The degree-diameter problem for claw-free graphs and hypergraphs

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Abstract

We study the degree-diameter problem for claw-free graphs and 2-regular hypergraphs. Let $\mathrm{cf}_{\Delta,D}$ be the largest order of a claw-free graph of maximum degree Δ and diameter D. We show that $\mathrm{cf}_{\Delta,D} \le 1 + 2\sum_{i=1}^{D}(\frac{\Delta}{2})^i - c'_\Delta \sum_{i=0}^{D-2}(\frac{\Delta}{2})^i$, where $c'_\Delta = 2\frac{(\Delta/2)^2}{(\Delta/2)^2+\Delta/2+2}$, for any D and any even $\Delta \ge 4$. So for claw-free graphs the well-known Moore bound can be strengthened considerably. We further show that $\mathrm{cf}_{\Delta,2} \ge \frac{5}{16}(\Delta+2)^2$ for $\Delta \ge 6$ with $\Delta \equiv 2 \pmod 4$. We also give an upper bound on the order of $K_{1,p}$-free graphs of given maximum degree and diameter for $p \ge 3$. We prove similar results for the hypergraph version of the degree-diameter problem. The hypergraph Moore bound states that the order of a hypergraph of maximum degree Δ, rank k and diameter D is at most $1+\Delta\sum_{i=1}^{D}(\Delta-1)^{i-1}(k-1)^i$. For 2-regular hypergraph of rank $k \ge 3$ and any diameter D, we improve this bound to $1 + 2\sum_{i=1}^{D}(k-1)^i - c_k\sum_{i=0}^{D-2}(k-1)^i$, where $c_k = 2\frac{k^2-2k+1}{k^2-k+2}$. Our construction of claw-free graphs of diameter 2 yields a similar result for hypergraphs of diameter 2, degree 2 and any even rank $k \ge 4$.

Keywords: claw-free graph; hypergraph; degree; diameter; Moore geometry

1. Introduction and results

The degree-diameter problem is to determine the largest order $n_{\Delta,D}$ of a graph of given maximum degree Δ and diameter D. It is well-known that the number of vertices in a graph of maximum degree Δ and diameter D cannot exceed the Moore bound $M_{\Delta,D} = 1+\Delta+\Delta(\Delta-1)+\ldots+\Delta(\Delta-1)^{D-1}$. Bannai and Ito [1] improved the upper bound and showed that $n_{\Delta,D} \leq M_{\Delta,D} - 2$ for any $\Delta, D \geq 3$, and we have $n_{\Delta,D} \leq M_{\Delta,D} - 3$ if $\Delta \geq 4$ is even; see [8]. There is a slightly better bound if $\Delta = 3$, since Miller and Pineda-Villavicencio [11] proved that $n_{3,D} \leq M_{3,D} - 6$ for any $D \geq 5$. A construction of Canale and Gómez [4] gives the best known lower bound on $n_{\Delta,D}$ for large Δ and D. They showed that there is a constant D_0 such that for each $D \geq D_0$ congruent with $-1, 0$ or 1 modulo 8, and for infinitely many values of Δ, we have $n_{\Delta,D} \geq (\frac{\Delta}{1.45})^D$. For a survey on the problem we refer the reader to [12].

In this paper we study the degree-diameter problem for claw-free graphs and hypergraphs. This is motivated by the observation that graphs whose order is close to the Moore bound do not have small cycles, hence such graphs have many claws. It is therefore natural to expect that the Moore bound can be significantly improved if restricted to claw-free graphs, or more general to graphs that contain no induced subgraph $K_{1,p}$. We show in Theorem 1 that this is indeed the case, and we derive a bound that we term $K_{1,p}$-free Moore bound:

$$n \leq 1 + \sum_{i=1}^{D} \left(\frac{p-2}{p-1}\right)^{i-1} \Delta^i. \tag{1}$$

Our investigations focus particularly on claw-free graphs of even maximum degree. Let $\mathrm{cf}_{\Delta,D}$ be the largest order of a claw-free graph of maximum degree Δ and diameter D. The special case $p = 3$ of (1), which we term claw-free Moore bound, is the upper bound

$$\mathrm{cf}_{\Delta,D} \leq 1 + 2\sum_{i=1}^{D} \left(\frac{\Delta}{2}\right)^{i}, \tag{2}$$

which differs from the Moore bound $M_{\Delta,D}$ by a factor of approximately $(\frac{1}{2})^{D-1}$. In Theorem 2 we show that claw-free graphs whose order is close to this bound are underlying graphs of 2-regular, $(\frac{\Delta}{2}+1)$-uniform hypergraphs.

Having established this link between claw-free graphs and hypergraphs, we consider the degree-diameter problem for 2-regular hypergraphs. Note that hypergraphs are often modelled by bipartite graphs with the vertices and edges of the hypergraphs as partite sets. This representation facilitates structural analysis on the hypergraphs and relates the degree-diameter problem for hypergraphs to the degree-diameter problem for bipartite graphs.

The general degree-diameter problem for hypergraphs is to determine the largest order of a connected hypergraph of given maximum degree Δ (the maximum number of hyperedges containing any vertex), rank k (the maximum number of vertices in any hyperedge) and diameter D. The well-known hypergraph Moore bound on the order of such a hypergraph is

$$n \leq 1 + \Delta \sum_{i=1}^{D} (\Delta - 1)^{i-1}(k-1)^i. \tag{3}$$

The hypergraphs attaining the upper bound are called Moore geometries. Moore geometries with $k = 2$ are Moore graphs. Damerell and Georgiacodis [6] and Damerell [5] showed that there are no non-trivial Moore geometries for $D \geq 5$, and Fuglister [9, 10] proved the non-existence for diameters 3 and 4. Except for odd cycles, all Moore geometries have diameter $D \leq 2$, and there are no known Moore geometries with $D = 2$ and $k \geq 3$, see [2].

The hypergraph Moore bound for 2-regular hypergraphs is $n \leq 1 + 2\sum_{i=1}^{D}(k-1)^i$. In Theorem 3 we improve this bound and show that the order n of a 2-regular hypergraph of rank $k \geq 3$ and diameter D is bounded by

$$n \leq 1 + 2\sum_{i=1}^{D}(k-1)^i - c_k \sum_{i=1}^{D-2}(k-1)^i.$$

where $c_k = 2\frac{k^2-2k+1}{k^2-k+2}$. So, unlike for graphs, where the Moore bound has been improved only by a small constant, the Moore bound for 2-regular hypergraphs can be improved significantly. As a corollary to Theorem 3 we obtain the non-existence of Moore geometries of degree 2 of any diameter and rank $k \geq 3$.

In the last section of the paper we give lower bounds on $\mathrm{cf}_{\Delta,D}$. These are derived by taking line graphs of known constructions of large graphs of given maximum degree and diameter. We also give a new construction of claw-free graphs of diameter 2, degree $\Delta \geq 6$, where $\Delta \equiv 2 \pmod 4$, and order $\frac{5}{16}(\Delta+2)^2$. We show that these bounds give rise to lower bounds for the maximal order of 2-regular, k-uniform hypergraphs of given diameter.

Our notation is as follows. The vertex set of a graph or hypergraph G is denoted by $V(G)$, and the edge (for hypergraphs we sometimes use the term hyperedge) set by $E(G)$. We use $n(G)$ for the number of vertices of G. We write $\deg_G(v)$ for the degree of vertex v, i.e., the number of edges incident with v, and the maximum degree is $\Delta(G)$. If G is understood then we often drop the argument or subscript G. The rank of a hypergraph is the largest cardinality of its edges. If all edges have the same cardinality k, then we say G is k-uniform. Every hypergraph H gives rise to a graph on the same vertex set, where two vertices are adjacent in G if some edge of H contains them both. We say that in this case G is the underlying graph of H. The diameter of a hypergraph is the diameter of its underlying graph. Let $N(v)$ be the neighbourhood of v. If G is a graph and U_1, U_2 are two disjoint subsets of $V(G)$, then $E(U_1, U_2)$ is the set of edges which have one end in U_1 and the other end in U_2. The graph $G - U_1$ is obtained from G by removing the vertices in U_1 and all edges incident with a vertex in U_1. The subgraph of G induced by U_1 is denoted by $G[U_1]$.

