

# Mathematical Analysis of West Nile Virus Model with Discrete Delays

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## Abstract

The paper presents the basic model for the transmission dynamics of West Nile virus (WNV). The model, which consists of seven mutually-exclusive compartments representing the birds and vector dynamics, has a locally-asymptotically stable disease-free equilibrium whenever the associated reproduction number ( $\mathcal{R}_0$ ) is less than unity. As reveal in [3, 20], the analyses of the model show the existence of the phenomenon of backward bifurcation (where the stable disease-free equilibrium of the model co-exists with a stable endemic equilibrium when the reproduction number of the disease is less than unity). It is shown, that the backward bifurcation phenomenon can be removed by substituting the associated standard incidence function with a mass action incidence. Analysis of the reproduction number of the model shows that, the disease will persist, whenever  $\mathcal{R}_0 > 1$ , and increase in the length of incubation period can help reduce WNV burden in the community if a certain threshold quantities, denoted by  $\Delta_b$  and  $\Delta_v$  are negative. On the other hand, increasing the length of the incubation period increases disease burden if  $\Delta_b > 0$  and  $\Delta_v > 0$ . Furthermore, it is shown that adding time delay to the corresponding autonomous model with standard incidence (considered in [2]) does not alter the qualitative dynamics of the autonomous system (with respect to the elimination or persistence of the disease).

**Keywords:** West Nile virus (WNV), equilibria, stability, persistent, reproduction number.

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# 1 Introduction

West Nile virus (WNV) was detected for the first time in North America in 1999, during an outbreak involving humans, horses, and birds in the New York City [5, 14]. Since then WNV has spread rapidly across the continent resulting in numerous human infections and death [2]. The virus is widespread in Africa, the Middle East, west and central Asia, with occasional outbreaks in Europe introduced by migrating birds [17, 28]. It is believe that birds are the natural reservoir, and humans, horses and probably other vertebrates are circumstantial hosts (that is, they can be infected by an infectious mosquito but they do not transmit the infection). Thus, WNV is maintain in nature in mosquito-bird-mosquito transmission cycle [8, 15, 17]. The disease is spread to humans and other animals via mosquito bites. Recent evidence show that WNV can be transmitted through blood transfusions, organ/tissue transplants, needle stick injury, exposure to infected laboratory specimen and mouth to mouth transmission [1, 2, 6, 18, 27].

Although many WNV-infected people remain asymptomatic, and some show mild flu-like symptoms such as fever, headache, body aches, nausea, vomiting etc., 75% of infected individuals develop severe illness, such as high fever, meningitis, encephalitis, disorientation, coma tremors, convulsions, muscle weakness, vision loss, numbness and paralysis which typically last for weeks [2, 18].

There is no specific treatment for WNV other than supportive therapy (such as hospitalization, intravenous fluids, and respiratory support) for severe cases. Antibiotics will not work because a virus, not bacteria, causes West Nile disease. No vaccine for the virus is currently available. In the absence of effective anti-WNV therapeutic treatment and vaccine, WNV control strategies are based on taking appropriate preventive measures. These measures include the use of mosquito repellent when outdoors. Mosquitoes may bite through thin clothing, so spraying clothes with repellent containing permethrin or another EPA-registered repellent will give extra protection. These repellents are the most effective and the most studied [5, 6]. Culling has also been a widely adopted tool to control vector-borne diseases, for instance, larvicides and insecticides sprays as intervention strategies against mosquito [14].

A number of authors have put up some mathematical modelling work about the transmission dynamics of WNV [2, 3, 8, 9, 14, 15, 21, 22, 23, 24, 32, 33]. The work on modelling transmission dynamics of WNV can be divided into two categories. The first category consists of models that assume immediate transition from an infected to an infectious state [2, 3, 8]. The second category includes models with delay in which the assumption that there is some time lapse (delay) from an infected to an infectious state in one or both human/birds and mosquito populations is made [14, 15]. The work in this paper is builds on the second category of models, that is, it assumes delay from an exposed and infected to an infectious state in both mosquito and birds populations.

The following aspects considered in this paper differentiate our work from some of those that have been previously done, for instance [2, 3, 8, 9, 21, 22, 23, 24, 32].

- (i) assumes delay from an exposed and infected to an infectious state in both mosquito and birds populations;
- (ii) assumes transmission by exposed mosquitoes and birds.

The model to be considered in the current study is assumed to be an extensions to the models considered in [2, 3]. The main objective of this research work is to use mathematical modeling and analysis to gain insight into the transmission dynamics of WNV in a population, with particular emphasis in delay on the transmission process (i.e., one of the main aims of this study is to determine whether or not incorporating time delay alters the qualitative dynamics of the autonomous models considered in [2, 3]).

The paper is organized as follows. The governing time-delay model is given in Section 2. Some of its basic dynamical features are also presented. The model is analysed in Section 3.

## 2 Model formulation

The WNV model to be considered is given by the following non-autonomous system of differential equations:

$$\begin{aligned}
\frac{dS_b(t)}{dt} &= \Pi_b - \frac{C_{vb}(N_b(t), N_v(t))S_b(t)[\eta_v E_v(t) + I_v(t)]}{N_v(t)} - \mu_b S_b(t), \\
E_b &= \int_{t-\tau_b}^t \frac{C_{vb}(N_b(t), N_v(t))S_b(y)[\eta_v E_v(y) + I_v(y)]e^{-\mu_b(t-y)}}{N_v(y)} dy, \\
\frac{dI_b(t)}{dt} &= \frac{C_{vb}(N_b(t), N_v(t))S_b(t-\tau_b)[\eta_v E_v(t-\tau_b) + I_v(t-\tau_b)]e^{-\tau_b\mu_b}}{N_v(t-\tau_b)} - (\gamma_b + \mu_b + \delta_b)I_b(t), \\
\frac{dR_b(t)}{dt} &= \gamma_b I_b(t) - \mu_b R_b(t), \\
\frac{dS_v(t)}{dt} &= \Pi_v - \frac{C_{bv}S_v(t)[\eta_b E_b(t) + I_b(t)]}{N_b(t)} - \mu_v S_v(t), \\
E_v &= \int_{t-\tau_v}^t \frac{C_{bv}S_b(y)[\eta_b E_b(y) + I_b(y)]e^{-\mu_v(t-y)}}{N_b(y)} dy, \\
\frac{dI_v(t)}{dt} &= \frac{C_{bv}S_v(t-\tau_v)[\eta_b E_b(t-\tau_v) + I_b(t-\tau_v)]e^{-\tau_v\mu_v}}{N_b(t-\tau_v)} - (\mu_v + \delta_v)I_v(t).
\end{aligned} \tag{1}$$

The  $S_b$ ,  $E_b$ ,  $I_b$  and  $R_b$  denote, respectively, the population of susceptible, exposed, infectious and recovered birds. Similarly, the  $S_v$ ,  $E_v$  and  $I_v$  denote, respectively, the

population of susceptible, exposed and infectious mosquitoes. So that the total number of birds and mosquitoes, at, time  $t$ , are respectively, given by,

$$N_b(t) = S_b(t) + E_b(t) + I_b(t) + R_b(t)$$

and

$$N_v(t) = S_v(t) + E_v(t) + I_v(t).$$

Furthermore,  $\Pi_b$  is the recruitment rate into the susceptible birds population,  $C_{vb}(N_b, N_v) = \rho_{vb}\beta_i$  is the rate at which birds acquire infection from infected mosquitoes (exposed or infectious),  $\beta_i$  is the biting rate of infectious mosquitoes and  $\rho_{vb}$  is the transmission probability from infectious mosquitoes to susceptible birds. Similarly,  $C_{bv} = \rho_{bv}\beta_s$  is the rate at which mosquitoes acquire infection from infected birds (exposed or infectious),  $\beta_s$  is the biting rate of susceptible mosquitoes and  $\rho_{bv}$  is the transmission probability from infectious birds to susceptible mosquitoes.  $\mu_b$  and  $\delta_b$  are the natural and disease induced death rate for birds.  $\gamma_b$  is the birds recovery rate.  $\Pi_v$  is the birth rate for the susceptible mosquitoes,  $\mu_v$  and  $\delta_v$  are the natural and disease induced death rate for mosquitoes. Finally, transmission of the disease is through birds and/or mosquitoes at a fixed latent period  $\tau_b$  and  $\tau_v$ , respectively.

## 2.1 Incidence Functions

In this section, the functional forms of the incidence functions associated with the transmission dynamics of WNV disease will be derived. The derivation is based on the basic fact that for mosquito-borne diseases (such as WNV), the total number of bites made by mosquitoes must equal the total number of bites received by birds (see also [2, 11]). Consequently, we define the mosquito biting rate to be a function of these total populations (i.e.,  $\beta_s = \beta_s(N_v, N_b)$ ).

Since mosquitoes bite both susceptible and infected birds, it is assumed that the average number of mosquito bites received by birds depends on the total sizes of the populations of mosquitoes and birds in the community. It is assumed that each susceptible mosquito bites an infected bird at an average biting rate,  $\beta_s$ , and the birds reservoir are always sufficient in abundance; so that it is reasonable to assume that the biting rate,  $\beta_s$ , is constant. As defined earlier,

$$C_{bv} = \rho_{bv}\beta_s,$$

$C_{bv}$  is a constant. Similarly,

$$C_{vb}(N_b, N_v) = \rho_{vb}\beta_i.$$

Thus, for the number of bites to be conserved, the following equation must hold,

$$C_{bv}N_v = C_{vb}(N_b, N_v)N_b, \quad (2)$$

so that,

$$N_v = \frac{C_{vb}(N_b, N_v)}{C_{bv}}N_b. \quad (3)$$

Using (3) in (1) gives,

$$\begin{aligned} \frac{dS_b(t)}{dt} &= \Pi_b - \frac{C_{bv}S_b(t)[\eta_v E_v(t) + I_v(t)]}{N_b(t)} - \mu_b S_b(t), \\ E_b &= \int_{t-\tau_b}^t \frac{C_{bv}S_b(y)[\eta_v E_v(y) + I_v(y)]e^{-\mu_b(t-y)}}{N_b(y)} dy, \\ \frac{dI_b(t)}{dt} &= \frac{C_{bv}S_b(t-\tau_b)[\eta_v E_v(t-\tau_b) + I_v(t-\tau_b)]e^{-\tau_b\mu_b}}{N_b(t-\tau_b)} - (\gamma_b + \mu_b + \delta_b)I_b(t), \\ \frac{dR_b(t)}{dt} &= \gamma_b I_b(t) - \mu_b R_b(t), \\ \frac{dS_v(t)}{dt} &= \Pi_v - \frac{C_{bv}S_v(t)[\eta_b E_b(t) + I_b(t)]}{N_b(t)} - \mu_v S_v(t), \\ E_v &= \int_{t-\tau_v}^t \frac{C_{bv}S_v(y)[\eta_b E_b(y) + I_b(y)]e^{-\mu_v(t-y)}}{N_b(y)} dy, \\ \frac{dI_v(t)}{dt} &= \frac{C_{bv}S_v(t-\tau_v)[\eta_b E_b(t-\tau_v) + I_b(t-\tau_v)]e^{-\tau_v\mu_v}}{N_b(t-\tau_v)} - (\mu_v + \delta_v)I_v(t). \end{aligned} \quad (4)$$

Since the model (1) monitors birds and mosquito populations, it is assumed that all the state variables and parameters of the model are non-negative. The initial data for the model (1) is given by

$$\begin{aligned} S_b(t) &= \phi_1(\theta), E_b(t) = \phi_2(\theta), I_b(t) = \phi_3(\theta), R_b(t) = \phi_4(\theta), \\ S_v(t) &= \phi_5(\theta), E_v(t) = \phi_6(\theta), I_v(t) = \phi_7(\theta), \text{ and } \theta \in [-\tau, 0], \end{aligned} \quad (5)$$

where,  $\phi = [\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6, \phi_7]^T \in \mathcal{C}$ ,  $\tau = \max(\tau_b, \tau_v)$ , such that  $\phi_i(\theta) \geq \phi_i(0)$  for  $(\theta \in [-\tau, 0], i = 1, 2, 3, 4, 5, 6, 7)$ ,  $\phi_2(\theta) \geq 0$ ,  $(\theta \in [-\tau, 0])$  and  $\mathcal{C}$  denotes the Banach space  $\mathcal{C}([-\tau, 0], \mathbb{R}^7)$  of continuous functions mapping the interval  $[-\tau, 0]$  into  $\mathbb{R}^7$ , equipped with the uniform norm defined by  $\|\phi\| = \sup_{\theta \in [-\tau, 0]} |\phi(\theta)|$ . Furthermore, it is assumed that  $\phi_i(0) > 0$  (for  $i = 1, 2, \dots, 7$ ).

