Bimatrix variate distributions of Wishart ratios with application

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by

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Submitted in fulfilment of
the requirements for the degree

Philosphiae Doctor (Mathematical Statistics)

In the Faculty of Natural & Agricultural Sciences
University of Pretoria
Pretoria

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February 2011

UNIVERSITEIT VAN PRETORIA
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Declaration of originality

I, René Ehlers, declare that the thesis, which I hereby submit for the degree Philosphiae Doctor (Mathematical Statistics) at the University of Pretoria, is my own work and has not previously been submitted by me for a degree at this or any other tertiary institution.

Signature ____________________ Date ____________________

Acknowledgements

I would like to thank my supervisor, Prof. Andriëtte Bekker, for the privilege of working with her. Her guidance, encouragement, support and expertise were invaluable throughout, ensuring that the whole process was an excellent learning experience for me.

I would also like to thank my co-supervisor, Prof. J.J.J. Roux, for his valuable input. Once again his expertise and especially the ability to place this study within the context of the discipline as a whole, were incalculable.

Prof. M. Arashi deserves heartfelt thanks for his input and encouragement.

The financial assistance of the National Research Foundation (NRF) towards this research is hereby acknowledged.

Summary

The focus of this study is the development of a group of bimatrix variate beta distributions with bounded domain for the central and noncentral cases. They are constructed from different dependent Wishart ratios. Exact expressions are derived for the density functions by using symmetrisation. The role of the parameters of these new distributions is also highlighted. These new bimatrix variate beta distributions add value to multivariate statistical analysis and an application in this field is the link to statistics that are functions of the product of determinants of bimatrix beta variates. Exact expressions are derived for the density functions of the product of determinants of these bimatrix beta variates.

Contents

1 Introduction

In this thesis a bimatrix group of beta distributions is derived in explicit form from different Wishart ratios such that the bimatrix beta variates $(\boldsymbol{Y}_1, \boldsymbol{Y}_2)$ are defined on the region $\boldsymbol{0} < \boldsymbol{Y}_i < \boldsymbol{I}_p$, $i = 1, 2$, that is $\boldsymbol{Y}_i$ and $\boldsymbol{I}_p - \boldsymbol{Y}_i$, $i = 1, 2$, are positive definite matrices. Exact expressions are derived for the density functions of the product of determinants of these bimatrix beta variates.

1.1 Wishart ratios

Abbreviations and Notation

$\boldsymbol{S}$, $\boldsymbol{S}_1$, $\boldsymbol{S}_2$ and $\boldsymbol{B}$ are $p \times p$ positive definite matrices.

$W_p(n, \boldsymbol{\Sigma})$	:	Wishart distribution with parameters n and $\boldsymbol{\Sigma}$.
$W_p(n, \boldsymbol{\Sigma}; \boldsymbol{\Theta})$	:	Noncentral Wishart distribution with parameters n, $\boldsymbol{\Sigma}$ and $\boldsymbol{\Theta}$.
$B_p^I(n, m)$	:	Matrix beta type I distribution with parameters n and m.
$B_p^I(n, m; \boldsymbol{\Theta})$	:	Noncentral matrix beta type I distribution with parameters n, m and $\boldsymbol{\Theta}$.

$(\boldsymbol{U}_1, \boldsymbol{U}_2)$, $(\boldsymbol{W}_1, \boldsymbol{W}_2)$, $(\boldsymbol{X}_1, \boldsymbol{X}_2)$ and $(\boldsymbol{Q}_1, \boldsymbol{Q}_2)$ are bimatrix variates where $\boldsymbol{U}_i$, $\boldsymbol{I}_p - \boldsymbol{U}_i$, $\boldsymbol{W}_i$, $\boldsymbol{I}_p - \boldsymbol{W}_i$, $\boldsymbol{X}_i$, $\boldsymbol{I}_p - \boldsymbol{X}_i$, $\boldsymbol{Q}_i$, $\boldsymbol{I}_p - \boldsymbol{Q}_i$, $i = 1, 2$, $\boldsymbol{I}_p - \sum_{i=1}^{2} \boldsymbol{U}_i$, $\boldsymbol{I}_p - \sum_{i=1}^{2} \boldsymbol{W}_i$ and $\boldsymbol{I}_p - \sum_{i=1}^{2} \boldsymbol{Q}_i$ are $p \times p$ positive definite matrices.

$BB_p^I(n_1, n_2, m)$	:	Bimatrix beta type I distribution with parameters n_1, n_2 and m.
$BB_p^I(n_1, n_2, m; \boldsymbol{\Theta})$	:	Noncentral bimatrix beta type I distribution with parameters n_1, n_2, m and $\boldsymbol{\Theta}$.
$BB_p^{III}(n_1, n_2, m, c)$	:	Bimatrix beta type III distribution with parameters n_1, n_2, m and c.
$BB_p^{III}(n_1, n_2, m, c; \boldsymbol{\Theta})$	:	Noncentral bimatrix beta type III distribution with parameters n_1, n_2, m, c and $\boldsymbol{\Theta}$.
$BB_p^{IV}(n_1, n_2, m)$	:	Bimatrix beta type IV distribution with parameters n_1, n_2 and m.
$BB_p^{IV}(n_1, n_2, m; \boldsymbol{\Theta})$	:	Noncentral bimatrix beta type IV distribution with parameters n_1, n_2, m and $\boldsymbol{\Theta}$.
$BB_p^{V}(n_1, n_2, m, \alpha_1, \alpha_2, c)$	:	Bimatrix beta type V distribution with parameters n_1, n_2, m, α_1, α_2 and c.
$BB_p^{V}(n_1, n_2, m, \alpha_1, \alpha_2, c; \boldsymbol{\Theta})$	:	Noncentral bimatrix beta type V distribution with parameters n_1, n_2, m, α_1, α_2, c and $\boldsymbol{\Theta}$.

The different Wishart ratios that are focused on in this study, each with its own characteristic and associated distribution, are described here in short. A detailed discussion of these ratios together with the necessary derivations are given in the chapters that follow. The first Wishart ratio is defined analogous to the chi-square ratio in the univariate case and follows as

$$\boldsymbol{U} = (\boldsymbol{S} + \boldsymbol{B})^{-\frac{1}{2}} \boldsymbol{S} (\boldsymbol{S} + \boldsymbol{B})^{-\frac{1}{2}} \tag{1.1}$$

where $\boldsymbol{S} \sim W_p(n, \boldsymbol{\Sigma})$, independent of $\boldsymbol{B} \sim W_p(m, \boldsymbol{\Sigma})$ and $\boldsymbol{C}^{\frac{1}{2}} \boldsymbol{C}^{\frac{1}{2}} = \boldsymbol{C}$ is a reasonable nonsingular factorization of $\boldsymbol{C}$. The distribution of this ratio (1.1) was derived by Khatri (1959) and is denoted by $\boldsymbol{U} \sim$

$B_p^I(n,m)$ with the probability density function (pdf) given in Chapter 3. In the noncentral case where $\boldsymbol{S} \sim W_p(n, \boldsymbol{I}_p)$ and $\boldsymbol{B} \sim W_p(m, \boldsymbol{I}_p; \boldsymbol{\Theta})$ the corresponding matrix variate distribution of $\boldsymbol{U}$, denoted by $B_p^I(n, m; \boldsymbol{\Theta})$, was derived by de Waal (1968) in a non-explicit form and an exact expression for the pdf was derived by Díaz-García and Gutiérrez-Jáimez (2006a). The pdf is given in Chapter 8 and also derived for $\boldsymbol{\Sigma} \neq \boldsymbol{I}_p$.

The bimatrix beta distributions in this study are all constructed from ratios of three independent Wishart matrix variates $\boldsymbol{S}_i \sim W_p(n_i, \boldsymbol{\Sigma})$, $i = 1, 2$, and $\boldsymbol{B} \sim W_p(m, \boldsymbol{\Sigma})$. The noncentral bimatrix variate beta distributions in this study are considered for the case where $\boldsymbol{B} \sim W_p(m, \boldsymbol{\Sigma}; \boldsymbol{\Theta})$.

The *bimatrix variate beta type I* distribution is obtained from the Wishart ratios

$$\boldsymbol{U}_i = (\boldsymbol{S}_1 + \boldsymbol{S}_2 + \boldsymbol{B})^{-\frac{1}{2}} \boldsymbol{S}_i (\boldsymbol{S}_1 + \boldsymbol{S}_2 + \boldsymbol{B})^{-\frac{1}{2}}, \; i = 1, 2, \tag{1.2}$$

denoted by $(\boldsymbol{U}_1, \boldsymbol{U}_2) \sim BB_p^I(n_1, n_2, m)$ and the pdf is given in Chapter 4. The corresponding Dirichlet distribution, that is for $i = 1, \ldots, r$, was derived by Olkin and Rubin (1964). For the noncentral case where $(\boldsymbol{U}_1, \boldsymbol{U}_2) \sim BB_p^I(n_1, n_2, m; \boldsymbol{\Theta})$, the pdf is derived in Chapter 9.

The *bimatrix variate beta type III* distribution is obtained from the Wishart ratios

$$\boldsymbol{W}_i = (\boldsymbol{S}_1 + \boldsymbol{S}_2 + c\boldsymbol{B})^{-\frac{1}{2}} \boldsymbol{S}_i (\boldsymbol{S}_1 + \boldsymbol{S}_2 + c\boldsymbol{B})^{-\frac{1}{2}}, \quad i = 1, 2, \tag{1.3}$$

denoted by $(\boldsymbol{W}_1, \boldsymbol{W}_2) \sim BB_p^{III}(n_1, n_2, m, c)$. The pdf is derived in Chapter 5 (see also Ehlers, Bekker and Roux, 2009). The role of the parameter c in (1.3) will be demonstrated in Chapter 5. In the noncentral case where $(\boldsymbol{W}_1, \boldsymbol{W}_2) \sim BB_p^{III}(n_1, n_2, m, c; \boldsymbol{\Theta})$, the pdf is derived in Chapter 10.

The *bimatrix variate beta type IV* distribution is obtained from the Wishart ratios

$$\boldsymbol{X}_i = \left(\boldsymbol{I}_p + \boldsymbol{B}^{-\frac{1}{2}} \boldsymbol{S}_i \boldsymbol{B}^{-\frac{1}{2}}\right)^{-\frac{1}{2}} \boldsymbol{B}^{-\frac{1}{2}} \boldsymbol{S}_i \boldsymbol{B}^{-\frac{1}{2}} \left(\boldsymbol{I}_p + \boldsymbol{B}^{-\frac{1}{2}} \boldsymbol{S}_i \boldsymbol{B}^{-\frac{1}{2}}\right)^{-\frac{1}{2}}, \quad i = 1, 2, \tag{1.4}$$

denoted by $(\boldsymbol{X}_1, \boldsymbol{X}_2) \sim BB_p^{IV}(n_1, n_2, m)$, and the pdf is derived in Chapter 6 (see also Bekker, Roux, Ehlers and Arashi, 2010). For the noncentral case where $(\boldsymbol{X}_1, \boldsymbol{X}_2) \sim BB_p^{IV}(n_1, n_2, m; \boldsymbol{\Theta})$, the pdf is derived in Chapter 11. The latter was derived by Díaz-García and Gutiérrez-Jáimez (2009) for $\boldsymbol{\Sigma} = \boldsymbol{I}_p$.

The *bimatrix variate beta type V* distribution is obtained from the Wishart ratios

$$\boldsymbol{Q}_i = (\alpha_1 \boldsymbol{S}_1 + \alpha_2 \boldsymbol{S}_2 + c\boldsymbol{B})^{-\frac{1}{2}} (\alpha_i \boldsymbol{S}_i) (\alpha_1 \boldsymbol{S}_1 + \alpha_2 \boldsymbol{S}_2 + c\boldsymbol{B})^{-\frac{1}{2}}, \quad i = 1, 2, \tag{1.5}$$

denoted by $(\boldsymbol{Q}_1, \boldsymbol{Q}_2) \sim BB_p^V(n_1, n_2, m, \alpha_1, \alpha_2, c)$ and the pdf is derived in Chapter 7. The effect of the parameters α_1 and α_2 will be illustrated in Chapter 7. In the noncentral case where $(\boldsymbol{Q}_1, \boldsymbol{Q}_2) \sim BB_p^V(n_1, n_2, m, \alpha_1, \alpha_2, c; \boldsymbol{\Theta})$ the pdf is derived in Chapter 12.

1.2 Application of Wishart ratios

In this Section 1.2 we link the exact expressions for the pdfs of the determinants of the Wishart ratios defined in Section 1.1 to the pdfs of Wilks' statistics. From (1.1) the determinant of $\boldsymbol{U}$ is

$$\Lambda_1 = \frac{|\boldsymbol{S}|}{|\boldsymbol{S} + \boldsymbol{B}|} \tag{1.6}$$

which is known as the Wilks' statistic proposed by Wilks (1932) testing the population mean equal to zero. The distribution of the Wilks' statistic is of importance in multivariate analysis of variance and covariance since several test statistics are functions of the beta matrix (Troskie, 1972). The exact expression for the pdf of Λ_1 for $\boldsymbol{\Theta} = \mathbf{0}$ in terms of the G-function (see Chapter 2) is given in Chapter 3. The exact pdf of Λ_1 for $\boldsymbol{\Theta} \neq \mathbf{0}$ is given in Chapter 8. The latter gives the exact expression for the pdf of the Wilks' statistic under the nonnull hypothesis and is important because it is used to calculate the power function.

From (1.2) the determinant

$$\Lambda_2 = |\boldsymbol{U}_1|^{\frac{1}{2}n_1} \, |\boldsymbol{U}_2|^{\frac{1}{2}n_2} \tag{1.7}$$

is the test statistic given by Anderson (1984, page 410) for testing whether two multivariate normal distributions are identical. As a generalisation of (1.7) consider the following statistics from (1.3) and (1.5):

$$\Lambda_3 = |\boldsymbol{W}_1|^{\frac{1}{2}n_1} \, |\boldsymbol{W}_2|^{\frac{1}{2}n_2} \tag{1.8}$$

and

$$\Lambda_5 = |\boldsymbol{Q}_1|^{\frac{1}{2}n_1} \, |\boldsymbol{Q}_2|^{\frac{1}{2}n_2} \tag{1.9}$$

respectively. The use of (1.8) and (1.9) as test statistics in multivariate analysis involving multivariate normal distributions with different covariance structures will be studied in future research. For $\boldsymbol{\Theta} = \mathbf{0}$ the exact expressions for the pdfs of Λ_2, Λ_3 and Λ_5 are derived in Chapter 4, Chapter 5 and Chapter 7 respectively. Further, for $\boldsymbol{\Theta} \neq \mathbf{0}$ the exact expressions for the pdfs of Λ_2, Λ_3 and Λ_5 are derived in Chapters 9, 10 and 12 respectively.

From (1.4) the determinant is

$$\Lambda_4 = \frac{|\boldsymbol{S}_1|}{|\boldsymbol{S}_1 + \boldsymbol{B}|} \frac{|\boldsymbol{S}_2|}{|\boldsymbol{S}_2 + \boldsymbol{B}|} \tag{1.10}$$

which is the product of factors of Wilks' statistics. The exact expression for the pdf of Λ_4 for $\boldsymbol{\Theta} = \mathbf{0}$ is derived in Chapter 6 and for $\boldsymbol{\Theta} \neq \mathbf{0}$ in Chapter 11.

Firstly, a collection of some fundamental mathematical results are given in Chapter 2 for use in later chapters. As a background to this study the well known matrix variate beta type I distribution is discussed in Chapters 3 and 8 for the central and noncentral cases respectively. This distribution plays an important role in multivariate analysis because of its link to the Wilks' statistic (see (1.6)). The development of the bimatrix group of beta distributions is presented in Chapters 4 to 7 and 9 to 12. Some concluding remarks are given at the end.

2 Special functions and theory

2.1 Jacobians of matrix transformations

In this study new distributions of matrix variates are derived by making transformations on known matrix variates. The transformation technique is explained in this section and the Jacobians are given of transformations used.

[2.1.1] *(Greenacre, 1972, Definition 2.1.1, page 4; Gupta and Nagar, 2000b, page 12)*
Let $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_k$ be matrix variates with n independent elements $(x_1, \ldots, x_n)$ and joint pdf $f(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_k)$. Also, let $\mathbf{Y}_1, \ldots, \mathbf{Y}_m$ be matrix variates with n independent elements $(y_1, \ldots, y_n)$. Consider the matrix transformations

$$\mathbf{Y}_i = t_i(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_k), \ \ i = 1, \ldots, m$$

where the inverse set of transformations exists, that is

$$\boldsymbol{X}_j = s_j(\mathbf{Y}_1, \ldots, \mathbf{Y}_m), \ \ j = 1, \ldots, k.$$

Then the pdf of $(\mathbf{Y}_1, \ldots, \mathbf{Y}_m)$ is

$$g(\mathbf{Y}_1, \ldots, \mathbf{Y}_m) = f(s_1(\mathbf{Y}_1, \ldots, \mathbf{Y}_m), \ldots, s_k(\mathbf{Y}_1, \ldots, \mathbf{Y}_m)) \cdot J((\boldsymbol{X}_1, \ldots, \boldsymbol{X}_k) \to (\mathbf{Y}_1, \ldots, \mathbf{Y}_m)),$$

where the Jacobian of the transformation is $J((\boldsymbol{X}_1, \ldots, \boldsymbol{X}_k) \to (\mathbf{Y}_1, \ldots, \mathbf{Y}_m)) = \operatorname{mod}\det \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \vdots & & \vdots \\ \frac{\partial x_n}{\partial y_1} & \cdots & \frac{\partial x_n}{\partial y_n} \end{pmatrix}$.

The following results for Jacobians are used in this study (see Deemer and Olkin, 1951; Olkin and Rubin, 1964; Perlman, 1977; Gupta and Nagar, 2000b).

[2.1.2] If $J(\boldsymbol{X} \to \mathbf{Y}) \neq 0$, then

$$J(\mathbf{Y} \to \boldsymbol{X}) = \{J(\boldsymbol{X} \to \mathbf{Y})\}^{-1}.$$

[2.1.3] If $\boldsymbol{Y} = F(\boldsymbol{Z})$ and $\boldsymbol{Z} = G(\boldsymbol{X})$, then

$$J(\boldsymbol{X} \to \mathbf{Y}) = J(\boldsymbol{X} \to \mathbf{Z}) J(\boldsymbol{Z} \to \mathbf{Y}).$$

[2.1.4] If $\boldsymbol{Y} = F(\boldsymbol{X})$ and $\boldsymbol{Z} = G(\boldsymbol{W})$, then

$$J(\boldsymbol{X}, \boldsymbol{W} \to \mathbf{Y}, \boldsymbol{Z}) = J(\boldsymbol{X} \to \mathbf{Y}) J(\boldsymbol{W} \to \boldsymbol{Z}).$$

[2.1.5] If $\boldsymbol{Y}\ (p \times p)$ and $\boldsymbol{X}\ (p \times p)$ are symmetric matrices and $\boldsymbol{Y} = \alpha \boldsymbol{X}$ where α is a constant scalar $\neq 0$, then

$$J(\boldsymbol{Y} \to \boldsymbol{X}) = \alpha^{\frac{1}{2}p(p+1)}.$$

[**2.1.6**] If $\boldsymbol{Y}\ (p\times p)$ and $\boldsymbol{X}\ (p\times p)$ are symmetric matrices, $\boldsymbol{A}\ (p\times p)$ and $\boldsymbol{Y}=\boldsymbol{A}\boldsymbol{X}\boldsymbol{A}'$, then

$$J\left(\boldsymbol{Y}\to\boldsymbol{X}\right)=\det\left(\boldsymbol{A}\right)^{p+1}=\left|\boldsymbol{A}\right|^{p+1}.$$

[**2.1.7**] If $\boldsymbol{Y}\ (p\times p)$ and $\boldsymbol{X}\ (p\times p)$ are nonsingular symmetric matrices and $\boldsymbol{Y}=\boldsymbol{X}^{-1}$, then

$$J\left(\boldsymbol{Y}\to\boldsymbol{X}\right)=\left|\boldsymbol{X}\right|^{-(p+1)}.$$

[**2.1.8**] If $\boldsymbol{Y}\,(p\times p)\,,\boldsymbol{X}\,(p\times p)\,,\boldsymbol{U}\,(p\times p)$ and $\boldsymbol{V}\,(p\times p)$ are symmetric positive definite matrices, $\boldsymbol{U}=\boldsymbol{X}+\boldsymbol{Y}$ and $\boldsymbol{V}=\boldsymbol{U}^{-\frac{1}{2}}\boldsymbol{Y}\left(\boldsymbol{U}^{-\frac{1}{2}}\right)'$ where $\boldsymbol{U}=\boldsymbol{U}^{\frac{1}{2}}\left(\boldsymbol{U}^{\frac{1}{2}}\right)'$, then

$$J\left(\boldsymbol{X},\boldsymbol{Y}\to\boldsymbol{U},\boldsymbol{V}\right)=\left|\boldsymbol{U}\right|^{\frac{1}{2}(p+1)}.$$

[**2.1.9**] If $\boldsymbol{Y}\,(p\times p)$ and $\boldsymbol{X}\,(p\times p)$ are symmetric positive definite matrices and $\boldsymbol{Y}=\left(\boldsymbol{I}_p+\boldsymbol{X}\right)^{-\frac{1}{2}}\boldsymbol{X}\left(\left(\boldsymbol{I}_p+\boldsymbol{X}\right)^{-\frac{1}{2}}\right)'$ where $\left(\boldsymbol{I}_p+\boldsymbol{X}\right)=\left(\boldsymbol{I}_p+\boldsymbol{X}\right)^{\frac{1}{2}}\left(\left(\boldsymbol{I}_p+\boldsymbol{X}\right)^{\frac{1}{2}}\right)'$, then

$$J\left(\boldsymbol{Y}\to\boldsymbol{X}\right)=\left|\boldsymbol{I}_p+\boldsymbol{X}\right|^{-(p+1)}.$$

2.2 Liouville distribution and integration

In this section the definition of the Liouville distribution of the first and second kind is given. Some results for integrals with matrix arguments are also given.

[**2.2.1**] *(Gupta and Richards, 1987, Equations 1.5 and 2.1)*
The $p\times p$ symmetric positive definite random matrices $\boldsymbol{X}_1,\ldots,\boldsymbol{X}_r$ are said to have the Liouville distribution of the *first kind* if their joint pdf exists and is proportional to

$$f\left(\sum_{i=1}^{r}\boldsymbol{X}_i\right)\prod_{i=1}^{r}\left|\boldsymbol{X}_i\right|^{a_i-\frac{1}{2}(p+1)},$$

$\boldsymbol{X}_i>\boldsymbol{0}$, $a_i>\frac{1}{2}\left(p-1\right),\ i=1,\ldots,r$. Here $f\left(\cdot\right)$ is positive, continuous, supported on $\boldsymbol{S}=\left\{\boldsymbol{X}\ (p\times p):\boldsymbol{X}>\boldsymbol{0}\right\}$ such that

$$\int_{\boldsymbol{T}>\boldsymbol{0}}\left|\boldsymbol{T}\right|^{a-\frac{1}{2}(p+1)}f\left(\boldsymbol{T}\right)\mathrm{d}\boldsymbol{T}<\infty,$$

where $a=\sum_{i=1}^{r}a_i$. The matrix Liouville distributions of the *second kind* are those for which $\boldsymbol{I}_p-\sum_{i=1}^{r}\boldsymbol{X}_i$ is also positive definite. The properties of this family of distributions are discussed by Gupta and Richards (1987).

[**2.2.2**] *(Díaz-García, 2009, Equation 2.3)*
The multivariate gamma function is defined as

$$\begin{aligned}\Gamma_p\left(a\right)&=\int_{\boldsymbol{S}>\boldsymbol{0}}\operatorname{etr}\left(-\boldsymbol{S}\right)\left|\boldsymbol{S}\right|^{a-\frac{1}{2}(p+1)}\mathrm{d}\boldsymbol{S}\\&=\pi^{\frac{1}{4}p(p-1)}\prod_{i=1}^{p}\Gamma\left[a-\tfrac{1}{2}\left(i-1\right)\right]\\&=\pi^{\frac{1}{4}p(p-1)}\prod_{i=1}^{p}\Gamma\left[a-\tfrac{1}{2}\left(p-i\right)\right],\end{aligned}$$

where $\operatorname{Re}(a) > \frac{1}{2}(p-1)$, and the integral is over the space of $p \times p$ symmetric positive definite matrices.

For $p = 1$ the gamma function is defined by

$$\Gamma(a) = \int_0^\infty e^{-s} s^{a-1} \mathrm{d}s.$$

[2.2.3] *(Herz, 1955, Equation 1.1)*
For $\operatorname{Re}(\boldsymbol{Z}) > \boldsymbol{0}$ and $\operatorname{Re}(a) > \frac{1}{2}(p-1)$, then

$$\int_{\boldsymbol{S}>\boldsymbol{0}} \operatorname{etr}(-\boldsymbol{S}\boldsymbol{Z}) |\boldsymbol{S}|^{a-\frac{1}{2}(p+1)} \mathrm{d}\boldsymbol{S} = \Gamma_p(a) |\boldsymbol{Z}|^{-a},$$

where $\Gamma_p(\cdot)$ is the multivariate gamma function given in [2.2.2].

[2.2.4] *(Gupta and Nagar, 2000b, Equation 1.4.7 and 1.4.8, page 20)*
The multivariate beta function, denoted by $\beta_p(a,b)$, is defined by

$$\begin{aligned}\beta_p(a,b) &= \int_{\boldsymbol{0}<\boldsymbol{S}<\boldsymbol{I}_p} |\boldsymbol{S}|^{a-\frac{1}{2}(p+1)} |\boldsymbol{I}_p - \boldsymbol{S}|^{b-\frac{1}{2}(p+1)} \mathrm{d}\boldsymbol{S} \\ &= \tfrac{\Gamma_p(a)\Gamma_p(b)}{\Gamma_p(a+b)} \\ &= \beta_p(b,a),\end{aligned}$$

where $\operatorname{Re}(a) > \frac{1}{2}(p-1)$, $\operatorname{Re}(b) > \frac{1}{2}(p-1)$ and $\Gamma_p(\cdot)$ is the multivariate gamma function given in [2.2.2].

For $p = 1$ the beta function is given by

$$\beta(a,b) = \int_0^1 s^{a-1}(1-s)^{b-1} \mathrm{d}s = \tfrac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

[2.2.5] *(Gupta and Nagar, 2000b, Equation 1.4.10 and 1.4.11, page 21)*
The multivariate Dirichlet function, denoted by $\beta_p(a_1, \ldots, a_r; b)$, is defined by

$$\begin{aligned}\beta_p(a_1,\ldots,a_r;b) &= \int \cdots \int_{\substack{\boldsymbol{0}<\sum_{i=1}^r \boldsymbol{Z}_i<\boldsymbol{I}_p \\ \boldsymbol{Z}_i>\boldsymbol{0}}} \prod_{i=1}^r |\boldsymbol{Z}_i|^{a_i-\frac{1}{2}(p+1)} \left|\boldsymbol{I}_p - \sum_{i=1}^r \boldsymbol{Z}_i\right|^{b-\frac{1}{2}(p+1)} \prod_{i=1}^r \mathrm{d}\boldsymbol{Z}_i \\ &= \tfrac{\Gamma_p(b)\prod_{i=1}^r \Gamma_p(a_i)}{\Gamma_p(a+b)}\end{aligned}$$

where $\operatorname{Re}(a_i) > \frac{1}{2}(p-1)$, $i = 1, \ldots, r$, $\operatorname{Re}(b) > \frac{1}{2}(p-1)$ and $a = \sum_{i=1}^r a_i$.