K_n and $K_{m,n}$ stand for the complete graph on n vertices and the complete bipartite graph with partite sets of cardinality m and n, respectively. If H is a fixed graph, and if G does not have H as an induced subgraph, then we say that G is H-free. A $K_{1,3}$-free graph is commonly referred to as a claw-free graph. The independence number of G and the complement of G are denoted by $\alpha(G)$ and G^c, respectively.

Assume that G is rooted at a vertex v and that $k \in \mathbb{N}_0$. Then $V_k(v)$ is the set of vertices of G at distance exactly k from v. If v is understood, then we often drop the argument v and write V_k. We also denote $\bigcup_{i=1}^{k} V_i$ by $V_{\leq k}$. If u and w are distinct vertices of G, and u is on a $v-w$ geodesic, then we say that w is a descendant of u, and that u is a predecessor of w.

2. Upper bounds

It is well-known that graphs of given maximum degree and diameter with order close to the Moore bound cannot have small cycles, specifically triangles. Hence the neighbourhood of every vertex is an independent set, so such a graph contains many claws, or generally many induced subgraphs $K_{1,p}$ for $p \leq \Delta$. Theorem 1 below shows that for graphs that do not contain an induced subgraph $K_{1,p}$ the Moore bound can be improved significantly. We will call the bound in Theorem 1 the $K_{1,p}$-free Moore bound.

In our proofs we will make use of a slightly weaker form of Turán's Theorem (Theorem A) and its complement version (Corollary A). The Turán graph $T_{n,s}$ is the complete s-partite graph of order n, whose partite sets have cardinalities $\lceil \frac{n}{s} \rceil$ or $\lfloor \frac{n}{s} \rfloor$. Note that $|E(T_{n,s})| \leq (\frac{s-1}{s})\frac{n^2}{2}$, with equality only if n is a multiple of s.

Theorem A. *Let H be a K_{s+1}-free graph of order n. Then $|E(H)| \leq |E(T_{n,s})| \leq \frac{s-1}{2s}n^2$, where $|E(H)| = \frac{s-1}{2s}n^2$ if and only if n is divisible by s and $H = T_{n,s}$.*

Corollary A. *Let H be a graph of order n and let $\alpha(G) \leq s$. Then $|E(H)| \geq \frac{n}{2}(\frac{n}{s} - 1)$ with equality if and only if n is divisible by s and $G = T^c_{n,s}$.*

The complement version of Theorem A implies that only graphs consisting of two disjoint cliques have minimum size among the graphs of independence number 2. The following lemma, which we will need in the proof of Theorem 2, gives a lower bound on the size of graphs of independence number at most 2 that are not the union of two cliques.

Lemma 1 *Let H be a graph of order n and independence number at most 2. If $V(H)$ cannot be partitioned into two sets, such that each set induces a clique, then $n \geq 5$ and*

$$|E(H)| \geq \begin{cases} \frac{n^2}{4} - 1 & \text{if } n \text{ is even,} \\ \frac{n^2}{4} - \frac{5}{4} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Since $V(H)$ cannot be partitioned into two sets, each inducing a clique, the complement graph H^c is not bipartite. Hence H^c contains an odd cycle. Among the odd cycles in H^c let C_k be one of minimum length k. Then C_k is an induced cycle of H^c. We have $k \geq 5$ since otherwise, if $k = 3$, the 3-cycle vertices would form an independent set in H, a contradiction to $\alpha(H) \leq 2$. This implies $n \geq 5$. Since C_k is an induced cycle in H^c, we have

$$|E(H^c[V(C_k)])| = k.$$

Since $H^c - V(C_k)$ is triangle-free, by Theorem A,

$$|E(H^c - V(C_k))| \leq \begin{cases} \frac{(n-k)^2-1}{4} & \text{if } n-k \text{ is odd,} \\ \frac{(n-k)^2}{4} & \text{if } n-k \text{ is even.} \end{cases}$$

Since a vertex in $H^c - V(C_k)$ cannot be adjacent to two consecutive vertices of C_k, it has at most $\frac{k-1}{2}$ neighbours on C_k. Hence,

$$|E(V(H^c - V(C_k)), V(C_k))| \leq (n-k)\frac{k-1}{2}.$$

In total we obtain

$$\begin{array}{rcl} |E(H^c)| & = & |E(H^c[V(C_k)])| + |E(H^c - V(C_k))| + |E(V(H^c - V(C_k)), V(C_k))| \\ & \leq & \begin{cases} k + \frac{(n-k)^2-1}{4} + (n-k)\frac{k-1}{2} & \text{if } n-k \text{ is odd,} \\ k + \frac{(n-k)^2}{4} + (n-k)\frac{k-1}{2} & \text{if } n-k \text{ is even.} \end{cases} \end{array}$$

It is easy to verify that the right hand side of the inequality is decreasing in k. Substituting $k = 5$ yields

$$|E(H^c)| \leq \begin{cases} \frac{n^2}{4} - \frac{n}{2} + 1 & \text{if } n \text{ is even,} \\ \frac{n^2}{4} - \frac{n}{2} + \frac{5}{4} & \text{if } n \text{ is odd,} \end{cases}$$

which implies

$$|E(H)| \geq \begin{cases} \frac{n^2}{4} - 1 & \text{if } n \text{ is even,} \\ \frac{n^2}{4} - \frac{5}{4} & \text{if } n \text{ is odd.} \end{cases}$$

□

Lemma 2 *Let $p \geq 3$ and let G be a $K_{1,p}$-free graph of maximum degree Δ, rooted at a vertex v. Then $|V_2| \leq \frac{p-2}{p-1}\Delta^2$.*

Proof. Since G has no induced $K_{1,p}$, the graph $G[V_1]$ has independence number at most $p-1$. By Corollary A, we have $|E(G[V_1])| \geq \frac{|V_1|}{2}(\frac{|V_1|}{p-1} - 1)$. Hence

$$\begin{array}{rcl} |V_2| & \leq & |E(V_1, V_2)| \;=\; \sum_{u \in V_1} \deg_G(u) - 2|E(G[V_1])| - |E(V_0, V_1)| \\ & \leq & |V_1|\Delta - |V_1|\Big(\frac{|V_1|}{p-1} - 1\Big) - |V_1| \;=\; |V_1|\Big(\Delta - \frac{|V_1|}{p-1}\Big). \end{array} \quad (4)$$

It is easy to verify that, subject to $|V_1| \leq \Delta$, the right hand side of (4) is maximized if $|V_1| = \Delta$. Substituting this yields $|V_2| \leq \frac{p-2}{p-1}\Delta^2$. □

Theorem 1 *Let $p \geq 3$. Let G be a connected graph of order n, maximum degree Δ and diameter D, which contains no $K_{1,p}$ as an induced subgraph. Then*

$$n \leq 1 + \sum_{i=1}^{D} \left(\frac{p-2}{p-1}\right)^{i-1} \Delta^i.$$

Proof. Let G be a $K_{1,p}$-free graph of degree at most Δ and diameter D, rooted at a vertex v. From Lemma 2 it follows that $|V_2| \le \frac{p-2}{p-1}\Delta^2$. Since G is $K_{1,p}$-free, we have $\alpha(G[N(x) \cap V_i]) \le p-2$ for all $x \in V_{i-1}$, $2 \le i \le D$. This implies

$$\alpha(G[V_i]) \le |V_{i-1}|(p-2).$$

As above, applying Corollary A to $G[V_i]$ we obtain

$$\begin{aligned} |V_{i+1}| &\le \sum_{u \in V_i} \deg_G(u) - 2|E(G[V_i])| - |E(V_{i-1}, V_i)| \\ &\le |V_i|\Delta - |V_i|\Big(\frac{|V_i|}{|V_{i-1}|(p-2)} - 1\Big) - |V_i| \\ &= |V_i|\Big(\Delta - \frac{|V_i|}{|V_{i-1}|(p-2)}\Big). \end{aligned} \tag{5}$$

We now show that, for $i = 1, 2, \ldots, D$, we have $|V_i| \le (\frac{p-2}{p-1})^{i-1}\Delta^i$. The proof is by induction on i. For $i = 1, 2$ the inequality has already been shown, so we assume that $i \ge 3$. Denote the right hand side of (5) by f. We consider two cases.