## 2.2 Basic Properties

Since  $R_b$  does not feature in any of the other equation of the model (4), we can easily remove it from the model. Further, using the generalized Leibnitz rule of differentiation,

the model (4) can be re-written as a system of delayed differential difference equation given by:

$$\begin{aligned}
\frac{dS_b(t)}{dt} &= \Pi_b - \frac{C_{bv}S_b(t)[\eta_v E_v(t) + I_v(t)]}{N_b(t)} - \mu_b S_b(t), \\
\frac{dE_b(t)}{dt} &= \frac{C_{bv}S_b(t)[\eta_v E_v(t) + I_v(t)]}{N_b(t)} - \frac{C_{bv}S_b(t - \tau_b)[\eta_v E_v(t - \tau_b) + I_v(t - \tau_b)]e^{-\tau_b \mu_b}}{N_b(t - \tau_b)} - \mu_b E_b(t), \\
\frac{dI_b(t)}{dt} &= \frac{C_{bv}S_b(t - \tau_b)[\eta_v E_v(t - \tau_b) + I_v(t - \tau_b)]e^{-\tau_b \mu_b}}{N_b(t - \tau_b)} - (\gamma_b + \mu_b + \delta_b)I_b(t), \\
\frac{dS_v(t)}{dt} &= \Pi_v - \frac{C_{bv}S_v(t)[\eta_b E_b(t) + I_b(t)]}{N_b(t)} - \mu_v S_v(t), \\
\frac{dE_v(t)}{dt} &= \frac{C_{bv}S_v(t)[\eta_b E_b(t) + I_b(t)]}{N_b(t)} - \frac{C_{bv}S_v(t - \tau_v)[\eta_b E_b(t - \tau_v) + I_b(t - \tau_v)]e^{-\tau_v \mu_v}}{N_b(t - \tau_v)} - \mu_v E_v(t), \\
\frac{dI_v(t)}{dt} &= \frac{C_{bv}S_v(t - \tau_v)[\eta_b E_b(t - \tau_v) + I_b(t - \tau_v)]e^{-\tau_v \mu_v}}{N_b(t - \tau_v)} - (\mu_v + \delta_v)I_v(t).
\end{aligned} \tag{6}$$

The basic properties of the model (6) will now be investigated.

### 2.3 Positivity and boundedness of solutions

For the model (6) to be epidemiologically meaningful, it is important to prove that all its state variables are non-negative at all time. In other words solutions of the model system (6) with positive initial data will remain positive for all time. We claim the following.

**Theorem 1** *The solutions  $S_b(t)$ ,  $E_b(t)$ ,  $I_b(t)$ ,  $S_v(t)$ ,  $E_v(t)$ ,  $I_v(t)$  of the system (6), with initial data (5), exists, for all  $t \geq 0$  and is unique. Furthermore,  $S_b(t) > 0$ ,  $E_b(t) > 0$ ,  $I_b(t) > 0$ ,  $S_v(t) > 0$ ,  $E_v(t) > 0$  and  $I_v(t) > 0$  for all  $t \geq 0$ . In addition,*

$$\limsup_{t \rightarrow \infty} N_b(t) \leq \frac{\Pi_b}{\mu_b} \quad \text{and} \quad \limsup_{t \rightarrow \infty} N_v(t) \leq \frac{\Pi_v}{\mu_v}.$$

*Proof.* System (1) can be written as

$$\dot{X} = f(t, X_t),$$

where  $X_t = [S_b(t), E_b(t), I_b(t), S_v(t), E_v(t), I_v(t)] \in \mathcal{C}$ . Since  $f(t, X_t)$  is continuous and Lipschitz in  $X_t$ , it follows then, by the fundamental theory functional differential equation [16], the system (6) has a unique solution  $[S_b(t), E_b(t), I_b(t), S_v(t), E_v(t), I_v(t)]$  satisfying the initial data (5). It is clear from the first equation of (6) that

$$\frac{dS_b}{dt} \geq - \left\{ \frac{C_{bv}[\eta_v E_v(t) + I_v(t)]}{N_b(t)} + \mu_b \right\} S_b(t),$$

so that

$$S_b(t) \geq S_b(0) \exp \left[ - \int_0^t \left( \frac{C_{bv}[\eta_v E_v(u) + I_v(u)]}{N_b(u)} + \mu_b \right) du \right] > 0, \text{ for all } t > 0.$$

Similarly, it follows from the third equation of the system (6), that  $I_b(t) > 0$  for all  $t \geq 0$ . Since the second equation in (6) is equivalent to the second equation in (1), we have

$$E_b(t) = \int_{t-\tau_b}^t \frac{C_{bv} S_b(y) [\eta_v E_v(y) + I_v(y)] e^{-\mu_b(t-y)}}{N_b(y)} dy > 0.$$

Furthermore, from the third equation of system (6), it can be shown that

$$I_b(t) > I_b(0) e^{-(\gamma_b + \mu_b + \delta_b)t} > 0.$$

Using the same approach as that for  $S_b(t)$ ,  $E_b(t)$  and  $I_b(t)$  it is easy to show that  $S_v(t) > 0$ ,  $E_v(t) > 0$  and  $I_v(t) > 0$ , for all  $t > 0$ . Adding the first three equations and the last three equations in the system (6) gives, respectively,

$$\frac{dN_b}{dt} = \Pi_b - \mu_b N_b - \delta_b I_b \quad \text{and} \quad \frac{dN_v}{dt} = \Pi_v - \mu_v N_v - \delta_v I_v. \quad (7)$$

It follows that

$$\begin{aligned} \Pi_b - (\mu_b + \delta_b) N_b &\leq \frac{dN_b}{dt} < \Pi_b - \mu_b N_b, \\ \Pi_v - (\mu_v + \delta_v) N_v &\leq \frac{dN_v}{dt} < \Pi_v - \mu_v N_v, \end{aligned} \quad (8)$$

so that

$$\begin{aligned} \frac{\Pi_b}{\mu_b + \delta_b} &\leq \liminf_{t \rightarrow \infty} N_b(t) \leq \limsup_{t \rightarrow \infty} N_b(t) \leq \frac{\Pi_b}{\mu_b}, \\ \frac{\Pi_v}{\mu_v + \delta_v} &\leq \liminf_{t \rightarrow \infty} N_v(t) \leq \limsup_{t \rightarrow \infty} N_v(t) \leq \frac{\Pi_v}{\mu_v}. \end{aligned}$$

Hence

$$\limsup_{t \rightarrow \infty} N_b(t) \leq \frac{\Pi_b}{\mu_b} \quad \text{and} \quad \limsup_{t \rightarrow \infty} N_v(t) \leq \frac{\Pi_v}{\mu_v}. \quad (9)$$

□

## 2.4 Invariant region

From (7), following the terminology in [25], the conservation law

$$\frac{dN_b}{dt} \leq \Pi_b - \mu_b N_b \quad \text{and} \quad \frac{dN_v}{dt} \leq \Pi_v - \mu_v N_v \quad (10)$$

holds.

It follows from (10) and the Gronwall inequality, that

$$N_b(t) \leq N_b(0)e^{-\mu_b t} + \frac{\Pi_b}{\mu_b} \left(1 - e^{-\mu_b t}\right) \text{ and } N_v(t) \leq N_v(0)e^{-\mu_v t} + \frac{\Pi_v}{\mu_v} \left(1 - e^{-\mu_v t}\right).$$

Hence,

$$N_b(t) \leq \Pi_b/\mu_b \text{ if } N_b(0) \leq \Pi_b/\mu_b \text{ and } N_v(t) \leq \Pi_v/\mu_v \text{ if } N_v(0) \leq \Pi_v/\mu_v. \quad (11)$$

This result is summarized below.

**Lemma 1** *The following biologically-feasible region of the model (6) is positively-invariant:*

$$\mathcal{D} = \left\{ (S_b, E_b, I_b, S_v, E_v, I_v) \in \mathbb{R}_+^6 : S_b + E_b + I_b \leq \frac{\Pi_b}{\mu_b}; S_v + E_v + I_v \leq \frac{\Pi_v}{\mu_v} \right\}.$$

It should be noted that Lemma 1 means that the model (6) is a dynamical system on the region  $\mathcal{D}$  [31]. Thus, in the region  $\mathcal{D}$ , the model is well-posed epidemiologically and mathematically [19]. Hence, it is sufficient to study the qualitative dynamics of the model (6) in  $\mathcal{D}$ .

### 3 Analysis of the model

The disease-free equilibrium point (DFE) of the system (6), is given by

$$\mathcal{E}_0 = (S_b^*, E_b^*, I_b^*, S_v^*, E_v^*, I_v^*) = \left( \frac{\Pi_b}{\mu_b}, 0, 0, \frac{\Pi_v}{\mu_v}, 0, 0 \right). \quad (12)$$

It follows then that the *associated reproduction number*, denoted by,  $\mathcal{R}_0$ , is given by

$$\mathcal{R}_0 = \sqrt{\frac{C_{bv}^2 \Pi_v [\eta_b K_1 (1 - e^{-\tau_b \mu_b}) + \mu_b e^{-\tau_b \mu_b}] [\eta_v K_2 (1 - e^{-\tau_v \mu_v}) + \mu_v e^{-\tau_v \mu_v}]}{\Pi_b \mu_v^2 K_1 K_2}}$$

where  $K_1 = \gamma_b + \mu_b + \delta_b$  and  $K_2 = \mu_v + \delta_v$ . The threshold quantity,  $\mathcal{R}_0$ , is the reproduction number of the model (6), which is defined as the average number of secondary cases that one infected case can generate when introduced into a completely susceptible population [10, 19]. Since 1911, control and intervention efforts have been based on the concept of the basic reproduction number, introduced in [29].



### 3.1 Local Stability of the DFE

We claim the following.

**Lemma 2** *The DFE,  $\mathcal{E}_0$ , of the model (6), is locally-asymptotically stable (LAS) if  $\mathcal{R}_0 < 1$ , and unstable if  $\mathcal{R}_0 > 1$ .*

*Proof.* The linearised form for the model system (6) may be written in matrix form as

$$\frac{dz}{dt} = J_1 z(t) + J_2 z(t - \tau_b) + J_3 z(t - \tau_v), \quad (13)$$

where  $z$  is a vector with components  $z_{ij}$  and  $J_1 = (a_{ij})$ ,  $J_2 = (b_{ij})$ ,  $J_3 = (c_{ij})$  are as given below

$$J_1 = \begin{bmatrix} -\mu_b & 0 & 0 & 0 & -C_{bv}\eta_v & -C_{bv} \\ 0 & -\mu_b & 0 & 0 & C_{bv}\eta_v & C_{bv} \\ 0 & 0 & -K_1 & 0 & 0 & 0 \\ 0 & -\frac{C_{bv}\eta_b\mu_b\Pi_v}{\Pi_b\mu_v} & -\frac{C_{bv}\mu_b\Pi_v}{\Pi_b\mu_v} & -\mu_v & 0 & 0 \\ 0 & \frac{C_{bv}\eta_b\mu_b\Pi_v}{\Pi_b\mu_v} & \frac{C_{bv}\mu_b\Pi_v}{\Pi_b\mu_v} & 0 & -\mu_v & 0 \\ 0 & 0 & 0 & 0 & 0 & -K_2 \end{bmatrix},$$

$$J_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -C_{bv}\eta_v e^{-\tau_b\mu_b} & -C_{bv} e^{-\tau_b\mu_b} \\ 0 & 0 & 0 & 0 & C_{bv}\eta_v e^{-\tau_b\mu_b} & C_{bv} e^{-\tau_b\mu_b} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$J_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{C_{bv}\eta_b\Pi_v\mu_b e^{-\tau_v\mu_v}}{\Pi_b\mu_v} & -\frac{C_{bv}\Pi_v\mu_b e^{-\tau_v\mu_v}}{\Pi_b\mu_v} & 0 & 0 & 0 & 0 \\ 0 & \frac{C_{bv}\eta_b\Pi_v\mu_b e^{-\tau_v\mu_v}}{\Pi_b\mu_v} & \frac{C_{bv}\Pi_v\mu_b e^{-\tau_v\mu_v}}{\Pi_b\mu_v} & 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is known that the zero solution of (6) is asymptotically stable if and only if the zero solution of the linearisation (13) is asymptotically stable. If  $z(t) = e^{\lambda t}u$  is a solution of (13), where  $u = (u_1, u_2, \dots, u_6)^T$  is a constant vector, then

$$(\lambda I - J_1 - J_2 e^{-\lambda \tau_b} - J_3 e^{-\lambda \tau_v})u = 0. \quad (14)$$