For $p = 1$ the Dirichlet function is given by

$$\beta(a_1,\ldots,a_r;b) = \tfrac{\Gamma(b)\prod_{i=1}^r \Gamma(a_i)}{\Gamma(a+b)}.$$

[2.2.6] *(Olkin, 1959, Equation 5.2; Gupta and Nagar, 2000b, Theorem 1.4.4, page 22)*
Let $f(\boldsymbol{V})$ be a continuous scalar function of the symmetric matrix $\boldsymbol{V}$ $(p \times p)$. Then for $\boldsymbol{B}$ $(p \times p) > \boldsymbol{A}$ $(p \times p) > \boldsymbol{0}$, $\mathrm{Re}(a_i) > \frac{1}{2}(p-1)$, $i = 1, \ldots, r$, and $a = \sum_{i=1}^{r} a_i$,

$$\begin{aligned} &\int \cdots \int_{\substack{\boldsymbol{A} < \sum_{i=1}^{r} \boldsymbol{Z}_i < \boldsymbol{B} \\ \boldsymbol{Z}_i > \boldsymbol{0}}} \prod_{i=1}^{r} |\boldsymbol{Z}_i|^{a_i - \frac{1}{2}(p+1)} f\left(\sum_{i=1}^{r} \boldsymbol{Z}_i\right) \prod_{i=1}^{r} \mathrm{d}\boldsymbol{Z}_i \\ = \;& \beta_p(a_1, \ldots a_{r-1}; a_r) \int_{\boldsymbol{A} < \boldsymbol{Z} < \boldsymbol{B}} |\boldsymbol{Z}|^{a - \frac{1}{2}(p+1)} f(\boldsymbol{Z})\, \mathrm{d}\boldsymbol{Z} \end{aligned}$$

where $\beta_p(a_1, \ldots a_{r-1}; a_r)$ is given in [2.2.5].

2.3 Zonal polynomials

A brief description of zonal polynomials and results involving zonal polynomials are given in this section. For a more detailed discussion see James (1960, 1961, 1964), Constantine (1963) and Khatri (1966).

[2.3.1] *(James, 1964)*
Let $\boldsymbol{S}$ be a $(p \times p)$ symmetric matrix and let V_k be the vector space of homogeneous polynomials $\varphi(\boldsymbol{S})$ of degree k in the $\frac{1}{2}p(p+1)$ distinct elements of $\boldsymbol{S}$. The dimension of V_k is $\binom{\frac{1}{2}p(p+1)-1}{k}$, the number of monomials

$$\sum_{i \leq j}^{p} s_{ij}^{k_{ij}}, \quad \text{of degree} \quad \sum_{i \leq j}^{p} k_{ij}.$$

Let $\boldsymbol{L}$ $(p \times p)$ be a nonsingular matrix. Corresponding to any congruence transformation $\boldsymbol{S} \rightarrow \boldsymbol{LSL}'$, a linear transformation of the space V_k of polynomials $\varphi(\boldsymbol{S})$ can be defined, that is

$$\varphi \rightarrow \boldsymbol{L}\varphi : (\boldsymbol{L}\varphi)(\boldsymbol{S}) = \varphi\left(\boldsymbol{L}^{-1}\boldsymbol{S}\boldsymbol{L}^{-1}\right).$$

A subspace $V' \subset V_k$ is called *invariant* if $\boldsymbol{L}V' \subset V'$ for all $\boldsymbol{L}$ and is called an *irreducible* invariant subspace if it has no proper invariant subspace. The space V_k decomposes into a direct sum of irreducible invariant subspaces V_κ corresponding to each partition κ of k into not more than p parts

$$V_k = \bigoplus_{\kappa} V_\kappa,$$

where $\kappa = (k_1, \ldots, k_p)$, $k_1 \geq \cdots \geq k_p \geq 0$, $k_1 + \cdots + k_p = k$. The polynomial $(\mathrm{tr}\boldsymbol{S})^k \in V_k$ has a unique decomposition

$$(\mathrm{tr}\boldsymbol{S})^k = \sum_{\kappa} C_\kappa(\boldsymbol{S})$$

into polynomials, $C_\kappa(\boldsymbol{S}) \in V_\kappa$, belonging to the respective invariant subspaces.

The zonal polynomial $C_\kappa(\boldsymbol{S})$ is defined as the component of $(\mathrm{tr}\boldsymbol{S})^k$ in the subspace V_κ. It is a symmetric homogeneous polynomial of degree k in the latent roots of $\boldsymbol{S}$ and holds for all p. If the partition κ has more than p parts, the corresponding zonal polynomial will be identically zero.

[2.3.2] *(Díaz-García, 2009)*
Let $\kappa = (k_1, \ldots, k_p)$, $k_1 \geq \cdots \geq k_p \geq 0$, $k_1 + \cdots + k_p = k$. The generalised hypergeometric coefficient $(a)_\kappa$, also known as the generalised Pochhammer symbol of weight κ, is defined as

$$\begin{aligned}(a)_\kappa &= \prod_{i=1}^{p} \left(a - \tfrac{1}{2}(i-1)\right)_{k_i} \\ &= \frac{\pi^{\frac{1}{4}p(p-1)} \prod_{j=1}^{p} \Gamma\left[a + k_j - \frac{1}{2}(j-1)\right]}{\Gamma_p(a)},\end{aligned}$$

where $\mathrm{Re}(a) > \frac{1}{2}(p-1) - k_p$, $\Gamma(\cdot)$ is the gamma function defined in [2.2.2].

For $p = 1$,

$$(a)_k = a(a+1)\ldots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}, \quad (a)_0 = 1, \quad a \neq 0,$$

is the standard Pochhammer symbol.

[2.3.3] *(Díaz-García, 2009, Equations 2.4 and 2.6)*
Let $\kappa = (k_1, \ldots, k_p)$, $k_1 \geq \cdots \geq k_p \geq 0$, $k_1 + \cdots + k_p = k$. The generalised gamma function of weight κ is defined by

$$\begin{aligned}\Gamma_p(a, \kappa) &= \int_{\boldsymbol{S}>\boldsymbol{0}} \mathrm{etr}(-\boldsymbol{S}) \left|\boldsymbol{S}\right|^{a - \frac{1}{2}(p+1)} q_\kappa(\boldsymbol{S}) \mathrm{d}\boldsymbol{S} \\ &= \pi^{\frac{1}{4}p(p-1)} \prod_{j=1}^{p} \Gamma\left[a + k_j - \tfrac{1}{2}(j-1)\right] \\ &= \pi^{\frac{1}{4}p(p-1)} \prod_{j=1}^{p} \Gamma\left[a + k_j - \tfrac{1}{2}(p-j)\right] \\ &= (a)_\kappa \Gamma_p(a),\end{aligned}$$

where the integral is over the space of $p \times p$ symmetric positive definite matrices, q_κ is a homogeneous polynomial of degree k, $(a)_\kappa$ is given in [2.3.2], $\mathrm{Re}(a) \geq \frac{1}{2}(p-1) - k_p$ and $\Gamma_p(a, 0) = \Gamma_p(a)$.
Also,

$$\begin{aligned}\Gamma_p(a, -\kappa) &= \int_{\boldsymbol{S}>\boldsymbol{0}} \mathrm{etr}(-\boldsymbol{S}) \left|\boldsymbol{S}\right|^{a - \frac{1}{2}(p+1)} q_\kappa\left(\boldsymbol{S}^{-1}\right) \mathrm{d}\boldsymbol{S} \\ &= \pi^{\frac{1}{4}p(p-1)} \prod_{j=1}^{p} \Gamma\left[a - k_j - \tfrac{1}{2}(p-j)\right] \\ &= \pi^{\frac{1}{4}p(p-1)} \prod_{j=1}^{p} \Gamma\left[a - k_j - \tfrac{1}{2}(j-1)\right] \\ &= \frac{(-1)^k \Gamma_p(a)}{\left[-a + \frac{1}{2}(p-1) + 1\right]_\kappa},\end{aligned}$$

where $\mathrm{Re}(a) \geq \frac{1}{2}(p-1) + k_1$.

[2.3.4] *(James, 1961, Equation 29; Muirhead, 1982, Theorem 7.2.5, page 243)*
If $\boldsymbol{S}\ (p \times p)$ is a symmetric matrix and $\boldsymbol{R}\ (p \times p) > \boldsymbol{0}$, then

$$\int_{O(p)} C_\kappa\left(\boldsymbol{RHSH}'\right) \mathrm{d}\boldsymbol{H} = \frac{C_\kappa(\boldsymbol{R}) C_\kappa(\boldsymbol{S})}{C_\kappa(\boldsymbol{I}_p)},$$

where $O(p) = \left\{\boldsymbol{H}\ (p \times p) \,|\, \boldsymbol{HH}' = \boldsymbol{H}'\boldsymbol{H} = \boldsymbol{I}_p\right\}$ and $\mathrm{d}\boldsymbol{H}$ denotes the normalised Haar invariant measure on the orthogonal group $O(p)$ (see Muirhead, 1982, page 72).

[2.3.5] *(Gupta and Nagar, 2000b, Equation 1.5.3, page 30)*
If $\boldsymbol{S}\ (p \times p)$ is a symmetric matrix and $\boldsymbol{R}\ (p \times p) > \boldsymbol{0}$ then

$$C_\kappa(\boldsymbol{RS}) = C_\kappa\left(\boldsymbol{R}^{\frac{1}{2}}\boldsymbol{S}\boldsymbol{R}^{\frac{1}{2}}\right).$$

[2.3.6]
If $\boldsymbol{S}\ (p \times p)$ is a symmetric matrix, $\boldsymbol{R}\ (p \times p) > \boldsymbol{0}$ and $\boldsymbol{T}\ (p \times p) > \boldsymbol{0}$, then

$$\int_{O(p)} C_\kappa\left(\boldsymbol{R}^{\frac{1}{2}}\boldsymbol{T}\boldsymbol{R}^{\frac{1}{2}}\boldsymbol{H}\boldsymbol{S}\boldsymbol{H}'\right)\mathrm{d}\boldsymbol{H} = \int_{O(p)} C_\kappa\left(\boldsymbol{T}^{\frac{1}{2}}\boldsymbol{R}\boldsymbol{T}^{\frac{1}{2}}\boldsymbol{H}\boldsymbol{S}\boldsymbol{H}'\right)\mathrm{d}\boldsymbol{H}.$$

Proof:
From [2.3.4] and [2.3.5] it follows that

$$\begin{aligned}\int_{O(p)} C_\kappa\left(\boldsymbol{R}^{\frac{1}{2}}\boldsymbol{T}\boldsymbol{R}^{\frac{1}{2}}\boldsymbol{H}\boldsymbol{S}\boldsymbol{H}'\right)\mathrm{d}\boldsymbol{H} &= \frac{C_\kappa\left(\boldsymbol{R}^{\frac{1}{2}}\boldsymbol{T}\boldsymbol{R}^{\frac{1}{2}}\right)C_\kappa(\boldsymbol{S})}{C_\kappa(\boldsymbol{I}_p)} \\ &= \frac{C_\kappa\left(\boldsymbol{T}^{\frac{1}{2}}\boldsymbol{R}\boldsymbol{T}^{\frac{1}{2}}\right)C_\kappa(\boldsymbol{S})}{C_\kappa(\boldsymbol{I}_p)} \\ &= \int_{O(p)} C_\kappa\left(\boldsymbol{T}^{\frac{1}{2}}\boldsymbol{R}\boldsymbol{T}^{\frac{1}{2}}\boldsymbol{H}\boldsymbol{S}\boldsymbol{H}'\right)\mathrm{d}\boldsymbol{H}. \quad \blacksquare\end{aligned}$$

[2.3.7] *(In analogy to Khatri, 1966, Equation 13)*
If $\boldsymbol{R}\ (p \times p)$ is a symmetric matrix, then

$$\int_{\boldsymbol{S}>\boldsymbol{0}} |\boldsymbol{S}|^{a-\frac{1}{2}(p+1)} |\boldsymbol{I}_p + \boldsymbol{S}|^{-(a+b)} C_\kappa(\boldsymbol{RS})\mathrm{d}\boldsymbol{S} = \frac{\Gamma_p(a,\kappa)\Gamma_p(b,-\kappa)}{\Gamma_p(a+b)} C_\kappa(\boldsymbol{R}),$$

where $\mathrm{Re}\,(a) > \frac{1}{2}(p-1) - k_p$, $\mathrm{Re}\,(b) > \frac{1}{2}(p-1) + k_1$ and where $\Gamma_p(a,\kappa)$ and $\Gamma_p(b,-\kappa)$ are given in [2.3.3].

[2.3.8] *(Davis, 1979, Equation 3.2)*
If $\boldsymbol{S}\ (p \times p)$ is a symmetric matrix, then

$$\int_{\boldsymbol{0}<\boldsymbol{S}<\boldsymbol{I}_p} |\boldsymbol{S}|^{a-\frac{1}{2}(p+1)} \frac{C_\phi(\boldsymbol{S})}{C_\phi(\boldsymbol{I}_p)} d\boldsymbol{S} = \frac{\Gamma_p(a,\phi)\Gamma_p\left(\frac{p+1}{2}\right)}{\Gamma_p\left(a+\frac{p+1}{2},\phi\right)}.$$

2.4 Invariant polynomials

Davis (1979, 1980, 1981), Chikuse (1980) and Chikuse and Davis (1986) introduced a class of homogeneous invariant polynomials with two or more matrix arguments. The definition as well as some properties of invariant polynomials are given in this section.

[2.4.1] *(Davis, 1979, 1980, 1981)*
A class of invariant polynomials $C_\phi^{\kappa,\tau}(\boldsymbol{X},\boldsymbol{Y})$ in the elements of the $p \times p$ symmetric matrices $\boldsymbol{X}$ and $\boldsymbol{Y}$ is defined when having the property of invariance under the simultaneous transformations

$$\boldsymbol{X} \to \boldsymbol{H}\boldsymbol{X}\boldsymbol{H}', \quad \boldsymbol{Y} \to \boldsymbol{H}\boldsymbol{Y}\boldsymbol{H}', \quad \boldsymbol{H} \in O(p),$$

where $O(p)$ is the group of $p \times p$ orthogonal matrices. The following relationship is satisfied:

$$\int_{O(p)} C_\kappa \left(\boldsymbol{A}\boldsymbol{H}'\boldsymbol{X}\boldsymbol{H}\right) C_\tau \left(\boldsymbol{B}\boldsymbol{H}'\boldsymbol{Y}\boldsymbol{H}\right) \mathrm{d}\boldsymbol{H} = \sum_{\phi \in \kappa \cdot \tau} C_\phi^{\kappa,\tau}\left(\boldsymbol{A},\boldsymbol{B}\right) C_\phi^{\kappa,\tau}\left(\boldsymbol{X},\boldsymbol{Y}\right) / C_\phi\left(\boldsymbol{I}_p\right),$$

where C_κ, C_τ and C_ϕ are zonal polynomials indexed by the ordered partitions κ, τ and ϕ of the nonnegative integers k, t and $f = k + t$ respectively into $\leq p$ parts. Furthermore $\phi \in \kappa \cdot \tau$ signifies that the irreducible representation of the group of $p \times p$ real nonsingular matrices, indexed by 2ϕ occurs in the decomposition of the Kronecker product $2\kappa \otimes 2\tau$ of the irreducible representations indexed by 2κ and 2τ.

Some properties of invariant polynomials are given in [2.4.2] to [2.4.5].

[2.4.2] *(Davis, 1979, Equation 2.8)*
If $\boldsymbol{X}$ $(p \times p)$ and $\boldsymbol{Y}$ $(p \times p)$ are symmetric matrices, then

$$C_\kappa\left(\boldsymbol{X}\right) C_\tau\left(\boldsymbol{Y}\right) = \sum_{\phi \in \kappa \cdot \tau} \theta_\phi^{\kappa,\tau} C_\phi^{\kappa,\tau}(\boldsymbol{X},\boldsymbol{Y}),$$

where $\theta_\phi^{\kappa,\tau} = \frac{C_\phi^{\kappa,\tau}(\boldsymbol{I}_p,\boldsymbol{I}_p)}{C_\phi(\boldsymbol{I}_p)}$.

[2.4.3] *(Davis, 1979, Equation 2.10)*
If $\boldsymbol{X}$ $(p \times p)$ is a symmetric matrix, then

$$C_\kappa\left(\boldsymbol{X}\right) C_\tau\left(\boldsymbol{X}\right) = \sum_{\phi^* \in \kappa \cdot \tau} g_{\kappa,\tau}^{\phi^*} C_{\phi^*}(\boldsymbol{X}), \qquad g_{\kappa,\tau}^{\phi^*} = \sum_{\phi \equiv \phi^*} \left(\theta_\phi^{\kappa,\tau}\right)^2,$$

where $\sum\limits_{\phi^* \in \kappa \cdot \tau}$ implies that we sum over the inequivalent representations $2\phi^*$ occurring in $2\kappa \otimes 2\tau$, and $\sum\limits_{\phi \equiv \phi^*}$ denotes summation over the representations equivalent to $2\phi^*$ in $2\kappa \otimes 2\tau$.

[2.4.4] *(Díaz-García, 2009, Equation 5.13)*
If $\boldsymbol{A}$ $(p \times p)$ and $\boldsymbol{B}$ $(p \times p)$ are symmetric matrices and $\mathrm{Re}\left(\boldsymbol{Z}\right) > \boldsymbol{0}$, then

$$\int_{\boldsymbol{X}>\boldsymbol{0}} \mathrm{etr}\left(-\boldsymbol{X}\boldsymbol{Z}\right) \left|\boldsymbol{X}\right|^{a-\frac{1}{2}(p+1)} C_\phi^{\kappa,\tau}(\boldsymbol{A}\boldsymbol{X},\boldsymbol{B}\boldsymbol{X}) \mathrm{d}\boldsymbol{X} = \Gamma_p\left(a,\phi\right) \left|\boldsymbol{Z}\right|^{-a} C_\phi^{\kappa,\tau}(\boldsymbol{A}\boldsymbol{Z}^{-1},\boldsymbol{B}\boldsymbol{Z}^{-1}),$$

where $\mathrm{Re}\left(a\right) > \frac{1}{2}\left(p-1\right) + \left(k+t\right)_1$.

[2.4.5] *(Díaz-García, 2009, Equation 5.21)*
If $\boldsymbol{A}$ $(p \times p)$ and $\boldsymbol{B}$ $(p \times p)$ are symmetric matrices, then

$$\int_{\boldsymbol{0}<\boldsymbol{X}<\boldsymbol{I}_p} \left|\boldsymbol{X}\right|^{a-\frac{1}{2}(p+1)} \left|\boldsymbol{I}_p - \boldsymbol{X}\right|^{b-\frac{1}{2}(p+1)} C_\phi^{\kappa,\tau}(\boldsymbol{A}\boldsymbol{X},\boldsymbol{B}\boldsymbol{X}) \mathrm{d}\boldsymbol{X} = \frac{\Gamma_p(a,\phi)\Gamma_p(b)}{\Gamma_p(a+b,\phi)} C_\phi^{\kappa,\tau}(\boldsymbol{A},\boldsymbol{B}),$$

where $\mathrm{Re}\left(a\right) > \frac{1}{2}\left(p-1\right) - \left(k+t\right)_p$.

2.5 Hypergeometric function

In this section the definitions of the hypergeometric and Appell functions are given.

[2.5.1] *(Mathai, 1993, Definition 2.2, page 96)*

The hypergeometric series with m upper parameters and n lower parameters is defined as

$$ {}_mF_n\left(a_1,\ldots,a_m;b_1,\ldots,b_n;z\right)=\sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_m)_k}{(b_1)_k\cdots(b_n)_k}\frac{z^k}{k!} $$

where $(a)_k$ is the standard Pochhammer symbol given in [2.3.2], that is $(a)_k=\frac{\Gamma(a+k)}{\Gamma(a)}$.

The following holds for the series in [2.5.1]:

(i) if any a_i, $i=1,\ldots m$, is a negative integer or zero the series terminates and ${}_mF_n$ becomes a polynomial in z provided none of b_j, $j=1,\ldots,n$, is zero or a negative integer;

(ii) if any b_j, $j=1,\ldots n$, is zero or a negative integer then the series is not defined unless there is an a_i, $i=1,\ldots m$, such that $(a_i)_k$ becomes zero first. That is, suppose a_i and b_j are two negative integers such that $(a_i)_k=0$ for $k\geq r$ and $(b_j)_k=0$ for $k\geq t$. Then in order for ${}_mF_n$ to be defined r must be less than t;

(iii) the series converges for all z if $m\leq n$ and for $|z|<1$ if $m=n+1$;

(iv) the series diverges for all z, $z\neq 0$ for $m>n+1$;

(v) if $m=n+1$ and $|z|=1$, the series is absolutely convergent if $\operatorname{Re}(\beta)<0$ where $\beta=\sum_{j=1}^{m}a_j-\sum_{j=1}^{n}b_j$; divergent if $\operatorname{Re}(\beta)\geq 1$; and if $m=n+1$ and $|z|=1$, $z\neq 1$, the series is conditionally convergent if $0\leq\operatorname{Re}(\beta)<1$.

[2.5.2] *(Mathai, 1993, page 97)*

$$ {}_1F_0\left(\alpha;z\right)=(1-z)^{-\alpha}. $$

where $|z|<1$.

[2.5.3] *(Mathai, 1993, page 97)*

$$ {}_2F_1\left(\alpha,\beta;\gamma;z\right)=\frac{\Gamma_p(\gamma)}{\Gamma_p(\alpha)\Gamma_p(\gamma-\alpha)}\int_0^1 x^{\alpha-1}\left(1-x\right)^{\gamma-\alpha-1}\left(1-zx\right)^{-\beta}\mathrm{d}x, $$

where $|z|<1$, $\operatorname{Re}(\gamma)>\operatorname{Re}(\alpha)>0$. This is known as the *Gauss hypergeometric function.*

[2.5.4] *(Gradshteyn and Ryzhik, 2000, Equation 9.131(1), page 998)*

$$ {}_2F_1\left(\alpha,\beta;\gamma;z\right)=(1-z)^{-\beta}\ {}_2F_1\left(\gamma-\alpha,\beta;\gamma;-\frac{z}{1-z}\right). $$

[2.5.5] *(Gradshteyn and Ryzhik, 2000, Equation 7.512(12), page 807)*

$$\begin{aligned}&\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\ {}_mF_n(a_1,\ldots,a_m;b_1,\ldots b_n;zx)\,\mathrm{d}x\\ =\ &\tfrac{\Gamma_p(\alpha)\Gamma_p(\beta)}{\Gamma_p(\alpha+\beta)}\ {}_{m+1}F_{n+1}(a_1,\ldots,a_m,\alpha;b_1,\ldots b_n,\alpha+\beta;z)\,.\end{aligned}$$

[2.5.6] *(Gradshteyn and Ryzhik, 2000, Equation 9.180(1), page 1008)*

The Appell function of the first kind is defined by

$$F_1\left(\alpha,\beta,\beta';\gamma;x,y\right)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\tfrac{(\alpha)_{m+n}(\beta)_m\left(\beta'\right)_n}{(\gamma)_{m+n}m!n!}x^my^n,$$

where $(f)_m=f\,(f+1)\cdots(f+m-1)$ denotes the ascending factorial, $|x|<1$, $|y|<1$.

[2.5.7] *(Gradshteyn and Ryzhik, 2000, Equation 9.184(1), page 1011)*

$$F_1\left(\alpha,\beta,\beta',\gamma;x,y\right)=\tfrac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta-\beta')}\int\int_{\substack{u\geq 0,v\geq 0\\ u+v\leq 1}} u^{\beta-1}v^{\beta'-1}(1-u-v)^{\gamma-\beta-\beta'-1}(1-ux-vy)^{-\alpha}\,\mathrm{d}u\mathrm{d}v,$$

$\mathrm{Re}\,(\beta)>0$, $\mathrm{Re}\left(\beta'\right)>0$, $\mathrm{Re}\left(\gamma-\beta-\beta'\right)>0$.

[2.5.8] *(Gradshteyn and Ryzhik, 2000 Equation 9.182 (11), page 1010)*

$$F_1\left(\alpha,\beta,\beta',\gamma;x,x\right)=\ {}_2F_1\left(\alpha,\beta+\beta';\gamma;x\right).$$

2.6 Hypergeometric function of matrix argument

The definition of the hypergeometric function of matrix argument is given in this section as well as some results involving this function (see Herz, 1955; Constantine, 1963; Gupta and Nagar, 2000b).

[2.6.1] *(Constantine, 1963; Gupta and Nagar, 2000b, Definition 1.6.1, page 34)*
The hypergeometric function of matrix argument is defined by

$${}_mF_n(a_1,\ldots,a_m;b_1,\ldots,b_n;\boldsymbol{S})=\sum_{k=0}^{\infty}\sum_{\kappa}\tfrac{(a_1)_\kappa\cdots(a_m)_\kappa}{(b_1)_\kappa\cdots(b_n)_\kappa}\tfrac{1}{k!}C_\kappa(\boldsymbol{S}),$$

where a_i, $i=1,\ldots,m$; b_j, $j=1,\ldots,n$ are arbitrary numbers, $\boldsymbol{S}$ $(p\times p)$ is a real symmetric matrix, $\sum_\kappa$ denotes summation over all partitions κ and $C_\kappa(\boldsymbol{S})$ is the zonal polynomial of $\boldsymbol{S}$ (see [2.3.1]). The generalised hypergeometric coefficient $(a)_\kappa$ is given in [2.3.2] and from [2.3.3] it can be written as $(a)_\kappa=\frac{\Gamma_p(a,\kappa)}{\Gamma_p(a)}$.

The following are conditions for the convergence of the series in [2.6.1]:

(i) the series converges for all $\boldsymbol{S}$ $(p\times p)$ if $m<n+1$, otherwise the series may only converge for $\boldsymbol{S}=\mathbf{0}$;

(ii) for $m=n+1$ the series converges for $||\boldsymbol{S}||<1$ (where $||\boldsymbol{S}||$ denotes the maximum of the absolute values of the characteristic roots of $\boldsymbol{S}$);

(iii) for $m \leq n$ the series converges for all $\boldsymbol{S}$;

(iv) for $m > n+1$ the series diverges for all $\boldsymbol{S} \neq \boldsymbol{0}$ unless the series terminates;

(v) none of the b_j is zero, an integer or half integer $\leq \frac{1}{2}(p-1)$ (otherwise some of the denominators in [2.6.1] will vanish);

(vi) if a_i is a negative integer, say $-r$, then for $k \geq pr+1$, all coefficients in [2.6.1] vanish and the function reduces to a finite polynomial of degree pr.

The results in [2.6.2] to [2.6.5] are some special cases of [2.6.1].

[2.6.2] *(Constantine, 1963, Equation 30)*

$$\begin{aligned} {}_0F_0(\boldsymbol{S}) &= \sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa(\boldsymbol{S})}{k!} \\ &= \sum_{k=0}^{\infty}\frac{(\operatorname{tr}\boldsymbol{S})^k}{k!} \\ &= \operatorname{etr}(\boldsymbol{S}). \end{aligned}$$

[2.6.3] *(Herz, 1955)*
If $\boldsymbol{Z}$ $(p \times p)$ is a symmetric matrix where $\operatorname{Re}(\boldsymbol{Z}) < \boldsymbol{I}_p$, then

$$\begin{aligned} {}_1F_0(\alpha; \boldsymbol{Z}) &= \frac{1}{\Gamma_p(\alpha)}\int_{\boldsymbol{S}>\boldsymbol{0}} \operatorname{etr}\left[-\boldsymbol{S}(\boldsymbol{I}_p - \boldsymbol{Z})\right]|\boldsymbol{S}|^{\alpha-\frac{1}{2}(p+1)}\,\mathrm{d}\boldsymbol{S} \\ &= |\boldsymbol{I}_p - \boldsymbol{Z}|^{-\alpha}, \end{aligned}$$

where $\operatorname{Re}(\alpha) > \frac{1}{2}(p-1)$.