Case 1: $p = 3$.
Simple maximisation shows that f, as a function of $|V_i|$ attains its maximum if $|V_i| = \frac{\Delta}{2}|V_{i-1}|$. Hence

$$|V_{i+1}| \le \frac{\Delta}{2}|V_{i-1}|\Big(\Delta - \frac{\Delta|V_{i-1}|}{2|V_{i-1}|}\Big) = \frac{\Delta^2}{4}|V_{i-1}|.$$

Case 2: $p \ge 4$.
f is increasing in $|V_{i-1}|$. Moreover, since $|V_i| < \Delta|V_{i-1}|$, f is also increasing in $|V_i|$. Hence

$$\begin{aligned} |V_{i+1}| &\le |V_i|\Big(\Delta - \frac{|V_i|}{|V_{i-1}|(p-2)}\Big) \\ &\le \Big(\frac{p-2}{p-1}\Big)^{i-1}\Delta^i\Big(\Delta - \frac{(\frac{p-2}{p-1})^{i-1}\Delta^i}{(\frac{p-2}{p-1})^{i-2}\Delta^{i-1}(p-2)}\Big) = \Big(\frac{p-2}{p-1}\Big)^i\Delta^{i+1}, \end{aligned}$$

as desired. □

We now focus on the case $p = 3$, i.e., claw-free graphs. Theorem 1 shows that for a claw-free, connected graph of maximum degree Δ and diameter

D, the order n can be bounded by

$$n \leq 1 + 2\sum_{i=1}^{D} \left(\frac{\Delta}{2}\right)^i.$$

We refer to this as the claw-free Moore bound. We improve this bound for even Δ in two steps. In Theorem 2 below we show that every claw-free graph of even maximum degree Δ and diameter D whose order is close to the claw-free Moore bound is the underlying graph of some 2-regular, $(\frac{\Delta}{2}+1)$-uniform hypergraph. Then we improve the hypergraph Moore bound for 2-regular hypergraphs in Theorem 3.

We call a vertex u partitionable if the neighbourhood of u can be partitioned into two disjoint sets such that each set induces a complete graph. We call u equipartitionable if the neighbourhood of u can be partitioned into two disjoint sets of equal cardinality such that each set induces a complete graph.

Theorem 2 *Let G be a connected, claw-free graph of order n, diameter D and even maximum degree Δ. If*

$$n > 1 + 2\sum_{i=1}^{D} \left(\frac{\Delta}{2}\right)^i - 2\left(\frac{\Delta}{2}\right)^{D-2},$$

then G is Δ-regular and every vertex of G is equipartitionable. Moreover, G is the underlying graph of some 2-regular, $(\frac{\Delta}{2}+1)$-uniform hypergraph.

Proof. Let G be rooted at an arbitrary but fixed vertex v, which we will determine later. For $i = 0, 1, \ldots, D$ we denote the ith distance layer, i.e., the set of vertices at distance exactly i from v, by V_i.

Claim 1: Let $v_i \in V_i$, $1 \leq i \leq D-2$. If $|N(v_i) \cap V_{\leq i}| = \frac{\Delta}{2} + s$, then v_i has in V_{i+2} at most $\frac{\Delta^2}{4} - s^2$ descendants if $s > 0$, and at most $\frac{\Delta^2}{4}$ descendants if $s \leq 0$.

Let $|N(v_i) \cap V_{i+1}| = \frac{\Delta}{2} - p$. Then $p \geq s$. Since G is claw-free, and since v_i is adjacent to some vertex in V_{i-1}, the set $N(v_i) \cap V_{i+1}$ induces a complete graph $K_{\frac{\Delta}{2}-p}$. Hence each neighbour of v_i in V_{i+1} is adjacent to at most $\frac{\Delta}{2}+p$ vertices in V_{i+2}. Therefore, v_i has at most

$$\left(\frac{\Delta}{2} - p\right)\left(\frac{\Delta}{2} + p\right) = \left(\frac{\Delta}{2}\right)^2 - p^2 \qquad (6)$$

descendants in V_{i+2}. Clearly, (6) is at most $(\frac{\Delta}{2})^2 - s^2$ if $s > 0$ and (6) is at most $(\frac{\Delta}{2})^2$ if $s \le 0$.

As an immediate consequence of Claim 1 we obtain the following claim:

Claim 2: Let $1 \le i \le D-2$. Then $|V_{i+2}| \le \frac{\Delta^2}{4}|V_i|$.

Claim 3: $|V_2| > \frac{\Delta^2}{2} - 2$ or $|V_3| > \frac{\Delta^3}{4} - \Delta + 2$.

Suppose to the contrary that $|V_2| \le \frac{\Delta^2}{2} - 2$ and $|V_3| \le \frac{\Delta^3}{4} - \Delta + 2$. Then $|V_2| \le \frac{\Delta^2}{2} - 2 + \frac{4}{\Delta}$. Repeated application of Claim 2 yields that,

$$|V_i| \le \frac{\Delta^i}{2^{i-1}} - \frac{\Delta^{i-2}}{2^{i-3}} + \frac{\Delta^{i-3}}{2^{i-4}} \text{ for } i = 2, 3, \ldots, D.$$

Summation over $i = 2, 3, \ldots, D$ yields

$$\begin{aligned} n &= 1 + \sum_{i=1}^{D} |V_i| \\ &\le 1 + \sum_{i=1}^{D} \frac{\Delta^i}{2^{i-1}} - \sum_{i=1}^{D} \frac{\Delta^{i-2}}{2^{i-3}} + \sum_{i=2}^{D} \frac{\Delta^{i-3}}{2^{i-4}} \\ &= 1 + \sum_{i=1}^{D} \frac{\Delta^i}{2^{i-1}} - 2\Big(\frac{\Delta}{2}\Big)^{D-2}, \end{aligned}$$

a contradiction to the hypothesis of the lemma.

Claim 4: Every vertex of G is partitionable.

Suppose to the contrary that G contains a vertex that is not partitionable. Denote this vertex by v and root G at v. Then $G[V_1]$ cannot be partitioned into two sets such that each set induces a clique. Since Δ is even, it follows by Lemma 1 that $\Delta \ge 6$ and the graph $G[V_1]$ has at least $\frac{|V_1|^2}{4} - \frac{5}{4}$ edges. Let $|V_1| = \Delta - p$. Then we bound $|V_2|$ as follows.

$$\begin{aligned} |V_2| &\le |E(V_1, V_2)| \qquad (7) \\ &\le \sum_{u \in V_1} \deg_G(u) - 2|E(G[V_1])| - |E(V_0, V_1)| \\ &\le |V_1|\Delta - 2\Big(\frac{|V_1|^2}{4} - \frac{5}{4}\Big) - |V_1| \\ &= \frac{\Delta^2}{2} - \Delta + \frac{5}{2} + p - \frac{p^2}{2}, \end{aligned}$$

which is at most $\frac{\Delta^2}{2} - 3$ since $p - \frac{p^2}{2} \leq \frac{1}{2}$ for any p.

We now show that $|V_3| \leq \frac{\Delta^3}{4} - \Delta + 2$, which yields a contradiction to Claim 3. If $|V_1| \leq \Delta - 1$, then $|V_3| \leq (\Delta - 1)\frac{\Delta^2}{4} < \frac{\Delta^3}{4} - \Delta + 2$, so we assume that $|V_1| = \Delta$. Let $u_1, u_2, \ldots, u_\Delta$ be the neighbours of v, and define s_i by $|N(u_i) \cap V_{\leq 1}| = \frac{\Delta}{2} + s_i$ for $i = 1, 2, \ldots, \Delta$. We first bound $\sum_{i=1}^{\Delta} s_i$ from below. Since $|V_1|$ is even, by Lemma 1, the graph $G[V_1]$ has at least $\frac{\Delta^2}{4} - 1$ edges. Hence

$$\sum_{i=1}^{\Delta} |N(u_i) \cap V_{\leq 1}| = 2|E(G[V_1])| + \Delta \geq 2\Big(\frac{\Delta^2}{4} - 1\Big) + \Delta = \frac{\Delta^2}{2} + \Delta - 2.$$

On the other hand we have

$$\sum_{i=1}^{\Delta} |N(u_i) \cap V_{\leq 1}| = \sum_{i=1}^{\Delta} \Big(\frac{\Delta}{2} + s_i\Big) = \frac{\Delta^2}{2} + \sum_{i=1}^{\Delta} s_i.$$

The last two inequalities imply $\sum_{i=1}^{\Delta} s_i \geq \Delta - 2$.

