Noncommutative Ricci flow in a matrix geometry

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Abstract

We study noncommutative Ricci flow in a finite-dimensional representation of a noncommutative torus. It is shown that the flow exists and converges to the flat metric. We also consider the evolution of entropy and a definition of scalar curvature in terms of the Ricci flow.

1 Introduction

Recently three different approaches to Ricci flow in noncommutative geometry were initiated in [36], [6] and [10] respectively, the latter two focussing on the case of a noncommutative torus. In this paper we develop a more elementary approach in the case of a simple matrix geometry, namely a finite-dimensional representation of the rational noncommutative torus often called the fuzzy torus.

The setting we use, in particular the type of the metric we consider, is of the same form as that of [11] which also forms the basis for [6] and [10], although due to our finite-dimensional setup the situation is simpler and we do not make direct use of results in [11]. Our formulation of the Ricci flow is more direct than the spectral approach of [6], and similar to that of [10].

We note that since the introduction of Ricci flow by Hamilton [17] as an approach to the Thurston geometrization conjecture and in particular the Poincaré conjecture, it has been very useful in geometry and topology, as is by now well-known, in particular due to the work of Perelman [29], [30], [31] proving the Poincaré conjecture. It should also be mentioned though, that at about the same time that Hamilton introduced Ricci flow, it appeared independently and for very different reasons in the study of renormalization of sigma models in physics by Friedan [15], [16]. Considering the importance of Ricci flow in geometry, topology and physics on the one hand, and the development and potential physical applications of noncommutative geometry on the other, it is natural to explore Ricci flow in the noncommutative framework. This is the general motivation for this paper.

More specifically we are motivated by the goal to have a simple version of noncommutative Ricci flow. The approach followed here is concrete enough that one can prove certain results that in the other approaches appear difficult, for example we show that in our setting the metric flows to the flat metric as it does in the case of a classical torus [18]. It should be kept in mind though that this does not solve the corresponding, but more difficult problem for the noncommutative torus mentioned in [6]. Part of the reason that we can prove results like these, is that in this approach in terms of a finite-dimensional representation, the Ricci flow is given by a system of first order ordinary differential equations (rather than a partial differential equation as in the classical case).

The latter point will become clear in Section 2, where we also briefly review the ideas from matrix geometry that we need, and describe the metrics that we consider. In Section 3 the existence and convergence to the flat metric of the Ricci flow is shown. In Section 4 we explain how the metric can be viewed as a density matrix and study the evolution of the resulting von Neumann entropy under the Ricci flow. Section 5 briefly considers how the scalar curvature of the matrix geometry can be unambiguously defined via the Ricci flow, and is closely related to the work in [10]. The relevance of this is that, as discussed for example in [10], [14] and [1], defining scalar curvature is somewhat more subtle in noncommutative geometry than in the classical case. Noncommutative Ricci flow provides a particularly elegant approach to defining scalar curvature, at least in our framework. Finally, in Section 6, we outline possible further work that can be explored.

2   Basic ideas and definitions

Here we review relevant background and set up the framework used in the rest of the paper.

Recall that Ricci flow is given by [17]

$$\frac{\partial g_{\mu\nu}}{\partial t} = -2R_{\mu\nu}$$

where $g_{\mu\nu}$ is a metric on some differentiable manifold, $R_{\mu\nu}$ is the corresponding Ricci tensor, and t is a real variable ("time"). If we restrict ourselves to surfaces and to metrics of the form $g_{\mu\nu} = c\delta_{\mu\nu}$ where $\delta_{\mu\nu}$ is the Kronecker delta and c is some strictly positive function on the surface (a conformal rescaling factor), then it can be shown that the above Ricci flow equation reduces to

$$\frac{\partial c}{\partial t} = \partial_\mu \partial_\mu \log c$$

where ∂_μ is the partial derivative with respect to the coordinate x^μ on the surface, and we sum over the repeated index $\mu = 1, 2$. An interesting metric

of this form is the cigar metric of Hamilton [18], also known as the (euclidean version of) the Witten black hole [38], [27], [37], given by

$$c = \frac{1}{M + x^\mu x^\mu}$$

where the parameter $M > 0$ is independent of x^μ and can be interpreted as describing the mass of the black hole. In the case of the manifold $\mathbb{R}^2$, this metric is a soliton for the Ricci flow, however as one might expect from the general theory of Ricci flow for compact surfaces (see in particular [18, Theorem 10.1]) solitons other than the flat metric do not appear in the noncommutative case below where we consider a noncommutative version of the torus, which classically is a compact manifold.