[2.6.4] *(Gupta and Nagar, 2000b, Equation 1.6.7, page 36)*
If $\boldsymbol{R}$ $(p \times p)$ is a symmetric matrix, then

$${}_1F_1(\alpha; \beta; \boldsymbol{R}) = \frac{\Gamma_p(\beta)}{\Gamma_p(\alpha)\Gamma_p(\beta-\alpha)}\int_{\boldsymbol{0}<\boldsymbol{S}<\boldsymbol{I}_p} |\boldsymbol{S}|^{\alpha-\frac{1}{2}(p+1)}\,|\boldsymbol{I}_p - \boldsymbol{S}|^{\beta-\alpha-\frac{1}{2}(p+1)}\operatorname{etr}(\boldsymbol{R}\boldsymbol{S})\,\mathrm{d}\boldsymbol{S},$$

where $\operatorname{Re}(\alpha) > \frac{1}{2}(p-1)$, $\operatorname{Re}(\beta) > \frac{1}{2}(p-1)$ and $\operatorname{Re}(\beta-\alpha) > \frac{1}{2}(p-1)$. This is the *confluent hypergeometric function* of matrix argument.

[2.6.5] *(Gupta and Nagar, 2000b, Equation 1.6.8, page 36)*
If $\boldsymbol{R}$ $(p \times p)$ is a symmetric matrix where $\operatorname{Re}(\boldsymbol{R}) < \boldsymbol{I}_p$, then

$${}_2F_1(\alpha, \beta; \gamma; \boldsymbol{R}) = \frac{\Gamma_p(\gamma)}{\Gamma_p(\alpha)\Gamma_p(\gamma-\alpha)}\int_{\boldsymbol{0}<\boldsymbol{S}<\boldsymbol{I}_p} |\boldsymbol{S}|^{\alpha-\frac{1}{2}(p+1)}\,|\boldsymbol{I}_p - \boldsymbol{S}|^{\gamma-\alpha-\frac{1}{2}(p+1)}\,|\boldsymbol{I}_p - \boldsymbol{R}\boldsymbol{S}|^{-\beta}\,\mathrm{d}\boldsymbol{S},$$

where $\operatorname{Re}(\gamma) > \frac{1}{2}(p-1)$ and $\operatorname{Re}(\gamma-\alpha) > \frac{1}{2}(p-1)$. This is known as the *Gauss hypergeometric function* of matrix argument.

Further results involving hypergeometric functions of matrix argument are given in [2.6.6] to [2.6.8].

[2.6.6] *(In analogy to Gupta and Nagar, 2000b, Equation 1.6.4, page 35)*

If $\boldsymbol{R}\,(p\times p)>\mathbf{0}$ and $\boldsymbol{T}\,(p\times p)$ is a symmetric matrix, then

$$\begin{aligned}&\int_{\boldsymbol{S}>\mathbf{0}}\operatorname{etr}\left(-\boldsymbol{RS}\right)|\boldsymbol{S}|^{\alpha-\frac{1}{2}(p+1)}\ {}_mF_n\left(a_1,\ldots,a_m;b_1,\ldots b_n;\boldsymbol{S}\boldsymbol{T}\right)\mathrm{d}\boldsymbol{S}\\ =\ &\Gamma_p\left(\alpha\right)|\mathbf{R}|^{-\alpha}\ {}_{m+1}F_n\left(a_1,\ldots,a_m,\alpha;b_1,\ldots b_n;\boldsymbol{TR}^{-1}\right),\end{aligned}$$

where $\operatorname{Re}\left(\alpha\right)>\frac{1}{2}\left(p-1\right).$

[2.6.7] *(Gupta and Nagar, 2000b, Equation 1.6.10, page 37)*

$${}_2F_1\left(\alpha,\beta;\gamma;\mathbf{S}\right)=|\boldsymbol{I}_p-\mathbf{S}|^{-\beta}\ {}_2F_1\left(\gamma-\alpha,\beta;\gamma;-\boldsymbol{S}\left(\boldsymbol{I}_p-\mathbf{S}\right)^{-1}\right)$$

[2.6.8] *(Gupta and Nagar, 2000b, Equation 1.6.6, page 36)*

If $\boldsymbol{R}\,(p\times p)$ is a symmetric matrix, then

$$\begin{aligned}&\int_{\mathbf{0}<\boldsymbol{S}<\boldsymbol{I}_p}|\boldsymbol{S}|^{\alpha-\frac{1}{2}(p+1)}\,|\boldsymbol{I}_p-\mathbf{S}|^{\beta-\frac{1}{2}(p+1)}\ {}_mF_n\left(a_1,\ldots,a_m;b_1,\ldots b_n;\boldsymbol{RS}\right)\mathrm{d}\boldsymbol{S}\\ =\ &\tfrac{\Gamma_p(\alpha)\Gamma_p(\beta)}{\Gamma_p(\alpha+\beta)}\ {}_{m+1}F_{n+1}\left(a_1,\ldots,a_m,\alpha;b_1,\ldots b_n,\alpha+\beta;\mathbf{R}\right).\end{aligned}$$

[2.6.9] *(Gupta and Nagar, 2000b, Definition 1.6.2, page 34)*

The hypergeometric function of two symmetric matrices, $\boldsymbol{S}\,(p\times p)$ and $\boldsymbol{T}\,(p\times p)$ is defined by

$${}_mF_n^{(p)}\left(a_1,\ldots,a_m;b_1,\ldots,b_n;\mathbf{S},\boldsymbol{T}\right)=\sum_{k=0}^{\infty}\sum_{\kappa}\tfrac{(a_1)_\kappa\cdots(a_m)_\kappa}{(b_1)_\kappa\cdots(b_n)_\kappa}\tfrac{1}{k!}\tfrac{C_\kappa(\boldsymbol{S})C_\kappa(\boldsymbol{T})}{C_\kappa(\boldsymbol{I}_p)}.$$

Conditions of convergence are similar to those given in [2.6.1] with the exception that for $m=n+1$ the series converges for $\|\boldsymbol{S}\|<1$ or $\|\boldsymbol{T}\|<1$.

2.7 Laguerre polynomials

Herz (1955) introduced Laguerre polynomials of matrix argument and Constantine (1966) gave integral representations.

[2.7.1] *(Constantine, 1966)*

The Laguerre polynomial $L_\kappa^\gamma\left(\boldsymbol{S}\right)$ of a symmetric matrix $\boldsymbol{S}\,(p\times p)$ corresponding to the partition κ of k is defined as

$$L_\kappa^\gamma\left(\boldsymbol{S}\right)=\operatorname{etr}\left(\boldsymbol{S}\right)\int_{\boldsymbol{S}>\mathbf{0}}\operatorname{etr}\left(-\mathbf{R}\right)|\mathbf{R}|^{\gamma}\,C_\kappa\left(\mathbf{R}\right)A_\gamma\left(\boldsymbol{RS}\right)\mathrm{d}\boldsymbol{R},$$

where $A_\gamma\left(\boldsymbol{RS}\right)=\frac{1}{\Gamma_p\left(\gamma+\frac{p+1}{2}\right)}\ {}_0F_1\left(\gamma+\frac{p+1}{2};-\boldsymbol{RS}\right)$ and $\operatorname{Re}\left(\gamma\right)>-1.$

[2.7.2] *(Muirhead, 1982, Theorem 7.6.3, page 283)*
If $\boldsymbol{S}\,(p \times p) > \boldsymbol{0}$, then

$$|\boldsymbol{I}_p - \boldsymbol{Z}|^{-\gamma-\frac{1}{2}(p+1)}\ {}_0F_0^{(p)}\left(-\boldsymbol{S}, \boldsymbol{Z}\,(\boldsymbol{I}_p - \boldsymbol{Z})^{-1}\right) = \sum_{k=0}^{\infty}\sum_{\kappa}\frac{1}{k!}\frac{L_\kappa^\gamma(\boldsymbol{S})C_\kappa(\boldsymbol{Z})}{C_\kappa(\boldsymbol{I}_p)},$$

where $\|\boldsymbol{Z}\| < 1$.

[2.7.3] *(Gupta and Nagar, 2000b, Equation 1.7.4, page 41)*
If $\boldsymbol{S}\,(p \times p)$ is a symmetric matrix, then

$$L_\kappa^\gamma(\boldsymbol{S}) = \left(\gamma + \tfrac{p+1}{2}\right)_\kappa C_\kappa(\boldsymbol{I}_p)\sum_{t=0}^{k}\sum_{\tau}\binom{\kappa}{\tau}\frac{C_\tau(-\boldsymbol{S})}{\left(\gamma+\frac{p+1}{2}\right)_\tau C_\tau(\boldsymbol{I}_p)}.$$

2.8 Mellin transform, Meijer's G-function and Fox's H-function

In this section the definitions of the Mellin transform and inverse Mellin transform are given as well as the definitions for Meijer's G-function and Fox's H-function. Some properties of Meijer's G-function are also given.

[2.8.1] *(Mathai, 1993, Definition 1.8, page 23)*
If $f(x)$ is a real function which is single valued almost everywhere for $x \geq 0$ and if the integral

$$\int_0^\infty x^{k-1}\,|f(x)|\,\mathrm{d}x$$

converges for some value of k then the Mellin transform of $f(x)$ is defined as follows:

$$M_f(s) \equiv \int_0^\infty x^{s-1} f(x)\,\mathrm{d}x$$

where $M_f(s)$ is the Mellin transform of f with respect to the parameter s and s is a complex number. The inverse Mellin transform is given by the inverse integral

$$f(x) = \frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty} M_f(s)\,x^{-s}\mathrm{d}s$$

where $i = \sqrt{-1}$ and ω is a real number in the strip of analyticity of $M_f(s)$.

[2.8.2] *(Mathai, 1993, Definition 2.1, page 60)*
Meijer's G-function with the parameters $a_1, \ldots, a_p$ and $b_1, \ldots, b_q$ is defined as follows:

$$G_{p,q}^{m,n}\left(z\middle|_{b_1,\ldots,b_q}^{a_1,\ldots,a_p}\right) = \frac{1}{2\pi i}\int_L g(s)\,z^{-s}\mathrm{d}s$$

where $i = \sqrt{-1}$, L is a suitable contour, $z \neq 0$,

$$g(s) = \frac{\prod_{j=1}^m \Gamma(b_j+s)\prod_{j=1}^n \Gamma(1-a_j-s)}{\prod_{j=m+1}^q \Gamma(1-b_j-s)\prod_{j=n+1}^p \Gamma(a_j+s)}$$

where m, n, p and q are integers with $0 \leq n \leq p$ and $0 \leq m \leq q$.

The parameters $a_1,\ldots,a_p$ *and* $b_1,\ldots,b_q$ are complex numbers such that no pole of $\Gamma(b_j+s), j=1,\ldots,m$ coincides with any pole of $\Gamma(1-a_k-s)$, $k=1,\ldots,n$.
The empty product is interpreted as 0.

[2.8.3] *(Springer, 1979, page 195; Mathai, 1993, Definition 3.1, page 140)*
Fox's H-function is defined as follows:

$$H_{p,q}^{m,n}\left(z|_{(b_1,\beta_1),\ldots,(b_q,\beta_q)}^{(a_1,\alpha_1),\ldots,(a_p,\alpha_p)}\right)=\frac{1}{2\pi i}\int_C g(s)\,z^{-s}\mathrm{d}s$$

where

$$g(s)=\frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j+\alpha_j s)}$$

and where $0\leq m\leq q, 0\leq n\leq p, \alpha_j>0$ for $j=1,2,\ldots,p$, $\beta_j>0$ for $j=1,2,\ldots,q$, and a_j $(j=1,2,\ldots,p)$ and b_j $(j=1,2,\ldots,q)$ are complex numbers such that no pole of $\Gamma\left(b_j+\beta_j s\right)$ for $j=1,2,\ldots,m$ coincides with any pole of $\Gamma(1-a_j-\alpha_j s)$ for $j=1,2,\ldots,n$. Furthermore, C is a contour in the complex s-plane running from $\omega-i\infty$ to $\omega+i\infty$ for some real number ω.

[2.8.4] *(Mathai, 1993, Equation 2.2.1, page 69)*

$$z^{\alpha}G_{p,q}^{m,n}\left(z|_{b_1,\ldots,b_q}^{a_1,\ldots,a_p}\right)=G_{p,q}^{m,n}\left(z|_{b_1+\alpha,\ldots,b_q+\alpha}^{a_1+\alpha,\ldots,a_p+\alpha}\right).$$

[2.8.5] *(Mathai, 1993, Equation 3, page 131)*

$$G_{2,2}^{2,0}\left(z|_{\alpha_1-1,\alpha_2-1}^{\alpha_1+\beta_1-1,\alpha_2+\beta_2-1}\right)=\frac{z^{\alpha_2-1}(1-z)^{\beta_1+\beta_2-1}}{\Gamma(\beta_1+\beta_2)}\,{}_2F_1(\alpha_2+\beta_2-\alpha_1,\beta_1;\beta_1+\beta_2;1-z),\ |z|<1.$$

2.9 Symmetrised density function

[2.9.1]
Given a density function $f(\boldsymbol{X})$, $\boldsymbol{X}:p\times p$, $\boldsymbol{X}>0$, the symmetrised density function is defined by Greenacre (1973) as

$$f_s(\boldsymbol{X})\equiv\int_{O(p)}f\left(\boldsymbol{HXH}'\right)\mathrm{d}\boldsymbol{H},\qquad \boldsymbol{H}\in O(p)$$

where $O(p)=\left\{\boldsymbol{H}\ (p\times p)\,|\boldsymbol{HH}'=\boldsymbol{H}'\boldsymbol{H}=\boldsymbol{I}_p\right\}$ and $\mathrm{d}\boldsymbol{H}$ denotes the normalised Haar invariant measure on the orthogonal group $O(p)$ (Muirhead, 1982, page 72).

[2.9.2]
Díaz-García and Gutiérrez-Jáimez (2006b) propose to apply this idea from Greenacre (1973) in an inverse way. That is from [2.9.1] the pdf $f(\boldsymbol{X})$ can be identified from $f\left(\boldsymbol{HXH}'\right)$.

[2.9.3]
Note that $f_s(\boldsymbol{X})=f(\boldsymbol{X})$ if $f\left(\boldsymbol{HXH}'\right)=f(\boldsymbol{X})$.

2.10 Wishart distribution

The definitions for the central and noncentral Wishart distributions are given in this section. The relationship between a random sample from a multivariate normal distribution and the Wishart distribution is also given.

[2.10.1] *(Gupta and Nagar, 2000b, Definition 3.2.1, page 87)*
A $p \times p$ random symmetric positive definite matrix $\boldsymbol{S}$ is said to have a Wishart distribution with parameters p, n, and $\boldsymbol{\Sigma}$ $(p \times p) > \mathbf{0}$, written as $\boldsymbol{S} \sim W_p(n, \boldsymbol{\Sigma})$, if its pdf is given by

$$\left\{ 2^{\frac{1}{2}np} \Gamma_p\left(\tfrac{n}{2}\right) |\boldsymbol{\Sigma}|^{\frac{1}{2}n} \right\}^{-1} |\boldsymbol{S}|^{\frac{1}{2}(n-p-1)} \operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}\right), \quad \boldsymbol{S} > \mathbf{0}, \quad n \geq p.$$

[2.10.2] *(Gupta and Nagar, 2000b, Definition 3.5.1, page 113)*
A $p \times p$ random symmetric positive definite matrix $\boldsymbol{S}$ is said to have a noncentral Wishart distribution with parameters p, n, $\boldsymbol{\Sigma} > \mathbf{0}$ and $\boldsymbol{\Theta}$, written as $\boldsymbol{S} \sim W_p(n, \boldsymbol{\Sigma}; \boldsymbol{\Theta})$, if its pdf is given by

$$\left\{ 2^{\frac{1}{2}np} \Gamma_p\left(\tfrac{n}{2}\right) |\boldsymbol{\Sigma}|^{\frac{1}{2}n} \right\}^{-1} \operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) |\boldsymbol{S}|^{\frac{1}{2}(n-p-1)} \operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}\right) \ {}_0F_1\left(\tfrac{n}{2}; \tfrac{1}{4}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}\right), \quad \boldsymbol{S} > \mathbf{0}, \quad n \geq p$$

where ${}_0F_1(\cdot)$ is the hypergeometric function of matrix argument given in [2.6.1].

[2.10.3] *(Gupta and Nagar, 2000b, Theorems 3.3.1 and 3.2.2, page 88)*
If $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ are independent $N_p(\mathbf{0}, \boldsymbol{\Sigma})$, $\boldsymbol{X} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) \sim N_{p,n}(\mathbf{0}, \boldsymbol{\Sigma} \otimes \boldsymbol{I}_n)$. Further, if $n \geq p$, then $\boldsymbol{X}\boldsymbol{X}' > \mathbf{0}$ with probability one and $\boldsymbol{S} = \sum_{i=1}^{n} \boldsymbol{x}_i \boldsymbol{x}_i' = \boldsymbol{X}\boldsymbol{X}' \sim W_p(n, \boldsymbol{\Sigma})$.

[2.10.4] *(Gupta and Nagar, 2000b, Theorem 3.5.1, page 114)*
If $\boldsymbol{X} \sim N_{p,n}(\boldsymbol{M}, \boldsymbol{\Sigma} \otimes \boldsymbol{I}_n)$, $n \geq p$, then $\boldsymbol{S} = \boldsymbol{X}\boldsymbol{X}' \sim W_p(n, \boldsymbol{\Sigma}; \boldsymbol{\Theta})$, where $\boldsymbol{\Theta} = \boldsymbol{\Sigma}^{-1}\boldsymbol{M}\boldsymbol{M}'$.

2.11 Wilks' statistic

The Wilks' statistic in (1.6) was proposed by Wilks (1932) and a definition is given in this section.

[2.11.1] Definition of Wilks' statistic *(Kshirsagar, 1972, page 291)*
If a matrix $\boldsymbol{S}$ has the $W_p(n, \boldsymbol{\Sigma})$ distribution and is independent of $\boldsymbol{x}_i$, $i = 1, \ldots, m$, which themselves are independently distributed as $N_p(\mathbf{0}, \boldsymbol{\Sigma})$ then Wilks' Λ_1 criterion, defined by Wilks (1932), is

$$\Lambda_1 = \frac{|\boldsymbol{S}|}{|\boldsymbol{S} + \boldsymbol{B}|}$$

where $\boldsymbol{B} = \sum_{i=1}^{m} \boldsymbol{x}_i \boldsymbol{x}_i'$. Furthermore $\boldsymbol{B} = \boldsymbol{X}\boldsymbol{X}' \sim W_p(m, \boldsymbol{\Sigma})$ where $\boldsymbol{X} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m) \sim N_{p,m}(\mathbf{0}, \boldsymbol{\Sigma} \otimes \boldsymbol{I}_m)$. The Wilks' Λ_1 is used to test any hypothesis which is equivalent to

$$H_0 : E(\boldsymbol{x}_i) = \mathbf{0}, \quad i = 1, \ldots, m.$$

In [2.11.1] $\boldsymbol{S}$ is the matrix of sums of squares and sums of products due to the error and $\boldsymbol{B}$ is the matrix of sums of squares and sums of products due to the hypothesis. The parameters of the Wilks' Λ_1 statistic are $n + m$ (the degrees of freedom of $\boldsymbol{S} + \boldsymbol{B}$), p (the order of $\boldsymbol{S}$ and $\boldsymbol{B}$) and m (the degrees of freedom of $\boldsymbol{B}$).

I

Central distributions

3 Matrix variate beta type I distribution

In the literature there are three definitions of Wishart ratios to find the matrix variate beta type I distribution (Srivastava and Khatri, 1979; Muirhead, 1982; Gupta and Nagar, 2000b). The pdf of the first ratio in (1.1) is given in Theorem 3.1. The other two definitions of the Wishart ratios are given in Theorem 3.2 and Theorem 3.3 respectively. The determinant of these Wishart ratios is the same and is named the Wilks' statistic in multivariate analysis (see [2.11.1]). Further, in this section the moment of the determinant and the pdf of the Wilks' statistic are given.

3.1 Probability density function

The covariance matrices of the Wishart distributions in Theorems 3.1 to 3.3 are $\mathbf{\Sigma}$. In Theorem 3.4 the covariance matrices of the Wishart distributions differ with constant factors, and in Theorem 3.5 the covariance matrices are $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$.

Theorem 3.1 *(Hsu, 1939; Khatri, 1959; Olkin and Rubin, 1964)*
Let $\boldsymbol{S} \sim W_p(n, \mathbf{\Sigma})$ *and* $\boldsymbol{B} \sim W_p(m, \mathbf{\Sigma})$ *be independently distributed. Define*

$$\boldsymbol{U} = (\boldsymbol{S}+\boldsymbol{B})^{-\frac{1}{2}} \boldsymbol{S} (\boldsymbol{S}+\boldsymbol{B})^{-\frac{1}{2}},$$

where $\boldsymbol{S}+\boldsymbol{B} = (\boldsymbol{S}+\boldsymbol{B})^{\frac{1}{2}} (\boldsymbol{S}+\boldsymbol{B})^{\frac{1}{2}}$ *(see* (1.1)*). The pdf of* $\boldsymbol{U} \sim B_p^I(n,m)$ *is given by*

$$\left\{\beta_p\left(\tfrac{n}{2}, \tfrac{m}{2}\right)\right\}^{-1} |\boldsymbol{U}|^{\frac{1}{2}(n-p-1)} |\boldsymbol{I}_p - \boldsymbol{U}|^{\frac{1}{2}(m-p-1)}, \quad \mathbf{0} < \boldsymbol{U} < \boldsymbol{I}_p, \tag{3.1}$$

where $\beta_p\left(\frac{n}{2}, \frac{m}{2}\right)$ *is the multivariate beta function given in [2.2.4],* $n > (p-1)$ *and* $m > (p-1)$.

Theorem 3.2 *(Srivastava and Khatri, 1979)*
Let $\boldsymbol{S} \sim W_p(n, \mathbf{\Sigma})$ *and* $\boldsymbol{B} \sim W_p(m, \mathbf{\Sigma})$ *be independently distributed. Define*

$$\boldsymbol{U} = \boldsymbol{S}^{\frac{1}{2}} (\boldsymbol{S}+\boldsymbol{B})^{-1} \boldsymbol{S}^{\frac{1}{2}}. \tag{3.2}$$

The pdf of $\boldsymbol{U}$ *is given by* (3.1).

Theorem 3.3 *(Gupta and Nagar, 2000b, Theorem 5.3.7(ii), page 173)*
Let $\boldsymbol{S} \sim W_p(n, \mathbf{\Sigma})$ *and* $\boldsymbol{B} \sim W_p(m, \mathbf{\Sigma})$ *be independently distributed. Define*

$$\boldsymbol{U} = \left(\boldsymbol{I}_p + \boldsymbol{B}^{-\frac{1}{2}} \boldsymbol{S} \boldsymbol{B}^{-\frac{1}{2}}\right)^{-\frac{1}{2}} \boldsymbol{B}^{-\frac{1}{2}} \boldsymbol{S} \boldsymbol{B}^{-\frac{1}{2}} \left(\boldsymbol{I}_p + \boldsymbol{B}^{-\frac{1}{2}} \boldsymbol{S} \boldsymbol{B}^{-\frac{1}{2}}\right)^{-\frac{1}{2}}. \tag{3.3}$$

The pdf of $\boldsymbol{U}$ *is given by* (3.1).

Remark 3.1
Let $\boldsymbol{Y} \sim N_{p,n}(\mathbf{0}, \mathbf{\Sigma} \otimes \boldsymbol{I}_n)$ and $\boldsymbol{B} \sim W_p(m, \mathbf{\Sigma})$ be independent, $\boldsymbol{S} = \boldsymbol{Y}\boldsymbol{Y}'$ and $\boldsymbol{Z} = (\boldsymbol{S}+\boldsymbol{B})^{-\frac{1}{2}} \boldsymbol{Y}$, where $(\boldsymbol{S}+\boldsymbol{B})^{-\frac{1}{2}}$ is a nonsingular square root of $(\boldsymbol{S}+\boldsymbol{B})$. Then $\boldsymbol{Z}\boldsymbol{Z}' = (\boldsymbol{S}+\boldsymbol{B})^{-\frac{1}{2}} \boldsymbol{S} (\boldsymbol{S}+\boldsymbol{B})^{-\frac{1}{2}} \sim B_p^I(n,m)$ and $\boldsymbol{Z}'\boldsymbol{Z} = \boldsymbol{Y}'(\boldsymbol{S}+\boldsymbol{B})^{-1}\boldsymbol{Y} \sim B_n^I(p, n+m-p)$ (see [2.10.3], Khatri, 1959, Corollary 2; Gupta and Nagar, 2000b, Theorem 5.2.3, page 167 and Theorem 5.2.4, page 168).

Theorem 3.4
Let $\boldsymbol{S}^* \sim W_p(n, \alpha\boldsymbol{\Sigma})$ *and* $\boldsymbol{B}^* \sim W_p(m, c\boldsymbol{\Sigma})$ *be independently distributed. Define*

$$\boldsymbol{U} = (\boldsymbol{S}^* + \boldsymbol{B}^*)^{-\frac{1}{2}}\boldsymbol{S}^*(\boldsymbol{S}^* + \boldsymbol{B}^*)^{-\frac{1}{2}}, \tag{3.4}$$

where $\boldsymbol{S}^* + \boldsymbol{B}^* = (\boldsymbol{S}^* + \boldsymbol{B}^*)^{\frac{1}{2}}(\boldsymbol{S}^* + \boldsymbol{B}^*)^{\frac{1}{2}}$. *The pdf of* $\boldsymbol{U}$ *is given by*

$$\left\{\beta\left(\tfrac{n}{2}, \tfrac{m}{2}\right)\right\}^{-1}\left(\tfrac{c}{\alpha}\right)^{\frac{1}{2}np}|\boldsymbol{U}|^{\frac{1}{2}n-\frac{1}{2}(p+1)}|\boldsymbol{I}_p - \boldsymbol{U}|^{\frac{1}{2}m-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p + \tfrac{c-\alpha}{\alpha}\boldsymbol{U}\right|^{-\frac{1}{2}(n+m)}, \quad \boldsymbol{0} < \boldsymbol{U} < \boldsymbol{I}_p, \tag{3.5}$$

where $\beta_p\left(\frac{n}{2}, \frac{m}{2}\right)$ *is the multivariate beta function given in [2.2.4],* $n > (p-1)$ *and* $m > (p-1)$. *This is said to have the matrix variate beta type V distribution and is denoted by* $\boldsymbol{U} \sim B_p^V(n, m, \alpha, c)$.

Proof:
The pdf of $(\boldsymbol{S}^*, \boldsymbol{B}^*)$ is given by

$$K\mathrm{etr}\left(-\tfrac{1}{2\alpha}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}^*\right)|\boldsymbol{S}^*|^{\frac{1}{2}(n-p-1)}\mathrm{etr}\left(-\tfrac{1}{2c}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}^*\right)|\boldsymbol{B}^*|^{\frac{1}{2}(m-p-1)} \tag{3.6}$$

where $K^{-1} = \Gamma_p\left(\frac{n}{2}\right)\Gamma_p\left(\frac{m}{2}\right)|2\boldsymbol{\Sigma}|^{\frac{1}{2}(n+m)}\alpha^{\frac{1}{2}np}c^{\frac{1}{2}mp}$ (see [2.10.1]) and where $\Gamma_p(\cdot)$ is the multivariate gamma function given in [2.2.2].

