

and define $\Psi_{j,\varepsilon} = \varphi_{0j} + \varepsilon\varphi_{1j}^\varepsilon$ and $\Phi_\varepsilon = \psi_0 + \varepsilon\psi_1^\varepsilon$ as in (5.4). We have

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} \|Du_\varepsilon - D\Phi_\varepsilon\|_{L^p(Q_T \times \Omega)^N} &\leq \limsup_{\varepsilon \rightarrow 0} \|Du_\varepsilon - D\Psi_{j,\varepsilon}\|_{L^p(Q_T \times \Omega)^N} + \\ &\limsup_{\varepsilon \rightarrow 0} \|D\Psi_{j,\varepsilon} - D\Phi_\varepsilon\|_{L^p(Q_T \times \Omega)^N} \\ &\leq \bar{\nu} \left(\|\mathbb{D}_y \mathbf{u} - \mathbb{D}_y \Psi_j\|_{L^p(Q_T \times \Omega; \mathcal{B}_A^p)^N} \right) + \limsup_{\varepsilon \rightarrow 0} \|D\Psi_{j,\varepsilon} - D\Phi_\varepsilon\|_{L^p(Q_T \times \Omega)^N}. \end{aligned}$$

Now, since $\Psi_j \rightarrow \Phi$ in F_0^1 , we get $\mathbb{D}\Psi_j \rightarrow \mathbb{D}\Phi$ in $L^p(Q_T \times \Omega \times \Delta(A))^N$ as $j \rightarrow \infty$. On the other hand, it can be easily shown that $\lim_{j \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \|D\Psi_{j,\varepsilon} - D\Phi_\varepsilon\|_{L^p(Q_T \times \Omega)^N} = 0$. Hence, taking the limit (as $j \rightarrow \infty$) of both sides of the last inequality above, we are led to (7.1). ■

All the ingredients are now available to state the corrector result.

Corollary 2. *Let the hypotheses be as in Theorem 6. Assume moreover that*

$$u_1 \in L^p(\Omega \times (0, T); W_0^{1,p}(Q)) \otimes [\varrho(A_\tau \otimes (A_y^1/\mathbb{R}))].$$

Then, as $\varepsilon \rightarrow 0$,

$$u_\varepsilon - u_0 - \varepsilon u_1^\varepsilon \rightarrow 0 \text{ in } L^p(\Omega \times (0, T); H^1(Q)).$$

Proof. It is clear that, on one hand, $\varepsilon u_1^\varepsilon \rightarrow 0$ in $L^p(Q_T \times \Omega)$ as $\varepsilon \rightarrow 0$; and on the other hand, due to the tightness property, it can be shown that the convergence result (6.1) still holds with $L^2(Q_T \times \Omega)$ replaced by $L^p(\Omega \times (0, T); L^2(Q))$, so that we have $u_\varepsilon - u_0 \rightarrow 0$ in $L^p(0, T; L^2(Q))$ a.s., and hence $u_\varepsilon - u_0 \rightarrow 0$ in $L^p(\Omega \times (0, T); L^2(Q))$. Thus $u_\varepsilon - u_0 - \varepsilon u_1^\varepsilon \rightarrow 0$ in $L^p(\Omega \times (0, T); L^2(Q))$.

as $\varepsilon \rightarrow 0$. It remains to show that $D(u_\varepsilon - u_0 - \varepsilon u_1^\varepsilon) \rightarrow 0$ in $L^p(\Omega \times (0, T); L^2(Q))^N$ as $\varepsilon \rightarrow 0$. But, if we set $\Phi_\varepsilon = u_0 + \varepsilon u_1^\varepsilon$, then applying (7.1), we get

$$\limsup_{\varepsilon \rightarrow 0} \|Du_\varepsilon - D\Phi_\varepsilon\|_{L^p(Q_T \times \Omega)^N} = 0$$

since $\bar{\nu} \left(\|\mathbb{D}_y \mathbf{u} - \mathbb{D}_y \Phi\|_{L^p(Q_T \times \Omega; \mathcal{B}_A^p)^N} \right) = \bar{\nu}(0) = 0$, and so $\lim_{\varepsilon \rightarrow 0} \|Du_\varepsilon - D\Phi_\varepsilon\|_{L^p(Q_T \times \Omega)^N} = 0$. Thus $D(u_\varepsilon - u_0 - \varepsilon u_1^\varepsilon) \rightarrow 0$ in $L^p(Q_T \times \Omega)^N$, and the result follows from the continuous embedding $L^p(\Omega \times (0, T); L^2(Q)) \rightarrow L^p(Q_T \times \Omega)$. We are therefore done. ■

Remark 6. If we assume that $u_\varepsilon \rightarrow u_0$ in $L^p(Q_T \times \Omega)$, then the corrector result is finer and expresses as follows:

$$u_\varepsilon - u_0 - \varepsilon u_1^\varepsilon \rightarrow 0 \text{ in } L^p(\Omega \times (0, T); W^{1,p}(Q)) \text{ as } \varepsilon \rightarrow 0.$$

This result holds especially in the deterministic setting since we have in that case the strong convergence result $u_\varepsilon \rightarrow u_0$ in $L^p(Q_T)$. In the stochastic framework, the above result fails in general, and we can not have a better result than the one in Corollary 2.

8. SOME CONCRETE APPLICATIONS OF THE ABSTRACT HOMOGENIZATION RESULT

In this section we give some applications of the results of Section 6 to concrete situations that occurred in some physical setting.