Next we recall the matrix geometry we are going to work with. The basic ideas regarding matrix geometries originated in [20] and [25], the latter including a brief discussion of fuzzy tori. Also see for example [26, Section 3.1], [13] and [21] for work more directly related to fuzzy tori. We consider two unitary $n \times n$ matrices u and v satisfying the commutation relation

$$vu = quv$$

where

$$q = e^{2\pi i m/n}$$

for an $m \in \{1, 2, ..., n-1\}$ such that m and n are relatively prime; note in particular that $q^n = 1$, but $q^j \neq 1$ for $j = 1, ..., n-1$. This commutation relation is similar to the Weyl form of the commutation relation between position and momentum of a quantum particle in one dimension. Indeed, the analogy is very close, since as in the case of position and momentum, u and v are connected by a Fourier transform, but for the cyclic group $\mathbb{Z}_n = \{0, 1, 2, ..., n-1\}$ instead of the group $\mathbb{R}$ as for position and momentum. It is worth mentioning that matrices of this form already appeared long ago in the physics literature, for example in [35], [19] and [5]. Concretely we can use

$$u = \begin{bmatrix} 1 & & & \\ & q & & \\ & & \ddots & \\ & & & q^{n-1} \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{bmatrix}$$

where the blanks spaces are filled with zeroes. It is then an easy exercise to verify that as mentioned above, v is obtained by the Fourier transform of $\mathbb{Z}_n$ from u, i.e. $v = F^* u F$ where F is the Fourier transform on $\mathbb{Z}_n$ in a suitable basis, namely the $n \times n$ unitary matrix $F_{jk} = q^{-jk}/n^{1/2}$ for $j, k = 0, 1, ..., n-1$, and F^* denotes its Hermitian adjoint. It is also easily checked that u and v generate the algebra M_n of all $n \times n$ complex matrices,

namely the commutant of $\{u, v\}$ consists of scalar multiples of the identity matrix, $\{u, v\}' = \mathbb{C}1$, where here and later we denote the $n \times n$ identity matrix simply as 1, since no confusion will arise.

We now take any two Hermitian matrices x and y such that

$$u = e^{\frac{2\pi i}{n} x} \quad \text{and} \quad v = e^{\frac{2\pi i}{n} y}$$

and define derivations δ_1 and δ_2 on the algebra M_n by the commutators

$$\delta_1 := [y, \cdot] \quad \text{and} \quad \delta_2 := -[x, \cdot]$$

which are analogues of the derivatives $\frac{1}{i}\partial_\mu$ in the classical case above. Note that we have made a specific choice of real multiplicative constants in these derivations, and moreover x and y are not uniquely determined, but our analysis will not depend on the exact choices made.

To gain some insight about these derivations, we calculate $\delta_1 u$ and $\delta_2 v$, though only for a specific choice of x and y. Note that in general from the definitions of u and F above, we have $FuF^* = \left(F^{*T} u^T F^T\right)^T = (F^* u F)^T = v^T$ where a^T denotes the transpose of a matrix a. Also note that x is a diagonal matrix (with real values on its diagonal). As part of our specific choice we choose $y := F^* x F$, given a choice of x. It then follows that

$$\delta_1 u = F^* [x, FuF^*] F = F^* \left[x^T, v^T\right] F = F^* [v, x]^T F = F^* (\delta_2 v)^T F$$

If we now consider the specific (and most obvious) choice

$$x = \begin{bmatrix} 0 & & & & \\ & m & & & \\ & & 2m & & \\ & & & \ddots & \\ & & & & (n-1)m \end{bmatrix}$$

it is easily verified that
$$\delta_2 v = mv - mn e_{n1}$$

where e_{jk} denotes the $n \times n$ matrix with 1 in the (j, k)-position and zeroes elsewhere. From the formulas above it then follows that

$$\delta_1 u = mu - mn F^* e_{1n} F$$

These formulas are analogous to the classical formulas $\frac{1}{i}\partial_2 e^{imx^2} = me^{imx^2}$ and $\frac{1}{i}\partial_1 e^{imx^1} = me^{imx^1}$ respectively in terms of classical real coordinates x^μ, but with terms included to compensate for finite-dimensionality. Similarly one can consider other choices of x. For example taking all the entries of our current choice modulo n, leads to similar formulas for $\delta_1 u$ and $\delta_2 v$, but with different "compensation" terms which are notationally a bit more difficult

to write down than in the formulas for $\delta_1 u$ and $\delta_2 v$ above. Of course, for general choices of x and y we obviously also have $\delta_1 v = 0$ and $\delta_2 u = 0$ in perfect analogy to the classical case.

Returning to general x and y, from these derivations we can define a noncommutative analogue of a Laplacian as an operator on M_n:

$$\triangle := \delta_1^2 + \delta_2^2 = \delta_\mu \delta_\mu$$

The importance of this for our purposes is clear from the classical Ricci flow equation above where $\partial_\mu \partial_\mu$ corresponds to $-\delta_\mu \delta_\mu$. We are using conventions ensuring that $\triangle$ is a positive operator (see Proposition 2.1 below) and that's why $\triangle$ corresponds to $-\partial_\mu \partial_\mu$ rather than $\partial_\mu \partial_\mu$.

