The product method for finding bivariate Kummer beta distributions

by
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Declaration

I, Rianne Jacobs, declare that this dissertation, which I hereby submit for the degree Master of Science (Mathematical Statistics) at the University of Pretoria, is my own work and has not previously been submitted by me for a degree at this or any other tertiary institution.

SIGNATURE: Jacobs

DATE: 28 November 2011

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Abstract

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This study systematically derives bivariate Kummer beta distributions for the first time using the product method. Although the definition of the bivariate product distribution is stated in general as the product of any two functions, $f(.)$ and $h(.)$, this study looks at a special case where $f(.)$ and $h(.)$, are taken to be kernels of known distributions. Particularly, this study considers the case where $f(x_1, x_2)$ is taken to be kernels from various bivariate beta distributions and $h(x_1, x_2)$ is taken to be the product of two exponential kernels, i.e. $h(x_1, x_2) = e^{-\psi x_1} e^{-\psi x_2} = e^{-\psi(x_1+x_2)}$. The bivariate beta distributions that are considered include: the bivariate beta type I, bivariate generalized beta type I, bivariate beta type III, bivariate beta type IV, bivariate extended beta type IV and the bivariate beta type V distribution.

The new bivariate product distributions that are constructed in this way are referred to as bivariate Kummer beta distributions. The word *Kummer* originates from the fact that the normalizing constant, K, of the pdf's contain the Kummer function (also referred to as the confluent hypergeometric function) or a related form of it. These new bivariate Kummer beta distributions have the original bivariate beta distribution parameters as well as the parameter, ψ. When ψ is set equal to 0, the bivariate Kummer beta type distributions simplify to the bivariate beta distribution whose kernel was used in its construction.

This study derives the joint, marginal and conditional pdf's of these distributions. The effect of the parameter, ψ, on the correlation between X_1 and X_2, the joint pdf and the marginal pdf is investigated graphically. Finally, two examples of possible applications are provided.

Keywords: beta-binomial, bivariate beta distribution, Kummer function, product method, stress-strength

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Chapter 1

Introduction

1.1 Background and Motivation

Consider the scenario where two observers are gathering information about some variable, X. Each observer gathers information about X independently from the other and obtains some knowledge about the possible values that X can attain. Arguably, the two observers together have obtained more information than each observer individually. These two independent pieces of knowledge can both be expressed as functions of X. If these information sources can be combined into one probability density function (pdf), a distribution can be obtained that contains the information from both observers. This product method of constructing a univariate distribution is defined in the following definition:

Definition 1 *A continuous random variable X is said to have a univariate product distribution if its pdf is given by*

$$g(x) = Kf(x)h(x) \tag{1.1}$$

where $f(.)$ and $h(.)$ are two integrable functions of x and K the normalizing constant ensuring that the pdf integrates to 1. The parameters are such that $g(x)$ is a valid pdf.

Many well-known distributions can be viewed as a product distribution even though not originally so constructed. Consider, for example, the gamma distribution with pdf $g(x) \propto x^{a-1}e^{-bx}$, $x > 0$. Here is an example of a pdf that can be viewed as being constructed from the product of two integrable functions of x, namely $f(x) = x^{a-1}$ and $h(x) = e^{-bx}$.

The next step is to consider a special case of this definition, whereby one or both of the functions, $f(.)$ and $h(.)$, are the *kernel* (see Definition B.1) of some known univariate distribution. In this way, new distributions can be constructed from other known distributions. Considering again the gamma distribution example above, note that $h(x)$ can be viewed as the kernel of an exponential distribution. The exponential distribution is, therefore, often referred to as a special case of the gamma distribution.

As an example of using Definition 1, consider $f(x) = x^{a-1}(1-x)^{b-1}$ and $h(x) = e^{\psi x}$, i.e. $f(x)$ is the kernel of the beta type I distribution and $h(x)$ is the kernel of the exponential distribution. The product distribution is then defined by the pdf

$$g(x|a, b, \psi) \propto x^{a-1}(1-x)^{b-1}e^{\psi x}$$

for $0 \leq x \leq 1, 0 < a, b$ and $-\infty < \psi < \infty$ and is referred to as the Kummer beta type I distribution. This distribution was studied by Ng and Kotz [47]. The study of this distribution was motivated by the structure of the Kummer gamma distribution studied by Armero and Bayarri ([2] and [3]) which forms a conjugate prior of the waiting and inter-arrival times of a queueing system, both of which are exponentially distributed.

Nadarajah and Xu [44] and Nadarajah and Gupta [40] considered an example of the univariate product distribution where both $f(.)$ and $h(.)$ are the kernels of the Pareto distribution. They referred to the new distribution as the product Pareto distribution. Pauw et al. [50] also considered the univariate case, calling it the kernel approach. Pauw et al. [50] took $h(x)$ to be the kernel of a gamma distribution and $f(x)$ the kernel of (i) a beta type II kernel and (ii) a Pearson type I kernel, thereby constructing generalized gamma distributions. See also [38], [43], [39] and [42] for other examples of product distributions.

Using similar arguments as given above, one can extend Definition 1 to the bivariate case. Two variables then each have two independent sources of information:

Definition 2 *The continuous random variables X_1 and X_2 are said to have a bivariate product distribution if their joint pdf is given by*

$$g(x_1, x_2) = Kf(x_1, x_2)h(x_1, x_2) \tag{1.2}$$

where $f(.)$ and $h(.)$ are two integrable functions of x_1 and x_2 and K the normalizing constant ensuring that the pdf integrates to 1. The parameters are such that $g(x_1, x_2)$ is a valid pdf.

As in the case of the univariate product distribution, one can have one or both of the functions, $f(.)$ and $h(.)$, in Definition 2, be a kernel of some known bivariate distribution.

1.2 Objective and Scope

The emphasis of this study is deriving new bivariate product distributions using Definition 2. Both $f(x_1, x_2)$ and $h(x_1, x_2)$ are taken to be kernels of known distributions. The function $f(x_1, x_2)$ is assumed to be the kernel of various known bivariate beta distributions with bounded support space and $h(x_1, x_2)$ is the product of the kernels of two *independent* univariate exponential distributions, namely

$$\begin{aligned} h(x_1, x_2) &= h(x_1)h(x_2) \\ &= e^{-\psi x_1}e^{-\psi x_2} \\ &= e^{-\psi(x_1+x_2)}. \end{aligned} \tag{1.3}$$

Note that it is possible to have an extenstion of (1.3) by letting $\psi_1 \neq \psi_2$, i.e. $h(x_1, x_2) = e^{-\psi_1 x_1}e^{-\psi_2 x_2}$. More is said about this in Chapter 9.

The new bivariate product distributions that are being constructed using this product method are defined as bivariate Kummer beta distributions and the components that are used to obtain these distributions are illustrated in Figure 1.1 below. Note that these new bivariate Kummer beta distributions have the original bivariate beta distribution parameters as well as a new parameter, ψ, coming from the exponential kernels (see (1.3)). As can be seen from Figure 1.1, a systematic approach in building up this group

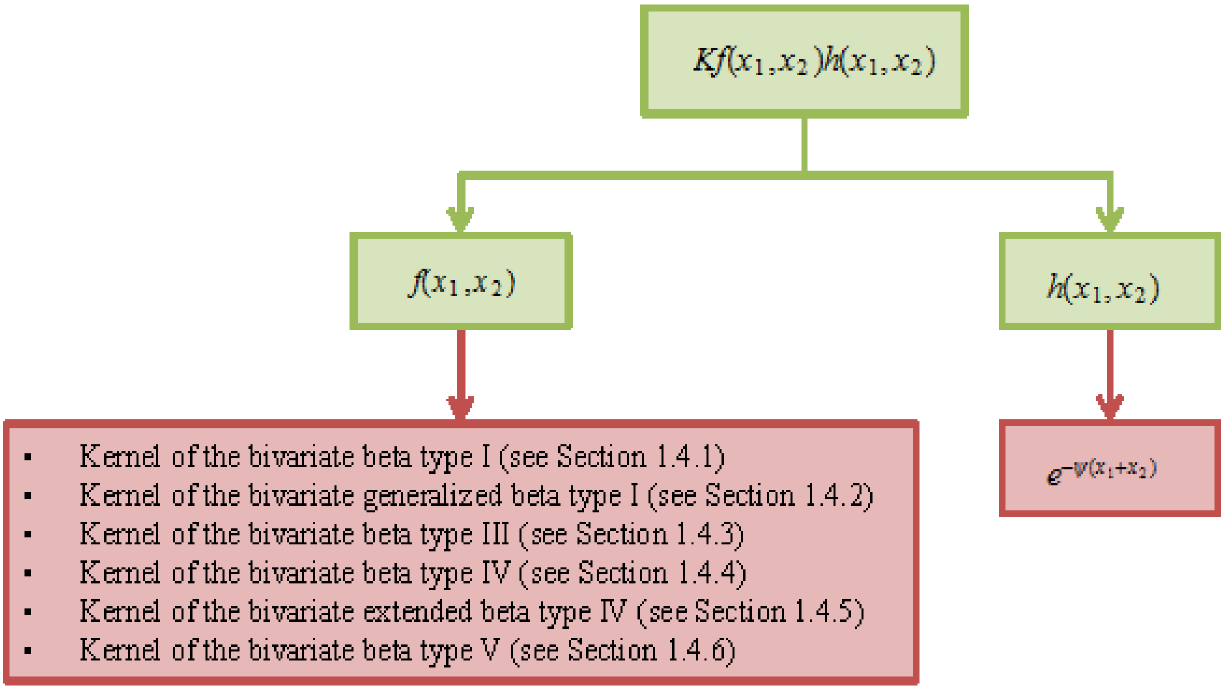


Figure 1.1: Bounded bivariate Kummer beta distributions

of bivariate Kummer beta distributions is followed where it is assumed that $f(x_1, x_2)$

are kernels of bivariate beta distributions with bounded support space and $h(x_1, x_2) = h_1(x_1)h_2(x_2) = e^{-\psi(x_1+x_2)}$. Some properties of these newly proposed distributions are derived and the role of the additional parameter, ψ, is investigated.

This study ends with examples of possible applications of the newly proposed Kummer beta distributions; demonstrating the contribution of the proposed models.

1.3 Naming Conventions for Kummer Beta Distributions

Different authors have developed and adopted different naming conventions. It is, therefore, essential that the naming convention used in this study is clearly defined from the beginning. The name *Kummer beta* consists of two parts, namely *Kummer* and *beta*. Both of these parts are discussed in this section.

The use of the word Kummer is explained by Armero and Bayarri ([2] and [3]), i.e. the Kummer gamma or Kummer beta distributions contain the word Kummer, because the Kummer function (also known as the confluent hypergeometric function) appears in the normalizing constant of the pdf. This is similar to the naming convention of the gamma and the beta distributions, where the gamma function and beta function appear in the normalizing constant of the respective pdf's.

In order to understand why the word *beta* is used, one has to consider some of the research done on the multivariate and matrix variate versions of these Kummer distributions. Gupta et al. [20] studied matrix variate Kummer Dirichlet distributions as an extension of the multivariate Kummer Dirichlet distributions. They called the multivariate case of the Kummer beta distribution, the Kummer *Dirichlet type I* distribution and the multivariate case of the Kummer *gamma* distribution the Kummer *Dirichlet type II*. However, it is possible to call the Kummer *Dirichlet type II* distribution the multivariate Kummer *gamma* distribution. This is exactly what Nagar and Cardeño [45] do, when they refer to the matrix variate Kummer *gamma* distribution. This brings some confusion about which distribution is referred to. In this study, a uniform naming method is used by referring to the various Kummer distributions by the name of the kernel used in $f(x_1, x_2)$ (see Figure 1.1). The function, $f(x_1, x_2)$, is taken to be the building block from which the naming is obtained. Moreover, $h(x_1, x_2)$ contains the new parameter, ψ, and when this parameter is set equal to 0, i.e. $\psi = 0$, the bivariate Kummer beta distribution reduces to the distribution described by $f(x_1, x_2)$. Consequently, in this study the word *beta* is used, since $f(x_1, x_2)$ is assumed to be a bivariate beta kernel.

1.4 Known Bivariate Beta Distributions with bounded support space

This section provides some background to bivariate beta distributions in general and then focusses on the group of bivariate beta distributions with bounded support space. The bivariate beta distributions that are being used in the construction of the bivariate Kummer beta distributions are given and a short description is provided.

The comprehensive work of Balakrishnan and Lai [5] describes many of the methods currently used or available in the literature on the construction of *bivariate distributions*. One such technique is the variables in common or trivariate reduction technique; this method is commonly used in the construction of bivariate beta distributions. This technique creates two dependent random variables from three or more independent random variables. A general definition is given by

$$\begin{aligned} X_1 &= \tau_1(S_i, i = 1, 2, 3...) \\ X_2 &= \tau_2(S_i, i = 1, 2, 3...) \end{aligned}$$

where the functions τ_1 and τ_2, connecting the random variables $S_i, i = 1, 2, 3...$ to the two dependent variables, X_1 and X_2, are generally elementary ones. Some examples include

$$X_1 = S_1 + S_3 \text{ and } X_2 = S_2 + S_3,$$
$$X_1 = \frac{S_1}{S_3} \text{ and } X_2 = \frac{S_2}{S_3},$$
$$X_1 = \frac{S_1}{S_1 + S_3} \text{ and } X_2 = \frac{S_2}{S_2 + S_3}.$$

For other methods of construction of bivariate distributions see Balakrishnan and Lai [5].

The bivariate Kummer beta distributions proposed in this study are constructed using the product method with bivariate beta kernels as the function, $f(x_1, x_2)$, (see 1.2). In this study, the *bivariate beta distributions* are classified according to their domain. The three basic domains on which bivariate beta distributions are defined are as follows: (i) $0 \leq x_1, x_2 \leq 1$, (ii) $0 \leq x_1 + x_2 \leq 1$ and (iii) $x_1, x_2 \geq 0$. These three domains are displayed in Figure 1.2. Although there are many bivariate beta type distributions, we will be focussing on some of the bivariate beta distributions which have a bounded support space on the domain $0 \leq x_1, x_2 \leq 1$ and $0 \leq x_1 + x_2 \leq 1$, i.e. the first two panels in Figure 1.2.

In the following sections a brief overview is given regarding the following bivariate beta distributions that are considered as the building block kernels, $f(x_1, x_2)$, of this study:

- Bivariate beta type I (1.5) - see Section 1.4.1

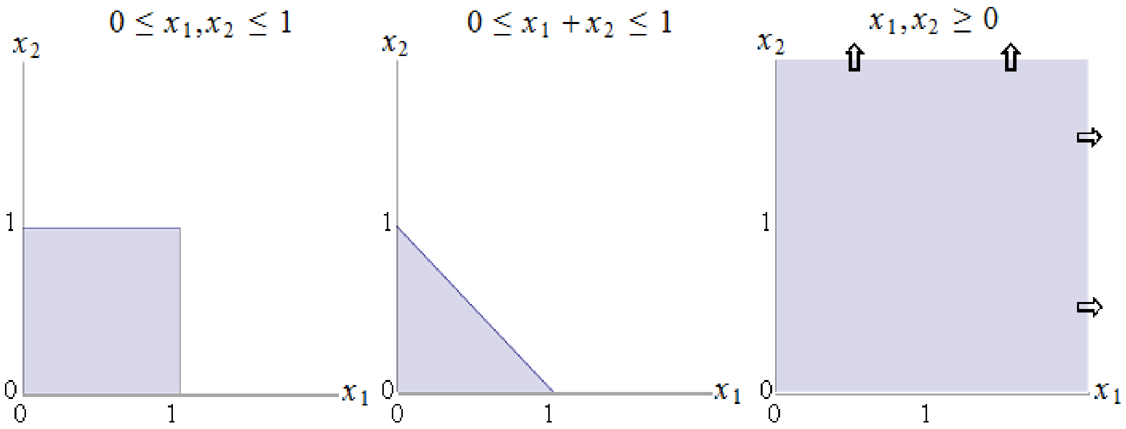


Figure 1.2: Domains for bivariate beta distributions

- Bivariate generalized beta type I (1.7) - see Section 1.4.2
- Bivariate beta type III (1.9) - see Section 1.4.3
- Bivariate beta type IV (1.11) - see Section 1.4.4
- Bivariate extended beta type IV (1.13) - see Section 1.4.5
- Bivariate beta type V (1.15) - see Section 1.4.6

1.4.1 Bivariate Beta Type I

The bivariate beta type I distribution is probably one of the most well-known bivariate beta distributions in the literature. It is often referred to as the bivariate Dirichlet distribution ([5], p374). Its pdf is given by

$$(B(a,b,c))^{-1}\, x_1^{a-1} x_2^{b-1} (1 - x_1 - x_2)^{c-1} \tag{1.4}$$

with kernel

$$f_{BBI}(x_1, x_2) = x_1^{a-1} x_2^{b-1} (1 - x_1 - x_2)^{c-1} \tag{1.5}$$

for $x_1, x_2 \geq 0$, $x_1 + x_2 \leq 1$, a, b, $c > 0$ and where $B(.,.,.)$ is the three-parameter beta function as defined in (B.2) in Definition B.5.

The bivariate beta type I distribution is obtained by means of the trivariate reduction method ([5], p375). If $S_1 \sim Gamma(a,1) \equiv \chi^2(2a)$, $S_2 \sim Gamma(b,1) \equiv \chi^2(2b)$ and $S_3 \sim Gamma(c,1) \equiv \chi^2(2c)$, then $X_1 = \frac{S_1}{S_1+S_2+S_3}$ and $X_2 = \frac{S_2}{S_1+S_2+S_3}$, conditional on $S_1 + S_2 + S_3 \leq 1$, have a bivariate beta type I distribution. The marginal distributions of X_1 and X_2 are $Beta^I(a, b+c)$ and $Beta^I(b, a+c)$, respectively.

1.4.2 Bivariate Generalized Beta Type I

The generalized Dirichlet distribution originates from the concept of "neutrality", which arises naturally in the context of eliminating a single proportion from a set of proportions. Applying this concept of neutrality to the Dirichlet distribution, the generalized Dirichlet distribution [12] is obtained. The pdf of the bivariate case is given by

$$(B(a,d)B(b,c))^{-1}\, x_1^{a-1}x_2^{b-1}(1-x_1)^{d-b-c}(1-x_1-x_2)^{c-1} \tag{1.6}$$

with kernel

$$f_{BGBI}(x_1,x_2) = x_1^{a-1}x_2^{b-1}(1-x_1)^{d-b-c}(1-x_1-x_2)^{c-1} \tag{1.7}$$

for $x_1, x_2 \geq 0$, $x_1 + x_2 \leq 1$ and a, b, c, $d > 0$.

The bivariate generalized beta type I distribution is obtained by means of a simple transformation from univariate beta variates [12]. If $S_1 \sim Beta^I(a,d)$ and $S_2 \sim Beta^I(b,c)$, then $X_1 = S_1$ and $X_2 = S_2(1-S_1)$ have a bivariate generalized beta distribution. The bivariate generalized beta type I distribution simplifies to the bivariate beta type I distribution in (1.4) when $d = b + c$.

1.4.3 Bivariate Beta Type III

The matrix variate beta type III distribution was studied by Ehlers et al. [14]. From this the pdf of the bivariate beta type III distribution is obtained and is given by

$$\left(\beta^{-(a+b)}B(a,b,c)\right)^{-1} x_1^{a-1}x_2^{b-1}(1-x_1-x_2)^{c-1}(1+(\beta-1)x_1+(\beta-1)x_2)^{-(a+b+c)} \tag{1.8}$$

with kernel

$$f_{BBIII}(x_1,x_2) = x_1^{a-1}x_2^{b-1}(1-x_1-x_2)^{c-1}(1+(\beta-1)x_1+(\beta-1)x_2)^{-(a+b+c)} \tag{1.9}$$

for $x_1, x_2 \geq 0$, $x_1 + x_2 \leq 1$ and a, b, c, $\beta > 0$. A special case ($\beta = 2$) of the matrix variate beta type III distribution was studied by Gupta and Nagar [21] and some more of its properties were studied by the same authors [22] more recently. Cardeño et al. [11] studied the univariate beta type III and its multivariate generalization for the special case ($\beta = 2$).

The bivariate beta type III distribution is obtained by means of the trivariate reduction method [14]. If $S_1 \sim Gamma(a,1) \equiv \chi^2(2a)$, $S_2 \sim Gamma(b,1) \equiv \chi^2(2b)$ and $S_3 \sim Gamma(c,1) \equiv \chi^2(2c)$, then $X_1 = \frac{S_1}{S_1+S_2+\beta S_3}$ and $X_2 = \frac{S_2}{S_1+S_2+\beta S_3}$ have a bivariate beta type III distribution. When $\beta = 2$, we obtain the special case of the bivariate beta type III distribution which was studied by Cardeño et al. [11]. For $\beta = 1$, the bivariate beta type III distribution reduces to the bivariate beta type I distribution given in (1.4).

1.4.4 Bivariate Beta Type IV

The bivariate beta type IV distribution was derived independently by Jones [30] and Olkin and Liu [48]. It is also a special case ($\lambda_1 = \lambda_2 = 1$) of the model proposed by Libby and Novick [33]. It is often referred to as the Jones' model or Jones' bivariate beta distribution ([5], p379). The matrix variate beta type IV distribution was studied by Bekker et al. [7] (See also [18] and [23]). The pdf of the bivariate beta type IV distribution is given by

$$(B(a,b,c))^{-1}\, x_1^{a-1} x_2^{b-1} (1-x_1)^{b+c-1} (1-x_2)^{a+c-1} (1-x_1x_2)^{-(a+b+c)} \tag{1.10}$$

with kernel

$$f_{BBIV}(x_1,x_2) = x_1^{a-1} x_2^{b-1} (1-x_1)^{b+c-1} (1-x_2)^{a+c-1} (1-x_1x_2)^{-(a+b+c)} \tag{1.11}$$

for $0 \leq x_1, x_2 \leq 1$ and $a,\ b,\ c > 0$.

The bivariate beta type IV distribution is obtained by means of the trivariate reduction method (see [48]). If $S_1 \sim Gamma(a,1) \equiv \chi^2(2a)$, $S_2 \sim Gamma(b,1) \equiv \chi^2(2b)$ and $S_3 \sim Gamma(c,1) \equiv \chi^2(2c)$, then $X_1 = \frac{S_1}{S_1+S_3}$ and $X_2 = \frac{S_2}{S_2+S_3}$ have a bivariate beta type IV distribution. The marginal distributions of X_1 and X_2 are $Beta^I(a,c)$ and $Beta^I(b,c)$, respectively.

1.4.5 Bivariate Extended Beta Type IV

The bivariate extended beta type IV distribution was proposed by El-Bassiouny and Jones [16]. The matrix variate case was studied by Bekker et al. [6]. The pdf of the bivariate extended beta type IV distribution is given by

$$\begin{aligned}&(B(a,c)B(b,a+c+d))^{-1}\, x_1^{a-1} x_2^{b-1} (1-x_1)^{b+c+d-1} (1-x_2)^{a+c+d-1}\\ &\times (1-x_1x_2)^{-(a+b+c+d)}\, {}_2F_1\left(a+b+c+d, d; a+c+d; \frac{x_1(1-x_2)}{1-x_1x_2}\right)\end{aligned} \tag{1.12}$$

with kernel

$$\begin{aligned}f_{BEBIV}(x_1,x_2) &= x_1^{a-1} x_2^{b-1} (1-x_1)^{b+c+d-1} (1-x_2)^{a+c+d-1} (1-x_1x_2)^{-(a+b+c+d)}\\ &\quad \times {}_2F_1\left(a+b+c+d, d; a+c+d; \frac{x_1(1-x_2)}{1-x_1x_2}\right)\end{aligned} \tag{1.13}$$

for $0 \leq x_1, x_2 \leq 1$, a, b, $c > 0$, $d \geq 0$ and where ${}_2F_1(.)$ denotes the Gauss hypergeometric function defined in (B.6) in Definition B.8.

The bivariate extended beta type IV distribution is obtained by means of the trivariate reduction method [16]. If $S_1 \sim Gamma(a, 1) \equiv \chi^2(2a)$, $S_2 \sim Gamma(b, 1) \equiv \chi^2(2b)$, $S_3 \sim Gamma(c, 1) \equiv \chi^2(2c)$ and $S_4 \sim Gamma(d, 1) \equiv \chi^2(2d)$ then $X_1 = \frac{S_1}{S_1+S_3}$ and $X_2 = \frac{S_2}{S_2+S_3+S_4}$ have a bivariate extended beta type IV distribution. The marginal distributions of X_1 and X_2 are $Beta^I(a, c)$ and $Beta^I(b, c+d)$, respectively. When $d = 0$, the bivariate extended beta type IV distribution reduces to the bivariate beta type IV distribution given in (1.10).

1.4.6 Bivariate Beta Type V

The bivariate beta type V distribution was introduced by Ehlers et al. [15]. They studied both the central and triply non-central cases. The pdf of the bivariate beta type V distribution is given by

$$\left(B(a,b,c)\left(\frac{\beta}{\alpha_1}\right)^{-a}\left(\frac{\beta}{\alpha_2}\right)^{-b}\right)^{-1} x_1^{a-1}x_2^{b-1}(1-x_1-x_2)^{c-1}\left(1+\left(\frac{\beta-\alpha_1}{\alpha_1}\right)x_1+\left(\frac{\beta-\alpha_2}{\alpha_2}\right)x_2\right)^{-(a+b+c)} \tag{1.14}$$

with kernel

$$f_{BBV}(x_1,x_2) = x_1^{a-1}x_2^{b-1}(1-x_1-x_2)^{c-1}\left(1+\left(\frac{\beta-\alpha_1}{\alpha_1}\right)x_1+\left(\frac{\beta-\alpha_2}{\alpha_2}\right)x_2\right)^{-(a+b+c)} \tag{1.15}$$

for $x_1, x_2 \geq 0$, $x_1 + x_2 \leq 1$, a, b, c, β, α_1, $\alpha_2 > 0$.

The bivariate beta type V distribution is obtained by means of the trivariate reduction method [15]. If $S_1 \sim Gamma(a, 1) \equiv \chi^2(2a)$, $S_2 \sim Gamma(b, 1) \equiv \chi^2(2b)$ and $S_3 \sim Gamma(c, 1) \equiv \chi^2(2c)$, then $X_1 = \frac{\alpha_1 S_1}{\alpha_1 S_1+\alpha_2 S_2+\beta S_3}$ and $X_2 = \frac{\alpha_2 S_2}{\alpha_1 S_1+\alpha_2 S_2+\beta S_3}$ have a bivariate beta type V distribution. When $\alpha_1 = \alpha_2 = 1$, the bivariate beta type V distribution reduces to the bivariate beta type III distribution in (1.8) and when $\alpha_1 = \alpha_2 = \beta = 1$, the bivariate beta type V distribution reduces to the bivariate beta type I distribution in (1.4).

1.4.7 The Importance of the Bivariate Beta Distributions

The motivation for using bivariate beta kernels in the construction of bivariate Kummer beta distributions, is the fact that bivariate beta distributions are used in many areas of research. Bivariate beta distributions receive a lot of attention in the literature lately (see for example [4], [24], [37], [41]). Some examples of applications of bivariate beta distributions are given below:

- It is a popular distribution in Bayesian statistics. The bivariate beta type I distribution forms a conjugate prior for the multinomial distribution. (see [9]).
- The bivariate beta distribution is a very flexible distribution, because of its construction. Many different bivariate beta distributions can be constructed by using different chi-squared ratios (as seen in the Sections 1.4.1 through 1.4.6). In this way, a bivariate beta distribution can be constructed that suits the situation or application.
- In the compounding context, the beta distribution is also well-known. Consider for example the beta-binomial distribution. Here a beta distribution is compounded with a binomial distribution. This distribution along with an example of a possible application is discussed in more detail in Chapter 8.
- One example of an application in which the bivariate beta distribution is frequently used is reliability. Consider for example the work by Nadarajah [36] on the stress-strength model in reliability. The ratio of bivariate beta variables are often used to calculate the reliability of systems. This is discussed in more detail in Chapter 8.
- Some other applications where the bivariate beta distribution is used includes utility assessment [33] and drought data [37]. Balakrishnan and Lai [5] also provide many applications for the bivariate beta distribution.

1.5 Dissertation Outline

This section briefly discusses the main structure of this study.

Utilizing Definition 2, new bivariate Kummer beta distributions are proposed in Chapters 2 to 7. The outline of each of the Chapters 2 to 7 is presented in Figure 1.3. The effect of the new parameter, ψ, is investigated by means of graphs.

In each of Chapters 2 to 7, one of the bivariate Kummer beta types is proposed as follows:

- Chapter 2: Bivariate Kummer beta type I with $f(x_1, x_2)$ given by (1.5).
- Chapter 3: Bivariate Kummer generalized beta type I with $f(x_1, x_2)$ given by (1.7).
- Chapter 4: Bivariate Kummer beta type III with $f(x_1, x_2)$ given by (1.9).
- Chapter 5: Bivariate Kummer beta type IV with $f(x_1, x_2)$ given by (1.11).

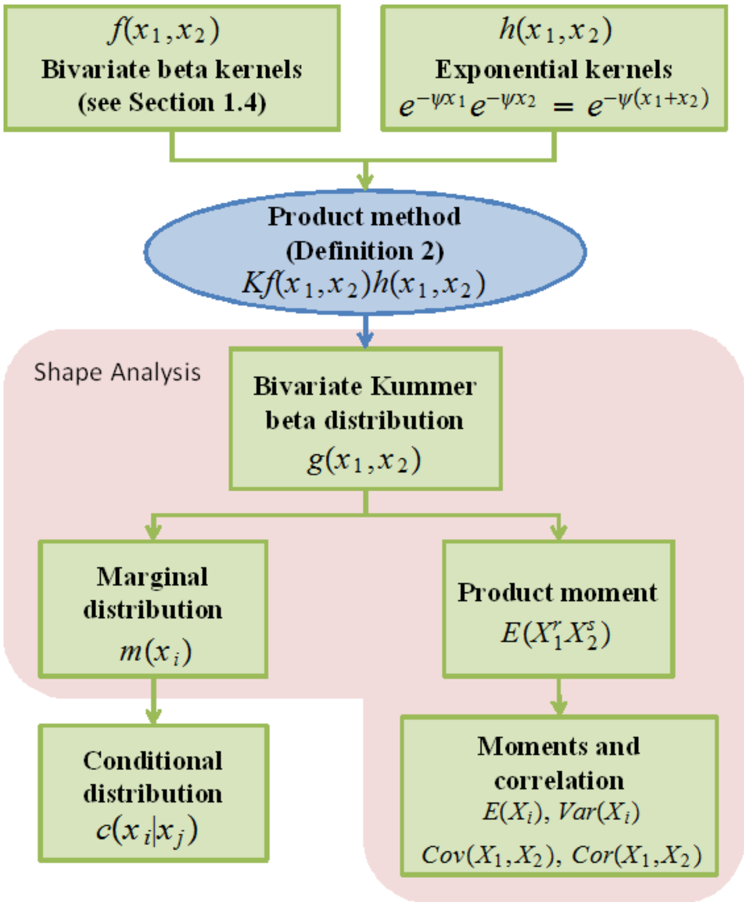


Figure 1.3: Structure of Chapters 2 to 7

- Chapter 6: Bivariate Kummer extended beta type IV with $f(x_1, x_2)$ given by (1.13).
- Chapter 7: Bivariate Kummer beta type V with $f(x_1, x_2)$ given by (1.15).

Chapter 8 concludes by showing two examples of possible applications. Firstly, the stress-strength model in reliability is discussed and applied to the bivariate Kummer beta type IV distribution. Secondly, the beta-binomial distribution is discussed and subsequently the marginal distribution of the bivariate Kummer beta type I is used to construct the Kummer beta-binomial distribution. Both the beta-binomial and the Kummer beta-binomial distributions are fitted to a real dataset and compared.

Chapter 9 gives some conclusive remarks and suggestions for further research.

The following appendices are included:

- Appendix A: Definition of notation used
- Appendix B: Background mathematical results
- Appendix C: Computer programs.

Chapter 2

Bivariate Kummer Beta Type I

In this chapter the product method is used to construct the bivariate Kummer beta type I distribution, utilizing Definition 2 (see (1.2)) with $f(x_1, x_2)$ the bivariate beta type I kernel (see (1.5)). The joint distribution is considered as well as the marginal and conditional distribution functions. An expression is derived for the product moment and expressions for the means, variances, covariance and correlation are provided. Finally, the effect of the parameter, ψ, is investigated by means of a shape analysis; this parameter is introduced via the function $h(x_1, x_2) = e^{-\psi(x_1+x_2)}$ (see Figure 1.1).

2.1 Joint Distribution

In this section, the bivariate Kummer beta type I distribution is derived.

Theorem 2.1 *The pdf of the bivariate Kummer beta type I distribution is given by*

$$g_{BKBI}(x_1, x_2) = Kx_1^{a-1}x_2^{b-1}(1 - x_1 - x_2)^{c-1}e^{-\psi(x_1+x_2)} \tag{2.1}$$

where $0 \leq x_1, x_2,\ x_1 + x_2 \leq 1,\ a, b, c > 0,\ -\infty < \psi < \infty,\ {}_1F_1(.)$ *denotes the confluent hypergeometric function (see Definition B.8) and the normalizing constant,* K*, is given by*

$$K^{-1} = B(a, b, c)_1F_1(a + b; a + b + c; -\psi). \tag{2.2}$$

This distribution is denoted as $(X_1, X_2) \sim BKB^I(a, b, c, \psi)$ *.*

Proof. Definition 2 (see (1.2)) is used to construct the new bivariate Kummer beta type I distribution from the bivariate beta type I kernel (see (1.5)). The pdf of the bivariate

Kummer beta type I distribution is then given by

$$\begin{aligned} g_{BKBI}(x_1,x_2) &= Kf_{BBI}(x_1,x_2)e^{-\psi(x_1+x_2)} \\ &= Kx_1^{a-1}x_2^{b-1}(1-x_1-x_2)^{c-1}e^{-\psi(x_1+x_2)} \qquad \text{(see (1.5))} \end{aligned}$$

with K the normalizing constant. In order to obtain K, the following well-known property of pdf's is used:

$$\begin{aligned} \int_0^1 \int_0^{1-x_2} g_{BKBI}(x_1,x_2)dx_1dx_2 &= \int_0^1 \int_0^{1-x_2} Kx_1^{a-1}x_2^{b-1}(1-x_1-x_2)^{c-1}e^{-\psi(x_1+x_2)}dx_1dx_2 \\ &= 1. \end{aligned}$$

From this the normalizing constant is obtained as:

$$\begin{aligned} K^{-1} &= \int_0^1 \int_0^{1-x_2} x_1^{a-1}x_2^{b-1}(1-x_1-x_2)^{c-1}e^{-\psi(x_1+x_2)}dx_1dx_2 \\ &= \int_0^1 x_2^{b-1}e^{-\psi x_2} \int_0^{1-x_2} x_1^{a-1}e^{-\psi x_1}(1-x_1-x_2)^{c-1}dx_1dx_2. \end{aligned} \tag{2.3}$$

Since by Relation B.3

$$\int_0^{1-x_2} x_1^{a-1}e^{-\psi x_1}(1-x_1-x_2)^{c-1}dx_1 = B(a,c)(1-x_2)^{a+c-1}{}_1F_1(a;a+c;-\psi(1-x_2)),$$

the expression in (2.3) becomes

$$\begin{aligned} K^{-1} &= \int_0^1 x_2^{b-1}e^{-\psi x_2}B(a,c)(1-x_2)^{a+c-1}{}_1F_1(a;a+c;-\psi(1-x_2))dx_2 \\ &= B(a,c)\int_0^1 x_2^{b-1}(1-x_2)^{a+c-1}e^{-\psi x_2}{}_1F_1(a;a+c;-\psi(1-x_2))dx_2. \end{aligned}$$

Using Relation B.4, it follows that

$$\begin{aligned} K^{-1} &= B(a,c)\int_0^1 x_2^{b-1}(1-x_2)^{a+c-1}e^{-\psi x_2}e^{-\psi(1-x_2)}{}_1F_1(c;a+c;\psi(1-x_2))dx_2 \\ &= B(a,c)e^{-\psi}\int_0^1 x_2^{b-1}(1-x_2)^{a+c-1}{}_1F_1(c;a+c;\psi(1-x_2))dx_2. \end{aligned}$$

Writing the function, ${}_1F_1(.)$, out in its series representation (see (B.5)) and using (B.3) the following is obtained:

$$\begin{aligned}
K^{-1} &= B(a,c)e^{-\psi}\int_0^1 x_2^{b-1}(1-x_2)^{a+c-1}\sum_{k=0}^{\infty}\frac{(\psi(1-x_2))^k}{k!}\frac{(c)_k}{(a+c)_k}dx_2 \\
&= B(a,c)e^{-\psi}\sum_{k=0}^{\infty}\frac{\psi^k}{k!}\frac{(c)_k}{(a+c)_k}\int_0^1 x_2^{b-1}(1-x_2)^{a+c+k-1}dx_2 \\
&= B(a,c)e^{-\psi}\sum_{k=0}^{\infty}\frac{\psi^k}{k!}\frac{(c)_k}{(a+c)_k}B(b,a+c+k).
\end{aligned}$$

Using (B.1), (B.2), (B.5) and Relation B.4, the required result (2.2) is obtained:

$$\begin{aligned}
K^{-1} &= \frac{\Gamma(a)\Gamma(c)}{\Gamma(a+c)}e^{-\psi}\sum_{k=0}^{\infty}\frac{\psi^k}{k!}\frac{\Gamma(c+k)\Gamma(a+c)\Gamma(a+c+k)\Gamma(b)}{\Gamma(c)\Gamma(a+c+k)\Gamma(a+b+c+k)} \\
&= \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(a+b+c)}e^{-\psi}\sum_{k=0}^{\infty}\frac{\psi^k}{k!}\frac{\Gamma(c+k)\Gamma(a+b+c)}{\Gamma(c)\Gamma(a+b+c+k)} \\
&= B(a,b,c)e^{-\psi}\sum_{k=0}^{\infty}\frac{\psi^k}{k!}\frac{\Gamma(c+k)\Gamma(a+b+c)}{\Gamma(c)\Gamma(a+b+c+k)} \\
&= B(a,b,c)e^{-\psi}{}_1F_1(c;a+b+c;\psi) \\
&= B(a,b,c)e^{-\psi}e^{\psi}{}_1F_1(a+b;a+b+c;-\psi) \\
&= B(a,b,c){}_1F_1(a+b;a+b+c;-\psi).
\end{aligned}$$

■

Remark 2.1 *Note that the bivariate Kummer beta type I distribution may also be obtained by setting $p = 2$ in the pdf of the multivariate Kummer dirichlet type I distribution defined by Ng and Kotz [47].*

Remark 2.2 *The non-central Kummer beta type I distribution can be obtained by using the non-central beta type I distribution, the latter which was studied by Troskie [55] with pdf given by*

$$(B(a,b,c))^{-1}\, x_1^{a-1}x_2^{b-1}(1-x_1-x_2)^{c-1}e^{-\delta}{}_1F_1(a+b+c;c;\delta(1-x_1-x_2)) \tag{2.4}$$

with kernel

$$f_{NCBBI}(x_1,x_2) = x_1^{a-1}x_2^{b-1}(1-x_1-x_2)^{c-1}{}_1F_1(a+b+c;c;\delta(1-x_1-x_2)) \tag{2.5}$$

for $x_1, x_2 \geq 0$, $x_1 + x_2 \leq 1$, a, b, $c > 0$ *and where* $\delta \geq 0$ *denotes the non-centrality parameter. The pdf of the non-central Kummer beta type I distribution is constructed using Definition 2 with* $f(x_1, x_2)$ *given by (2.5) and is given by*

$$g_{NCBKBI}(x_1, x_2) = K_{NC} x_1^{a-1} x_2^{b-1} (1 - x_1 - x_2)^{c-1} {}_1F_1(a + b + c; c; \delta(1 - x_1 - x_2)) e^{-\psi(x_1+x_2)} \tag{2.6}$$

where $x_1, x_2 \geq 0$, $x_1 + x_2 \leq 1$, $a, b, c > 0$, $-\infty < \psi < \infty$, $\delta \geq 0$ *denotes the non-centrality parameter and the normalizing constant,* K_{NC}*, is given by*

$$\begin{aligned} K_{NC}^{-1} &= B(a, b, c) \sum_{j=0}^{\infty} \frac{\delta^j}{j!} {}_1F_1(a + b; a + b + c + j; -\psi) \\ &= B(a, b, c) \Phi_2(a + b, a + b + c; a + b + c; -\psi, \delta) \end{aligned}$$

where $\Phi_2(.)$ *denotes the confluent hypergeometric series of two variables as defined in Definition B.10. This distribution is denoted as* $(X_1, X_2) \sim NCBKB^I(a, b, c, \psi)$.