Example 1. The homogenization of (1.1) can be achieved under the periodicity assumption:

(5.1)₁ The functions $a_i(x, t, \cdot, \cdot, \mu, \lambda)$, $a_0(x, t, \cdot, \cdot, \mu)$ and $M_k(\cdot, \cdot, \mu)$ are both periodic of period 1 in each scalar coordinate, for any fixed $(x, t) \in \overline{Q_T}$, $(\mu, \lambda) \in \mathbb{R} \times \mathbb{R}^N$, and $1 \leq i \leq N$ and $k \geq 1$.

This leads to (5.1) with $A = \mathcal{C}_{\text{per}}(Y \times Z) = \mathcal{C}_{\text{per}}(Y) \odot \mathcal{C}_{\text{per}}(Z)$ (the product algebra, with $Y = (0, 1)^N$ and $Z = (0, 1)$), and hence $B_A^r = L_{\text{per}}^r(Y \times Z)$ for $1 \leq r \leq \infty$.

Example 2. The above functions in (5.1)₁ are both Besicovitch almost periodic in (y, τ) . This amounts to (5.1) with $A = AP(\mathbb{R}_{y, \tau}^{N+1}) = AP(\mathbb{R}_y^N) \odot AP(\mathbb{R}_\tau)$ ($AP(\mathbb{R}_y^N)$ the Bohr almost periodic functions on \mathbb{R}_y^N).

Example 3. The homogenization problem for (1.1) can be considered under the assumption

(5.1)₂ $a_i(x, t, \cdot, \cdot, \mu, \lambda)$ is weakly almost periodic while the functions $a_0(x, t, \cdot, \cdot, \mu)$ and $M_k(\cdot, \cdot, \mu)$ are almost periodic in the Besicovitch sense. This yields (5.1) with $A = WAP(\mathbb{R}_y^N) \odot WAP(\mathbb{R}_\tau)$ ($WAP(\mathbb{R}_y^N)$, the algebra of continuous weakly almost periodic functions on \mathbb{R}_y^N ; see e.g., [12]).

Example 4 (Homogenization in the Fourier-Stieltjes algebra). Let us first and foremost define the Fourier-Stieltjes algebra on \mathbb{R}^N . The Fourier-Stieltjes algebra on \mathbb{R}^N is defined as the closure in $BUC(\mathbb{R}^N)$ (the bounded uniformly continuous functions on \mathbb{R}^N) of the space

$$FS_*(\mathbb{R}^N) = \left\{ f : \mathbb{R}^N \rightarrow \mathbb{R}, f(x) = \int_{\mathbb{R}^N} \exp(ix \cdot y) d\nu(y) \text{ for some } \nu \in \mathcal{M}_*(\mathbb{R}^N) \right\}$$

where $\mathcal{M}_*(\mathbb{R}^N)$ denotes the space of complex valued measures ν with finite total variation: $|\nu|(\mathbb{R}^N) < \infty$. We denote it by $FS(\mathbb{R}^N)$.

Since by [12] any function in $FS_*(\mathbb{R}^N)$ is a weakly almost periodic continuous function, we have that $FS(\mathbb{R}^N) \subset WAP(\mathbb{R}^N)$. It is a well known fact that $FS(\mathbb{R}^N)$ is an ergodic algebra which is

translation invariant (this follows from the fact that $FS_*(\mathbb{R}^N)$ is translation invariant), so that all the hypotheses of Theorem 3 are satisfied with any algebra $A = FS(\mathbb{R}^N) \odot A_\tau$, A_τ being any algebra wmv on \mathbb{R}_τ .

This being so, we aim at solve homogenization problem for (1.1) under the assumption

$$(5.1)_3 \quad a_i(x, t, \cdot, \cdot, \mu, \lambda) \in B_{A_\tau}^{p'}(\mathbb{R}_\tau; B_{FS}^{p'}(\mathbb{R}_y^N)), \quad a_0(x, t, \cdot, \cdot, \mu), \quad M_k(\cdot, \cdot, \mu) \in B_{A_\tau}^2(\mathbb{R}_\tau; B_{FS}^2(\mathbb{R}_y^N))$$

for any $(\mu, \lambda) \in \mathbb{R} \times \mathbb{R}^N$, and for all $(x, t) \in \overline{Q}_T$, $(1 \leq i \leq N)$
 where $B_{FS}^{p'}(\mathbb{R}_y^N)$ denotes the closure of the algebra $FS(\mathbb{R}_y^N)$ with respect to the seminorm $\|\cdot\|_{p'}$, and A_τ is any arbitrary algebra wmv on \mathbb{R}_τ . We are then led to (5.1) with $A = FS(\mathbb{R}^N) \odot A_\tau$.

Remark 7. It should be stressed that the problems solved in Examples 3 and 4 are new in the literature as far as the homogenization of SPDEs is concerned.

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DEPARTMENT OF MATHEMATICS AND APPLIED MATHEMATICS, UNIVERSITY OF PRETORIA, PRETORIA 0002, SOUTH AFRICA (P.A. RAZAFIMANDIMBY)

E-mail address: paulrazafi@gmail.com

DEPARTMENT OF MATHEMATICS AND APPLIED MATHEMATICS, UNIVERSITY OF PRETORIA, PRETORIA 0002, SOUTH AFRICA (M. SANGO)

E-mail address: mamadou.sango@up.ac.za

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, UNIVERSITY OF DSCHANG, P.O. BOX 67, DSCHANG, CAMEROON (J.L. WOUKENG)

E-mail address: jwoukeng@yahoo.fr