Let V_1' be the subset of V_1 containing those u_i for which $s_i > 0$. We may assume that $V_1' = \{u_1, u_2, \ldots, u_{\Delta'}\}$ where $\Delta' \leq \Delta$. Clearly, $\sum_{i=1}^{\Delta'} s_i \geq \Delta - 2$. From Claim 1 it follows that any vertex u_i, where $i = 1, 2, \ldots, \Delta'$ (where $i = \Delta'+1, \Delta'+2, \ldots, \Delta$) has at most $\frac{\Delta^2}{4} - s_i^2$ (at most $\frac{\Delta^2}{4}$) descendants in V_3. Then $|V_3| \leq \frac{\Delta^3}{4} - \sum_{i=1}^{\Delta'} s_i^2$. Since $\sum_{i=1}^{\Delta'} s_i^2 \geq \Delta - 2$, we have $|V_3| \leq \frac{\Delta^3}{4} - \Delta + 2$. This, in conjunction with $|V_2| \leq \frac{\Delta^2}{2} - 3$ yields a contradiction to Claim 3, and so Claim 4 follows.

From now on we will denote the two cliques in which a vertex u is contained by $C_1(u)$ and $C_2(u)$, and their cardinalities as $c_1(u)$ and $c_2(u)$, respectively. (If there is more than one way to partition the neighbours of u into cliques then chose any.) Note that $c_1(u) + c_2(u) = \deg_G(u) + 2$.

Claim 5: For each vertex $u \in V_1$ we have

$$\bigcup_{x \in V_1 - \{u\}} N(x) \cap V_2 \quad \subset \bigcup_{x \in V_1} N(x) \cap V_2.$$

Suppose to the contrary that there exists a vertex $u \in V_1$ with $\bigcup_{x \in V_1} N(x) \cap V_2 = \bigcup_{x \in V_1 - \{u\}} N(x) \cap V_2$. For $i = 1, 2$, every vertex in $C_i(v) - \{v\}$ is adjacent to at least $c_i(v) - 1$ vertices in $V_0 \cup V_1$, and has thus at most $\Delta - c_i(v) + 1$

descendants in V_2. We may assume that $u \in C_1(v)$. Then, by our hypothesis,

$$\begin{aligned} |V_2| &\leq (c_1(v)-2)(\Delta - c_1(v)+1) + (c_2(v)-1)(\Delta - c_2(v)+1) \\ &= \sum_{i=1}^{2}(c_i(v)-1)(\Delta - c_i(v)+1) - (\Delta - c_1(v)+1). \end{aligned}$$

It is now easy to see that $(c_2(v)-1)(\Delta - c_2(v)+1) \leq \frac{\Delta^2}{4}$ and $(c_1(v)-2)(\Delta - c_1(v)+1) \leq \frac{\Delta}{2}(\frac{\Delta}{2}-1)$, implying that $|V_2| \leq \frac{\Delta^2}{4} - \frac{\Delta}{2} \leq \frac{\Delta^2}{4} - 2$. We bound $|V_3|$ as follows. Since every vertex in V_3 is a descendant of a vertex in $V_1 - \{u\}$, we have by Claim 2 that $|V_3| \leq (\Delta - 1)\frac{\Delta^2}{4} \leq \frac{\Delta^3}{4} - \Delta + 2$. These bounds on $|V_2|$ and $|V_3|$ yield a contradiction to Claim 3, so Claim 5 follows.

Claim 6: There is no edge joining a vertex in $C_1(v) - \{v\}$ to a vertex in $C_2(v) - \{v\}$.

Suppose to the contrary that there exists an edge uw with $u \in C_1(v) - \{v\}$ and $w \in C_2(v) - \{v\}$. Without loss of generality we may assume that $v \in C_1(u)$ and $v \in C_1(w)$, so $(C_1(u) \cup C_1(w)) \cap V_2 = \emptyset$. Then $C_2(u)$ does not contain any vertex in $V_1 - \{u\}$ since otherwise, if there was a vertex $x \in C_2(u) \cap (V_1 - \{u\})$, every neighbour of u in V_2 would also be a neighbour of x, contradicting Claim 5. Similarly $C_2(w)$ does not contain any vertex in $V_1 - \{w\}$. Hence $w \in C_1(u)$, and so w is adjacent to every vertex in $C_1(u)$ and thus to every vertex in V_1. This implies $C_1(w) = V_0 \cup V_1$, and so the neighbourhood of v induces a clique. If now v has degree k, then every vertex in V_1 is adjacent to at most $\Delta - k$ vertices in V_2. Hence

$$|V_2| \leq k(\Delta - k) \leq \frac{\Delta^2}{4} \leq \frac{\Delta^2}{2} - 2.$$

Furthermore, it follows from Claim 1 that

$$|V_3| \leq \begin{cases} k\frac{\Delta^2}{4} & \text{if } k \leq \frac{\Delta}{2}, \\ k(\frac{\Delta^2}{4} - (k - \frac{\Delta}{2})^2) & \text{if } k > \frac{\Delta}{2}, \end{cases}$$

which is easily seen to be less than $\frac{\Delta^3}{4} - \Delta + 2$, and so we obtain a contradiction to Claim 3. This proves Claim 6.

Since v is arbitrary, Claim 6 implies that the neighbourhood of every vertex induces a graph isomorphic to the disjoint union of two complete graphs. This yields the following claim.

Claim 7: Let $uw \in E(G)$. If $u \in C_i(w)$ and $w \in C_j(u)$, then $C_i(w) = C_j(u)$. Furthermore, if K is an inclusion-maximal clique of G, then for each $u \in K$ we have $C_i(u) = K$ for some $i \in \{1, 2\}$.

Claim 8: For all $u \in V$ and $i \in \{1, 2\}$ we have $c_i(u) \neq \frac{\Delta}{2}$.

Suppose to the contrary that there exists a vertex u with, say, $c_1(u) = \frac{\Delta}{2}$. Then $C_1(u)$ forms a maximal clique of order $\frac{\Delta}{2}$. We consider three cases.
Case 1: There exists a maximal clique K of G of order $\frac{\Delta}{2}$, such that every vertex of K has degree Δ.

It follows from Claim 7 that for every $w \in K$, we have $C_i(w) = K$ for some $i \in \{1, 2\}$. We may assume that $C_1(w) = K$ for all $w \in K$. Then each $C_2(w)$ is a clique of order $\frac{\Delta}{2} + 2$. Choose v to be one of the vertices in K and root G at v. For any vertex $u \in C_1(v) - \{v\}$ we have $|N(u) \cap V_2| \leq \frac{\Delta}{2} + 1$, and for any $u \in C_2(v) - \{v\}$ we have $|N(u) \cap V_2| \leq \frac{\Delta}{2} - 1$. Therefore,

$$|V_2| \leq \sum_{u \in V_1} |N(u) \cap V_2| \leq \left(\frac{\Delta}{2} - 1\right)\left(\frac{\Delta}{2} + 1\right) + \left(\frac{\Delta}{2} + 1\right)\left(\frac{\Delta}{2} - 1\right) = \frac{\Delta^2}{2} - 2.$$

Let $u \in C_1(v) - \{v\}$. Every vertex in $N(u) \cap V_2$ has at least $\frac{\Delta}{2} + 1$ neighbours in $V_1 \cup V_2$, and thus at most $\frac{\Delta}{2} - 1$ neighbours in V_3. Hence, u has at most $(\frac{\Delta}{2} + 1)(\frac{\Delta}{2} - 1)$ descendants in V_3. Now let $u \in C_2(v) - \{v\}$. Then u is adjacent to $\frac{\Delta}{2} + 1$ vertices in $V_0 \cup V_1$, and thus has at most $\frac{\Delta^2}{4} - 1$ descendants in V_3 by Claim 1. Hence we obtain

$$|V_3| \leq (c_1(v) - 1)\left(\frac{\Delta^2}{4} - 1\right) + (c_2(v) - 1)\left(\frac{\Delta^2}{4} - 1\right) = \frac{\Delta^3}{4} - \Delta.$$

This bound on $|V_3|$ together with the above bound on $|V_2|$ yield a contradiction to Claim 3.