2.2 Marginal and Conditional Distributions

In this section the marginal, $m(x_i)$, and the conditional, $c(x_i|x_j)$, pdf's of the bivariate Kummer beta type I distribution are derived.

Theorem 2.2 *If* $(X_1, X_2) \sim BKB^I(a, b, c, \psi)$*, the marginal pdf of* X_1 *is given by*

$$m(x_1) = \frac{x_1^{a-1} (1 - x_1)^{b+c-1} {}_1F_1(c; b + c; \psi(1 - x_1)}{B(a, b + c) {}_1F_1(c; a + b + c; \psi)} \tag{2.7}$$

where $0 \leq x_1 \leq 1$, $a, b, c > 0$ *and* $-\infty < \psi < \infty$.

Proof. Using (2.1), $m(x_1)$ given in (2.7), is obtained by using Relation B.2:

$$\begin{aligned} m(x_1) &= \int_0^{1-x_1} K x_1^{a-1} x_2^{b-1} (1 - x_1 - x_2)^{c-1} e^{-\psi(x_1+x_2)} dx_2 \\ &= K x_1^{a-1} e^{-\psi x_1} \int_0^{1-x_1} x_2^{b-1} (1 - x_1 - x_2)^{c-1} e^{-\psi x_2} dx_2 \\ &= K B(b, c) x_1^{a-1} e^{-\psi x_1} (1 - x_1)^{b+c-1} {}_1F_1(b; b + c; -\psi(1 - x_1)). \end{aligned}$$

The expression is simplified using Relation B.4, (B.1) and (B.2):

$$\begin{aligned} m(x_1) &= \frac{B(b,c)x_1^{a-1}e^{-\psi x_1}(1-x_1)^{b+c-1}{}_1F_1(b;b+c;-\psi(1-x_1)}{B(a,b,c){}_1F_1(a+b;a+b+c;-\psi)} \\ &= \frac{B(b,c)x_1^{a-1}e^{-\psi}(1-x_1)^{b+c-1}{}_1F_1(c;b+c;\psi(1-x_1)}{B(a,b,c)e^{-\psi}{}_1F_1(c;a+b+c;\psi)} \\ &= \frac{x_1^{a-1}(1-x_1)^{b+c-1}{}_1F_1(c;b+c;\psi(1-x_1)}{B(a,b+c){}_1F_1(c;a+b+c;\psi)}. \end{aligned}$$

■

Remark 2.3 *Note that the marginal pdf of X_2 is obtained by substituting x_2 for x_1 in (2.7) and interchanging the parameters a and b.*

Theorem 2.3 *If $(X_1, X_2) \sim BKB^I(a,b,c,\psi)$, the conditional pdf of $X_2|X_1 = x_1$ is given by*

$$c(x_2|x_1) = Dx_2^{b-1}(1-x_1-x_2)^{c-1}e^{-\psi x_2} \tag{2.8}$$

where $0 \leq x_2 \leq 1-x_1$, $a,b,c > 0$, $-\infty < \psi < \infty$ and the normalizing constant D is defined as

$$D^{-1} = B(b,c)(1-x_1)^{b+c-1}{}_1F_1(b;b+c;-\psi(1-x_1)).$$

Proof. Using the joint pdf, $g_{BKBI}(x_1,x_2)$, in (2.1) and the marginal pdf, $m(x_1)$, in (2.7), expression (2.8) for the conditional pdf of $X_2|X_1 = x_1$ follows directly:

$$\begin{aligned} c(x_2|x_1) &= \frac{g_{BKBI}(x_1,x_2)}{m(x_1)} \\ &= \frac{Kx_1^{a-1}x_2^{b-1}(1-x_1-x_2)^{c-1}e^{-\psi(x_1+x_2)}}{Kx_1^{a-1}e^{-\psi x_1}B(b,c)(1-x_1)^{b+c-1}{}_1F_1(b;b+c;-\psi(1-x_1)} \\ &= \frac{x_2^{b-1}(1-x_1-x_2)^{c-1}e^{-\psi x_2}}{B(b,c)(1-x_1)^{b+c-1}{}_1F_1(b;b+c;-\psi(1-x_1)} \\ &= Dx_2^{b-1}(1-x_1-x_2)^{c-1}e^{-\psi x_2}. \end{aligned}$$

■

Remark 2.4 *Note that the conditional pdf of $X_1|X_2 = x_2$ is obtained by interchanging the variables x_1 and x_2 and the parameters a and b in (2.8).*

2.3 Moments and Correlation

In this section the product moment is derived from which expressions for the mean and variance of X_1 and X_2 as well as the covariance and correlation of (X_1, X_2) are obtained.

Furthermore, the effect of the parameter ψ on the correlation between X_1 and X_2 is investigated.

Theorem 2.4 *If* $(X_1, X_2) \sim BKB^I(a,b,c,\psi)$, *the product moment, i.e.* $E(X_1^r X_2^s)$, *equals*

$$\begin{aligned} & KB(a+r,b+s,c)_1F_1(a+b+r+s;a+b+c+r+s;-\psi) \\ = \ & (A(a,b,c,0,0))^{-1} \times A(a,b,c,r,s) \end{aligned} \tag{2.9}$$

where

$$A(a,b,c,r,s) = B(a+r,b+s,c)_1F_1(a+b+r+s;a+b+c+r+s;-\psi) \tag{2.10}$$

with $A(a,b,c,0,0)^{-1} = K$ *as defined in (2.2). Note that the definition of* $A(.)$ *must be read in context as its definition depends on the distribution used.*

Proof. Using the pdf of the bivariate Kummer beta type I distribution in (2.1), the expected value of $X_1^r X_2^s$ is taken:

$$\begin{aligned} E(X_1^r X_2^s) &= \int_0^1 \int_0^{1-x_2} x_1^r x_2^s g_{BKBI}(x_1,x_2) dx_1 dx_2 \\ &= \int_0^1 \int_0^{1-x_2} x_1^r x_2^s K x_1^{a-1} x_2^{b-1} (1-x_1-x_2)^{c-1} e^{-\psi(x_1+x_2)} dx_1 dx_2 \\ &= K \int_0^1 \int_0^{1-x_2} x_1^{a+r-1} x_2^{b+s-1} (1-x_1-x_2)^{c-1} e^{-\psi(x_1+x_2)} dx_1 dx_2. \end{aligned} \tag{2.11}$$

Because the integrals in (2.11) are similar to those found in the proof of Theorem 2.1, the desired expression can be obtained as

$$\begin{aligned} E(X_1^r X_2^s) &= KB(a+r,b+s,c)_1F_1(a+b+r+s;a+b+c+r+s;-\psi) \\ &= (A(a,b,c,0,0))^{-1} \times A(a,b,c,r,s) \end{aligned}$$

with $A(a,b,c,r,s)$ as defined in (2.10). ■

It follows from (2.9):

$$\begin{aligned} E(X_1) &= \frac{a}{a+b+c} \frac{{}_1F_1(a+b+1;a+b+c+1;-\psi)}{{}_1F_1(a+b;a+b+c;-\psi)}, \\ E(X_1^2) &= \frac{(a+1)a}{(a+b+c+1)(a+b+c)} \frac{{}_1F_1(a+b+2;a+b+c+2;-\psi)}{{}_1F_1(a+b;a+b+c;-\psi)}, \end{aligned}$$

$$\begin{aligned}
E(X_2) &= \frac{b}{a+b+c}\frac{{}_1F_1(a+b+1;a+b+c+1;-\psi)}{{}_1F_1(a+b;a+b+c;-\psi)},\\
E(X_2^2) &= \frac{(b+1)b}{(a+b+c+1)(a+b+c)}\frac{{}_1F_1(a+b+2;a+b+c+2;-\psi)}{{}_1F_1(a+b;a+b+c;-\psi)},\\
E(X_1X_2) &= \frac{ab}{(a+b+c+1)(a+b+c)}\frac{{}_1F_1(a+b+2;a+b+c+2;-\psi)}{{}_1F_1(a+b;a+b+c;-\psi)},\\
Var(X_2) &= \frac{(b+1)(b)}{(a+b+c+1)(a+b+c)}\frac{{}_1F_1(a+b+2;a+b+c+2;-\psi)}{{}_1F_1(a+b;a+b+c;-\psi)}\\
&\quad -\left(\frac{b}{a+b+c}\frac{{}_1F_1(a+b+1;a+b+c+1;-\psi)}{{}_1F_1(a+b;a+b+c;-\psi)}\right)^2\\
\text{and} \qquad Cov(X_1,X_2) &= \frac{ab}{{}_1F_1(a+b;a+b+c;-\psi)}\left(\frac{{}_1F_1(a+b+2;a+b+c+2;-\psi)}{(a+b+c+1)(a+b+c)}\right.\\
&\quad \left.-\frac{{}_1F_1(a+b+1;a+b+c+1;-\psi)}{(a+b+c)}\right).
\end{aligned}$$

Figure 2.1 shows the correlation between X_1 and X_2 as ψ varies where $(X_1,X_2)\sim BKB^I(a,b,c,\psi)$. The graph shows the correlation as a function of $\psi\in[-10,20]$ for certain combinations of values for the parameters a, b and c. Note that positive correlation is obtained when $0<c<1$ and some positive values of ψ. The value of ψ from which the correlation becomes positive depends on the values of a, b and c. Note that the correlations for $\psi=0$ are those of the bivariate beta type I distribution which are always negative (see [31], p488). The new parameter, ψ, therefore, brings in positive correlation which is a significant improvement over the bivariate beta type I distribution as the negative correlation of the bivariate beta type I distribution is often deemed a draw-back. For the effect of the parameters a, b and c on the correlation of the bivariate beta type I distribution the reader is referred to Ehlers ([13]).

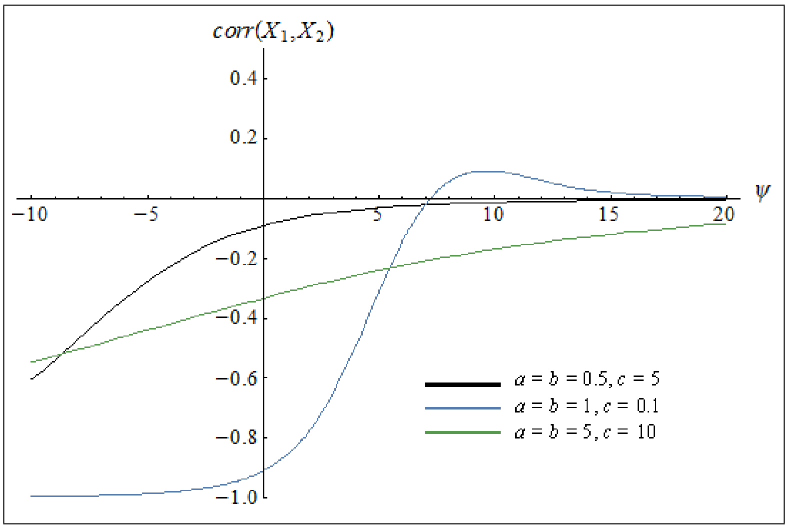


Figure 2.1: Correlation of the bivariate Kummer beta type I distribution as ψ varies

Remark 2.5 *The definition for the bivariate Kummer beta type I distribution to be positive quadrant dependent (PQD) (see [34], p34) is*

$$G(x_1, x_2) \geq M(x_1)M(x_2) \quad \forall x_1, \forall x_2 \tag{2.12}$$

with

$$G(x_1,x_2) = \begin{cases} 0 & \text{for } x_1 < 0, x_2 < 0 \\ \int_0^{1-x_1} \int_0^{x_1} g_{BKBI}(t_1,t_2)dt_1dt_2 & \\ \quad + \int_{1-x_1}^{y} \int_0^{1-t_2} g_{BKBI}(t_1,t_2)dt_1dt_2 & \text{for } 0 \leq x_1 + x_2 \leq 1 \\ \int_0^{x_2} \int_0^{1-t_2} g_{BKBI}(t_1,t_2)dt_1dt_2 & \text{for } 1 - x_2 < x_1 \leq 1, 0 \leq x_2 \leq 1 \\ \int_0^{x_1} \int_0^{1-t_1} g_{BKBI}(t_1,t_2)dt_2dt_1 & \text{for } x_1 > 1, 0 \leq x_2 \leq 1 \\ \int_0^{1} \int_0^{1-t_1} g_{BKBI}(t_1,t_2)dt_1dt_2 & \text{for } x_2 > 1, 0 \leq x_1 \leq 1 \\ 1 & \text{for } x_1 > 1, x_2 > 1 \end{cases}$$

and

$$M(x_1)M(x_2) = \int_0^1 m(t_1)dt_1 \int_0^1 m(t_2)dt_2$$

for $0 \leq x_i \leq 1$, $i = 1, 2$, *where* $g_{BKBI}(t_1, t_2)$, $m(t_i)$, $i = 1, 2$ *are given in (2.1), (2.7) and Remark 2.3. For negative quadrant dependence (NQD), the inequality sign in (2.12) reverses ([34], p34). By substituting various values of the parameters in (2.12) it can be determined whether the bivariate Kummer beta type I distribution has positive or negative correlation since PQD (NQD) implies positive correlation (negative correlation) ([34], p34, 46).*

2.4 Shape Analysis

The effect of the parameters a, b and c was studied by Ehlers ([13]) for the bivariate beta type I distribution. In this study, therefore, only the effect of the new parameter, ψ, is considered.

Figure 2.2 displays the joint pdf and contour plots for the bivariate Kummer beta type I distribution whose pdf is given by (2.1). The effect of ψ is illustrated by setting $\psi = -3$, 0 and 3 and keeping the other parameters fixed at $a = b = c = 2$. Note that ψ pushes the pdf towards the $x_1 + x_2 = 1$ line for negative values of ψ and towards the origin for positive values of ψ. The middle graph (for $\psi = 0$), is the graph of the bivariate beta type I whose pdf is given by (1.4).

Figure 2.3 displays the marginal pdf (see (2.7)) for the bivariate Kummer beta type

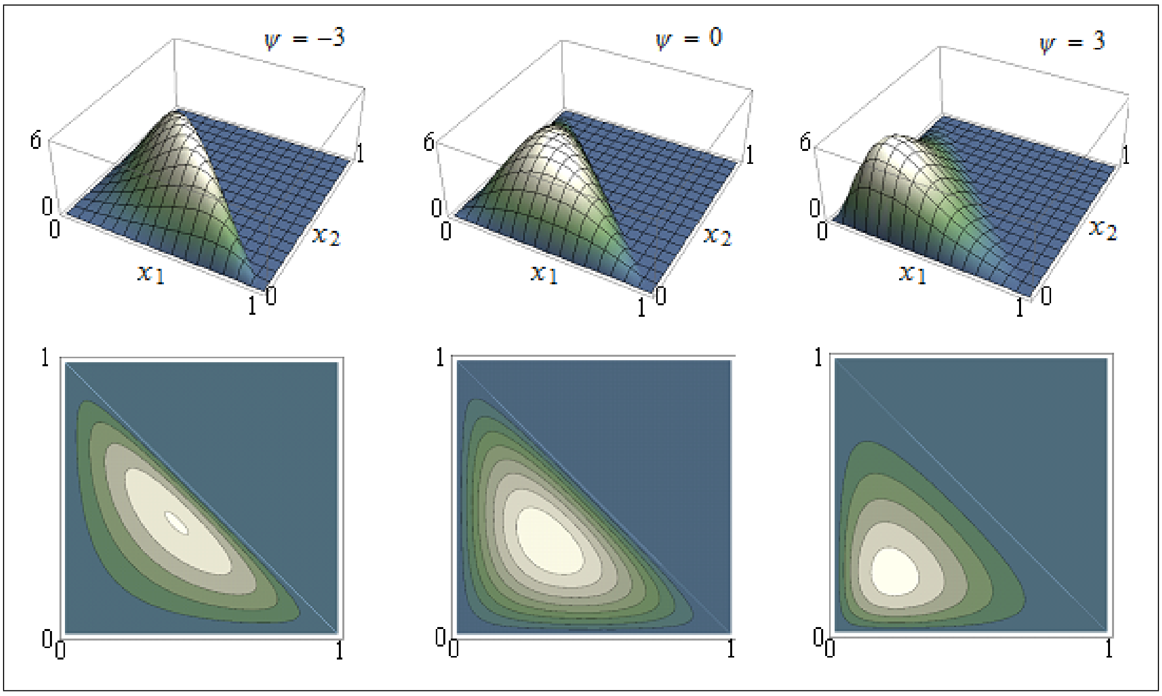


Figure 2.2: Joint pdf of the bivariate Kummer beta type I distribution

I distribution. The four graphs represent the four basic shapes of a univariate beta distribution, namely, symmetric, u-shaped, negatively skewed and positively skewed. For each of the four shapes the effect of ψ is studied. Note that in the cases where $a = 3$,

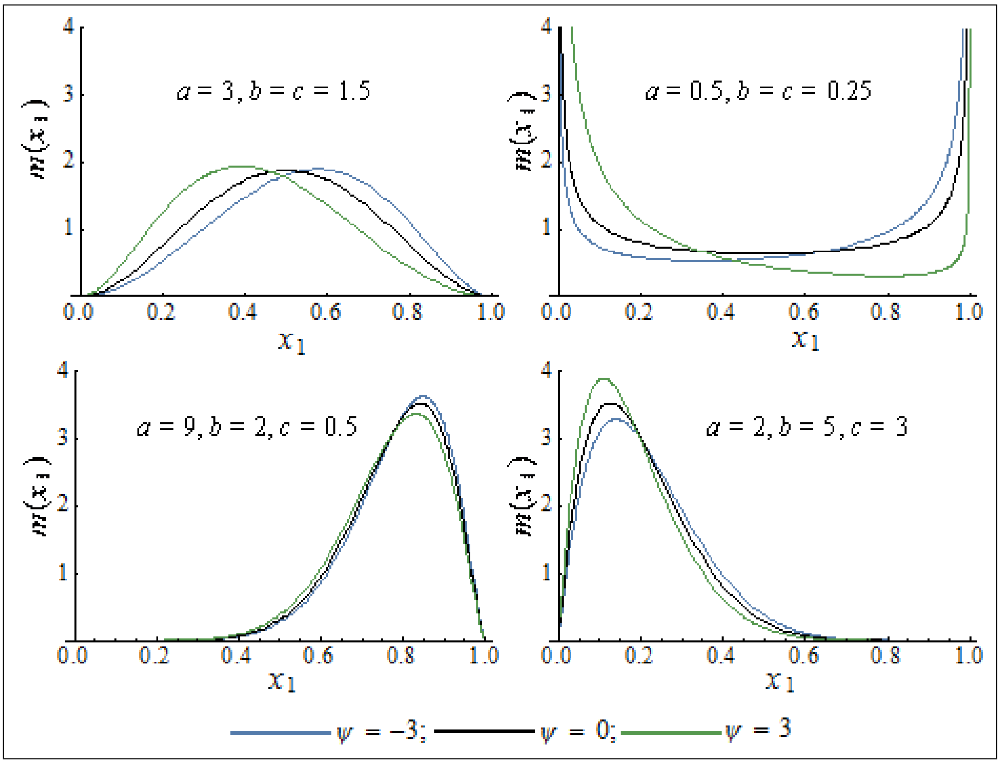


Figure 2.3: Marginal pdf of the bivariate Kummer beta type I distribution

$b = c = 1.5$ and $a = 0.5$, $b = c = 0.25$, ψ changes the skewness of the pdf. In the bottom two graphs, ψ changes the kurtosis of the graphs. For the negatively skewed graph, a negative value of ψ increases the kurtosis while a positive value of ψ decreases the kurtosis. In the positively skewed graph, the exact opposite is seen, with a negative ψ decreasing the kurtosis and a positive ψ increasing the kurtosis.

2.5 Summary

A summary of the newly derived pdf's of Chapter 2 is given in Table 2.1.

Type	**pdf**	**Equation number**
BKB^I	$Kx_1^{a-1}x_2^{b-1}(1-x_1-x_2)^{c-1}e^{-\psi(x_1+x_2)}$	(2.1)
$NCBKB^I$	$K_{NC}x_1^{a-1}x_2^{b-1}(1-x_1-x_2)^{c-1}{}_1F_1(a+b+c;c;\delta(1-x_1-x_2))e^{-\psi(x_1+x_2)}$	(2.6)
$m(x_1)$ of the BKB^I	$m(x_1)=(B(a,b+c){}_1F_1(c;a+b+c;\psi))^{-1}x_1^{a-1}(1-x_1)^{b+c-1}{}_1F_1(c;b+c;\psi(1-x_1)$	(2.7)
$m(x_2)$ of the BKB^I	$m(x_2)=(B(b,a+c){}_1F_1(c;a+b+c;\psi))^{-1}x_2^{b-1}(1-x_2)^{a+c-1}{}_1F_1(c;a+c;\psi(1-x_2)$	(2.7)
$c(x_1\|x_2)$ of the BKB^I	$c(x_1\|x_2)=Dx_1^{a-1}(1-x_1)^{1-a-c}(1-x_1-x_2)^{c-1}e^{-\psi x_1}$	(2.8)
$c(x_2\|x_1)$ of the BKB^I	$c(x_2\|x_1)=Dx_2^{b-1}(1-x_2)^{1-b-c}(1-x_2-x_1)^{c-1}e^{-\psi x_2}$	(2.8)

Table 2.1: Pdf's derived in Chapter 2

Chapter 3

The Bivariate Kummer Generalized Beta Type I

In this chapter the product method is used to construct the bivariate Kummer generalized beta type I distribution, utilizing Definition 2 (see 1.2) with $f(x_1, x_2)$ the bivariate generalized beta type I kernel (see (1.7)). The joint distribution is considered as well as the marginal and conditional distribution functions. An expression is derived for the product moment and expressions for the means, variances, covariance and correlation are provided. Finally, the effect of the parameter, ψ, is investigated by means of a shape analysis; this parameter is introduced via the function $h(x_1, x_2) = e^{-\psi(x_1+x_2)}$ (see Figure 1.1).

3.1 Joint Distribution

In this section, the bivariate Kummer generalized beta type I distribution is derived.

Theorem 3.1 *The pdf of the bivariate Kummer generalized beta type I distribution is given by*

$$g_{BKBIG}(x_1, x_2) = Kx_1^{a-1}x_2^{b-1}(1-x_1)^{d-b-c}(1-x_1-x_2)^{c-1}e^{-\psi(x_1+x_2)} \tag{3.1}$$

where $0 \leq x_1, x_2,\ x_1 + x_2 \leq 1,\ a, b, c, d > 0,\ -\infty < \psi < \infty$ *and the normalizing constant,* K*, is given by*

$$K^{-1} = B(a, d)B(b, c)e^{-\psi}{}_2F_2\left(c, d; b + c, a + d; \psi\right). \tag{3.2}$$

where ${}_2F_2(.)$ *denotes the hypergeometric function (see Definition B.8). This distribution is denoted as* $(X_1, X_2) \sim BKGB^I\left(a, b, c, d, \psi\right)$.

Proof. Definition 2 (see (1.2)) is used to construct the new bivariate Kummer generalized beta type I distribution from the bivariate generalized beta type I kernel (see (1.7)). The pdf of the bivariate Kummer generalized beta type I distribution is then given by

$$\begin{aligned} g_{BKGBI}(x_1, x_2) &= K f_{BGBI}(x_1, x_2) e^{-\psi(x_1+x_2)} \\ &= K x_1^{a-1} x_2^{b-1} (1-x_1)^{d-b-c} (1-x_1-x_2)^{c-1} e^{-\psi(x_1+x_2)} \qquad \text{(see (1.7))} \end{aligned}$$

with K the normalizing constant. In order to obtain K, the following well-known property of pdf's is used:

$$\begin{aligned} & \int_0^1 \int_0^{1-x_2} g_{BKGBI}(x_1, x_2) dx_1 dx_2 \\ = & \int_0^1 \int_0^{1-x_2} K x_1^{a-1} x_2^{b-1} (1-x_1)^{d-b-c} (1-x_1-x_2)^{c-1} e^{-\psi(x_1+x_2)} dx_1 dx_2 \\ = & \ 1. \end{aligned}$$

From this the normalizing constant is obtained as:

$$\begin{aligned} K^{-1} &= \int_0^1 \int_0^{1-x_2} x_1^{a-1} x_2^{b-1} (1-x_1)^{d-b-c} (1-x_1-x_2)^{c-1} e^{-\psi(x_1+x_2)} dx_1 dx_2 \\ &= \int_0^1 x_2^{b-1} e^{-\psi x_2} \int_0^{1-x_2} x_1^{a-1} (1-x_1)^{d-b-c} (1-x_1-x_2)^{c-1} e^{-\psi x_1} dx_1 dx_2. \end{aligned}$$

Use Definition B.7 to obtain

$$K^{-1} = \sum_{k=0}^{\infty} \frac{(-\psi)^k}{k!} \int_0^1 x_2^{b-1} e^{-\psi x_2} \int_0^{1-x_2} x_1^{a+k-1} (1-x_1)^{d-b-c} (1-x_1-x_2)^{c-1} dx_1 dx_2. \qquad (3.3)$$

Since by Relation B.5

$$\begin{aligned} & \int_0^{1-x_2} x_1^{a+k-1} (1-x_1)^{d-b-c} (1-x_1-x_2)^{c-1} dx_1 \\ = & \ (1-x_2)^{a+c+k-1} B(a+k, c) {}_2F_1(a+k, b+c-d; a+c+k; 1-x_2), \end{aligned}$$

the expression in (3.3) becomes

$$\begin{aligned} & K^{-1} \\ = & \sum_{k=0}^{\infty} \frac{(-\psi)^k}{k!} \int_0^1 x_2^{b-1} e^{-\psi x_2} (1-x_2)^{a+c+k-1} B(a+k,c) {}_2F_1(a+k, b+c-d; a+c+k; 1-x_2) dx_2 \\ = & \sum_{k=0}^{\infty} \frac{(-\psi)^k}{k!} B(a+k,c) \int_0^1 x_2^{b-1} (1-x_2)^{a+c+k-1} e^{-\psi x_2} {}_2F_1(a+k, b+c-d; a+c+k; 1-x_2) dx_2. \end{aligned}$$

Using Relation B.13, (B.1), Definition B.4 and Relation B.14, the required result (3.2) is obtained:

$$\begin{aligned} K^{-1} &= \sum_{k=0}^{\infty} \frac{(-\psi)^k}{k!} B(a+k,c) \frac{\Gamma(a+c+k)\Gamma(b)\Gamma(d)}{\Gamma(b+c)\Gamma(a+d+k)} {}_2F_2\left(b,d;b+c,a+d+k;-\psi\right) \\ &= \sum_{k=0}^{\infty} \frac{(-\psi)^k}{k!} \frac{\Gamma(a+k)\Gamma(c)\Gamma(b)\Gamma(d)}{\Gamma(b+c)\Gamma(a+d+k)} {}_2F_2\left(b,d;b+c,a+d+k;-\psi\right) \\ &= \frac{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(d)}{\Gamma(b+c)\Gamma(a+d)} \sum_{k=0}^{\infty} \frac{(-\psi)^k}{k!} \frac{(a)_k}{(a+d)_k} {}_2F_2\left(b,d;b+c,a+d+k;-\psi\right) \\ &= \frac{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(d)}{\Gamma(b+c)\Gamma(a+d)} e^{-\psi} {}_2F_2\left(c,d;b+c,a+d;\psi\right) \\ &= B(a,d)B(b,c)e^{-\psi} {}_2F_2\left(c,d;b+c,a+d;\psi\right). \end{aligned}$$

■

Remark 3.1 *For $d = b+c$, the pdf of the bivariate Kummer generalized beta type I reduces to that of the bivariate Kummer beta type I. This can be shown by setting $d = b+c$ in (3.2) and subsequently in (3.1):*

$$\begin{aligned} K^{-1} &= B(a,b+c)B(b,c)e^{-\psi} {}_2F_2\left(c,b+c;b+c,a+b+c;\psi\right) \\ &= B(a,b+c)B(b,c)e^{-\psi} {}_2F_2\left(c,b+c;b+c,a+b+c;\psi\right) \\ &= \frac{\Gamma(a)\Gamma(b+c)\Gamma(b)\Gamma(c)}{\Gamma(a+b+c)\Gamma(b+c)} e^{-\psi} {}_2F_2\left(c,b+c;b+c,a+b+c;\psi\right) \end{aligned}$$

Using Relation B.4, (B.2) and the fact that by definition ${}_2F_2\left(a,b;b,c;x\right) = {}_1F_1\left(a;c;x\right)$ the normalizing constant in (2.2) is obtained:

$$\begin{aligned} K^{-1} &= \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(a+b+c)} e^{-\psi} {}_1F_1\left(c;a+b+c;\psi\right) \\ &= B(a,b,c)e^{-\psi}e^{\psi} {}_1F_1\left(a+b;a+b+c;-\psi\right) \\ &= B(a,b,c) {}_1F_1\left(a+b;a+b+c;-\psi\right) \end{aligned}$$

The pdf in (3.1) then becomes

$$\begin{aligned} & (B(a,b,c)_1F_1\left(a+b;a+b+c;-\psi\right))^{-1}x_1^{a-1}x_2^{b-1}(1-x_1)^{b+c-b-c}(1-x_1-x_2)^{c-1}e^{-\psi(x_1+x_2)} \\ = & (B(a,b,c)_1F_1\left(a+b;a+b+c;-\psi\right))^{-1}x_1^{a-1}x_2^{b-1}(1-x_1-x_2)^{c-1}e^{-\psi(x_1+x_2)} \end{aligned}$$

which is the pdf of the bivariate Kummer beta type I in (2.1).

3.2 Marginal and Conditional Distributions

In this section the marginal, $m(x_i)$, and the conditional, $c(x_i|x_j)$, pdf's of the bivariate Kummer generalized beta type I distribution are derived.

Theorem 3.2 *If* $(X_1, X_2) \sim BKGB^I\,(a,b,c,d,\psi)$*, the marginal pdf of*

1. X_1 *is given by*

$$m(x_1) = Cx_1^{a-1}(1-x_1)^{d-1}{}_1F_1(c;b+c;\psi(1-x_1)) \tag{3.4}$$

where $0 \leq x_1 \leq 1$, $a,b,c > 0$, $-\infty < \psi < \infty$ *and the normalizing constant,* C*, is given by*

$$C^{-1} = B(a,d)_2F_2\left(c,d;b+c,a+d;\psi\right).$$

2. X_2 *is given by*

$$m(x_2) = K\sum_{k=0}^{\infty}\frac{(-\psi)^k}{k!}B(a+k,c)x_2^{b-1}(1-x_2)^{a+c+k-1}e^{-\psi x_2}{}_2F_1(a+k,b+c-d;a+c+k;1-x_2) \tag{3.5}$$

where $0 \leq x_2 \leq 1$, $a,b,c > 0$, $-\infty < \psi < \infty$ *and* K *as defined in* (3.2)*.*

Proof. The marginal pdf's of X_1 and X_2 are obtained by integrating the pdf of the bivariate Kummer generalized beta type I distributions over X_2 and X_1, respectively.

1. Using (3.1), $m(x_1)$ given in (3.4) is obtained by using Relation B.3:

$$\begin{aligned} m(x_1) &= K\int_0^{1-x_1} x_1^{a-1}x_2^{b-1}(1-x_1)^{d-b-c}(1-x_1-x_2)^{c-1}e^{-\psi(x_1+x_2)}dx_2 \\ &= Kx_1^{a-1}(1-x_1)^{d-b-c}e^{-\psi x_1}\int_0^{1-x_1} x_2^{b-1}(1-x_1-x_2)^{c-1}e^{-\psi x_2}dx_2 \\ &= Kx_1^{a-1}(1-x_1)^{d-b-c}e^{-\psi x_1}B(b,c)(1-x_1)^{b+c-1}{}_1F_1(b;b+c;-\psi(1-x_1)). \end{aligned}$$

The expression is simplified using Relation B.4:

$$\begin{aligned}
m(x_1) &= \frac{B(b,c)x_1^{a-1}(1-x_1)^{d-1}e^{-\psi x_1}{}_1F_1(b;b+c;-\psi(1-x_1))}{B(a,d)B(b,c)e^{-\psi}{}_2F_2\left(c,d;b+c,a+d;\psi\right)} \\
&= \frac{x_1^{a-1}(1-x_1)^{d-1}e^{-\psi x_1}{}_1F_1(b;b+c;-\psi(1-x_1))}{B(a,d)e^{-\psi}{}_2F_2\left(c,d;b+c,a+d;\psi\right)} \\
&= \frac{x_1^{a-1}(1-x_1)^{d-1}e^{-\psi x_1}e^{-\psi(1-x_1)}{}_1F_1(c;b+c;\psi(1-x_1))}{B(a,d)e^{-\psi}{}_2F_2\left(c,d;b+c,a+d;\psi\right)} \\
&= \frac{x_1^{a-1}(1-x_1)^{d-1}{}_1F_1(c;b+c;\psi(1-x_1))}{B(a,d){}_2F_2\left(c,d;b+c,a+d;\psi\right)} \\
&= Cx_1^{a-1}(1-x_1)^{d-1}{}_1F_1(c;b+c;\psi(1-x_1)).
\end{aligned}$$

2. Using (3.1), $m(x_2)$ given in (3.5) is obtained by using Definition B.7 and Relation B.5:

$$\begin{aligned}
m(x_2) &= K\int_0^{1-x_2} x_1^{a-1}x_2^{b-1}(1-x_1)^{d-b-c}(1-x_1-x_2)^{c-1}e^{-\psi(x_1+x_2)}dx_1 \\
&= Kx_2^{b-1}e^{-\psi x_2}\int_0^{1-x_2} x_1^{a-1}(1-x_1)^{d-b-c}(1-x_1-x_2)^{c-1}e^{-\psi x_1}dx_1 \\
&= K\sum_{k=0}^{\infty}\frac{(-\psi)^k}{k!}x_2^{b-1}e^{-\psi x_2}\int_0^{1-x_2} x_1^{a+k-1}(1-x_1)^{d-b-c}(1-x_1-x_2)^{c-1}dx_1 \\
&= K\sum_{k=0}^{\infty}\frac{(-\psi)^k}{k!}B(a+k,c)x_2^{b-1}(1-x_2)^{a+c+k-1}e^{-\psi x_2}{}_2F_1(a+k,b+c-d;a+c+k;1-x_2).
\end{aligned}$$

■

Theorem 3.3 *If $(X_1, X_2) \sim BKGB^I(a,b,c,\psi)$, the conditional pdf of*

1. *$X_2|X_1 = x_1$ is given by*

$$c(x_2|x_1) = D_1x_2^{b-1}\left(1-x_1-x_2\right)^{c-1}e^{-\psi x_2} \tag{3.6}$$

where $0 \leq x_2 \leq 1-x_1$, $b,c>0$, $-\infty<\psi<\infty$ and the normalizing constant D_1 is defined as

$$D_1^{-1} = B\left(b,c\right)\left(1-x_1\right)^{b+c-1}{}_1F_1\left(b;b+c;-\psi(1-x_1)\right).$$

2. *$X_1|X_2 = x_2$ is given by*

$$c(x_1|x_2) = D_2x_1^{a-1}(1-x_1)^{d-b-c}(1-x_1-x_2)^{c-1}e^{-\psi x_1} \tag{3.7}$$

where $0 \leq x_1 \leq 1 - x_2$, $a, b, c, d > 0$, $-\infty < \psi < \infty$ *and the normalizing constant* D_2 *is defined as*

$$D_2^{-1} = \sum_{k=0}^{\infty} \frac{(-\psi)^k}{k!} B(a+k, c)(1-x_2)^{a+c+k-1} {}_2F_1(a+k, b+c-d; a+c+k; (1-x_2)).$$

Proof. 1. Using the joint pdf, $g_{BKGBI}(x_1, x_2)$, in (3.1) and the marginal pdf, $m(x_1)$, in (3.4), expression (3.6) for the conditional pdf of $X_2|X_1 = x_1$ follows directly:

$$\begin{aligned}
c(x_2|x_1) &= \frac{g_{BKGBI}(x_1, x_2)}{m(x_1)} \\
&= \frac{x_1^{a-1} x_2^{b-1} (1-x_1)^{d-b-c} (1-x_1-x_2)^{c-1} e^{-\psi(x_1+x_2)}}{B(a,d)B(b,c)e^{-\psi} {}_2F_2(c, d; b+c, a+d; \psi)} \\
&\quad \div \frac{x_1^{a-1}(1-x_1)^{d-1} {}_1F_1(c; b+c; \psi(1-x_1))}{B(a,d) {}_2F_2(c, d; b+c, a+d; \psi)} \\
&= \frac{x_2^{b-1}(1-x_1-x_2)^{c-1} e^{-\psi(x_1+x_2)}}{B(b,c)e^{-\psi}(1-x_1)^{b+c-1} {}_1F_1(c; b+c; \psi(1-x_1))} \\
&= \frac{x_2^{b-1}(1-x_1-x_2)^{c-1} e^{-\psi(x_1+x_2)}}{B(b,c)e^{-\psi x_1}(1-x_1)^{b+c-1} {}_1F_1(b; b+c; -\psi(1-x_1))} \\
&= \frac{x_2^{b-1}(1-x_1-x_2)^{c-1} e^{-\psi x_2}}{B(b,c)(1-x_1)^{b+c-1} {}_1F_1(b; b+c; -\psi(1-x_1))} \\
&= D_1 x_2^{b-1}(1-x_1-x_2)^{c-1} e^{-\psi x_2}.
\end{aligned}$$

2. Using the joint pdf, $g_{BKGBI}(x_1, x_2)$, in (3.1) and the marginal pdf, $m(x_2)$, in (3.5), expression (3.7) for the conditional pdf of $X_1|X_2 = x_2$ follows directly:

$$\begin{aligned}
c(x_1|x_2) &= \frac{g_{BKGBI}(x_1, x_2)}{m(x_2)} \\
&= \frac{K x_1^{a-1} x_2^{b-1}(1-x_1)^{d-b-c}(1-x_1-x_2)^{c-1} e^{-\psi(x_1+x_2)}}{K \sum_{k=0}^{\infty} \frac{(-\psi)^k}{k!} B(a+k, c) x_2^{b-1}(1-x_2)^{a+c+k-1} e^{-\psi x_2}} \\
&\quad \times \frac{1}{{}_2F_1(a+k, b+c-d; a+c+k; (1-x_2))} \\
&= \frac{x_1^{a-1}(1-x_1)^{d-b-c}(1-x_1-x_2)^{c-1} e^{-\psi x_1}}{\sum_{k=0}^{\infty} \frac{(-\psi)^k}{k!} B(a+k, c)(1-x_2)^{a+c+k-1} {}_2F_1(a+k, b+c-d; a+c+k; (1-x_2))} \\
&= D_2 x_1^{a-1}(1-x_1)^{d-b-c}(1-x_1-x_2)^{c-1} e^{-\psi x_1}.
\end{aligned}$$

■

3.3 Moments and Correlation

In this section the product moment is derived from which expressions for the mean and variance of X_1 and X_2 as well as the covariance and correlation of (X_1, X_2) are obtained. Furthermore, the effect of the parameter ψ on the correlation between X_1 and X_2 is investigated.