Making the transformation (3.4) (see [2.1.1]) and letting $\boldsymbol{Y} = \boldsymbol{S}^* + \boldsymbol{B}^*$ gives $\boldsymbol{S}^* = \boldsymbol{Y}^{\frac{1}{2}}\boldsymbol{U}\boldsymbol{Y}^{\frac{1}{2}}$ and $\boldsymbol{B}^* = \boldsymbol{Y}^{\frac{1}{2}}(\boldsymbol{I}_p - \boldsymbol{U})\boldsymbol{Y}^{\frac{1}{2}}$ with Jacobian $J(\boldsymbol{S}^*, \boldsymbol{B}^* \to \boldsymbol{U}, \boldsymbol{Y}) = |\boldsymbol{Y}|^{\frac{1}{2}(p+1)}$ (see [2.1.8]). Substituting in (3.6) gives

$$\begin{aligned} f(\boldsymbol{U}, \boldsymbol{Y}) &= K\left|\boldsymbol{Y}^{\frac{1}{2}}\boldsymbol{U}\boldsymbol{Y}^{\frac{1}{2}}\right|^{\frac{1}{2}n-\frac{1}{2}(p+1)}\left|\boldsymbol{Y}^{\frac{1}{2}}(\boldsymbol{I}_p - \boldsymbol{U})\boldsymbol{Y}^{\frac{1}{2}}\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)}|\boldsymbol{Y}|^{\frac{1}{2}(p+1)} \\ &\quad \cdot\mathrm{etr}\left[-\tfrac{1}{2\alpha}\boldsymbol{\Sigma}^{-1}\boldsymbol{Y}^{\frac{1}{2}}\boldsymbol{U}\boldsymbol{Y}^{\frac{1}{2}} - \tfrac{1}{2c}\boldsymbol{\Sigma}^{-1}\boldsymbol{Y}^{\frac{1}{2}}(\boldsymbol{I}_p - \boldsymbol{U})\boldsymbol{Y}^{\frac{1}{2}}\right] \\ &= K|\boldsymbol{U}|^{\frac{1}{2}n-\frac{1}{2}(p+1)}|\boldsymbol{I}_p - \boldsymbol{U}|^{\frac{1}{2}m-\frac{1}{2}(p+1)}|\boldsymbol{Y}|^{\frac{1}{2}(n+m)-\frac{1}{2}(p+1)} \\ &\quad \cdot\mathrm{etr}\left[-\tfrac{1}{2c}\boldsymbol{\Sigma}^{-1}\boldsymbol{Y}^{\frac{1}{2}}\left(\boldsymbol{I}_p + \tfrac{c-\alpha}{\alpha}\boldsymbol{U}\right)\boldsymbol{Y}^{\frac{1}{2}}\right]. \end{aligned} \tag{3.7}$$

We consider the symmetrised density function of $(\boldsymbol{U}, \boldsymbol{Y})$ (see [2.9.1]), that is

$f_s(\boldsymbol{U}, \boldsymbol{Y}) \equiv \int_{O(p)} f(\boldsymbol{H}\boldsymbol{U}\boldsymbol{H}', \boldsymbol{H}\boldsymbol{Y}\boldsymbol{H}')\mathrm{d}\boldsymbol{H}$ where $\boldsymbol{H}$ $(p \times p)$ is orthogonal, $\mathrm{d}\boldsymbol{H}$ is the normalised Haar invariant measure on $O(p)$ and from (3.7)

$$\begin{aligned} & f(\boldsymbol{H}\boldsymbol{U}\boldsymbol{H}', \boldsymbol{H}\boldsymbol{Y}\boldsymbol{H}') \\ &= K|\boldsymbol{H}\boldsymbol{U}\boldsymbol{H}'|^{\frac{1}{2}n-\frac{1}{2}(p+1)}|\boldsymbol{I}_p - \boldsymbol{H}\boldsymbol{U}\boldsymbol{H}'|^{\frac{1}{2}m-\frac{1}{2}(p+1)}|\boldsymbol{H}\boldsymbol{Y}\boldsymbol{H}'|^{\frac{1}{2}(n+m)-\frac{1}{2}(p+1)} \\ &\quad \cdot\mathrm{etr}\left[-\tfrac{1}{2c}\boldsymbol{\Sigma}^{-1}(\boldsymbol{H}\boldsymbol{Y}\boldsymbol{H}')^{\frac{1}{2}}\left(\boldsymbol{I}_p + \tfrac{c-\alpha}{\alpha}\boldsymbol{H}\boldsymbol{U}\boldsymbol{H}'\right)(\boldsymbol{H}\boldsymbol{Y}\boldsymbol{H}')^{\frac{1}{2}}\right] \\ &= K|\boldsymbol{U}|^{\frac{1}{2}n-\frac{1}{2}(p+1)}|\boldsymbol{I}_p - \boldsymbol{U}|^{\frac{1}{2}m-\frac{1}{2}(p+1)}|\boldsymbol{Y}|^{\frac{1}{2}(n+m)-\frac{1}{2}(p+1)}\mathrm{etr}\left[-\tfrac{1}{2c}\boldsymbol{\Sigma}^{-1}\boldsymbol{H}\boldsymbol{Y}^{\frac{1}{2}}\left(\boldsymbol{I}_p + \tfrac{c-\alpha}{\alpha}\boldsymbol{U}\right)\boldsymbol{Y}^{\frac{1}{2}}\boldsymbol{H}'\right]. \end{aligned} \tag{3.8}$$

Hence, from (3.8) and [2.3.6] we get

$$f_s(\boldsymbol{U},\boldsymbol{Y}) = K|\boldsymbol{U}|^{\frac{1}{2}n-\frac{1}{2}(p+1)}|\boldsymbol{I}_p-\boldsymbol{U}|^{\frac{1}{2}m-\frac{1}{2}(p+1)}|\boldsymbol{Y}|^{\frac{1}{2}(n+m)-\frac{1}{2}(p+1)} \cdot\int_{O(p)}\operatorname{etr}\left[-\tfrac{1}{2c}\boldsymbol{H}'\boldsymbol{\Sigma}^{-1}\boldsymbol{H}\left(\boldsymbol{I}_p+\tfrac{c-\alpha}{\alpha}\boldsymbol{U}\right)^{\frac{1}{2}}\boldsymbol{Y}\left(\boldsymbol{I}_p+\tfrac{c-\alpha}{\alpha}\boldsymbol{U}\right)^{\frac{1}{2}}\right]\mathrm{d}\boldsymbol{H}. \tag{3.9}$$

Integrating $f_s(\boldsymbol{U},\boldsymbol{Y})$ in (3.9) with respect to $\boldsymbol{Y}$ by using [2.2.3] gives

$$\begin{aligned} & f_s(\boldsymbol{U}) \\ = \; & K|\boldsymbol{U}|^{\frac{1}{2}n-\frac{1}{2}(p+1)}|\boldsymbol{I}_p-\boldsymbol{U}|^{\frac{1}{2}m-\frac{1}{2}(p+1)} \\ & \cdot\int_{O(p)}\int_{\boldsymbol{Y}>\boldsymbol{0}}|\boldsymbol{Y}|^{\frac{1}{2}(n+m)-\frac{1}{2}(p+1)}\operatorname{etr}\left[-\tfrac{1}{2c}\left(\boldsymbol{I}_p+\tfrac{c-\alpha}{\alpha}\boldsymbol{U}\right)^{\frac{1}{2}}\boldsymbol{H}'\boldsymbol{\Sigma}^{-1}\boldsymbol{H}\left(\boldsymbol{I}_p+\tfrac{c-\alpha}{\alpha}\boldsymbol{U}\right)^{\frac{1}{2}}\boldsymbol{Y}\right]\mathrm{d}\boldsymbol{Y}\mathrm{d}\boldsymbol{H} \\ = \; & \left[\Gamma_p\left(\tfrac{n}{2}\right)\Gamma_p\left(\tfrac{m}{2}\right)|2\boldsymbol{\Sigma}|^{\frac{1}{2}(n+m)}\alpha^{\frac{1}{2}np}c^{\frac{1}{2}mp}\right]^{-1}\Gamma_p\left(\tfrac{n+m}{2}\right)|\boldsymbol{U}|^{\frac{1}{2}n-\frac{1}{2}(p+1)}|\boldsymbol{I}_p-\boldsymbol{U}|^{\frac{1}{2}m-\frac{1}{2}(p+1)} \\ & \cdot\int_{O(p)}\left|\tfrac{1}{2c}\left(\boldsymbol{I}_p+\tfrac{c-\alpha}{\alpha}\boldsymbol{U}\right)^{\frac{1}{2}}\boldsymbol{H}'\boldsymbol{\Sigma}^{-1}\boldsymbol{H}\left(\boldsymbol{I}_p+\tfrac{c-\alpha}{\alpha}\boldsymbol{U}\right)^{\frac{1}{2}}\right|^{-\frac{1}{2}(n+m)}\mathrm{d}\boldsymbol{H} \\ = \; & \left\{\beta\left(\tfrac{n}{2},\tfrac{m}{2}\right)\right\}^{-1}\left(\tfrac{c}{\alpha}\right)^{\frac{1}{2}np}|\boldsymbol{U}|^{\frac{1}{2}n-\frac{1}{2}(p+1)}|\boldsymbol{I}_p-\boldsymbol{U}|^{\frac{1}{2}m-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p+\tfrac{c-\alpha}{\alpha}\boldsymbol{U}\right|^{-\frac{1}{2}(n+m)}\int_{O(p)}\mathrm{d}\boldsymbol{H}. \end{aligned} \tag{3.10}$$

The result follows from (3.10) and [2.9.3]. ■

Remark 3.2
If $\boldsymbol{S}\sim W_p(n,\boldsymbol{\Sigma})$ and $\boldsymbol{B}\sim W_p(m,\boldsymbol{\Sigma})$ independent and

$$\boldsymbol{U}=(\alpha\boldsymbol{S}+c\boldsymbol{B})^{-\frac{1}{2}}\alpha\boldsymbol{S}(\alpha\boldsymbol{S}+c\boldsymbol{B})^{-\frac{1}{2}} \tag{3.11}$$

then the pdf of $\boldsymbol{U}$ is the same as (3.5), that is $\boldsymbol{U}\sim B_p^V(n,m,\alpha,c)$. Therefore, further in this study we build the constant factors of the covariance matrices of the Wishart distributions (see Theorem 3.4) into the Wishart ratio(s) (see (3.11)).

Theorem 3.5
Let $\boldsymbol{S}\sim W_p(n,\boldsymbol{\Sigma}_1)$ *and* $\boldsymbol{B}\sim W_p(m,\boldsymbol{\Sigma}_2)$ *be independently distributed. Define*

$$\boldsymbol{U}=(\boldsymbol{S}+\boldsymbol{B})^{-\frac{1}{2}}\boldsymbol{S}(\boldsymbol{S}+\boldsymbol{B})^{-\frac{1}{2}},$$

where $\boldsymbol{S}+\boldsymbol{B}=(\boldsymbol{S}+\boldsymbol{B})^{\frac{1}{2}}(\boldsymbol{S}+\boldsymbol{B})^{\frac{1}{2}}$ *(see (1.1)). The pdf of* $\boldsymbol{U}$ *is given by*

$$\begin{aligned} & \left\{\beta\left(\tfrac{n}{2},\tfrac{m}{2}\right)\right\}^{-1}|\boldsymbol{\Sigma}_1|^{-\frac{1}{2}n}|\boldsymbol{\Sigma}_2|^{-\frac{1}{2}m}|\boldsymbol{U}|^{\frac{1}{2}n-\frac{1}{2}(p+1)}|\boldsymbol{I}_p-\boldsymbol{U}|^{\frac{1}{2}m-\frac{1}{2}(p+1)} \\ & \cdot\left|\boldsymbol{\Sigma}_2^{-1}-\left(\boldsymbol{\Sigma}_2^{-1}-\boldsymbol{\Sigma}_1^{-1}\right)^{\frac{1}{2}}\boldsymbol{U}\left(\boldsymbol{\Sigma}_2^{-1}-\boldsymbol{\Sigma}_1^{-1}\right)^{\frac{1}{2}}\right|^{-\frac{1}{2}(n+m)},\quad \boldsymbol{0}<\boldsymbol{U}<\boldsymbol{I}_p, \end{aligned} \tag{3.12}$$

where $\beta_p\left(\frac{n}{2},\frac{m}{2}\right)$ *is the multivariate beta function given in [2.2.4],* $n>(p-1)$ *and* $m>(p-1)$. *This is the matrix variate beta type V distribution and is denoted by* $\boldsymbol{U}\sim B_p^V(n,m,\boldsymbol{\Sigma}_1,\boldsymbol{\Sigma}_2)$.

Proof:

Let $\boldsymbol{Y} = \boldsymbol{S} + \boldsymbol{B}$. From De Waal (1970, Equation 3.1) the pdf of $\boldsymbol{U}$ is given by

$$\begin{aligned} f(\boldsymbol{U}) = & \left[\Gamma_p\left(\tfrac{n}{2}\right)\Gamma_p\left(\tfrac{m}{2}\right)|2\boldsymbol{\Sigma}_1|^{\frac{1}{2}n}|2\boldsymbol{\Sigma}_2|^{\frac{1}{2}m}\right]^{-1}|\boldsymbol{U}|^{\frac{1}{2}n-\frac{1}{2}(p+1)}|\boldsymbol{I}_p-\boldsymbol{U}|^{\frac{1}{2}m-\frac{1}{2}(p+1)} \\ & \cdot\int_{\boldsymbol{Y}>\boldsymbol{0}}|\boldsymbol{Y}|^{\frac{1}{2}(n+m)-\frac{1}{2}(p+1)}\operatorname{etr}\left[-\tfrac{1}{2}\boldsymbol{\Sigma}_2^{-1}\boldsymbol{Y}-\tfrac{1}{2}\left(\boldsymbol{\Sigma}_1^{-1}-\boldsymbol{\Sigma}_2^{-1}\right)\boldsymbol{Y}^{\frac{1}{2}}\boldsymbol{U}\boldsymbol{Y}^{\frac{1}{2}}\right]\mathrm{d}\boldsymbol{Y}, \end{aligned} \tag{3.13}$$

$\boldsymbol{0} < \boldsymbol{U} < \boldsymbol{I}_p$.

We consider the symmetrised density function of $\boldsymbol{U}$ (see [2.9.1]), that is $f_s(\boldsymbol{U}) \equiv \int_{O(p)} f\left(\boldsymbol{H}\boldsymbol{U}\boldsymbol{H}'\right)\mathrm{d}\boldsymbol{H}$ where $\boldsymbol{H}$ $(p\times p)$ is orthogonal and $\mathrm{d}\boldsymbol{H}$ is the normalised Haar invariant measure on $O(p)$. From (3.13) and [2.6.2]

$$\begin{aligned} f\left(\boldsymbol{H}\boldsymbol{U}\boldsymbol{H}'\right) = & \left[\Gamma_p\left(\tfrac{n}{2}\right)\Gamma_p\left(\tfrac{m}{2}\right)|2\boldsymbol{\Sigma}_1|^{\frac{1}{2}n}|2\boldsymbol{\Sigma}_2|^{\frac{1}{2}m}\right]^{-1}\left|\boldsymbol{H}\boldsymbol{U}\boldsymbol{H}'\right|^{\frac{1}{2}n-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\boldsymbol{H}\boldsymbol{U}\boldsymbol{H}'\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)} \\ & \cdot\int_{\boldsymbol{Y}>\boldsymbol{0}}|\boldsymbol{Y}|^{\frac{1}{2}(n+m)-\frac{1}{2}(p+1)}\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Sigma}_2^{-1}\boldsymbol{Y}\right){}_0F_0\left(\tfrac{1}{2}\boldsymbol{Y}^{\frac{1}{2}}\left(\boldsymbol{\Sigma}_2^{-1}-\boldsymbol{\Sigma}_1^{-1}\right)\boldsymbol{Y}^{\frac{1}{2}}\boldsymbol{H}\boldsymbol{U}\boldsymbol{H}'\right)\mathrm{d}\boldsymbol{Y}, \end{aligned} \tag{3.14}$$

where ${}_0F_0(\cdot)$ is the hypergeometric function of matrix argument given in [2.6.2]. Hence, from (3.14), [2.3.6] and [2.6.2] we get

$$\begin{aligned} & f_s(\boldsymbol{U}) \\ = & \left[\Gamma_p\left(\tfrac{n}{2}\right)\Gamma_p\left(\tfrac{m}{2}\right)|2\boldsymbol{\Sigma}_1|^{\frac{1}{2}n}|2\boldsymbol{\Sigma}_2|^{\frac{1}{2}m}\right]^{-1}|\boldsymbol{U}|^{\frac{1}{2}n-\frac{1}{2}(p+1)}|\boldsymbol{I}_p-\boldsymbol{U}|^{\frac{1}{2}m-\frac{1}{2}(p+1)} \\ & \cdot\int_{\boldsymbol{Y}>\boldsymbol{0}}|\boldsymbol{Y}|^{\frac{1}{2}(n+m)-\frac{1}{2}(p+1)}\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Sigma}_2^{-1}\boldsymbol{Y}\right) \\ & \cdot\int_{O(p)}{}_0F_0\left(\tfrac{1}{2}\left(\boldsymbol{\Sigma}_2^{-1}-\boldsymbol{\Sigma}_1^{-1}\right)^{\frac{1}{2}}\boldsymbol{H}\boldsymbol{U}\boldsymbol{H}'\left(\boldsymbol{\Sigma}_2^{-1}-\boldsymbol{\Sigma}_1^{-1}\right)^{\frac{1}{2}}\boldsymbol{Y}\right)\mathrm{d}\boldsymbol{H}\mathrm{d}\boldsymbol{Y} \\ = & \left[\Gamma_p\left(\tfrac{n}{2}\right)\Gamma_p\left(\tfrac{m}{2}\right)|2\boldsymbol{\Sigma}_1|^{\frac{1}{2}n}|2\boldsymbol{\Sigma}_2|^{\frac{1}{2}m}\right]^{-1}|\boldsymbol{U}|^{\frac{1}{2}n-\frac{1}{2}(p+1)}|\boldsymbol{I}_p-\boldsymbol{U}|^{\frac{1}{2}m-\frac{1}{2}(p+1)} \\ & \cdot\int_{O(p)}\int_{\boldsymbol{Y}>\boldsymbol{0}}|\boldsymbol{Y}|^{\frac{1}{2}(n+m)-\frac{1}{2}(p+1)}\operatorname{etr}\left\{-\tfrac{1}{2}\left[\boldsymbol{\Sigma}_2^{-1}-\left(\boldsymbol{\Sigma}_2^{-1}-\boldsymbol{\Sigma}_1^{-1}\right)^{\frac{1}{2}}\boldsymbol{H}\boldsymbol{U}\boldsymbol{H}'\left(\boldsymbol{\Sigma}_2^{-1}-\boldsymbol{\Sigma}_1^{-1}\right)^{\frac{1}{2}}\right]\boldsymbol{Y}\right\}\mathrm{d}\boldsymbol{Y}\mathrm{d}\boldsymbol{H}. \end{aligned} \tag{3.15}$$

Integrating $f_s(\boldsymbol{U})$ in (3.15) with respect to $\boldsymbol{Y}$ by using [2.2.3] gives

$$\begin{aligned} & f_s(\boldsymbol{U}) \\ = & \left[\Gamma_p\left(\tfrac{n}{2}\right)\Gamma_p\left(\tfrac{m}{2}\right)|2\boldsymbol{\Sigma}_1|^{\frac{1}{2}n}|2\boldsymbol{\Sigma}_2|^{\frac{1}{2}m}\right]^{-1}\Gamma_p\left(\tfrac{n+m}{2}\right)|\boldsymbol{U}|^{\frac{1}{2}n-\frac{1}{2}(p+1)}|\boldsymbol{I}_p-\boldsymbol{U}|^{\frac{1}{2}m-\frac{1}{2}(p+1)} \\ & \cdot\int_{O(p)}\left|\tfrac{1}{2}\left[\boldsymbol{\Sigma}_2^{-1}-\left(\boldsymbol{\Sigma}_2^{-1}-\boldsymbol{\Sigma}_1^{-1}\right)^{\frac{1}{2}}\boldsymbol{H}\boldsymbol{U}\boldsymbol{H}'\left(\boldsymbol{\Sigma}_2^{-1}-\boldsymbol{\Sigma}_1^{-1}\right)^{\frac{1}{2}}\right]\right|^{-\frac{1}{2}(n+m)}\mathrm{d}\boldsymbol{H} \\ = & \left\{\beta\left(\tfrac{n}{2},\tfrac{m}{2}\right)\right\}^{-1}|\boldsymbol{\Sigma}_1|^{-\frac{1}{2}n}|\boldsymbol{\Sigma}_2|^{-\frac{1}{2}m}|\boldsymbol{U}|^{\frac{1}{2}n-\frac{1}{2}(p+1)}|\boldsymbol{I}_p-\boldsymbol{U}|^{\frac{1}{2}m-\frac{1}{2}(p+1)} \\ & \int_{O(p)}\left|\boldsymbol{\Sigma}_2^{-1}-\left(\boldsymbol{\Sigma}_2^{-1}-\boldsymbol{\Sigma}_1^{-1}\right)^{\frac{1}{2}}\boldsymbol{H}\boldsymbol{U}\boldsymbol{H}'\left(\boldsymbol{\Sigma}_2^{-1}-\boldsymbol{\Sigma}_1^{-1}\right)^{\frac{1}{2}}\right|^{-\frac{1}{2}(n+m)}\mathrm{d}\boldsymbol{H}. \end{aligned} \tag{3.16}$$

Since $f_s(\boldsymbol{U}) \equiv \int_{O(p)} f(\boldsymbol{HUH}')\mathrm{d}\boldsymbol{H}$ it follows from (3.16) that

$$
\begin{aligned}
& f(\boldsymbol{HUH}') \\
= \ & \left\{\beta\left(\tfrac{n}{2},\tfrac{m}{2}\right)\right\}^{-1} |\boldsymbol{\Sigma}_1|^{-\frac{1}{2}n} |\boldsymbol{\Sigma}_2|^{-\frac{1}{2}m} \left|\boldsymbol{HUH}'\right|^{\frac{1}{2}n-\frac{1}{2}(p+1)} \left|\boldsymbol{I}_p - \boldsymbol{HUH}'\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)} \\
& \left|\boldsymbol{\Sigma}_2^{-1} - \left(\boldsymbol{\Sigma}_2^{-1} - \boldsymbol{\Sigma}_1^{-1}\right)^{\frac{1}{2}} \boldsymbol{HUH}' \left(\boldsymbol{\Sigma}_2^{-1} - \boldsymbol{\Sigma}_1^{-1}\right)^{\frac{1}{2}}\right|^{-\frac{1}{2}(n+m)}.
\end{aligned}
$$

From [2.9.2] the pdf of $\boldsymbol{U}$, given by (3.12), is identified. ■

3.2 Moments of the determinant

The h^{th} moment of the determinant of $\boldsymbol{U}$ where $\boldsymbol{U}$ is distributed as $B_p^I(n,m)$, $B_p^V(n,m,\alpha,c)$ and $B_p^V(n,m,\boldsymbol{\Sigma}_1,\boldsymbol{\Sigma}_2)$ are respectively given in Theorem 3.6 (i), (ii) and (iii).

Theorem 3.6

(i) *If* $\boldsymbol{U} \sim B_p^I(n,m)$ *as given by* (3.1) *then*

$$
E\left(|\boldsymbol{U}|^h\right) = \frac{\Gamma_p\left(\frac{n+m}{2}\right)}{\Gamma_p\left(\frac{n}{2}\right)} \frac{\Gamma_p\left(\frac{n}{2}+h\right)}{\Gamma_p\left(\frac{n+m}{2}+h\right)}, \tag{3.17}
$$

where $\mathrm{Re}\left(\frac{n}{2}+h\right) > \frac{1}{2}(p-1)$ *(Gupta and Nagar, 2000b, Theorem 5.3.15(i), page 176).*

(ii) *If* $\boldsymbol{U} \sim B_p^V(n,m,\alpha,c)$ *as given by* (3.5) *then*

$$
\begin{aligned}
E\left(|\boldsymbol{U}|^h\right) = \ & \frac{\Gamma_p\left(\frac{n+m}{2}\right)}{\Gamma_p\left(\frac{n}{2}\right)} \frac{\Gamma_p\left(\frac{n}{2}+h\right)}{\Gamma_p\left(\frac{n+m}{2}+h\right)} \\
& \cdot \left(\tfrac{c}{\alpha}\right)^{\frac{1}{2}np} {}_2F_1\left(\tfrac{n}{2}+h, \tfrac{n+m}{2}; \tfrac{n+m}{2}+h; \tfrac{\alpha-c}{\alpha}\boldsymbol{I}_p\right),
\end{aligned} \tag{3.18}
$$

where $\mathrm{Re}\left(\frac{n}{2}+h\right) > \frac{1}{2}(p-1)$ *and where* ${}_2F_1(\cdot)$ *is the Gauss hypergeometric function of matrix argument given in [2.6.5].*

(iii) *If* $\boldsymbol{U} \sim B_p^V(n,m,\boldsymbol{\Sigma}_1,\boldsymbol{\Sigma}_2)$ *as given by* (3.12) *then*

$$
\begin{aligned}
E\left(|\boldsymbol{U}|^h\right) = \ & \frac{\Gamma_p\left(\frac{n+m}{2}\right)}{\Gamma_p\left(\frac{n}{2}\right)} \frac{\Gamma_p\left(\frac{n}{2}+h\right)}{\Gamma_p\left(\frac{n+m}{2}+h\right)} \\
& \cdot \left|\boldsymbol{\Sigma}_1^{-1}\boldsymbol{\Sigma}_2\right|^{\frac{1}{2}n} {}_2F_1\left(\tfrac{n}{2}+h, \tfrac{n+m}{2}; \tfrac{n+m}{2}+h; \boldsymbol{I}_p - \boldsymbol{\Sigma}_1^{-\frac{1}{2}}\boldsymbol{\Sigma}_2\boldsymbol{\Sigma}_1^{-\frac{1}{2}}\right),
\end{aligned} \tag{3.19}
$$

where $\mathrm{Re}\left(\frac{n}{2}+h\right) > \frac{1}{2}(p-1)$ *(De Waal, 1970, Equation 3.2; Gupta and Nagar, 2000b, Theorem 5.4.5, page 186).*

Proof:
Only result (3.18) is proven.

From (3.5)

$$\begin{aligned}
&E\left(|\boldsymbol{U}|^h\right) \\
&= \frac{\Gamma_p\left(\frac{n+m}{2}\right)}{\Gamma_p\left(\frac{n}{2}\right)\Gamma_p\left(\frac{m}{2}\right)}\left(\frac{c}{\alpha}\right)^{\frac{1}{2}np}\int_{\mathbf{0}<\boldsymbol{U}<\boldsymbol{I}_p}|\boldsymbol{U}|^{\frac{1}{2}n+h-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\boldsymbol{U}\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p+\frac{c-\alpha}{\alpha}\boldsymbol{U}\right|^{-\frac{1}{2}(n+m)}\mathrm{d}\boldsymbol{U}.
\end{aligned}$$

Using [2.6.5] to solve the integral over $\boldsymbol{U}$ gives the result (3.18). Alternatively the result follows from (3.19) with $\boldsymbol{\Sigma}_1 = \alpha\boldsymbol{\Sigma}$ and $\boldsymbol{\Sigma}_2 = c\boldsymbol{\Sigma}$. ■

3.3 Probability density function of the Wilks' statistic

The Wilks' statistic, (1.6) proposed by Wilks (1932), plays the same role in multivariate analysis that the F-statistic plays in univariate analysis. Wilks (1932) derived an expression for the pdf of (1.6) in a non-explicit form under the null hypothesis (a correction was made by Díaz-García and Caro-Lopera, 2007). Anderson (1984, pages 298-301) showed that the Wilks' Λ_1 is distributed as the product of independent beta type I random variables. Exact expressions for the pdfs of Wilks' Λ_1 under the null hypothesis if $\boldsymbol{U} \sim B_p^I(n,m)$, $\boldsymbol{U} \sim B_p^V(n,m,\alpha,c)$ and $\boldsymbol{U} \sim B_p^V(n,m,\boldsymbol{\Sigma}_1,\boldsymbol{\Sigma}_2)$ are given in Theorems 3.7, 3.8 and 3.9 respectively in terms of Meijer's G-function (see [2.8.2]).