Case 2: Every maximal clique of order $\frac{\Delta}{2}$ contains a vertex of degree at most $\Delta - 1$ and a vertex of degree Δ.

Choose v to be a vertex of degree Δ which is contained in a maximum clique, $C_1(v)$ say, of order $\frac{\Delta}{2}$, and root G at v. As in the previous case it follows that $|V_2| \leq \frac{\Delta^2}{2} - 2$.
To bound $|V_3|$, we bound the number of descendants of the vertices in $C_1(v) - \{v\}$ and in $C_2(v) - \{v\}$ separately. By Claim 2 every vertex in $C_1(v) - \{v\}$ has at most $\frac{\Delta^2}{4}$ descendants in V_3.

Now consider a vertex $u \in C_2(v)$. We show that u has at most $\frac{\Delta^2}{4} - 2$ descendants in V_3. By Claim 7, $C_2(v) = C_i(u)$ for some $i \in \{1, 2\}$, we may assume that $C_2(v) = C_1(u)$. Then $N(u) \cap V_2 \subset C_2(V_2) - \{u\}$, and so u has at most $\frac{\Delta}{2} - 1$ neighbours in V_2. We consider two cases: (a) u has $\frac{\Delta}{2} - 1$ neighbours in V_2, and (b) u has at most $\frac{\Delta}{2} - 2$ neighbours in V_2.
In case (a), $\deg u = \Delta$, $C_2(u) - \{u\} \subseteq V_2$, and $C_2(u)$ is a maximal clique of order $\frac{\Delta}{2}$. By the defining condition of Case 2, $C_2(u)$ contains a vertex of degree at most $\Delta - 1$, which thus has at most $\frac{\Delta}{2}$ neighbours in V_3, while the other $\frac{\Delta}{2} - 2$ vertices of $C_2(u) - \{u\}$ have at most $\frac{\Delta}{2} + 1$ neighbours in V_3. Hence u has at most $(\frac{\Delta}{2} - 2)(\frac{\Delta}{2} + 1) + \frac{\Delta}{2} = \frac{\Delta^2}{4} - 2$ descendants in V_3. In case (b), if u has $\frac{\Delta}{2} - p$ neighbours in V_2, where $p \geq 2$, then u has at most $(\frac{\Delta}{2} - p)(\frac{\Delta}{2} + p) \leq \frac{\Delta^2}{4} - 4$ descendants in V_3, as desired.
In total it follows that

$$|V_3| \leq (c_1(v) - 1)\frac{\Delta^2}{4} + (c_2(v) - 1)\Big(\frac{\Delta^2}{4} - 2\Big) = \frac{\Delta^3}{4} - \Delta - 2.$$

This, in conjunction with the bound on $|V_2|$ yields a contradiction to Claim 3.

Case 3: There exists a maximal clique of order $\frac{\Delta}{2}$, whose vertices all have degree at most $\Delta - 1$.

Choose a vertex v of such a clique, and root G at v. Let the clique of order $\frac{\Delta}{2}$ be, say, $C_1(v)$. Since $\deg(v) \leq \Delta - 1$, we have $c_2(v) \leq \frac{\Delta}{2} + 1$. Since each vertex in $C_1(v) - \{v\}$ has at most $\frac{\Delta}{2}$ neighbours in V_2, and every vertex in $C_2(v) - \{v\}$ has at most $\Delta - c_2(v) + 1$ neighbours in V_2, we have

$$|V_2| \leq (c_1(v) - 1)\frac{\Delta}{2} + (c_2(v) - 1)(\Delta - c_2(v) + 1).$$

Now $c_1(v) = \frac{\Delta}{2}$, and since $c_2(v) \leq \frac{\Delta}{2} + 1$, we have

$$|V_2| \leq \Big(\frac{\Delta}{2} - 1\Big)\frac{\Delta}{2} + \frac{\Delta^2}{4} = \frac{\Delta^2}{4} - \frac{\Delta}{2} \leq \frac{\Delta^2}{4} - 2.$$

Since each of the $\Delta - 1$ vertices in V_1 has at most $\frac{\Delta^2}{4}$ descendants in V_3 by Claim 2, we obtain

$$|V_3| \leq (\Delta - 1)\frac{\Delta^2}{4} \leq \frac{\Delta^3}{4} - \Delta + 2.$$

The bounds on $|V_2|$ and $|V_3|$ contradict Claim 3. Claim 8 follows.

Claim 9: G is Δ-regular and every vertex of G is equitably partitionable.

Suppose to the contrary that G has a vertex v for which $c_1(v) \neq \frac{\Delta}{2} + 1$ or $c_2(v) \neq \frac{\Delta}{2} + 1$. Root G at v. We may assume that $c_1(v) \leq c_2(v)$, so $c_1(v) \leq \frac{\Delta}{2} + 1$. If $c_1(v) = \frac{\Delta}{2} + 1$ then $c_1(v) = c_2(v)$, contradicting the choice of v. If $c_1(v) = \frac{\Delta}{2}$ then we have a contradiction to Claim 7, therefore $c_1(v) = \frac{\Delta}{2} + 1 - p$ for some integer $p \geq 2$, and thus $c_2(v) = \frac{\Delta}{2} + 1 + s$ for some integer $s \leq p$. Consider $u \in V_1$. If $u \in C_1(v) - \{v\}$, then u has at most $\frac{\Delta}{2} + p$ neighbours in V_2, and if $u \in C_2(v) - \{v\}$, then u has at most $\frac{\Delta}{2} - s$ neighbours in V_2. Hence, since $p \geq 2$,

$$|V_2| \leq \Big(\frac{\Delta}{2} - p\Big)\Big(\frac{\Delta}{2} + p\Big) + \Big(\frac{\Delta}{2} - s\Big)\Big(\frac{\Delta}{2} + s\Big) = \frac{\Delta^2}{2} - p^2 - s^2 < \frac{\Delta^2}{2} - 2.$$

Now bound $|V_3|$. If $|V_1| \leq \Delta - 1$, then by Claim 2, $|V_3| \leq \frac{\Delta^3}{4} - \frac{\Delta^2}{4} < \frac{\Delta^3}{4} - \Delta + 2$, so we may assume that $|V_1| = \Delta$ and $p = s \geq 2$. By Claim 1, any vertex in $V(C_1(v)) - \{v\}$ (in $V(C_2(v)) - \{v\}$) has at most $\frac{\Delta^2}{4}$ (at most $\frac{\Delta^2}{4} - p^2$) descendants in V_3. Thus

$$|V_3| \leq \Big(\frac{\Delta}{2} - p\Big)\frac{\Delta^2}{4} + \Big(\frac{\Delta}{2} + p\Big)\Big(\frac{\Delta^2}{4} - p^2\Big) = \frac{\Delta^3}{4} - \frac{\Delta}{2}p^2 - p^3 < \frac{\Delta^3}{4} - \Delta + 2,$$

and we obtain a contradiction to Claim 3.

Claim 10: G is the underlying graph of a 2-regular, $(\frac{\Delta}{2} + 1)$-uniform hypergraph.

Let K_i, $i = 1, 2, \ldots, m$ be the (inclusion) maximal cliques of G. Define H to be the hypergraph on the vertex set $V(G)$ whose hyperedges are the maximal cliques K_i, $i = 1, 2, \ldots, m$ of G. By Claim 7, each K_i is of the form $C_j(u)$ for some vertex u of G, and so H is 2-regular, and G is the underlying graph of H. By Claim 9 we have $|K_i| = \frac{\Delta}{2} + 1$, so H is $(\frac{\Delta}{2} + 1)$-uniform, as desired. $\square$

Theorem 3 *Let $k \geq 3$ and let G be a 2-regular, k-uniform hypergraph of diameter D. Let $c_k := 2\frac{k^2-2k+1}{k^2-k+2}$. Then*

$$n \leq 1 + 2\sum_{i=1}^{D}(k-1)^i - c_k \sum_{i=0}^{D-2}(k-1)^i.$$

Proof. It suffices to show that for every real c with $0 < c < c_k$ we have

$$n < 1 + 2\sum_{i=1}^{D}(k-1)^i - c\sum_{i=0}^{D-2}(k-1)^i.$$

We prove this by contradiction. Suppose to the contrary that there exists a 2-regular, k-uniform hypergraph G of order n and diameter D with

$$n \geq 1 + 2\sum_{i=1}^{D}(k-1)^i - c\sum_{i=0}^{D-2}(k-1)^i, \tag{8}$$

for some real c with $0 < c < c_k$.