Solving (14) gives a characteristic quasi-polynomial equation

$$\lambda^4 + (\mu_b + \mu_v + K_1 + K_2)\lambda^3 + Q_1\lambda^2 + Q_2\lambda + \mu_b\mu_v K_1 K_2 = (P_1\lambda^2 + P_2\lambda + \mu_b\mu_v K_1 K_2 R_0^2)e^{-\lambda(\tau_b + \tau_v)}$$

where,

$$\begin{aligned} P_1 &= \frac{C_{bv}^2 \Pi_v \mu_b}{\Pi_b \mu_v} \left[ \eta_b \eta_v e^{-(\tau_b + \tau_v)} + (\eta_b + e^{-\tau_b} e^{-\tau_b \mu_b})(\eta_v + e^{-\tau_v} e^{-\tau_v \mu_v}) \right], \\ P_2 &= \frac{C_{bv}^2 \Pi_v \mu_b}{\Pi_b \mu_v} \left\{ e^{-(\tau_b + \tau_v)} e^{-\tau_b \mu_b} e^{-\tau_v \mu_v} [\mu_b + \mu_v + \eta_b \eta_v (K_1 + K_2)] + e^{-\tau_b} e^{-\tau_b \mu_b} (K_2 + \mu_b) \right. \\ &\quad \left. + e^{-\tau_v} e^{-\tau_v \mu_v} (K_1 + \mu_v) + \eta_b \eta_v (K_1 + K_2) \right\}, \\ Q_1 &= \frac{C_{bv}^2 \Pi_v \mu_b}{\Pi_b \mu_v} \left[ e^{-(\tau_b + \tau_v)} e^{-\tau_b \mu_b} e^{-\tau_v \mu_v} (\eta_b + \eta_v) + \eta_b \eta_v (e^{-\tau_b} e^{-\tau_b \mu_b} + e^{-\tau_v} e^{-\tau_v \mu_v}) \right] \\ &\quad + (\mu_b + \mu_v)(K_1 + K_2) + K_1 K_2 + \mu_b \mu_v, \\ Q_2 &= \frac{C_{bv}^2 \Pi_v \mu_b}{\Pi_b \mu_v} \left\{ e^{-(\tau_b + \tau_v)} e^{-\tau_b \mu_b} e^{-\tau_v \mu_v} [\eta_b (K_1 + \mu_v) + \eta_v (K_2 + \mu_b)] \right. \\ &\quad \left. + \eta_b \eta_v (e^{-\tau_b} e^{-\tau_b \mu_b} + e^{-\tau_v} e^{-\tau_v \mu_v})(K_1 + K_2) \right\} + \mu_b \mu_v (K_1 + K_2) + K_1 K_2 (\mu_b + \mu_v). \end{aligned}$$

Thus,

$$\left| \frac{(P_1\lambda^2 + P_2\lambda + \mu_b\mu_v K_1 K_2 R_0^2)e^{-\lambda(\tau_b + \tau_v)}}{\lambda^4 + (\mu_b + \mu_v + K_1 + K_2)\lambda^3 + Q_1\lambda^2 + Q_2\lambda + \mu_b\mu_v K_1 K_2} \right| = 1. \quad (15)$$

Define

$$g(\lambda) = \frac{(P_1\lambda^2 + P_2\lambda + \mu_b\mu_v K_1 K_2 R_0^2)e^{-\lambda(\tau_b + \tau_v)}}{\lambda^4 + (\mu_b + \mu_v + K_1 + K_2)\lambda^3 + Q_1\lambda^2 + Q_2\lambda + \mu_b\mu_v K_1 K_2}.$$

Let  $Re(\lambda) > 0$ . Since the function is continuous on any closed subset  $[0, a]$  of  $[0, \infty)$ , and differentiable on the closed subset  $[0, a]$  of  $[0, \infty)$ , it follows, by the Maximum Modulus Theorem [7], that  $|g(\lambda)|$  attains its maximum on the boundary. Thus, on the closed subset  $[0, a]$  of  $[0, \infty)$ , its maximum is either  $|g(0)|$  or  $\lim_{a \rightarrow \infty} |g(a)|$ . Since  $|g(0)| > \lim_{a \rightarrow \infty} |g(a)| = 0$  then the maximum must occur at  $\lambda = 0$ . Hence,

$$1 = |g(\lambda)| = \left| \frac{(P_1\lambda^2 + P_2\lambda + \mu_b\mu_v K_1 K_2 R_0^2)e^{-\lambda(\tau_b + \tau_v)}}{\lambda^4 + (\mu_b + \mu_v + K_1 + K_2)\lambda^3 + Q_1\lambda^2 + Q_2\lambda + \mu_b\mu_v K_1 K_2} \right| \leq |g(0)| = \mathcal{R}_0^2.$$

Thus, for  $\tau_b + \tau_v > 0$  and  $Re(\lambda) > 0$  we have

$$\left| \frac{(P_1\lambda^2 + P_2\lambda + \mu_b\mu_v K_1 K_2 R_0^2)e^{-\lambda(\tau_b + \tau_v)}}{\lambda^4 + (\mu_b + \mu_v + K_1 + K_2)\lambda^3 + Q_1\lambda^2 + Q_2\lambda + \mu_b\mu_v K_1 K_2} \right| \leq \mathcal{R}_0^2. \quad (16)$$

Therefore, it is shown that  $\mathcal{R}_0^2 \geq 1$  whenever  $Re(\lambda) > 0$ , which implies that if  $\mathcal{R}_0^2 < 1$  then  $Re(\lambda) \leq 0$ . Hence,  $\mathcal{E}_0$  is a locally asymptotically stable equilibrium if  $\mathcal{R}_0 < 1$ .  $\square$

### 3.2 Endemic equilibria and backward bifurcation

In order to establish the existence of endemic equilibria of the model (6) (that is, equilibria where at least one of the infected components of the model is non-zero), the following steps are taken. Let  $E_1 = (S_b^{**}, E_b^{**}, I_b^{**}, S_v^{**}, E_v^{**}, I_v^{**})$ , represents any arbitrary endemic equilibrium of the model (6). We also note that at equilibrium  $f(t) = f(t - \tau)$ , where  $f$  is a state variable and  $\tau$  is delay time. Solving the equations of the model (6) at steady-state gives

$$\begin{aligned} S_b^{**} &= \frac{\Pi_b}{\lambda_b^{**} + \mu_b}, & E_b^{**} &= \frac{\Pi_b \lambda_b^{**} (1 - e^{-\tau_b \mu_b})}{\mu_b (\lambda_b^{**} + \mu_b)}, & I_b^{**} &= \frac{\Pi_b \lambda_b^{**} e^{-\tau_b \mu_b}}{K_1 (\lambda_b^{**} + \mu_b)}, \\ S_v^{**} &= \frac{\Pi_v}{\lambda_v^{**} + \mu_v}, & E_v^{**} &= \frac{\Pi_v \lambda_v^{**} (1 - e^{-\tau_v \mu_v})}{\mu_v (\lambda_v^{**} + \mu_v)}, & I_v^{**} &= \frac{\Pi_v \lambda_v^{**} e^{-\tau_v \mu_v}}{K_2 (\lambda_v^{**} + \mu_v)}, \end{aligned} \quad (17)$$

where

$$\lambda_b^{**} = \frac{C_{bv} S_b^{**} [\eta_v E_v^{**} + I_v^{**}]}{N_b^{**}}, \quad \lambda_v^{**} = \frac{C_{bv} S_v^{**} [\eta_b E_b^{**} + I_b^{**}]}{N_b^{**}}. \quad (18)$$

Substituting (17) in (18), and simplifying, gives, respectively,

$$\lambda_b^{**} = \frac{C_{bv} \Pi_v \mu_b K_1 [\eta_v K_2 \lambda_b^{**} (\lambda_v^{**} + \mu_b) (1 - e^{-\tau_v \mu_v}) + e^{-\tau_v \mu_v} \mu_v e^{-\tau_v \mu_v}]}{K_2 \Pi_b \mu_v (\lambda_b^{**} + \mu_v) \{K_1 [\mu_b + \lambda_v^{**} (1 - e^{-\tau_b \mu_b})] + e^{-\tau_b \mu_b}\}}, \quad (19)$$

$$\lambda_v^{**} = \frac{C_{bv} \lambda_v^{**} [\eta_b K_1 (1 - e^{-\tau_b \mu_b}) + \mu_b e^{-\tau_b \mu_b}]}{K_1 [\mu_b + \lambda_v^{**} (1 - e^{-\tau_v \mu_v})] + \mu_b e^{-\tau_b \mu_b}}. \quad (20)$$

By substituting (20) in (19), it can be shown that the non-zero equilibria of the model (6) satisfy the following equation (in terms of  $\lambda_b^{**}$ )

$$\lambda_b^{**} [a_1 (\lambda_b^{**})^2 + a_2 \lambda_b^{**} + a_3] = 0, \quad (21)$$

where

$$\begin{aligned} a_1 &= \mu_v K_2 \Pi_b \left\{ \left[ K_1 (1 - e^{-\tau_b \mu_b}) + \mu_b e^{-\tau_b \mu_b} \right] \left[ K_1 (1 - e^{-\tau_b \mu_b}) (C_{bv} \eta_b + \mu_v) + \mu_b e^{-\tau_b \mu_b} (c_{bv} + \mu_v) \right] \right\}, \\ a_2 &= \mu_v \mu_b K_1 K_2 \Pi_b \left\{ C_{bv} \left[ K_1 \eta_b (1 - e^{-\tau_b \mu_b}) + \mu_b e^{-\tau_b \mu_b} \right] + 2\mu_v \mu_b e^{-\tau_b \mu_b} + K_1 \mu_v \left[ 2(1 - e^{-\tau_b \mu_b}) - \mathcal{R}_0^2 \right] \right\}, \\ a_3 &= \mu_v^2 K_1^2 K_2 \Pi_b (1 - \mathcal{R}_0^2). \end{aligned}$$

Clearly  $\lambda_b^{**} = 0$  is a solution of (6) which corresponds to the disease-free equilibrium  $\mathcal{E}_0$ . Furthermore, the coefficient  $a_1$ , of (21), is always positive, and  $a_3$  is positive (negative) if  $\mathcal{R}_0$  is less than (greater than) unity, respectively. Thus, the following result is established.

**Theorem 2** *The WNV model (6) has:*

- (i) *a unique endemic equilibrium if  $a_3 < 0 \Leftrightarrow \mathcal{R}_0 > 1$ ;*
- (ii) *unique endemic equilibrium if  $a_2 < 0$ , and  $a_3 = 0$  or  $a_2^2 - 4a_1a_3 = 0$ ;*
- (iii) *two endemic equilibria if  $a_3 > 0$ ,  $a_2 < 0$  and  $a_2^2 - 4a_1a_3 > 0$ ;*
- (iv) *no endemic equilibrium otherwise.*

It is clear from Theorem 2 (Case i) that the model has a unique endemic equilibrium whenever  $\mathcal{R}_0 > 1$ . Further, Case (iii) indicates the possibility of backward bifurcation (where the stable DFE co-exists with a stable endemic equilibrium when  $\mathcal{R}_0 < 1$ ; see, for instance, [3, 4, 11, 12, 13, 30]) in the model (6) when  $\mathcal{R}_0 < 1$ .