Regarding notation, when we write $\delta_\mu \delta_\mu$, we are summing over μ, but when we write δ_μ^2 there is no sum over μ.

Keep in mind that M_n is an involutive algebra, i.e. a $*$-algebra, with the involution given by the usual Hermitian adjoint of a matrix a, which we denote by a^* to fit into the standard C*-algebraic notation (in conventional quantum mechanical notation it is denoted by $a^\dagger$). Since the algebra is finite-dimensional, all norms on it are equivalent (so they give the same topology on M_n) and complete. In the operator norm M_n is indeed a unital C*-algebra, however the theory of C*-algebras will not be needed in this paper, though it is conceptually useful to think in terms of C*-algebras, since a unital C*-algebra is a noncommutative analogue of the C*-algebra of continuous complex-valued functions on a compact topological space. The equivalence of norms will be useful in a technical sense, since it will allow us to use whichever norm is most convenient in any given situation. In particular the Hilbert-Schmidt norm obtained from the inner product

$$\langle a, b \rangle := \tau \left(a^* b \right)$$

on M_n will come into play in the next section. Here τ denotes the usual trace on M_n, i.e. the sum of the diagonal elements of a matrix. We think of $\tau(a)$ as a noncommutative integral of the complex-valued "function" a, corresponding in the classical case to the integral with respect to Haar measure on the torus.

We have the following simple proposition which mimics properties of partial derivatives of complex-valued functions of two real variables on the classical torus. For example, the first property corresponds to the fact that if both the first partial derivatives of a function are zero globally, then the function is constant.

Proposition 2.1. *The following properties hold (where $a, b \in M_n$ are arbitrary):*

(a) If $\delta_1 a = \delta_2 a = 0$, then $a \in \mathbb{C}1$. Conversely, $\delta_\mu 1 = 0$.

(b) We can integrate by parts, i.e. $\tau \left(a \delta_\mu b \right) = -\tau \left(b \delta_\mu a \right)$.

(c) The derivations δ_1 and δ_2 are Hermitian, i.e. $\langle a, \delta_\mu b \rangle = \langle \delta_\mu a, b \rangle$, and furthermore $\left(\delta_\mu a \right)^* = -\delta_\mu \left(a^* \right)$.

(d) The operators δ_1^2, δ_2^2 and $\triangle$ on the Hilbert space M_n are positive, i.e.
$\langle a, \delta_\mu^2 a \rangle \geq 0$ *and* $\langle a, \triangle a \rangle \geq 0$.
(e) If $\langle a, \delta_\mu^2 a \rangle = 0$, then $\delta_\mu a = 0$, for each value of μ separately.
(f) $\ker \triangle = \mathbb{C}1$
(g) $\tau(\triangle a) = 0$

Proof. (a) Clearly if a matrix commutes with x then it commutes with u, and if it commutes with y it commutes with v, so if it commutes with both x and y it is in $\{u, v\}' = \mathbb{C}1$. The converse is trivial.

(b) This is verified easily from the definition of the derivations and the fact that $\tau(ab) = \tau(ba)$.

(c) The first part is just a different way of stating (b) using the second part, which in turn follows from the definition of the derivations.

(d) and (e) follow from (c) and the fact that τ is faithful, i.e. $\tau(a^*a) = 0$ implies that $a = 0$.

(f) Clearly $\triangle 1 = 0$. On the other hand, if $\triangle a = 0$, it follows that $0 = \langle a, \triangle a \rangle = \langle a, \delta_1^2 a \rangle + \langle a, \delta_2^2 a \rangle$ which means that $\langle a, \delta_\mu^2 a \rangle = 0$ by (d) so $\delta_\mu a = 0$ by (e). By (a) it then follows that $a \in \mathbb{C}1$.

(g) This follows directly from the fact that $\triangle$ is defined in terms of commutators while τ is the trace. $\qquad\square$

As will be seen, we really just need the abstract properties of the derivations listed in this proposition in the rest of the paper, rather than the explicit definitions of u and v, and δ_1 and δ_2, given above. Because of this and for notational convenience, we will from now on work with the more abstract notation

$$A = M_n$$

Indeed, even in the proof of the proposition the explicit definitions of u and v are not used directly, though we did use the fact that they generate the algebra, and their relation to x and y. The finite-dimensionality of A will be used as well in the rest of the paper.

We now have an algebra A with derivations, so we have the basic elements of a noncommutative geometry. Next we need to discuss a class of metrics. Following the idea in [11], we describe a metric by a strictly positive element c of the C*-algebra A, i.e. a Hermitian element of A whose eigenvalues are strictly positive. (We write $a > 0$ to indicate that a Hermitian $a \in A$ is strictly positive, and $a \geq 0$ that it is positive, meaning that its eigenvalues are larger or equal to zero.) This c of course is the noncommutative counterpart of the c in $g_{\mu\nu} = c\delta_{\mu\nu}$ in the classical case discussed above. Since we only consider metrics of this form, we refer to such a c itself simply as a *noncommutative metric*. Note that if c is a scalar multiple of the identity matrix, then it can be interpreted as a *flat* metric, since it corresponds to a constant function c in the classical case.