Theorem 3.7 *(Pham-Gia, 2008, Equation 14)*
Let $\boldsymbol{U} \sim B_p^I(n,m)$ *with pdf given by* (3.1) *and* $\Lambda_1 = |\boldsymbol{U}|$, *then the pdf of* Λ_1 *is given by*

$$\frac{\Gamma_p\left(\frac{n+m}{2}\right)}{\Gamma_p\left(\frac{n}{2}\right)}G_{p,p}^{p,0}\left(\lambda_1\middle|_{b_1,\ldots,b_p}^{a_1,\ldots,a_p}\right),\quad 0<\lambda_1<1,$$

where $G(\cdot)$ *denotes Meijer's G-function (see [2.8.2]) and*

$a_j = \frac{n+m}{2} - \frac{1}{2}(j+1)$ *for* $j = 1,2,\ldots,p$,

$b_j = \frac{n}{2} - \frac{1}{2}(j+1)$ *for* $j = 1,2,\ldots,p$.

Theorem 3.8
Let $\boldsymbol{U} \sim B_p^V(n,m,\alpha,c)$ *with pdf given by* (3.5) *and* $\Lambda_1 = |\boldsymbol{U}|$, *then the pdf of* Λ_1 *is given by*

$$\frac{1}{\Gamma_p\left(\frac{n}{2}\right)}\left(\frac{c}{\alpha}\right)^{\frac{1}{2}np}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{1}{k!}\Gamma_p\left(\frac{n+m}{2},\kappa\right)C_\kappa\left(\frac{\alpha-c}{\alpha}\boldsymbol{I}_p\right)G_{p,p}^{p,0}\left(\lambda_1\middle|_{b_1,\ldots,b_p}^{a_1,\ldots,a_p}\right),\quad 0<\lambda_1<1, \tag{3.20}$$

where $C_\kappa(\cdot)$ *is the zonal polynomial defined in [2.3.1],* $G(\cdot)$ *denotes Meijer's G-function (see [2.8.2]) and*

$a_j = \frac{n+m}{2} + k_j - \frac{1}{2}(j+1)$ *for* $j = 1,2,\ldots,p$,

$b_j = \frac{n}{2} + k_j - \frac{1}{2}(j+1)$ *for* $j = 1,2,\ldots,p$.

Proof:

From (3.18) and [2.6.1] the Mellin transform (see [2.8.1]) of $f(\lambda_1)$ is

$$\begin{aligned} M_f(h) &\equiv E\left(\Lambda_1^{h-1}\right) \\ &= E\left(|\boldsymbol{U}|^{h-1}\right) \\ &= \frac{\Gamma_p\left(\frac{n+m}{2}\right)}{\Gamma_p\left(\frac{n}{2}\right)\Gamma_p\left(\frac{m}{2}\right)} \frac{\Gamma_p\left(\frac{n}{2}+h-1\right)\Gamma_p\left(\frac{m}{2}\right)}{\Gamma_p\left(\frac{n+m}{2}+h-1\right)} \left(\frac{c}{\alpha}\right)^{\frac{1}{2}np} {}_2F_1\left(\frac{n}{2}+h-1, \frac{n+m}{2}; \frac{n+m}{2}+h-1; \frac{\alpha-c}{\alpha}\boldsymbol{I}_p\right) \\ &= \frac{\Gamma_p\left(\frac{n+m}{2}\right)}{\Gamma_p\left(\frac{n}{2}\right)} \frac{\Gamma_p\left(\frac{n}{2}+h-1\right)}{\Gamma_p\left(\frac{n+m}{2}+h-1\right)} \left(\frac{c}{\alpha}\right)^{\frac{1}{2}np} \sum_{k=0}^{\infty}\sum_{\kappa} \frac{1}{k!} \frac{\Gamma_p\left(\frac{n}{2}+h-1,\kappa\right)}{\Gamma_p\left(\frac{n}{2}+h-1\right)} \frac{\Gamma_p\left(\frac{n+m}{2},\kappa\right)}{\Gamma_p\left(\frac{n+m}{2}\right)} \frac{\Gamma_p\left(\frac{n+m}{2}+h-1\right)}{\Gamma_p\left(\frac{n+m}{2}+h-1,\kappa\right)} C_\kappa\left(\frac{\alpha-c}{\alpha}\boldsymbol{I}_p\right) \\ &= \frac{1}{\Gamma_p\left(\frac{n}{2}\right)} \left(\frac{c}{\alpha}\right)^{\frac{1}{2}np} \sum_{k=0}^{\infty}\sum_{\kappa} \frac{1}{k!}\Gamma_p\left(\frac{n+m}{2},\kappa\right) \frac{\Gamma_p\left(\frac{n}{2}+h-1,\kappa\right)}{\Gamma_p\left(\frac{n+m}{2}+h-1,\kappa\right)} C_\kappa\left(\frac{\alpha-c}{\alpha}\boldsymbol{I}_p\right). \end{aligned} \tag{3.21}$$

From [2.3.3] the generalised gamma functions of weight κ in (3.21) can be written as

$$\Gamma_p\left(\frac{n+m}{2}+h-1,\kappa\right) = \pi^{\frac{1}{4}p(p-1)} \prod_{j=1}^{p} \Gamma\left(a_j+h\right), \tag{3.22}$$

where $a_j = \frac{n+m}{2} + k_j - \frac{1}{2}(j+1)$ for $j = 1, 2, \ldots, p$,

and

$$\Gamma_p\left(\frac{n}{2}+h-1,\kappa\right) = \pi^{\frac{1}{4}p(p-1)} \prod_{j=1}^{p} \Gamma\left(b_j+h\right) \tag{3.23}$$

where $b_j = \frac{n}{2} + k_j - \frac{1}{2}(j+1)$ for $j = 1, 2, 3, \ldots, p$.

Substituting (3.22) and (3.23) in (3.21) gives

$$M_f(h) \equiv \frac{1}{\Gamma_p\left(\frac{n}{2}\right)} \left(\frac{c}{\alpha}\right)^{\frac{1}{2}np} \sum_{k=0}^{\infty}\sum_{\kappa} \frac{1}{k!}\Gamma_p\left(\frac{n+m}{2},\kappa\right) C_\kappa\left(\frac{\alpha-c}{\alpha}\boldsymbol{I}_p\right) \frac{\prod_{j=1}^{p}\Gamma(b_j+h)}{\prod_{j=1}^{p}\Gamma(a_j+h)}. \tag{3.24}$$

The pdf of Λ_1 in (3.20) is uniquely obtained from the inverse Mellin transform (see [2.8.1]) of (3.24) and the definition of the G-function (see [2.8.2]) and is given by

$$\begin{aligned} f(\lambda_1) &= \frac{1}{2\pi i} \int_{\omega-i\infty}^{\omega+i\infty} M_f(h)\, \lambda_1^{-h} \mathrm{d}h \\ &= \frac{1}{\Gamma_p\left(\frac{n}{2}\right)} \left(\frac{c}{\alpha}\right)^{\frac{1}{2}np} \sum_{k=0}^{\infty}\sum_{\kappa} \frac{1}{k!}\Gamma_p\left(\frac{n+m}{2},\kappa\right) C_\kappa\left(\frac{\alpha-c}{\alpha}\boldsymbol{I}_p\right) \left[\frac{1}{2\pi i} \int_{\omega-i\infty}^{\omega+i\infty} \frac{\prod_{j=1}^{p}\Gamma(b_j+h)}{\prod_{j=1}^{p}\Gamma(a_j+h)} \lambda_1^{-h} \mathrm{d}h\right] \\ &= \frac{1}{\Gamma_p\left(\frac{n}{2}\right)} \left(\frac{c}{\alpha}\right)^{\frac{1}{2}np} \sum_{k=0}^{\infty}\sum_{\kappa} \frac{1}{k!}\Gamma_p\left(\frac{n+m}{2},\kappa\right) C_\kappa\left(\frac{\alpha-c}{\alpha}\boldsymbol{I}_p\right) G_{p,p}^{p,0}\left(\lambda_1 \middle|_{b_1,\ldots,b_p}^{a_1,\ldots,a_p}\right). \quad \blacksquare \end{aligned}$$

Theorem 3.9

Let $\boldsymbol{U} \sim B_p^V(n, m, \boldsymbol{\Sigma}_1, \boldsymbol{\Sigma}_2)$ *with pdf given by* (3.12) *and* $\Lambda_1 = |\boldsymbol{U}|$, *then the pdf of* Λ_1 *is given by*

$$\frac{1}{\Gamma_p\left(\frac{n}{2}\right)} \left|\boldsymbol{\Sigma}_1^{-1}\boldsymbol{\Sigma}_2\right|^{\frac{1}{2}n} \sum_{k=0}^{\infty}\sum_{\kappa} \frac{1}{k!}\Gamma_p\left(\frac{n+m}{2},\kappa\right) C_\kappa\left(\boldsymbol{I}_p - \boldsymbol{\Sigma}_1^{-\frac{1}{2}}\boldsymbol{\Sigma}_2\boldsymbol{\Sigma}_1^{-\frac{1}{2}}\right) G_{p,p}^{p,0}\left(\lambda_1\Big|_{b_1,\ldots,b_p}^{a_1,\ldots,a_p}\right), \quad 0 < \lambda_1 < 1, \tag{3.25}$$

where $a_j = \frac{n+m}{2} + k_j - \frac{1}{2}(j+1)$ *for* $j = 1, 2, \ldots, p$,

$b_j = \frac{n}{2} + k_j - \frac{1}{2}(j+1)$ *for* $j = 1, 2, \ldots, p$.

Proof:

The Mellin transform (see [2.8.1]) of $f(\lambda_1)$ is obtained from (3.19) and [2.6.1] as

$$\begin{aligned}
M_f(h) &\equiv E\left(\Lambda_1^{h-1}\right) \\
&= E\left(|\boldsymbol{U}|^{h-1}\right) \\
&= \frac{\Gamma_p\left(\frac{n+m}{2}\right)}{\Gamma_p\left(\frac{n}{2}\right)\Gamma_p\left(\frac{m}{2}\right)} \frac{\Gamma_p\left(\frac{n}{2}+h-1\right)\Gamma_p\left(\frac{m}{2}\right)}{\Gamma_p\left(\frac{n+m}{2}+h-1\right)} \left|\boldsymbol{\Sigma}_1^{-1}\boldsymbol{\Sigma}_2\right|^{\frac{1}{2}n} {}_2F_1\left(\frac{n}{2}+h-1, \frac{n+m}{2}; \frac{n+m}{2}+h-1; \boldsymbol{I}_p - \boldsymbol{\Sigma}_1^{-\frac{1}{2}}\boldsymbol{\Sigma}_2\boldsymbol{\Sigma}_1^{-\frac{1}{2}}\right) \\
&= \frac{1}{\Gamma_p\left(\frac{n}{2}\right)} \left|\boldsymbol{\Sigma}_1^{-1}\boldsymbol{\Sigma}_2\right|^{\frac{1}{2}n} \sum_{k=0}^{\infty}\sum_{\kappa} \frac{1}{k!}\Gamma_p\left(\frac{n+m}{2},\kappa\right) \frac{\Gamma_p\left(\frac{n}{2}+h-1,\kappa\right)}{\Gamma_p\left(\frac{n+m}{2}+h-1,\kappa\right)} C_\kappa\left(\boldsymbol{I}_p - \boldsymbol{\Sigma}_1^{-\frac{1}{2}}\boldsymbol{\Sigma}_2\boldsymbol{\Sigma}_1^{-\frac{1}{2}}\right).
\end{aligned}$$

The proof of (3.25) is similar to the proof given in Theorem 3.8. ■

4 Bimatrix variate beta type I distribution

The pdf of the ratios (1.2) is given in Theorem 4.1. The marginal and conditional pdfs are given in Theorem 4.2 and the product moment of the determinants in Theorem 4.3. The latter result is used in Theorem 4.4 to derive an exact expression for the pdf of the product of the determinants of matrix variates having a bimatrix beta type I distribution. Finally, in Section 4.5 the role of the parameters is studied.

4.1 Probability density function

The bimatrix variate beta type I distribution is constructed from ratios of three independent Wishart variates, $\boldsymbol{S}_1 \sim W_p(n_1, \boldsymbol{\Sigma})$, $\boldsymbol{S}_2 \sim W_p(n_2, \boldsymbol{\Sigma})$ and $\boldsymbol{B} \sim W_p(m, \boldsymbol{\Sigma})$ (see Olkin and Rubin, 1964).

Theorem 4.1 *(Olkin and Rubin, 1964)*
Let $\boldsymbol{S}_1 \sim W_p(n_1, \boldsymbol{\Sigma})$, $\boldsymbol{S}_2 \sim W_p(n_2, \boldsymbol{\Sigma})$ *and* $\boldsymbol{B} \sim W_p(m, \boldsymbol{\Sigma})$ *be independently distributed. Consider the definition given by* (1.2), *that is*

$$\boldsymbol{U}_i = \boldsymbol{S}^{-\frac{1}{2}} \boldsymbol{S}_i \boldsymbol{S}^{-\frac{1}{2}}, \ i = 1, 2, \tag{4.1}$$

where $\boldsymbol{S} = \boldsymbol{S}_1 + \boldsymbol{S}_2 + \boldsymbol{B}$, $\boldsymbol{S} = \boldsymbol{S}^{\frac{1}{2}} \boldsymbol{S}^{\frac{1}{2}}$.

The pdf of $(\boldsymbol{U}_1, \boldsymbol{U}_2) \sim BB_p^I(n_1, n_2, m)$ *is given by*

$$\left\{\beta_p\left(\tfrac{n_1}{2}, \tfrac{n_2}{2}; \tfrac{m}{2}\right)\right\}^{-1} \prod_{i=1}^{2} |\boldsymbol{U}_i|^{\frac{1}{2}n_i - \frac{1}{2}(p+1)} \left|\boldsymbol{I}_p - \sum_{i=1}^{2} \boldsymbol{U}_i\right|^{\frac{1}{2}m - \frac{1}{2}(p+1)}, \tag{4.2}$$

$\boldsymbol{0} < \boldsymbol{U}_i < \boldsymbol{I}_p$, $i = 1, 2$, $\boldsymbol{0} < \sum_{i=1}^{2} \boldsymbol{U}_i < \boldsymbol{I}_p$, *where* $\beta_p\left(\frac{n_1}{2}, \frac{n_2}{2}; \frac{m}{2}\right)$ *is the multivariate Dirichlet function given in [2.2.5],* $n_i > (p-1)$, $i = 1, 2$, *and* $m > (p-1)$.

Remark 4.1
Similar to the matrix variate beta type I distribution (see (3.2) and (3.3)) the following definitions of Wishart ratios will give matrix variates having the bimatrix variate beta type I distribution with pdf given by (4.2):

$$\boldsymbol{U}_i = \boldsymbol{S}_i^{\frac{1}{2}} (\boldsymbol{S}_1 + \boldsymbol{S}_2 + \boldsymbol{B})^{-1} \boldsymbol{S}_i^{\frac{1}{2}}, \ i = 1, 2,$$

and

$$\boldsymbol{U}_i = \left(\boldsymbol{I}_p + \sum_{i=1}^{2} \boldsymbol{B}^{-\frac{1}{2}} \boldsymbol{S}_i \boldsymbol{B}^{-\frac{1}{2}}\right)^{-\frac{1}{2}} \boldsymbol{B}^{-\frac{1}{2}} \boldsymbol{S}_i \boldsymbol{B}^{-\frac{1}{2}} \left(\boldsymbol{I}_p + \sum_{i=1}^{2} \boldsymbol{B}^{-\frac{1}{2}} \boldsymbol{S}_i \boldsymbol{B}^{-\frac{1}{2}}\right)^{-\frac{1}{2}} \ i = 1, 2. \tag{4.3}$$

Remark 4.2
The matrix variate Dirichlet I distribution (Olkin and Rubin, 1964), denoted by $(\boldsymbol{U}_1, \ldots, \boldsymbol{U}_r) \sim D_p^I(n_1, \ldots, n_r, m)$, results by extending (4.1) to r independent Wishart matrix variates, $\boldsymbol{S}_i \sim W_p(n_i, \boldsymbol{\Sigma})$, $i = 1, \ldots, r$, all independent of $\boldsymbol{B} \sim W_p(m, \boldsymbol{\Sigma})$. The pdf of $(\boldsymbol{U}_1, \ldots, \boldsymbol{U}_r)$ is given by

$$\left\{\beta_p\left(\tfrac{n_1}{2}, \ldots, \tfrac{n_r}{2}; \tfrac{m}{2}\right)\right\}^{-1} \prod_{i=1}^{r} |\boldsymbol{U}_i|^{\frac{1}{2}n_i - \frac{1}{2}(p+1)} \left|\boldsymbol{I}_p - \sum_{i=1}^{r} \boldsymbol{U}_i\right|^{\frac{1}{2}m - \frac{1}{2}(p+1)}, \tag{4.4}$$

$\mathbf{0} < \boldsymbol{U}_i < \boldsymbol{I}_p$, $i = 1, \ldots, r$, $\mathbf{0} < \sum_{i=1}^{r} \boldsymbol{U}_i < \boldsymbol{I}_p$.

Remark 4.3
The pdfs of the bimatrix variate beta type I distribution (4.2) and the matrix variate Dirichlet type I distribution (4.4) are members of the Liouville family of distributions of the second kind (see [2.2.1]).

4.2 Marginal property and conditional density

In this section the marginal and conditional pdfs of the bimatrix variate beta type I distribution are given.

Theorem 4.2 *(In analogy to Gupta and Nagar, 2000b, Theorem 6.3.2, page 204 and Corollary 6.3.2.1, page 205)*
If $(\boldsymbol{U}_1, \boldsymbol{U}_2) \sim BB_p^I(n_1, n_2, m)$, *then the pdf of* $\boldsymbol{U}_1$ *is given by*

$$\left\{\beta_p\left(\tfrac{n_1}{2}, \tfrac{n_2+m}{2}\right)\right\}^{-1} |\boldsymbol{U}_1|^{\frac{1}{2}n_1 - \frac{1}{2}(p+1)} |\boldsymbol{I}_p - \boldsymbol{U}_1|^{\frac{1}{2}(n_2+m) - \frac{1}{2}(p+1)}, \quad \mathbf{0} < \boldsymbol{U}_1 < \boldsymbol{I}_p, \tag{4.5}$$

that is, $\boldsymbol{U}_1 \sim B_p^I(n_1, n_2 + m)$.

Furthermore the pdf of $\boldsymbol{U}_2 | \boldsymbol{U}_1$ *is*

$$\left\{\beta_p\left(\tfrac{n_2}{2}, \tfrac{m}{2}\right)\right\}^{-1} |\boldsymbol{U}_2|^{\frac{1}{2}n_2 - \frac{1}{2}(p+1)} \left|\boldsymbol{I}_p - \sum_{i=1}^{2} \boldsymbol{U}_i\right|^{\frac{1}{2}m - \frac{1}{2}(p+1)} |\boldsymbol{I}_p - \boldsymbol{U}_1|^{-\frac{1}{2}(n_2+m) + \frac{1}{2}(p+1)}, \tag{4.6}$$

$\mathbf{0} < \boldsymbol{U}_2 < \boldsymbol{I}_p - \boldsymbol{U}_1$.

Remark 4.4
If $(\boldsymbol{U}_1, \ldots, \boldsymbol{U}_r) \sim D_p^I(n_1, \ldots, n_r, m)$, then $(\boldsymbol{U}_1, \ldots, \boldsymbol{U}_s) \sim D_p^I(n_1, \ldots, n_s, n_{s+1} + \cdots + n_r + m)$, $s \leq r$, and $\boldsymbol{U}_i \sim B_p^I\left(n_i, \sum_{j=1(\neq i)}^{r} n_i + m\right)$, $i = 1, \ldots, r$. The pdf of $(\boldsymbol{U}_{s+1}, \ldots, \boldsymbol{U}_r) \,|\, (\boldsymbol{U}_1, \ldots, \boldsymbol{U}_s)$ is given by

$$\left\{\beta_p\left(\tfrac{n_{s+1}}{2}, \ldots, \tfrac{n_r}{2}; \tfrac{m}{2}\right)\right\}^{-1} \prod_{i=s+1}^{r} |\boldsymbol{U}_i|^{\frac{1}{2}n_i - \frac{1}{2}(p+1)} \left|\boldsymbol{I}_p - \sum_{i=1}^{r} \boldsymbol{U}_i\right|^{\frac{1}{2}m - \frac{1}{2}(p+1)} \left|\boldsymbol{I}_p - \sum_{i=1}^{s} \boldsymbol{U}_i\right|^{-\frac{1}{2}(n_{s+1} + \cdots + n_r + m) + \frac{1}{2}(p+1)}, \tag{4.7}$$

$\mathbf{0} < \boldsymbol{U}_i < \boldsymbol{I}_p - \sum_{i=1}^{s} \boldsymbol{U}_i$, $i = s+1, \ldots, r$, $\sum_{i=s+1}^{r} \boldsymbol{U}_i < \boldsymbol{I}_p - \sum_{i=1}^{s} \boldsymbol{U}_i$ (see Gupta and Nagar, 2000b, Theorem 6.3.2 and Corollary 6.3.2.1, pages 204 and 205).

Remark 4.5
The marginal and conditional pdfs of the matrix variate Dirichlet I distribution (see Remark 4.4 and equation (4.7)) are members of the Liouville family of distributions of the second and first kind respectively (see [2.2.1]).

4.3 Product moment of the determinants

The $(h_1, h_2)^{th}$ product moment, $E\left(|\boldsymbol{U}_1|^{h_1} |\boldsymbol{U}_2|^{h_2}\right)$, where $(\boldsymbol{U}_1, \boldsymbol{U}_2) \sim BB_p^I(n_1, n_2, m)$, is given in Theorem 4.3.

Theorem 4.3 *(In analogy to Gupta and Nagar, 2000b, Theorem 6.5.3, page 220)*
If $(\boldsymbol{U}_1, \boldsymbol{U}_2) \sim BB_p^I(n_1, n_2, m)$ *as given by* (4.2) *then*

$$E\left(|\boldsymbol{U}_1|^{h_1} |\boldsymbol{U}_2|^{h_2}\right) = \frac{\Gamma_p\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)} \frac{\Gamma_p\left(\frac{n_1}{2}+h_1\right)\Gamma_p\left(\frac{n_2}{2}+h_2\right)}{\Gamma_p\left(\frac{n_1+n_2+m}{2}+h_1+h_2\right)}, \tag{4.8}$$

where $\operatorname{Re}\left(\frac{1}{2}n_i + h_i\right) > \frac{1}{2}(p-1), \, i = 1, 2.$

4.4 Distribution of the product of determinants

De Waal (1970) derived an asymptotic distribution of a suitable function of the product of determinants of the bimatrix beta type I variates, $\Lambda_2 = \prod_{i=1}^{2} |\boldsymbol{U}_i|^{\frac{1}{2}n_i} = \prod_{i=1}^{2} \left|\frac{\boldsymbol{S}_i}{\boldsymbol{S}_1+\boldsymbol{S}_2+\boldsymbol{B}}\right|^{\frac{1}{2}n_i}$ in (1.7) where $(\boldsymbol{U}_1, \boldsymbol{U}_2) \sim BB_p^I(n_1, n_2, m)$, $\boldsymbol{S}_1 \sim W_p(n_1, \boldsymbol{\Sigma}_1)$, $\boldsymbol{S}_2 \sim W_p(n_2, \boldsymbol{\Sigma}_1)$ and $\boldsymbol{B} \sim W_p(m, \boldsymbol{\Sigma}_2)$. An exact expression for the pdf of Λ_2 in (1.7) is derived in Theorem 4.4.

Theorem 4.4
Let $\boldsymbol{S}_1 \sim W_p(n_1, \boldsymbol{\Sigma})$, $\boldsymbol{S}_2 \sim W_p(n_2, \boldsymbol{\Sigma})$ *and* $\boldsymbol{B} \sim W_p(m, \boldsymbol{\Sigma})$. *The ratios given by* (4.1),

$$\boldsymbol{U}_i = (\boldsymbol{S}_1 + \boldsymbol{S}_2 + \boldsymbol{B})^{-\frac{1}{2}} \boldsymbol{S}_i (\boldsymbol{S}_1 + \boldsymbol{S}_2 + \boldsymbol{B})^{-\frac{1}{2}}, \quad i = 1, 2,$$

give $(\boldsymbol{U}_1, \boldsymbol{U}_2) \sim BB_p^I(n_1, n_2, m)$. *Let* $\Lambda_2 = |\boldsymbol{U}_1|^{\frac{1}{2}n_1} |\boldsymbol{U}_2|^{\frac{1}{2}n_2}$.
The pdf of Λ_2 *is given by*

$$\frac{\pi^{\frac{1}{4}p(p-1)}\Gamma_p\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)} H_{p,2p}^{2p,0}\left(\lambda_2\middle|_{(b_1,\beta_1),\ldots,(b_{2p},\beta_{2p})}^{(a_1,\alpha_1),\ldots,(a_p,\alpha_p)}\right), \quad 0 < \lambda_2 < 1, \tag{4.9}$$

where $H(\cdot)$ *denotes Fox's H-function (see [2.8.3]) and*

$a_j = \frac{m}{2} - \frac{1}{2}(j-1)$ *for* $j = 1, 2, \ldots, p,$

$\alpha_j = \frac{n_1+n_2}{2}$ *for* $j = 1, 2, \ldots, p,$

$$b_j = \begin{cases} -\frac{1}{4}(j-1) & \text{for } j = 1, 3, 5, \ldots, 2p-1 \\ -\frac{1}{4}(j-2) & \text{for } j = 2, 4, 6, \ldots, 2p, \end{cases}$$

$$\beta_j = \begin{cases} \frac{n_1}{2} & \text{for } j = 1, 3, 5, \ldots, 2p-1 \\ \frac{n_2}{2} & \text{for } j = 2, 4, 6, \ldots, 2p. \end{cases}$$

Proof:

From (4.8) the Mellin transform (see [2.8.1]) of $f(\lambda_2)$ is

$$\begin{aligned} M_f(h) &\equiv E\left(\Lambda_2^{h-1}\right) \\ &= E\left[\left(|\boldsymbol{U}_1|^{\frac{1}{2}n_1}|\boldsymbol{U}_2|^{\frac{1}{2}n_2}\right)^{h-1}\right] \\ &= \frac{\Gamma_p\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)}\frac{\Gamma_p\left[\frac{n_1}{2}+\frac{n_1}{2}(h-1)\right]\Gamma_p\left[\frac{n_2}{2}+\frac{n_2}{2}(h-1)\right]}{\Gamma_p\left[\frac{n_1+n_2+m}{2}+\frac{n_1+n_2}{2}(h-1)\right]} \\ &= \frac{\Gamma_p\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)}\frac{\Gamma_p\left(\frac{n_1}{2}h\right)\Gamma_p\left(\frac{n_2}{2}h\right)}{\Gamma_p\left(\frac{m}{2}+\frac{n_1+n_2}{2}h\right)} \end{aligned} \tag{4.10}$$

Using [2.2.2] the multivariate gamma function in (4.10) can be written as

$$\begin{aligned} &\Gamma_p\left(\frac{m}{2}+\frac{n_1+n_2}{2}h\right) \\ &= \pi^{\frac{1}{4}p(p-1)}\prod_{j=1}^{p}\Gamma\left(a_j+\alpha_j h\right), \end{aligned} \tag{4.11}$$

where $a_j = \frac{m}{2} - \frac{1}{2}(j-1)$ for $j = 1, 2, \ldots, p$,

and $\alpha_j = \frac{n_1+n_2}{2}$ for $j = 1, 2, \ldots, p$.