To check for this, the discriminant  $a_2^2 - 4a_1a_3$  is set to zero and solved for the critical value of  $\mathcal{R}_0$ , denoted by  $\mathcal{R}_c$ , given by

$$\mathcal{R}_c = \sqrt{1 - \frac{a_2^2}{4\mu_v^2\mu_b^2K_1^2K_2\Pi_b}}. \quad (22)$$

Thus, backward bifurcation would occur for values of  $\mathcal{R}_0$  such that  $\mathcal{R}_c < \mathcal{R}_0 < 1$ . This is illustrated by simulating the model with the following set of parameter values (it should be stated that these parameters are chosen for illustrative purpose only, and may not necessarily be realistic epidemiologically):  $\mu_b = 0.0099$ ,  $\mu_v = 0.0714$ ,  $\delta_b = 0.599$ ,  $\delta_v = 0.0575$ ,  $\Pi_b = 10$ ,  $\Pi_v = 30$ ,  $\eta_b = 0.799$ ,  $\eta_v = 0.799$ ,  $C_{bv} = 0.12$ ,  $\gamma_b = 0.53$ ,  $\tau_b = 15$ ,  $\tau_v = 25$  (see Table 1 for the units of the aforementioned parameters). With this set of parameters,  $\mathcal{R}_c = 0.8008553758 < 1$  and  $\mathcal{R}_0 = 0.8698334365 < 1$  (so that,  $\mathcal{R}_c < \mathcal{R}_0 < 1$ ). The associated bifurcation diagram is depicted in Figure 2. This clearly shows the co-existence of two locally-asymptotically stable equilibria when  $\mathcal{R}_0 < 1$ , confirming that the model (6) undergoes the phenomenon of backward bifurcation. Thus, the following result is established.

**Lemma 3** *The basic model (6) undergoes backward bifurcation when Case (iii) of Theorem 2 holds and  $\mathcal{R}_c < \mathcal{R}_0 < 1$ .*

The epidemiological significance of the phenomenon of backward bifurcation is that the classical requirement of  $\mathcal{R}_0 < 1$  is, although necessary, no longer sufficient for disease

elimination. In such a scenario, disease elimination would depend on the initial sizes of the sub-populations (state variables) of the model.

This result is consistent with the finding in [3, 20] that reveal the existence of the phenomenon of backward bifurcation in transmission dynamics of West Nile virus.

### 3.3 Analysis of reduced model with mass action incidence

Another interesting aspect to note is that it was shown in [3, 11] that some disease transmission models with standard incidence can lose their backward bifurcation property if the standard incidence formulation is replaced by mass action incidence. To explore this in the context of the model (6), we consider the mass action equivalent of the model (6) (note that, here, we assume the disease induced death rate of birds  $\delta_b$  and mosquitoes  $\delta_v$  are negligible, so that the total birds population is constant,  $N_b(t) = N_b = \text{constant}$ ), given by:

$$\begin{aligned}
\frac{dS_b(t)}{dt} &= \Pi_b - C_{bv}S_b(t)[\eta_v E_v(t) + I_v(t)] - \mu_b S_b(t), \\
E_b &= \int_{t-\tau_b}^t C_{bv}S_b(y)[\eta_v E_v(y) + I_v(y)]e^{-\mu_b(t-y)} dy, \\
\frac{dI_b(t)}{dt} &= C_{bv}S_b(t - \tau_b)[\eta_v E_v(t - \tau_b) + I_v(t - \tau_b)]e^{-\tau_b\mu_b} - (\gamma_b + \mu_b)I_b(t), \\
\frac{dR_b(t)}{dt} &= \gamma_b I_b(t) - \mu_b R_b(t), \\
\frac{dS_v(t)}{dt} &= \Pi_v - C_{bv}S_v(t)[\eta_b E_b(t) + I_b(t)] - \mu_v S_v(t), \\
E_v &= \int_{t-\tau_v}^t C_{bv}S_v(y)[\eta_b E_b(y) + I_b(y)]e^{-\mu_v(t-y)} dy, \\
\frac{dI_v(t)}{dt} &= C_{bv}S_v(t - \tau_v)[\eta_b E_b(t - \tau_v) + I_b(t - \tau_v)]e^{-\tau_v\mu_v} - \mu_v I_v(t).
\end{aligned} \tag{23}$$

Using the method as in model (6), the mass action model (23) can be re-written as a

system of a delayed differential difference equation given by:

$$\begin{aligned}
\frac{dS_b(t)}{dt} &= \Pi_b - C_{bv}S_b(t)[\eta_v E_v(t) + I_v(t)] - \mu_b S_b(t), \\
\frac{dE_b(t)}{dt} &= C_{bv}S_b(t)[\eta_v E_v(t) + I_v(t)] - C_{bv}S_b(t - \tau_b)[\eta_v E_v(t - \tau_b) + I_v(t - \tau_b)]e^{-\tau_b \mu_b} - \mu_b E_b(t), \\
\frac{dI_b(t)}{dt} &= C_{bv}S_b(t - \tau_b)[\eta_v E_v(t - \tau_b) + I_v(t - \tau_b)]e^{-\tau_b \mu_b} - (\gamma_b + \mu_b)I_b(t), \\
\frac{dS_v(t)}{dt} &= \Pi_v - C_{bv}S_v(t)[\eta_b E_b(t) + I_b(t)] - \mu_v S_v(t), \\
\frac{dE_v(t)}{dt} &= C_{bv}S_v(t)[\eta_b E_b(t) + I_b(t)] - C_{bv}S_v(t - \tau_v)[\eta_b E_b(t - \tau_v) + I_b(t - \tau_v)]e^{-\tau_v \mu_v} - \mu_v E_v(t), \\
\frac{dI_v(t)}{dt} &= C_{bv}S_v(t - \tau_v)[\eta_b E_b(t - \tau_v) + I_b(t - \tau_v)]e^{-\tau_v \mu_v} - \mu_v I_v(t).
\end{aligned} \tag{24}$$

The resulting (mass action) model (24), has the same DFE,  $\mathcal{E}_0$ , given by (12). For this model, the *associated reproduction number* is given by

$$\mathcal{R}_0^m = \sqrt{\frac{C_{bv}^2 \Pi_v \Pi_b [\eta_b \hat{K}_1 (1 - e^{-\tau_b \mu_b}) + \mu_b e^{-\tau_b \mu_b}] [\eta_v (1 - e^{-\tau_v \mu_v}) + e^{-\tau_v \mu_v}]}{\mu_b^2 \mu_v^2 \hat{K}_1}}$$

where  $\hat{K}_1 = \gamma_b + \mu_b$ , so that the following local stability result is established (using Theorem 2 of [33]).

**Lemma 4** *The DFE,  $\mathcal{E}_0$ , of the mass action model (24), is LAS if  $\mathcal{R}_0^m < 1$ , and unstable if  $\mathcal{R}_0^m > 1$ .*

### 3.3.1 Non-existence of endemic equilibria for $\mathcal{R}_0^m \leq 1$

We claim the following.

**Theorem 3** *The mass action WNV model (24), has no endemic equilibrium when  $\mathcal{R}_0^m \leq 1$ , and has a unique endemic equilibrium otherwise.*

*Proof.* Solving the equations of the model (24) at steady-state gives

$$\begin{aligned}
S_b^{**} &= \frac{\Pi_b}{(\lambda_b^m)^{**} + \mu_b}, & E_b^{**} &= \frac{\Pi_b (\lambda_b^m)^{**} (1 - e^{-\tau_b \mu_b})}{\mu_b ((\lambda_b^m)^{**} + \mu_b)}, & I_b^{**} &= \frac{\Pi_b (\lambda_b^m)^{**} e^{-\tau_b \mu_b}}{\hat{K}_1 ((\lambda_b^m)^{**} + \mu_b)}, \\
S_v^{**} &= \frac{\Pi_v}{(\lambda_v^m)^{**} + \mu_v}, & E_v^{**} &= \frac{\Pi_v (\lambda_v^m)^{**} (1 - e^{-\tau_v \mu_v})}{\mu_v ((\lambda_v^m)^{**} + \mu_v)}, & I_v^{**} &= \frac{\Pi_v (\lambda_v^m)^{**} e^{-\tau_v \mu_v}}{\mu_v ((\lambda_v^m)^{**} + \mu_v)},
\end{aligned} \tag{25}$$

where

$$(\lambda_b^m)^{**} = C_{bv} S_b^{**} (\eta_v E_v^{**} + I_v^{**}), \quad (\lambda_v^m)^{**} = C_{bv} S_v^{**} (\eta_b E_b^{**} + I_b^{**}). \tag{26}$$

Substituting (25) in (26), and simplifying, gives, respectively,

$$(\lambda_b^m)^{**} = \frac{C_{bv}\Pi_v(\lambda_b^m)^{**}[\eta_v(1 - e^{-\tau_v\mu_v}) + e^{-\tau_v\mu_v}]}{\mu_v((\lambda_b^m)^{**} + \mu_v)}, \quad (27)$$

$$(\lambda_v^m)^{**} = \frac{C_{bv}\Pi_b(\lambda_v^m)^{**}[\eta_b\hat{K}_1(1 - e^{-\tau_b\mu_b}) + \mu_b e^{-\tau_b\mu_b}]}{\mu_b\hat{K}_1((\lambda_v^m)^{**} + \mu_b)}. \quad (28)$$

By substituting (28) in (27), it can be shown that the non-zero equilibria of the model (24) satisfy the following quadratic (in terms of  $(\lambda_b^m)^{**}$ )

$$(\lambda_b^m)^{**}[b_1(\lambda_b^m)^{**} + b_2] = 0, \quad (29)$$

where

$$\begin{aligned} b_1 &= C_{bv}\Pi_b[\eta_b\hat{K}_1(1 - e^{-\tau_b\mu_b})] + \mu_v\mu_b\hat{K}_1 \\ b_2 &= \mu_v^2\mu_b\hat{K}_1(1 - (\mathcal{R}_0^m)^2). \end{aligned}$$

Clearly,  $b_1 > 0$  and  $b_2 \geq 0$  whenever  $\mathcal{R}_0^m \leq 1$ , so that  $(\lambda_b^m)^{**} = \frac{-b_2}{b_1} \leq 0$ . Therefore, the mass action model, (24), has no endemic equilibrium whenever  $\mathcal{R}_0^m \leq 1$ .  $\square$

The above result suggests the impossibility of backward bifurcation in the mass action model (24), since no endemic equilibrium exists when  $\mathcal{R}_0^m \leq 1$  (and backward bifurcation requires the presence of at least two endemic equilibria when  $\mathcal{R}_0^m \leq 1$ ). A global stability result is established for the DFE of the mass action model (24) below (to completely rule out backward bifurcation).

### 3.3.2 Global Stability of the DFE

We claim the following.