Let G be rooted at a vertex v which we will determine later. Every vertex w of G is incident with exactly two hyperedges, which we denote by $e(w)$ and $f(w)$, respectively. For the root v we assign the names $e(v)$ and $f(v)$ to the hyperedges containing v in an arbitrary way. For $w \in V_i, i = 1, 2, \ldots, D$, we assign the names $e(w)$ and $f(w)$ such that $e(w)$ always contains a vertex in V_{i-1}. Clearly such a vertex always exists. Hence $e(w) \subseteq V_{i-1} \cup V_i$, and $f(w) \subseteq V_{i-1} \cup V_i$ or $f(w) \subseteq V_i \cup V_{i+1}$. This implies that, for all $i \geq 1$,

$$|V_{i+1}| \leq |\bigcup_{w \in V_i}(f(w) - \{w\}| \leq \sum_{w \in V_i}|f(w) - \{w\}| = (k-1)|V_i|. \tag{9}$$

This in turn implies that $|V_i| \leq 2(k-1)^i$ for $i = 1, 2, \ldots, D$, and by summation over all $i \in \{1, 2, \ldots, D\}$ we obtain $n \leq 1 + 2\sum_{i=1}^{D}(k-1)^i$, which is the Moore bound for 2-regular hypergraphs.

To improve this bound we need some further notation. For $i \in \{1, 2, \ldots, D\}$ we define B_i to be the set of vertices $u \in V_i$ for which $|e(u) \cap V_{i-1}| + |f(u) \cap V_{i-1}| \geq 2$. Further let B_i' be the set of vertices $u \in B_i$ for which $f(u) \cap V_{i-1} \neq \emptyset$ (and thus $f(u) \subseteq V_{i-1} \cup V_i$), and let B_i'' be the set of vertices of B_i for which $f(u) \cap V_{i-1} = \emptyset$ (and thus $f(u) \subseteq V_i \cup V_{i+1}$). Note that B_1'' is empty. Denote the cardinalities of B_i, B_i' and B_i'' by b_i, b_i' and b_i'', respectively. Clearly, $b_i = b_i' + b_i''$. We further let $\hat{B}_i = N[B_i] \cap V_i$, $\hat{B}_i' = N[B_i'] \cap V_i$ and $\hat{B}_i'' = N[B_i''] \cap V_i$ for $i \in \{1, 2, \ldots, D\}$, $B = \bigcup_{i=2}^{D-1} B_i$ and $\hat{B} = \bigcup_{i=2}^{D-1} \hat{B}_i$.

Note that we are not assuming that these sets are nonempty. In the next three claims of the proof we show that our assumption on the order of G implies that the sets B_i are so small that there exists a hyperedge $f \subset V_D - \hat{B}_D$ such that f has no ancestor in $\hat{B}$. This in turn is used to obtain a contradiction and to complete the proof of Theorem 3.

Claim 1: Let $1 \leq i \leq D-1$. Then

$$n \leq 1 + 2\sum_{i=1}^{D}(k-1)^i - \sum_{i=1}^{D-1}(k-1)^{D-i}(2b'_i + b''_i) - b_D.$$

Clearly, we have $|V_1| = 2(k-1) - b'_1 = 2(k-1) - b_1$. Since for $j \in \{1, 2, \ldots, D-1\}$ the vertices in B'_j have no neighbour in V_{j+1}, we have

$$V_{j+1} = \bigcup_{u \in V_j - B'_j} (f(u) \cap V_{j+1}).$$

Since each $f(u) \cap V_{j+1}$ contains at most $k-1$ vertices, and since b_{j+1} vertices in V_{j+1} appear in two of the sets $f(u) \cap V_{j+1}$, we have $|V_{j+1}| \leq (k-1)(|V_j| - b'_j) - b_{j+1}$. Repeated application of this inequality yields that, for $1 \leq j \leq D$,

$$|V_j| \leq 2(k-1)^j - \sum_{i=1}^{j-1}(k-1)^{j-i}(b_i + b'_i) - b_j.$$

Summation over all $j \in \{1, 2, \ldots, D\}$, in conjunction with $b_i + b'_i = 2b'_i + b''_i$, yields

$$\begin{aligned} n &\leq 1 + 2\sum_{j=1}^{D}(k-1)^j - \sum_{j=3}^{D}\sum_{i=1}^{j-1}(k-1)^{j-i}(b_i + b'_i) - \sum_{j=1}^{D} b_j \\ &\leq 1 + 2\sum_{i=1}^{D}(k-1)^i - \sum_{i=1}^{D-1}(k-1)^{D-i}(2b'_i + b''_i) - b_D, \end{aligned} \tag{10}$$

as desired.

Let $F(\hat{B})$ denote the number of vertices in V_D which are descendants of vertices in $\hat{B}$.

Claim 2: $F(\hat{B}) \leq (2k-4)\sum_{i=1}^{D-1} b'_i(k-1)^{D-i} + \sum_{i=1}^{D-1} b''_i(k-1)^{D-i}$.

We bound the number of descendants of vertices in $\hat{B}_i$ for $1 \leq i \leq D-1$.

First consider the vertices in B'_i. For each $u \in B'_i$ both hyperedges, $e(u)$ and $f(u)$ contain a vertex of V_{i-1}, and no vertex of V_{i+1}. Hence, $|\hat{B}'_i| \leq b'_i(2k-3)$. Since any vertex of $\hat{B}'_i - B'_i$ is adjacent to at most $k-1$ vertices in V_{i+1}, there are at most $b'_i(2k-4)(k-1)^{D-i}$ vertices in V_D which are descendants of vertices in $\hat{B}'_i$.

Now consider the vertices in $\hat{B}''_i$. Each $u \in B''_i$ has at most $(k-1)^{D-i}$ descendants in V_D. We claim that for each neighbour w of u in $\hat{B}''_i - B''_i$, every descendant of w is also a descendant of u. Indeed, since $u \in B''_i$, we have $f(u) \subseteq V_i \cup V_{i+1}$. Each neighbour of u in $e(u) \cap V_i$ is also in B''_i, so $w \in f(u)$. Hence $f(w) = f(u)$, and so $N(u) \cap V_{i+1} = f(u) \cap V_{i+1} = f(w) \cap V_{i+1} = N(w) \cap V_{i+1}$, as desired. Hence the descendants of vertices in $\hat{B}''_i$ are descendants of vertices in B''_i, and so there are in total at most $b''_i(k-1)^{D-i}$ descendants of vertices in $\hat{B}''_i$ in V_D.

It follows from the above that the total number of descendants in V_D of vertices in $\hat{B}_i$ is at most $(2k-4)b'_i(k-1)^{D-i} + b''_i(k-1)^{D-i}$. Summation over all $i \in \{1, 2, \ldots, D-1\}$ now yields the claim.

Claim 3: $F(\hat{B}) \leq c(k-1)^{D-1} - c - (k-2)b_D$.

From Claim 1 and our assumption $|V(G)| \geq 1 + 2\sum_{i=1}^{D}(k-1)^i - c\sum_{i=0}^{D-2}(k-1)^i$ we get that

$$c\sum_{i=0}^{D-2}(k-1)^i \geq \sum_{i=1}^{D-1}(k-1)^{D-i}(2b'_i + b''_i) + b_D,$$

or equivalently,

$$\sum_{i=1}^{D-1}(k-1)^{D-i}b'_i \leq \frac{c}{2}\sum_{i=0}^{D-2}(k-1)^i - \frac{1}{2}\sum_{i=1}^{D-1}(k-1)^{D-i}b''_i - \frac{1}{2}b_D. \qquad (11)$$

Substituting the right hand side of (11) in Claim 2 yields

$$\begin{aligned}
F(\hat{B}) &\leq (2k-4)\sum_{i=1}^{D-1} b'_i(k-1)^{D-i} + \sum_{i=1}^{D-1} b''_i(k-1)^{D-i} \\
&\leq (k-2)\Big[c\sum_{i=0}^{D-2}(k-1)^i - \sum_{i=1}^{D-1}(k-1)^{D-i}b''_i - b_D\Big] + \sum_{i=1}^{D-1} b''_i(k-1)^{D-i} \\
&\leq (k-2)\Big[c\sum_{i=0}^{D-2}(k-1)^i - b_D\Big] \\
&= c(k-1)^{D-1} - c - (k-2)b_D.
\end{aligned}$$

Claim 4: There exists a hyperedge $f \subset V_D - \hat{B}_D$, such that no vertex in f has an ancestor in $\hat{B}$.