Also, from [2.2.2] and (4.10),

$$\begin{aligned} &\Gamma_p\left(\frac{n_1}{2}h\right)\Gamma_p\left(\frac{n_2}{2}h\right) \\ &\quad= \pi^{\frac{1}{4}p(p-1)}\prod_{j=1}^{p}\Gamma\left[\frac{n_1}{2}h-\frac{1}{2}(j-1)\right]\pi^{\frac{1}{4}p(p-1)}\prod_{j=1}^{p}\Gamma\left[\frac{n_2}{2}h-\frac{1}{2}(j-1)\right] \\ &\quad= \pi^{\frac{1}{2}p(p-1)}\prod_{j=1}^{2p}\Gamma\left(b_j+\beta_j h\right) \end{aligned} \tag{4.12}$$

$$\text{where } b_j = \begin{cases} -\frac{1}{4}(j-1) & \text{for } j = 1, 3, 5, \ldots, 2p-1 \\ -\frac{1}{4}(j-2) & \text{for } j = 2, 4, 6, \ldots, 2p, \end{cases}$$

$$\text{and } \beta_j = \begin{cases} \frac{n_1}{2} & \text{for } j = 1, 3, 5, \ldots, 2p-1 \\ \frac{n_2}{2} & \text{for } j = 2, 4, 6, \ldots, 2p. \end{cases}$$

Substituting (4.11) and (4.12) in (4.10) gives

$$M_f(h) \equiv \frac{\pi^{\frac{1}{4}p(p-1)}\Gamma_p\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)}\frac{\prod_{j=1}^{2p}\Gamma\left(b_j+\beta_j h\right)}{\prod_{j=1}^{p}\Gamma\left(a_j+\alpha_j h\right)}. \tag{4.13}$$

The pdf of Λ_2 is uniquely obtained from the inverse Mellin transform (see [2.8.1]) of (4.13) and is given in terms of Fox's H-function (see [2.8.3]) as

$$\begin{aligned} f(\lambda_2) &= \frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty} M_f(h)\,\lambda_2^{-h}\mathrm{d}h \\ &= \frac{\pi^{\frac{1}{4}p(p-1)}\Gamma_p\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)}\left[\frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty}\frac{\prod_{j=1}^{2p}\Gamma\left(b_j+\beta_j h\right)}{\prod_{j=1}^{p}\Gamma\left(a_j+\alpha_j h\right)}\lambda_2^{-h}\mathrm{d}h\right] \\ &= \frac{\pi^{\frac{1}{4}p(p-1)}\Gamma_p\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)} H_{p,2p}^{2p,0}\left(\lambda_2\middle|_{(b_1,\beta_1),\ldots,(b_{2p},\beta_{2p})}^{(a_1,\alpha_1),\ldots,(a_p,\alpha_p)}\right). \qquad \blacksquare \end{aligned}$$

4.5 Role of the parameters

In this section we study the effect of the parameters n_1, n_2 and m.

Firstly, we consider the bivariate case, $p = 1$, to illustrate the effect of the parameters n_1, n_2 and m on

- (i) the form of the pdf of (U_1, U_2);
- (ii) the correlation between U_1 and U_2;
- (iii) the graphs of $E(U_2|u_1)$ and $var(U_2|u_1)$ plotted against u_1;
- (iv) the form of the pdf of Λ_2.

From (4.2) the joint pdf of U_1 and U_2 simplifies to

$$f(u_1, u_2) = \frac{\Gamma\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{m}{2}\right)} u_1^{\frac{1}{2}n_1-1} u_2^{\frac{1}{2}n_2-1} (1-u_1-u_2)^{\frac{1}{2}m-1}, \tag{4.14}$$

$0 < u_i < 1,\ i = 1, 2,\ 0 < u_1 + u_2 < 1$ (Balakrishnan and Lai, 2009).

Figures 4.1a and 4.1b show graphs of the pdfs of $BB_1^I(10, n_2, 10)$ and $BB_1^I(10, 10, m)$ distributions respectively (see (4.14)). In Figure 4.1a, as n_2 increases with all the other parameters constant, the pdf shifts towards smaller values of U_1 and larger values of U_2. The opposite will be observed for increasing values of n_1. Increasing the value of the parameter m in Figure 4.1b with all other parameters constant, causes the pdf $f(u_1, u_2)$ to shift towards smaller values of both U_1 and U_2.

Figure 4.1a: Effect of n_2 on $f(u_1,u_2)$, $(U_1,U_2) \sim BB_1^I(10,n_2,10)$

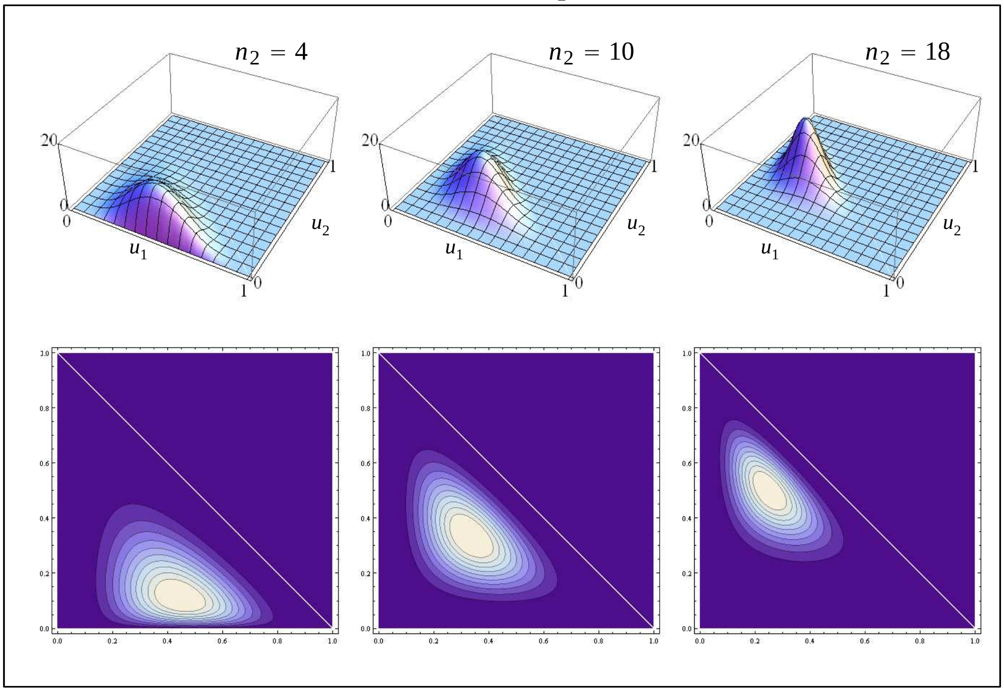
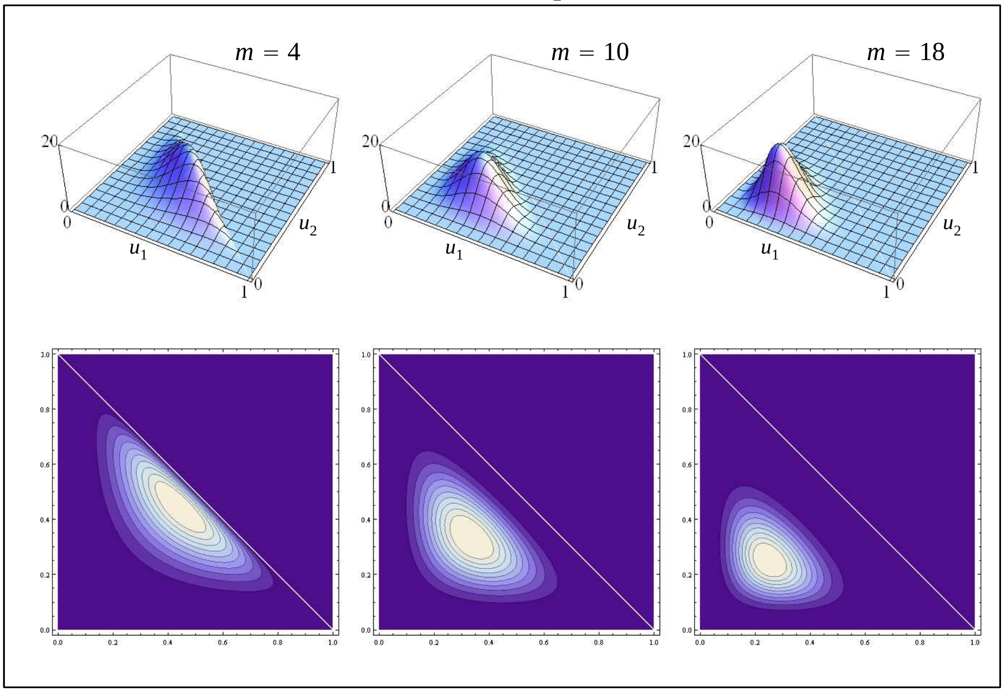


Figure 4.1b: Effect of m on $f(u_1,u_2)$, $(U_1,U_2) \sim BB_1^I(10,10,m)$

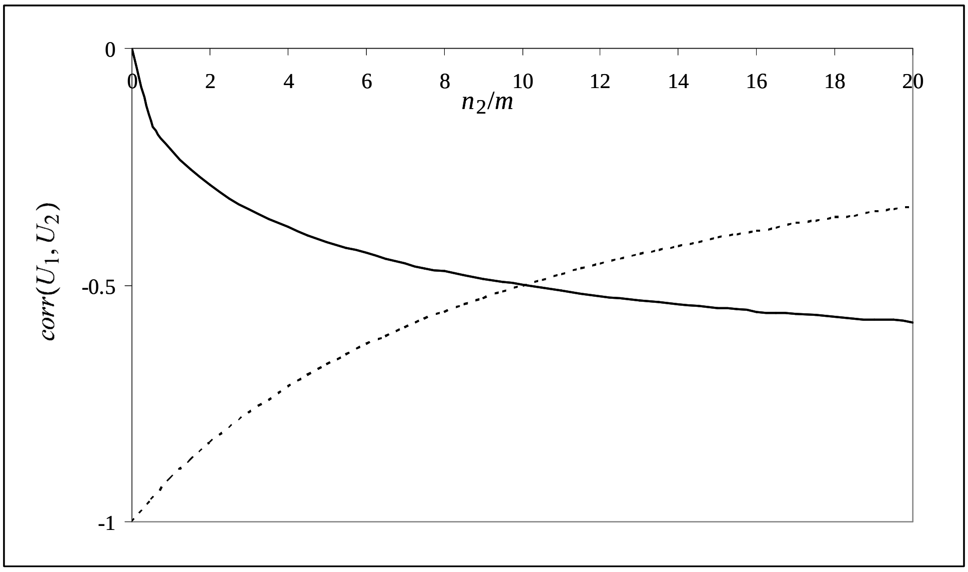


From (4.8) the $(h_1,h_2)^{th}$ product moment, $E\left(U_1^{h_1}U_2^{h_2}\right)$, associated with (4.14) is given by

$$E\left(U_1^{h_1}U_2^{h_2}\right) = \frac{\Gamma\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} \frac{\Gamma\left(\frac{n_1}{2}+h_1\right)\Gamma\left(\frac{n_2}{2}+h_2\right)}{\Gamma\left(\frac{n_1+n_2+m}{2}+h_1+h_2\right)}. \tag{4.15}$$

The correlation coefficient, $corr\,(U_1, U_2)$, was calculated by using (4.15). Figure 4.2 shows graphs of $corr\,(U_1, U_2)$ for increasing values of n_2 and m. The correlation is negative and as n_2 increases or m decreases the negative relationship shifts towards -1. The effect of n_1 on $corr\,(U_1, U_2)$ will be the same as that of n_2.

Figure 4.2: Effect of n_2 and m on $corr(U_1, U_2)$
(i) $(U_1, U_2) \sim BB_1^I(10, n_2, 10)$ ———
(ii) $(U_1, U_2) \sim BB_1^I(10, 10, m)$ - - - - - -



From (4.5), the marginal pdf of U_1 for $p = 1$ is given by

$$f(u_1) = \frac{\Gamma\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2+m}{2}\right)} u_1^{\frac{1}{2}n_1-1}(1-u_1)^{\frac{1}{2}(n_2+m)-1}, \quad 0 < u_1 < 1,$$

and the conditional pdf of $(U_2|U_1 = u_1)$ given by (4.6) simplifies to

$$f(u_2|u_1) = \frac{\Gamma\left(\frac{n_2+m}{2}\right)}{\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{m}{2}\right)} u_2^{\frac{1}{2}n_2-1}(1-u_1)^{-\frac{1}{2}(n_2+m)+1}(1-u_1-u_2)^{\frac{1}{2}m-1}, \quad 0 < u_2 < 1-u_1. \tag{4.16}$$

The variables U_1 and U_2 are dependent and the nature of the relationship between these two variables is studied by drawing graphs of $E\,(U_2|u_1)$ and $var\,(U_2|u_1)$ against u_1.

Theorem 4.5
If $(U_1, U_2) \sim BB_1^I(n_1, n_2, m)$ *then*

$$E\left(U_2^h | U_1 = u_1\right) = \frac{\Gamma\left(\frac{n_2+m}{2}\right)}{\Gamma\left(\frac{n_2}{2}\right)} \frac{\Gamma\left(\frac{n_2}{2}+h\right)}{\Gamma\left(\frac{n_2+m}{2}+h\right)} (1-u_1)^h . \tag{4.17}$$

Proof:
From (4.16) it follows that

$$\begin{aligned} & E\left(U_2^h | U_1 = u_1\right) \\ = \; & \frac{\Gamma\left(\frac{n_2+m}{2}\right)}{\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{m}{2}\right)} (1-u_1)^{-\frac{1}{2}(n_2+m)+1} \int_0^{1-u_1} u_2^{\frac{1}{2}n_2+h-1} (1-u_1-u_2)^{\frac{1}{2}m-1} \mathrm{d}u_2 \\ = \; & \frac{\Gamma\left(\frac{n_2+m}{2}\right)}{\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{m}{2}\right)} (1-u_1)^{-\frac{1}{2}(n_2+m)+1} \\ & \cdot (1-u_1)^{\frac{1}{2}n_2+h-1} (1-u_1)^{\frac{1}{2}m-1} (1-u_1) \int_0^1 z^{\frac{1}{2}n_2+h-1} (1-z)^{\frac{1}{2}m-1} \mathrm{d}z, \quad \text{where} \quad z = \tfrac{u_2}{1-u_1}. \end{aligned} \tag{4.18}$$

The integral in (4.18) is solved by using the well known beta function and from this (4.17) follows. ■

Figures 4.3a and 4.3b show graphs of $E(U_2|u_1)$ given by (4.17) plotted against u_1 for different values of the parameters n_2 and m where $(U_1, U_2) \sim BB_1^I(10, n_2, 10)$ and $(U_1, U_2) \sim BB_1^I(10, 10, m)$ respectively. From (4.17) it can be seen that for given values of the parameters the relationship between $E(U_2|u_1)$ and u_1 is a straight line with a slope equal to $-\frac{n_2}{n_2+m}$. As n_2 increases or m decreases the slope becomes more negative. This can also be seen from Figure 4.2 where the correlation between U_1 and U_2 shifts towards -1 as n_2 increases or m decreases. The value of the parameter n_1 has no effect on the relationship between $E(U_2|u_1)$ and u_1. For a given value u_1 of U_1, $E(U_2|u_1)$ increases as n_2 increases or m decreases. This also corresponds to what is observed in Figures 4.1a and 4.1b.

Figure 4.3a: Effect of n_2 on $E(U_2|u_1)$, $(U_1, U_2) \sim BB_1^I(10, n_2, 10)$

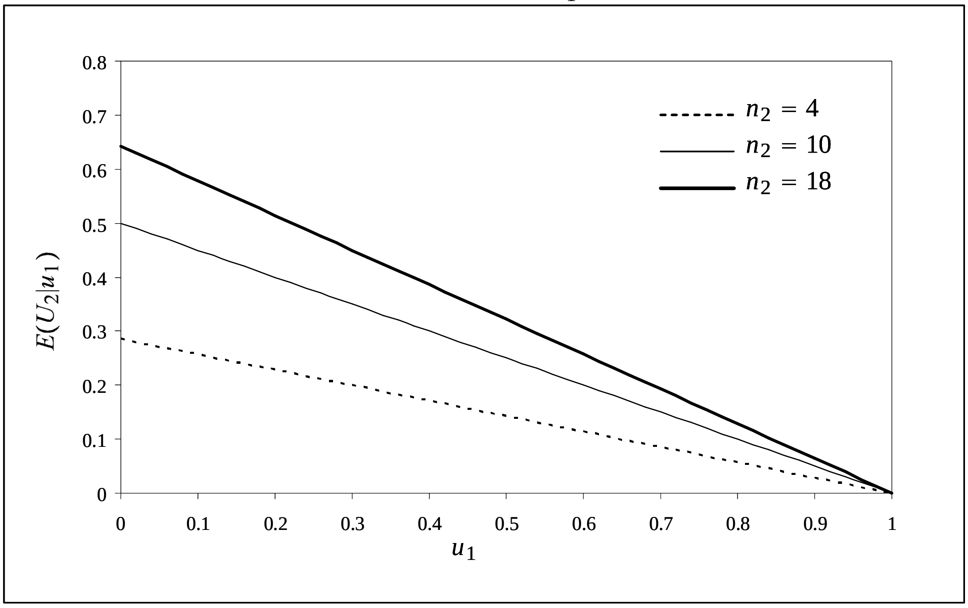


Figure 4.3b: Effect of m on $E(U_2|u_1)$, $(U_1, U_2) \sim BB_1^I(10, 10, m)$

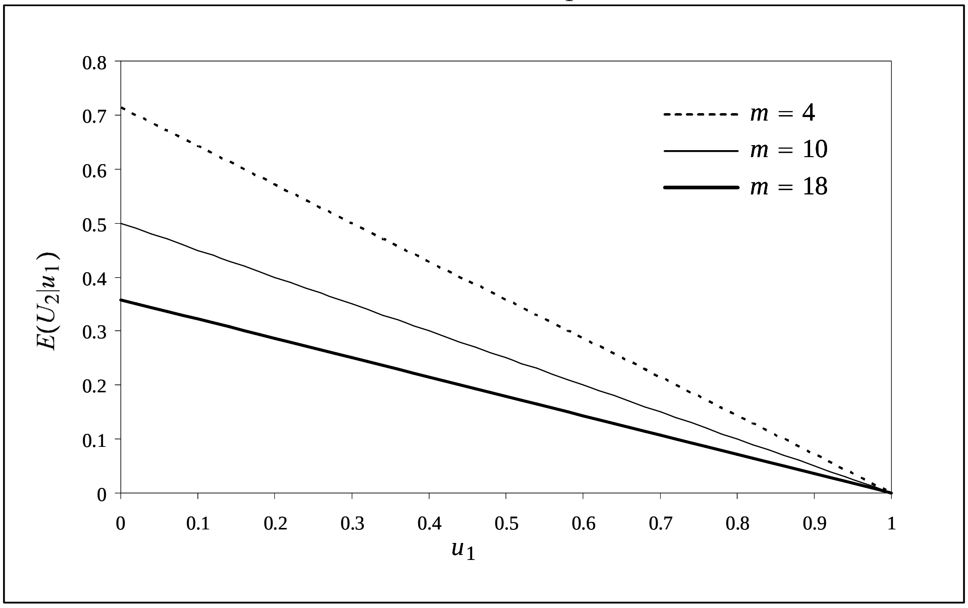
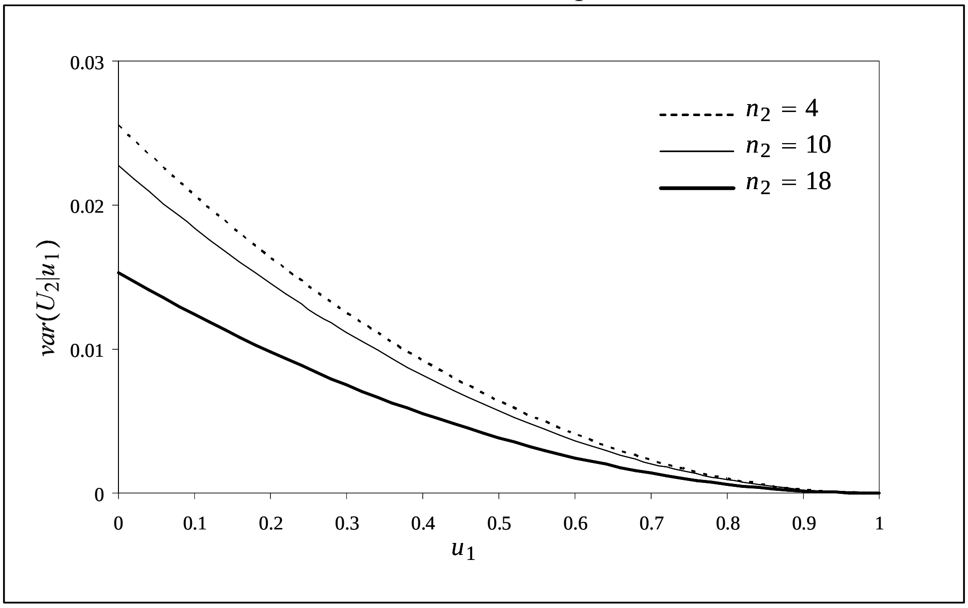


Figure 4.4 shows graphs of $var(U_2|u_1)$ plotted against u_1 for different values of n_2 where $(U_1, U_2) \sim BB_1^I(10, n_2, 10)$. Since $var(U_2|u_1) = \frac{2n_2m}{(n_2+m)^2(n_2+m+2)}(1-u_1)^2$, the effect of m on the form of the graph is exactly the same as that of n_2 and the parameter n_1 does not play a role. For given values of the parameters $var(U_2|u_1)$ decreases as u_1 increases. For a given value u_1 of U_1, $var(U_2|u_1)$ decreases as n_2 (or m) increases.

Figure 4.4: Effect of n_2 on $var(U_2|u_1)$, $(U_1, U_2) \sim BB_1^I(10, n_2, 10)$



Finally, for the bivariate case, the effect of the parameters n_1, n_2 and m on the pdf of $\Lambda_2 = U_1^{\frac{1}{2}n_1} U_2^{\frac{1}{2}n_2}$ was studied where $(U_1, U_2) \sim BB_1^I(n_1, n_2, m)$. The result (4.9) simplifies to

$$f(\lambda_2) = \frac{\Gamma\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} H_{1,2}^{2,0}\left(\lambda_2 \middle| \begin{smallmatrix} \left(\frac{m}{2}, \frac{n_1+n_2}{2}\right) \\ \left(0, \frac{n_1}{2}\right), \left(0, \frac{n_2}{2}\right) \end{smallmatrix}\right), \quad 0 < \lambda_2 < 1. \tag{4.19}$$

Figures 4.5a and 4.5b show the effect of n_2 and m on $f(\lambda_2)$ given by (4.19) where $(U_1, U_2) \sim BB_1^I(2, n_2, 2)$ and $(U_1, U_2) \sim BB_1^I(2, 2, m)$ respectively. The parameters n_2 and m have the same effect on the form of the pdf. That is, at smaller values of Λ_2 the pdf increases as n_2 or m increases. The effect of n_1 will be the same as that of n_2.

Figure 4.5a: Effect of n_2 on $f(\lambda_2)$, $\Lambda_2 = U_1 U_2^{\frac{1}{2}n_2}$, $(U_1, U_2) \sim BB_1^I(2, n_2, 2)$

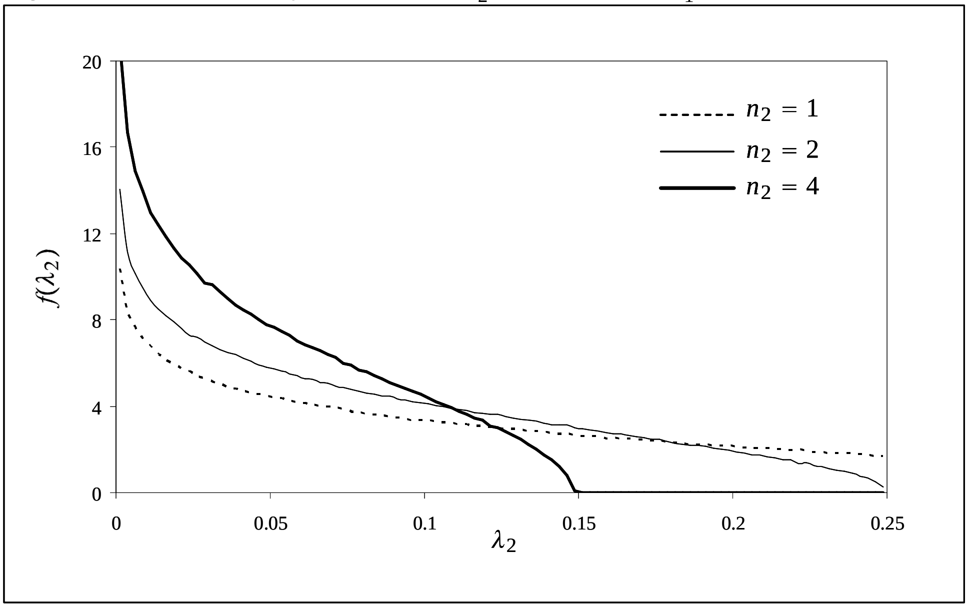
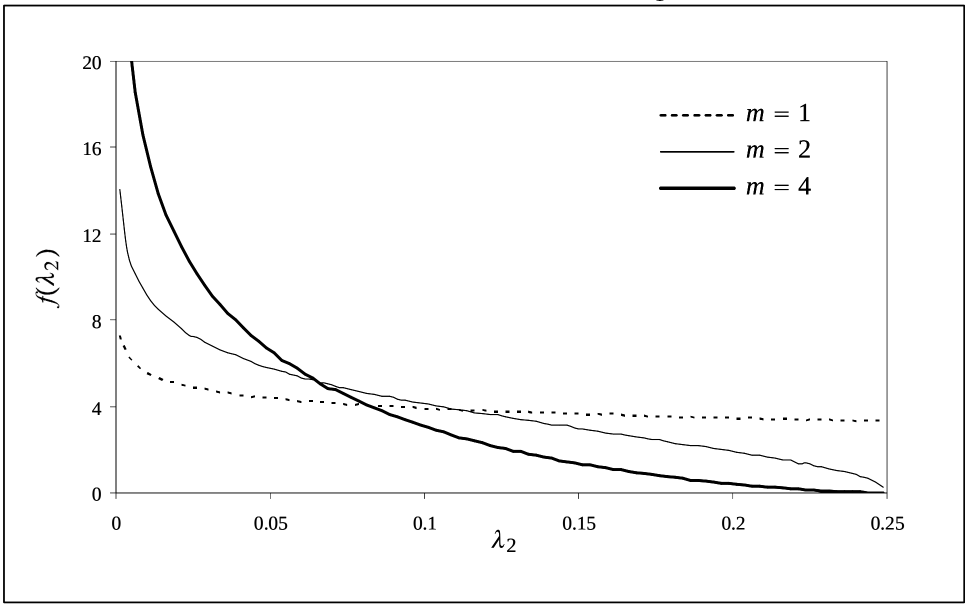


Figure 4.5b: Effect of m on $f(\lambda_2)$, $\Lambda_2 = U_1 U_2$, $(U_1, U_2) \sim BB_1^I(2, 2, m)$



Secondly, we consider the bimatrix case, $p = 2$, to illustrate the effect of the parameters n_1, n_2 and m on the pdf of Λ_2. From (4.9), the pdf of Λ_2 for $p = 2$ simplifies to

$$f(\lambda_2) = \frac{\Gamma_2\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma_2\left(\frac{n_1}{2}\right)\Gamma_2\left(\frac{n_2}{2}\right)} \pi^{\frac{1}{2}} H_{2,4}^{4,0}\left(\lambda_2 \middle| \begin{smallmatrix}(a_1,\alpha_1),(a_2,\alpha_2)\\(b_1,\beta_1),\ldots,(b_4,\beta_4)\end{smallmatrix}\right), \quad 0 < \lambda_2 < 1,$$

where $a_1 = \frac{m}{2}$, $a_2 = \frac{m}{2} - \frac{1}{2}$, $\alpha_1 = \alpha_2 = \frac{n_1+n_2}{2}$, $b_1 = b_2 = 0$, $b_3 = b_4 = -\frac{1}{2}$, $\beta_1 = \beta_3 = \frac{n_1}{2}$ and $\beta_2 = \beta_4 = \frac{n_2}{2}$.