**Theorem 4** *The DFE,  $\mathcal{E}_0$ , of the model (24), given by (12), is globally-asymptotically stable (GAS) in  $\mathcal{D}$  if  $\mathcal{R}_0^m < 1$ .*

*Proof.* Taking the lim sup of both sides of  $E_b(t)$  in system (23) and apply the fact that  $\limsup \int g \leq \int \limsup g$  (Kribs-Zaleta and Velasco Hernandez, 2000; Kribs-Zaleta, 1999) to get

$$\begin{aligned} \limsup_{t \rightarrow \infty} E_b &= \limsup_{t \rightarrow \infty} \int_{t-\tau_b}^t C_{bv}S_b(y)[\eta_v E_v(y) + I_v(y)]e^{-\mu_b(t-y)} dy, \\ &\leq \int_{t-\tau_b}^t \limsup_{t \rightarrow \infty} C_{bv}S_b(y) \limsup_{t \rightarrow \infty} [\eta_v E_v(y) + I_v(y)]e^{-\mu_b(t-y)} dy, \\ &\leq \limsup_{t \rightarrow \infty} C_{bv}S_b(t) \limsup_{t \rightarrow \infty} [\eta_v E_v(t) + I_v(t)] \int_{t-\tau_b}^t e^{-\mu_b(t-y)} dy, \\ &= \frac{C_{bv}\Pi_b(1 - e^{-\tau_b\mu_b})}{\mu_b^2} \limsup_{t \rightarrow \infty} [\eta_v E_v(t) + I_v(t)]. \end{aligned} \quad (30)$$

Similarly, for the fifth integral in (23), we get

$$\limsup_{t \rightarrow \infty} E_v = \frac{C_{bv}\Pi_v(1 - e^{-\tau_v\mu_v})}{\mu_v^2} \limsup_{t \rightarrow \infty} [\eta_b E_b(t) + I_b(t)]. \quad (31)$$

Furthermore, rewriting the infective compartments of both populations in system (24) as a set of integral equations:

$$\begin{aligned} I_b(t) &= \int_{-\infty}^t C_{bv} S_b(x - \tau_b) [\eta_v E_v(x - \tau_b) + I_v(x - \tau_b)] e^{-\tau_b\mu_b} e^{-\hat{K}_1(t-x)} dx, \\ I_v(t) &= \int_{-\infty}^t C_{bv} S_v(x - \tau_v) [\eta_b E_b(x - \tau_v) + I_b(x - \tau_v)] e^{-\tau_v\mu_v} e^{-\mu_v(t-x)} dx. \end{aligned} \quad (32)$$

Using the substitution  $w = t - x$ , we take the lim sup of both sides of the equation of  $I_b(t)$  to get

$$\begin{aligned} \limsup_{t \rightarrow \infty} I_b(t) &= \limsup_{t \rightarrow \infty} \int_0^\infty C_{bv} S_b(t - w - \tau_b) [\eta_v E_v(t - w - \tau_b) + I_v(t - w - \tau_b)] e^{-\tau_b\mu_b} e^{-\hat{K}_1(w)} dw, \\ &\leq \int_0^\infty \limsup_{t \rightarrow \infty} C_{bv} e^{-\tau_b\mu_b} S_b(t - w - \tau_b) \times \\ &\quad \limsup_{t \rightarrow \infty} [\eta_v E_v(t - w - \tau_b) + I_v(t - w - \tau_b)] e^{-\hat{K}_1(w)} dw, \\ &\leq \limsup_{t \rightarrow \infty} C_{bv} e^{-\tau_b\mu_b} S_b(t) \limsup_{t \rightarrow \infty} [\eta_v E_v(t) + I_v(t)] \int_0^\infty e^{-\hat{K}_1(w)} dw, \\ &= \frac{C_{bv}\Pi_b e^{-\tau_b\mu_b}}{\mu_b \hat{K}_1} \limsup_{t \rightarrow \infty} [\eta_v E_v(t) + I_v(t)]. \end{aligned} \quad (33)$$

Similarly, for the second integral in (32), we get

$$\limsup_{t \rightarrow \infty} I_v(t) \leq \frac{C_{bv}\Pi_v e^{-\tau_v\mu_v}}{\mu_v^2} \limsup_{t \rightarrow \infty} [\eta_b E_b(t) + I_b(t)]. \quad (34)$$

Thus, from (30) - (33) we have

$$\limsup_{t \rightarrow \infty} [\eta_b E_b + I_b(t)] \leq \frac{C_{bv}\Pi_b \left[ \eta_b \hat{K}_1 (1 - e^{-\tau_b\mu_b}) + \mu_b e^{-\tau_b\mu_b} \right]}{\mu_b^2 \hat{K}_1} \limsup_{t \rightarrow \infty} [\eta_v E_v(t) + I_v(t)], \quad (35)$$

and

$$\limsup_{t \rightarrow \infty} [\eta_v E_v + I_v(t)] \leq \frac{C_{bv}\Pi_v [\eta_v (1 - e^{-\tau_v\mu_v}) + e^{-\tau_v\mu_v}]}{\mu_v^2} \limsup_{t \rightarrow \infty} [\eta_b E_b(t) + I_b(t)]. \quad (36)$$

Substituting (36) into (35), we have

$$\limsup_{t \rightarrow \infty} [\eta_b E_b + I_b(t)] \leq (\mathcal{R}_0^m)^2 \limsup_{t \rightarrow \infty} [\eta_b E_b(t) + I_b(t)]. \quad (37)$$



If  $\mathcal{R}_0^m < 1$ , we have the strict inequality (and contradiction)  $\limsup_{t \rightarrow \infty} [\eta_b E_b + I_b(t)] < \limsup_{t \rightarrow \infty} [\eta_b E_b + I_b(t)]$ , unless  $\limsup_{t \rightarrow \infty} [\eta_b E_b + I_b(t)] = 0$ . Thus, the DFE,  $\mathcal{E}_0$ , of the model (1), is GAS in  $\mathcal{D}$  if  $\mathcal{R}_0^m < 1$ .  $\square$

Theorem 4 guarantees that the solutions of the model (24) will always converge to the DFE,  $\mathcal{E}_0$ , whenever  $\mathcal{R}_0^m < 1$ . This result is numerically illustrated by simulating the model (24) with parameter values such that  $\mathcal{R}_0^m < 1$  (Figure 3).

### 3.4 Permanence of Disease

Permanence (or persistence) address the long-term survival of some or all components of a given epidemiological system. In this section, the conditions needed for persistence of WNV in a population (for the case when  $\mathcal{R}_0 > 1$ ) will be explored. In other words, it will be shown that whenever an EEP of the model (6) exists (i.e.,  $\mathcal{R}_0 > 1$ ), then the number of people infected with WNV will remain above certain positive level. The procedure described in [26, 34] will be used to established the permanence of WNV in the population when  $\mathcal{R}_0 > 1$ . Assumed  $\mathcal{R}_0 > 1$ . It should be recalled from equation (8) that

$$\begin{aligned} \Pi_b - (\mu_b + \delta_b)N_b &\leq \frac{dN_b}{dt} < \Pi_b - \mu_b N_b, \\ \Pi_v - (\mu_v + \delta_v)N_v &\leq \frac{dN_v}{dt} < \Pi_v - \mu_v N_v. \end{aligned}$$

Let

$$M_1 = \frac{\Pi_b}{\mu_b + \delta_b}, M_2 = \frac{\Pi_b}{\mu_b}, N_1 = \frac{\Pi_v}{\mu_v + \delta_v} \text{ and } N_2 = \frac{\Pi_v}{\mu_v}.$$

Setting

$$I_1 = \left(1 - \frac{1}{\mathcal{R}_0}\right)M_1 \text{ and } I_2 = \left(1 - \frac{1}{\mathcal{R}_0}\right)N_1. \quad (38)$$

Furthermore, suppose

$$0 < \eta_1 < I_1 \text{ and } 0 < \eta_2 < I_2. \quad (39)$$

It follows that

$$\mathcal{R}_0 \left[1 - \frac{1}{M_1} \left(I_1 - \frac{\eta_1}{2}\right)\right] > 1 \text{ and } \mathcal{R}_0 \left[1 - \frac{1}{N_1} \left(I_2 - \frac{\eta_2}{2}\right)\right] > 1,$$

so that

$$\mathcal{R}_0^2 \left[1 - \frac{1}{M_1} \left(I_1 - \frac{\eta_1}{2}\right)\right] \left[1 - \frac{1}{N_1} \left(I_2 - \frac{\eta_2}{2}\right)\right] > 1. \quad (40)$$

We claim the following.

**Lemma 5** For any  $t_0 > 0$ , it is impossible that  $S_b(t) < I_1 - \eta_1, E_b(t) < I_1 - \eta_1, I_b(t) < I_1 - \eta_1, S_v(t) < I_2 - \eta_2, E_v(t) < I_2 - \eta_2$  and  $I_v(t) < I_2 - \eta_2$ , for all  $t \geq t_0$ .

*Proof.*

See Appendix A. □

Furthermore, we claim the following.

**Theorem 5** Consider the model (6). WNV will persist in the population whenever  $\mathcal{R}_0 > 1$ . That is whenever  $\mathcal{R}_0 > 1$ , there are positive constants,  $q_i$  and  $Q_i$ , ( $i = 1, 2, \dots, 6$ ), such that the following inequalities hold for any solutions  $(S_b(t), E_b(t), I_b(t), S_v(t), E_v(t), I_v(t))$  of the model (6) with initial condition in  $\mathcal{D}$ :

$$\begin{aligned} q_1 &\leq \liminf_{t \rightarrow \infty} S_b(t) \leq \limsup_{t \rightarrow \infty} S_b(t) \leq Q_1, & q_2 &\leq \liminf_{t \rightarrow \infty} E_b(t) \leq \limsup_{t \rightarrow \infty} E_b(t) \leq Q_2, \\ q_3 &\leq \liminf_{t \rightarrow \infty} I_b(t) \leq \limsup_{t \rightarrow \infty} I_b(t) \leq Q_3, & q_4 &\leq \liminf_{t \rightarrow \infty} S_v(t) \leq \limsup_{t \rightarrow \infty} S_v(t) \leq Q_4, \\ q_5 &\leq \liminf_{t \rightarrow \infty} E_v(t) \leq \limsup_{t \rightarrow \infty} E_v(t) \leq Q_5, & q_6 &\leq \liminf_{t \rightarrow \infty} I_v(t) \leq \limsup_{t \rightarrow \infty} I_v(t) \leq Q_6. \end{aligned} \quad (41)$$

*Proof.* See Appendix B. □

### 3.5 Threshold Analysis: Effect of Time Delay

In order to qualitatively measure the impact of time delay (incubation period) on the transmission dynamics of WNV, a threshold analysis is carried out on the delay parameters ( $\tau_b$  and  $\tau_v$ ) of the model (6) by computing the partial derivatives of  $\mathcal{R}_0$  with respect to these parameters. Setting  $\mathcal{R}_0 = 1$ , implies that

$$\frac{C_{bv}^2 \Pi_v [\eta_b K_1 (1 - e^{-\tau_b \mu_b}) + \mu_b e^{-\tau_b \mu_b}] [\eta_v K_2 (1 - e^{-\tau_v \mu_v}) + \mu_v e^{-\tau_v \mu_v}]}{\Pi_b \mu_v^2 h a t K_1 K_2} = 1, \quad (42)$$

let

$$\Delta_b = K_1 \left( \eta_b - \frac{\Pi_b K_2 \mu_v^2}{C_{bv}^2 \Pi_v [\eta_v K_2 (1 - e^{-\tau_v \mu_v}) + \mu_v e^{-\tau_v \mu_v}]} \right),$$

$$\text{and } \Delta_v = K_2 \left( \eta_v - \frac{\Pi_b K_1 \mu_b^2}{C_{bv}^2 \Pi_v [\eta_b K_1 (1 - e^{-\tau_b \mu_b}) + \mu_b e^{-\tau_b \mu_b}]} \right).$$

Solving for  $\tau_b$  in (42) gives

$$\tau_b = \tau_b^c = -\frac{1}{\mu_b} \ln \left( \frac{\Delta_b}{\eta_b K_1 - \mu_b} \right).$$

Similarly, solving for  $\tau_v$  in (42) gives

$$\tau_v = \tau_v^c = -\frac{1}{\mu_v} \ln\left(\frac{\Delta_v}{\eta_v K_2 - \mu_v}\right).$$

Since  $\tau_b^c$  and  $\tau_v^c$  represent biological process (incubation period),  $\tau_b^c > 0$  and  $\tau_v^c > 0$ . Hence,  $\tau_b^c$  and  $\tau_v^c$  will be biologically feasible whenever the following conditions hold:

- (i)  $sign(\Delta_b) = sign(\eta_b K_1 - \mu_b)$ ;
- (ii)  $sign(\Delta_v) = sign(\eta_v K_2 - \mu_v)$ ;
- (iii)  $0 < \left(\frac{\Delta_b}{\eta_b K_1 - \mu_b}\right) < 1$ ;
- (iv)  $0 < \left(\frac{\Delta_v}{\eta_v K_2 - \mu_v}\right) < 1$ .

Thus,  $\tau_i^c$  ( $i = b, v$ ) is a sharp epidemiological threshold above (below) which disease persistence (elimination) is possible. Hence, applying Lemma 2, we have the following result.

**Corollary 1** *Suppose Conditions (i) to (iv) holds. The DFE  $\mathcal{E}_0$ , of the model (6), is LAS if any of the following conditions is satisfied:*

- (v)  $\tau_b < \tau_b^c(\Delta_b > 0)$ ,  $\tau_v < \tau_v^c(\Delta_v > 0)$ ;
- (vi)  $\tau_b > \tau_b^c(\Delta_b < 0)$ ,  $\tau_v > \tau_v^c(\Delta_v < 0)$ ;
- (vii)  $\tau_b < \tau_b^c(\Delta_b > 0)$ ,  $\tau_v > \tau_v^c(\Delta_v < 0)$ ;
- (viii)  $\tau_b > \tau_b^c(\Delta_b < 0)$ ,  $\tau_v < \tau_v^c(\Delta_v > 0)$ .