For any $u \in V_D - \hat{B}_D$, $e(u)$ contains exactly one vertex of V_{D-1} and $f(u)$ contains only vertices of $V_D - \hat{B}_D$. Let N be the number of hyperedges in $G[V_D - \hat{B}_D]$. In order to prove the claim, it suffices to show that $N > F(\hat{B})$. Clearly, $N = |V_D - \hat{B}_D|/k$, so we first bound $|V_D - \hat{B}_D|$ from below.

Since $|V(G)| \geq 1 + 2\sum_{i=1}^{D}(k-1)^i - c\sum_{i=0}^{D-2}(k-1)^i$ and $\sum_{i=0}^{D-1}|V_i| \leq 1 + 2\sum_{i=1}^{D-1}(k-1)^i$, we get $|V_D| \geq 2(k-1)^D - c\sum_{i=0}^{D-2}(k-1)^i$. Since $|\hat{B}_D| \leq (2k-3)b_D$, it follows that

$$\begin{aligned} |V_D - \hat{B}_D| &\geq 2(k-1)^D - c\sum_{i=0}^{D-2}(k-1)^i - (2k-3)b_D \\ &\geq 2(k-1)^D - 2c(k-1)^{D-2} - (2k-3)b_D. \end{aligned}$$

Considering the right hand side of the last inequality, we note that

$$[2(k-1)^D - 2c(k-1)^{D-2}]/k > c(k-1)^{D-1}$$

This follows from the fact that the equivalent inequalities

$$2(k-1)^D - 2c(k-1)^{D-2} > c(k-1)^D + c(k-1)^{D-1}$$

and

$$(2-c)(k-1)^2 - c(k-1) - 2c > 0$$

hold for our choice of c. Hence

$$N = \frac{|V_D - \hat{B}_D|}{k} \geq c(k-1)^{D-1} - \frac{2k-3}{k}b_D \geq c(k-1)^{D-1} - (k-2)b_D > F(\hat{B}),$$

the last inequality following from Claim 3 and the fact that $c > 0$. This proves Claim 4.

We are now in a position to complete the proof. By Claim 4 there exists a hyperedge $f \subseteq V_D$, such that none of its vertices has an ancestor in $\hat{B}$.

We show that for every vertex $x \in f$ there exists a unique vertex $\overline{x} \in V_1$ at distance $D-1$ from x. Let $x \in f$ be arbitrary, and let $v, x_1, x_2, \ldots, x_{D-1}, x$ be a $v - x$ path of length D, so $x_i \in V_i$ for $i = 1, 2, \ldots, D-1$. Then $e(x_i) \cap V_{i-1} = \{x_{i-1}\}$ since $x_i \notin B$, and $f(x_i) = e(x_{i+1})$ for $i = 1, 2, \ldots, D-1$. So every vertex on the $v - x$ path has a unique neighbour which is closer to

v, and so the $v-w$ path of length D is unique, which implies that $\overline{x} := x_1$ is the unique vertex in V_1 at distance $D-1$ from x.

Now fix a vertex $w \in f$. Without loss of generality we assume that $\overline{w} \in e(v)$.

We claim that for each $u \in V_1$ there exists $x \in f$ such that $\overline{x} = u$. First let $u \in f(v)$. Consider a shortest $u-w$ path $P(u,w)$. Clearly, $P(u,w)$ does not contain v. Also, $P(u,w)$ cannot contain vertex v_{D-1}, since otherwise, if $P(u,w)$ contained v_{D-1}, it would be of the form $u_1, u_2, \ldots, u_j, v_j, v_{j+1}, \ldots, v_{D-1}, w$, where $u_i \in V_i$ for $i = 1, 2, \ldots, j$, $u_1 = u$ and $v_l \in V_l, l = j, j+1, \ldots, D-1$. Now $u_j \in e(v_j) \cup f(v_j)$, but since $f(v_j)$ has only vertex v_j in V_j, we have $u_j \in e(v_j)$. But since $e(v_j)$ has only one vertex in V_{j-1}, viz. v_{j-1}, we have, $u_{j-1} = v_{j-1}$, and so P is not a shortest path since v_j and u_{j-1} are adjacent, a contradiction. Therefore $P(u,w)$ does not contain v_{D-1}, and so $P(u,w)$ contains a vertex $x \in f - \{w\}$. Hence $P(u,w)$ has the form $u_1, \ldots, x, w$, implying $\overline{x} = u$, as desired. Now let $u \in e(v)$. Fixing a vertex $w' \in f$ for which $\overline{w'} \in f(v)$ and applying the same argument as above, with w' replacing w, we show there exists $x \in f$ such that $\overline{x} = u$. In total we conclude that for each vertex in $u \in V_1$ we have $\overline{x}$ for some $x \in f$.

Hence the mapping $x \to \overline{x}$ from f to V_1 is surjective, and so $|f| \geq |V_1|$. But $|f| = k$ and $|V_1| = 2k-2$, which implies $k \leq 2$. This contradiction to $k \geq 3$ completes the proof of Theorem 3. $\square$

Corollary 1 *There are no Moore geometries of degree* 2*, diameter* $D \geq 2$ *and rank* $k \geq 3$*.*

Theorem 4 *Let* Δ, D *be positive integers such that* $\Delta \geq 4$ *is even. Let* $c'_\Delta := 2\frac{(\Delta/2)^2}{(\Delta/2)^2+\Delta/2+2}$*. Then*

$$\mathrm{cf}_{\Delta,D} \leq 1 + 2\sum_{i=1}^{D}\left(\frac{\Delta}{2}\right)^i - c'_\Delta \sum_{i=0}^{D-2}\left(\frac{\Delta}{2}\right)^i.$$

Proof. We prove the result by contradiction. Suppose that there exists a claw-free graph G of even maximum degree $\Delta \geq 4$, diameter $D \geq 2$ and order n, where

$$n > 1 + 2\sum_{i=1}^{D}\left(\frac{\Delta}{2}\right)^i - c'_\Delta \sum_{i=0}^{D-2}\left(\frac{\Delta}{2}\right)^i,$$

and c'_Δ is as defined above. In order to show that G satisfies the hypothesis of Theorem 2 it suffices to prove that

$$1+2\sum_{i=1}^{D}\Big(\frac{\Delta}{2}\Big)^i - c'_\Delta\sum_{i=0}^{D-2}\Big(\frac{\Delta}{2}\Big)^i > 1+2\sum_{i=1}^{D}\Big(\frac{\Delta}{2}\Big)^i - 2\Big(\frac{\Delta}{2}\Big)^{D-2},$$

or, equivalently,

$$c'_\Delta < \frac{2\Big(\frac{\Delta}{2}\Big)^{D-2}}{\sum_{i=0}^{D-2}(\frac{\Delta}{2})^i},$$

which is easy to verify for our choice of c'_Δ.

Hence, by Theorem 2, G is the underlying graph of a 2-regular, k-uniform hypergraph H, where $k=\frac{\Delta}{2}+1$. But then, since $c_k = c'_\Delta$,

$$n(H)=n(G)>1+2\sum_{i=1}^{D}\Big(\frac{\Delta}{2}\Big)^i - c'_\Delta\sum_{i=0}^{D-2}\Big(\frac{\Delta}{2}\Big)^i = 1+2\sum_{i=1}^{D}(k-1)^i - c_k\sum_{i=0}^{D-2}(k-1)^i,$$

a contradiction to Theorem 3. □

3. Lower bounds

In this section we obtain lower bounds on $\mathrm{cf}_{\Delta,D}$. In view of Theorem 2, underlying graphs of 2-regular, uniform, linear hypergraphs (or, equivalently, graphs whose vertices are equipartitionable) are likely candidates for large claw-free graphs of given degree and diameter. Note that graphs whose vertices are equipartitionable are line graphs, because the neighbouhood of every vertex of a line graph can be divided into (at most) 2 cliques. It also follows that no vertex of a line graph has 3 neighbours such that all of them are non-adjacent. Hence all line graphs are claw-free. We mainly consider line graphs of known constructions to obtain lower bounds on $\mathrm{cf}_{\Delta,D}$.