We can now immediately and unambiguously write down a noncommutative version of the classical Ricci flow equation above:

$$\frac{d}{dt}c(t) = -\triangle \log c\,(t)$$

where $c\,(t) \in A$ denotes the metric at "time" t. We say that this is unambiguous, since, given the Laplacian $\triangle$ on A, the equation does not depend on an arbitrary choice of order of noncommuting elements of A. We interpret the time derivative in the equation in terms of any norm on A, since these are equivalent as already mentioned. In particular we can therefore view it as component-wise differentiation, but equivalently we can view the derivative as being defined in terms of the operator norm (i.e. the C*-algebraic norm) or the Hilbert-Schmidt norm on A. For any strictly positive $a \in A$, $\log a$ is defined by the usual functional calculus, namely transform unitarily to an orthonormal basis in which a is diagonal, take log of the diagonal elements, and then transform back to the original orthonormal basis.

As an analogue of the cigar metric mentioned above, we have

$$c = \left(M + x^2 + y^2\right)^{-1}$$

as an example of a noncommutative metric of the form above, where $M > 0$ is a real number. (Note that by $M + x^2 + y^2$ we mean $M1 + x^2 + y^2$, but here and later on we often drop the identity matrix 1 when we add a scalar multiple of 1 to another matrix.) This is well defined because $M + x^2 + y^2 > 0$ implying that $M + x^2 + y^2$ is indeed invertible, since $M > 0$ while $x^2, y^2 \geq 0$ due to the fact that x and y are Hermitian. This is not a soliton for Ricci flow though, since as will be seen in the next section all metrics flow to a constant metric as is the case for metrics on the classical torus.

3   Existence and convergence

In this section we study the mathematical properties of the noncommutative Ricci flow equation introduced in the previous section. Note that this equation is in fact a system of first order ordinary differential equations, and the theory of such systems will therefore play a central role in this section (see for example [7] and [28]). In particular we show the existence of the flow on any interval $[t_0, \infty)$, and show that in the $t \to \infty$ limit the Ricci flow takes any initial metric to a flat metric. We continue with the notation introduced in the previous section.

The following inequality will shortly be used in the proof of our main result. In it we use the exponential e^a of a Hermitian element a of A, which can equivalently be defined by the functional calculus as for log above, or by a power series, or by the analytic functional calculus.

Proposition 3.1. *For any Hermitian $a \in A$ we have*

$$\tau\left(e^a \triangle a\right) \geq 0$$

with equality if and only if $a \in \mathbb{C}1$, i.e. if and only if a is a scalar multiple of the identity matrix.

Proof. Let δ denote either of the two derivations δ_1 or δ_2. By Proposition 2.1(b)

$$\tau\left(e^a \delta^2 a\right) = -\tau\left((\delta a)\delta e^a\right) = -e^\lambda \tau\left((\delta\left(a - \lambda\right))\delta e^{a-\lambda}\right)$$

where λ is the smallest eigenvalue of a, so $h := a - \lambda \geq 0$. We consider

$$\tau\left((\delta h)\delta e^h\right) = \sum_{j=0}^{\infty} \frac{1}{j!}\tau\left((\delta h)\delta h^j\right)$$

and show that each term in this series is negative (or zero). By the product rule, $\tau\left((\delta h)\delta h^j\right)$ is a sum of terms of the form

$$
\begin{aligned}
\tau\left((\delta h)h^p(\delta h)h^q\right) &= \tau\left(h^{q/2}(\delta h)h^{p/2}h^{p/2}(\delta h)h^{q/2}\right) \\
&= -\tau\left((h^{p/2}(\delta h)h^{q/2})^* h^{p/2}(\delta h)h^{q/2}\right) \\
&\leq 0
\end{aligned}
$$

by Proposition 2.1(c) and the fact that $h^{p/2}$ and $h^{q/2}$ are Hermitian since $h \geq 0$, where $p, q \in \{0, 1, 2, ...\}$ with $p+1+q = j$. Thus indeed $\tau\left((\delta h)\delta h^j\right) \leq 0$ and therefore $\tau\left((\delta h)\delta e^h\right) \leq 0$, hence $\tau\left(e^a \delta^2 a\right) \geq 0$, so $\tau\left(e^a \triangle a\right) \geq 0$.