Theorem 3.4 *If* $(X_1, X_2) \sim BKGB^I(a, b, c, \psi)$, *the product moment, i.e.* $E(X_1^r X_2^s)$, *equals*

$$\begin{aligned} & KB(a+r, d+s)B(b+s, c)e^{-\psi}{}_2F_2(c, d+s; b+c+s, a+d+r+s; \psi) \\ = \ & (A(a,b,c,d,0,0))^{-1} \times A(a,b,c,d,r,s) \end{aligned} \tag{3.8}$$

where

$$A(a,b,c,d,r,s) = B(a+r, d+s)B(b+s, c)e^{-\psi}{}_2F_2(c, d+s; b+c+s, a+d+r+s; \psi) \tag{3.9}$$

with $A(a,b,c,d,0,0)^{-1} = K$ *as defined in (3.2). Note that the definition of* $A(.)$ *must be read in context as its definition depends on the distribution used.*

Proof. Using the pdf of the bivariate Kummer generalized beta type I distribution in (3.1), the expected value of $X_1^r X_2^s$ is taken:

$$\begin{aligned} & E(X_1^r X_2^s) \\ = & \int_0^1 \int_0^{1-x_2} x_1^r x_2^s g_{BKGBI}(x_1, x_2) dx_1 dx_2 \\ = & \int_0^1 \int_0^{1-x_2} x_1^r x_2^s K x_1^{a-1} x_2^{b-1} (1-x_1)^{d-b-c} (1-x_1-x_2)^{c-1} e^{-\psi(x_1+x_2)} dx_1 dx_2 \\ = & K \int_0^1 \int_0^{1-x_2} x_1^{a+r-1} x_2^{b+s-1} (1-x_1)^{d-b-c} (1-x_1-x_2)^{c-1} e^{-\psi(x_1+x_2)} dx_1 dx_2. \end{aligned} \tag{3.10}$$

Because the integrals in (3.10) are similar to those found in the proof of Theorem 3.1, the desired expression can be obtained as

$$\begin{aligned} E(X_1^r X_2^s) &= KB(a+r, d+s)B(b+s, c)e^{-\psi}{}_2F_2(c, d+s; b+c+s, a+d+r+s; \psi) \\ &= (A(a,b,c,d,0,0))^{-1} \times A(a,b,c,d,r,s) \end{aligned}$$

with $A(a,b,c,r,s)$ as defined in (3.9). ■

It follows from (3.8):

$$\begin{aligned}
E(X_1) &= \frac{a}{a+d}\frac{{}_2F_2\left(c,d;b+c,a+d+1;\psi\right)}{{}_2F_2\left(c,d;b+c,a+d;\psi\right)}, \\
E(X_1^2) &= \frac{(a+1)a}{(a+d+1)(a+d)}\frac{{}_2F_2\left(c,d;b+c,a+d+2;\psi\right)}{{}_2F_2\left(c,d;b+c,a+d;\psi\right)}, \\
E(X_2) &= \frac{bc}{(a+d)(b+c)}\frac{{}_2F_2\left(c,d+1;b+c+1,a+d+1;\psi\right)}{{}_2F_2\left(c,d;b+c,a+d;\psi\right)}, \\
E(X_2^2) &= \frac{(b+1)b(d+1)d}{(a+d+1)(a+c)(b+c+1)(b+c)}\frac{{}_2F_2\left(c,d+2;b+c+2,a+d+2;\psi\right)}{{}_2F_2\left(c,d;b+c,a+d;\psi\right)}, \\
E(X_1X_2) &= \frac{abd}{(a+d+1)(a+d)(b+c)}\frac{{}_2F_2\left(c,d+1;b+c+1,a+d+2;\psi\right)}{{}_2F_2\left(c,d;b+c,a+d;\psi\right)}, \\
Var(X_1) &= \frac{(a+1)a}{(a+d+1)(a+d)}\frac{{}_2F_2\left(c,d;b+c,a+d+2;\psi\right)}{{}_2F_2\left(c,d;b+c,a+d;\psi\right)} \\
&\quad -\left(\frac{a}{a+d}\frac{{}_2F_2\left(c,d;b+c,a+d+1;\psi\right)}{{}_2F_2\left(c,d;b+c,a+d;\psi\right)}\right), \\
Var(X_2) &= \frac{(b+1)b(d+1)d{}_2F_2\left(c,d+2;b+c+2,a+d+2;\psi\right)}{(a+d+1)(a+c)(b+c+1)(b+c){}_2F_2\left(c,d;b+c,a+d;\psi\right)} \\
&\quad -\left(\frac{bc}{(a+d)(b+c)}\frac{{}_2F_2\left(c,d+1;b+c+1,a+d+1;\psi\right)}{{}_2F_2\left(c,d;b+c,a+d;\psi\right)}\right)^2 \\
\text{and }\ Cov(X_1,X_2) &= \frac{abd{}_2F_2\left(c,d+1;b+c+1,a+d+2;\psi\right)}{(a+d+1)(a+d)(b+c){}_2F_2\left(c,d;b+c,a+d;\psi\right)} \\
&\quad -\frac{abc{}_2F_2\left(c,d;b+c,a+d+1;\psi\right){}_2F_2\left(c,d+1;b+c+1,a+d+1;\psi\right)}{(a+d)^2(b+c)\left({}_2F_2\left(c,d;b+c,a+d;\psi\right)\right)^2}.
\end{aligned}$$

Figure 3.1 shows the correlation between X_1 and X_2 as ψ varies where $(X_1, X_2) \sim BKGB^I(a,b,c,d,\psi)$. The graph shows the correlation as a function of $\psi \in [-10, 20]$ for certain combinations of the parameters a, b, c and d. Note that positive correlation is obtained when $0 < c < 1$ and some positive values of ψ. The specific value of ψ for which the correlation becomes positive depends on the values of a, b, c and d. Note that the correlations for $\psi = 0$ are those of the bivariate generalized beta type I distribution which are always negative (see [31], p520). The new parameter, ψ, therefore, brings in positive correlation which is a significant improvement over the negative correlation of the bivariate generalized beta type I distribution. For the effect of the parameters a, b, c and d on the correlation of the bivariate generalized beta distribution the reader is referred to Bodvin [8].

Remark 3.2 *Similar to Remark 2.5, the definition for PQD (see (2.12)) can be used to determine for which parameter values the bivariate Kummer generalized beta type I distribution has positive correlation.*

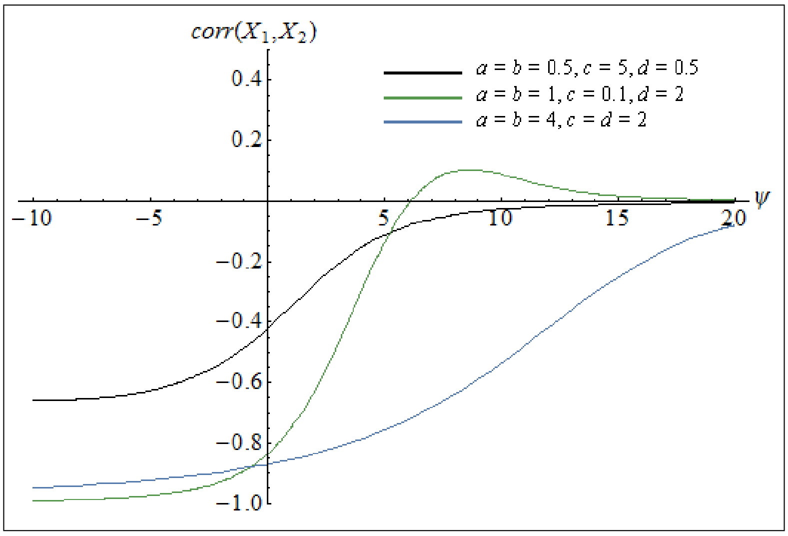


Figure 3.1: Correlation of the bivariate Kummer generalized beta type I distribution as ψ varies

3.4 Shape Analysis

The effect of the parameters a, b, c and d was studied by Bodvin ([8]) for the bivariate generalized beta type I distribution. In this study, therefore, only the effect of the new parameter, ψ, is investigated.

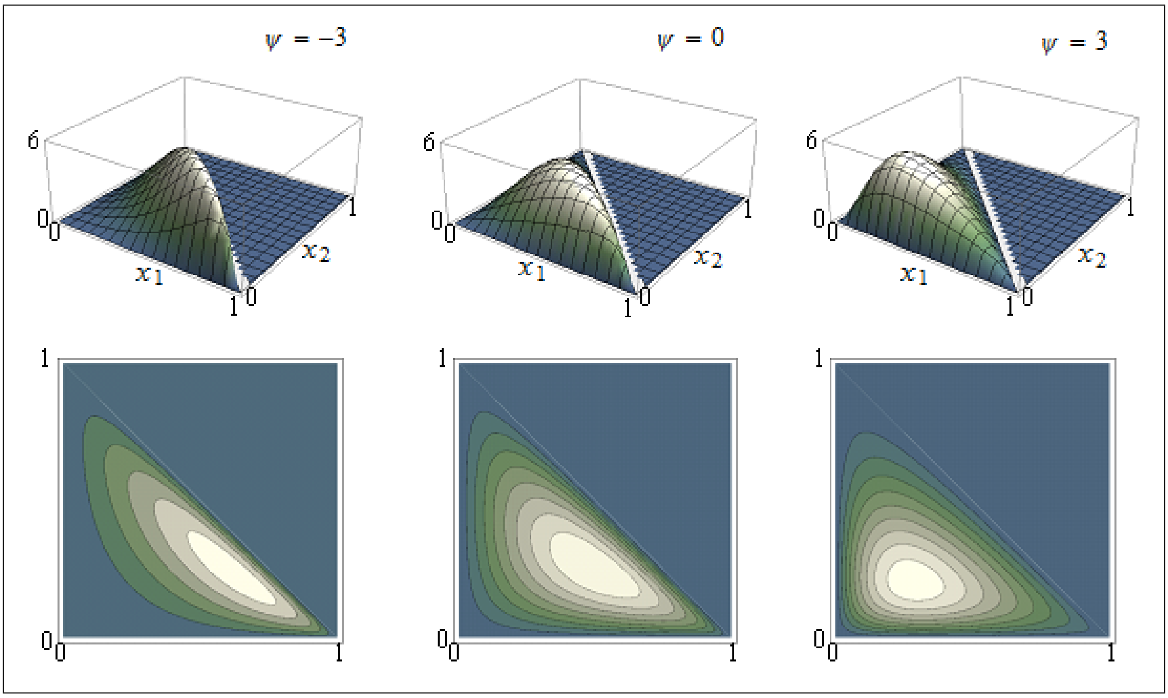


Figure 3.2: Joint pdf of the bivariate Kummer generalized beta type I distribution

In Figure 3.2 the joint pdf and contour plots for the bivariate Kummer generalized beta type I distribution whose pdf is given by (3.1) are displayed. The effect of ψ is illustrated by setting $\psi = -3$, 0 and 3 and keeping the other parameters fixed at $a = 2$, $b = 5$, $c = 3$ and $d = 3$. Note that a negative value of ψ pushes the pdf towards the $x_1 + x_2 = 1$ line, while a positive value of ψ brings the pdf closer to the origin. The middle graph (for $\psi = 0$), is the graph of the bivariate generalized beta type I whose pdf is given by (1.4).

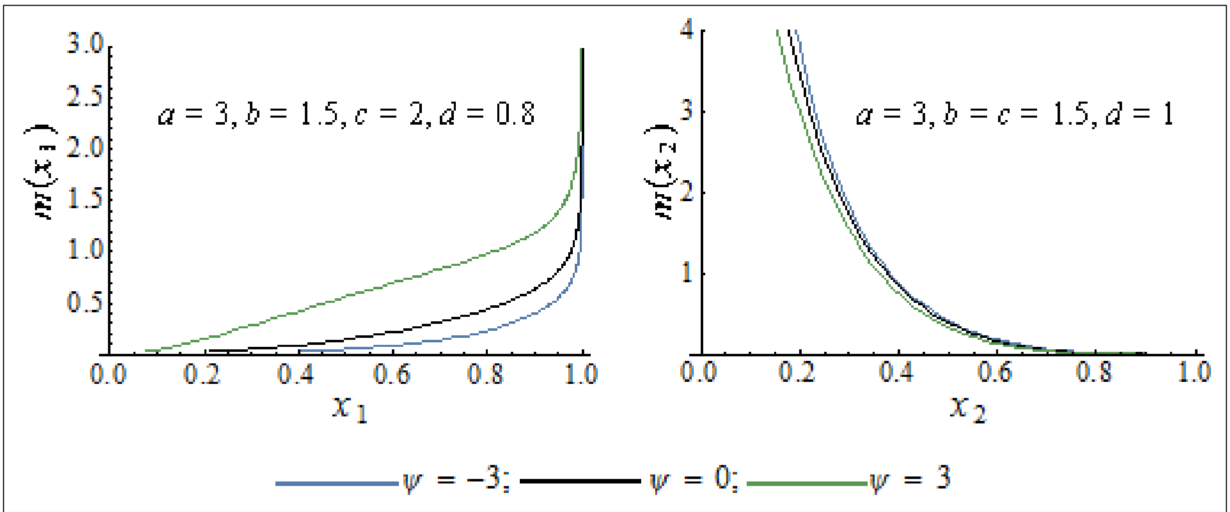


Figure 3.3: Marginal pdf's of the bivariate Kummer generalized beta type I distribution

Figure 3.3 displays the marginal pdf's (see (3.4) and (3.5)) for the bivariate Kummer generalized beta type I distribution. The left and right hand graphs are the marginal pdf's for X_1 and X_2, respectively. For both marginals the effect of ψ is illustrated. It is interesting to note that for the marginal pdf of X_1 a positive value of ψ results in higher values of the pdf and a negative value of ψ results in lower values of the pdf, while for the marginal pdf of X_2 the exact opposite is observed, although not as pronounced.

3.5 Summary

A summary of the newly derived pdf's of Chapter 3 is given in Table 3.1.

Type	**pdf**	**Equation number**
$BKGB^I$	$Kx_1^{a-1}x_2^{b-1}(1-x_1)^{d-b-c}(1-x_1-x_2)^{c-1}e^{-\psi(x_1+x_2)}$	(3.1)
$m(x_1)$ of the $BKGB^I$	$m(x_1) = Cx_1^{a-1}(1-x_1)^{d-1}{}_1F_1(c; b+c; \psi(1-x_1))$	(3.4)
$m(x_2)$ of the $BKGB^I$	$m(x_2) = K\sum_{k=0}^{\infty}\frac{(-\psi)^k}{k!}B(a+k,c)x_2^{b-1}(1-x_2)^{a+c+k-1}e^{-\psi x_2}$	
	$\times {}_2F_1(a+k, b+c-d; a+c+k; (1-x_2))$	(3.5)
$c(x_1\|x_2)$ of the $BKGB^I$	$c(x_1\|x_2) = D_2x_1^{a-1}(1-x_1)^{d-b-c}(1-x_1-x_2)^{c-1}e^{-\psi x_1}$	(3.7)
$c(x_2\|x_1)$ of the $BKGB^I$	$c(x_2\|x_1) = D_1x_2^{b-1}\left(1-x_2-x_1\right)^{c-1}e^{-\psi x_2}$	(3.6)

Table 3.1: Pdf's derived in Chapter 3

Chapter 4

The Bivariate Kummer Beta Type III

In this chapter the product method is used to construct the bivariate Kummer beta type III distribution, utilizing Definition 2 (see (1.2)) with $f(x_1, x_2)$ the bivariate beta type III kernel (see (1.9)). The joint distribution is considered as well as the marginal and conditional distribution functions. An expression is derived for the product moment and expressions for the means, variances, covariance and correlation are provided. Finally, the effect of the parameter, ψ, is investigated by means of a shape analysis; this parameter is introduced via the function $h(x_1, x_2) = e^{-\psi(x_1+x_2)}$ (see Figure 1.1).

4.1 Joint Distribution

In this section, the bivariate Kummer beta type III distribution is derived.

Theorem 4.1 *The pdf of the bivariate Kummer beta type III distribution is given by*

$$g_{BKBIII}(x_1, x_2) = Kx_1^{a-1}x_2^{b-1}(1 - x_1 - x_2)^{c-1}(1 + (\beta - 1)x_1 + (\beta - 1)x_2)^{-(a+b+c)}e^{-\psi(x_1+x_2)} \tag{4.1}$$

where $0 \leq x_1, x_2,\ 0 \leq x_1 + x_2 < 1,\ a, b, c, \beta > 0,\ -\infty < \psi < \infty$ *and the normalizing constant,* K*, is given by*

$$\begin{aligned} K^{-1} &= B(a, b, c) \sum_{k=0}^{\infty} \frac{(-\psi)^k}{k!} \frac{(a)_k}{(a + b + c)_k} \sum_{j=0}^{\infty} \frac{(-\psi)^j}{j!} \frac{(b)_j}{(a + b + c + k)_j} \\ &\quad \times {}_2F_1 \left(a + b + c, a + b + k + j; a + b + c + k + j; -(\beta - 1)\right) \end{aligned} \tag{4.2}$$

where ${}_2F_1(.)$ *denotes the Gauss hypergeometric function (see Definition B.8). This distribution is denoted as* $(X_1, X_2) \sim BKB^{III}(a, b, c, \beta, \psi)$.

Proof. Definition 2 (see (1.2)) is used to construct the new bivariate Kummer beta type III distribution from the bivariate beta type III kernel (see (1.9)). The pdf of the bivariate Kummer beta type III distribution is then given by

$$\begin{aligned} & g_{BKBIII}(x_1, x_2) \\ = \; & K \; f_{BBIII}(x_1, x_2) e^{-\psi(x_1+x_2)} \\ = \; & K x_1^{a-1} x_2^{b-1} (1-x_1-x_2)^{c-1} (1+(\beta-1)x_1+(\beta-1)x_2)^{-(a+b+c)} e^{-\psi(x_1+x_2)} \qquad \text{(see (1.9))} \end{aligned}$$

with K the normalizing constant. In order to obtain K, the following well-known property of pdf's is used:

$$\begin{aligned} & \int_0^1 \int_0^{1-x_2} g_{BKBIII}(x_1, x_2) dx_1 dx_2 \\ = \; & \int_0^1 \int_0^{1-x_2} K x_1^{a-1} x_2^{b-1} (1-x_1-x_2)^{c-1} (1+(\beta-1)x_1+(\beta-1)x_2)^{-(a+b+c)} e^{-\psi(x_1+x_2)} dx_1 dx_2 \\ = \; & 1. \end{aligned}$$

From this the normalizing constant follows as:

$$\begin{aligned} & K^{-1} \\ = \; & \int_0^1 \int_0^{1-x_2} x_1^{a-1} x_2^{b-1} (1-x_1-x_2)^{c-1} (1+(\beta-1)x_1+(\beta-1)x_2)^{-(a+b+c)} e^{-\psi(x_1+x_2)} dx_1 dx_2 \\ = \; & \int_0^1 x_2^{b-1} e^{-\psi x_2} \int_0^{1-x_2} x_1^{a-1} (1-x_1-x_2)^{c-1} (1+(\beta-1)x_1+(\beta-1)x_2)^{-(a+b+c)} e^{-\psi x_1} dx_1 dx_2. \end{aligned}$$

The exponential function is written as an infinite sum using Definition B.7:

$$K^{-1} = \sum_{k=0}^{\infty} \frac{(-\psi)^k}{k!} \int_0^1 x_2^{b-1} e^{-\psi x_2} \int_0^{1-x_2} x_1^{a+k-1} (1-x_1-x_2)^{c-1} (1+(\beta-1)x_1+(\beta-1)x_2)^{-(a+b+c)} dx_1 dx_2. \tag{4.3}$$

Since by Relation B.5

$$\begin{aligned} & \int_0^{1-x_2} x_1^{a+k-1} (1-x_1-x_2)^{c-1} (1+(\beta-1)x_1+(\beta-1)x_2)^{-(a+b+c)} dx_1 \\ = \; & (1-x_2)^{a+c+k-1} (1+(\beta-1)x_2)^{-(a+b+c)} \\ & \times B(a+k, c) {}_2F_1 \left(a+k, a+b+c; a+c+k; \frac{-(\beta-1)(1-x_2)}{1+(\beta-1)x_2} \right), \end{aligned}$$

the expression in (4.3) becomes

$$
\begin{aligned}
K^{-1} &= \sum_{k=0}^{\infty} \frac{(-\psi)^k}{k!} \int_0^1 x_2^{b-1} e^{-\psi x_2} (1-x_2)^{a+c+k-1} (1+(\beta-1)x_2)^{-(a+b+c)} \\
&\quad \times B(a+k,c)_2F_1\left(a+k, a+b+c; a+c+k; \frac{-(\beta-1)(1-x_2)}{1+(\beta-1)x_2}\right) dx_2.
\end{aligned}
$$

Writing out the exponential function and the Gauss hypergeometric function as infinite sums (see Definition B.7 and Definition B.8, respectively):

$$
\begin{aligned}
K^{-1} &= \sum_{k=0}^{\infty} \frac{(-\psi)^k}{k!} B(a+k,c) \sum_{j=0}^{\infty} \frac{(-\psi)^j}{j!} \sum_{m=0}^{\infty} \frac{(-(\beta-1))^m}{m!} \\
&\quad \times \frac{(a+k)_m (a+b+c)_m}{(a+c+k)_m} \int_0^1 x_2^{b+j-1} (1-x_2)^{a+c+k+m-1} (1+(\beta-1)x_2)^{-(a+b+c+m)} dx_2.
\end{aligned}
$$

Using Relation B.5 to write the above integral in terms of the Gauss hypergeometric function, ${}_2F_1(.)$ leads to:

$$
\begin{aligned}
K^{-1} &= \sum_{k=0}^{\infty} \frac{(-\psi)^k}{k!} B(a+k,c) \sum_{j=0}^{\infty} \frac{(-\psi)^j}{j!} \sum_{m=0}^{\infty} \frac{(-(\beta-1))^m}{m!} \frac{(a+k)_m (a+b+c)_m}{(a+c+k)_m} \\
&\quad \times B(b+j, a+c+k+m)_2F_1\left(b+j, a+b+c+m; a+b+c+k+j+m; -(\beta-1)\right).
\end{aligned}
$$

Using Definition B.4 and (B.1) the expression is simplified:

$$
\begin{aligned}
& K^{-1} \\
&= \sum_{k=0}^{\infty} \frac{(-\psi)^k}{k!} \frac{\Gamma(a+k)\Gamma(c)}{\Gamma(a+c+k)} \sum_{j=0}^{\infty} \frac{(-\psi)^j}{j!} \\
&\quad \times \sum_{m=0}^{\infty} \frac{\Gamma(a+k+m)\Gamma(a+b+c+m)\Gamma(a+c+k)\Gamma(b+j)\Gamma(a+c+k+m)}{\Gamma(a+k)\Gamma(a+b+c)\Gamma(a+c+k+m)\Gamma(a+b+c+k+j+m)} \\
&\quad \times \frac{(-(\beta-1))^m}{m!} {}_2F_1\left(b+j, a+b+c+m; a+b+c+k+j+m; -(\beta-1)\right) \\
&= \frac{\Gamma(c)}{\Gamma(a+b+c)} \sum_{k=0}^{\infty} \frac{(-\psi)^k}{k!} \sum_{j=0}^{\infty} \frac{(-\psi)^j}{j!} \Gamma(b+j) \sum_{m=0}^{\infty} \frac{(-(\beta-1))^m}{m!} \frac{\Gamma(a+k+m)\Gamma(a+b+c+m)}{\Gamma(a+b+c+k+j+m)} \\
&\quad \times {}_2F_1\left(b+j, a+b+c+m; a+b+c+k+j+m; -(\beta-1)\right) \\
&= \Gamma(c) \sum_{k=0}^{\infty} \frac{(-\psi)^k}{k!} \Gamma(a+k) \sum_{j=0}^{\infty} \frac{(-\psi)^j}{j!} \frac{\Gamma(b+j)}{\Gamma(a+b+c+k+j)} \sum_{m=0}^{\infty} \frac{(-(\beta-1))^m}{m!} \\
&\quad \times \frac{(a+k)_m (a+b+c)_m}{(a+b+c+k+j)_m} {}_2F_1\left(b+j, a+b+c+m; a+b+c+k+j+m; -(\beta-1)\right).
\end{aligned}
$$

Using Relation B.6, the required result (4.2) is obtained:

$$
\begin{aligned}
K^{-1} &= \Gamma(c)\sum_{k=0}^{\infty}\frac{(-\psi)^k}{k!}\Gamma(a+k)\sum_{j=0}^{\infty}\frac{(-\psi)^j}{j!}\frac{\Gamma(b+j)}{\Gamma(a+b+c+k+j)} \\
&\quad \times {}_2F_1\left(a+b+c, a+b+k+j; a+b+c+k+j; -(\beta-1)\right) \\
&= \frac{\Gamma(c)\Gamma(a)\Gamma(b)}{\Gamma(a+b+c)}\sum_{k=0}^{\infty}\frac{(-\psi)^k}{k!}\frac{(a)_k}{(a+b+c)_k}\sum_{j=0}^{\infty}\frac{(-\psi)^j}{j!}\frac{(b)_j}{(a+b+c+k)_j} \\
&\quad \times {}_2F_1\left(a+b+c, a+b+k+j; a+b+c+k+j; -(\beta-1)\right) \\
&= B(a,b,c)\sum_{k=0}^{\infty}\frac{(-\psi)^k}{k!}\frac{(a)_k}{(a+b+c)_k}\sum_{j=0}^{\infty}\frac{(-\psi)^j}{j!}\frac{(b)_j}{(a+b+c+k)_j} \\
&\quad \times {}_2F_1\left(a+b+c, a+b+k+j; a+b+c+k+j; -(\beta-1)\right).
\end{aligned}
$$

■

Remark 4.1 *For $\beta = 1$, the pdf of the bivariate Kummer beta type III reduces to that of the bivariate Kummer beta type I. This can be shown by setting $\beta = 1$ in (4.2) and subsequently in (4.1):*

$$
\begin{aligned}
K^{-1} &= B(a,b,c)\sum_{k=0}^{\infty}\frac{(-\psi)^k}{k!}\frac{(a)_k}{(a+b+c)_k}\sum_{j=0}^{\infty}\frac{(-\psi)^j}{j!}\frac{(b)_j}{(a+b+c+k)_j} \\
&\quad \times {}_2F_1\left(a+b+c, a+b+k+j; a+b+c+k+j; 0)\right).
\end{aligned}
$$

Use Relation B.12, Definition B.10 and Relation B.8 to obtain the normalizing constant in (2.2):

$$
\begin{aligned}
K^{-1} &= B(a,b,c)\sum_{k=0}^{\infty}\frac{(-\psi)^k}{k!}\frac{(a)_k}{(a+b+c)_k}\sum_{j=0}^{\infty}\frac{(-\psi)^j}{j!}\frac{(b)_j}{(a+b+c+k)_j} \\
&= B(a,b,c)\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-\psi)^k(-\psi)^j}{k!j!}\frac{(a)_k(b)_j}{(a+b+c)_k(a+b+c+k)_j} \\
&= B(a,b,c)\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-\psi)^k(-\psi)^j}{k!j!}\frac{(a)_k(b)_j}{(a+b+c)_{k+j}} \\
&= B(a,b,c)\Phi_2(a,b;a+b+c;-\psi,-\psi) \\
&= B(a,b,c)\,{}_1F_1(a+b;a+b+c;-\psi).
\end{aligned}
$$

For $\beta = 1$, the pdf in (4.1) then becomes

$$
\left(B(a,b,c)\,{}_1F_1(a+b;a+b+c;-\psi)\right)^{-1} x_1^{a-1}x_2^{b-1}(1-x_1-x_2)^{c-1}e^{-\psi(x_1+x_2)}
$$

which is the pdf of the bivariate Kummer beta type I in (2.1).

Remark 4.2 *The non-central Kummer beta type III distribution can be obtained by using the non-central beta type III distribution, the latter which was studied by Ehlers et al. [14] with pdf given by*

$$\begin{aligned}&\left(\beta^{-(a+b)}B(a,b,c)\right)^{-1} x_1^{a-1}x_2^{b-1}(1-x_1-x_2)^{c-1}(1+(\beta-1)x_1+(\beta-1)x_2)^{-(a+b+c)}\\&\times e^{-\delta}{}_1F_1\left(a+b+c;c;\delta\frac{1-x_1-x_2}{1+(\beta-1)x_1+(\beta-1)x_2}\right)\end{aligned} \tag{4.4}$$

with kernel

$$\begin{aligned}f_{NCBBIII}(x_1,x_2) &= x_1^{a-1}x_2^{b-1}(1-x_1-x_2)^{c-1}(1+(\beta-1)x_1+(\beta-1)x_2)^{-(a+b+c)}\\&\quad\times {}_1F_1\left(a+b+c;c;\delta\frac{1-x_1-x_2}{1+(\beta-1)x_1+(\beta-1)x_2}\right)\end{aligned} \tag{4.5}$$

for $x_1,x_2\geq 0$, $x_1+x_2\leq 1$, a, b, c, $\beta>0$ and where $\delta\geq 0$ denotes the non-centrality parameter. The pdf of the non-central Kummer beta type III distribution is constructed using Definition 2 with $f(x_1,x_2)$ given by (4.5) and is given by

$$\begin{aligned}&g_{NCBKBIII}(x_1,x_2)\\&= K_{NC}\sum_{j=0}^{\infty}\frac{\delta^j}{j!}x_1^{a-1}x_2^{b-1}(1-x_1-x_2)^{c-1}(1+(\beta-1)x_1+(\beta-1)x_2)^{-(a+b+c)}e^{-\psi(x_1+x_2)}\end{aligned} \tag{4.6}$$

where $0\leq x_1,x_2$, $x_1+x_2\leq 1$, $a,b,c,\beta>0$, $-\infty<\psi<\infty$, $\delta\geq 0$ denotes the non-centrality parameter and the normalizing constant, K_{NC}, is given by

$$\begin{aligned}&K_{NC}^{-1}\\&= B(a,b,c)\sum_{j=0}^{\infty}\frac{\delta^j}{j!}\sum_{k=0}^{\infty}\frac{(-\psi)^k}{k!}\frac{(a)_k}{(a+b+c+j)_k}\sum_{l=0}^{\infty}\frac{(-\psi)l}{l!}\frac{(b)_l}{(a+b+c+k+j)_l}\\&\quad\times {}_2F_1\left(a+b+c+j,a+b+k+l;a+b+c+k+l+j;-(\beta-1)\right).\end{aligned} \tag{4.7}$$

This distribution is denoted as $(X_1,X_2)\sim NCBKB^{III}(a,b,c,\beta,\psi)$.

4.2 Marginal and Conditional Distributions

In this section the marginal, $m(x_i)$, and the conditional, $c(x_i|x_j)$, pdf's of the bivariate Kummer beta type III distribution are derived.

Theorem 4.2 *If* $(X_1, X_2) \sim BKB^{III}(a, b, c, \beta, \psi)$, *the marginal pdf of* X_1 *is given by*

$$\begin{aligned} m(x_1) &= Kx_1^{a-1}e^{-\psi x_1}(1+(\beta-1)x_1)^{-(a+b+c)}\sum_{j=0}^{\infty}B(b+j,c)\frac{(-\psi)^j}{j!}(1-x_1)^{b+c+j-1} \\ &\quad \times {}_2F_1\left(b+j, a+b+c; b+c+j; \frac{-(\beta-1)(1-x_1)}{1+(\beta-1)x_1}\right) \end{aligned} \tag{4.8}$$

where $0 \leq x_1 \leq 1$, $a, b, c, \beta > 0$, $-\infty < \psi < \infty$ *and* K *as defined in (4.2).*

Proof. Using (4.1), $m(x_1)$ given in (4.8), is obtained by using Definition B.7:

$$\begin{aligned} & m(x_1) \\ &= \int_0^{1-x_1} Kx_1^{a-1}x_2^{b-1}(1-x_1-x_2)^{c-1}(1+(\beta-1)x_1+(\beta-1)x_2)^{-(a+b+c)}e^{-\psi(x_1+x_2)}dx_2 \\ &= Kx_1^{a-1}e^{-\psi x_1}\int_0^{1-x_1} x_2^{b-1}(1-x_1-x_2)^{c-1}(1+(\beta-1)x_1+(\beta-1)x_2)^{-(a+b+c)}e^{-\psi x_2}dx_2 \\ &= Kx_1^{a-1}e^{-\psi x_1}\sum_{j=0}^{\infty}\frac{(-\psi)^j}{j!}\int_0^{1-x_1} x_2^{b+j-1}(1-x_1-x_2)^{c-1}(1+(\beta-1)x_1+(\beta-1)x_2)^{-(a+b+c)}dx_2. \end{aligned}$$

By Relation B.5, the required result is obtained:

$$\begin{aligned} m(x_1) &= Kx_1^{a-1}e^{-\psi x_1}\sum_{j=0}^{\infty}\frac{(-\psi)^j}{j!}(1-x_1)^{b+c+j-1}(1+(\beta-1)x_1)^{-(a+b+c)}B(b+j,c) \\ &\quad \times {}_2F_1\left(b+j, a+b+c; b+c+j; \frac{-(\beta-1)(1-x_1)}{1+(\beta-1)x_1}\right) \\ &= Kx_1^{a-1}e^{-\psi x_1}(1+(\beta-1)x_1)^{-(a+b+c)}\sum_{j=0}^{\infty}B(b+j,c)\frac{(-\psi)^j}{j!}(1-x_1)^{b+c+j-1} \\ &\quad \times {}_2F_1\left(b+j, a+b+c; b+c+j; \frac{-(\beta-1)(1-x_1)}{1+(\beta-1)x_1}\right). \end{aligned}$$

■

Remark 4.3 *Note that the marginal pdf of* X_2 *is obtained by substituting* x_2 *for* x_1 *in (4.8) and interchanging the parameters* a *and* b.

Theorem 4.3 *If* $(X_1, X_2) \sim BKB^{III}(a, b, c, \beta, \psi)$, *the conditional pdf of* $X_2|X_1 = x_1$ *is given by*

$$c(x_2|x_1) = Dx_2^{b-1}(1-x_1-x_2)^{c-1}(1+(\beta-1)x_1+(\beta-1)x_2)^{-(a+b+c)}e^{-\psi x_2} \tag{4.9}$$

where $0 \leq x_2 \leq 1 - x_1,\ a, b, c, \beta > 0,\ -\infty < \psi < \infty$ *and the normalizing constant* D *is defined as*

$$\begin{aligned} D^{-1} &= (1+(\beta-1)x_1)^{-(a+b+c)} \sum_{j=0}^{\infty} B(b+j,c) \frac{(-\psi)^j}{j!} (1-x_1)_2^{b+c+j-1} \\ &\quad \times {}_2F_1\left(b+j, a+b+c; b+c+j; \frac{-(\beta-1)(1-x_1)}{1+(\beta-1)x_1}\right). \end{aligned}$$

Proof. Using the joint pdf, $g_{BKBIII}(x_1, x_2)$, in (4.1) and the marginal pdf, $m(x_1)$, in (4.8), expression (4.9) for the conditional pdf of $X_2|X_1 = x_1$ follows directly:

$$\begin{aligned} c(x_2|x_1) &= \frac{g_{BKBIII}(x_1, x_2)}{m(x_1)} \\ &= \frac{K x_1^{a-1} x_2^{b-1} (1-x_1-x_2)^{c-1} (1+(\beta-1)x_1+(\beta-1)x_2)^{-(a+b+c)} e^{-\psi(x_1+x_2)}}{K x_1^{a-1} e^{-\psi x_1} (1+(\beta-1)x_1)^{-(a+b+c)} \sum_{j=0}^{\infty} B(b+j,c) \frac{(-\psi)^j}{j!} (1-x_1)^{b+c+j-1}} \\ &\quad \times \frac{1}{{}_2F_1\left(b+j, a+b+c; b+c+j; \frac{-(\beta-1)(1-x_1)}{1+(\beta-1)x_1}\right)} \\ &= \frac{x_2^{b-1} (1-x_1-x_2)^{c-1} (1+(\beta-1)x_1+(\beta-1)x_2)^{-(a+b+c)} e^{-\psi x_2}}{(1+(\beta-1)x_1)^{-(a+b+c)} \sum_{j=0}^{\infty} B(b+j,c) \frac{(-\psi)^j}{j!} (1-x_1)^{b+c+j-1}} \\ &\quad \times \frac{1}{{}_2F_1\left(b+j, a+b+c; b+c+j; \frac{-(\beta-1)(1-x_1)}{1+(\beta-1)x_1}\right)} \\ &= D x_2^{b-1} (1-x_1-x_2)^{c-1} (1+(\beta-1)x_1+(\beta-1)x_2)^{-(a+b+c)} e^{-\psi x_2}. \end{aligned}$$

■

Remark 4.4 *Note that the conditional pdf of* $X_1|X_2 = x_2$ *is obtained by interchanging the variables* x_1 *and* x_2 *and the parameters* a *and* b *in (4.9).*

4.3 Moments and Correlation

In this section the product moment is derived from which expressions for the mean and variance of X_1 and X_2 as well as the covariance and correlation of (X_1, X_2) are obtained. Furthermore, the effect of the parameter ψ on the correlation between X_1 and X_2 is investigated.