Numerical simulation of the model (6) using the parameter values in Table 3 and various initial conditions for the case when  $\mathcal{R}_0 = 0.1802 < 1$  ( $\tau_b > \tau_b^c(\Delta_b < 0)$ ) is depicted in Figure 3. It is evident from this figure that all solutions converged to the DFE,  $\mathcal{E}_0$ . On the other hand the disease persist when  $\mathcal{R}_0 = 1.4014 > 1$  ( $\tau_b < \tau_b^c(\Delta_b > 0)$ ) as shown in Figure 4. Further simulation show that, increase in the incubation period (time delay) reduces the transmission rate of WNV (Figure 5).

## Conclusion

A deterministic model for the transmission dynamics of WNV with time delay in a population is designed and rigorously analysed. The model has a locally asymptotically stable DFE whenever the associated reproduction number is less than unity. As reveal

in [3, 20], the analyses of the model show the existence of the phenomenon of backward bifurcation (where the stable disease-free equilibrium of the model co-exists with a stable endemic equilibrium when the reproduction number of the disease is less than unity) in the transmission dynamics of WNV. It is shown, that the backward bifurcation phenomenon can be removed by substituting the associated standard incidence function with a mass action formulation. Analysis of the reproduction number show that, the disease will persist when  $\mathcal{R}_0 > 1$ . Furthermore, an increase in the length of incubation period can help reduce WNV burden in the community if a certain threshold quantities, denoted by  $\Delta_b$  and  $\Delta_v$  are negative. On the other hand, increasing the length of the incubation period increases disease burden if  $\Delta_b > 0$  and  $\Delta_v > 0$ .

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## Appendix A: Proof of Lemma 5

*Proof.* The proof is based on the approach in [26, 34], by contradiction. Suppose Lemma 3 does not hold. Then, there exists  $t_0 > 0$  and  $t_1 \geq \max\{t_0 + \tau_b, t_0 + \tau_v\}$  such that

$$\begin{aligned} E_b(t) < I_1 - \eta_1 \quad \text{and} \quad \frac{E_b(t)}{N_b(t)} &\leq \frac{1}{M_1} \left( I_1 - \frac{\eta_1}{2} \right), \\ I_b(t) < I_1 - \eta_1 \quad \text{and} \quad \frac{I_b(t)}{N_b(t)} &\leq \frac{1}{M_1} \left( I_1 - \frac{\eta_1}{2} \right), \\ E_v(t) < I_2 - \eta_2 \quad \text{and} \quad \frac{E_v(t)}{N_v(t)} &\leq \frac{1}{M_2} \left( I_2 - \frac{\eta_2}{2} \right), \\ I_v(t) < I_2 - \eta_2 \quad \text{and} \quad \frac{I_v(t)}{N_v(t)} &\leq \frac{1}{M_2} \left( I_2 - \frac{\eta_2}{2} \right), \text{ for } t \geq \max\{t_1 - \tau_b, t_1 - \tau_v\}. \end{aligned} \quad (43)$$

Since the infection period for infectious birds is  $\tau_b$  and infectious mosquitoes is  $\tau_v$ . It follows from (30) and (32) that

$$\begin{aligned} E_b(t) &\geq C_{bv} \int_{t-\tau_b}^t [\eta_v E_v(y) + I_v(y)] \frac{S_b(y)}{N_b(y)} e^{-\mu_b(t-y)} dy, \\ I_b(t) &\geq C_{bv} e^{-\tau_b \mu_b} \int_{-\infty}^t [\eta_v E_v(y - \tau_b) + I_v(y - \tau_b)] \frac{S_b(y - \tau_b)}{N_b(y - \tau_b)} e^{-K_1(t-y)} dy, \text{ for } t \geq \tau_b. \end{aligned} \quad (44)$$

Similarly,

$$\begin{aligned} E_v(t) &\geq C_{bv} \int_{t-\tau_b}^t [\eta_b E_b(y) + I_b(y)] \frac{S_v(y)}{N_b(y)} e^{-\mu_v(t-y)} dy, \\ I_v(t) &\geq C_{bv} e^{-\tau_v \mu_v} \int_{-\infty}^t [\eta_b E_b(y - \tau_v) + I_b(y - \tau_v)] \frac{S_v(y - \tau_v)}{N_b(y - \tau_v)} e^{-K_2(t-y)} dy, \text{ for } t \geq \tau_v, \end{aligned} \quad (45)$$

Letting

$$G_b(t) = \eta_b E_b(t) + I_b(t) \quad \text{and} \quad G_v(t) = \eta_v E_v(t) + I_v(t), \quad (46)$$

and noting from  $N_b(t) = S_b(t) + E_b(t) + I_b(t)$ , that,

$$\frac{S_b(t)}{N_b(t)} = 1 - \frac{E_b(t)}{N_b(t)} - \frac{I_b(t)}{N_b(t)}, \quad (47)$$

it follows from using the bounds given in (43) into (47) that

$$\frac{S_b(t)}{N_b(t)} \geq 1 - \frac{2}{M_1} \left( I_1 - \frac{\eta_1}{2} \right). \quad (48)$$

Hence, using (44) and (48), the first equation of (46) for  $t \geq t_1$ , gives

$$G_b(t) \geq C_{bv} \left[ 1 - \frac{2}{M_1} \left( I_1 - \frac{\eta_1}{2} \right) \right] \left[ \eta_b \int_{t-\tau_b}^t G_v(y) e^{-\mu_b(t-y)} dy + e^{-\tau_b \mu_b} \int_{-\infty}^t G_v(y - \tau_b) e^{-K_1(t-y)} dy \right]. \quad (49)$$

Similarly, using (44) and (48), the second equation of (46) for  $t \geq t_1$ , gives

$$G_v(t) \geq \frac{C_{bv}\Pi_v\mu_b}{\Pi_b\mu_v} \left[ 1 - \frac{2}{M_2} \left( I_2 - \frac{\eta_2}{2} \right) \right] \left[ \eta_v \int_{t-\tau_v}^t G_b(y) e^{-\mu_v(t-y)} dy \right. \\ \left. + e^{-\tau_v\mu_v} \int_{-\infty}^t G_b(y - \tau_b) e^{-K_2(t-y)} dy \right]. \quad (50)$$

Let  $J_1 = \min_{t \in [t_1, t_1 + \tau_b]} G_b(t)$  and  $J_2 = \min_{t \in [t_1, t_1 + \tau_v]} G_v(t)$ . We claim that  $G_b(t) \geq J_1$  and  $G_v(t) \geq J_2$  for all  $t \geq t_1$ . If not, then there is a  $t_2 \geq \max\{t_1 + \tau_b, t_1 + \tau_v\}$  such that  $G_b(t) = J_1$  and  $G_v(t) = J_2$  and  $G_b(t) \geq G_b(t_2)$ ,  $G_v(t) \geq G_v(t_2)$  for  $t_1 \leq t \leq t_2$ . It follows from (46) that:

$$G_b(t) \geq C_{bv} \left[ 1 - \frac{2}{M_1} \left( I_1 - \frac{\eta_1}{2} \right) \right] \left[ \eta_b G_v(t_2) \frac{1 - e^{-\mu_b\tau_b}}{\mu_b} + G_v(t_2) \frac{e^{-\mu_b\tau_b}}{K_1} \right], \\ = C_{bv} G_v(t_2) \left[ 1 - \frac{2}{M_1} \left( I_1 - \frac{\eta_1}{2} \right) \right] \left[ \eta_b \frac{1 - e^{-\mu_b\tau_b}}{\mu_b} + \frac{e^{-\mu_b\tau_b}}{K_1} \right]. \quad (51)$$

Similarly, it follows from (46) that

$$G_v(t) \geq \frac{C_{bv}\Pi_v\mu_b}{\Pi_b\mu_v} \left[ 1 - \frac{2}{N_1} \left( I_2 - \frac{\eta_2}{2} \right) \right] \left[ \eta_v G_b(t_2) \frac{1 - e^{-\mu_v\tau_v}}{\mu_v} + G_b(t_2) \frac{e^{-\mu_v\tau_v}}{K_2} \right], \\ = \frac{C_{bv}\Pi_v\mu_b}{\Pi_b\mu_v} G_b(t_2) \left[ 1 - \frac{2}{N_1} \left( I_2 - \frac{\eta_2}{2} \right) \right] \left[ \eta_v \frac{1 - e^{-\mu_v\tau_v}}{\mu_v} + \frac{e^{-\mu_v\tau_v}}{K_2} \right]. \quad (52)$$

Combining (51) and (52) gives

$$G_b(t_2) \geq \mathcal{R}_0^2 G_b(t_2) \left[ 1 - \frac{2}{M_1} \left( I_1 - \frac{\eta_1}{2} \right) \right] \left[ 1 - \frac{2}{N_1} \left( I_2 - \frac{\eta_2}{2} \right) \right], \quad (53)$$

which implies

$$\mathcal{R}_0^2 \left[ 1 - \frac{2}{M_1} \left( I_1 - \frac{\eta_1}{2} \right) \right] \left[ 1 - \frac{2}{N_1} \left( I_2 - \frac{\eta_2}{2} \right) \right] < 1, \quad (54)$$

contradicting (40). This proves the claim that  $G_b(t) \geq J_1$  and  $G_v(t) \geq J_2$  for all  $t \geq t_1$ . We next choose constants  $C_1$  and  $C_2$ , such that

$$1 < C_1 < \mathcal{R}_0 \left[ 1 - \frac{2}{M_1} \left( I_1 - \frac{\eta_1}{2} \right) \right], \\ 1 < C_2 < \mathcal{R}_0 \left[ 1 - \frac{2}{N_1} \left( I_2 - \frac{\eta_2}{2} \right) \right]. \quad (55)$$

We claim now that  $G_b(t) > J_1 C_1$  and  $G_v(t) > J_2 C_2$  for all  $t = \max\{t_1 + \tau_b, t_1 + \tau_v\}$ . It should be noted that

$$G_b(t + \tau_b) \geq C_{bv} \left[ 1 - \frac{2}{M_1} \left( I_1 - \frac{\eta_1}{2} \right) \right] \left[ \eta_b \int_t^{t+\tau_b} G_v(y) e^{-\mu_b(t+\tau_b-y)} dy \right. \\ \left. + e^{-\tau_b\mu_b} \int_{-\infty}^{t+\tau_b} G_v(y - \tau_b) e^{-K_1(t+\tau_b-y)} dy \right] = \left[ 1 - \frac{2}{M_1} \left( I_1 - \frac{\eta_1}{2} \right) \right] J_1 \mathcal{R}_0 > J_1 C_1. \quad (56)$$

Similarly, using (44) and (48), the second equation of (46) for  $t \geq t_1$ , gives

$$G_v(t + \tau_v) \geq \frac{C_{bv}\Pi_v\mu_b}{\Pi_b\mu_v} \left[ 1 - \frac{2}{M_2} \left( I_2 - \frac{\eta_2}{2} \right) \right] \left[ \eta_v \int_t^{t+\tau_v} G_b(y) e^{-\mu_v(t+\tau_v-y)} dy + e^{-\tau_v\mu_v} \int_{-\infty}^{t+\tau_v} G_b(y - \tau_b) e^{-K_2(t+\tau_v-y)} dy \right] = \left[ 1 - \frac{2}{N_1} \left( I_2 - \frac{\eta_2}{2} \right) \right] J_2 \mathcal{R}_0 > J_2 C_2. \quad (57)$$

If the above claim is not valid, then there exists a  $t_3 \geq \max\{t_1 + \tau_b, t_1 + \tau_v\}$  such that  $G_b(t_3) = J_1 C_1$ ,  $G_v(t_3) = J_2 C_2$  and  $G_b(t) \geq J_1 C_1$ ,  $G_v(t) \geq J_2 C_2$  for  $\max\{t_1 + \tau_b, t_1 + \tau_v\} \leq t \leq t_3$ . On the other hand, it follows from (49).

$$G_b(t) \geq C_{bv} \left[ 1 - \frac{2}{M_1} \left( I_1 - \frac{\eta_1}{2} \right) \right] \left[ \eta_b \int_{t-\tau_b}^t J_1 e^{-\mu_b(t-y)} dy + e^{-\tau_b\mu_b} \int_{-\infty}^t J_1 e^{-K_1(t-y)} dy \right], \quad (58)$$

$$= \left[ 1 - \frac{2}{M_1} \left( I_1 - \frac{\eta_1}{2} \right) \right] J_1 \mathcal{R}_0 > J_1 C_1.$$