Now suppose $\tau\left(e^a \triangle a\right) = 0$. Since $\tau\left(e^a \delta_\mu^2 a\right) \geq 0$, it follows that $\tau\left(e^a \delta_\mu^2 a\right) = 0$ for $\mu = 1, 2$. Again in terms of the notation above, it follows from $\tau\left((\delta h)\delta h^j\right) \leq 0$ that $\tau\left((\delta h)\delta h^j\right) = 0$, in particular $0 = \tau((\delta h)\delta h) = -\langle \delta h, \delta h \rangle$. So $\delta_\mu a = \delta_\mu\left(h + \lambda\right) = 0$, hence $a \in \mathbb{C}1$ by Proposition 2.1(a). The converse is trivial. $\qquad\square$

The central result of this paper is the following (note that in its proof finite-dimensionality plays an essential role):

Theorem 3.2. *Let c_0 be any initial noncommutative metric at the initial time $t_0 \in \mathbb{R}$. Then the noncommutative Ricci flow equation has a unique solution c on the interval $[t_0, \infty)$. Furthermore, the flow preserves the trace, i.e. $\tau\left(c\left(t\right)\right) = \tau\left(c_0\right)$ for all $t \geq t_0$, and*

$$\lim_{t \to \infty} c\left(t\right) = c_\infty 1$$

in any of the equivalent norms on A, where

$$c_\infty := \frac{1}{n}\tau\left(c_0\right)$$

with n the dimension as before. Lastly, the determinant $\det c\left(t\right)$ is strictly increasing in t, unless $c_0 = c_\infty$ in which case $c\left(t\right) = c_0$ for all $t \in \mathbb{R}$.

Proof. The set A_{sa} of all Hermitian elements of A is a real vector space (not that its elements have real entries, but it is a vector space with respect to real scalar multiplication). If $c(t) > 0$, then $\log c(t)$ is Hermitian, hence so is $\triangle \log c(t)$ because of Proposition 2.1(c). So the Ricci flow equation is expressed wholly in terms of the space A_{sa}. In particular $\frac{d}{dt}c(t)$ is Hermitian, hence $c(t)$ at least remains in A_{sa} under the flow, so the eigenvalues of $c(t)$ remain real under the Ricci flow. However, due to the log, the equation is only defined on the set of strictly positive matrices. (In conventional ordinary differential equation terms, the equation is formulated for some domain in $\mathbb{R}^{n^2}$, the latter representing A_{sa}.) Since the set of strictly positive matrices is open in A_{sa}, we have solutions on open intervals which are short enough, and in order to show that the Ricci flow exists for all $t \geq t_0$, we need to show that $c(t) > 0$ remains true under the flow, and that the solution does not blow up in finite time. The solution will necessarily be unique, since it is a system of first order ordinary differential equations.

First note that by Proposition 2.1(g),

$$\frac{d}{dt}\tau(c) = \tau\left(\frac{dc}{dt}\right) = -\tau(\triangle \log c) = 0$$

on any interval on which the solution exists, i.e. the trace is preserved under Ricci flow. By Proposition 3.1 and the Ricci flow equation, we have

$$0 \leq \tau\left(e^{-\log c}\triangle(-\log c)\right) = \tau\left(c^{-1}\frac{dc}{dt}\right) = \frac{d}{dt}\log(\det c)$$

with equality to 0 if and only if $c(t) \in \mathbb{C}1$, and where the last equality is a standard identity from matrix analysis (see for example [22, Section 6.5] where the proof is outlined using properties of $\det c$). It follows that $\det c$ is strictly increasing in t, or constant in the case $c(t) \in \mathbb{C}1$, therefore $\det c$ remains larger than some strictly positive real number under the Ricci flow. However, since Ricci flow preserves the trace, none of the eigenvalues of c can become bigger than nc_∞ under the flow while they are all positive, so should one of the eigenvalues go to zero under the flow, it would follow that $\det c(t)$ would go to zero, which is a contradiction. Since none of the eigenvalues go to zero, this then also proves that all the eigenvalues remain strictly positive and bounded by nc_∞, which means that $c(t) > 0$ remains true under the flow and the solution c does not blow up. Therefore we have shown that the flow exists for all $t \in [t_0, \infty)$.

Also note that $c_0 \neq c_\infty$ along with the uniqueness of the solution, implies that $c(t)$ is not in $\mathbb{C}1$ for any t. For if $c(t)$ was in $\mathbb{C}1$ for some t, it would be there for all $t \in [t_0, \infty)$, since it is clearly a fixed point of the Ricci flow.