Theorem 4.4 *If* $(X_1, X_2) \sim BKB^{III}(a, b, c, \beta, \psi)$, *the product moment, i.e.* $E(X_1^r X_2^s)$, *equals*

$$\begin{aligned} & KB(a+r,b+s,c)\sum_{k=0}^{\infty}\frac{(-\psi)^k}{k!}\frac{(a+r)_k}{(a+b+c+r+s)_k}\sum_{j=0}^{\infty}\frac{(-\psi)^j}{j!}\frac{(b+s)_j}{(a+b+c+k+r+s)_j} \\ & \times {}_2F_1\left(a+b+c,a+b+k+j+r+s;a+b+c+k+j+r+s;-(\beta-1)\right) \\ = \; & (A(a,b,c,\beta,0,0))^{-1}\times A(a,b,c,\beta,r,s) \end{aligned} \tag{4.10}$$

where

$$\begin{aligned} & A(a,b,c,\beta,r,s) \\ = \; & B(a+r,b+s,c)\sum_{k=0}^{\infty}\frac{(-\psi)^k}{k!}\frac{(a+r)_k}{(a+b+c+r+s)_k}\sum_{j=0}^{\infty}\frac{(-\psi)^j}{j!}\frac{(b+s)_j}{(a+b+c+k+r+s)_j} \\ & \times {}_2F_1\left(a+b+c,a+b+k+j+r+s;a+b+c+k+j+r+s;-(\beta-1)\right) \end{aligned} \tag{4.11}$$

with $A(a,b,c,\beta,0,0)^{-1}=K$ *as defined in (4.2). Note that the definition of* $A(.)$ *must be read in context as its definition depends on the distribution used.*

Proof. Using the pdf of the bivariate Kummer beta type III distribution in (4.1), the expected value of $X_1^rX_2^s$ is taken:

$$\begin{aligned} & E(X_1^rX_2^s) \\ = \; & \int_0^1\int_0^{1-x_2} x_1^rx_2^sg_{BKBIII}(x_1,x_2)dx_1dx_2 \\ = \; & \int_0^1\int_0^{1-x_2} x_1^rx_2^sKx_1^{a-1}x_2^{b-1}(1-x_1-x_2)^{c-1}(1+(\beta-1)x_1+(\beta-1)x_2)^{-(a+b+c)}e^{-\psi(x_1+x_2)}dx_1dx_2 \\ = \; & K\int_0^1\int_0^{1-x_2} x_1^{a+r-1}x_2^{b+s-1}(1-x_1-x_2)^{c-1}(1+(\beta-1)x_1+(\beta-1)x_2)^{-(a+b+c)}e^{-\psi(x_1+x_2)}dx_1dx_2. \end{aligned} \tag{4.12}$$

Because the integrals in (4.12) are similar to those found in the proof of Theorem 4.1, the desired expression can be obtained as

$$\begin{aligned} & E(X_1^rX_2^s) \\ = \; & KB(a+r,b+s,c)\sum_{k=0}^{\infty}\frac{(-\psi)^k}{k!}\frac{(a+r)_k}{(a+b+c+r+s)_k}\sum_{j=0}^{\infty}\frac{(-\psi)^j}{j!}\frac{(b+s)_j}{(a+b+c+k+r+s)_j} \\ & \times {}_2F_1\left(a+b+c,a+b+k+j+r+s;a+b+c+k+j+r+s;-(\beta-1)\right) \\ = \; & (A(a,b,c,\beta,0,0))^{-1}\times A(a,b,c,\beta,r,s) \end{aligned}$$

with $A(a,b,c,r,s)$ as defined in (4.11). ■

It follows from (4.10):

$$\begin{aligned}
E(X_1) &= (A(a,b,c,\beta,0,0))^{-1} \times A(a,b,c,\beta,1,0),\\
E(X_1^2) &= (A(a,b,c,\beta,0,0))^{-1} \times A(a,b,c,\beta,2,0),\\
E(X_2) &= (A(a,b,c,\beta,0,0))^{-1} \times A(a,b,c,\beta,0,1),\\
E(X_2^2) &= (A(a,b,c,\beta,0,0))^{-1} \times A(a,b,c,\beta,0,2),\\
E(X_1X_2) &= (A(a,b,c,\beta,0,0))^{-1} \times A(a,b,c,\beta,1,1),\\
Var(X_1) &= E(X_1^2) - (E(X_1))^2,\\
Var(X_2) &= E(X_2^2) - (E(X_2))^2\\
\text{and} \qquad Cov(X_1,X_2) &= E(X_1X_2) - E(X_1)E(X_2).
\end{aligned}$$

Figure 4.1 shows the correlation between X_1 and X_2 as ψ varies where $(X_1,X_2) \sim BKB^{III}(a,b,c,\beta,\psi)$. The graph shows the correlation as a function of $\psi \in [-10,10]$ for certain combinations of the parameters a, b and c. Note that positive correlation is obtained when $0 < c < 1$ and some positive values of ψ. The value of ψ from which the correlation becomes positive depends on the values of a, b, c and β. Note that the correlations for $\psi = 0$ are those of the bivariate beta type III distribution. For the effect of the parameters a, b, c and β for the bivariate beta type III the reader is referred to Ehlers ([13]). More specifically, Ehlers et al. [14] and Bodvin et al. [9] discussed the role of β in the bivariate beta type III model. One can observe here that ψ also has the effect of positive correlation between the dependent components X_1 and X_2 for this model (4.1).

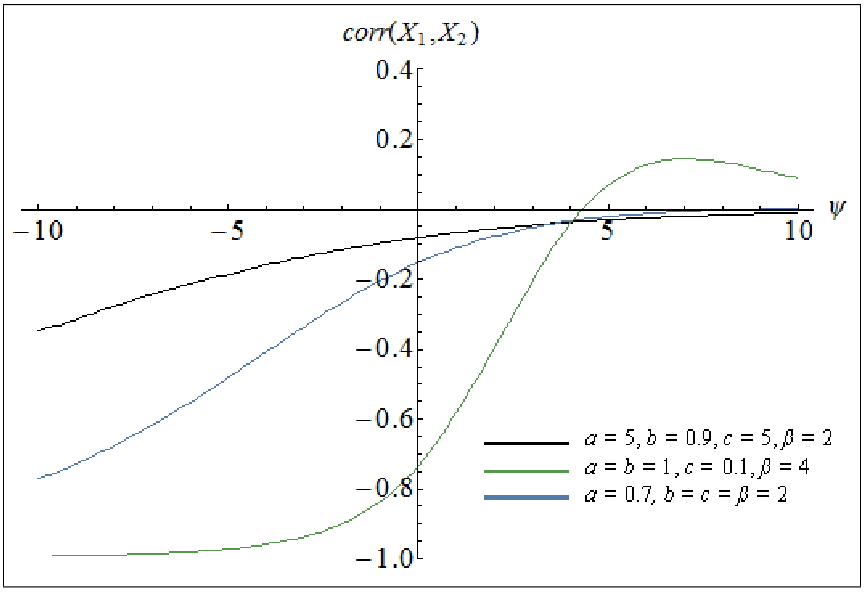


Figure 4.1: Correlation of the bivariate Kummer beta type III distribution as ψ varies

Remark 4.5 *Similar to Remark 2.5, the definition for PQD (see (2.12)) can be used to determine for which parameter values the bivariate Kummer beta type III has positive correlation.*

4.4 Shape Analysis

The effect of the parameters a, b, c and β on the bivariate beta type III distribution was studied by Ehlers ([13]) in detail, therefore, in this study, only the effect of the new parameter, ψ, is considered.

In Figure 4.2 the joint pdf and contour plots for the bivariate Kummer beta type III distribution whose pdf is given by (4.1) are shown. The effect of ψ is illustrated by setting $\psi = -1.1$, 0 and 1.1 and keeping the other parameters fixed at $a = b = c = \beta = 2$. Note that a negative value of ψ produces a slightly flatter pdf while a positive value of ψ gives a higher graph. It is clearly seen in the contour plots that the pdf is pushed towards the origin as ψ increases from being negative to becoming positive. Note that the middle graph (for $\psi = 0$), is the graph of the bivariate beta type III whose pdf is given by (1.8).

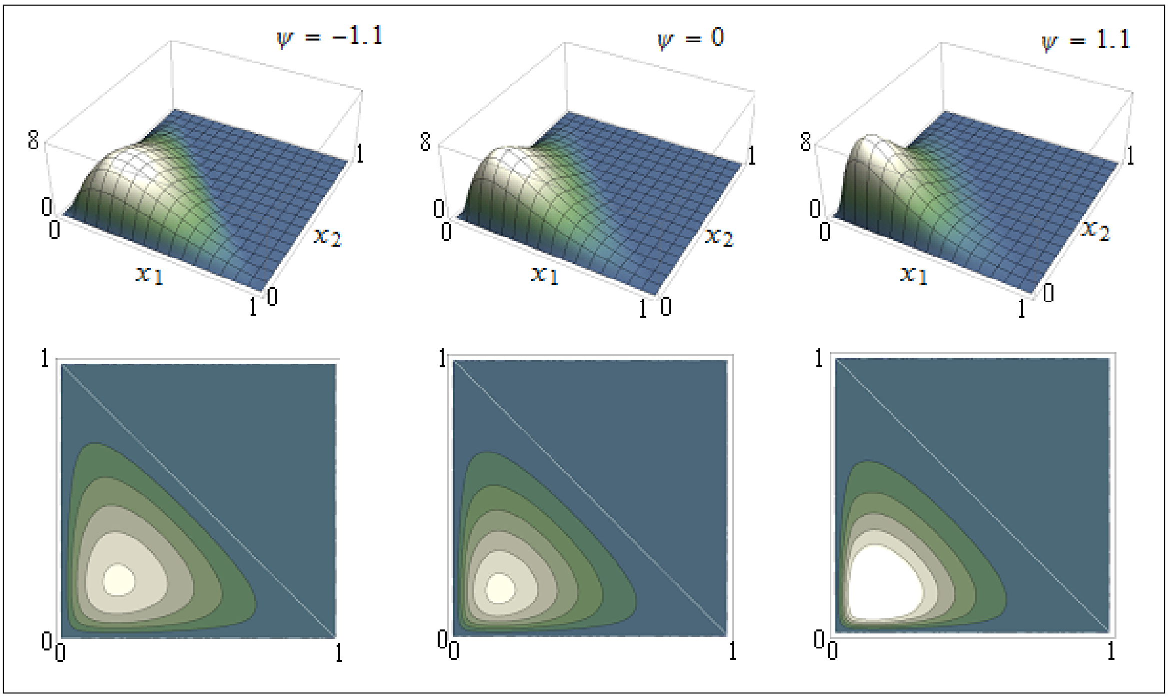


Figure 4.2: Joint pdf of the bivariate Kummer beta type III distribution

Figure 4.3 displays the marginal pdf (see (4.8)) of the bivariate Kummer beta type III distribution. The four graphs illustrate four different shapes of a univariate beta type III distribution. For each of the four shapes the effect of ψ is illustrated. Note that in the top two graphs, ψ changes the skewness of the pdf. In the bottom two graphs, ψ changes the

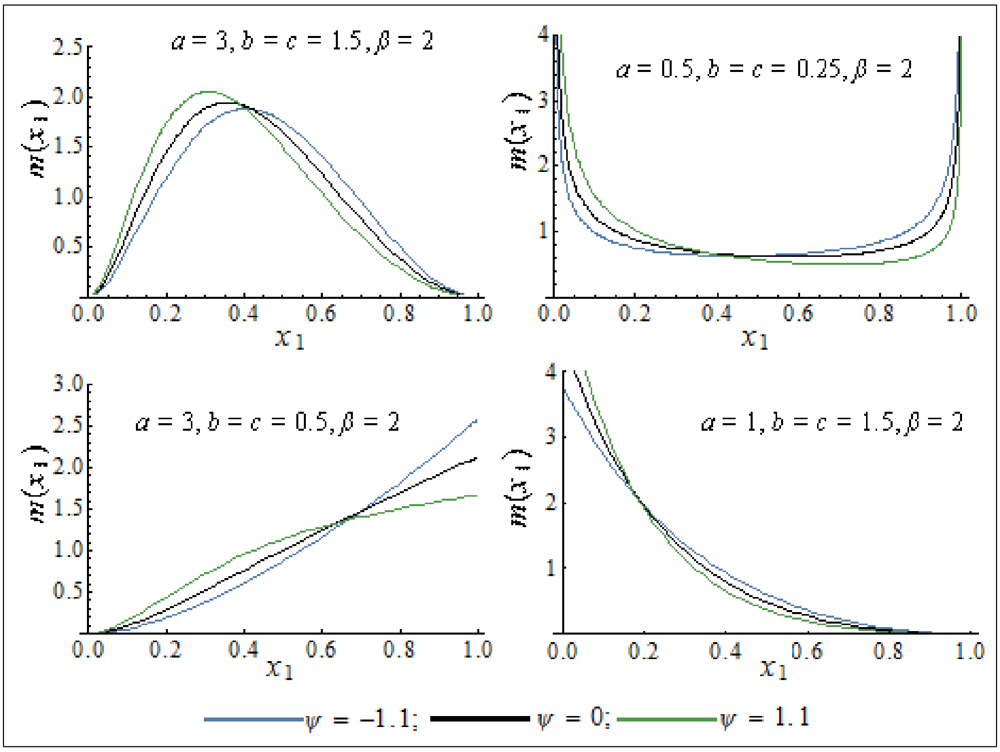


Figure 4.3: Marginal pdf of the bivariate Kummer beta type III distribution

shape of the graphs. For the negatively skewed graph, a negative value of ψ gives the pdf a concave shape and a positive value of ψ gives the pdf a convex shape. In the positively skewed graph, the graph remains concave, but ψ changes the intensity of the concavity.

4.5 Summary

A summary of the newly derived pdf's of Chapter 4 is given in Table 4.1.

Type	pdf	Equation number
BKB^{III}	$Kx_1^{a-1}x_2^{b-1}(1-x_1-x_2)^{c-1}(1+(\beta-1)x_1+(\beta-1)x_2)^{-(a+b+c)}e^{-\psi(x_1+x_2)}$	(4.1)
$NCBKB^{III}$	$K_{NC}\sum_{j=0}^{\infty}\frac{\delta^j}{j!}x_1^{a-1}x_2^{b-1}(1-x_1-x_2)^{c-1}(1+(\beta-1)x_1+(\beta-1)x_2)^{-(a+b+c)}e^{-\psi(x_1+x_2)}$	(4.6)
$m(x_1)$ of the BKB^{III}	$m(x_1)=Kx_1^{a-1}e^{-\psi x_1}(1+(\beta-1)x_1)^{-(a+b+c)}\sum_{j=0}^{\infty}B(b+j,c)\frac{(-\psi)^j}{j!}(1-x_1)^{b+c+j-1}$ $\times {}_2F_1\left(b+j,a+b+c;b+c+j;\frac{-(\beta-1)(1-x_1)}{1+(\beta-1)x_1}\right)$	(4.8)
$m(x_2)$ of the BKB^{III}	$m(x_2)=Kx_2^{b-1}e^{-\psi x_1}(1+(\beta-1)x_2)^{-(a+b+c)}\sum_{j=0}^{\infty}B(a+j,c)\frac{(-\psi)^j}{j!}(1-x_2)^{a+c+j-1}$ $\times {}_2F_1\left(a+j,a+b+c;a+c+j;\frac{-(\beta-1)(1-x_2)}{1+(\beta-1)x_2}\right)$	(4.8)
$c(x_1\|x_2)$ of the BKB^{III}	$c(x_1\|x_2)=Dx_1^{a-1}(1-x_1-x_2)^{c-1}(1+(\beta-1)x_1+(\beta-1)x_2)^{-(a+b+c)}e^{-\psi x_1}$	(4.9)
$c(x_2\|x_1)$ of the BKB^{III}	$c(x_2\|x_1)=Dx_2^{b-1}(1-x_2-x_1)^{c-1}(1+(\beta-1)x_2+(\beta-1)x_1)^{-(a+b+c)}e^{-\psi x_2}$	(4.9)

Table 4.1: Pdf's derived in Chapter 4

Chapter 5

The Bivariate Kummer beta Type IV

In this chapter the product method is used to construct the bivariate Kummer beta type IV distribution, utilizing Definition 2 (see (1.2)) with $f(x_1, x_2)$ the bivariate beta type IV kernel (see (1.11)). The joint distribution is considered as well as the marginal and conditional distribution functions. An expression is derived for the product moment and expressions for the means, variances, covariance and correlation are provided. Finally, the effect of the parameter, ψ, is investigated by means of a shape analysis; this parameter is introduced via the function $h(x_1, x_2) = e^{-\psi(x_1+x_2)}$ (see Figure 1.1).

5.1 Joint Distribution

In this section, the bivariate Kummer beta type IV distribution is derived.

Theorem 5.1 *The pdf of the bivariate Kummer beta type IV distribution is given by*

$$g_{BKBIV}(x_1, x_2) = Kx_1^{a-1}x_2^{b-1}(1-x_1)^{b+c-1}(1-x_2)^{a+c-1}(1-x_1x_2)^{-(a+b+c)}e^{-\psi(x_1+x_2)} \tag{5.1}$$

where $0 \leq x_1, x_2 \leq 1$, $a, b, c > 0$, $-\infty < \psi < \infty$, *and the normalizing constant,* K, *is given by*

$$K^{-1} = \sum_{k=0}^{\infty} \frac{(a+b+c)_k}{k!} \frac{{}_1F_1(b+k; a+b+c+k; -\psi)\, {}_1F_1(a+k; a+b+c+k; -\psi)}{(B(a+c, b+k)\, B(b+c, a+k))^{-1}}. \tag{5.2}$$

This distribution is denoted as $(X_1, X_2) \sim BKB^{IV}(a, b, c, \psi)$.

Proof. Definition 2 (see (1.2)) is used to construct the new bivariate Kummer beta type IV distribution from the bivariate beta type IV kernel (see (1.11)). The pdf of the bivariate Kummer beta type IV distribution is then given by

$$\begin{aligned}
& g_{BKBIV}(x_1, x_2) \\
= \;& K \; f_{BBIV}(x_1, x_2) e^{-\psi(x_1+x_2)} \\
= \;& K x_1^{a-1} x_2^{b-1} (1-x_1)^{b+c-1} (1-x_2)^{a+c-1} (1-x_1x_2)^{-(a+b+c)} e^{-\psi(x_1+x_2)} \qquad \text{(see (1.11))}
\end{aligned}$$

with K the normalizing constant. In order to obtain K, the following well-known property of pdf's is used:

$$\begin{aligned}
& \int_0^1 \int_0^1 g_{BKBIV}(x_1, x_2) dx_1 dx_2 \\
= \;& \int_0^1 \int_0^1 K x_1^{a-1} x_2^{b-1} (1-x_1)^{b+c-1} (1-x_2)^{a+c-1} (1-x_1x_2)^{-(a+b+c)} e^{-\psi(x_1+x_2)} dx_1 dx_2 \\
= \;& 1.
\end{aligned}$$

From this the normalizing constant is obtained as:

$$\begin{aligned}
K^{-1} &= \int_0^1 \int_0^1 x_1^{a-1} x_2^{b-1} (1-x_1)^{b+c-1} (1-x_2)^{a+c-1} (1-x_1x_2)^{-(a+b+c)} e^{-\psi(x_1+x_2)} dx_1 dx_2 \\
&= \int_0^1 x_2^{b-1} (1-x_2)^{a+c-1} e^{-\psi x_2} \int_0^1 x_1^{a-1} (1-x_1)^{b+c-1} (1-x_1x_2)^{-(a+b+c)} e^{-\psi x_1} dx_1 dx_2.
\end{aligned}$$

By Relation B.1 it follows:

$$K^{-1} = \sum_{k=0}^{\infty} \frac{(a+b+c)_k}{k!} \int_0^1 x_2^{b+k-1}(1-x_2)^{a+c-1} e^{-\psi x_2} \int_0^1 x_1^{a+k-1}(1-x_1)^{b+c-1} e^{-\psi x_1} dx_1 dx_2. \tag{5.3}$$

Since by Relation B.3

$$\int_0^1 x_1^{a+k-1}(1-x_1)^{b+c-1} e^{-\psi x_1} dx_1 = \frac{{}_1F_1\left(a+k; a+b+c+k; -\psi\right)}{\left(B\left(b+c, a+k\right)\right)^{-1}},$$

the expression in (5.3) becomes

$$\begin{aligned}
K^{-1} &= \sum_{k=0}^{\infty} \frac{(a+b+c)_k}{k!} \int_0^1 x_2^{b+k-1}(1-x_2)^{a+c-1} e^{-\psi x_2} \frac{{}_1F_1\left(a+k; a+b+c+k; -\psi\right)}{\left(B\left(b+c, a+k\right)\right)^{-1}} dx_2 \\
&= \sum_{k=0}^{\infty} \frac{(a+b+c)_k}{k!} \frac{{}_1F_1\left(a+k; a+b+c+k; -\psi\right)}{\left(B\left(b+c, a+k\right)\right)^{-1}} \int_0^1 x_2^{b+k-1}(1-x_2)^{a+c-1} e^{-\psi x_2} dx_2.
\end{aligned}$$

Once again, using Relation B.3 to solve the remaining integral, the required result (5.2) is obtained:

$$K^{-1}=\sum_{k=0}^{\infty}\frac{(a+b+c)_k}{k!}\frac{{}_1F_1\left(b+k;a+b+c+k;-\psi\right){}_1F_1\left(a+k;a+b+c+k;-\psi\right)}{\left(B\left(a+c,b+k\right)B\left(b+c,a+k\right)\right)^{-1}}.$$

■

Remark 5.1 *The bivariate Kummer beta type IV distribution may also be obtained by substituting $p=1$ in the pdf of the bimatrix variate Kummer beta type IV distribution defined by Bekker et al. [7]. (See Section 5.3 of their article.)*

Remark 5.2 *The non-central Kummer beta type IV distribution can be obtained by using the non-central beta type IV distribution, the latter which was studied by Ehlers [13] and Gupta et al. [24] with pdf given by*

$$\begin{aligned}&\left(B(a,b,c)\right)^{-1}x_1^{a-1}x_2^{b-1}(1-x_1)^{b+c-1}(1-x_2)^{a+c-1}(1-x_1x_2)^{-(a+b+c)}\\&\times e^{-\delta}{}_1F_1\left(a+b+c;c;\delta\frac{(1-x_1)(1-x_2)}{(1-x_1x_2)}\right)\end{aligned}\tag{5.4}$$

with kernel

$$\begin{aligned}f_{NCBBIV}(x_1,x_2)&=x_1^{a-1}x_2^{b-1}(1-x_1)^{b+c-1}(1-x_2)^{a+c-1}(1-x_1x_2)^{-(a+b+c)}\\&\quad\times{}_1F_1\left(a+b+c;c;\delta\frac{(1-x_1)(1-x_2)}{(1-x_1x_2)}\right)\end{aligned}\tag{5.5}$$

for $0\leq x_1,x_2\leq 1$, a, b, $c>0$ and where $\delta\geq 0$ denotes the non-centrality parameter. The pdf of the non-central bivariate Kummer beta type IV distribution is constructed using Definition 2 with $f(x_1,x_2)$ given by (5.5) and is given by

$$\begin{aligned}g_{NCBKBIV}(x_1,x_2)&=K_{NC}x_1^{a-1}x_2^{b-1}(1-x_1)^{b+c-1}(1-x_2)^{a+c-1}(1-x_1x_2)^{-(a+b+c)}\\&\quad\times{}_1F_1\left(a+b+c;c;\delta\frac{(1-x_1)(1-x_2)}{(1-x_1x_2)}\right)e^{-\delta\psi(x_1+x_2)}\end{aligned}\tag{5.6}$$

where $0\leq x_1,x_2\leq 1$, $a,b,c,>0$, $-\infty<\psi<\infty$, $\delta\geq 0$ denotes the non-centrality parameter and the normalizing constant, K_{NC}, is given by

$$\begin{aligned}K_{NC}^{-1}&=\sum_{k=0}^{\infty}\frac{\Gamma(a+b+c+k)\Gamma(c)\delta^k}{\Gamma(a+b+c)\Gamma(c+k)k!}\left(\sum_{j=0}^{\infty}\frac{(a+b+c+k)_j}{j!\left(B\left(a+c+k,b+j\right)B\left(b+c+k,a+j\right)\right)^{-1}}\right.\\&\quad\left.\times{}_1F_1\left(b+j;a+b+c+k+j;-\psi\right){}_1F_1\left(a+j;a+b+c+k+j;-\psi\right)\right).\end{aligned}\tag{5.7}$$

This distribution is denoted as $(X_1,X_2)\sim NCBKB^{IV}(a,b,c,\psi)$.

5.2 Marginal and Conditional Distributions

In this section the marginal, $m(x_i)$, and the conditional, $c(x_i|x_j)$, pdf's of the bivariate Kummer beta type IV distribution are derived.

Theorem 5.2 *If* $(X_1, X_2) \sim BKB^{IV}(a, b, c, \psi)$, *the marginal pdf of* X_1 *is given by*

$$m(x_1) = KB(a+c,b)x_1^{a-1}(1-x_1)^{b+c-1}e^{-\psi x_1}\sum_{k=0}^{\infty}\frac{(b)_k x_1^k}{k!}{}_1F_1(b+k; a+b+c+k; -\psi) \tag{5.8}$$

where $0 \leq x_1 \leq 1$, $a, b, c > 0$, $-\infty < \psi < \infty$ *and* K *as defined in (5.2).*

Proof. Using (5.1), $m(x_1)$ given in (5.8), is obtained by using Relation B.1:

$$\begin{aligned}
m(x_1) &= \int_0^1 Kx_1^{a-1}x_2^{b-1}(1-x_1)^{b+c-1}(1-x_2)^{a+c-1}(1-x_1x_2)^{-(a+b+c)}e^{-\psi(x_1+x_2)}dx_2 \\
&= Kx_1^{a-1}(1-x_1)^{b+c-1}e^{-\psi x_1}\int_0^1 x_2^{b-1}(1-x_2)^{a+c-1}(1-x_1x_2)^{-(a+b+c)}e^{-\psi x_2}dx_2 \\
&= K\sum_{k=0}^{\infty}\frac{(a+b+c)_k}{k!}x_1^{a+k-1}(1-x_1)^{b+c-1}e^{-\psi x_1}\int_0^1 x_2^{b+k-1}(1-x_2)^{a+c-1}e^{-\psi x_2}dx_2 \\
&= Kx_1^{a-1}(1-x_1)^{b+c-1}e^{-\psi x_1}\sum_{k=0}^{\infty}\frac{(a+b+c)_k x_1^k}{k!}\int_0^1 x_2^{b+k-1}(1-x_2)^{a+c-1}e^{-\psi x_2}dx_2.
\end{aligned}$$

Relation B.3 and (B.1) are used to solve the integral and the desired result (5.8) is obtained:

$$\begin{aligned}
&m(x_1) \\
&= Kx_1^{a-1}(1-x_1)^{b+c-1}e^{-\psi x_1}\sum_{k=0}^{\infty}\frac{(a+b+c)_k x_1^k}{k!}B(b+k, a+c)\,{}_1F_1(b+k; a+b+c+k; -\psi) \\
&= Kx_1^{a-1}(1-x_1)^{b+c-1}e^{-\psi x_1}\sum_{k=0}^{\infty}\frac{(a+b+c)_k x_1^k}{k!}\frac{{}_1F_1(b+k; a+b+c+k; -\psi)}{B(b+k, a+c)^{-1}} \\
&= K\frac{\Gamma(a+c)\Gamma(b)}{\Gamma(a+b+c)}x_1^{a-1}(1-x_1)^{b+c-1}e^{-\psi x_1}\sum_{k=0}^{\infty}\frac{(b)_k x_1^k}{k!}{}_1F_1(b+k; a+b+c+k; -\psi) \\
&= KB(a+c,b)x_1^{a-1}(1-x_1)^{b+c-1}e^{-\psi x_1}\sum_{k=0}^{\infty}\frac{(b)_k x_1^k}{k!}{}_1F_1(b+k; a+b+c+k; -\psi).
\end{aligned}$$

■

Remark 5.3 *Note that the marginal pdf of* X_2 *is obtained by substituting* x_2 *for* x_1 *in (5.8) and interchanging the parameters* a *and* b*.*

Theorem 5.3 *If* $(X_1, X_2) \sim BKB^{IV}(a, b, c, \psi)$, *the conditional pdf of* $X_2|X_1 = x_1$ *is given by*

$$c(x_2|x_1) = Dx_2^{b-1}(1-x_2)^{a+c-1}(1-x_1x_2)^{-(a+b+c)}e^{-\psi x_2} \tag{5.9}$$

where $0 \leq x_2 \leq 1$, $a, b, c, \beta > 0$, $-\infty < \psi < \infty$ *and the normalizing constant* D *is defined as*

$$D^{-1} = B(a+c, b)\sum_{k=0}^{\infty}\frac{(b)_k x_1^k}{k!}{}_1F_1(b+k; a+b+c+k; -\psi).$$

Proof. Using the joint pdf, $g_{BKBIV}(x_1, x_2)$, in (5.1) and the marginal pdf, $m(x_1)$, in (5.8), expression (5.9) for the conditional pdf of $X_2|X_1 = x_1$ follows directly:

$$\begin{aligned}
c(x_2|x_1) &= \frac{g_{BKBIV}(x_1, x_2)}{m(x_1)} \\
&= \frac{Kx_1^{a-1}x_2^{b-1}(1-x_1)^{b+c-1}(1-x_2)^{a+c-1}(1-x_1x_2)^{-(a+b+c)}e^{-\psi(x_1+x_2)}}{KB(a+c,b)x_1^{a-1}(1-x_1)^{b+c-1}e^{-\psi x_1}\sum_{k=0}^{\infty}\frac{(b)_k x_1^k}{k!}{}_1F_1(b+k; a+b+c+k; -\psi)} \\
&= \frac{x_2^{b-1}(1-x_2)^{a+c-1}(1-x_1x_2)^{-(a+b+c)}e^{-\psi x_2}}{B(a+c,b)\sum_{k=0}^{\infty}\frac{(b)_k x_1^k}{k!}{}_1F_1(b+k; a+b+c+k; -\psi)} \\
&= Dx_2^{b-1}(1-x_2)^{a+c-1}(1-x_1x_2)^{-(a+b+c)}e^{-\psi x_2}.
\end{aligned}$$

■

Remark 5.4 *Note that the conditional pdf of* $X_1|X_2 = x_2$ *is obtained by interchanging the variables* x_1 *and* x_2 *and the parameters* a *and* b *in (5.9).*

5.3 Moments and Correlation

In this section the product moment is derived from which expressions for the mean and variance of X_1 and X_2 as well as the covariance and correlation of (X_1, X_2) are obtained. Furthermore, the effect of the parameter ψ on the correlation between X_1 and X_2 is investigated.

Theorem 5.4 *If* $(X_1, X_2) \sim BKB^{IV}(a, b, c, \psi)$, *the product moment, i.e.* $E(X_1^r X_2^s)$, *equals*

$$\begin{aligned}
&K\sum_{k=0}^{\infty}\frac{(a+b+c)_k}{k!}\frac{{}_1F_1(a+k+r, a+b+c+k+r; -\psi)\,{}_1F_1(b+k+s, a+b+c+k+s; -\psi)}{(B(a+c, b+k+s)\,B(b+c, a+k+r))^{-1}} \\
&= (A(a,b,c,0,0))^{-1} \times A(a,b,c,r,s) \qquad (5.10)
\end{aligned}$$

where

$$\begin{aligned} & A(a,b,c,r,s) \\ = & \sum_{k=0}^{\infty} \frac{(a+b+c)_k}{k!} \frac{{}_1F_1\left(a+k+r, a+b+c+k+r; -\psi\right) {}_1F_1\left(b+k+s, a+b+c+k+s; -\psi\right)}{\left(B\left(a+c, b+k+s\right) B\left(b+c, a+k+r\right)\right)^{-1}} \end{aligned} \tag{5.11}$$

with $A(a,b,c,0,0)^{-1} = K$ as defined in (5.2). Note that the definition of $A(.)$ must be read in context as its definition depends on the distribution used.

Proof. Using the pdf of the bivariate Kummer beta type IV distribution (5.1), the expected value of $X_1^r X_2^s$ is taken:

$$\begin{aligned} & E(X_1^r X_2^s) \\ = & \int_0^1 \int_0^1 x_1^r x_2^s g_{BKBIV}(x_1, x_2) dx_1 dx_2 \\ = & \int_0^1 \int_0^1 x_1^r x_2^s K x_1^{a-1} x_2^{b-1} \left(1-x_1\right)^{b+c-1} \left(1-x_2\right)^{a+c-1} \left(1-x_1x_2\right)^{-(a+b+c)} e^{-\psi(x_1+x_2)} dx_1 dx_2 \\ = & K \int_0^1 \int_0^1 x_1^{a+r-1} x_2^{b+s-1} \left(1-x_1\right)^{b+c-1} \left(1-x_2\right)^{a+c-1} \left(1-x_1x_2\right)^{-(a+b+c)} e^{-\psi(x_1+x_2)} dx_1 dx_2. \end{aligned} \tag{5.12}$$

Because the integrals in (5.12) are similar to those found in the proof of Theorem 5.1, the desired expression is obtained as

$$\begin{aligned} & E(X_1^r X_2^s) \\ = & K \sum_{k=0}^{\infty} \frac{(a+b+c)_k}{k!} \frac{{}_1F_1\left(a+k+r; a+b+c+k+r; -\psi\right) {}_1F_1\left(b+k+s; a+b+c+k+s; -\psi\right)}{\left(B\left(b+c, a+k+r\right) B\left(a+c, b+k+s\right)\right)^{-1}} \\ = & \left(A(a,b,c,0,0)\right)^{-1} \times A(a,b,c,r,s) \end{aligned}$$

with $A(a,b,c,r,s)$ as defined in (5.11). ■

It follows from (5.10):

$$\begin{aligned} E(X_1) &= (A(a,b,c,0,0))^{-1} \times A(a,b,c,1,0), \\ E(X_1^2) &= (A(a,b,c,0,0))^{-1} \times A(a,b,c,2,0), \\ E(X_2) &= (A(a,b,c,0,0))^{-1} \times A(a,b,c,0,1), \\ E(X_2^2) &= (A(a,b,c,0,0))^{-1} \times A(a,b,c,0,2), \end{aligned}$$

$$\begin{aligned} E(X_1X_2) &= (A(a,b,c,0,0))^{-1} \times A(a,b,c,1,1), \\ Var(X_1) &= E(X_1^2) - (E(X_1))^2, \\ Var(X_2) &= E(X_2^2) - (E(X_2))^2 \\ \text{and} \quad Cov(X_1,X_2) &= E(X_1X_2) - E(X_1)E(X_2). \end{aligned}$$

Figure 5.1 shows the correlation between X_1 and X_2 as ψ varies where $(X_1, X_2) \sim BKB^{IV}(a,b,c,\psi)$. The graph shows the correlation as a function of $\psi \in [-10, 10]$ for certain combinations of the parameters a, b and c. Note that the correlations for $\psi = 0$ are those of the bivariate beta type IV distribution which are always positive (see [48]). The new parameter, ψ, gives us a wide range of values giving correlations ranging from 0 to 1. It is observed that ψ can weaken and strengthen the correlation, depending on the values of a, b and c.