Figures 4.6a and 4.6b illustrate the shape of the pdf, $f(\lambda_2)$, for increasing values of n_2 and m. We note that as n_2 or m increases the pdf shifts towards smaller values of Λ_2. The same will be observed as n_1 increases.

Figure 4.6a: Effect of n_2 on $f(\lambda_2)$, $\Lambda_2 = |\boldsymbol{U}_1||\boldsymbol{U}_2|^{\frac{1}{2}n_2}$, $(\boldsymbol{U}_1, \boldsymbol{U}_2) \sim BB_2^I(2, n_2, 2)$

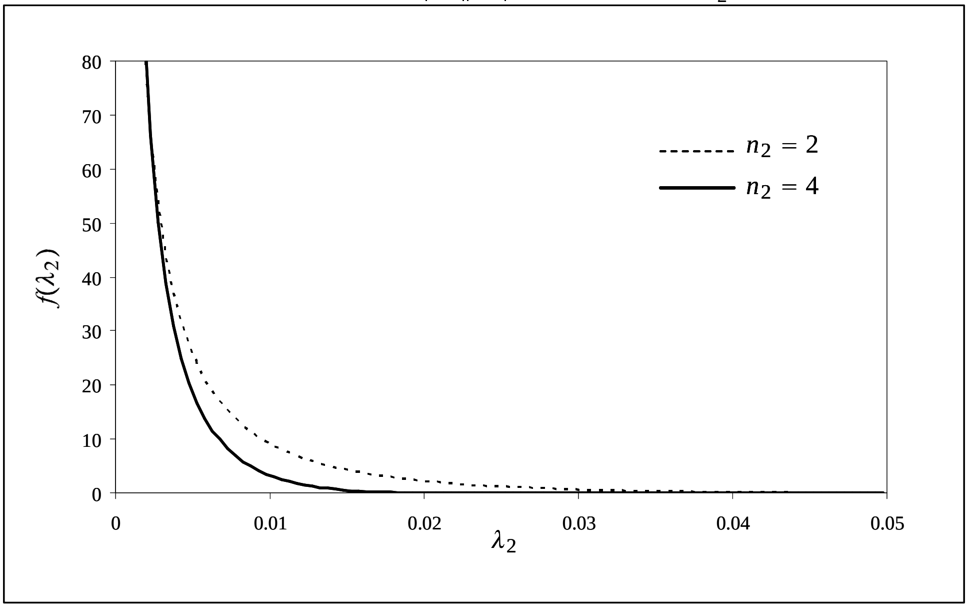
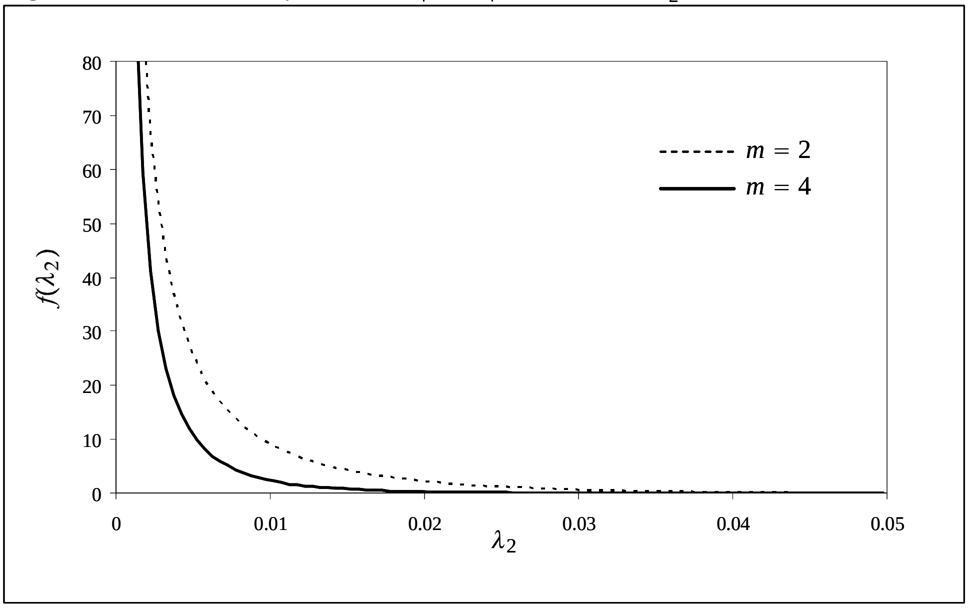


Figure 4.6b: Effect of m on $f(\lambda_2)$, $\Lambda_2 = |\boldsymbol{U}_1\boldsymbol{U}_2|$, $(\boldsymbol{U}_1, \boldsymbol{U}_2) \sim BB_2^I(2, 2, m)$



5 Bimatrix variate beta type III distribution

By introducing the additional parameter c in the ratios (1.2) we derive the pdf of (1.3) in Theorem 5.1 and call it the bimatrix variate beta type III distribution that can be used as an alternative to the bimatrix variate beta type I distribution (Ehlers, Bekker and Roux, 2009). In Theorem 5.2 the marginal and conditional pdfs are given. Theorem 5.3 derives the product moment of the determinants and the latter is used in Theorem 5.4 to derive an exact expression for the pdf of $\Lambda_3 = |\boldsymbol{W}_1|^{\frac{1}{2}n_1} |\boldsymbol{W}_2|^{\frac{1}{2}n_1}$ (see (1.8)). The role of the additional parameter c is studied in Section 5.5.

5.1 Probability density function

The pdf of the bimatrix variate beta type III distribution is derived from Wishart ratios in Theorem 5.1.

Theorem 5.1
Let $\boldsymbol{S}_1 \sim W_p(n_1, \boldsymbol{\Sigma})$, $\boldsymbol{S}_2 \sim W_p(n_2, \boldsymbol{\Sigma})$ *and* $\boldsymbol{B} \sim W_p(m, \boldsymbol{\Sigma})$ *be independently distributed. Consider the definition in* (1.3), *that is*

$$\boldsymbol{W}_i = \boldsymbol{S}^{-\frac{1}{2}} \boldsymbol{S}_i \boldsymbol{S}^{-\frac{1}{2}},\ i = 1, 2, \tag{5.1}$$

where $\boldsymbol{S} = \boldsymbol{S}_1 + \boldsymbol{S}_2 + c\boldsymbol{B}$, $c > 0$ *and* $\boldsymbol{S} = \boldsymbol{S}^{\frac{1}{2}} \boldsymbol{S}^{\frac{1}{2}}$.

The pdf of $(\boldsymbol{W}_1, \boldsymbol{W}_2) \sim BB_p^{III}(n_1, n_2, m, c)$ *is given by*

$$\begin{aligned} & f(\boldsymbol{W}_1, \boldsymbol{W}_2) \\ = \;& \left\{\beta_p\left(\tfrac{n_1}{2}, \tfrac{n_2}{2}; \tfrac{m}{2}\right)\right\}^{-1} \prod_{i=1}^{2} |\boldsymbol{W}_i|^{\frac{1}{2}n_i - \frac{1}{2}(p+1)} \left|\boldsymbol{I}_p - \sum_{i=1}^{2} \boldsymbol{W}_i\right|^{\frac{1}{2}m - \frac{1}{2}(p+1)} \\ & \cdot c^{\frac{1}{2}(n_1+n_2)p} \left|\boldsymbol{I}_p + (c-1)\sum_{i=1}^{2} \boldsymbol{W}_i\right|^{-\frac{1}{2}(n_1+n_2+m)} \\ = \;& g(\boldsymbol{W}_1, \boldsymbol{W}_2)\, c^{\frac{1}{2}(n_1+n_2)p} \left|\boldsymbol{I}_p + (c-1)\sum_{i=1}^{2} \boldsymbol{W}_i\right|^{-\frac{1}{2}(n_1+n_2+m)}, \end{aligned} \tag{5.2}$$

$\boldsymbol{0} < \boldsymbol{W}_i < \boldsymbol{I}_p$, $i = 1, 2$, $\boldsymbol{0} < \sum_{i=1}^{2} \boldsymbol{W}_i < \boldsymbol{I}_p$, *where* $n_i > (p-1)$, $i = 1, 2$, $m > (p-1)$ *and* $g(\cdot)$ *is the pdf of* $BB_p^I(n_1, n_2, m)$ *given by* (4.2).

Proof:
The pdf of $(\boldsymbol{S}_1, \boldsymbol{S}_2, \boldsymbol{B})$ is given by

$$K \prod_{i=1}^{2} \left[\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}_i\right) |\boldsymbol{S}_i|^{\frac{1}{2}(n_i-p-1)}\right] \left[\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\right) |\boldsymbol{B}|^{\frac{1}{2}(m-p-1)}\right] \tag{5.3}$$

where $K^{-1} = \Gamma_p\left(\frac{n_1}{2}\right) \Gamma_p\left(\frac{n_2}{2}\right) \Gamma_p\left(\frac{m}{2}\right) |2\boldsymbol{\Sigma}|^{\frac{1}{2}(n_1+n_2+m)}$ (see [2.10.1]).
The ratios given by (5.1) give $\boldsymbol{S}_i = \boldsymbol{S}^{\frac{1}{2}} \boldsymbol{W}_i \boldsymbol{S}^{\frac{1}{2}}$ for $i = 1, 2$, and
$\boldsymbol{B} = \frac{1}{c}\left(\boldsymbol{S} - \sum_{i=1}^{2} \boldsymbol{S}^{\frac{1}{2}} \boldsymbol{W}_i \boldsymbol{S}^{\frac{1}{2}}\right) = \frac{1}{c}\boldsymbol{S}^{\frac{1}{2}}\left(\boldsymbol{I}_p - \sum_{i=1}^{2} \boldsymbol{W}_i\right)\boldsymbol{S}^{\frac{1}{2}}$. From [2.1.3], [2.1.5] and [2.1.8] the Jacobian of the transformations is $J(\boldsymbol{S}_1, \boldsymbol{S}_2, \boldsymbol{B} \to \boldsymbol{W}_1, \boldsymbol{W}_2, \boldsymbol{S}) = |\boldsymbol{S}|^{(p+1)} c^{-\frac{1}{2}p(p+1)}$. Now, substituting in (5.3) we get

$$
\begin{aligned}
& f\left(\boldsymbol{W}_1, \boldsymbol{W}_2, \boldsymbol{S}\right) \\
= \; & K \prod_{i=1}^{2}\left[\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{W}_i\boldsymbol{S}^{\frac{1}{2}}\right)\left|\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{W}_i\boldsymbol{S}^{\frac{1}{2}}\right|^{\frac{1}{2}n_i-\frac{1}{2}(p+1)}\right] \\
& \cdot \operatorname{etr}\left[-\tfrac{1}{2c}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}^{\frac{1}{2}}\left(\boldsymbol{I}_p-\sum_{i=1}^{2}\boldsymbol{W}_i\right)\boldsymbol{S}^{\frac{1}{2}}\right]\left|\tfrac{1}{c}\boldsymbol{S}^{\frac{1}{2}}\left(\boldsymbol{I}_p-\sum_{i=1}^{2}\boldsymbol{W}_i\right)\boldsymbol{S}^{\frac{1}{2}}\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)}\left|\boldsymbol{S}\right|^{(p+1)} c^{-\frac{1}{2}p(p+1)} \\
= \; & K c^{-\frac{1}{2}mp}\prod_{i=1}^{2}\left|\boldsymbol{W}_i\right|^{\frac{1}{2}n_i-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\sum_{i=1}^{2}\boldsymbol{W}_i\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)} \\
& \cdot\left|\boldsymbol{S}\right|^{\frac{1}{2}(n_1+n_2+m)-\frac{1}{2}(p+1)}\operatorname{etr}\left\{-\tfrac{1}{2c}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}^{\frac{1}{2}}\left[\boldsymbol{I}_p+(c-1)\sum_{i=1}^{2}\boldsymbol{W}_i\right]\boldsymbol{S}^{\frac{1}{2}}\right\}.
\end{aligned}
\tag{5.4}
$$

We consider the symmetrised density function of $(\boldsymbol{W}_1, \boldsymbol{W}_2)$ (see [2.9.1]), that is $f_s\left(\boldsymbol{W}_1, \boldsymbol{W}_2\right) \equiv \int_{\boldsymbol{S}>\boldsymbol{0}}\int_{O(p)} f\left(\boldsymbol{H}\boldsymbol{W}_1\boldsymbol{H}', \boldsymbol{H}\boldsymbol{W}_2\boldsymbol{H}', \boldsymbol{H}\boldsymbol{S}\boldsymbol{H}'\right)\mathrm{d}\boldsymbol{H}\,\mathrm{d}\boldsymbol{S}$ where $\boldsymbol{H}$ $(p\times p)$ is orthogonal and $\mathrm{d}\boldsymbol{H}$ is the normalised Haar invariant measure on $O(p)$. Note that $\mathrm{d}\boldsymbol{S} = \mathrm{d}\boldsymbol{H}\boldsymbol{S}\boldsymbol{H}'$ (Díaz-García and Gutiérrez-Jáimez, 2006b). From (5.4)

$$
\begin{aligned}
& f\left(\boldsymbol{H}\boldsymbol{W}_1\boldsymbol{H}', \boldsymbol{H}\boldsymbol{W}_2\boldsymbol{H}', \boldsymbol{H}\boldsymbol{S}\boldsymbol{H}'\right) \\
= \; & K c^{-\frac{1}{2}mp}\prod_{i=1}^{2}\left|\boldsymbol{H}\boldsymbol{W}_i\boldsymbol{H}'\right|^{\frac{1}{2}n_i-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\sum_{i=1}^{2}\boldsymbol{H}\boldsymbol{W}_i\boldsymbol{H}'\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)} \\
& \cdot\left|\boldsymbol{H}\boldsymbol{S}\boldsymbol{H}'\right|^{\frac{1}{2}(n_1+n_2+m)-\frac{1}{2}(p+1)}\operatorname{etr}\left\{-\tfrac{1}{2c}\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{H}\boldsymbol{S}\boldsymbol{H}'\right)^{\frac{1}{2}}\left[\boldsymbol{I}_p+(c-1)\sum_{i=1}^{2}\boldsymbol{H}\boldsymbol{W}_i\boldsymbol{H}'\right]\left(\boldsymbol{H}\boldsymbol{S}\boldsymbol{H}'\right)^{\frac{1}{2}}\right\} \\
= \; & K c^{-\frac{1}{2}mp}\prod_{i=1}^{2}\left|\boldsymbol{W}_i\right|^{\frac{1}{2}n_i-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\sum_{i=1}^{2}\boldsymbol{W}_i\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)} \\
& \cdot\left|\boldsymbol{S}\right|^{\frac{1}{2}(n_1+n_2+m)-\frac{1}{2}(p+1)}\operatorname{etr}\left\{-\tfrac{1}{2c}\boldsymbol{\Sigma}^{-1}\boldsymbol{H}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{H}'\left[\boldsymbol{I}_p+(c-1)\sum_{i=1}^{2}\boldsymbol{H}\boldsymbol{W}_i\boldsymbol{H}'\right]\boldsymbol{H}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{H}'\right\}.
\end{aligned}
\tag{5.5}
$$

Then, from (5.5)

$$
\begin{aligned}
& f_s\left(\boldsymbol{W}_1, \boldsymbol{W}_2\right) \\
= \; & K c^{-\frac{1}{2}mp}\prod_{i=1}^{2}\left|\boldsymbol{W}_i\right|^{\frac{1}{2}n_i-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\sum_{i=1}^{2}\boldsymbol{W}_i\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)} \\
& \cdot\int_{\boldsymbol{S}>\boldsymbol{0}}\left|\boldsymbol{S}\right|^{\frac{1}{2}(n_1+n_2+m)-\frac{1}{2}(p+1)}\int_{O(p)}\operatorname{etr}\left\{-\tfrac{1}{2c}\boldsymbol{H}'\boldsymbol{\Sigma}^{-1}\boldsymbol{H}\boldsymbol{S}^{\frac{1}{2}}\left[\boldsymbol{I}_p+(c-1)\sum_{i=1}^{2}\boldsymbol{W}_i\right]\boldsymbol{S}^{\frac{1}{2}}\right\}\mathrm{d}\boldsymbol{H}\,\mathrm{d}\boldsymbol{S}.
\end{aligned}
\tag{5.6}
$$

Using [2.3.6] and integrating (5.6) with respect to $\boldsymbol{S}$ by using [2.2.3] gives the symmetrised density as

$$\begin{aligned}
& f_s(\boldsymbol{W}_1, \boldsymbol{W}_2) \\
= \; & Kc^{-\frac{1}{2}mp} \prod_{i=1}^{2} |\boldsymbol{W}_i|^{\frac{1}{2}n_i - \frac{1}{2}(p+1)} \left| \boldsymbol{I}_p - \sum_{i=1}^{2} \boldsymbol{W}_i \right|^{\frac{1}{2}m - \frac{1}{2}(p+1)} \\
& \cdot \int_{O(p)} \int_{\boldsymbol{S}>\boldsymbol{0}} |\boldsymbol{S}|^{\frac{1}{2}(n_1+n_2+m) - \frac{1}{2}(p+1)} \operatorname{etr}\left\{ -\frac{1}{2c} \left[\boldsymbol{I}_p + (c-1)\sum_{i=1}^{2} \boldsymbol{W}_i \right]^{\frac{1}{2}} \boldsymbol{H}'\boldsymbol{\Sigma}^{-1}\boldsymbol{H} \left[\boldsymbol{I}_p + (c-1)\sum_{i=1}^{2} \boldsymbol{W}_i \right]^{\frac{1}{2}} \boldsymbol{S} \right\} \mathrm{d}\boldsymbol{S}\mathrm{d}\boldsymbol{H} \\
= \; & K\Gamma_p\left(\tfrac{n_1+n_2+m}{2}\right) c^{-\frac{1}{2}mp} \prod_{i=1}^{2} |\boldsymbol{W}_i|^{\frac{1}{2}n_i - \frac{1}{2}(p+1)} \left| \boldsymbol{I}_p - \sum_{i=1}^{2} \boldsymbol{W}_i \right|^{\frac{1}{2}m - \frac{1}{2}(p+1)} \\
& \cdot \int_{O(p)} \left| \frac{1}{2c} \left[\boldsymbol{I}_p + (c-1)\sum_{i=1}^{2} \boldsymbol{W}_i \right]^{\frac{1}{2}} \boldsymbol{H}'\boldsymbol{\Sigma}^{-1}\boldsymbol{H} \left[\boldsymbol{I}_p + (c-1)\sum_{i=1}^{2} \boldsymbol{W}_i \right]^{\frac{1}{2}} \right|^{-\frac{1}{2}(n_1+n_2+m)} \mathrm{d}\boldsymbol{H} \\
= \; & \left\{ \beta_p\left(\tfrac{n_1}{2}, \tfrac{n_2}{2}; \tfrac{m}{2}\right) \right\}^{-1} c^{\frac{1}{2}(n_1+n_2)p} \\
& \cdot \int_{O(p)} \prod_{i=1}^{2} |\boldsymbol{H}\boldsymbol{W}_i\boldsymbol{H}'|^{\frac{1}{2}n_i - \frac{1}{2}(p+1)} \left| \boldsymbol{I}_p - \sum_{i=1}^{2} \boldsymbol{H}\boldsymbol{W}_i\boldsymbol{H}' \right|^{\frac{1}{2}m - \frac{1}{2}(p+1)} \left| \boldsymbol{I}_p + (c-1)\sum_{i=1}^{2} \boldsymbol{H}\boldsymbol{W}_i\boldsymbol{H}' \right|^{-\frac{1}{2}(n_1+n_2+m)} \mathrm{d}\boldsymbol{H}.
\end{aligned} \tag{5.7}$$

The result follows from (5.7) and [2.9.3]. ■

Remark 5.1

Independent of this study, Gupta and Nagar (2009a) derived the pdf of $(\boldsymbol{W}_1, \boldsymbol{W}_2) \sim BB_p^{III}(n_1, n_2, m, 2)$ in (5.2) from the ratios in (5.1) when $\boldsymbol{S}_1 \sim W_p(n_1, \boldsymbol{I}_p)$, $\boldsymbol{S}_2 \sim W_p(n_2, \boldsymbol{I}_p)$ and $\boldsymbol{B} \sim W_p(m, \boldsymbol{I}_p)$ are independent.

Remark 5.2

The following definitions of Wishart ratios will also give matrix variates having the bimatrix variate beta type III distribution with pdf given by (5.2):

$$\boldsymbol{W}_i = \boldsymbol{S}_i^{\frac{1}{2}} (\boldsymbol{S}_1 + \boldsymbol{S}_2 + c\boldsymbol{B})^{-1} \boldsymbol{S}_i^{\frac{1}{2}}, \; i = 1, 2,$$

and

$$\boldsymbol{W}_i = \left(c\boldsymbol{I}_p + \sum_{i=1}^{2} \boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{S}_i\boldsymbol{B}^{-\frac{1}{2}} \right)^{-\frac{1}{2}} \boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{S}_i\boldsymbol{B}^{-\frac{1}{2}} \left(c\boldsymbol{I}_p + \sum_{i=1}^{2} \boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{S}_i\boldsymbol{B}^{-\frac{1}{2}} \right)^{-\frac{1}{2}}, \quad i = 1, 2. \tag{5.8}$$

Remark 5.3

The ratio (1.3) (or (5.1)) can also be written as

$$\boldsymbol{Q}_i = (\boldsymbol{S}_1 + \boldsymbol{S}_2 + \boldsymbol{B}^*)^{-\frac{1}{2}} \boldsymbol{S}_i (\boldsymbol{S}_1 + \boldsymbol{S}_2 + \boldsymbol{B}^*)^{-\frac{1}{2}}, \quad i = 1, 2,$$

where $\boldsymbol{B}^* = c\boldsymbol{B} \sim W_p(m, c\boldsymbol{\Sigma})$, all independent (see Theorem 3.4 and Remark 3.2).

Remark 5.4
Consider the ratio in (3.11) with $\alpha = 1$,

$$\boldsymbol{W} = (\boldsymbol{S} + c\boldsymbol{B})^{-\frac{1}{2}} \boldsymbol{S} (\boldsymbol{S} + c\boldsymbol{B})^{-\frac{1}{2}}, \tag{5.9}$$

From (3.5) the pdf of $\boldsymbol{W}$ is

$$\begin{aligned} f(\boldsymbol{W}) &= \left\{\beta_p\left(\tfrac{n}{2}, \tfrac{m}{2}\right)\right\}^{-1} c^{\frac{1}{2}np} |\boldsymbol{W}|^{\frac{1}{2}n-\frac{1}{2}(p+1)} |\boldsymbol{I}_p - \boldsymbol{W}|^{\frac{1}{2}m-\frac{1}{2}(p+1)} |\boldsymbol{I}_p + (c-1)\boldsymbol{W}|^{-\frac{1}{2}(n+m)} \\ &= g(\boldsymbol{W}) c^{\frac{1}{2}np} |\boldsymbol{I}_p + (c-1)\boldsymbol{W}|^{-\frac{1}{2}(n+m)}, \ \ \boldsymbol{0} < \boldsymbol{W} < \boldsymbol{I}_p, \end{aligned} \tag{5.10}$$

where $n > (p-1)$, $m > (p-1)$ and $g(\cdot)$ is the pdf of $B_p^I(n, m)$ given by (3.1). This is the matrix variate beta type III distribution and is denoted by $\boldsymbol{W} \sim B_p^{III}(n, m, c)$. This distribution was defined by Gupta and Nagar (2000a) for $c = 2$. They derived the pdf of the ratio (5.9) where $\boldsymbol{S} \sim W_p(n, \boldsymbol{I}_p)$ independent of $\boldsymbol{B} \sim W_p(m, \boldsymbol{I}_p)$ and obtained (5.10) for $c = 2$ (Gupta and Nagar, 2009a).

Remark 5.5
The matrix variate Dirichlet type III distribution, denoted by $(\boldsymbol{W}_1, \ldots, \boldsymbol{W}_r) \sim D_p^{III}(n_1, \ldots, n_r, m, c)$, results by extending (5.1) to r independent Wishart matrix variates, $\boldsymbol{S}_i \sim W_p(n_i, \boldsymbol{\Sigma})$, $i = 1, \ldots, r$, all independent of $\boldsymbol{B} \sim W_p(m, \boldsymbol{\Sigma})$. The pdf of $(\boldsymbol{W}_1, \ldots, \boldsymbol{W}_r)$ is given by

$$\begin{aligned} &\left\{\beta_p\left(\tfrac{n_1}{2}, \ldots, \tfrac{n_r}{2}; \tfrac{m}{2}\right)\right\}^{-1} c^{\frac{1}{2}(n_1+\cdots+n_r)p} \\ &\cdot \prod_{i=1}^r |\boldsymbol{W}_i|^{\frac{1}{2}n_i-\frac{1}{2}(p+1)} \left|\boldsymbol{I}_p - \sum_{i=1}^r \boldsymbol{W}_i\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)} \left|\boldsymbol{I}_p + (c-1)\sum_{i=1}^r \boldsymbol{W}_i\right|^{-\frac{1}{2}(n_1+\cdots+n_r+m)}, \end{aligned} \tag{5.11}$$

$\boldsymbol{0} < \boldsymbol{W}_i < \boldsymbol{I}_p$, $i = 1, \ldots, r$, $\boldsymbol{0} < \sum_{i=1}^r \boldsymbol{W}_i < \boldsymbol{I}_p$.

Remark 5.6
The pdfs of the bimatrix variate beta type III distribution in (5.2) and the matrix variate Dirichlet type III distribution in (5.11) are members of the Liouville family of distributions of the second kind (see [2.2.1]).

5.2 Marginal property and conditional density

In this section the marginal and conditional pdfs of the bimatrix variate beta type III distribution are derived.

Theorem 5.2
If $(\boldsymbol{W}_1, \boldsymbol{W}_2) \sim BB_p^{III}(n_1, n_2, m, c)$*, then the pdf of* $\boldsymbol{W}_1$ *is given by*

$$\begin{aligned} f(\boldsymbol{W}_1) &= \left\{\beta_p\left(\tfrac{n_1}{2}, \tfrac{n_2+m}{2}\right)\right\}^{-1} c^{-\frac{1}{2}mp} |\boldsymbol{W}_1|^{\frac{1}{2}n_1-\frac{1}{2}(p+1)} |\boldsymbol{I}_p - \boldsymbol{W}_1|^{\frac{1}{2}(n_2+m)-\frac{1}{2}(p+1)} \\ &\quad \cdot {}_2F_1\left(\tfrac{m}{2}, \tfrac{n_1+n_2+m}{2}; \tfrac{n_2+m}{2}; \tfrac{c-1}{c}(\boldsymbol{I}_p - \boldsymbol{W}_1)\right) \\ &= g(\boldsymbol{W}_1) c^{-\frac{1}{2}mp} {}_2F_1\left(\tfrac{m}{2}, \tfrac{n_1+n_2+m}{2}; \tfrac{n_2+m}{2}; \tfrac{c-1}{c}(\boldsymbol{I}_p - \boldsymbol{W}_1)\right), \end{aligned} \tag{5.12}$$

$\boldsymbol{0} < \boldsymbol{W}_1 < \boldsymbol{I}_p$*, where* $g(\cdot)$ *is the pdf of* $B_p^I\left(\frac{n_1}{2}, \frac{n_2+m}{2}\right)$ *given by* (3.1).