All that remains is to show that $c(t)$ converges to $c_\infty 1$. It is going to be convenient to work in terms of the Hilbert-Schmidt norm $\|\cdot\|_2$ given by $\|a\|_2 := \langle a, a \rangle^{1/2} = \tau(a^*a)^{1/2}$ for all $a \in A$. Consider the function v on A

defined by

$$v(a) := \|a - c_\infty\|_2^2$$

and note that since Ricci flow preserves the trace τ, we have

$$\frac{d}{dt}v(c) = \tau\left(\frac{d}{dt}(c - c_\infty)^2\right) = 2\tau\left(c\frac{dc}{dt}\right) = -2\tau(c\triangle\log c) \leq 0$$

by Proposition 3.1. In other words, $v(c(t))$ is decreasing in t and bounded from below by 0, therefore

$$L := \lim_{t\to\infty} v(c(t))$$

exists. We show that $L = 0$, since that will imply that $c(t)$ converges to $c_\infty 1$. Set

$$V := \left\{a \in A_{sa} : \frac{1}{n}\tau(e^a) = c_\infty \quad \text{and} \quad L \leq v(e^a) \leq L + 1\right\}$$

which is norm closed, since A_{sa} is. But V is also bounded and by finite-dimensionality of A it follows that V is compact. By definition of L and the conservation of $\tau(c(t))$ under Ricci flow, there exists a $t_1 \geq t_0$ such that $\log c(t) \in V$ for all $t > t_1$. Also note that $\frac{d}{dt}v(c(t))$ comes arbitrarily close to 0 for $t > t_1$, since suppose that $\frac{d}{dt}v(c(t)) \leq -\varepsilon$ for all $t > t_1$ for some $\varepsilon > 0$, then $v(c(t)) \leq v_0 - \varepsilon t$ for all $t > t_1$ for some $v_0 \in \mathbb{R}$, contradicting the fact that $v(c(t)) \geq 0$ for all $t > t_0$ by the definition of v. Furthermore, the function

$$f : A_{sa} \to \mathbb{C} : a \mapsto -2\tau(e^a\triangle a)$$

is continuous, so $f(V)$ is compact and therefore closed. But $f(\log c) = \frac{d}{dt}v(c)$ comes arbitrarily close to 0 for $t > t_1$ therefore $0 \in f(V)$. In other words there is an $a \in A_{sa}$ such that $\frac{1}{n}\tau(e^a) = c_\infty$, $v(e^a) \geq L \geq 0$ and $\tau(e^a\triangle a) = 0$. However, the last condition implies that $a \in \mathbb{C}1$ by Proposition 3.1, so $e^a = c_\infty 1$ by the first condition, which means that $v(e^a) = 0$. We conclude that $L = 0$. $\qquad\square$

Note that $c(t) = c_\infty 1$ is the unique fixed point of the Ricci flow which has trace nc_∞, since if $dc/dt = 0$, then $\triangle\log c = 0$, so by Proposition 2.1(f) $c(t) \in \mathbb{C}1$ from which it follows that $c(t) = c_\infty 1$. The theorem above says that the metric flows to this fixed point. Furthermore, the preservation of the trace corresponds the the preservation of the total area in the classical case. Also keep in mind that the solution referred to in the theorem is continuously differentiable by the general theory of systems of first order ordinary differential equations, although we can also see it from the fact that $\triangle\log c$ is differentiable, since c is (see for example [22, Section 6.6]), which means that d^2c/dt^2 exists by the noncommutative Ricci flow equation.

4 Density matrices and entropy

In Theorem 3.2 we found that the Ricci flow exists and converges to a flat metric. It is now natural to study further qualitative features of the flow. In particular we study a monotonicity property of the flow in terms of an entropy. In the classical case monotonicity properties are often useful in a qualitative analysis of geometric flows. See [18, Section 8] for a case in point. One can therefore expect monotonicity properties to be useful in the noncommutative case as well. In fact, we already saw a simple instance of this as part of the proof of Theorem 3.2 where the monotonicity of $\det c$, or equivalently of $\log(\det c) = \tau(\log c)$, helped us to show the existence of the Ricci flow on $[t_0, \infty)$. Here we consider a particularly simple entropy similar in form to the function $\tau(\log c)$ just mentioned, which can likewise be defined directly in terms of the metric and appears to fit very naturally into the theory we have set up so far, but is interestingly enough also well known from usual quantum mechanics.

Note that if in the noncommutative Ricci flow equation we set $t' = \kappa t$ and $\rho(t') = \kappa c(t'/\kappa)$ for any constant $\kappa > 0$, then

$$\frac{d}{dt}\rho(t) = -\triangle \log \rho(t)$$

(where we have written t instead of t') so we have simply scaled the Ricci flow. In particular, by the conservation of trace given by Theorem 3.2, we can choose κ such that $\rho(t)$ is a strictly positive density matrix, meaning $\rho(t) > 0$ and $\tau(\rho(t)) = 1$, for all $t \geq t_0$. By the previous section we then know that under the Ricci flow the density matrix flows to the unique fixed point of the flow, which in this case is the density matrix of maximum von Neumann entropy, namely $\frac{1}{n}\mathbf{1}$. But we can say more:

Theorem 4.1. *Consider the noncommutative Ricci flow of a strictly positive density matrix $\rho(t)$, i.e. $c = \rho$ in Theorem 3.2. Let*

$$S := -\tau(\rho \log \rho)$$

be the von Neumann entropy of ρ. Then either $\rho(t)$ is the density matrix of maximum entropy (the fixed point of the Ricci flow), or S is a strictly increasing function of t converging to the maximum entropy as $t \to \infty$.