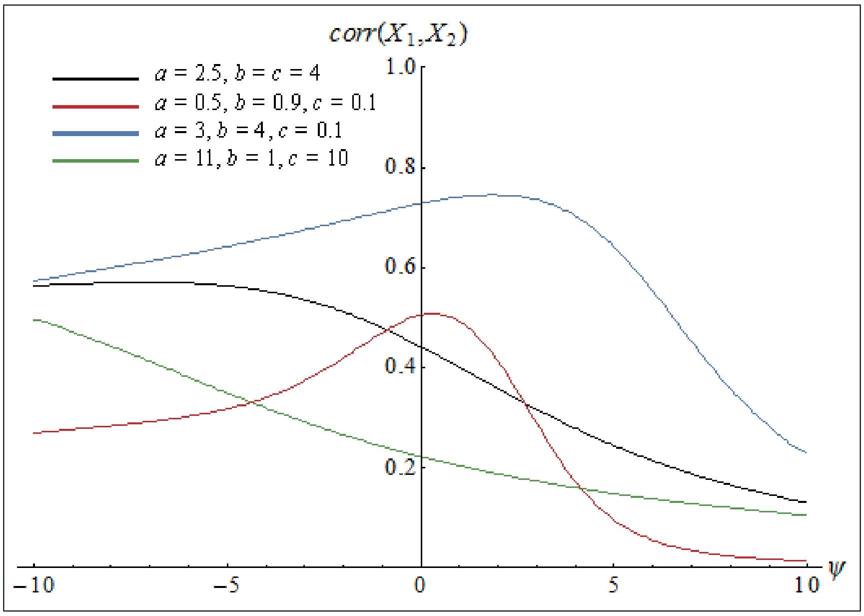


Figure 5.1: Correlation of the bivariate Kummer beta type IV distribution as ψ varies

Remark 5.5 *As illustrated in Figure 5.1, the bivariate Kummer beta type IV distribution has positive correlation. This can be proven by showing that the bivariate Kummer beta type IV distribution is positively likelihood ratio dependent (also denoted as totally positive of order 2 (TP_2)), since TP_2 implies positive quadrant dependence which implies positive correlation (see [5], p116). In order to prove that the bivariate Kummer beta type IV distribution is TP_2, Definition B.2 is used. It is required to prove that*

$$g_{BKBIV}(x_1, y_1) g_{BKBIV}(x_2, y_2) \geq g_{BKBIII}(x_1, y_2) g_{BKBIII}(x_2, y_1)$$

for all $x_1 < x_2$, $y_1 < y_2$, *which is similar to:*

$$\begin{aligned}
&\left(Kx_1^{a-1}y_1^{b-1}\left(1-x_1\right)^{b+c-1}\left(1-y_1\right)^{a+c-1}\left(1-x_1y_1\right)^{-(a+b+c)}e^{-\psi(x_1+y_1)}\right)\\
&\times\left(Kx_2^{a-1}y_2^{b-1}\left(1-x_2\right)^{b+c-1}\left(1-y_2\right)^{a+c-1}\left(1-x_2y_2\right)^{-(a+b+c)}e^{-\psi(x_2+y_2)}\right)\\
\geq\ &\left(Kx_1^{a-1}y_2^{b-1}\left(1-x_1\right)^{b+c-1}\left(1-y_2\right)^{a+c-1}\left(1-x_1y_2\right)^{-(a+b+c)}e^{-\psi(x_1+y_2)}\right)\\
&\times\left(Kx_2^{a-1}y_1^{b-1}\left(1-x_2\right)^{b+c-1}\left(1-y_1\right)^{a+c-1}\left(1-x_2y_1\right)^{-(a+b+c)}e^{-\psi(x_2+y_1)}\right)
\end{aligned}$$

and simplifies to

$$\left(1-x_1y_1\right)^{-(a+b+c)}\left(1-x_2y_2\right)^{-(a+b+c)} \geq \left(1-x_1y_2\right)^{-(a+b+c)}\left(1-x_2y_1\right)^{-(a+b+c)}$$

$$\left(1-x_1y_1\right)\left(1-x_2y_2\right) \leq \left(1-x_1y_2\right)\left(1-x_2y_1\right). \tag{5.13}$$

The left hand side (LHS) and right hand side (RHS) of (5.13) reduce to $1-x_1y_1-x_2y_2+x_1y_1x_2y_2$ *and* $1-x_1y_2-x_2y_1+x_1y_2x_2y_1$, *respectively.*

Cancelling out similar terms on the LHS and the RHS we obtain $-x_1y_1-x_2y_2$ *and* $-x_1y_2-x_2y_1$, *respectively.*

Hence, (5.13) becomes

$$\begin{aligned}
&-x_1y_1-x_2y_2 \leq -x_1y_2-x_2y_1\\
\Rightarrow\quad & x_1y_1+x_2y_2 \geq x_1y_2+x_2y_1\\
\Rightarrow\quad & x_1y_1-x_1y_2 \geq x_2y_1-x_2y_2\\
\Rightarrow\quad & x_1(y_1-y_2) \geq x_2(y_1-y_2).
\end{aligned}$$

Since $(y_1-y_2)<0$, *we can divide on both sides by reversing the inequality sign to obtain* $x_1 \leq x_2$, *which clearly holds.*

Similarly,

$$\begin{aligned}
&-x_1y_1-x_2y_2 \leq -x_1y_2-x_2y_1\\
\Rightarrow\quad & x_1y_1+x_2y_2 \geq x_1y_2+x_2y_1\\
\Rightarrow\quad & x_1y_1-x_2y_1 \geq x_1y_2-x_2y_2\\
\Rightarrow\quad & y_1(x_1-x_2) \geq y_2(x_1-x_2).
\end{aligned}$$

Since $(x_1-x_2)<0$, *it follows that* $y_1 \leq y_2$, *which clearly holds. Therefore, the bivariate Kummer beta type IV distribution is* TP_2.

5.4 Shape Analysis

The effect of the parameters a, b and c for the bivariate beta type IV distribution was studied by Olkin and Liu [48]. In this study, therefore, only the effect of the new parameter, ψ, is illustrated.

In Figure 5.2 the joint pdf and contour plots for the bivariate Kummer beta type IV distribution whose pdf is given by (5.1) are displayed. The effect of ψ is illustrated by setting $\psi = -1.1$, 0 and 1.1 and keeping the other parameters fixed at $a = b = c = 2$. Note that a negative value of ψ pushes the pdf away from the origin and a positive value of ψ pushes the pdf towards the origin. Note that the middle graph (for $\psi = 0$), is the graph of the bivariate beta type IV whose pdf is given by (1.10).

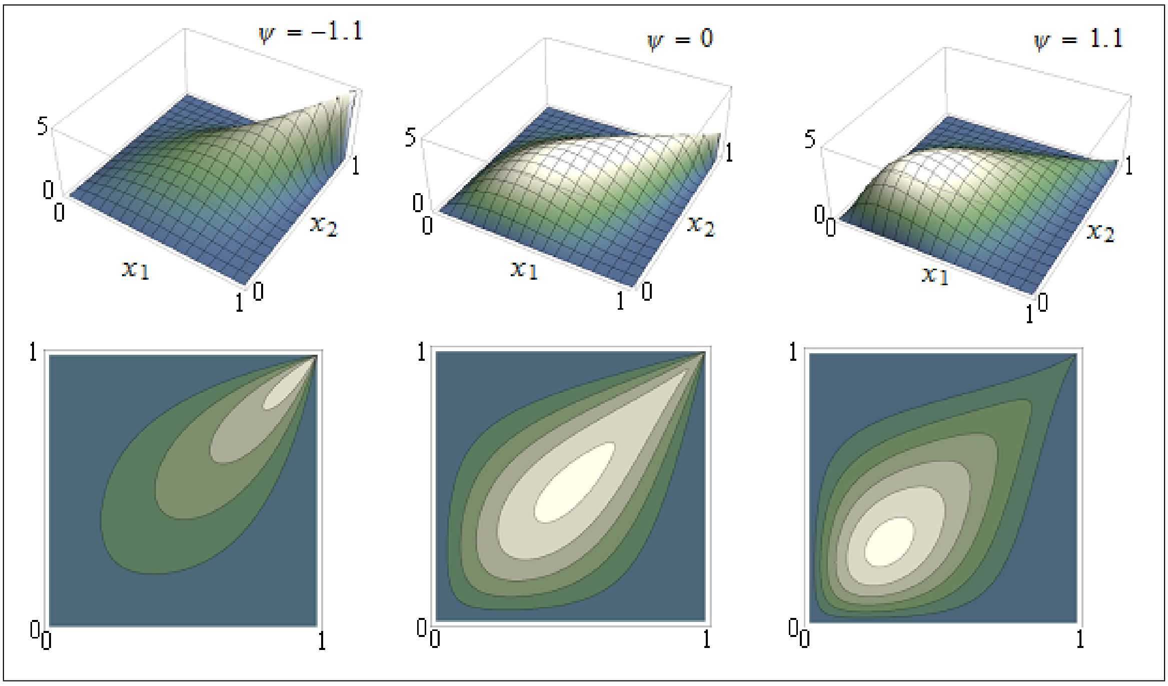


Figure 5.2: Joint pdf of the bivariate Kummer beta type IV distribution

Figure 5.3 shows the marginal pdf (see (5.8)) for the bivariate Kummer beta type IV distribution. The four graphs represent the four basic shapes of a univariate beta distribution, namely, symmetric, u-shaped, negatively skewed and positively skewed. For each of the four shapes the effect of ψ is studied. Note that in the top two graphs, ψ changes the skewness of the pdf. In the two skewed graphs, ψ changes the kurtosis of the graphs. For both the skewed graphs, a negative value of ψ decreasing the kurtosis and a positive value of ψ increasing the kurtosis.

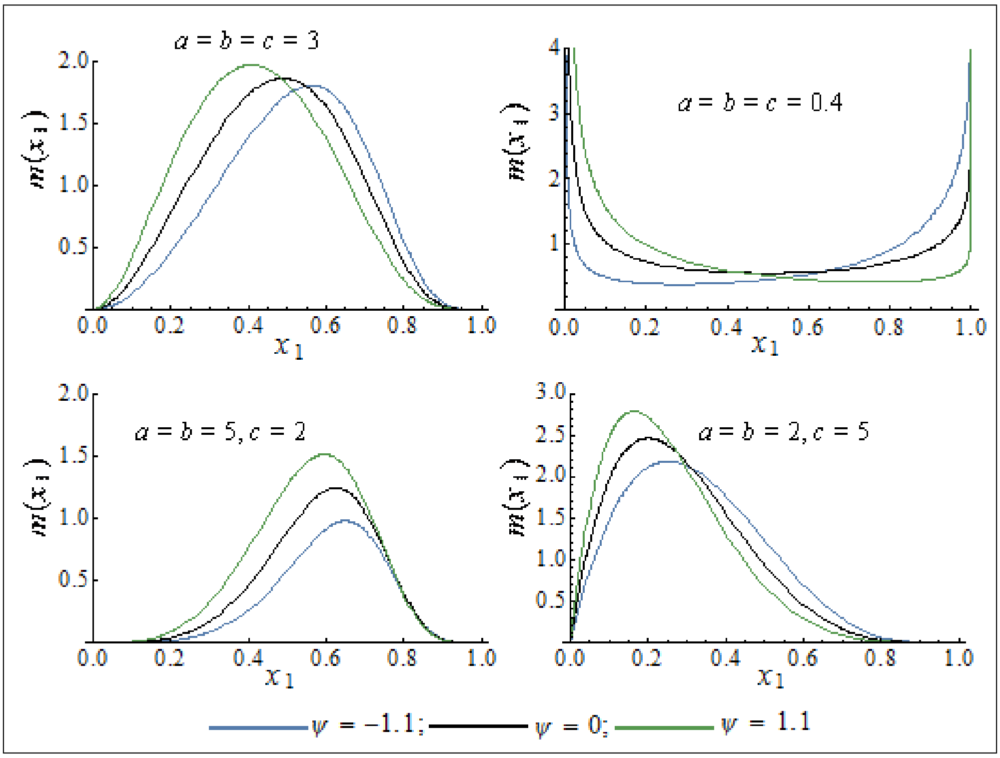


Figure 5.3: Marginal pdf of the bivariate Kummer beta type IV distribution

5.5 Summary

A summary of the newly derived pdf's of Chapter 5 is given in Table 5.1.

Type	pdf	Equation number
BKB^{IV}	$Kx_1^{a-1}x_2^{b-1}\left(1-x_1\right)^{b+c-1}\left(1-x_2\right)^{a+c-1}\left(1-x_1x_2\right)^{-(a+b+c)}e^{-\psi(x_1+x_2)}$	(5.1)
$NCBKB^{IV}$	$K_{NC}x_1^{a-1}x_2^{b-1}(1-x_1)^{b+c-1}(1-x_2)^{a+c-1}(1-x_1x_2)^{-(a+b+c)}$ $\times {}_1F_1\left(a+b+c;c;\delta\frac{(1-x_1)(1-x_2)}{(1-x_1x_2)}\right)e^{-\delta\psi(x_1+x_2)}$	(5.6)
$m(x_1)$ of the BKB^{IV}	$m(x_1)=K\frac{\Gamma(a+c)\Gamma(b)}{\Gamma(a+b+c)}x_1^{a-1}(1-x_1)^{b+c-1}e^{-\psi x_1}\sum_{k=0}^{\infty}\frac{(b)_k x_1^k}{k!}$ $\times {}_1F_1\left(b+k;a+b+c+k;-\psi\right)$	(5.8)
$m(x_2)$ of the BKB^{IV}	$m(x_2)=K\frac{\Gamma(b+c)\Gamma(a)}{\Gamma(a+b+c)}x_2^{b-1}(1-x_2)^{a+c-1}e^{-\psi x_2}\sum_{k=0}^{\infty}\frac{(a)_k x_2^k}{k!}$ $\times {}_1F_1\left(a+k;a+b+c+k;-\psi\right)$	(5.8)
$c(x_1\|x_2)$ of the BKB^{IV}	$c(x_1\|x_2)=Dx_1^{a-1}(1-x_1)^{b+c-1}(1-x_1x_2)^{-(a+b+c)}e^{-\psi x_1}$	(5.9)
$c(x_2\|x_1)$ of the BKB^{IV}	$c(x_2\|x_1)=Dx_2^{b-1}(1-x_2)^{a+c-1}(1-x_2x_1)^{-(a+b+c)}e^{-\psi x_2}$	(5.9)

Table 5.1: Pdf's derived in Chapter 5

Chapter 6

The Bivariate Kummer Extended Beta Type IV

In this chapter the product method is used to construct the bivariate Kummer extended beta type IV distribution, utilizing Definition 2 (see (1.2)) with $f(x_1, x_2)$ the bivariate extended beta type IV kernel (see(1.13)). The joint distribution is considered as well as the marginal and conditional distribution functions. An expression is derived for the product moment and expressions for the means, variances, covariance and correlation are provided. Finally, the effect of the parameter, ψ, is investigated by means of a shape analysis; this parameter is introduced via the function $h(x_1, x_2) = e^{-\psi(x_1+x_2)}$ (see Figure 1.1).

6.1 Joint Distribution

In this section, the bivariate Kummer extended beta type IV distribution is derived.

Theorem 6.1 *The pdf of the bivariate Kummer extended beta type IV distribution is given by*

$$\begin{aligned} g_{BKEBIV}(x_1, x_2) &= K x_1^{a-1} x_2^{b-1} (1-x_1)^{b+c+d-1} (1-x_2)^{a+c+d-1} e^{-\psi(x_1+x_2)} \\ &\quad \times (1-x_1x_2)^{-(a+b+c+d)} {}_2F_1\left(a+b+c+d, d; a+c+d; \frac{x_1(1-x_2)}{1-x_1x_2}\right) \end{aligned} \tag{6.1}$$

where $0 \leq x_1, x_2 \leq 1,\ a, b, c > 0,\ d \geq 0,\ -\infty < \psi < \infty$ *and the normalizing constant,* K*, is given by*

$$K^{-1} = B(a,b+c+d)B(b,a+c+d)\sum_{j=0}^{\infty}\frac{(d)_j(a)_j}{j!(a+b+c+d)_j}\sum_{k=0}^{\infty}\frac{(a+j)_k(b)_k}{k!(a+b+c+d+j)_k}$$
$$\times {}_1F_1(a+j+k;a+b+c+d+j+k;-\psi){}_1F_1(b+k;a+b+c+d+j+k;-\psi). \quad (6.2)$$

This distribution is denoted as $(X_1,X_2)\sim BKEB^{IV}(a,b,c,\psi)$.

Proof. Definition 2 (see (1.2)) is used to construct the new bivariate Kummer extended beta type IV distribution from the bivariate beta type IV kernel (see (1.13)). The pdf of the bivariate Kummer extended beta type IV distribution is then given by

$$\begin{aligned} g_{BKEBIV}(x_1,x_2) &= Kf_{BEBIV}(x_1,x_2)e^{-\psi(x_1+x_2)} \\ &= Kx_1^{a-1}x_2^{b-1}(1-x_1)^{b+c+d-1}(1-x_2)^{a+c+d-1}(1-x_1x_2)^{-(a+b+c+d)} \\ &\times {}_2F_1\left(a+b+c+d,d;a+c+d;\frac{x_1(1-x_2)}{1-x_1x_2}\right)e^{-\psi(x_1+x_2)} \qquad \text{(see (1.13)).} \end{aligned}$$

with K the normalizing constant. In order to obtain K, the following well-known property of pdf's is used:

$$\begin{aligned} &\int_0^1\int_0^1 g_{BKEBIV}(x_1,x_2)dx_1dx_2 \\ = &\int_0^1\int_0^1 Kx_1^{a-1}x_2^{b-1}(1-x_1)^{b+c+d-1}(1-x_2)^{a+c+d-1}(1-x_1x_2)^{-(a+b+c+d)} \\ &\times {}_2F_1\left(a+b+c+d,d;a+c+d;\frac{x_1(1-x_2)}{1-x_1x_2}\right)e^{-\psi(x_1+x_2)}dx_1dx_2 \\ = &\ 1. \end{aligned}$$

From this, the normalizing constant is obtained as:

$$\begin{aligned} K^{-1} = &\int_0^1\int_0^1 x_1^{a-1}x_2^{b-1}(1-x_1)^{b+c+d-1}(1-x_2)^{a+c+d-1}(1-x_1x_2)^{-(a+b+c+d)} \\ &\times {}_2F_1\left(a+b+c+d,d;a+c+d;\frac{x_1(1-x_2)}{1-x_1x_2}\right)e^{-\psi(x_1+x_2)}dx_1dx_2. \end{aligned}$$

Definition B.8 is used to obtain:

$$\begin{aligned} K^{-1} &= \sum_{j=0}^{\infty}\frac{(a+b+c+d)_j(d)_j}{j!(a+c+d)_j}\int_0^1\int_0^1 x_1^{a+j-1}x_2^{b-1}(1-x_1)^{b+c+d-1}(1-x_2)^{a+c+d+j-1} \\ &\times(1-x_1x_2)^{-(a+b+c+d+j)}e^{-\psi(x_1+x_2)}dx_1dx_2. \end{aligned}$$

Use Relation B.1.to obtain:

$$
\begin{aligned}
& K^{-1} \\
= & \sum_{j=0}^{\infty} \frac{(a+b+c+d)_j (d)_j}{j!(a+c+d)_j} \sum_{k=0}^{\infty} \frac{(a+b+c+d+j)_k}{k!} \int_0^1 \int_0^1 x_1^{a+j+k-1} x_2^{b+k-1} (1-x_1)^{b+c+d-1} \\
& \times (1-x_2)^{a+c+d+j-1} e^{-\psi(x_1+x_2)} dx_1 dx_2 \\
= & \sum_{j=0}^{\infty} \frac{(a+b+c+d)_j (d)_j}{j!(a+c+d)_j} \sum_{k=0}^{\infty} \frac{(a+b+c+d+j)_k}{k!} \int_0^1 x_2^{b+k-1} (1-x_2)^{a+c+d+j-1} e^{-\psi x_2} \\
& \times \int_0^1 x_1^{a+j+k-1} (1-x_1)^{b+c+d-1} e^{-\psi x_1} dx_1 dx_2. \qquad (6.3)
\end{aligned}
$$

Since by Relation B.3

$$
\int_0^1 x_1^{a+j+k-1} (1-x_1)^{b+c+d-1} e^{-\psi x_1} dx_1 = B(a+j+k, b+c+d) {}_1F_1(a+j+k; a+b+c+d+j+k; -\psi),
$$

the expression in (6.3) becomes

$$
\begin{aligned}
K^{-1} = & \sum_{j=0}^{\infty} \frac{(a+b+c+d)_j (d)_j}{j!(a+c+d)_j} \sum_{k=0}^{\infty} \frac{(a+b+c+d+j)_k}{k!} \int_0^1 x_2^{b+k-1} (1-x_2)^{a+c+d+j-1} e^{-\psi x_2} \\
& \times B(a+j+k, b+c+d) {}_1F_1(a+j+k; a+b+c+d+j+k; -\psi) dx_2 \\
= & \sum_{j=0}^{\infty} \frac{(a+b+c+d)_j (d)_j}{j!(a+c+d)_j} \sum_{k=0}^{\infty} \frac{(a+b+c+d+j)_k}{k!} B(a+j+k, b+c+d) \\
& \times {}_1F_1(a+j+k; a+b+c+d+j+k; -\psi) \int_0^1 x_2^{b+k-1} (1-x_2)^{a+c+d+j-1} e^{-\psi x_2} dx_2.
\end{aligned}
$$

Definition B.4 and (B.1) is used to obtain:

$$
\begin{aligned}
K^{-1} = & \frac{\Gamma(b+c+d)}{\Gamma(a+b+c+d)} \sum_{j=0}^{\infty} \frac{(d)_j}{j!(a+c+d)_j} \sum_{k=0}^{\infty} \frac{\Gamma(a+j+k)}{k!} \\
& \times {}_1F_1(a+j+k; a+b+c+d+j+k; -\psi) \int_0^1 x_2^{b+k-1} (1-x_2)^{a+c+d+j-1} e^{-\psi x_2} dx_2.
\end{aligned}
$$

Once again using Relation B.3 to solve the remaining integral, the required result (6.2) is obtained by using (B.1) and Definition B.4:

$$\begin{aligned}
&K^{-1}\\
=\ &\frac{\Gamma(b+c+d)}{\Gamma(a+b+c+d)}\sum_{j=0}^{\infty}\frac{(d)_j}{j!(a+c+d)_j}\sum_{k=0}^{\infty}\frac{\Gamma(a+j+k)}{k!}{}_1F_1(a+j+k;a+b+c+d+j+k;-\psi)\\
&\times B(b+k,a+c+d+j){}_1F_1(b+k;a+b+c+d+j+k;-\psi)\\
=\ &\frac{\Gamma(b+c+d)}{\Gamma(a+b+c+d)}\sum_{j=0}^{\infty}\frac{(d)_j}{j!(a+c+d)_j}\sum_{k=0}^{\infty}\frac{\Gamma(a+j+k)}{k!}\frac{\Gamma(b+k)\Gamma(a+c+d+j)}{\Gamma(a+b+c+d+j+k)}\\
&\times{}_1F_1(a+j+k;a+b+c+d+j+k;-\psi){}_1F_1(b+k;a+b+c+d+j+k;-\psi)\\
=\ &\frac{\Gamma(b+c+d)\Gamma(a+c+d)\Gamma(a)\Gamma(b)}{\Gamma(a+b+c+d)\Gamma(a+b+c+d)}\sum_{j=0}^{\infty}\frac{(d)_j(a)_j}{j!(a+b+c+d)_j}\sum_{k=0}^{\infty}\frac{(a+j)_k(b)_k}{k!(a+b+c+d+j)_k}\\
&\times{}_1F_1(a+j+k;a+b+c+d+j+k;-\psi){}_1F_1(b+k;a+b+c+d+j+k;-\psi)\\
=\ &B(a,b+c+d)B(b,a+c+d)\sum_{j=0}^{\infty}\frac{(d)_j(a)_j}{j!(a+b+c+d)_j}\sum_{k=0}^{\infty}\frac{(a+j)_k(b)_k}{k!(a+b+c+d+j)_k}\\
&\times{}_1F_1(a+j+k;a+b+c+d+j+k;-\psi){}_1F_1(b+k;a+b+c+d+j+k;-\psi).
\end{aligned}$$

■

Remark 6.1 *For $d=0$, the pdf of the bivariate Kummer extended beta type IV reduces to that of the bivariate Kummer beta type IV. This can be shown by setting $d=0$ in (6.2) and subsequently in (6.1):*

$$\begin{aligned}
K^{-1} &= B(a,b+c)B(b,a+c)\sum_{j=0}^{\infty}\frac{(0)_j(a)_j}{j!(a+b+c)_j}\sum_{k=0}^{\infty}\frac{(a+j)_k(b)_k}{k!(a+b+c+j)_k}\\
&\quad\times{}_1F_1(a+j+k;a+b+c+j+k;-\psi){}_1F_1(b+k;a+b+c+j+k;-\psi).
\end{aligned}$$

Use (B.1) and Definition B.8 to obtain

$$\begin{aligned}
K^{-1} &= \frac{\Gamma(a)\Gamma(b)\Gamma(a+c)\Gamma(b+c)}{\Gamma(a+b+c)\Gamma(a+b+c)}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(0)_j(a)_j(a+j)_k(b)_k(a+j+k)_m(b+k)_n}{j!k!m!n!(a+b+c)_j(a+b+c+j)_k}\\
&\quad\times\frac{(-\psi)^m(-\psi)^n}{(a+b+c+j+k)_m(a+b+c+j+k)_n}.
\end{aligned}$$

Simplify by using Definition B.4 to obtain

$$\begin{aligned}
K^{-1} &= \frac{\Gamma(a+c)\Gamma(b+c)}{\Gamma(a+b+c)}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(0)_j\Gamma(a+j+k+m)\Gamma(b+k+n)}{j!k!m!n!}\\
&\quad\times\frac{\Gamma(a+b+c+j+k)(-\psi)^m(-\psi)^n}{\Gamma(a+b+c+j+k+m)\Gamma(a+b+c+j+k+n)}
\end{aligned}$$

$$
\begin{aligned}
= \ & \frac{\Gamma(a+c)\Gamma(b+c)}{\Gamma(a+b+c)} \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{m=0}^{\infty} \frac{(-\psi)^m}{m!} \sum_{n=0}^{\infty} \frac{(-\psi)^n}{n!} \Gamma(b+k+n) \\
& \times \sum_{j=0}^{\infty} \frac{(0)_j \Gamma(a+j+k+m)\Gamma(a+b+c+j+k)}{j!\Gamma(a+b+c+j+k+m)\Gamma(a+b+c+j+k+n)}.
\end{aligned}
$$

Manipulate some of the terms to get the infinite sums into a standard form and obtain:

$$
\begin{aligned}
& K^{-1} \\
= \ & \sum_{k=0}^{\infty} \frac{1}{k!} \frac{B(a+c,b+k)B(b+c,a+k)(a+b+c)_k \Gamma(a+b+c+k)}{\Gamma(b+k)\Gamma(a+k)} \sum_{m=0}^{\infty} \frac{(-\psi)^m}{m!} \\
& \times \sum_{n=0}^{\infty} \frac{(-\psi)^n}{n!} \Gamma(b+k+n) \sum_{j=0}^{\infty} \frac{(0)_j \Gamma(a+j+k+m)\Gamma(a+b+c+j+k)}{j!\Gamma(a+b+c+j+k+m)\Gamma(a+b+c+j+k+n)} \\
= \ & \sum_{k=0}^{\infty} \frac{1}{k!} \frac{B(a+c,b+k)B(b+c,a+k)(a+b+c)_k}{\Gamma(b+k)} \sum_{m=0}^{\infty} \frac{(-\psi)^m}{m!} \frac{(a+k)_m \Gamma(a+b+c+k+m)}{(a+b+c+k)_m \Gamma(a+k+m)} \\
& \times \sum_{n=0}^{\infty} \frac{(-\psi)^n}{n!} \Gamma(b+k+n) \sum_{j=0}^{\infty} \frac{(0)_j \Gamma(a+j+k+m)\Gamma(a+b+c+j+k)}{j!\Gamma(a+b+c+j+k+m)\Gamma(a+b+c+j+k+n)} \\
= \ & \sum_{k=0}^{\infty} \frac{B(a+c,b+k)B(b+c,a+k)(a+b+c)_k}{k!} \sum_{m=0}^{\infty} \frac{(-\psi)^m}{m!} \frac{(a+k)_m \Gamma(a+b+c+k+m)}{(a+b+c+k)_m \Gamma(a+k+m)} \\
& \times \sum_{n=0}^{\infty} \frac{(-\psi)^n}{n!} \frac{(b+k)_n \Gamma(a+b+c+k+n)}{(a+b+c+k)_n \Gamma(a+b+c+k)} \sum_{j=0}^{\infty} \frac{(0)_j \Gamma(a+j+k+m)}{j!\Gamma(a+b+c+j+k+m)} \\
& \times \frac{\Gamma(a+b+c+j+k)}{\Gamma(a+b+c+j+k+n)} \\
= \ & \sum_{k=0}^{\infty} \frac{B(a+c,b+k)B(b+c,a+k)(a+b+c)_k}{k!} \sum_{m=0}^{\infty} \frac{(-\psi)^m}{m!} \frac{(a+k)_m}{(a+b+c+k)_m} \\
& \times \sum_{n=0}^{\infty} \frac{(-\psi)^n}{n!} \frac{(b+k)_n}{(a+b+c+k)_n} \sum_{j=0}^{\infty} \frac{(0)_j (a+k+m)_j (a+b+c+k)_j}{j!(a+b+c+k+m)_j (a+b+c+k+n)_j}
\end{aligned}
$$

Using Definition B.8 and Relation B.10:

$$
\begin{aligned}
&K^{-1}\\
&= \sum_{k=0}^{\infty} \frac{B(a+c,b+k)B(b+c,a+k)(a+b+c)_k}{k!} \sum_{m=0}^{\infty} \frac{(-\psi)^m}{m!}\\
&\times \frac{(a+k)_m}{(a+b+c+k)_m} \sum_{n=0}^{\infty} \frac{(-\psi)^n}{n!} \frac{(b+k)_n}{(a+b+c+k)_n}\\
&\times {}_3F_2(0,a+k+m,a+b+c+k;a+b+c+k+m,a+b+c+k+n;1)\\
&= \sum_{k=0}^{\infty} \frac{B(a+c,b+k)B(b+c,a+k)(a+b+c)_k}{k!} \sum_{m=0}^{\infty} \frac{(-\psi)^m}{m!} \frac{(a+k)_m}{(a+b+c+k)_m}\\
&\times \sum_{n=0}^{\infty} \frac{(-\psi)^n}{n!} \frac{(b+k)_n}{(a+b+c+k)_n} \frac{\Gamma(a+b+c+k+m)\Gamma(b+c+n)}{\Gamma(a+2b+2c+k+n)\Gamma(m)}\\
&\times {}_3F_2(a+b+c+k+n,b+c+n-m,a+b+c+k;a+2b+2c+k+n,a+b+c+k+n;1).
\end{aligned}
$$

Since by definition ${}_3F_2(a,b,c;d,a;x) = {}_2F_1(b,c;d;x)$ and using Relation B.11:

$$
\begin{aligned}
&K^{-1}\\
&= \sum_{k=0}^{\infty} \frac{B(a+c,b+k)B(b+c,a+k)(a+b+c)_k}{k!} \sum_{m=0}^{\infty} \frac{(-\psi)^m}{m!} \frac{(a+k)_m}{(a+b+c+k)_m}\\
&\times \sum_{n=0}^{\infty} \frac{(-\psi)^n}{n!} \frac{(b+k)_n}{(a+b+c+k)_n} \frac{\Gamma(a+b+c+k+m)\Gamma(b+c+n)}{\Gamma(a+2b+2c+k+n)\Gamma(m)}\\
&\times {}_2F_1(b+c+n-m,a+b+c+k;a+2b+2c+k+n;1)\\
&= \sum_{k=0}^{\infty} \frac{B(a+c,b+k)B(b+c,a+k)(a+b+c)_k}{k!} \sum_{m=0}^{\infty} \frac{(-\psi)^m}{m!} \frac{(a+k)_m}{(a+b+c+k)_m}\\
&\times \sum_{n=0}^{\infty} \frac{(-\psi)^n}{n!} \frac{(b+k)_n}{(a+b+c+k)_n} \frac{\Gamma(a+b+c+k+m)\Gamma(b+c+n)}{\Gamma(a+2b+2c+k+n)\Gamma(m)}\\
&\times \frac{\Gamma(a+2b+2c+k+n)\Gamma(m)}{\Gamma(a+b+c+k+m)\Gamma(b+c+n)}.
\end{aligned}
$$

Using (B.6), the normalizing constant in (5.2) is obtained:

$$
\begin{aligned}
&K^{-1}\\
&= \sum_{k=0}^{\infty} \frac{B(a+c,b+k)B(b+c,a+k)(a+b+c)_k}{k!} \sum_{m=0}^{\infty} \frac{(-\psi)^m}{m!} \frac{(a+k)_m}{(a+b+c+k)_m}\\
&\times \sum_{n=0}^{\infty} \frac{(-\psi)^n}{n!} \frac{(b+k)_n}{(a+b+c+k)_n}\\
&= \sum_{k=0}^{\infty} \frac{(a+b+c)_k}{k!} \frac{{}_1F_1\left(a+k,a+b+c+k;-\psi\right){}_1F_1\left(b+k,a+b+c+k;-\psi\right)}{\left(B\left(a+c,b+k\right)B\left(b+c,a+k\right)\right)^{-1}}.
\end{aligned}
$$

Using Relation B.12, the pdf in (6.1) then becomes:

$$\begin{aligned}
& K^{-1}x_1^{a-1}x_2^{b-1}(1-x_1)^{b+c+d-1}(1-x_2)^{a+c+d-1}e^{-\psi(x_1+x_2)} \\
& \times(1-x_1x_2)^{-(a+b+c+d)}{}_2F_1\left(a+b+c+d,d;a+c+d;\frac{x_1(1-x_2)}{1-x_1x_2}\right) \\
= \; & K^{-1}x_1^{a-1}x_2^{b-1}(1-x_1)^{b+c-1}(1-x_2)^{a+c-1}e^{-\psi(x_1+x_2)} \\
& \times(1-x_1x_2)^{-(a+b+c)}{}_2F_1\left(a+b+c,0;a+c;\frac{x_1(1-x_2)}{1-x_1x_2}\right) \\
= \; & K^{-1}x_1^{a-1}x_2^{b-1}(1-x_1)^{b+c-1}(1-x_2)^{a+c-1}e^{-\psi(x_1+x_2)} \\
& \times(1-x_1x_2)^{-(a+b+c)}{}_2F_1\left(a+b+c,0;a+c;\frac{x_1(1-x_2)}{1-x_1x_2}\right) \\
= \; & K^{-1}x_1^{a-1}x_2^{b-1}(1-x_1)^{b+c-1}(1-x_2)^{a+c-1}e^{-\psi(x_1+x_2)}(1-x_1x_2)^{-(a+b+c)}
\end{aligned}$$

which is the pdf of the bivariate Kummer beta type IV in (5.1).

6.2 Marginal and Conditional Distributions

In this section the marginal, $m(x_i)$, and the conditional, $c(x_i|x_j)$, pdf's of the bivariate Kummer extended beta type IV distribution are derived.

Theorem 6.2 *If $(X_1, X_2) \sim BKEB^{IV}(a,b,c,d,\psi)$, the marginal pdf of*

1. *X_1 is given by*

$$\begin{aligned}
m(x_1) &= K\frac{\Gamma(a+c+d)\Gamma(b)}{\Gamma(a+b+c+d)}x_1^{a-1}(1-x_1)^{b+c+d-1}e^{-\psi x_1}\sum_{j=0}^{\infty}\frac{(d)_j}{j!}x_1^j \\
& \quad \times\sum_{k=0}^{\infty}\frac{(b)_k}{k!}x_1^k{}_1F_1(b+k;a+b+c+d+j+k;-\psi) \qquad (6.4)
\end{aligned}$$

where $0 \leq x_1 \leq 1$, $a,b,c>0$, $d \geq 0$, $-\infty<\psi<\infty$ and K as defined in (6.2).

2. *X_2 is given by*

$$\begin{aligned}
m(x_2) &= K\frac{\Gamma(a)\Gamma(b+c+d)}{\Gamma(a+b+c+d)}x_2^{b-1}(1-x_2)^{a+c+d-1}e^{-\psi x_2}\sum_{j=0}^{\infty}\frac{(a)_j(d)_j}{j!(a+c+d)_j} \\
& \quad \times(1-x_2)^j\sum_{k=0}^{\infty}\frac{(a+j)_k}{k!}x_2^k{}_1F_1(a+j+k;a+b+c+d+j+k;-\psi) \\
& \qquad (6.5)
\end{aligned}$$

where $0 \leq x_2 \leq 1$, $a,b,c>0$, $d \geq 0$, $-\infty<\psi<\infty$ and K as defined in (6.2).

Proof. The marginal pdf's of X_1 and X_2 are obtained by integrating the pdf of the bivariate Kummer extended beta type IV distributions over X_2 and X_1, respectively.