Similarly, it can be shown from (49) that

$$G_v(t) \geq \frac{C_{bv}\Pi_v\mu_b}{\Pi_b\mu_v} \left[ 1 - \frac{2}{N_1} \left( I_2 - \frac{\eta_2}{2} \right) \right] \left[ \eta_v \int_{t-\tau_v}^t J_2 e^{-\mu_v(t-y)} dy + e^{-\tau_v\mu_v} \int_{-\infty}^t J_2 e^{-K_2(t-y)} dy \right], \quad (59)$$

$$= \left[ 1 - \frac{2}{N_1} \left( I_2 - \frac{\eta_2}{2} \right) \right] J_2 \mathcal{R}_0 > J_2 C_2,$$

for  $\max\{t_1 + \tau_b, t_1 + \tau_v\} \leq t \leq t_3$ . Thus,  $G_b(t_3) > J_1 C_1$ ,  $G_v(t_3) > J_2 C_2$  which leads to a contradiction. This proves the claim. Hence by induction, we conclude that  $G_b(t) > J_1 C_1^n$  and  $G_v(t) > J_2 C_2^n$  for  $t \geq \max\{t_1 + n\tau_b, t_1 + n\tau_v\}$ . It follows that  $S_b(t) \geq I_1 - \eta_1$ ,  $E_b(t) \geq I_1 - \eta_1$ ,  $I_b(t) \geq I_1 - \eta_1$ ,  $S_v(t) \geq I_2 - \eta_2$ ,  $E_v(t) \geq I_2 - \eta_2$  and  $I_v(t) \geq I_2 - \eta_2$ , if  $t$  is sufficiently large, contradicting  $S_b(t) < I_1 - \eta_1$ ,  $E_b(t) < I_1 - \eta_1$ ,  $I_b(t) < I_1 - \eta_1$ ,  $S_v(t) < I_2 - \eta_2$ ,  $E_v(t) < I_2 - \eta_2$  and  $I_v(t) < I_2 - \eta_2$ , for all  $t \geq t_1$ .  $\square$

## Appendix B: Proof of Theorem 5

*Proof.* The proof is based on the approach in [26, 34]. It should be noted from (6) that

$$\limsup_{t \rightarrow \infty} S_b(t) \leq \frac{\Pi_b}{\mu_b}, \limsup_{t \rightarrow \infty} E_b(t) \leq \frac{\Pi_b}{\mu_b}, \limsup_{t \rightarrow \infty} I_b(t) \leq \frac{\Pi_b}{\mu_b}, \quad (60)$$

$$\limsup_{t \rightarrow \infty} S_v(t) \leq \frac{\Pi_v}{\mu_v}, \limsup_{t \rightarrow \infty} E_v(t) \leq \frac{\Pi_v}{\mu_v}, \limsup_{t \rightarrow \infty} I_v(t) \leq \frac{\Pi_v}{\mu_v}.$$

Hence,  $Q_i = \frac{\Pi_b}{\mu_b}$ , ( $i = 1, 2, 3$ ) and  $Q_i = \frac{\Pi_v}{\mu_v}$ , ( $i = 4, 5, 6$ ) as required. Noticed that

$$\lambda_v(t) = \frac{C_{bv}[\eta_b E_b(t) + I_b(t)]}{N_b(t)} \leq C_{bv}(\eta_b + 1), \quad (61)$$

$$\lambda_b(t) = \frac{C_{vb}[\eta_v E_v(t) + I_v(t)]}{N_v(t)} = \frac{C_{bv}[\eta_v E_v(t) + I_v(t)]}{N_b(t)} \leq C_{bv}(\eta_v + 1),$$

so that,

$$\begin{aligned} \frac{dS_b(t)}{dt} &= \Pi_b - \lambda_b(t)S_b(t) - \mu_b(t)S_b(t), \\ &\geq \Pi_b - [C_{bv}(\eta_b + 1) + \mu_b]S_b(t), \\ \frac{dS_v(t)}{dt} &= \Pi_v - \lambda_v(t)S_v(t) - \mu_v(t)S_v(t), \\ &\geq \Pi_v - [C_{bv}(\eta_b + 1) + \mu_v]S_v(t). \end{aligned} \quad (62)$$

Hence,

$$\liminf_{t \rightarrow \infty} S_b(t) \leq \frac{\Pi_b}{C_{bv}(\eta_b + 1) + \mu_b} = q_1 \quad \text{and} \quad \liminf_{t \rightarrow \infty} S_v(t) \leq \frac{\Pi_v}{C_{bv}(\eta_b + 1) + \mu_v} = q_4. \quad (63)$$

Next assume that  $E_b(t^*) = E_b(t^* + q_b) = I_1 - \eta_1$ ,  $I_b(t^*) = I_b(t^* + q_b) = I_1 - \eta_1$ ,  $E_v(t^*) = E_v(t^* + q_v) = I_2 - \eta_2$  and  $I_v(t^*) = I_v(t^* + q_v) = I_2 - \eta_2$  for  $t^* \leq t \leq \min\{t^* + q_b, t^* + q_v\}$ , where  $t^*$  is sufficiently large such that

$$\frac{E_b(t)}{N_b(t)} \leq \frac{1}{M_1} \left( I_1 - \frac{\eta_1}{2} \right), \quad \frac{I_b(t)}{N_b(t)} \leq \frac{1}{M_1} \left( I_1 - \frac{\eta_1}{2} \right), \quad \frac{E_v(t)}{N_v(t)} \leq \frac{1}{M_2} \left( I_2 - \frac{\eta_2}{2} \right), \quad (64)$$

$$\frac{I_v(t)}{N_v(t)} \leq \frac{1}{M_2} \left( I_2 - \frac{\eta_2}{2} \right), \quad \text{for } t^* \leq t \leq \min\{t^* + q_b, t^* + q_v\}.$$

First of all, notice that each of the variables  $E_b(t), I_b(t), E_v(t), I_v(t)$  is uniformly continuous (since their positive solutions are bounded (Theorem 1)). Hence, there is a



$0 < \omega < \min\{\tau_b, \tau_v\}$  (independent of the choice of  $t^*$ ) such that  $E_b(t) \geq \frac{I_1 - \eta_1}{2}$ ,  $I_b(t) \geq \frac{I_1 - \eta_1}{2}$ ,  $E_v(t) \geq \frac{I_2 - \eta_2}{2}$  and  $I_v(t) \geq \frac{I_2 - \eta_2}{2}$  for  $t^* \leq t \leq \min\{t^* + q_b, t^* + q_v\}$ . Consider the case where  $\omega < \min\{q_b, q_v\} \leq \min\{\tau_b, \tau_v\}$ . For  $t^* + \omega \leq t \leq \min\{t^* + q_b, t^* + q_v\}$ , from equation (44) we have

$$\begin{aligned} E_b(t) &\geq C_{bv} \int_{t-\tau_b}^t [\eta_v E_v(y) + I_v(y)] \frac{S_b(y)}{N_b(y)} e^{-\mu_b(t-y)} dy, \\ &\geq C_{bv} \left[ 1 - \frac{2}{M_1} \left( I_1 - \frac{\eta_1}{2} \right) \right] \int_{t^*}^{t^* + \omega} [\eta_v E_v(y) + I_v(y)] e^{-\mu_b(t^* + \omega - y)} dy, \\ &> C_{bv} \left[ 1 - \frac{2}{M_1} \left( I_1 - \frac{\eta_1}{2} \right) \right] \left( \frac{I_1 - \eta_1}{2} \right) (\eta_b + 1) \omega e^{-\mu_b \omega} = q^2. \end{aligned} \quad (65)$$

Define  $q_2 = \min\left\{ \frac{I_1 - \eta_1}{2}, q^2 \right\}$ . It is easy to see that  $E_b(t) \geq q_2$  for  $t^* \leq t \leq \min\{t^* + q_b, t^* + q_v\}$ . Further, if  $\min\{q_b, q_v\} > \max\{\tau_b, \tau_v\}$ , it follows by similar argument as those in the proof of Lemma 3 that  $E_b(t) \geq q_2$  for  $t^* \leq t \leq \min\{t^* + q_b, t^* + q_v\}$ .

In view of Lemma 3, there are two cases to be considered for the positive solution. The first is that  $E_b(t) \geq q_2$  when  $t$  is sufficiently large. The second is that  $E_b(t)$  oscillates about  $I_1 - \eta_1$  when  $t$  is sufficiently large. In the first case,  $\liminf_{t \rightarrow \infty} E_b(t) \geq q_2$ . For the second case, the argument above shows that the minima of  $E_b(t)$  are greater than  $q_2$  for all large  $t$ . Notice that  $q_2$  is independent of the positive solution. Similarly, it can be shown that  $\liminf_{t \rightarrow \infty} E_v(t) \geq q_5$ , with  $q_5 = \min\left\{ \frac{I_2 - \eta_2}{2}, q^5 \right\}$  and

$$q_5 = C_{bv} \left[ 1 - \frac{2}{N_1} \left( I_2 - \frac{\eta_2}{2} \right) \right] \left( \frac{I_2 - \eta_2}{2} \right) (\eta_v + 1) \omega e^{-\mu_v \omega}.$$

Furthermore, it follows from (44) that

$$\begin{aligned} I_b(t) &\geq C_{bv} e^{-\tau_b \mu_b} \int_{-\infty}^t [\eta_v E_v(x - \tau_b) + I_v(x - \tau_b)] \frac{S_b(x - \tau_b)}{N_b(x - \tau_b)} e^{-K_1(t-x)} dx. \\ &C_{bv} e^{-\tau_b \mu_b} \left[ 1 - \frac{2}{M_1} \left( I_1 - \frac{\eta_1}{2} \right) \right] \left( \frac{I_1 - \eta_1}{2} \right) (\eta_b + 1) \omega e^{-K_1 \omega}. \end{aligned}$$

Similarly,

$$\begin{aligned} I_v(t) &\geq C_{bv} e^{-\tau_v \mu_v} \int_{-\infty}^t [\eta_b E_b(x - \tau_v) + I_b(x - \tau_v)] \frac{S_v(x - \tau_v)}{N_b(x - \tau_v)} e^{-K_2(t-x)} dx. \\ &C_{bv} e^{-\tau_v \mu_v} \left[ 1 - \frac{2}{N_1} \left( I_2 - \frac{\eta_2}{2} \right) \right] \left( \frac{I_2 - \eta_2}{2} \right) (\eta_v + 1) \omega e^{-K_2 \omega}. \end{aligned}$$

Hence, using similar argument as above,  $\liminf_{t \rightarrow \infty} I_b(t) \geq q_3$ ,  $\liminf_{t \rightarrow \infty} I_v(t) \geq q_6$ ,

$$q_3 = \min\left\{\frac{I_1 - \eta_1}{2}, q^3\right\}, q_6 = \min\left\{\frac{I_2 - \eta_2}{2}, q^6\right\} \text{ with}$$

$$q^3 = C_{bv}e^{-\tau_b\mu_b}\left[1 - \frac{2}{M_1}\left(I_1 - \frac{\eta_1}{2}\right)\right]\left(\frac{I_1 - \eta_1}{2}\right)(\eta_b + 1)\omega e^{-K_1\omega}$$

and

$$q^6 = C_{bv}e^{-\tau_v\mu_v}\left[1 - \frac{2}{N_1}\left(I_2 - \frac{\eta_2}{2}\right)\right]\left(\frac{I_2 - \eta_2}{2}\right)(\eta_v + 1)\omega e^{-K_2\omega}.$$

□

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Table 2: Description of variables and parameters of the models (1).