Proof. The first fact we need is that for any Hermitian $a \in A$ we have

$$\tau(a \triangle a) \geq 0$$

with equality if and only if $a \in \mathbb{C}\mathbf{1}$. The inequality follows from Proposition 2.1(d). If $\tau(a \triangle a) = 0$, it also follows from Proposition 2.1(d) that $\langle a, \delta_\mu^2 a \rangle = 0$, so $\delta_\mu a = 0$ by Proposition 2.1(e), hence $a \in \mathbb{C}\mathbf{1}$ by Proposition 2.1(a).

The second fact we prove is the following matrix analysis identity:

$$\tau\left(e^l\frac{dl}{dt}\right) = \frac{d}{dt}\tau\left(e^l\right)$$

where we have set $l := \log\rho$. To see this, begin by noticing that since $d\rho/dt$ exists and is continuous as mentioned in Section 3, the same is true for dl/dt (see for example [22, Section 6.6]). Regarding the series expansion

$$\tau\left(e^l\right) = \sum_{j=0}^{\infty}\frac{1}{j!}\tau\left(l^j\right)$$

we have

$$\frac{d}{dt}\tau\left(l^j\right) = \tau\left(\frac{dl}{dt}l^{j-1} + l\frac{dl}{dt}l^{j-2} + ... + l^{j-1}\frac{dl}{dt}\right) = j\tau\left(l^{j-1}\frac{dl}{dt}\right)$$

so on any time interval $[t_1, t_2]$ we have in terms of the operator norm $\|\cdot\|$ on A that for every $t \in [t_1, t_2]$,

$$\left\|\frac{d}{dt}\tau\left(l\left(t\right)^j\right)\right\| \le jnM^{j-1}N$$

where M and N are respectively the maxima of $t \mapsto \|l\left(t\right)\|$ and $t \mapsto \left\|\frac{d}{dt}l\left(t\right)\right\|$ on $[t_1, t_2]$ (these maxima exist, since l and dl/dt are continuous). Since

$$\sum_{j=0}^{\infty}\frac{1}{j!}jnM^{j-1}N$$

is convergent, it follows from the Weierstrass test that

$$\sum_{j=0}^{\infty}\frac{1}{j!}\frac{d}{dt}\tau\left(l^j\right)$$

converges uniformly on $[t_1, t_2]$, hence we can differentiate the series for $\tau\left(e^l\right)$ termwise:

$$\frac{d}{dt}\tau\left(e^l\right) = \sum_{j=0}^{\infty}\frac{1}{j!}\frac{d}{dt}\tau\left(l^j\right) = \sum_{j=1}^{\infty}\frac{1}{(j-1)!}\tau\left(l^{j-1}\frac{dl}{dt}\right) = \tau\left(e^l\frac{dl}{dt}\right)$$

as required. (Note that this works for general c instead of ρ as well.)

However, as ρ is a solution of the Ricci flow, it follows from this identity that

$$\tau\left(\rho\frac{d}{dt}\log\rho\right) = \frac{d}{dt}\tau\left(\rho\right) = 0$$

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by Theorem 3.2. Now, if S is not at its maximum and therefore ρ and thus l are not in $\mathbb{C}1$, it follows by again using the Ricci flow equation that

$$\frac{d}{dt}S = -\tau \left(\frac{d\rho}{dt} \log \rho \right) - \tau \left(\rho \frac{d}{dt} \log \rho \right) = \tau \left(l \triangle l \right) > 0$$

from the inequality above. That is to say, S is strictly increasing in t. That S converges to the maximum entropy follows from Theorem 3.2, as already mentioned. $\qquad\square$

Note that in the classical case various entropies related to Ricci flow have been studied extensively (see for example [18, Section 7], [8] and [29]) but they are quite different from the entropy we studied here, even aside from the noncommutative setting, as they are not defined directly in terms of the metric, but typically rather in terms of the scalar curvature.

5   Scalar curvature

If we define the classical scalar curvature R in terms of the Ricci tensor $R_{\mu\nu}$ by $R := R_\mu^\mu$ as usual, then in the case of the classical metric $g_{\mu\nu} = c\delta_{\mu\nu}$, we obtain

$$R = -\frac{1}{c} \partial_\mu \partial_\mu \log c$$

which leads to ambiguity when adapted directly to the noncommutative case, i.e. in general c^{-1} and $\triangle \log c$ will not commute in $c^{-1} \triangle \log c$. However, one can naturally obtain R from the Ricci flow without any such ambiguity in the noncommutative case. Indeed, in the classical case it is easily verified that given a solution c to the Ricci flow,

$$R(t) = -\frac{\partial}{\partial t} \log c(t)$$

is the scalar curvature corresponding to the metric $c(t)$.