1. Using (6.1), $m(x_1)$ given in (6.4), is obtained by using Definition B.8 to obtain:

$$\begin{aligned}
m(x_1) &= \int_0^1 K x_1^{a-1} x_2^{b-1} (1-x_1)^{b+c+d-1} (1-x_2)^{a+c+d-1} (1-x_1x_2)^{-(a+b+c+d)} \\
&\quad \times {}_2F_1\left(a+b+c+d, d; a+c+d; \frac{x_1(1-x_2)}{1-x_1x_2}\right) e^{-\psi(x_1+x_2)} dx_2 \\
&= K \sum_{j=0}^{\infty} \frac{(a+b+c+d)_j (d)_j}{j!(a+c+d)_j} \int_0^1 x_1^{a+j-1} x_2^{b-1} (1-x_1)^{b+c+d-1} (1-x_2)^{a+c+d+j-1} \\
&\quad \times (1-x_1x_2)^{-(a+b+c+d+j)} e^{-\psi(x_1+x_2)} dx_2.
\end{aligned}$$

Use Relation B.1 to obtain:

$$\begin{aligned}
m(x_1) &= K \sum_{j=0}^{\infty} \frac{(a+b+c+d)_j (d)_j}{j!(a+c+d)_j} \sum_{k=0}^{\infty} \frac{(a+b+c+d+j)_k}{k!} \int_0^1 x_1^{a+j+k-1} x_2^{b+k-1} \\
&\quad \times (1-x_1)^{b+c+d-1} (1-x_2)^{a+c+d+j-1} e^{-\psi(x_1+x_2)} dx_2 \\
&= K \sum_{j=0}^{\infty} \frac{(a+b+c+d)_j (d)_j}{j!(a+c+d)_j} \sum_{k=0}^{\infty} \frac{(a+b+c+d+j)_k}{k!} x_1^{a+j+k-1} \\
&\quad \times (1-x_1)^{b+c+d-1} e^{-\psi x_1} \int_0^1 x_2^{b+k-1} (1-x_2)^{a+c+d+j-1} e^{-\psi x_2} dx_2.
\end{aligned}$$

Use Relation B.3 to solve the integral to obtain the required result in (6.4).

$$\begin{aligned}
m(x_1) &= K \sum_{j=0}^{\infty} \frac{(a+b+c+d)_j (d)_j}{j!(a+c+d)_j} \sum_{k=0}^{\infty} \frac{(a+b+c+d+j)_k}{k!} x_1^{a+j+k-1} (1-x_1)^{b+c+d-1} \\
&\quad \times e^{-\psi x_1} B(b+k, a+c+d+j) {}_1F_1(b+k; a+b+c+d+j+k; -\psi) \\
&= K x_1^{a-1} (1-x_1)^{b+c+d-1} e^{-\psi x_1} \sum_{j=0}^{\infty} \frac{(a+b+c+d)_j (d)_j}{j!(a+c+d)_j} x_1^j \sum_{k=0}^{\infty} \frac{(a+b+c+d+j)_k}{k!} \\
&\quad \times x_1^k B(b+k, a+c+d+j) {}_1F_1(b+k; a+b+c+d+j+k; -\psi) \\
&= K \frac{\Gamma(a+c+d)\Gamma(b)}{\Gamma(a+b+c+d)} x_1^{a-1} (1-x_1)^{b+c+d-1} e^{-\psi x_1} \sum_{j=0}^{\infty} \frac{(d)_j}{j!} x_1^j \sum_{k=0}^{\infty} \frac{(b)_k}{k!} \\
&\quad \times x_1^k {}_1F_1(b+k; a+b+c+d+j+k; -\psi).
\end{aligned}$$

2. Using (6.1), $m(x_2)$ given in (6.5), is obtained by using Definition B.8 to obtain:

$$\begin{aligned} m(x_2) &= \int_0^1 K x_1^{a-1} x_2^{b-1} (1-x_1)^{b+c+d-1} (1-x_2)^{a+c+d-1} (1-x_1x_2)^{-(a+b+c+d)} \\ &\quad \times {}_2F_1\left(a+b+c+d, d; a+c+d; \frac{x_1(1-x_2)}{1-x_1x_2}\right) e^{-\psi(x_1+x_2)} dx_1 \\ &= K \sum_{j=0}^{\infty} \frac{(a+b+c+d)_j (d)_j}{j!(a+c+d)_j} \int_0^1 x_1^{a+j-1} x_2^{b-1} (1-x_1)^{b+c+d-1} (1-x_2)^{a+c+d+j-1} \\ &\quad \times (1-x_1x_2)^{-(a+b+c+d+j)} e^{-\psi(x_1+x_2)} dx_1. \end{aligned}$$

Use Relation B.1 to obtain:

$$\begin{aligned} m(x_2) &= K \sum_{j=0}^{\infty} \frac{(a+b+c+d)_j (d)_j}{j!(a+c+d)_j} \sum_{k=0}^{\infty} \frac{(a+b+c+d+j)_k}{k!} \int_0^1 x_1^{a+j+k-1} x_2^{b+k-1} \\ &\quad \times (1-x_1)^{b+c+d-1} (1-x_2)^{a+c+d+j-1} e^{-\psi(x_1+x_2)} dx_1 \\ &= K \sum_{j=0}^{\infty} \frac{(a+b+c+d)_j (d)_j}{j!(a+c+d)_j} \sum_{k=0}^{\infty} \frac{(a+b+c+d+j)_k}{k!} x_2^{b+k-1} \\ &\quad \times (1-x_2)^{a+c+d+j-1} e^{-\psi x_2} \int_0^1 x_1^{a+j+k-1} (1-x_1)^{b+c+d-1} e^{-\psi x_1} dx_1. \end{aligned}$$

Use Relation B.3 to solve the above integral to obtain the required result (6.5):

$$\begin{aligned} &m(x_2) \\ &= K \sum_{j=0}^{\infty} \frac{(a+b+c+d)_j (d)_j}{j!(a+c+d)_j} \sum_{k=0}^{\infty} \frac{(a+b+c+d+j)_k}{k!} x_2^{b+k-1} (1-x_2)^{a+c+d+j-1} e^{-\psi x_2} \\ &\quad \times B(a+j+k, b+c+d) {}_1F_1(a+j+k; a+b+c+d+j+k; -\psi) \\ &= K x_2^{b-1} (1-x_2)^{a+c+d-1} e^{-\psi x_2} \sum_{j=0}^{\infty} \frac{(a+b+c+d)_j (d)_j}{j!(a+c+d)_j} (1-x_2)^j \sum_{k=0}^{\infty} \frac{(a+b+c+d+j)_k}{k!} \\ &\quad \times x_2^k \, B(a+j+k, b+c+d) {}_1F_1(a+j+k; a+b+c+d+j+k; -\psi) \\ &= K \frac{\Gamma(a)\Gamma(b+c+d)}{\Gamma(a+b+c+d)} x_2^{b-1} (1-x_2)^{a+c+d-1} e^{-\psi x_2} \sum_{j=0}^{\infty} \frac{(a)_j (d)_j}{j!(a+c+d)_j} (1-x_2)^j \\ &\quad \times \sum_{k=0}^{\infty} \frac{(a+j)_k}{k!} x_2^k \, {}_1F_1(a+j+k; a+b+c+d+j+k; -\psi). \end{aligned}$$

■

Theorem 6.3 *If* $(X_1, X_2) \sim BKEB^{IV}(a,b,c,d,\psi)$*, the conditional pdf of*

1. $X_2|X_1 = x_1$ *is given by*

$$\begin{aligned} c(x_2|x_1) &= D_1 x_2^{b-1}(1-x_2)^{a+c+d-1}(1-x_1x_2)^{-(a+b+c+d)}e^{-\psi x_2} \\ &\quad \times {}_2F_1\left(a+b+c+d, d; a+c+d; \frac{x_1(1-x_2)}{1-x_1x_2}\right) \end{aligned} \tag{6.6}$$

where $0 \le x_2 \le 1$, $a,b,c > 0$, $d \ge 0$, $-\infty < \psi < \infty$ *and the normalizing constant* D_1 *is defined as*

$$\begin{aligned} D_1^{-1} &= \frac{\Gamma(a+c+d)\Gamma(b)}{\Gamma(a+b+c+d)} \sum_{j=0}^{\infty} \frac{(d)_j}{j!} x_1^j \sum_{k=0}^{\infty} \frac{(b)_k}{k!} x_1^k \\ &\quad \times {}_1F_1(b+k; a+b+c+d+j+k; -\psi). \end{aligned}$$

2. $X_1|X_2 = x_2$ *is given by*

$$\begin{aligned} c(x_1|x_2) &= D_2 x_1^{a-1}(1-x_1)^{b+c+d-1}(1-x_1x_2)^{-(a+b+c+d)}e^{-\psi x_1} \\ &\quad \times {}_2F_1\left(a+b+c+d, d; a+c+d; \frac{x_1(1-x_2)}{1-x_1x_2}\right) \end{aligned} \tag{6.7}$$

where $0 \le x_1 \le 1$, $a,b,c > 0$, $d \ge 0$, $-\infty < \psi < \infty$ *and the normalizing constant* D_2 *is defined as*

$$\begin{aligned} D_2^{-1} &= \frac{\Gamma(a)\Gamma(b+c+d)}{\Gamma(a+b+c+d)} \sum_{j=0}^{\infty} \frac{(a)_j(d)_j}{j!(a+c+d)_j}(1-x_2)^j \sum_{k=0}^{\infty} \frac{(a+j)_k}{k!} x_2^k \\ &\quad \times {}_1F_1(a+j+k; a+b+c+d+j+k; -\psi). \end{aligned}$$

Proof.

1. Using the joint pdf, $g_{BKEBIV}(x_1, x_2)$, in (6.1) and the marginal pdf, $m(x_1)$, in (6.4), expression (6.6) for the conditional pdf of $X_2|X_1 = x_1$ follows directly:

$$\begin{aligned} c(x_2|x_1) &= \frac{g_{BKEBIV}(x_1,x_2)}{m(x_1)} \\ &= \frac{K x_1^{a-1} x_2^{b-1}(1-x_1)^{b+c+d-1}(1-x_2)^{a+c+d-1}(1-x_1x_2)^{-(a+b+c+d)}e^{-\psi(x_1+x_2)}}{K\frac{\Gamma(a+c+d)\Gamma(b)}{\Gamma(a+b+c+d)} x_1^{a-1}(1-x_1)^{b+c+d-1} e^{-\psi x_1} \sum_{j=0}^{\infty} \frac{(d)_j}{j!} x_1^j \sum_{k=0}^{\infty} \frac{(b)_k}{k!} x_1^k} \\ &\quad \times \frac{{}_2F_1\left(a+b+c+d, d; a+c+d; \frac{x_1(1-x_2)}{1-x_1x_2}\right)}{{}_1F_1(b+k; a+b+c+d+j+k; -\psi)} \end{aligned}$$

$$
\begin{aligned}
&= \frac{x_2^{b-1}(1-x_2)^{a+c+d-1}(1-x_1x_2)^{-(a+b+c+d)}e^{-\psi x_2}}{\frac{\Gamma(a+c+d)\Gamma(b)}{\Gamma(a+b+c+d)}\sum_{j=0}^{\infty}\frac{(d)_j}{j!}x_1^j\sum_{k=0}^{\infty}\frac{(b)_k}{k!}x_1^k} \\
&\quad \times \frac{{}_2F_1\left(a+b+c+d,d;a+c+d;\frac{x_1(1-x_2)}{1-x_1x_2}\right)}{{}_1F_1(b+k;a+b+c+d+j+k;-\psi)} \\
&= D_1x_2^{b-1}(1-x_2)^{a+c+d-1}(1-x_1x_2)^{-(a+b+c+d)}e^{-\psi x_2} \\
&\quad \times {}_2F_1\left(a+b+c+d,d;a+c+d;\frac{x_1(1-x_2)}{1-x_1x_2}\right).
\end{aligned}
$$

2. Using the joint pdf, $g_{BKEBIV}(x_1,x_2)$, in (6.1) and the marginal pdf, $m(x_2)$, in (6.5), expression (6.7) for the conditional pdf of $(X_1|X_2=x_2)$ follows directly:

$$
\begin{aligned}
& c(x_1|x_2) \\
&= \frac{g_{BKEBIV}(x_1,x_2)}{m(x_2)} \\
&= \frac{Kx_1^{a-1}x_2^{b-1}(1-x_1)^{b+c+d-1}(1-x_2)^{a+c+d-1}(1-x_1x_2)^{-(a+b+c+d)}e^{-\psi(x_1+x_2)}}{K\frac{\Gamma(a)\Gamma(b+c+d)}{\Gamma(a+b+c+d)}x_2^{b-1}(1-x_2)^{a+c+d-1}e^{-\psi x_2}\sum_{j=0}^{\infty}\frac{(a)_j(d)_j}{j!(a+c+d)_j}(1-x_2)^j\sum_{k=0}^{\infty}\frac{(a+j)_k}{k!}x_2^k} \\
&\quad \times \frac{{}_2F_1\left(a+b+c+d,d;a+c+d;\frac{x_1(1-x_2)}{1-x_1x_2}\right)}{{}_1F_1(a+j+k;a+b+c+d+j+k;-\psi)} \\
&= \frac{x_1^{a-1}(1-x_1)^{b+c+d-1}(1-x_1x_2)^{-(a+b+c+d)}e^{-\psi x_1}}{\frac{\Gamma(a)\Gamma(b+c+d)}{\Gamma(a+b+c+d)}\sum_{j=0}^{\infty}\frac{(a)_j(d)_j}{j!(a+c+d)_j}(1-x_2)^j\sum_{k=0}^{\infty}\frac{(a+j)_k}{k!}x_2^k} \\
&\quad \times \frac{{}_2F_1\left(a+b+c+d,d;a+c+d;\frac{x_1(1-x_2)}{1-x_1x_2}\right)}{{}_1F_1(a+j+k;a+b+c+d+j+k;-\psi)} \\
&= D_2x_1^{a-1}(1-x_1)^{b+c+d-1}(1-x_1x_2)^{-(a+b+c+d)}e^{-\psi x_1} \\
&\quad \times {}_2F_1\left(a+b+c+d,d;a+c+d;\frac{x_1(1-x_2)}{1-x_1x_2}\right).
\end{aligned}
$$

■

6.3 Moments and Correlation

In this section the product moment is derived from which expressions for the mean and variance of X_1 and X_2 as well as the covariance and correlation of (X_1,X_2) are obtained. Furthermore, the effect of the parameter ψ on the correlation between X_1 and X_2 is investigated.

Theorem 6.4 *If* $(X_1,X_2)\sim BKEB^{IV}(a,b,c,d,\psi)$, *the product moment, i.e.* $E\left(X_1^rX_2^s\right)$, *equals*

$$\begin{aligned}
& K\sum_{j=0}^{\infty}\frac{(a+b+c+d)_j(d)_j}{j!(a+c+d)_j}\sum_{k=0}^{\infty}\frac{(a+b+c+d+j)_k}{k!}B(a+r+j+k,b+c+d) \\
& \times B(b+s+k,a+c+d+j)_1F_1(a+r+j+k;a+b+c+d+r+j+k;-\psi) \\
& \times {}_1F_1(b+s+k;a+b+c+d+s+j+k;-\psi) \\
= & \ (A(a,b,c,d,0,0))^{-1}\times A(a,b,c,d,r,s) \qquad (6.8)
\end{aligned}$$

where

$$\begin{aligned}
& A(a,b,c,d,r,s) \\
= & \sum_{j=0}^{\infty}\frac{(a+b+c+d)_j(d)_j}{j!(a+c+d)_j}\sum_{k=0}^{\infty}\frac{(a+b+c+d+j)_k}{k!}B(a+r+j+k,b+c+d) \\
& \times B(b+s+k,a+c+d+j)_1F_1(a+r+j+k;a+b+c+d+r+j+k;-\psi) \\
& \times {}_1F_1(b+s+k;a+b+c+d+s+j+k;-\psi) \qquad (6.9)
\end{aligned}$$

with $A(a,b,c,d,0,0)^{-1}=K$ *as defined in (6.2). Note that the definition of* $A(.)$ *must be read in context as its definition depends on the distribution used.*

Proof. Using the pdf of the bivariate Kummer extended beta type IV distribution (6.1), take the expected value of $X_1^rX_2^s$:

$$\begin{aligned}
E(X_1^rX_2^s) &= \int_0^1\int_0^1 x_1^rx_2^s g_{BKEBIV}(x_1,x_2)dx_1dx_2 \\
&= \int_0^1\int_0^1 x_1^rx_2^sKx_1^{a-1}x_2^{b-1}(1-x_1)^{b+c+d-1}(1-x_2)^{a+c+d-1}e^{-\psi(x_1+x_2)} \\
&\quad \times(1-x_1x_2)^{-(a+b+c+d)}{}_2F_1\left(a+b+c+d,d;a+c+d;\frac{x_1(1-x_2)}{1-x_1x_2}\right)dx_1dx_2 \\
&= K\int_0^1\int_0^1 x_1^{a+r-1}x_2^{b+s-1}(1-x_1)^{b+c+d-1}(1-x_2)^{a+c+d-1}e^{-\psi(x_1+x_2)} \\
&\quad \times(1-x_1x_2)^{-(a+b+c+d)}{}_2F_1\left(a+b+c+d,d;a+c+d;\frac{x_1(1-x_2)}{1-x_1x_2}\right)dx_1dx_2. \qquad (6.10)
\end{aligned}$$

Because the integrals in (6.10) are similar to those found in the proof of Theorem 6.1, the desired expression is obtained as

$$\begin{aligned}
E(X_1^r X_2^s) &= K\sum_{j=0}^{\infty}\frac{(a+b+c+d)_j(d)_j}{j!(a+c+d)_j}\sum_{k=0}^{\infty}\frac{(a+b+c+d+j)_k}{k!}B(a+r+j+k,b+c+d)\\
&\quad \times B(b+s+k,a+c+d+j)\,{}_1F_1(a+r+j+k;a+b+c+d+r+j+k;-\psi)\\
&\quad \times {}_1F_1(b+s+k;a+b+c+d+s+j+k;-\psi)\\
&= (A(a,b,c,d,0,0))^{-1}\times A(a,b,c,d,r,s)
\end{aligned}$$

with $A(a,b,c,r,s)$ as defined in (6.9). ■

It follows from (6.8) that:

$$\begin{aligned}
E(X_1) &= (A(a,b,c,d,0,0))^{-1}\times A(a,b,c,d,1,0),\\
E(X_1^2) &= (A(a,b,c,d,0,0))^{-1}\times A(a,b,c,d,2,0),\\
E(X_2) &= (A(a,b,c,d,0,0))^{-1}\times A(a,b,c,d,0,1),\\
E(X_2^2) &= (A(a,b,c,d,0,0))^{-1}\times A(a,b,c,d,0,2),\\
E(X_1X_2) &= (A(a,b,c,d,0,0))^{-1}\times A(a,b,c,d,1,1),\\
Var(X_1) &= E(X_1^2)-(E(X_1))^2\,,\\
Var(X_2) &= E(X_2^2)-(E(X_2))^2\\
\text{and}\qquad Cov(X_1,X_2) &= E(X_1X_2)-E(X_1)E(X_2).
\end{aligned}$$

Figure 6.1 shows the correlation between X_1 and X_2 as ψ varies where $(X_1,X_2)\sim BKEB^{IV}(a,b,c,d,\psi)$. The graph shows the correlation as a function of $\psi\in[-10,10]$ for

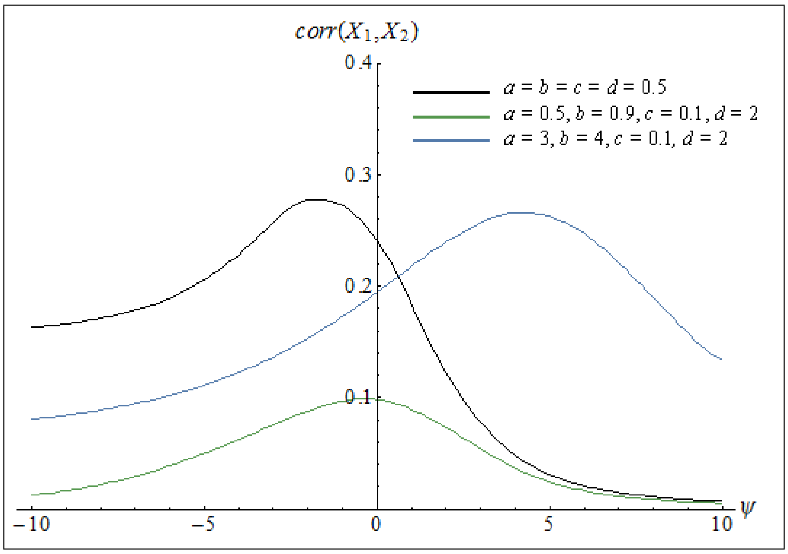


Figure 6.1: Correlation of the bivariate Kummer extended beta type IV distribution as ψ varies

certain combinations of values for the parameters a, b, c and d. Note that the correlations for $\psi = 0$ are those of the bivariate extended beta type IV distribution (see [16]). The new parameter, ψ, can weaken and strengthen the correlation, depending on the values of a, b, c and d. Also note that the correlations for $\psi = 0$ are those of the bivariate beta type IV distribution. For the effect of the parameters a, b and c for the bivariate extended beta type IV the reader is referred to El-Bassiouny and Jones [16].

Remark 6.2 *From Figure 6.1 the bivariate Kummer extended beta type IV distribution has positive correlation. To check if this holds for all parameter values, the definition for PQD (see (2.12)) given in Remark 2.5 can be used.*

6.4 Shape Analysis

As discussed in Section 1.4.5 - Chapter 1, the bivariate extended beta type IV distribution simplifies to the bivariate beta type IV distribution. The latter is graphically studied by Olkin and Liu [48]. For the bivariate extended beta type IV distribution, the reader is referred to El-Bassiouny and Jones [16] as well as Bekker et al. [6] . Therefore, in this study, only the effect of the new parameter, ψ, will be illustrated.

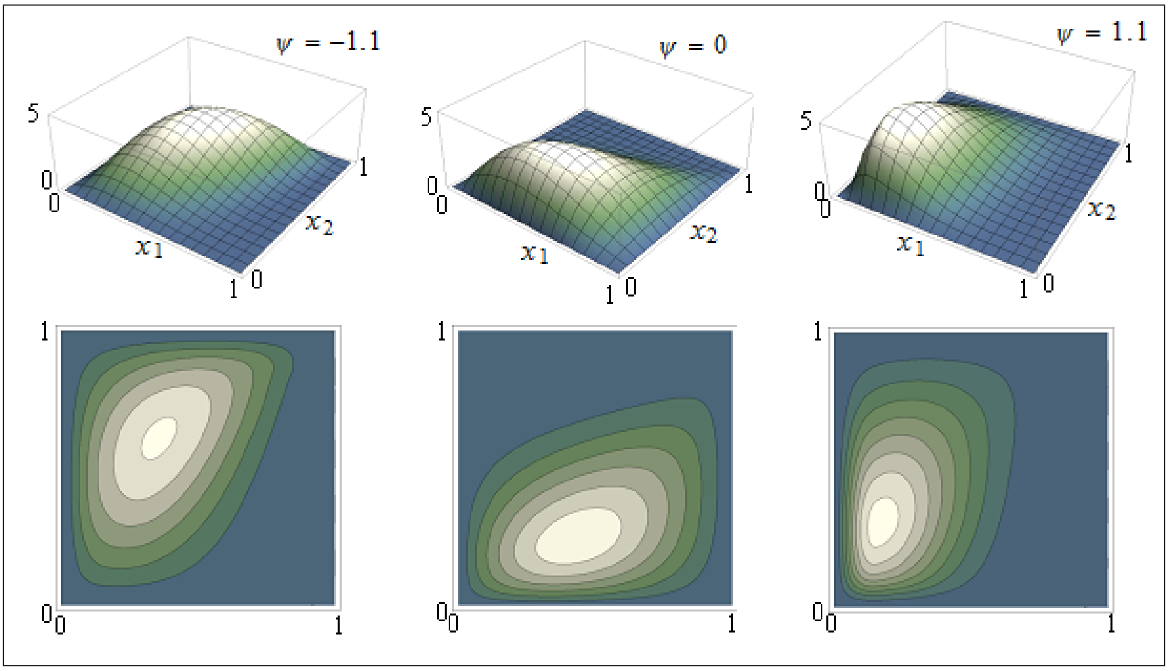


Figure 6.2: Joint pdf of the bivariate Kummer extended beta type IV distribution

In Figure 6.2 the joint pdf and contour plots for the bivariate Kummer extended beta type IV distribution whose pdf is given by (6.1) are displayed. The effect of ψ is illustrated by letting $\psi = -1.1$, 0 and 1.1 and keeping the other parameters fixed at

$a = b = c = d = 2$. Note that ψ moves the pdf around on the $[0,1] \times [0,1]$ plane. Note that the middle graph (for $\psi = 0$), is the graph of the bivariate extended beta type IV whose pdf is given by (1.12).

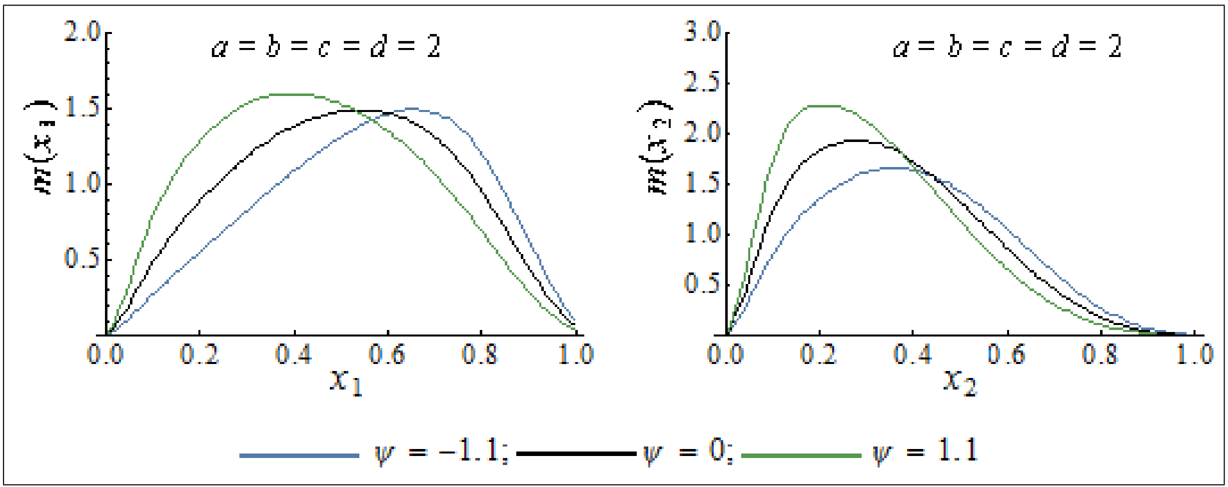


Figure 6.3: Marginal pdf's of the bivariate Kummer extended beta type IV distribution

Figure 6.3 displays the marginal pdf's (see (6.4) and (6.5)) for the bivariate Kummer extended beta type I distribution. The two graphs are the marginal pdf's for X_1 and X_2, respectively. For both marginals the effect of ψ is studied. In both graphs a positive value of ψ makes the pdf more positively skewed while a negative value of ψ makes it less positively and for the left graph more negatively skewed.

6.5 Summary

A summary of the newly derived pdf's of Chapter 6 is given in Table 6.1.

Type	**pdf**	**Equation number**
$BKEB^{IV}$	$Kx_1^{a-1}x_2^{b-1}(1-x_1)^{b+c+d-1}(1-x_2)^{a+c+d-1}e^{-\psi(x_1+x_2)}$ $\times(1-x_1x_2)^{-(a+b+c+d)}{}_2F_1\left(a+b+c+d,d;a+c+d;\frac{x_1(1-x_2)}{1-x_1x_2}\right)$	(6.1)
$m(x_1)$ of the $BKEB^{IV}$	$m(x_1)=K\frac{\Gamma(a+c+d)\Gamma(b)}{\Gamma(a+b+c+d)}x_1^{a-1}\left(1-x_1\right)^{b+c+d-1}e^{-\psi x_1}\sum_{j=0}^{\infty}\frac{(d)_j}{j!}x_1^j$ $\times\sum_{k=0}^{\infty}\frac{(b)_k}{k!}x_1^k{}_1F_1(b+k;a+b+c+d+j+k;-\psi)$	(6.4)
$m(x_2)$ of the $BKEB^{IV}$	$m(x_2)=K\frac{\Gamma(a)\Gamma(b+c+d)}{\Gamma(a+b+c+d)}x_2^{b-1}\left(1-x_2\right)^{a+c+d-1}e^{-\psi x_2}\sum_{j=0}^{\infty}\frac{(a)_j(d)_j}{j!(a+c+d)_j}$ $\times(1-x_2)^j\sum_{k=0}^{\infty}\frac{(a+j)_k}{k!}x_2^k{}_1F_1(a+j+k;a+b+c+d+j+k;-\psi)$	(6.5)
$c(x_1\|x_2)$ of the $BKEB^{IV}$	$c(x_1\|x_2)=D_1x_1^{a-1}(1-x_1)^{b+c+d-1}(1-x_1x_2)^{-(a+b+c+d)}e^{-\psi x_1}$ $\times{}_2F_1\left(a+b+c+d,d;a+c+d;\frac{x_1(1-x_2)}{1-x_1x_2}\right)$	(6.7)
$c(x_2\|x_1)$ of the $BKEB^{IV}$	$c(x_2\|x_1)=D_2x_2^{b-1}(1-x_2)^{a+c+d-1}(1-x_1x_2)^{-(a+b+c+d)}e^{-\psi x_2}$ $\times{}_2F_1\left(a+b+c+d,d;a+c+d;\frac{x_1(1-x_2)}{1-x_1x_2}\right)$	(6.6)

Table 6.1: Pdf's derived in Chapter 6

Chapter 7

The Bivariate Kummer Beta Type V

In this chapter the product method is used to construct the bivariate Kummer beta type V distribution, utilizing Definition 2 (see (1.2)) with $f(x_1, x_2)$ the bivariate beta type V kernel (see (1.15)). The joint distribution is considered as well as the marginal and conditional distribution functions. An expression is derived for the product moment and expressions for the means, variances, covariance and correlation are provided. Finally, the effect of the parameter, ψ, is investigated by means of a shape analysis; this parameter is introduced via the function $h(x_1, x_2) = e^{-\psi(x_1+x_2)}$ (see Figure 1.1).

7.1 Joint Distribution

In this section, the bivariate Kummer beta type V distribution is derived.

Theorem 7.1 *The pdf of the bivariate Kummer beta type V distribution is given by*

$$g_{BKBV}(x_1, x_2) = Kx_1^{a-1}x_2^{b-1}(1 - x_1 - x_2)^{c-1}(1 + \frac{\beta - \alpha_1}{\alpha_1}x_1 + \frac{\beta - \alpha_2}{\alpha_2}x_2)^{-(a+b+c)}e^{-\psi(x_1+x_2)} \tag{7.1}$$

where $0 \leq x_1, x_2,\ x_1 + x_2 \leq 1,\ a, b, c, \beta, \alpha_1, \alpha_2 > 0,\ -\infty < \psi < \infty$ *and the normalizing constant,* K*, is given by*

$$\begin{aligned} K^{-1} &= B(a,b,c)\sum_{k=0}^{\infty}\frac{(-\psi)^k}{k!}\frac{(a)_k}{(a+b+c)_k}\sum_{j=0}^{\infty}\frac{(-\psi)^j}{j!}\frac{(b)_j}{(a+b+c+k)j} \\ &\quad \times F_1\left(a+b+c, b+j, a+k; a+b+c+k+j; -\frac{\beta-\alpha_1}{\alpha_1}, -\frac{\beta-\alpha_2}{\alpha_2}\right) \end{aligned} \tag{7.2}$$

where $F_1(.)$ *denotes the hypergeometric function of two variables (see Definition B.9). This distribution is denoted as* $(X_1, X_2) \sim BKB^V(a, b, c, \beta, \alpha_1, \alpha_2, \psi)$.

Proof. Definition 2 (see (1.2)) is used to construct the new bivariate Kummer beta type V distribution from the bivariate beta type V kernel (see (1.15)). The pdf of the bivariate Kummer beta type V distribution is then given by

$$\begin{aligned} g_{BKBV}(x_1,x_2) &= K\ f_{BBV}(x_1,x_2)e^{-\psi(x_1+x_2)} \\ &= Kx_1^{a-1}x_2^{b-1}(1-x_1-x_2)^{c-1}(1+\frac{\beta-\alpha_1}{\alpha_1}x_1+\frac{\beta-\alpha_2}{\alpha_2}x_2)^{-(a+b+c)}e^{-\psi(x_1+x_2)} \\ &\qquad \text{(see (1.15))} \end{aligned}$$

with K the normalizing constant. In order to obtain K, the following well-known property of pdf's is used:

$$\begin{aligned} &\int_0^1\int_0^{1-x_2} g_{BKBV}(x_1,x_2)dx_1dx_2 \\ =& \int_0^1\int_0^{1-x_2} Kx_1^{a-1}x_2^{b-1}(1-x_1-x_2)^{c-1}(1+\frac{\beta-\alpha_1}{\alpha_1}x_1+\frac{\beta-\alpha_2}{\alpha_2}x_2)^{-(a+b+c)}e^{-\psi(x_1+x_2)}dx_1dx_2 \\ =& \ 1. \end{aligned}$$

From this, the normalizing constant is obtained as:

$$\begin{aligned} &K^{-1} \\ =& \int_0^1\int_0^{1-x_2} x_1^{a-1}x_2^{b-1}(1-x_1-x_2)^{c-1}(1+\frac{\beta-\alpha_1}{\alpha_1}x_1+\frac{\beta-\alpha_2}{\alpha_2}x_2)^{-(a+b+c)}e^{-\psi(x_1+x_2)}dx_1dx_2 \\ =& \int_0^1 x_2^{b-1}e^{-\psi x_2}\int_0^{1-x_2} x_1^{a-1}(1-x_1-x_2)^{c-1}(1+\frac{\beta-\alpha_1}{\alpha_1}x_1+\frac{\beta-\alpha_2}{\alpha_2}x_2)^{-(a+b+c)}e^{-\psi x_1}dx_1dx_2. \end{aligned}$$

The exponential function is written as an infinite sum using Definition B.7:

$$\begin{aligned} K^{-1} &= \sum_{k=0}^{\infty}\frac{(-\psi)^k}{k!}\int_0^1 x_2^{b-1}e^{-\psi x_2}\int_0^{1-x_2} x_1^{a+k-1}(1-x_1-x_2)^{c-1} \\ &\quad \times(1+\frac{\beta-\alpha_1}{\alpha_1}x_1+\frac{\beta-\alpha_2}{\alpha_2}x_2)^{-(a+b+c)}dx_1dx_2. \end{aligned} \tag{7.3}$$

Since by Relation B.5

$$\begin{aligned} &\int_0^{1-x_2} x_1^{a+k-1}(1-x_1-x_2)^{c-1}(1+\frac{\beta-\alpha_1}{\alpha_1}x_1+\frac{\beta-\alpha_2}{\alpha_2}x_2)^{-(a+b+c)}dx_1 \\ =& \ (1-x_2)^{a+c+k-1}(1+\frac{\beta-\alpha_2}{\alpha_2}x_2)^{-(a+b+c)}B(a+k,c) \\ &\times {}_2F_1\left(a+k,a+b+c;a+c+k;\frac{-\frac{\beta-\alpha_1}{\alpha_1}(1-x_2)}{1+\frac{\beta-\alpha_2}{\alpha_2}x_2}\right), \end{aligned}$$

the expression in (7.3) becomes:

$$\begin{aligned}K^{-1} &= \sum_{k=0}^{\infty}\frac{(-\psi)^k}{k!}\int_0^1 x_2^{b-1}e^{-\psi x_2}(1-x_2)^{a+c+k-1}(1+\frac{\beta-\alpha_2}{\alpha_2}x_2)^{-(a+b+c)}\\ &\quad\times B(a+k,c)_2F_1\left(a+k,a+b+c;a+c+k;\frac{-\frac{\beta-\alpha_1}{\alpha_1}(1-x_2)}{1+\frac{\beta-\alpha_2}{\alpha_2}x_2}\right)dx_2.\end{aligned}$$

Writing out the exponential function and the Gauss hypergeometric function as infinite sums (see Definition B.7 and Definition B.8):

$$\begin{aligned}K^{-1} &= \sum_{k=0}^{\infty}\frac{(-\psi)^k}{k!}B(a+k,c)\sum_{j=0}^{\infty}\frac{(-\psi)^j}{j!}\sum_{m=0}^{\infty}\frac{(a+k)_m(a+b+c)_m}{(a+c+k)_m m!}\\ &\quad\times\left(-\frac{\beta-\alpha_1}{\alpha_1}\right)^m\int_0^1 x_2^{b+j-1}(1-x_2)^{a+c+k+m-1}(1+\frac{\beta-\alpha_2}{\alpha_2}x_2)^{-(a+b+c+m)}dx_2.\end{aligned}$$

Use Relation B.5 to write the integral in terms of the Gauss hypergeometric function, ${}_2F_1(.)$:

$$\begin{aligned}K^{-1} &= \sum_{k=0}^{\infty}\frac{(-\psi)^k}{k!}B(a+k,c)\sum_{j=0}^{\infty}\frac{(-\psi)^j}{j!}\sum_{m=0}^{\infty}\frac{(a+k)_m(a+b+c)_m}{(a+c+k)_m m!}\\ &\quad\times\left(-\frac{\beta-\alpha_1}{\alpha_1}\right)^m B(b+j,a+c+k+m)\\ &\quad\times{}_2F_1\left(b+j,a+b+c+m;a+b+c+k+j+m;-\frac{\beta-\alpha_2}{\alpha_2}\right).\end{aligned}$$

Using Definition B.4 and (B.1) the expression is simplified:

$$\begin{aligned}K^{-1} &= \sum_{k=0}^{\infty}\frac{(-\psi)^k}{k!}\frac{\Gamma(a+k)\Gamma(c)}{\Gamma(a+c+k)}\sum_{j=0}^{\infty}\frac{(-\psi)^j}{j!}\\ &\quad\times\sum_{m=0}^{\infty}\frac{\Gamma(a+k+m)\Gamma(a+b+c+m)\Gamma(a+c+k)\Gamma(b+j)\Gamma(a+c+k+m)}{\Gamma(a+k)\Gamma(a+b+c)\Gamma(a+c+k+m)\Gamma(a+b+c+k+j+m)m!}\\ &\quad\times\left(-\frac{\beta-\alpha_1}{\alpha_1}\right)^m{}_2F_1\left(b+j,a+b+c+m;a+b+c+k+j+m;-\frac{\beta-\alpha_2}{\alpha_2}\right)\\ &= \frac{\Gamma(c)}{\Gamma(a+b+c)}\sum_{k=0}^{\infty}\frac{(-\psi)^k}{k!}\sum_{j=0}^{\infty}\frac{(-\psi)^j}{j!}\sum_{m=0}^{\infty}\frac{\Gamma(a+k+m)\Gamma(a+b+c+m)\Gamma(b+j)}{\Gamma(a+b+c+k+j+m)m!}\\ &\quad\times\left(-\frac{\beta-\alpha_1}{\alpha_1}\right)^m{}_2F_1\left(b+j,a+b+c+m;a+b+c+k+j+m;-\frac{\beta-\alpha_2}{\alpha_2}\right)\end{aligned}$$

$$= \Gamma(c)\sum_{k=0}^{\infty}\frac{(-\psi)^k}{k!}\Gamma(a+k)\sum_{j=0}^{\infty}\frac{(-\psi)^j}{j!}\frac{\Gamma(b+j)}{\Gamma(a+b+c+k+j)}\sum_{m=0}^{\infty}\left(-\frac{\beta-\alpha_1}{\alpha_1}\right)^m$$
$$\times\frac{(a+k)_m(a+b+c)_m}{(a+b+c+k+j)_m m!}{}_2F_1\left(b+j,a+b+c+m;a+b+c+k+j+m;-\frac{\beta-\alpha_2}{\alpha_2}\right).$$

Using Relation B.7, the required result (7.2) is obtained:

$$\begin{aligned}K^{-1} &= \Gamma(c)\sum_{k=0}^{\infty}\frac{(-\psi)^k}{k!}\Gamma(a+k)\sum_{j=0}^{\infty}\frac{(-\psi)^j}{j!}\frac{\Gamma(b+j)}{\Gamma(a+b+c+k+j)}\\ &\quad\times F_1\left(a+b+c,b+j,a+k;a+b+c+k+j;-\frac{\beta-\alpha_2}{\alpha_2},-\frac{\beta-\alpha_1}{\alpha_1}\right)\\ &= B(a,b,c)\sum_{k=0}^{\infty}\frac{(-\psi)^k}{k!}\frac{(a)_k}{(a+b+c)_k}\sum_{j=0}^{\infty}\frac{(-\psi)^j}{j!}\frac{(b)_j}{(a+b+c+k)j}\\ &\quad\times F_1\left(a+b+c,b+j,a+k;a+b+c+k+j;-\frac{\beta-\alpha_2}{\alpha_2},-\frac{\beta-\alpha_1}{\alpha_1}\right).\end{aligned}$$

■

Remark 7.1 *For $\alpha_1=\alpha_2=1$, the pdf of the bivariate Kummer beta type V reduces to that of the bivariate Kummer beta type III. This can be shown by setting $\alpha_1=\alpha_2=1$ in (7.2) and subsequently in (7.1):*

$$\begin{aligned}K^{-1} &= B(a,b,c)\sum_{k=0}^{\infty}\frac{(-\psi)^k}{k!}\frac{(a)_k}{(a+b+c)_k}\sum_{j=0}^{\infty}\frac{(-\psi)^j}{j!}\frac{(b)_j}{(a+b+c+k)_j}\\ &\quad\times F_1\left(a+b+c,b+j,a+k;a+b+c+k+j;-(\beta-1),-(\beta-1)\right).\end{aligned}$$

Using Relation B.9, the normalizing constant in (4.2) is obtained

$$\begin{aligned}K^{-1} &= B(a,b,c)\sum_{k=0}^{\infty}\frac{(-\psi)^k}{k!}\frac{(a)_k}{(a+b+c)_k}\sum_{j=0}^{\infty}\frac{(-\psi)^j}{j!}\frac{(b)_j}{(a+b+c+k)_j}\\ &\quad\times{}_2F_1\left(a+b+c,a+b+j+k;a+b+c+k+j;-(\beta-1)\right).\end{aligned}$$

The pdf in (7.1) then becomes

$$\begin{aligned}&Kx_1^{a-1}x_2^{b-1}(1-x_1-x_2)^{c-1}(1+\frac{\beta-\alpha_1}{\alpha_1}x_1+\frac{\beta-\alpha_2}{\alpha_2}x_2)^{-(a+b+c)}e^{-\psi(x_1+x_2)}\\ =\;&Kx_1^{a-1}x_2^{b-1}(1-x_1-x_2)^{c-1}(1+(\beta-1)x_1+(\beta-1)x_2)^{-(a+b+c)}e^{-\psi(x_1+x_2)}\end{aligned}$$

which is the pdf of the bivariate Kummer beta type III in (4.1).