Is is easy to see that the line graph of a connected graph of diameter D has diameter at most $D+1$, and that it is at most D if the graph is bipartite.

Delorme [7] showed that for any Δ and D, there exists a bipartite graph of order $2c(\frac{\Delta}{2})^{D-1}$, where $c\in\{1,2,\ldots,6\}$ depending on D (for example $c=6$ if $D\equiv 6$ or $26 \pmod{30}$). Taking line graphs we obtain graphs of order $2c(\frac{\Delta}{2})^D$ and degree $2\Delta-2$. This yields the following lower bound on $\mathrm{cf}_{\Delta,D}$.

Proposition 1 *For any D and any even Δ there exists a $c \in \{1,2,3,4,5,6\}$, depending only on D, such that*

$$\mathrm{cf}_{\Delta,D} \geq 2c\Big(\frac{\Delta+2}{4}\Big)^D.$$

From a construction of Canale and Gómez [4] we can get line graphs of maximum degree Δ, diameter D and order at least $\frac{29}{40}(\frac{\Delta+2}{2.9})^D$ for infinitely many values of Δ and D. Hence the above bound can be improved as follows.

Proposition 2 *For infinitely many values of Δ and D we have*

$$\mathrm{cf}_{\Delta,D} \geq \frac{29}{40}\Big(\frac{\Delta+2}{2.9}\Big)^D.$$

For diameters $D = 2, 3, 4$ and 6 we obtain large claw-free graphs from bipartite Moore graphs. For diameter $D = 2$, the bipartite Moore graphs are the complete bipartite graphs with partite sets of order Δ. If $\Delta - 1$ is a prime power, the bipartite Moore graphs of diameter 3 and degree Δ are the incidence graphs of projective planes, and for diameter 4 and degree Δ the graphs are called generalized quadrangles. For $D = 6$ generalized hexagons exist if $\Delta - 1$ is an odd power of 3, see [3]. The order of bipartite Moore graphs of degree Δ and diameter D is $\frac{2[(\Delta-1)^D-1]}{\Delta-2}$. Line graphs of the bipartite Moore graphs of diameter $D = 2, 3, 4$ and 6 give the bound $\mathrm{cf}_{\Delta',D} \geq \frac{\Delta'+2}{\Delta'-2}[(\frac{\Delta'}{2})^D - 1]$ where $\Delta' = 2\Delta - 2$.

Proposition 3 *If (i) $D = 2$ and Δ is even, (ii) $D = 3$ or 4, and $\frac{\Delta}{2}$ is a prime power, (iii) $D = 6$ and $\frac{\Delta}{2}$ is an odd power of 3, then we have*

$$\mathrm{cf}_{\Delta,D} \geq \frac{\Delta+2}{\Delta-2}\Big[\Big(\frac{\Delta}{2}\Big)^D - 1\Big].$$

We note that the above line graphs are underlying graphs of 2-regular, uniform hypergraphs. Hence they also yield lower bounds on the order of 2-regular, uniform hypergraphs of given rank and diameter.

For diameter 2 we construct claw-free graphs that improve the bound in Proposition 3.

Theorem 5 *Let $\Delta \geq 6$ such that $\Delta \equiv 2 \pmod 4$. Then $\mathrm{cf}_{\Delta,2} \geq \frac{5}{16}(\Delta+2)^2$.*

Proof. We construct a Δ-regular claw-free graph G of diameter 2 and order $\frac{5}{16}(\Delta+2)^2$, where $\Delta \geq 6$, $\Delta \equiv 2 \pmod 4$. Let X be a set of order $p \geq 2$ and let $\mathbb{Z}_5$ be the (additive) cyclic group of order 5. Let G be the graph with vertex set $V(G) = \mathbb{Z}_5 \times X \times X$, where vertex (a, x_1, x_2) is adjacent to the vertices of the form

$$(a, x_1, y_1), (a, y_2, x_2), (a-1, y_3, x_1), \text{ or } (a+1, x_2, y_4),$$

where $y_1 \in X \setminus \{x_2\}$, $y_2 \in X \setminus \{x_1\}$, and $y_3, y_4 \in X$. Clearly, the degree of every vertex is $\Delta = 4p - 2$ and the order of G is $5p^2 = \frac{5}{16}(\Delta+2)^2$. The neigbours of (a, x_1, x_2) can be divided into two cliques, one clique containing the neighbours of the form (a, x_1, y_1) and $(a-1, y_3, x_1)$, and the other clique containing the neighbours of the form (a, y_2, x_2) and $(a+1, x_2, y_4)$. Hence, G is a line graph and thus claw-free.

It remains to show that the diameter of G is 2. We show that any two different vertices (a, x_1, x_2) and (a', x_1', x_2') are either adjacent or there is a vertex v which is adjacent to both of them. If $a' = a$, then $v = (a, x_1', x_2)$ for $x_1 \neq x_1'$ and $x_2 \neq x_2'$, and (a, x_1, x_2) is adjacent to (a', x_1', x_2') for $x_1 = x_1'$ or $x_2 = x_2'$. If $a' = a+1$, then $v = (a, x_1, x_1')$ for $x_2 \neq x_1'$, otherwise (a, x_1, x_2) and (a', x_1', x_2') are adjacent. If $a' = a+2$, then $v = (a, x_2, x_1')$. The cases $a' = a-1$ or $a-2$ are analogous. Hence, $\mathrm{cf}_{\Delta,2} \geq \frac{5}{16}(\Delta+2)^2$. $\square$

The graph constructed above is an underlying graph of a 2-regular, $(\frac{\Delta}{2}+1)$-uniform hypergraph of diameter 2. Hence we obtain the following corollary.

Corollary 2 *For every even integer $k \geq 4$ there exists a hypergraph of degree 2, diameter 2, rank k and order $\frac{5}{4}k^2$.*

Open problems

In this paper we established bounds on the order of claw-free graphs of given diameter D and degree Δ, where Δ is even. It would be interesting to obtain similar results for odd Δ.

We established a connection between large claw-free graphs of given even degree and diameter and 2-regular hypergraphs. Are there similar connections between large $K_{1,p}$-free graphs and $(p-1)$-regular hypergraphs?

Much research has been undertaken on the construction of large graphs of given degree and diameter. Is it possible to give constructions of large hypergraphs of given rank, degree, and diameter?

References

[1] E. Bannai and T. Ito, Regular graphs with excess one. Discrete Math. **37** (1981), 147–158.

[2] J.C. Bermond, J. Bond, M. Paoli and C. Peyrat, Graphs and interconnection networks: diameter and vulnerability. Surveys in combinatorics, London Math. Soc. Lecture Note Ser. **82** (1983), 1-30.

[3] N.I. Biggs, Algebraic Graph Theory. Cambridge University Press, Second Edition (1993).

[4] E.A. Canale and J. Gómez, Asymptotically large (Δ, D)-graphs. Discrete Appl. Math. **152**, no. 1-3 (2005), 89–108.

[5] R.M. Damerell, On Moore geometries II. Math. Proc. Cambridge Philos. Soc. **90** (1981), 33–40.

[6] R.M. Damerell and M.A. Georgiacodis, On Moore geometries I. J. London Math. Soc. II, Ser. **23** (1981), 1–9.

[7] C. Delorme, Large bipartite graphs with given degree and diameter. J. Graph Theory **8** (1985), 325–334.

[8] R. Feria-Purón, M. Miller and G. Pineda-Villavicencio, On graphs of defect at most 2. Discrete Appl. Math. **159** no. 13 (2011), 1331-1344.

[9] F.J. Fuglister, On finite Moore geometries. J. Combin. Theory, Ser. A **23** (1977), 187–197.

[10] F.J. Fuglister, The nonexistence of Moore geometries of diameter 4. Discrete Math. **45** (1983), 229–238.

[11] M. Miller and G. Pineda-Villavicencio, Complete catalogue of graphs of maximum degree 3 and defect at most 4. Discrete Appl. Math. **157** no. 13 (2009), 2983–2996.

[12] M. Miller and J. Širáň. Moore graphs and beyond: A survey of the degree/diameter problem. Electr. J. Combin., Dynamic Survey D14 (2005).