The corresponding formula can be used without any ambiguity in the noncommutative case as the *definition* of the scalar curvature at any time t in the noncommutative Ricci flow:

$$R(t) := -\frac{d}{dt} \log c(t)$$

This formula corresponds to [10, eq. (121)], although there it is not a definition, but follows from an alternative definition of scalar curvature. In particular, given a noncommutative metric c_0, simply consider the Ricci flow c starting there, at time $t = 0$, and the scalar curvature of c_0 is then defined to be

$$R_0 := -\frac{d}{dt} \log c(t) \big|_{t=0}$$

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Since $\log c(t)$ is a Hermitian matrix, so is R_0, so it corresponds to a real-valued function on a classical surface as it should, in contrast to $c^{-1} \triangle \log c$ or $(\triangle \log c) c^{-1}$ which are not in general Hermitian.

The noncommutative scalar curvature will not be easy to evaluate analytically in general, although numerically it would be possible. Nevertheless we can still study some of properties of the noncommutative scalar curvature defined in this way, which is what we now turn to.

The basic property is that the scalar curvature has zero average, as it does on the classical torus. Keep in mind that in the classical case the average follows from an integral over the surface, where we need to include the metric as a factor to have the correct measure, i.e. the integral of a function f over the surface is given by $\int f c \, dx^1 dx^2$. Similarly we include the metric c in the noncommutative case as well, exactly as is done in [11]. We formulate this along with some related elementary properties:

Proposition 5.1. *If R_0 is the scalar curvature of the noncommutative metric c_0, then*

$$\tau (c_0 R_0) = 0$$

In particular, if R_0 is constant, i.e. $R_0 \in \mathbb{C}1$, then $R_0 = 0$, which in turn holds if and only if c_0 is constant, i.e. the metric is flat.

Proof. Using the identity derived in the proof of Theorem 4.1, we have

$$\tau (c_0 R_0) = -\tau \left(c \frac{d}{dt} \log c \right)\Big|_{t=0} = -\frac{d}{dt} \tau (c)\Big|_{t=0} = 0$$

by Theorem 3.2.

If R_0 is constant, say $R_0 = r1$ for some real number r, it follows that $r = \tau (c_0 R_0) / \tau (c_0) = 0$, so $R_0 = 0$.

If c_0 is constant, i.e. $c_0 \in \mathbb{C}1$, then it is a fixed point of the Ricci flow, so $R_0 = 0$. Conversely, if $R_0 = 0$, then using the Ricci flow equation as well as the matrix analysis identity appearing in the proof of Theorem 3.2, we have

$$
\begin{aligned}
\tau \left(e^{-\log c_0} \triangle (-\log c_0) \right) &= \tau \left(c^{-1} \frac{dc}{dt} \right)\Big|_{t=0} = \frac{d}{dt} \log (\det c)|_{t=0} \\
&= \frac{d}{dt} \tau (\log c)\Big|_{t=0} = -\tau (R_0) = 0
\end{aligned}
$$

from which it follows that $c_0 \in \mathbb{C}1$ by Proposition 3.1. $\qquad \square$

6   Further work

A number of points have not been addressed in this paper. One is the existence and behaviour of the noncommutative Ricci flow on $(-\infty, t_0]$. Another is whether the convergence of the noncommutative Ricci flow is exponential

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(see [18, Section 5] for the classical case). One could also consider generalizing the setting. For example in [12] and [34] more general setups for metrics on the noncommutative torus are considered. And of course one could attempt to study Ricci flow and the resulting scalar curvature on other matrix geometries, like the fuzzy sphere [25], or even more general frameworks like noncommutative Riemann surfaces [2], [3]. It would also be possible to study the noncommutative Ricci flow equation numerically.

The evolution of R under Ricci flow appears to be another worthwhile problem to explore. For example to find lower and upper bounds for the smallest and largest eigenvalue of $R(t)$ respectively, in analogy to the classical case [18] (also see for example [9, Chapter 5]). This would also open up the possibility to study entropy defined in terms of the scalar curvature as in for example [18] and [8].

A study of the connection between the noncommutative Ricci flow as studied in this paper and the other approaches mentioned in the Introduction might be insightful. This would somehow have to involve a large n "limit" of the approach in this paper, in which m/n in $q = e^{2\pi im/n}$ would have to converge to a specified value θ. Possibly ideas from [32], [23] and [4], or [33] and [24] would be relevant here.

More speculatively, the fact that c can be normalized to a density matrix by scaling the "time" parameter is suggestive that there may be an interpretation or use of the noncommutative Ricci flow in quantum mechanics. However, since the Ricci flow equation is not linear, it is not the physical time-evolution of a quantum system. Nevertheless it may be interesting to explore whether it has applications in for example quantum information. More broadly it may be fruitful to study if the geometric view of a density matrix as a metric in noncommutative geometry has any value in quantum mechanics and quantum information and if the Ricci scalar curvature has a quantum mechanical interpretation.

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