Remark 7.2 *The non-central Kummer beta type V distribution can be obtained by using* *the non-central beta type V distribution, the latter which was studied by Ehlers et al. [15] with pdf given by*

$$\left(\left(\frac{\beta}{\alpha_1}\right)^{-a}\left(\frac{\beta}{\alpha_2}\right)^{-b}B(a,b,c)\right)^{-1}x_1^{a-1}x_2^{b-1}(1-x_1-x_2)^{c-1}\left(1+\left(\frac{\beta-\alpha_1}{\alpha_1}\right)x_1\right.$$
$$\left.+\left(\frac{\beta-\alpha_2}{\alpha_2}\right)x_2\right)^{-(a+b+c)}e^{-\delta}{}_1F_1\left(a+b+c;c;\delta\frac{1-x_1-x_2}{\left(1+\left(\frac{\beta-\alpha_1}{\alpha_1}\right)x_1+\left(\frac{\beta-\alpha_2}{\alpha_2}\right)x_2\right)^{-(a+b+c)}}\right) \tag{7.4}$$

with kernel

$$\begin{aligned} f_{NCBBV}(x_1,x_2) &= x_1^{a-1}x_2^{b-1}(1-x_1-x_2)^{c-1}\left(1+\left(\frac{\beta-\alpha_1}{\alpha_1}\right)x_1+\left(\frac{\beta-\alpha_2}{\alpha_2}\right)x_2\right)^{-(a+b+c)} \\ &\quad \times{}_1F_1\left(a+b+c;c;\delta\frac{1-x_1-x_2}{\left(1+\left(\frac{\beta-\alpha_1}{\alpha_1}\right)x_1+\left(\frac{\beta-\alpha_2}{\alpha_2}\right)x_2\right)^{-(a+b+c)}}\right) \end{aligned} \tag{7.5}$$

for $x_1,x_2\geq 0$, $x_1+x_2\leq 1$, a, b, c, β, α_1, $\alpha_2>0$ and where $\delta\geq 0$ denotes the non-centrality parameter. The pdf of the non-central Kummer beta type V is constructed using Definition 2 with $f(x_1,x_2)$ given by (7.5) and is given by

$$\begin{aligned} & g_{NCBKBV}(x_1,x_2) \\ =\ & K_{NC}x_1^{a-1}x_2^{b-1}(1-x_1-x_2)^{c-1}\left(1+\left(\frac{\beta-\alpha_1}{\alpha_1}\right)x_1+\left(\frac{\beta-\alpha_2}{\alpha_2}\right)x_2\right)^{-(a+b+c)} \\ & \times{}_1F_1\left(a+b+c;c;\delta\frac{1-x_1-x_2}{\left(1+\left(\frac{\beta-\alpha_1}{\alpha_1}\right)x_1+\left(\frac{\beta-\alpha_2}{\alpha_2}\right)x_2\right)^{-(a+b+c)}}\right) \end{aligned} \tag{7.6}$$

where $0\leq x_1,x_2$, $x_1+x_2\leq 1$, $a,b,c,\beta,\alpha_1,\alpha_2>0$, $-\infty<\psi<\infty$, $\delta\geq 0$ denotes the non-centrality parameter and the normalizing constant, K_{NC}, is given by

$$\begin{aligned} K_{NC}^{-1} &= B(a,b,c)\sum_{j=0}^{\infty}\frac{\delta^j}{j!}\sum_{k=0}^{\infty}\frac{(-\psi)^k}{k!}\frac{(a)_k}{(a+b+c+j)_k}\sum_{l=0}^{\infty}\frac{(-\psi)l}{l!}\frac{(b)_l}{(a+b+c+k+j)_l} \\ &\quad \times F_1\left(a+b+c+j,b+l,a+k;a+b+c+k+j+l;-\frac{\beta-\alpha_2}{\alpha_2},-\frac{\beta-\alpha_1}{\alpha_1}\right). \end{aligned} \tag{7.7}$$

This distribution is denoted as $(X_1,X_2)\sim NCBKB^V(a,b,c,\beta,\alpha_1,\alpha_2,\psi)$.

7.2 Marginal and Conditional Distributions

In this section the marginal, $m(x_i)$, and the conditional, $c(x_i|x_j)$, pdf's of the bivariate Kummer beta type V distribution are derived.

Theorem 7.2 *If $(X_1, X_2) \sim BKB^V(a, b, c, \beta, \alpha_1, \alpha_2, \psi)$, the marginal pdf of X_1 is given by*

$$\begin{aligned} m(x_1) &= Kx_1^{a-1}e^{-\psi x_1}\left(1+\frac{\beta-\alpha_1}{\alpha_1}x_1\right)^{-(a+b+c)}\sum_{j=0}^{\infty}B(b+j,c)\frac{(-\psi)^j}{j!}(1-x_1)^{b+c+j-1} \\ &\times {}_2F_1\left(b+j, a+b+c; b+c+j; \frac{-\frac{\beta-\alpha_2}{\alpha_2}(1-x_1)}{1+\frac{\beta-\alpha_1}{\alpha_1}x_1}\right) \end{aligned} \tag{7.8}$$

where $0 \le x_1 \le 1$, $a, b, c, \beta, \alpha_1, \alpha_2 > 0$, $-\infty < \psi < \infty$ and K as defined in (7.2).

Proof. Using (7.1), $m(x_1)$ given in (7.8), is obtained by using Definition B.7 and Relation B.5:

$$\begin{aligned} &m(x_1) \\ &= \int_0^{1-x_1} Kx_1^{a-1}x_2^{b-1}(1-x_1-x_2)^{c-1}(1+\frac{\beta-\alpha_1}{\alpha_1}x_1+\frac{\beta-\alpha_2}{\alpha_2}x_2)^{-(a+b+c)}e^{-\psi(x_1+x_2)}dx_2 \\ &= Kx_1^{a-1}e^{-\psi x_1}\int_0^{1-x_1} x_2^{b-1}(1-x_1-x_2)^{c-1}(1+\frac{\beta-\alpha_1}{\alpha_1}x_1+\frac{\beta-\alpha_2}{\alpha_2}x_2)^{-(a+b+c)}e^{-\psi x_2}dx_2 \\ &= Kx_1^{a-1}e^{-\psi x_1}\sum_{j=0}^{\infty}\frac{(-\psi)^j}{j!}\int_0^{1-x_1} x_2^{b+j-1}(1-x_1-x_2)^{c-1}(1+\frac{\beta-\alpha_1}{\alpha_1}x_1+\frac{\beta-\alpha_2}{\alpha_2}x_2)^{-(a+b+c)}dx_2 \\ &= Kx_1^{a-1}e^{-\psi x_1}\sum_{j=0}^{\infty}\frac{(-\psi)^j}{j!}(1-x_1)^{b+c+j-1}\left(1+\frac{\beta-\alpha_1}{\alpha_1}x_1\right)^{-(a+b+c)}B(b+j,c) \\ &\times {}_2F_1\left(b+j, a+b+c; b+c+j; \frac{-\frac{\beta-\alpha_2}{\alpha_2}(1-x_1)}{1+\frac{\beta-\alpha_1}{\alpha_1}x_1}\right) \\ &= Kx_1^{a-1}e^{-\psi x_1}\left(1+\frac{\beta-\alpha_1}{\alpha_1}x_1\right)^{-(a+b+c)}\sum_{j=0}^{\infty}B(b+j,c)\frac{(-\psi)^j}{j!}(1-x_1)^{b+c+j-1} \\ &\times {}_2F_1\left(b+j, a+b+c; b+c+j; \frac{-\frac{\beta-\alpha_2}{\alpha_2}(1-x_1)}{1+\frac{\beta-\alpha_1}{\alpha_1}x_1}\right). \end{aligned}$$

■

Remark 7.3 *Note that the marginal pdf of X_2 is obtained by substituting x_2 for x_1 in (7.8), interchanging the parameters a and b and interchanging the parameters α_1 and α_2.*

Theorem 7.3 *If* $(X_1, X_2) \sim BKB^V(a, b, c, \beta, \alpha_1, \alpha_2, \psi)$, *the conditional pdf of* $X_2|X_1 = x_1$ *is given by*

$$c(x_2|x_1) = Dx_2^{b-1}(1 - x_1 - x_2)^{c-1}(1 + \frac{\beta - \alpha_1}{\alpha_1}x_1 + \frac{\beta - \alpha_2}{\alpha_2}x_2)^{-(a+b+c)}e^{-\psi x_2} \tag{7.9}$$

where $0 \leq x_2 \leq 1 - x_1$, $a, b, c, \beta, \alpha_1, \alpha_2 > 0$, $-\infty < \psi < \infty$ *and the normalizing constant* D *is defined as*

$$\begin{aligned} D^{-1} &= \left(1 + \frac{\beta - \alpha_1}{\alpha_1}x_1\right)^{-(a+b+c)} \sum_{j=0}^{\infty} B(b + j, c)\frac{(-\psi)^j}{j!}(1 - x_1)^{b+c+j-1} \\ &\quad \times {}_2F_1\left(b + j, a + b + c; b + c + j; \frac{-\frac{\beta-\alpha_2}{\alpha_2}(1 - x_1)}{1 + \frac{\beta-\alpha_1}{\alpha_1}x_1}\right). \end{aligned}$$

Proof. Using the joint pdf, $g_{BKBV}(x_1, x_2)$, in (7.1) and the marginal pdf, $m(x_1)$, in (7.8), expression (7.9) for the conditional pdf of $X_2|X_1 = x_1$ follows directly:

$$\begin{aligned} c(x_2|x_1) &= \frac{g_{BKBV}(x_1, x_2)}{m(x_1)} \\ &= \frac{Kx_1^{a-1}x_2^{b-1}(1 - x_1 - x_2)^{c-1}(1 + \frac{\beta-\alpha_1}{\alpha_1}x_1 + \frac{\beta-\alpha_2}{\alpha_2}x_2)^{-(a+b+c)}e^{-\psi(x_1+x_2)}}{Kx_1^{a-1}e^{-\psi x_1}\left(1 + \frac{\beta-\alpha_1}{\alpha_1}x_1\right)^{-(a+b+c)} \sum_{j=0}^{\infty} B(b + j, c)\frac{(-\psi)^j}{j!}(1 - x_1)^{b+c+j-1}} \\ &\quad \times \frac{1}{{}_2F_1\left(b + j, a + b + c; b + c + j; \frac{-\frac{\beta-\alpha_2}{\alpha_2}(1-x_1)}{1+\frac{\beta-\alpha_1}{\alpha_1}x_1}\right)} \\ &= \frac{x_2^{b-1}(1 - x_1 - x_2)^{c-1}(1 + \frac{\beta-\alpha_1}{\alpha_1}x_1 + \frac{\beta-\alpha_2}{\alpha_2}x_2)^{-(a+b+c)}e^{-\psi x_2}}{\left(1 + \frac{\beta-\alpha_1}{\alpha_1}x_1\right)^{-(a+b+c)} \sum_{j=0}^{\infty} B(b + j, c)\frac{(-\psi)^j}{j!}(1 - x_1)^{b+c+j-1}} \\ &\quad \times \frac{1}{{}_2F_1\left(b + j, a + b + c; b + c + j; \frac{-\frac{\beta-\alpha_2}{\alpha_2}(1-x_1)}{1+\frac{\beta-\alpha_1}{\alpha_1}x_1}\right)} \\ &= Dx_2^{b-1}(1 - x_1 - x_2)^{c-1}(1 + \frac{\beta - \alpha_1}{\alpha_1}x_1 + \frac{\beta - \alpha_2}{\alpha_2}x_2)^{-(a+b+c)}e^{-\psi x_2}. \end{aligned}$$

■

Remark 7.4 *Note that the conditional pdf of* $X_2|X_1 = x_1$ *is obtained by interchanging the variables* x_1 *and* x_2, *the parameters* a *and* b *and the parameters* α_1 *and* α_2 *in (7.9).*

7.3 Moments and Correlation

In this section the product moment is derived from which expressions for the mean and variance of X_1 and X_2 as well as the covariance and correlation of (X_1, X_2) are obtained. Furthermore, the effect of the parameter ψ on the correlation between X_1 and X_2 is investigated.

Theorem 7.4 *If* $(X_1, X_2) \sim BKB^V(a, b, c, \beta, \alpha_1, \alpha_2, \psi)$, *the product moment, i.e.* $E(X_1^r X_2^s)$, *equals*

$$\begin{aligned}
& KB(a+r, b+s, c) \sum_{k=0}^{\infty} \frac{(-\psi)^k}{k!} \frac{(a+r)_k}{(a+b+c+r+s)_k} \sum_{j=0}^{\infty} \frac{(-\psi)^j}{j!} \frac{(b+s)_j}{(a+b+c+k+r+s)_j} \\
& \times F_1\left(a+b+c, b+j+s, a+k+r, a+b+c+k+j+r+s; -\frac{\beta-\alpha_2}{\alpha_2}, -\frac{\beta-\alpha_1}{\alpha_1}\right) \\
= \; & (A(a, b, c, \beta, \alpha_1, \alpha_2, 0, 0))^{-1} \times A(a, b, c, \beta, \alpha_1, \alpha_2, r, s) \qquad (7.10)
\end{aligned}$$

where

$$\begin{aligned}
& A(a, b, c, \beta, \alpha_1, \alpha_2, r, s) \\
= \; & B(a+r, b+s, c) \sum_{k=0}^{\infty} \frac{(-\psi)^k}{k!} \frac{(a+r)_k}{(a+b+c+r+s)_k} \sum_{j=0}^{\infty} \frac{(-\psi)^j}{j!} \frac{(b+s)_j}{(a+b+c+k+r+s)_j} \\
& \times F_1\left(a+b+c, b+j+s, a+k+r, a+b+c+k+j+r+s; -\frac{\beta-\alpha_2}{\alpha_2}, -\frac{\beta-\alpha_1}{\alpha_1}\right) \qquad (7.11)
\end{aligned}$$

with $A(a, b, c, \beta, \alpha_1, \alpha_2, 0, 0)^{-1} = K$ *as defined in (7.2). Note that the definition of* $A(.)$ *must be read in context as its definition depends on the distribution used.*

Proof. Using the pdf of the bivariate Kummer beta type V distribution (7.1), the expected value of $X_1^r X_2^s$ is taken:

$$\begin{aligned}
& E(X_1^r X_2^s) \\
= \; & \int_0^1 \int_0^{1-x_2} x_1^r x_2^s g_{BKBV}(x_1, x_2) dx_1 dx_2 \\
= \; & \int_0^1 \int_0^{1-x_2} x_1^r x_2^s K x_1^{a-1} x_2^{b-1} (1-x_1-x_2)^{c-1} (1 + \frac{\beta-\alpha_1}{\alpha_1} x_1 + \frac{\beta-\alpha_2}{\alpha_2} x_2)^{-(a+b+c)} e^{-\psi(x_1+x_2)} dx_1 dx_2
\end{aligned}$$

$$= K\int_0^1\int_0^{1-x_2} x_1^{a+r-1}x_2^{b+s-1}(1-x_1-x_2)^{c-1}(1+\frac{\beta-\alpha_1}{\alpha_1}x_1+\frac{\beta-\alpha_2}{\alpha_2}x_2)^{-(a+b+c)}e^{-\psi(x_1+x_2)}dx_1dx_2. \tag{7.12}$$

Because the integrals in (7.12) are similar to those found in the proof of Theorem 7.1, the desired expression can be obtained as

$$\begin{aligned}
& E(X_1^rX_2^s) \\
= \ & KB(a+r,b+s,c)\sum_{k=0}^{\infty}\frac{(-\psi)^k}{k!}\frac{(a+r)_k}{(a+b+c+r+s)_k}\sum_{j=0}^{\infty}\frac{(-\psi)^j}{j!}\frac{(b+s)_j}{(a+b+c+k+r+s)_j} \\
& \times F_1\left(a+b+c,b+s+j,a+k+r;a+b+c+k+j+r+s;-\frac{\beta-\alpha_1}{\alpha_1},-\frac{\beta-\alpha_2}{\alpha_2}\right) \\
= \ & (A(a,b,c,\beta,\alpha_1,\alpha_2,0,0))^{-1}\times A(a,b,c,\beta,\alpha_1,\alpha_2,r,s)
\end{aligned}$$

with $A(a,b,c,\beta,\alpha_1,\alpha_2,r,s)$ as defined in (7.11). ■

It follows from (7.10):

$$\begin{aligned}
E(X_1) &= (A(a,b,c,\beta,\alpha_1,\alpha_2,0,0))^{-1}\times A(a,b,c,\beta,\alpha_1,\alpha_2,1,0), \\
E(X_1^2) &= (A(a,b,c,\beta,\alpha_1,\alpha_2,0,0))^{-1}\times A(a,b,c,\beta,\alpha_1,\alpha_2,2,0), \\
E(X_2) &= (A(a,b,c,\beta,\alpha_1,\alpha_2,0,0))^{-1}\times A(a,b,c,\beta,\alpha_1,\alpha_2,0,1), \\
E(X_2^2) &= (A(a,b,c,\beta,\alpha_1,\alpha_2,0,0))^{-1}\times A(a,b,c,\beta,\alpha_1,\alpha_2,0,2), \\
E(X_1X_2) &= (A(a,b,c,\beta,\alpha_1,\alpha_2,0,0))^{-1}\times A(a,b,c,\beta,\alpha_1,\alpha_2,1,1), \\
Var(X_1) &= E(X_1^2)-(E(X_1))^2, \\
Var(X_2) &= E(X_2^2)-(E(X_2))^2 \\
\text{and} \quad Cov(X_1,X_2) &= E(X_1X_2)-E(X_1)E(X_2).
\end{aligned}$$

Figure 7.1 shows the correlation between X_1 and X_2 as ψ varies where $(X_1,X_2)\sim BKB^V(a,b,c,\psi)$. The graph shows the correlation as a function of $\psi\in[-10,10]$ for certain combinations of values for the parameters a, b and c. Note that positive correlation is obtained when $0<c<1$ and some positive values of ψ. The value of ψ from which the correlation becomes positive depends on the values of a, b, c, β, α_1 and α_2. Note that the correlations for $\psi=0$ are those of the bivariate beta type V distribution. For the effect of the parameters a, b, c, β, α_1 and α_2 see Ehlers ([13]). More specifically, Ehlers et al. [15] discussed the role of α_1 and α_2 in the bivariate beta type V model. One can observe here that ψ also has the effect of positive correlation between the dependent components X_1 and X_2 for this model (7.1).

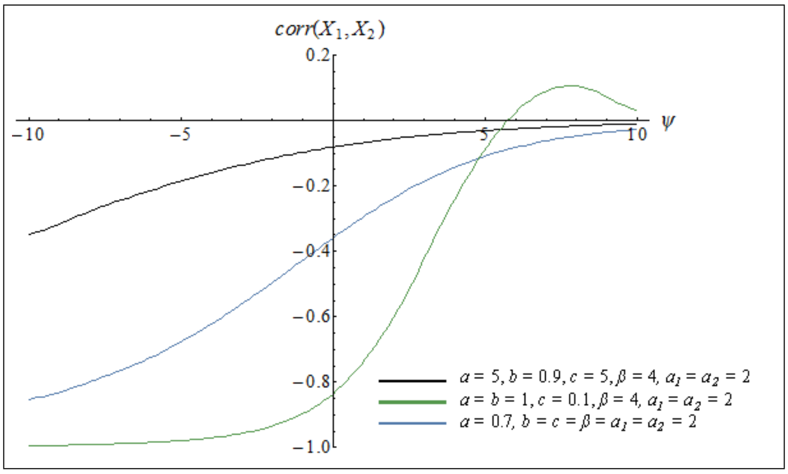


Figure 7.1: Correlation of the bivariate Kummer beta type V distribution as ψ varies

Remark 7.5 *Similar to Remark 2.5, the definition for PQD (see (2.12)) can be used to determine for which parameter values the bivariate Kummer beta type V has positive correlation.*

7.4 Shape Analysis

The effect of the parameters a, b, c, β, α_1 and α_2 on the bivariate beta type V distribution was studied by Ehlers [13]. In this study, therefore, only the effect of the new parameter, ψ, is considered.

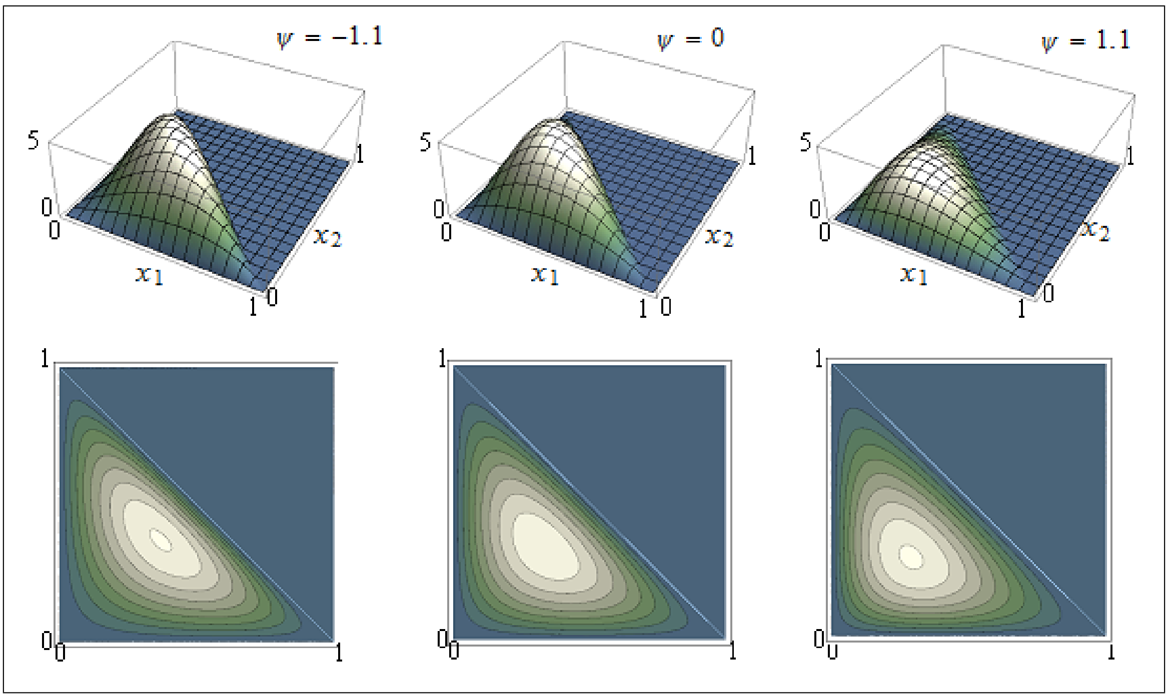


Figure 7.2: Joint pdf of the bivariate Kummer beta type V distribution

In Figure 7.2 the joint pdf and contour plots for the bivariate Kummer beta type V distribution whose pdf is given by (7.1) are shown. The effect of ψ is illustrated by setting $\psi = -1.1$, 0 and 1.1 and keeping the other parameters fixed at $a = b = c = \beta = \alpha_1 = \alpha_2 = 2$. Note that the value of ψ does not change the shape of the pdf. It does, however, change the kurtosis. Note that the middle graph (for $\psi = 0$), is the graph of the bivariate beta type V whose pdf is given by (1.14).

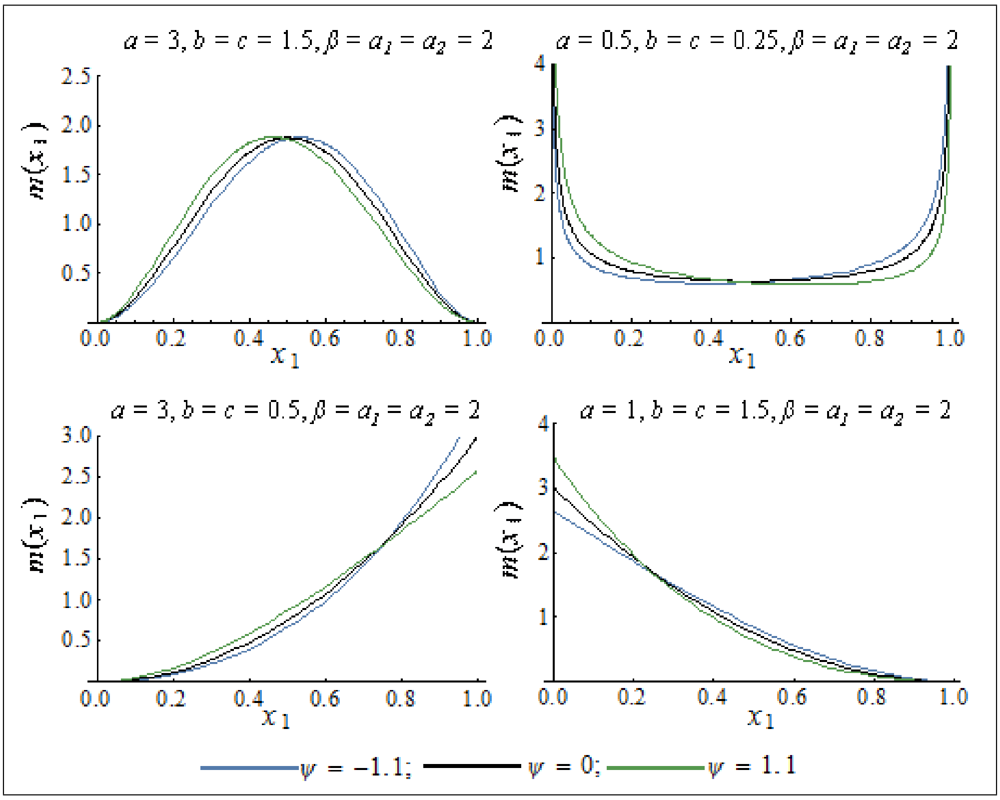


Figure 7.3: Marginal pdf of the bivariate Kummer beta type V distribution

Figure 7.3 displays the marginal pdf (see (7.8)) for the bivariate Kummer beta type V distribution. The four graphs illustrate four different shapes of a univariate beta type V distribution. For each of the four shapes the effect of ψ is studied. Note that in the top two graphs, ψ changes the skewness of the pdf. In the bottom two graphs, ψ changes the shape of the graphs. For the negatively skewed graph, a negative value of ψ gives the pdf a concave shape and a positive value of ψ gives the pdf a convex shape. In the positively skewed graph, the graph remains concave, but ψ changes the intensity of the concavity.

7.5 Summary

A summary of the newly derived pdf's of Chapter 7 is given in Table 7.1.

Type	pdf	Equation number
BKB^V	$Kx_1^{a-1}x_2^{b-1}(1-x_1-x_2)^{c-1}(1+\frac{\beta-\alpha_1}{\alpha_1}x_1+\frac{\beta-\alpha_2}{\alpha_2}x_2)^{-(a+b+c)}e^{-\psi(x_1+x_2)}$	(7.1)
$NCBKB^V$	$K_{NC}x_1^{a-1}x_2^{b-1}(1-x_1-x_2)^{c-1}\left(1+\left(\frac{\beta-\alpha_1}{\alpha_1}\right)x_1+\left(\frac{\beta-\alpha_2}{\alpha_2}\right)x_2\right)^{-(a+b+c)}$ $\times{}_1F_1\left(a+b+c;c;\delta\frac{1-x_1-x_2}{\left(1+\left(\frac{\beta-\alpha_1}{\alpha_1}\right)x_1+\left(\frac{\beta-\alpha_2}{\alpha_2}\right)x_2\right)^{-(a+b+c)}}\right)$	(7.6)
$m(x_1)$ of the BKB^V	$m(x_1)=Kx_1^{a-1}e^{-\psi x_1}\left(1+\frac{\beta-\alpha_1}{\alpha_1}x_1\right)^{-(a+b+c)}\sum_{j=0}^{\infty}B(b+j,c)\frac{(-\psi)^j}{j!}(1-x_1)^{b+c+j-1}$ $\times{}_2F_1\left(b+j,a+b+c;b+c+j;\frac{-\frac{\beta-\alpha_2}{\alpha_2}(1-x_1)}{1+\frac{\beta-\alpha_1}{\alpha_1}x_1}\right)$	(7.8)
$m(x_2)$ of the BKB^V	$m(x_2)=Kx_2^{b-1}e^{-\psi x_1}\left(1+\frac{\beta-\alpha_2}{\alpha_2}x_2\right)^{-(a+b+c)}\sum_{j=0}^{\infty}B(a+j,c)\frac{(-\psi)^j}{j!}(1-x_2)^{a+c+j-1}$ $\times{}_2F_1\left(a+j,a+b+c;a+c+j;\frac{-\frac{\beta-\alpha_1}{\alpha_1}(1-x_2)}{1+\frac{\beta-\alpha_2}{\alpha_2}x_2}\right)$	(7.8)
$c(x_1\|x_2)$ of the BKB^V	$c(x_1\|x_2)=Dx_1^{a-1}(1-x_1-x_2)^{c-1}(1+\frac{\beta-\alpha_1}{\alpha_1}x_1+\frac{\beta-\alpha_2}{\alpha_2}x_2)^{-(a+b+c)}e^{-\psi x_1}$	(7.9)
$c(x_2\|x_1)$ of the BKB^V	$c(x_2\|x_1)=Dx_2^{b-1}(1-x_2-x_1)^{c-1}(1+\frac{\beta-\alpha_2}{\alpha_2}x_2+\frac{\beta-\alpha_1}{\alpha_1}x_1)^{-(a+b+c)}e^{-\psi x_2}$	(7.9)

Table 7.1: Pdf's derived in Chapter 7

Chapter 8

Examples of Applications

In this chapter, two examples of possible applications are discussed. A well-known area of application for bivariate distributions is the stress-strength model in reliability; this is discussed using the bivariate Kummer beta type I distribution. The second example concerns the beta-binomial distribution; in this case a univariate distribution is needed and the marginal pdf of the Kummer beta type I is used in this application. The latter application provides a fusion between discrete and continuous distributions.

8.1 Reliability / Stress-Strength models

8.1.1 Background on Stress-Strength Model

Reliability is a term that arises frequently in discussions on bivariate distributions, because it provides an often used application for bivariate distributions. In this research, it is again used as an example of an application. First, some background is provided on the stress-strength model and then the reliability of the bivariate Kummer beta type IV distribution is derived.

The work of Kotz et al. [32] provides a comprehensive exposition on reliability and the stress-strength model. In order to explain what the stress-strength model is, consider the example given by Kotz et al. ([32], p210).

Suppose we have structural components of a mechanism which are mass produced. The strength of each individual component, X_2, i.e. the stress at which this component will fail, may be considered a random variable. The components are installed in assembly and exposed to a stress, with maximum value X_1. This maximum value of the stress, X_1 is a random variable. Then $P(X_2 < X_1)$ is the probability that failure will occur because, due to chance, a component with relatively low strength was subjected to a high stress.

Similarly, one could consider $P(X_1 < X_2)$, which is the probability that the component will not fail. In other words, $P(X_1 < X_2)$ is a measure of how reliable the component is. The probability $P(X_1 < X_2)$ is then often referred to as the reliability of the component. In the ideal case, when the distributions of the strength, X_2, and the stress, X_1, are known, one could calculate the theoretical reliability. The two random variables X_1 and X_2 are often taken to have a bivariate distribution. The reliability can, alternatively, be expressed as $P(X_1 < X_2) = P(\frac{X_1}{X_2} < 1) = P(R < 1)$. In this case, the distribution of the ratio of X_1 and X_2, $R = \dfrac{X_1}{X_2}$, is derived and the reliability can be easily calculated.

There are many examples where the stress-strength model is used. Examples mentioned by Johnson [25] include:

1. **Rocket engines:** Let X_1 represent the maximum chamber pressure generated by ignition of a solid propellant, and X_2 be the strength of the rocket chamber. Then the reliability is the probability of a successful firing of the engine without blowing up.

2. **Comparing two treatments:** Consider a design for comparing two drugs where Drug A is assigned to one group of subjects and Drug B is assigned to another independent group. Let X_1 and X_2 be the remission times with Drug A and Drug B, respectively. We can consider $P(X_1 < X_2)$ as an indication of the superiority of one drug over the other. Although the name "stress-strength" is not appropriate in this context, our target measure is the same, namely the reliability, $P(R < 1)$.

3. **Threshold response model:** A unit, say a receptor in an alarm sensor, operates only if it is stimulated by a source whose random magnitude, X_2, is greater than a random lower threshold for the unit. In this case the reliability is the probability that the unit operates, i.e. $P(X_1 < X_2) = P[\text{unit operates}]$.

8.1.2 Reliability using the Bivariate Kummer Beta Type IV

In this section, the pdf of $R = \dfrac{X_1}{X_2}$ where (X_1, X_2) has the bivariate Kummer beta type IV distribution is derived; followed by the calculation of spesific values of the reliability, $P(R < 1)$.

Theorem 8.1 *Let $R = \dfrac{X_1}{X_2}$ where $(X_1, X_2) \sim BKB^{IV}(a, b, c, \psi)$, then the pdf of R is given by*

$$w(r) = K\Gamma(a+c)\Gamma(b+c)\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{\infty}(a+b+c)_k\frac{(-\psi)^{j+l}}{j!k!l!}G_{2,2}^{1,1}\left(r\,\middle|\,\begin{matrix}\alpha_1,\alpha_2\\ \beta_1,\beta_2\end{matrix}\right) \quad \text{for } r \geq 0 \tag{8.1}$$

where

$$-\alpha_1 = b+k+j \qquad \alpha_2 = a+b+c+k+l-1$$
$$\beta_1 = a+k+l-1 \qquad -\beta_2 = a+b+c+k+j$$

and where $G^{1,1}_{2,2}\left(r \,\middle|\, \begin{matrix}\alpha_1,\alpha_2\\ \beta_1,\beta_2\end{matrix}\right)$ *is Meijer's G-function (see Definition B.11) and* K *as defined in (5.2).*

Proof. Setting $r = h-1$ and $s = 1-h$ in (5.10) and using (B.5) an expression for the Mellin transform of $g(x_1,x_2)$ (see Definition B.12) is obtained:

$$\begin{aligned}
& M_g(h) \\
= \;& E(R^{h-1}) = E\left(\left(\frac{X_1}{X_2}\right)^{h-1}\right) \\
= \;& (A(a,b,c,0,0))^{-1} \times A(a,b,c,h-1,1-h) \\
= \;& K\sum_{k=0}^{\infty}\frac{(a+b+c)_k}{k!}\frac{{}_1F_1\left(a+k+h-1,a+b+c+k+h-1;-\psi\right)}{\left(B\left(a+c,b+k+1-h\right)\right)^{-1}} \\
& \times\frac{{}_1F_1\left(b+k+1-h,a+b+c+k+1-h;-\psi\right)}{\left(B\left(b+c,a+k+h-1\right)\right)^{-1}} \\
= \;& K\sum_{k=0}^{\infty}\frac{\Gamma(a+b+c+k)}{\Gamma(a+b+c)k!}\sum_{l=0}^{\infty}\frac{(a+k+h-1)_l(-\psi)^l}{(a+b+c+k+h-1)_l l!} \\
& \times\frac{{}_1F_1\left(b+k-h+1;a+b+c+k-h+1;-\psi\right)}{\left(B\left(b+c,a+k+h-1\right)B\left(a+c,b+k-h+1\right)\right)^{-1}} \\
= \;& K\sum_{k=0}^{\infty}\frac{\Gamma(a+b+c+k)}{\Gamma(a+b+c)k!}\sum_{l=0}^{\infty}\frac{(a+k+h-1)_l(-\psi)^l}{(a+b+c+k+h-1)_l l!}\sum_{j=0}^{\infty}\frac{(b+k-h+1)_j(-\psi)^j}{(a+b+c+k-h+1)_j j!} \\
& \times B\left(b+c,a+k+h-1\right)B\left(a+c,b+k-h+1\right)).
\end{aligned}$$

Use (B.1) and (B.4):

$$\begin{aligned}
& M_g(h) \\
= \;& K\sum_{k=0}^{\infty}\frac{\Gamma(a+b+c+k)}{\Gamma(a+b+c)k!}\sum_{l=0}^{\infty}\frac{\Gamma(a+k+h-1+l)\Gamma(a+b+c+k+h-1)(-\psi)^l}{\Gamma(a+k+h-1)\Gamma(a+b+c+k+h-1+l)l!} \\
& \times\sum_{j=0}^{\infty}\frac{\Gamma(b+k-h+1+j)\Gamma(a+b+c+k-h+1)(-\psi)^j}{\Gamma(b+k-h+1)\Gamma(a+b+c+k-h+1+j)j!} \\
& \times\frac{\Gamma(b+c)\Gamma(a+k+h-1)}{\Gamma(a+b+c+k+h-1)}\frac{\Gamma(a+c)\Gamma(b+k-h+1)}{\Gamma(a+b+c+k-h+1)}.
\end{aligned}$$

After simplification the following expression follows:

$$\begin{aligned}
&M_g(h)\\
=\;& K\frac{\Gamma(b+c)\Gamma(a+c)}{\Gamma(a+b+c)}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{\infty}\frac{\Gamma(a+b+c+k)(-\psi)^{j+l}}{k!l!j!}\frac{\Gamma(a+k+h-1+l)}{\Gamma(a+b+c+k+h-1+l)}\\
&\times\frac{\Gamma(b+k-h+1+j)}{\Gamma(a+b+c+k-h+1+j)}\\
=\;& K\frac{\Gamma(b+c)\Gamma(a+c)}{\Gamma(a+b+c)}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{\infty}\Gamma(a+b+c+k)\\
&\times\frac{\Gamma(b+k-h+1+j)\Gamma(a+k+h-1+l)}{\Gamma(a+b+c+k-h+1+j)\Gamma(a+b+c+k+h-1+l)}\frac{(-\psi)^{j+l}}{j!k!l!}\\
=\;& K\frac{\Gamma(b+c)\Gamma(a+c)}{\Gamma(a+b+c)}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{\infty}\Gamma(a+b+c+k)\frac{\Gamma(1-\alpha_1-h)\,\Gamma(\beta_1+h)}{\Gamma(1-\beta_2-h)\,\Gamma(\alpha_2+h)}\frac{(-\psi)^{j+l}}{j!k!l!}
\end{aligned}$$

with

$$\begin{aligned}
-\alpha_1 &= b+k+j & \alpha_2 &= a+b+c+k+l-1\\
\beta_1 &= a+k+l-1 & -\beta_2 &= a+b+c+k+j.
\end{aligned}$$

Using the inverse Mellin transform (see Definition B.12), the pdf of the ratio, R, of the correlated components of the bivariate Kummer beta type IV distribution is given in terms of the Meijer's G-function (see Definition B.11) as follows:

$$\begin{aligned}
w(r) &= \frac{1}{2\pi i}\int E\left(R\right)^{h-1}\left(R\right)^{-h}dh\\
&= \frac{1}{2\pi i}\int K\frac{\Gamma(b+c)\Gamma(a+c)}{\Gamma(a+b+c)}\sum_{k}\sum_{l}\sum_{j}\Gamma(a+b+c+k)\\
&\quad\times\frac{\Gamma(1-\alpha_1-h)\,\Gamma(\beta_1+h)}{\Gamma(1-\beta_2-h)\,\Gamma(\alpha_2+h)}\frac{(-\psi)^{j+l}}{j!k!l!}R^{-h}dh\\
&= K\frac{\Gamma(b+c)\Gamma(a+c)}{\Gamma(a+b+c)}\sum_{k}\sum_{l}\sum_{j}\Gamma(a+b+c+k)\frac{(-\psi)^{j+l}}{j!k!l!}\frac{1}{2\pi i}\\
&\quad\times\int\frac{\prod_{j=1}^{1}\Gamma(1-\alpha_j-h)\prod_{j=1}^{1}\Gamma\left(\beta_j+h\right)}{\prod_{j=2}^{2}\Gamma\left(1-\beta_j-h\right)\prod_{j=2}^{2}\Gamma(\alpha_j+h)}\left(R\right)^{-h}dh\\
&= K\frac{\Gamma(b+c)\Gamma(a+c)}{\Gamma(a+b+c)}\sum_{k}\sum_{l}\sum_{j}\Gamma(a+b+c+k)\frac{(-\psi)^{j+l}}{j!k!l!}G_{2,2}^{1,1}\left(r\,\middle|\,\begin{matrix}\alpha_1,\alpha_2\\ \beta_1,\beta_2\end{matrix}\right).
\end{aligned}$$

■

Nagar et al. [46] studied the product and ratio of the bivariate beta type IV distribution. The effect of the parameters a, b and c on the ratio, R, of the correlated components of the bivariate beta type IV distribution was studied by Pienaar ([51] p25-31). Therefore, only the effect of the new parameter, ψ, is investigated. Figure 8.1 illustrates the shape of the pdf of $R = \frac{X_1}{X_2}$ (see (8.1)) for the case $a = 1, b = c = 2$ and $a = b = c = 2$ for different values of ψ. The domain for these graphs is $\mathbb{R} : [0, \infty]$. In the left graph we see that the value of ψ affects the shape of the pdf, especially in the left tail of the distribution. In the right graph we see that the value of ψ only changes the kurtosis with a positive value of ψ decreasing kurtosis and a negative value of ψ increasing kurtosis.