Variable	Interpretation	
$S_b$	Susceptible birds	
$E_b$	Exposed birds	
$I_b$	Infectious birds	
$R_b$	Recovered birds	
$S_v$	Susceptible Mosquitoes	
$E_v$	Exposed mosquitoes	
$I_v$	Infectious Mosquitoes	
Parameter	Interpretation	Unit
$\Pi_b$	Recruitment rate of birds	day <sup>-1</sup>
$\Pi_v$	Recruitment rate of mosquitoes	day <sup>-1</sup>
$\frac{1}{\mu_b}$	Average lifespan of bird	days
$\frac{1}{\mu_v}$	Average lifespan of mosquitoes	days
$\rho_{vb}$	Transmission probability from infectious mosquitoes to susceptible birds	day <sup>-1</sup>
$\rho_{bv}$	Transmission probability from infectious birds to susceptible mosquitoes	day <sup>-1</sup>
$\beta_i$	Biting rate of infectious mosquitoes	day <sup>-1</sup>
$\beta_s$	Biting rate of susceptible mosquitoes	day <sup>-1</sup>
$C_{vb}$	Infection rate of birds	day <sup>-1</sup>
$C_{bv}$	Infection rate of mosquitoes	day <sup>-1</sup>
$\gamma_b$	Birds recovery rate	day <sup>-1</sup>
$\tau_b$	Incubation period for birds	days
$\tau_v$	Incubation period for mosquitoes	days
$\eta_b, \eta_v$	Modification parameters	
$\delta_b$	disease induced death rate of birds	
$\delta_v$	disease induced death rate of mosquitoes	

Table 3: Parameter Values

Parameter	nominal value	references
$\Pi_b$	1000	[2]
$\Pi_v$	22000	[2]
$\frac{1}{\mu_b}$	1000	[2]
$\frac{1}{\mu_v}$	14	[2]
$\rho_{vb}$	0.88	[2]
$\rho_{bv}$	0.16	[2]
$\beta_i$	0.09	[2]
$\beta_s$	0.09	[2]
$\delta_b$	$5.0 \times 10^{-5}$	[2]
$\delta_v$	$5.0 \times 10^{-7}$	assumed
$\gamma_b$	0.5	assumed
$\eta_b$	0.5	assumed
$\eta_v$	0.5	assumed
$C_{vb}$	Variable	
$C_{bv}$	Variable	
$\tau_b$	Variable	
$\tau_v$	Variable	

Birds Population

Mosquitoes Population

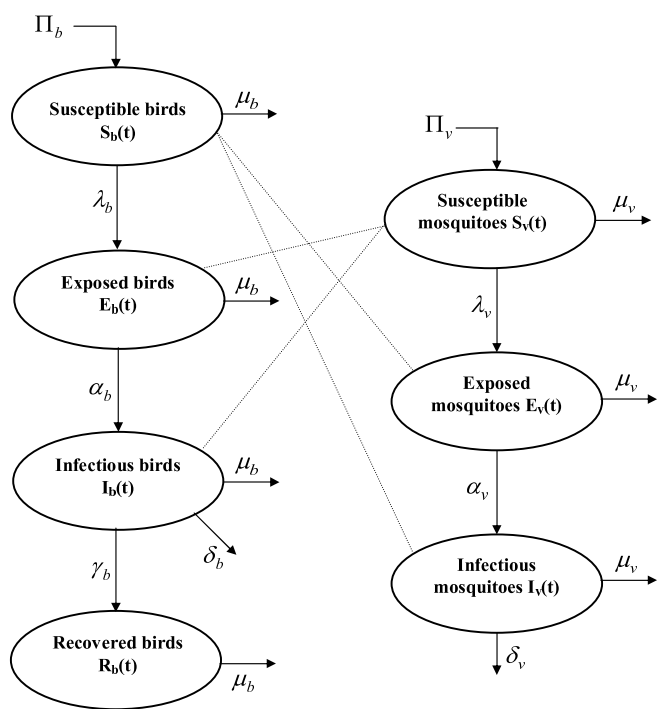


Figure 1: Schematic diagram of the model (1).



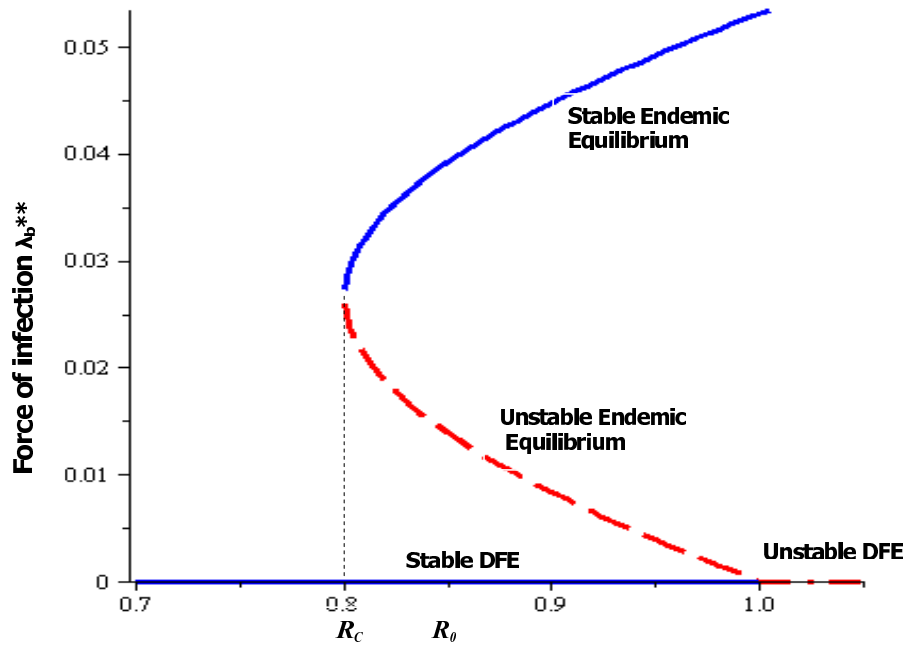


Figure 2: Bifurcation diagram for the force of infection ( $\lambda_b^{**}$ ) of the model (6). Parameter values used are:  $\mu_b = 0.0099$ ,  $\mu_v = 0.0714$ ,  $\delta_b = 0.599$ ,  $\delta_v = 0.0575$ ,  $\Pi_b = 10$ ,  $\Pi_v = 30$ ,  $\eta_b = 0.799$ ,  $\eta_v = 0.799$ ,  $C_{bv} = 0.12$ ,  $\gamma_b = 0.53$ ,  $\tau_b = 15$ ,  $\tau_v = 25$ .

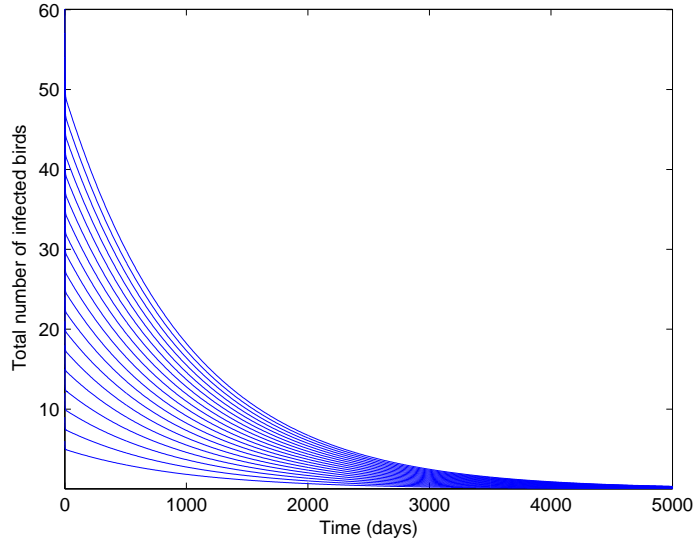


Figure 3: Simulations of the model using ODE45, showing the total number of infected birds as a function of time. Parameter values used are:  $\mu_b = 1/1000$ ,  $\mu_v = 1/14$ ,  $\delta_b = 5 \times 10^{-5}$ ,  $\delta_v = 5 \times 10^{-7}$ ,  $\Pi_b = 1000$ ,  $\Pi_v = 22000$ ,  $\eta_b = 0.5$ ,  $\eta_v = 0.5$ ,  $\beta_s = 0.09$ ,  $\rho_{bv} = 0.16$ ,  $C_{bv} = 0.0144$ ,  $\gamma_b = 0.5$ ,  $\tau_b = 1$ ,  $\tau_v = 1$  (so that  $\mathcal{R}_0^m = 0.1802 < 1$  ( $\tau_b > \tau_b^c$ )).

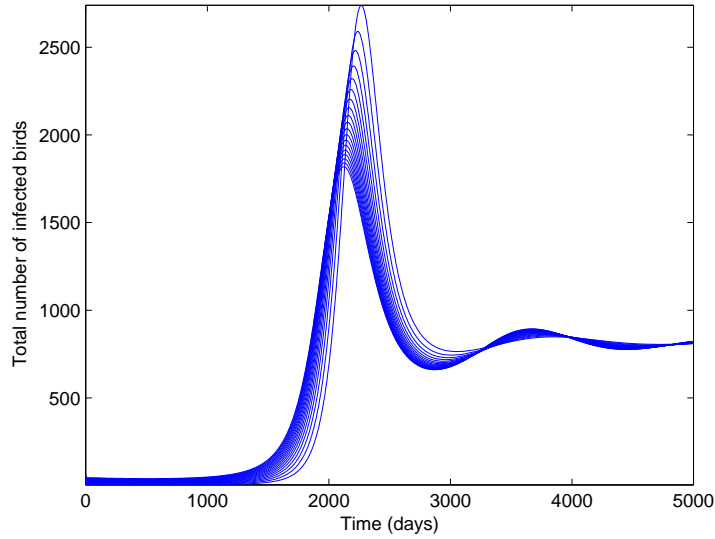


Figure 4: Simulations of the model using ODE45, showing the total number of infected birds as a function of time. Parameter values used are:  $\mu_b = 1/1000$ ,  $\mu_v = 1/14$ ,  $\delta_b = 5 \times 10^{-5}$ ,  $\delta_v = 5 \times 10^{-7}$ ,  $\Pi_b = 1000$ ,  $\Pi_v = 22000$ ,  $\eta_b = 0.5$ ,  $\eta_v = 0.5$ ,  $\beta_s = 0.7$ ,  $\rho_{bv} = 0.16$ ,  $C_{bv} = 0.112$ ,  $\gamma_b = 0.5$ ,  $\tau_b = 1$ ,  $\tau_v = 1$  (so that  $\mathcal{R}_0 = 1.4014 > 1$  ( $\tau_b < \tau_b^c$ )).

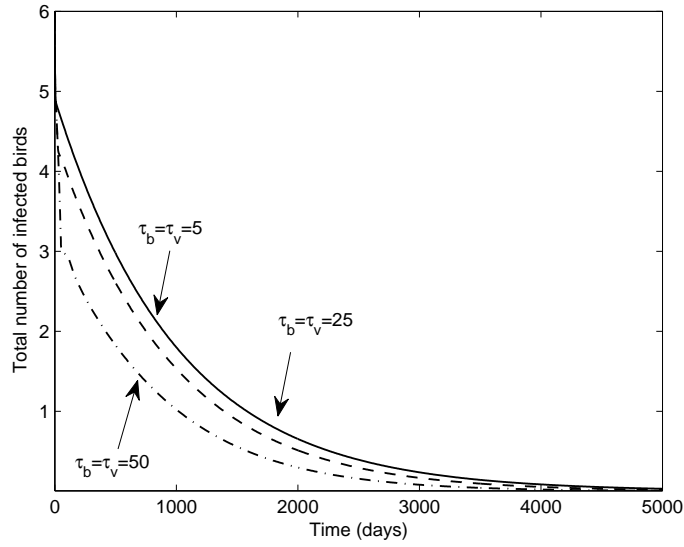


Figure 5: Simulations of the model using ODE45, showing the total number of infected birds as a function of time. Parameter values used are:  $\mu_b = 1/1000$ ,  $\mu_v = 1/14$ ,  $\delta_b = 5 \times 10^{-5}$ ,  $\delta_v = 5 \times 10^{-7}$ ,  $\Pi_b = 1000$ ,  $\Pi_v = 22000$ ,  $\eta_b = 0.5$ ,  $\eta_v = 0.5$ ,  $\beta_s = 0.09$ ,  $\rho_{bv} = 0.16$ ,  $C_{bv} = \beta_s \rho_{bv}$ ,  $\gamma_b = 0.5$ ,  $\mathcal{R}_0 = 0.1802 < 1$  With different values for delay (*i.e.*,  $\tau_b = \tau_v = 5, 25$  and  $50$ ).