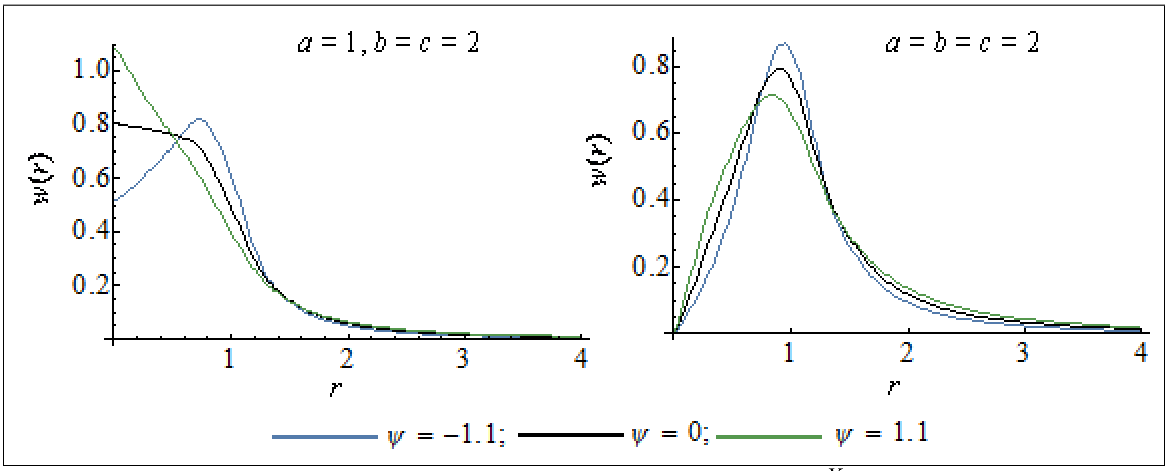


Figure 8.1: The pdf of the ratio $R = \frac{X_1}{X_2}$

Having obtained the distribution of R, i.e. the ratio, reliability values can be calculated. Using $P(R < 1) = \int_0^1 w(r)dr$ as the expression for the reliability, the computer software package Mathematica was used to obtain the values in Table 8.1 for $\psi = -1.1$, 0 and 1.1 with parameters $a = 1$, $b = c = 2$ and $a = b = c = 2$. As an example, when $a = 1, b = c = 2$ and $\psi = -1.1$, the reliability is $P(R < 1) = 0.68471$; this implies that the probability that the component will function satisfactorily is 0.68471; or, in other words, the component will fail with probability 0.31529.

a	**b**	**c**	ψ	$\mathbf{P(R < 1)}$
1	2	2	-1.1	0.68471
1	2	2	0	0.74904
1	2	2	1.1	0.75217
2	2	2	-1.1	0.42325
2	2	2	0	0.49782
2	2	2	1.1	0.48213

Table 8.1: Some reliability values

8.2 Beta-Binomial Distribution

8.2.1 The Beta-Binomial Distribution

A compound distribution is a special kind of mixture distribution where distributions are mixed by assigning a distribution to one or more parameters of another distribution [1]. Suppose, for example, one has a random variable $X|\theta$. Compounding would then be to let the parameter, θ, itself be a random variable. The distribution of X is then obtained by integrating over the distribution of θ; this is what we refer to as compounding the distribution of $X|\theta$ with that of θ, i.e $\int f(x|\theta)g(\theta)d\theta$ [17]. Such is the case with the beta-binomial distribution, which is discussed below.

The beta-binomial distribution is obtained by compounding the binomial distribution with the beta distribution. This process involves letting $X|P = p$ have a binomial distribution with parameters n and P. Also, P has a univariate beta type I distribution with parameters α and β. The unconditional distribution of X, known as the beta-binomial distribution, is then obtained by integrating over the distribution of P. The beta-binomial distribution [54] is denoted $B(n, P) \underset{P}{\wedge} beta(\alpha, \beta)$ with probability mass function (pmf) given by

$$\begin{aligned} f(x) &= P(X = x) \\ &= \binom{n}{x} \frac{B(\alpha + x, n + \beta - x)}{B(\alpha, \beta)} \end{aligned}$$

where $x = 0, 1, 2, \ldots n$. This distribution is a member of the family of Generalized Hypergeometric Probability Distributions (GHPD) [27]. This group of distributions is characterized by the form of the probability generating function (pgf) which consists of the quotient of two generalized hypergeometric functions. Specifically, in the case of the beta-binomial distribution, the pgf is given by

$$G(z) = \frac{{}_2F_1(-n, \alpha; -\beta - n + 1; z)}{{}_2F_1(-n, \alpha; -\beta - n + 1; 1)}$$

(see Rodriquez et al. [54]).

In the next section the Kummer beta-binomial distribution is constructed by letting the distribution of the binomial parameter, p, be the marginal of the Kummer beta type I distribution.

8.2.2 The Kummer Beta-Binomial Distribution

Consider a discrete random variable, X, which, conditional on the probability of success P, has a binomial distribution with pmf

$$g(x|P=p) = \binom{n}{x} p^x (1-p)^{n-x} \tag{8.2}$$

where $x = 0, 1, 2, \ldots, n$. Let the distribution of P be that of the marginal distribution of the bivariate Kummer beta type I distribution (see (2.7)). The pdf of P is then given by:

$$f(p) = \frac{p^{a-1}(1-p)^{b+c-1}{}_1F_1(c; b+c; \psi(1-p)}{B(a, b+c){}_1F_1(c; a+b+c; \psi)}. \tag{8.3}$$

The unconditional distribution of X is obtained by compounding (8.2) with (8.3) to obtain the Kummer beta-binomial distribution:

$$\begin{aligned} f(x) &= \int_0^1 g(x|p) f(p) dp \\ &= \int_0^1 \binom{n}{x} p^x (1-p)^{n-x} \frac{p^{a-1}(1-p)^{b+c-1}{}_1F_1(c; b+c; \psi(1-p))}{B(a, b+c){}_1F_1(c; a+b+c; \psi)} dp \\ &= \frac{\binom{n}{x}}{B(a, b+c){}_1F_1(c; a+b+c; \psi)} \int_0^1 p^{a+x-1}(1-p)^{b+c+n-x-1} \\ &\quad \times {}_1F_1(c; b+c; \psi(1-p)) dp. \end{aligned}$$

Use Definition B.8 and (B.3) to solve the integral:

$$\begin{aligned} f(x) &= \frac{\binom{n}{x}}{B(a, b+c){}_1F_1(c; a+b+c; \psi)} \sum_{k=0}^{\infty} \frac{(c)_k \psi^k}{(b+c)_k k!} \int_0^1 p^{a+x-1}(1-p)^{b+c+n-x+k-1} dp \\ &= \frac{\binom{n}{x}}{B(a, b+c){}_1F_1(c; a+b+c; \psi)} \sum_{k=0}^{\infty} \frac{(c)_k \psi^k}{(b+c)_k k!} B(a+x, b+c+n-x+k) \end{aligned}$$

and simplify using (B.1):

$$\begin{aligned} f(x) &= \frac{\binom{n}{x}}{B(a, b+c){}_1F_1(c; a+b+c; \psi)} \sum_{k=0}^{\infty} \frac{(c)_k \psi^k}{(b+c)_k k!} \frac{\Gamma(a+x)\Gamma(b+c+n-x+k)}{\Gamma(a+b+c+n+k)} \\ &= \frac{\Gamma(a+x)\Gamma(b+c+n-x)\binom{n}{x}}{B(a, b+c)\Gamma(a+b+c+n){}_1F_1(c; a+b+c; \psi)} \sum_{k=0}^{\infty} \frac{(c)_k \psi^k}{(b+c)_k k!} \frac{(b+c+n-x)_k}{(a+b+c+n)_k}. \end{aligned}$$

Use Definition B.8 to obtain the pmf of the Kummer beta-binomial distribution:

$$f(x) = \frac{B(a+x, b+c+n-x)\binom{n}{x}}{B(a,b+c)_1F_1(c; a+b+c; \psi)} {}_2F_2(c, b+c+n-x; b+c, a+b+c+n; \psi). \quad (8.4)$$

Remark 8.1 *Note that (8.4) reduces to the beta-binomial distribution with parameters a and $b+c$ when $\psi = 0$:*

$$\begin{aligned} f(x) &= \frac{B(a+x, b+c+n-x)\binom{n}{x}}{B(a,b+c)_1F_1(c; a+b+c; 0)} {}_2F_2(c, b+c+n-x; b+c, a+b+c+n; 0) \\ &= \frac{B(a+x, b+c+n-x)\binom{n}{x}}{B(a,b+c)} \\ &= \binom{n}{x} \frac{B(a+x, b+c+n-x)}{B(a,b+c)}. \end{aligned}$$

8.2.3 Numerical Example

Consider the example of the academic pass rates at the University of Jaén, Spain (see [54]). Here the random variable, X, is the number of subjects that were passed by every student in the University of Jaén, Spain during the 2003-2004 academic year. Rodríquez et al. [54] fitted the beta-binomial distribution and the generalized beta-binomial distribution to this data. In this study, the example is extended by fitting the Kummer beta-binomial distribution to the data. Maximum likelihood estimation is used to obtain the parameter estimates via a computer algorithm similar to that of Rodríquez et al. [54]. For this we need the log-likelihood function.

8.2.3.1 Log-Likelihood Function and Optimization

The likelihood function of the Kummer beta-binomial distribution is given by

$$\begin{aligned} & L(a,b,c,\psi) \\ = & \prod_{i=1}^{N} f(x_i) \\ = & \prod_{i=1}^{N} \frac{B(a+x_i, b+c+n-x_i)\binom{n}{x_i}}{B(a,b+c)_1F_1(c; a+b+c; \psi)} {}_2F_2(c, b+c+n-x_i; b+c, a+b+c+n; \psi) \\ = & \frac{1}{(B(a,b+c))^N \left({}_1F_1(c; a+b+c; \psi)\right)^N} \prod_{i=1}^{N} B(a+x_i, b+c+n-x_i) \\ & \times \binom{n}{x_i} {}_2F_2(c, b+c+n-x_i; b+c, a+b+c+n; \psi). \end{aligned}$$

Hence the log-likelihood function is given by

$$\begin{aligned}\ln(L(a,b,c,\psi)) &= \ln\left(\frac{1}{(B(a,b+c))^N\left({}_1F_1(c;a+b+c;\psi)\right)^N}\prod_{i=1}^{N}B(a+x_i,b+c+n-x_i)\right.\\ &\quad\left.\times\binom{n}{x_i}{}_2F_2(c,b+c+n-x_i;b+c,a+b+c+n;\psi)\right)\\ &= \sum_{i=1}^{N}\left(\ln\left[B(a+x_i,b+c+n-x_i)\right]+\ln\left[\binom{n}{x_i}\right]\right.\\ &\quad\left.+\ln\left[{}_2F_2(c,b+c+n-x_i;b+c,a+b+c+n;\psi)\right]\right)\\ &\quad -N\ln\left[B(a,b+c)\right]-N\ln\left[{}_1F_1(c;a+b+c;\psi)\right].\end{aligned}$$

Since the parameters a, b and c need to be positive, constraints are added to the optimization problem. These are given by

$$\begin{aligned}\ln a &= a_0,\\ \ln b &= b_0\\ \text{and} \qquad \ln c &= c_0.\end{aligned}$$

The software package Matlab is used to optimize the log-likelihood function. The two hypergeometric functions are calculated using their infinite sum representation (see Definition B.8). The Matlab optimization function *fminsearch* is used to optimize the log-likelihood function. This routine finds the minimum value of an unconstrained function. To find the maximum of the log-likelihood function, the negative log-likelihood function is fed into the routine. Appendix C provides the Matlab functions and programs used.

8.2.3.2 Results

This example only considers those students that registered for 19 subjects (i.e. in total there were $N = 71$ students) due to computer time constraints. Both the beta-binomial and the Kummer beta-binomial distributions were fitted to the data. The Akaike Information Criterion (AIC) was used to compare the fit of the two models:

$$AIC = -2\sum_{x_i}\{O_i\ln(E_i) - E_i - \ln(O_i!) + 2k\}$$

where each O_i is the observed frequency of the values x_i, E_i is its expected frequency under the model considered and k is the number of parameters in the model [54]. Note that, the lower the AIC, the better the fit. Table 8.2 shows the parameter estimates for the two models as well as the AIC for both models. Since the AIC for the Kummer beta-binomial model is slightly smaller than the AIC for the beta-binomial model, the Kummer beta-binomial distribution gives a slightly better fit. This is also apparent from Figure 8.2 where it is noted that the Kummer beta-binomial distribution follows the shape of the histogram better than the beta-binomial distribution.

Model	Kummer beta-binomial	Beta-binomial
Parameter	$\hat{a} = 0.4949$	$\hat{\alpha} = 1.4534$
Estimate	$\hat{b} = 0.0109$	$\hat{\beta} = 1.3034$
	$\hat{c} = 2.7724$	
	$\hat{\psi} = -6.9523$	
AIC	168.2557	169.0180

Table 8.2: Maximum Likelihood Estimation Results

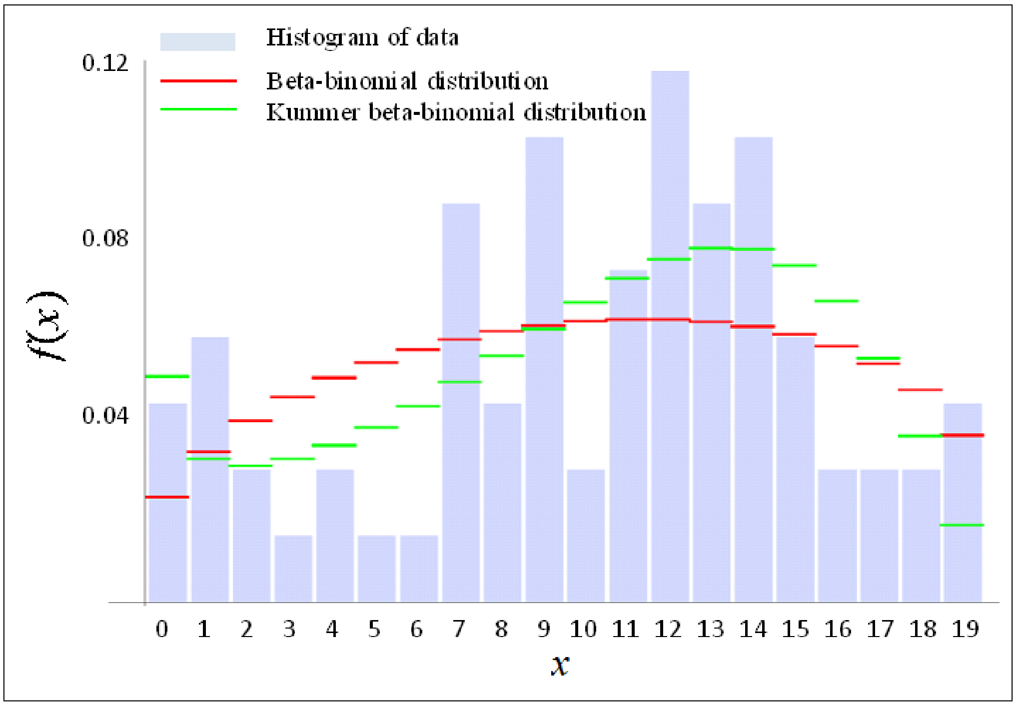
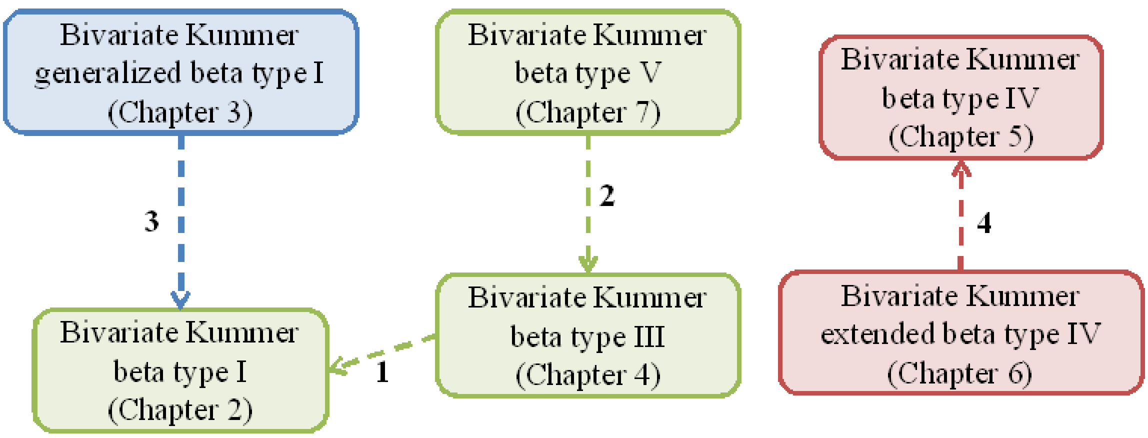


Figure 8.2: Beta-binomial and Kummer beta-binomial fitting to pass rate dataset

Chapter 9

Conclusion

In this study, new continuous bivariate distributions were constructed using the product method (see Definition 2). This method, although sporadically used in the literature to construct new univariate distributions (see [38], [39], [40], [42], [43], [44] and [50]), has not previously been used in a systematic way to construct bivariate distributions. This study proposed bivariate Kummer beta distributions by using bivariate beta kernels and univariate exponential kernels. In this way the following new distributions were constructed: bivariate Kummer beta type I, bivariate Kummer generalized beta type I, bivariate Kummer beta type III, bivariate Kummer beta type IV, bivariate Kummer extended beta type IV and the bivariate Kummer beta type V; some of the non-central distributions were also considered. Figure 9.1 shows how these newly derived bivariate Kummer beta distributions are interrelated by setting the parameters equal to specific values.



1. $\beta = 1$ 3. $d = b + c$

2. $\alpha_1 = \alpha_2 = 1$ 4. $d = 0$

Figure 9.1: Bivariate Kummer beta interrelations

For all the new distributions the joint, marginal and conditional pdf's were derived. The effect of the new parameter, ψ, was investigated and it was found that the overall tendency of ψ was to bring a greater sense of skewness into the pdf's. The correlation between X_1 and X_2 for the different distributions was also investigated with respect to ψ. Specifically, it was found that in the case of the bivariate Kummer beta type I and the bivariate Kummer generalized beta type I distributions, the parameter, ψ, introduced positive correlation. This is an improvement since it is a well-known fact that both the bivariate beta type I and the bivariate generalized beta type I distributions exhibit only negative correlation.

The study was concluded by two examples of possible applications. The first example considered the stress-strength model context where the reliability values were calculated for the case where (X_1, X_2) has the bivariate Kummer beta type IV distribution. Secondly, the beta-binomial and Kummer beta-binomial distributions were fitted to a dataset and the models compared. It was found that the Kummer beta-binomial distribution fits the data better than the beta-binomial distribution.

Areas for further research include, but are not limited to:

Constructing other bivariate product distributions using Definition 2 by assuming variations of the functions, $f(x_1, x_2)$ and $h(x_1, x_2)$ as follows:

- Using different ψ values, i.e. $h(x_1, x_2) = e^{-(\psi_1 x_1 + \psi_2 x_2)}$,
- Allowing for dependence in the function, $h(x_1, x_2)$, e.g.

$$h(x_1, x_2) = e^{-\psi_1 x_1 - \psi_2 x_2 - \psi_3 x_1 x_2} \neq h(x_1)h(x_2),$$

- Not limiting the function, $h(x_1, x_2)$ to the exponential function but allowing any bivariate kernel structure,
- Letting $f(x_1, x_2)$ be a kernel from other bivariate beta distributions not limited to a bounded support space and
- Assuming any bivariate kernel for the function, $f(x_1, x_2)$.

In final conclusion, this study developed new bivariate Kummer beta distributions using a product method approach, that contributed to the field of flexible bivariate distributions.

APPENDICES

A. NOTATION

~	Distributed like
$\equiv$	Is the same as
pdf	Probability Density Function
pgf	Probability Generating Function
pmf	Probability Mass Function
$Gamma(a, b)$	Gamma distribution with parameters a and b
$Beta^I(a, b)$	Beta type I distribution with parameters a and b
$\chi^2(a)$	χ^2 distribution with paramter a
X	Random variable
x	Observed value of the random variable X
$f(x_1, x_2)$	Kernel of the joint pdf of X_1 and X_2
$h(x_1, x_2)$	Product of exponential kernels
$g(x_1, x_2)$	Joint bivariate Kummer beta distribution of X_1 and X_2
$m(x_i)$	Marginal distribution of X_i
$c(x_i \vert x_j)$	Distribution of X_i, conditional on X_j
e^x	Exponential function
$\ln(x)$	Natural logarithmic function
$L(.)$	Likelihood function

B. MATHEMATICAL BACKGROUND

Definition B.1

The *kernel*, $f(x)$, of a distribution is defined as the part of the corresponding pdf without the normalizing constant. For example, the kernel of the univariate beta type I distribution is given by

$$f_{beta}(x) = x^{a-1}(1-x)^{b-1}.$$

Definition B.2 *(Balakrishnan and Lai, 2009, p115)*

A distribution is positively likelihood ratio dependent (also referred to as totally positive of order 2 (TP_2), see [5], p115) if

$$g(x_1, y_1)g(x_2, y_2) \geq g(x_1, y_2)g(x_2, y_1)$$

for all $x_1 < x_2$, $y_1 < y_2$.

Definition B.3 *(Gradshteyn and Ryzhik, 2007, p892,897)*

The gamma function is defined as

$$\Gamma(n) = (n-1)!$$

for n a natural number and with integral representation

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt.$$

Definition B.4 *(Prudnikov, Brychkov and Marichev, 1986, Vol.1, p772)*

The Pochhammer symbol is defined as

$$\begin{aligned}(a)_n &= a(a+1)\ldots(a+n-1) \\ &= \frac{\Gamma(a+n)}{\Gamma(a)}\end{aligned}$$

with $n = 1, 2, \ldots$ and $(a)_0 = 1$.

Definition B.5 *(Gradshteyn and Ryzhik, 2007, p909)*

The beta function is defined as

$$B(a_1, a_2, \ldots a_n) = \frac{\prod_{i=1}^{n} \Gamma(a_i)}{\Gamma(\sum_{i=1}^{n} a_i)}$$

from which we obtain the two-parameter beta function

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = B(y, x) \tag{B.1}$$

and the three-parameter beta function

$$B(x, y, z) = \frac{\Gamma(x)\Gamma(y)\Gamma(z)}{\Gamma(x+y+z)}. \tag{B.2}$$

Definition B.6 *(Gradshteyn and Ryzhik, 2007, p908)*

The integral representations of the beta function are

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt \tag{B.3}$$

and

$$B(x,y) = \int_0^\infty t^{x-1}(1+t)^{-(x+y)}dt. \tag{B.4}$$

Definition B.7 *(Gradshteyn and Ryzhik, 2007, p26)*

The series representation of the exponential function is given by

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

Definition B.8 *(Gradshteyn and Ryzhik, 2007, p1010)*

The generalized hypergeometric series is defined as

$$ {}_pF_q(a_1, a_2, ..., a_p; b_1, b_2, ..., b_q; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k...(a_p)_k}{(b_1)_k(b_2)_k...(b_q)_k}\frac{z^k}{k!}.$$

From this we obtain the confluent hypergeometric function (Kummer function)

$$ {}_1F_1(a_1; b_1; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k}{(b_1)_k}\frac{z^k}{k!} \tag{B.5}$$

and the Gauss hypergeometric function

$$ {}_2F_1(a_1, a_2; b_1; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k}{(b_1)_k}\frac{z^k}{k!}. \tag{B.6}$$

Definition B.9 *(Gradshteyn and Ryzhik, 2007, p1018)*

The hypergeometric function of two variables is defined as

$$F_1(\alpha, \beta, \beta', \gamma; x, y) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{(\alpha)_{k+j}(\beta)_k(\beta')_j}{(\gamma)_{k+j}k!j!}x^k y^j \qquad [|x| < 1, \;\; |y| < 1].$$

Definition B.10 *(Gradshteyn and Ryzhik, 2007, p1031)*

The confluent hypergeometric series of two variables is defined as

$$\Phi_2(\beta, \beta', \gamma, x, y) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{(\beta)_k(\beta')_k}{(\gamma)_{k+j}k!j!}x^k y^j.$$

Definition B.11 *(Mathai, 1993, p60)*

Meijer's G-function with parameters $a_1, ..., a_p$ and $b_1, ..., b_q$ is defined as

$$G^{m,n}_{p,q}\left(z \middle| \begin{matrix} a_1, ..., a_p \\ b_1, ..., b_q \end{matrix}\right) = (2\pi i)^{-1} \int_L g(s) z^{-s} ds$$

where $i = \sqrt{-1}$, L is a suitable contour, $z \neq 0$ and

$$g(s) = \frac{\prod\limits_{j=1}^{m} \Gamma(b_j + s) \prod\limits_{j=1}^{n} \Gamma(1 - a_j - s)}{\prod\limits_{j=m=1}^{q} \Gamma(1 - b_j - s) \prod\limits_{j=n+1}^{p} \Gamma(a_j + s)}.$$

Definition B.12 *(Mathai, 1993, p23)*

The Mellin transform of f with respect to the parameter s is given by

$$M_f(s) = \int_0^\infty x^{s-1} f(x) dx$$

where s is a complex number. The inverse Mellin transform is given by the inverse integral

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} M_f(s) x^{-s} ds$$

where $i = \sqrt{-1}$ and c is a real number in the strip of analyticity of $M_f(s)$.

Relation B.1 *(Prudnikov, Brychkov and Marichev, 1986, Vol.3, p453)*

$$\sum_{k=0}^{\infty} \frac{(a)_k}{k!} z^k = {}_1F_0(a; z) = (1 - z)^{-a}$$

Relation B.2 *(Gradshteyn and Ryzhik, 2007, p1023)*

The integral representation of the Kummer function, ${}_1F_1(.)$, is given by

$${}_1F_1(a; b; z) = \frac{z^{1-b}}{B(a, b-a)} \int_0^z e^t t^{a-1} (z - t)^{b-a-1} dt.$$

Relation B.3 *(Gradshteyn and Ryzhik, 2007, p347)*

$$\int_0^u x^{\nu-1} (u - x)^{\mu-1} e^{\beta x} dx = B(\mu, \nu) u^{\mu+\nu-1} {}_1F_1(\nu; \mu + \nu; \beta u) \qquad [Re\ \mu > 0,\ \text{Re}\ \nu > 0]$$

Relation B.4 *(Gradshteyn and Ryzhik, 2007, p1023)*

$$ {}_1F_1(a;b;z) = e^z {}_1F_1(b-a;b;-z) $$

Relation B.5 *(Prudnikov, Brychkov and Marichev, 1986, Vol.1, p301)*

$$ \int_a^b (x-a)^{\alpha-1}(b-x)^{\beta-1}(cx+d)^{\gamma}dx = (b-a)^{\alpha+\beta-1}(ac+d)^{\gamma}B(\alpha,\beta) \times {}_2F_1\left(\alpha,-\gamma;\alpha+\beta;\frac{c(a-b)}{ac+d}\right) $$
$$ [\text{Re }\alpha,\ \text{Re }\beta > 0;\ \ |\arg[(d+cb)/(d+ca)]| < \pi] $$

Relation B.6 *(Prudnikov, Brychkov and Marichev, 1986, Vol.3, p413)*

$$ \sum_{k=0}^{\infty} \frac{(a)_k(b')_k}{k!(c)_k} x^k {}_2F_1(a+k,b;c+k;x) = {}_2F_1(a,b+b';c;x) $$

Relation B.7 *(Prudnikov, Brychkov and Marichev, 1986, Vol.3, p413)*

$$ \sum_{k=0}^{\infty} \frac{(a)_k(b')_k}{k!(c)_k} t^k {}_2F_1(a+k,b;c+k;x) = F_1(a,b,b';c;x,t) \qquad [|t|,\ \ |x|<1] $$

Relation B.8 *(Burchnall, Chaundy, 1941)*

$$ \Phi_2(a,a';c;x,x) = {}_1F_1(a+a';c;x) $$

Relation B.9 *(Prudnikov, Brychkov and Marichev, 1986, Vol.3, p452)*

$$ F_1(a,b,b';c;z,z) = {}_2F_1(a,b+b';c;z) $$

Relation B.10 *(Prudnikov, Brychkov and Marichev, 1986, Vol.3, p533)*

$$ {}_3F_2(a,b,c;d,e;1) = \frac{\Gamma(d)\Gamma(d+e-a-b-c)}{\Gamma(d+e-a-b)\Gamma(d-c)} {}_3F_2(e-a,e-b,c;d+e-a-b,e;1) $$
$$ [\text{Re }(d+e-a-b-c),\ \text{Re }(d-c) > 0] $$

Relation B.11 *(Prudnikov, Brychkov and Marichev, 1986, Vol.3, p489)*

$$ {}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \qquad [\text{Re }(c-a-b) > 0] $$

Relation B.12 *(Prudnikov, Brychkov and Marichev, 1986, Vol.3, p431)*

$$ {}_2F_1(0,b;c;z) = {}_2F_1(a,0;c;z) = {}_2F_1(a,b;c;0) = 1 $$

Relation B.13 *(Prudnikov, Brychkov and Marichev, 1986, Vol.3, p320)*

$$\int_0^y x^{\alpha-1}(y-x)^{c-1}e^{-px}{}_2F_1\left(a,b;c;1-\frac{x}{y}\right)dx = y^{c+\alpha-1}\frac{\Gamma(c)\Gamma(\alpha)\Gamma(c-a-b+\alpha)}{\Gamma(c-a+\alpha)\Gamma(c-b+\alpha)}$$
$$\times\, {}_2F_2\left(\alpha, c-a-b+\alpha; c-a+\alpha, c-b+\alpha; -py\right)$$
$$[y,\ \mathrm{Re}\ c, \mathrm{Re}\ \alpha,\ \mathrm{Re}\ (-a-b+\alpha)>0]$$

Relation B.14 *(Paris, 2005)*

$$ {}_2F_2\left(a,d;b,c;x\right) = e^x \sum_{k=0}^{\infty}\frac{(c-d)_k}{k!(c)_k}(-x)^k {}_2F_2\left(b-a,d;b,c+k;-x\right)$$
$$[\mathrm{Re}\ b > \mathrm{Re}\ a > 0]$$

C. COMPUTER PROGRAMS

In this appendix, the Matlab computer programs used in this dissertation are given, together with some explanation. Text in italics following the % sign are comments and do not form part of the Matlab code.

Program: Hypergeometric1F1_sum.m

```
function [y] = Hypergeometric1F1_sum(a,b,z)
%This function calculates the 1F1 function using an infinite sum
s = 0; m = 0; old_z = 0; TOL = 0.0000000001;
while (s == 0)
    if m == 0,
        term = 1;
    else
        term = term * z * (a+(m-1)) / (b+(m-1)) / m ;
    end
    new_z = old_z + term;
    if abs(new_z - old_z) <= TOL,
        s = 1;
    else
        old_z = new_z;
    end
    m = m + 1;
end
y = new_z;
end
```

Program: Hypergeometric2F2_sum.m

```
function [y] = Hypergeometric2F2_sum(a,b,c,d,z)
%This function calculates the 2F2 function using an infinite sum
s = 0; m = 0; old_z = 0; TOL = 0.0000000001;
while (s == 0)
    if m == 0,
        term = 1;
    else
        term = term * z * (a+(m-1)) * (b+(m-1)) / m / (c+(m-1)) / (d+(m-1));
    end
    new_z = old_z + term;
    if abs(new_z - old_z) <= TOL,
        s = 1;
    else
        old_z = new_z;
    end
    m = m + 1;
end
y = new_z;
end
```

Program: Kummer_loglik.m

```
function y = Kummer_loglik(p0)
%This function calculates the log-likelihood function
load SubjectPassed
n = 19;%number of subjects taken by students
%For Beta-binomial fitting use only a0,b0,a,b
a0 = p0(1); b0 = p0(2); c0 = p0(3); psi = p0(4);
a = exp(a0);
b = exp(b0);
c = exp(c0);
dimen = size(SubjectPassed);
samplesize = dimen(1);%obtain sample size from data
x = SubjectPassed;
s = 0;
%Calculating logliklihood function, use either 1 or 2.
% 1.FOR BETA-BINOMIAL FITTING
for i = 1:samplesize
    ct_n = factorial(n);
    fact_x = factorial(x(i));
    fact_n_x = factorial(n-x(i));
    logn_x = log(fact_n)-(log(fact_x)+log(fact_n_x));
    s = s+logn_x+betaln(a+x(i),n+b-x(i))-betaln(a,b);
end
%END Beta-binomial log-likelihood
```

```
% 2.FOR KUMMER BETA-BINOMIAL FITTING
for i = 1:samplesize
    logn_x = log(nchoosek(n,x(i)));
    log2F2 = log(Hypergeometric2F2_sum(c,b+c+n-x(i),b+c,a+b+c+n,psi));
    s = s+logn_x+betaln(a+x(i),b+c+n-x(i))+log2F2;
end
log1F1 = log(Hypergeometric1F1_sum(c,a+b+c,psi));
s = s - samplesize*(log1F1 + betaln(a,b+c));
%END Kummer beta-binomial log-likelihood
y = -s;%function returns negative of log-likelihood function
```

Program: Kummer_maximize.m

```
function [parm] = Kummer_maximize(SubjectPassed)
%This function does the optimization of the log-likelihood function
%save dataset to be used in call of Kummer_loglik function
save SubjectPassed
%Initial values are obtained by user input
p0 = input('Initial values for the parameters = ');
%Maximum number of iterations is obtained by user input
niter = input('Maximum number of iterations = ');
options = optimset('MaxFunEvals',10000,'MaxIter',niter);
%fminsearch returns optimum parameter values
[parm] = fminsearch(@Kummer_loglik,p0,options);
end
```

Program: BBpdf.m

```
function prob = BBpdf(a,b,n,x)
%This function calculates the pmf of the Beta-binomial distribution
prob = Beta(a+x,b+n-x)*nchoosek(n,x)/Beta(a,b);
end
```

Program: KBBpdf.m

```
function prob = KBBpdf(a,b,c,psi,n,x)
%This function calculates the pmf of the Kummer Beta-binomial distribution
hyper1F1 = Hypergeometric1F1_sum(c,a+b+c,psi);
hyper2F2 = Hypergeometric2F2_sum(c,b+c+n-x,b+c,a+b+c+n,psi);
prob = Beta(a+x,b+c+n-x)*nchoosek(n,x)/(Beta(a,b+c)*hyper1F1)*hyper2F2;
end
```

Program: Fitting_data.m

```
%This program calculates the Akaike Information Criterion (AIC)
n = 19;
% Create empty vector for expected frequencies of Kummer beta-binomial
Exp_freq_KBB = zeros(n+1,1);
% Create empty vector for expected frequencies of beta-binomial
Exp_freq_BB = zeros(n+1,1);
% Vector of observed frequencies
Obs_freq = hist(SubjectPassed,n+1);
sum_KBB = 0;
sum_BB = 0;
for i = 0:n
% Calculate expected frequencies of Kummer beta-binomial
    Exp_freq_KBB(i+1) = n * KBBpdf(0.4949,0.0109,2.7724,-6.9523,n,i);
% Calculate expected frequencies of beta-binomial
    Exp_freq_BB(i+1) = n * BBpdf(1.4534,1.3034,n,i);
    sum_KBB = sum_KBB + Obs_freq(i+1)*log(Exp_freq_KBB(i+1)
        -Exp_freq_KBB(i+1)-log(factorial(Obs_freq(i+1)));
    sum_BB = sum_BB + Obs_freq(i+1)*log(Exp_freq_BB(i+1))
        -Exp_freq_BB(i+1)-log(factorial(Obs_freq(i+1)));
end
AIC_BB = -2*sum + 2*2
AIC_KBB = -2*sumK + 2*4
```

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