Furthermore the pdf of $\boldsymbol{W}_2|\boldsymbol{W}_1$ *is*

$$\left\{\beta_p\left(\tfrac{n_2}{2},\tfrac{m}{2}\right)\right\}^{-1} c^{\frac{1}{2}(n_1+n_2+m)p}\left|\boldsymbol{W}_2\right|^{\frac{1}{2}n_2-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\sum_{i=1}^{2}\boldsymbol{W}_i\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\boldsymbol{W}_1\right|^{-\frac{1}{2}(n_2+m)+\frac{1}{2}(p+1)}$$
$$\cdot\left|\boldsymbol{I}_p+(c-1)\sum_{i=1}^{2}\boldsymbol{W}_i\right|^{-\frac{1}{2}(n_1+n_2+m)}\left[{}_2F_1\left(\tfrac{m}{2},\tfrac{n_1+n_2+m}{2};\tfrac{n_2+m}{2};\tfrac{c-1}{c}\left(\boldsymbol{I}_p-\boldsymbol{W}_1\right)\right)\right]^{-1}, \tag{5.13}$$

$\boldsymbol{0}<\boldsymbol{W}_2<\boldsymbol{I}_p-\boldsymbol{W}_1$.

Proof:
The pdf of $(\boldsymbol{W}_1,\boldsymbol{W}_2)$, given by (5.2) can be rewritten as

$$\begin{aligned}
& f\left(\boldsymbol{W}_1,\boldsymbol{W}_2\right)\\
= \quad & K\left|\boldsymbol{W}_1\right|^{\frac{1}{2}n_1-\frac{1}{2}(p+1)}\left|\boldsymbol{W}_2\right|^{\frac{1}{2}n_2-\frac{1}{2}(p+1)}\\
& \cdot\left|\boldsymbol{I}_p-\boldsymbol{W}_1\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\left(\boldsymbol{I}_p-\boldsymbol{W}_1\right)^{-\frac{1}{2}}\boldsymbol{W}_2\left(\boldsymbol{I}_p-\boldsymbol{W}_1\right)^{-\frac{1}{2}}\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)}\\
& \cdot\left|\boldsymbol{I}_p+(c-1)\boldsymbol{W}_1+(c-1)\boldsymbol{W}_2\right|^{-\frac{1}{2}(n_1+n_2+m)},
\end{aligned}\tag{5.14}$$

where $K=\left\{\beta_p\left(\frac{n_1}{2},\frac{n_2}{2};\frac{m}{2}\right)\right\}^{-1}c^{\frac{1}{2}(n_1+n_2)p}$.

Consider the transformation $\boldsymbol{Z}_2=\left(\boldsymbol{I}_p-\boldsymbol{W}_1\right)^{-\frac{1}{2}}\boldsymbol{W}_2\left(\boldsymbol{I}_p-\boldsymbol{W}_1\right)^{-\frac{1}{2}}$ with Jacobian $J\left(\boldsymbol{W}_2\to\boldsymbol{Z}_2\right)=\left|\boldsymbol{I}_p-\boldsymbol{W}_1\right|^{\frac{1}{2}(p+1)}$ (see [2.1.6]) substituted in (5.14), then

$$\begin{aligned}
& f\left(\boldsymbol{W}_1,\boldsymbol{Z}_2\right)\\
= \quad & K\left|\boldsymbol{W}_1\right|^{\frac{1}{2}n_1-\frac{1}{2}(p+1)}\left|\left(\boldsymbol{I}_p-\boldsymbol{W}_1\right)^{\frac{1}{2}}\boldsymbol{Z}_2\left(\boldsymbol{I}_p-\boldsymbol{W}_1\right)^{\frac{1}{2}}\right|^{\frac{1}{2}n_2-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\boldsymbol{W}_1\right|^{\frac{1}{2}m}\left|\boldsymbol{I}_p-\boldsymbol{Z}_2\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)}\\
& \cdot\left|\boldsymbol{I}_p-\boldsymbol{W}_1+c\boldsymbol{W}_1+(c-1)\left(\boldsymbol{I}_p-\boldsymbol{W}_1\right)^{\frac{1}{2}}\boldsymbol{Z}_2\left(\boldsymbol{I}_p-\boldsymbol{W}_1\right)^{\frac{1}{2}}\right|^{-\frac{1}{2}(n_1+n_2+m)}\\
= \quad & K\left|\boldsymbol{W}_1\right|^{\frac{1}{2}n_1-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\boldsymbol{W}_1\right|^{\frac{1}{2}(n_2+m)-\frac{1}{2}(p+1)}\left|\boldsymbol{Z}_2\right|^{\frac{1}{2}n_2-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\boldsymbol{Z}_2\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)}\\
& \cdot\left|\boldsymbol{I}_p-\boldsymbol{W}_1\right|^{-\frac{1}{2}(n_1+n_2+m)}\left|\boldsymbol{I}_p+c\left(\boldsymbol{I}_p-\boldsymbol{W}_1\right)^{-\frac{1}{2}}\boldsymbol{W}_1\left(\boldsymbol{I}_p-\boldsymbol{W}_1\right)^{-\frac{1}{2}}+(c-1)\boldsymbol{Z}_2\right|^{-\frac{1}{2}(n_1+n_2+m)}\\
= \quad & K\left|\boldsymbol{W}_1\right|^{\frac{1}{2}n_1-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\boldsymbol{W}_1\right|^{\frac{1}{2}(n_2+m)-\frac{1}{2}(p+1)}\left|\boldsymbol{Z}_2\right|^{\frac{1}{2}n_2-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\boldsymbol{Z}_2\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)}\\
& \cdot\left|\boldsymbol{I}_p-\boldsymbol{W}_1\right|^{-\frac{1}{2}(n_1+n_2+m)}\left|\boldsymbol{I}_p+c\left(\boldsymbol{I}_p-\boldsymbol{W}_1\right)^{-1}\boldsymbol{W}_1+(c-1)\boldsymbol{Z}_2\right|^{-\frac{1}{2}(n_1+n_2+m)}\\
= \quad & K\left|\boldsymbol{W}_1\right|^{\frac{1}{2}n_1-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\boldsymbol{W}_1\right|^{\frac{1}{2}(n_2+m)-\frac{1}{2}(p+1)}\left|\boldsymbol{Z}_2\right|^{\frac{1}{2}n_2-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\boldsymbol{Z}_2\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)}\\
& \cdot\left|\boldsymbol{I}_p+(c-1)\boldsymbol{W}_1+(c-1)\left(\boldsymbol{I}_p-\boldsymbol{W}_1\right)\boldsymbol{Z}_2\right|^{-\frac{1}{2}(n_1+n_2+m)}\\
= \quad & K\left|\boldsymbol{W}_1\right|^{\frac{1}{2}n_1-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\boldsymbol{W}_1\right|^{\frac{1}{2}(n_2+m)-\frac{1}{2}(p+1)}\left|\boldsymbol{Z}_2\right|^{\frac{1}{2}n_2-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\boldsymbol{Z}_2\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)}\\
& \cdot\left|\boldsymbol{I}_p+(c-1)\boldsymbol{W}_1\right|^{-\frac{1}{2}(n_1+n_2+m)}\left|\boldsymbol{I}_p+(c-1)\left[\boldsymbol{I}_p+(c-1)\boldsymbol{W}_1\right]^{-1}\left(\boldsymbol{I}_p-\boldsymbol{W}_1\right)\boldsymbol{Z}_2\right|^{-\frac{1}{2}(n_1+n_2+m)}.
\end{aligned}\tag{5.15}$$

The marginal pdf of $\boldsymbol{W}_1$ is obtained by integrating $f(\boldsymbol{W}_1, \boldsymbol{Z}_2)$ with respect to $\boldsymbol{Z}_2$. For this we consider the following integral and use [2.6.5],

$$\begin{aligned}
&\int_{\mathbf{0}<\boldsymbol{Z}_2<\boldsymbol{I}_p} |\boldsymbol{Z}_2|^{\frac{1}{2}n_2-\frac{1}{2}(p+1)} |\boldsymbol{I}_p - \boldsymbol{Z}_2|^{\frac{1}{2}m-\frac{1}{2}(p+1)} \\
&\qquad \cdot \left|\boldsymbol{I}_p + (c-1)\left[\boldsymbol{I}_p + (c-1)\boldsymbol{W}_1\right]^{-1}(\boldsymbol{I}_p - \boldsymbol{W}_1)\boldsymbol{Z}_2\right|^{-\frac{1}{2}(n_1+n_2+m)} \mathrm{d}\boldsymbol{Z}_2 \\
&= \frac{\Gamma_p\left(\frac{n_2}{2}\right)\Gamma_p\left(\frac{m}{2}\right)}{\Gamma_p\left(\frac{n_2+m}{2}\right)} \, {}_2F_1\left(\frac{n_2}{2}, \frac{n_1+n_2+m}{2}; \frac{n_2+m}{2}; -(c-1)\left[\boldsymbol{I}_p + (c-1)\boldsymbol{W}_1\right]^{-1}(\boldsymbol{I}_p - \boldsymbol{W}_1)\right).
\end{aligned} \tag{5.16}$$

From (5.15) and (5.16) we get

$$\begin{aligned}
&f(\boldsymbol{W}_1) \\
&= \frac{\Gamma_p\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)\Gamma_p\left(\frac{m}{2}\right)} c^{\frac{1}{2}(n_1+n_2)p} \frac{\Gamma_p\left(\frac{n_2}{2}\right)\Gamma_p\left(\frac{m}{2}\right)}{\Gamma_p\left(\frac{n_2+m}{2}\right)} \\
&\quad \cdot |\boldsymbol{W}_1|^{\frac{1}{2}n_1-\frac{1}{2}(p+1)} |\boldsymbol{I}_p - \boldsymbol{W}_1|^{\frac{1}{2}(n_2+m)-\frac{1}{2}(p+1)} |\boldsymbol{I}_p + (c-1)\boldsymbol{W}_1|^{-\frac{1}{2}(n_1+n_2+m)} \\
&\quad \cdot {}_2F_1\left(\frac{n_2}{2}, \frac{n_1+n_2+m}{2}; \frac{n_2+m}{2}; -(c-1)\left[\boldsymbol{I}_p + (c-1)\boldsymbol{W}_1\right]^{-1}(\boldsymbol{I}_p - \boldsymbol{W}_1)\right).
\end{aligned} \tag{5.17}$$

Letting $\boldsymbol{S} = -(c-1)\left[\boldsymbol{I}_p + (c-1)\boldsymbol{W}_1\right]^{-1}(\boldsymbol{I}_p - \boldsymbol{W}_1)$ and using [2.6.7],

$$ {}_2F_1\left(\frac{n_2}{2}, \frac{n_1+n_2+m}{2}; \frac{n_2+m}{2}; \boldsymbol{S}\right) = |\boldsymbol{I}_p - \boldsymbol{S}|^{-\frac{1}{2}(n_1+n_2+m)} \; {}_2F_1\left(\frac{m}{2}, \frac{n_1+n_2+m}{2}; \frac{n_2+m}{2}; -\boldsymbol{S}(\boldsymbol{I}_p - \boldsymbol{S})^{-1}\right). \tag{5.18}$$

In (5.18),

$$\begin{aligned}
&-\boldsymbol{S}(\boldsymbol{I}_p - \boldsymbol{S})^{-1} \\
&= (c-1)\left[\boldsymbol{I}_p + (c-1)\boldsymbol{W}_1\right]^{-1}(\boldsymbol{I}_p - \boldsymbol{W}_1)\left\{\boldsymbol{I}_p + (c-1)\left[\boldsymbol{I}_p + (c-1)\boldsymbol{W}_1\right]^{-1}(\boldsymbol{I}_p - \boldsymbol{W}_1)\right\}^{-1} \\
&= (c-1)\left[\boldsymbol{I}_p + (c-1)\boldsymbol{W}_1\right]^{-1}(\boldsymbol{I}_p - \boldsymbol{W}_1)\left\{\left[\boldsymbol{I}_p + (c-1)\boldsymbol{W}_1\right]^{-1}\left[\boldsymbol{I}_p + (c-1)\boldsymbol{W}_1 + (c-1)(\boldsymbol{I}_p - \boldsymbol{W}_1)\right]\right\}^{-1} \\
&= \frac{c-1}{c}\left[\boldsymbol{I}_p + (c-1)\boldsymbol{W}_1\right]^{-1}(\boldsymbol{I}_p - \boldsymbol{W}_1)\left[\boldsymbol{I}_p + (c-1)\boldsymbol{W}_1\right]
\end{aligned}$$

and

$$\begin{aligned}
&|\boldsymbol{I}_p - \boldsymbol{S}| \\
&= \left|\boldsymbol{I}_p + (c-1)\left[\boldsymbol{I}_p + (c-1)\boldsymbol{W}_1\right]^{-1}(\boldsymbol{I}_p - \boldsymbol{W}_1)\right| \\
&= \left|\left[\boldsymbol{I}_p + (c-1)\boldsymbol{W}_1\right]^{-1}\left\{\left[\boldsymbol{I}_p + (c-1)\boldsymbol{W}_1\right] + (c-1)(\boldsymbol{I}_p - \boldsymbol{W}_1)\right\}\right| \\
&= \left|c\left[\boldsymbol{I}_p + (c-1)\boldsymbol{W}_1\right]^{-1}\right|.
\end{aligned}$$

Thus, it follows from (5.17) and (5.18) that

$$
\begin{aligned}
& f\left(\boldsymbol{W}_1\right) \\
= & \frac{\Gamma_p\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2+m}{2}\right)} c^{\frac{1}{2}(n_1+n_2)p}\left|\boldsymbol{W}_1\right|^{\frac{1}{2}n_1-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\boldsymbol{W}_1\right|^{\frac{1}{2}(n_2+m)-\frac{1}{2}(p+1)} \\
& \cdot\left|\boldsymbol{I}_p+(c-1)\boldsymbol{W}_1\right|^{-\frac{1}{2}(n_1+n_2+m)}\left|c\left[\boldsymbol{I}_p+(c-1)\boldsymbol{W}_1\right]^{-1}\right|^{-\frac{1}{2}(n_1+n_2+m)} \\
& \cdot {}_2F_1\left(\tfrac{m}{2}, \tfrac{n_1+n_2+m}{2}; \tfrac{n_2+m}{2}; \tfrac{c-1}{c}\left[\boldsymbol{I}_p+(c-1)\boldsymbol{W}_1\right]^{-1}\left(\boldsymbol{I}_p-\boldsymbol{W}_1\right)\left[\boldsymbol{I}_p+(c-1)\boldsymbol{W}_1\right]\right) \\
= & \left\{\beta_p\left(\tfrac{n_1}{2}, \tfrac{n_2+m}{2}\right)\right\}^{-1} c^{-\frac{1}{2}mp}\left|\boldsymbol{W}_1\right|^{\frac{1}{2}n_1-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\boldsymbol{W}_1\right|^{\frac{1}{2}(n_2+m)-\frac{1}{2}(p+1)} \\
& \cdot {}_2F_1\left(\tfrac{m}{2}, \tfrac{n_1+n_2+m}{2}; \tfrac{n_2+m}{2}; \tfrac{c-1}{c}\left(\boldsymbol{I}_p-\boldsymbol{W}_1\right)\right).
\end{aligned}
$$

The conditional pdf of $\boldsymbol{W}_2|\boldsymbol{W}_1$, given by (5.13), follows from (5.2) and (5.12). ■

Remark 5.7

Gupta and Nagar (2009a) derived the marginal pdf of $\boldsymbol{W}_1$ where $(\boldsymbol{W}_1, \boldsymbol{W}_2) \sim BB_p^{III}(n_1, n_2, m, 2)$ and obtained (5.17) for $c = 2$.

Remark 5.8

If $(\boldsymbol{W}_1, \ldots, \boldsymbol{W}_r) \sim D_p^{III}(n_1, \ldots, n_r, m, c)$ given by (5.11), then for $s \leq r$ the marginal pdf of $(\boldsymbol{W}_1, \ldots, \boldsymbol{W}_s)$ is

$$
\begin{aligned}
& f\left(\boldsymbol{W}_1, \ldots, \boldsymbol{W}_s\right) \\
= & \left\{\beta_p\left(\frac{n_1}{2}, \ldots, \frac{n_s}{2}; \frac{n_{s+1}+\cdots+n_r+m}{2}\right)\right\}^{-1} c^{-\frac{1}{2}mp}\prod_{i=1}^{s}\left|\boldsymbol{W}_i\right|^{\frac{1}{2}n_i-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\sum_{i=1}^{s}\boldsymbol{W}_i\right|^{\frac{1}{2}(n_{s+1}+\cdots+n_r+m)-\frac{1}{2}(p+1)} \\
& \cdot {}_2F_1\left(\frac{m}{2}, \frac{n_1+\cdots+n_r+m}{2}; \frac{n_{s+1}+\cdots+n_r+m}{2}; \frac{c-1}{c}\left(\boldsymbol{I}_p-\sum_{i=1}^{s}\boldsymbol{W}_i\right)\right),
\end{aligned} \tag{5.19}
$$

$\boldsymbol{0} < \boldsymbol{W}_i < \boldsymbol{I}_p$, $i = 1, \ldots, s$, $\boldsymbol{0} < \sum_{i=1}^{s}\boldsymbol{W}_i < \boldsymbol{I}_p$.

The pdf of $(\boldsymbol{W}_{s+1}, \ldots, \boldsymbol{W}_r) \,|\, (\boldsymbol{W}_1, \ldots, \boldsymbol{W}_s)$ is given by

$$
\begin{aligned}
& \left\{\beta_p\left(\tfrac{n_{s+1}}{2}, \ldots, \tfrac{n_r}{2}; \tfrac{m}{2}\right)\right\}^{-1} c^{\frac{1}{2}(n_1+\cdots+n_r+m)p}\prod_{i=s+1}^{r}\left|\boldsymbol{W}_i\right|^{\frac{1}{2}n_i-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\sum_{i=1}^{r}\boldsymbol{W}_i\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)} \\
& \cdot\left|\boldsymbol{I}_p+(c-1)\sum_{i=1}^{r}\boldsymbol{W}_i\right|^{\frac{1}{2}(n_1+\cdots+n_r+m)}\left|\boldsymbol{I}_p-\sum_{i=1}^{s}\boldsymbol{W}_i\right|^{-\frac{1}{2}(n_{s+1}+\cdots+n_r+m)+\frac{1}{2}(p+1)} \\
& \cdot\left[{}_2F_1\left(\frac{m}{2}, \frac{n_1+\cdots+n_r+m}{2}; \frac{n_{s+1}+\cdots+n_r+m}{2}; \frac{c-1}{c}\left(\boldsymbol{I}_p-\sum_{i=1}^{s}\boldsymbol{W}_i\right)\right)\right]^{-1},
\end{aligned} \tag{5.20}
$$

$\boldsymbol{0} < \boldsymbol{W}_i < \boldsymbol{I}_p - \sum_{i=1}^{s}\boldsymbol{W}_i$, $i = s+1, \ldots, r$, $\boldsymbol{0} < \sum_{i=s+1}^{r}\boldsymbol{W}_i < \boldsymbol{I}_p - \sum_{i=1}^{s}\boldsymbol{W}_i$.

Remark 5.9

In the case of the bimatrix variate beta type I distribution the marginal pdf is matrix variate beta type I (see (4.5)) and for the matrix variate Dirichlet type I distribution the marginal pdf of any subset of matrix variates is again matrix variate Dirichlet type I (see Remark 4.4). From (5.12) and (5.19) this property does not hold for the bimatrix variate beta type III and matrix variate Dirichlet type III distributions.

Remark 5.10

The marginal and conditional pdfs of the matrix variate Dirichlet III distribution given by (5.19) and (5.20) are members of the Liouville family of distributions of the second and first kind respectively (see [2.2.1]).

5.3 Product moment of the determinants

The $(h_1, h_2)^{th}$ product moment, $E\left(|\boldsymbol{W}_1|^{h_1}|\boldsymbol{W}_2|^{h_2}\right)$, where $(\boldsymbol{W}_1, \boldsymbol{W}_2) \sim BB_p^{III}(n_1, n_2, m, c)$ is derived in Theorem 5.3.

Theorem 5.3

If $(\boldsymbol{W}_1, \boldsymbol{W}_2) \sim BB_p^{III}(n_1, n_2, m, c)$ *as given by* (5.2) *then*

$$\begin{aligned} E\left(|\boldsymbol{W}_1|^{h_1}|\boldsymbol{W}_2|^{h_2}\right) &= \frac{\Gamma_p\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)}\frac{\Gamma_p\left(\frac{n_1}{2}+h_1\right)\Gamma_p\left(\frac{n_2}{2}+h_2\right)}{\Gamma_p\left(\frac{n_1+n_2+m}{2}+h_1+h_2\right)} \\ &\quad \cdot c^{-\frac{1}{2}mp}\ {}_2F_1\left(\tfrac{m}{2}, \tfrac{n_1+n_2+m}{2}; \tfrac{n_1+n_2+m}{2}+h_1+h_2; \tfrac{c-1}{c}\boldsymbol{I}_p\right) \\ &= E\left(|\boldsymbol{W}_1^*|^{h_1}|\boldsymbol{W}_2^*|^{h_2}\right) \\ &\quad \cdot c^{-\frac{1}{2}mp}\ {}_2F_1\left(\tfrac{m}{2}, \tfrac{n_1+n_2+m}{2}; \tfrac{n_1+n_2+m}{2}+h_1+h_2; \tfrac{c-1}{c}\boldsymbol{I}_p\right), \end{aligned} \tag{5.21}$$

where $\mathrm{Re}\left(\frac{1}{2}n_i + h_i\right) > \frac{1}{2}(p-1)$, $i = 1, 2$, $(\boldsymbol{W}_1^*, \boldsymbol{W}_2^*) \sim BB_p^I(n_1, n_2, m)$ *and* $E\left(|\boldsymbol{W}_1^*|^{h_1}|\boldsymbol{W}_2^*|^{h_2}\right)$ *is given by* (4.8).

Proof:

From (5.2),

$$\begin{aligned} &E\left(|\boldsymbol{W}_1|^{h_1}|\boldsymbol{W}_2|^{h_2}\right) \\ &= \left\{\beta_p\left(\tfrac{n_1}{2}, \tfrac{n_2}{2}; \tfrac{m}{2}\right)\right\}^{-1} c^{\frac{1}{2}(n_1+n_2)p} \\ &\quad \int\limits_{\substack{\boldsymbol{0}<\boldsymbol{W}_1+\boldsymbol{W}_2<\boldsymbol{I}_p \\ \boldsymbol{W}_i>\boldsymbol{0}}}\int \prod_{i=1}^{2}|\boldsymbol{W}_i|^{\frac{1}{2}n_i+h_i-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p - \sum_{i=1}^{2}\boldsymbol{W}_i\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p + (c-1)\sum_{i=1}^{2}\boldsymbol{W}_i\right|^{-\frac{1}{2}(n_1+n_2+m)} d\boldsymbol{W}_1 d\boldsymbol{W}_2. \end{aligned} \tag{5.22}$$

Let $f\left(\sum_{i=1}^{2}\boldsymbol{W}_i\right)=\left|\boldsymbol{I}_p-\sum_{i=1}^{2}\boldsymbol{W}_i\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p+(c-1)\sum_{i=1}^{2}\boldsymbol{W}_i\right|^{-\frac{1}{2}(n_1+n_2+m)}$ and use [2.2.6] to rewrite (5.22) as

$$\begin{aligned}
&E\left(|\boldsymbol{W}_1|^{h_1}|\boldsymbol{W}_2|^{h_2}\right)\\
&= \left\{\beta_p\left(\tfrac{n_1}{2},\tfrac{n_2}{2};\tfrac{m}{2}\right)\right\}^{-1}c^{\frac{1}{2}(n_1+n_2)p}\beta_p\left(\tfrac{n_1}{2}+h_1,\tfrac{n_2}{2}+h_2\right)\\
&\quad\cdot\int_{\boldsymbol{0}<\boldsymbol{Z}<\boldsymbol{I}_p}|\boldsymbol{Z}|^{\frac{1}{2}(n_1+n_2)+h_1+h_2-\frac{1}{2}(p+1)}|\boldsymbol{I}_p-\boldsymbol{Z}|^{\frac{1}{2}m-\frac{1}{2}(p+1)}|\boldsymbol{I}_p+(c-1)\boldsymbol{Z}|^{-\frac{1}{2}(n_1+n_2+m)}d\boldsymbol{Z}.
\end{aligned}\tag{5.23}$$

Now, using [2.6.5] to solve the integral in (5.23) and [2.6.7] to rewrite the expression gives

$$\begin{aligned}
&E\left(|\boldsymbol{W}_1|^{h_1}|\boldsymbol{W}_2|^{h_2}\right)\\
&= \frac{\Gamma_p\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)\Gamma_p\left(\frac{m}{2}\right)}\frac{\Gamma_p\left(\frac{n_1}{2}+h_1\right)\Gamma_p\left(\frac{n_2}{2}+h_2\right)}{\Gamma_p\left(\frac{n_1+n_2}{2}+h_1+h_2\right)}c^{\frac{1}{2}(n_1+n_2)p}\frac{\Gamma_p\left(\frac{n_1+n_2}{2}+h_1+h_2\right)\Gamma_p\left(\frac{m}{2}\right)}{\Gamma_p\left(\frac{n_1+n_2+m}{2}+h_1+h_2\right)}\\
&\quad\cdot{}_2F_1\left(\tfrac{n_1+n_2}{2}+h_1+h_2,\tfrac{n_1+n_2+m}{2};\tfrac{n_1+n_2+m}{2}+h_1+h_2;(1-c)\boldsymbol{I}_p\right)\\
&= \frac{\Gamma_p\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)}\frac{\Gamma_p\left(\frac{n_1}{2}+h_1\right)\Gamma_p\left(\frac{n_2}{2}+h_2\right)}{\Gamma_p\left(\frac{n_1+n_2+m}{2}+h_1+h_2\right)}c^{-\frac{1}{2}mp}\\
&\quad\cdot{}_2F_1\left(\tfrac{m}{2},\tfrac{n_1+n_2+m}{2};\tfrac{n_1+n_2+m}{2}+h_1+h_2;\tfrac{c-1}{c}\boldsymbol{I}_p\right). \qquad\blacksquare
\end{aligned}$$

Remark 5.11
Gupta and Nagar (2009a) derived the $(h_1,h_2)^{th}$ product moment, $E\left(|\boldsymbol{W}_1|^{h_1}|\boldsymbol{W}_2|^{h_2}\right)$, where $(\boldsymbol{W}_1,\boldsymbol{W}_2)\sim BB_p^{III}(n_1,n_2,m,2)$ and obtained the result (5.21) for $c=2$.

Remark 5.12
The h^{th} moment associated with (5.10) where $\boldsymbol{W}\sim B_p^{III}(n,m,c)$ is obtained from (3.18) with $\alpha=1$ as

$$\begin{aligned}
E\left(|\boldsymbol{W}|^h\right) &= \frac{\Gamma_p\left(\frac{n+m}{2}\right)}{\Gamma_p\left(\frac{n}{2}\right)}\frac{\Gamma_p\left(\frac{n}{2}+h\right)}{\Gamma_p\left(\frac{n+m}{2}+h\right)}\\
&\quad\cdot c^{\frac{1}{2}np}\,{}_2F_1\left(\tfrac{n}{2}+h,\tfrac{n+m}{2};\tfrac{n+m}{2}+h;(1-c)\boldsymbol{I}_p\right)\\
&= E\left(|\boldsymbol{W}^*|^h\right)c^{\frac{1}{2}np}\,{}_2F_1\left(\tfrac{n}{2}+h,\tfrac{n+m}{2};\tfrac{n+m}{2}+h;(1-c)\boldsymbol{I}_p\right),
\end{aligned}$$

where $\operatorname{Re}\left(\frac{n}{2}+h\right)>\frac{1}{2}(p-1)$, $\boldsymbol{W}^*\sim B_p^I(n,m)$ and $E\left(|\boldsymbol{W}^*|^h\right)$ is given by (3.17) (see Gupta and Nagar, 2000a and 2009a for the case where $c=2$).

5.4 Distribution of the product of determinants

An exact expression for the pdf of Λ_3 in (1.8) is derived in Theorem 5.4.

Theorem 5.4

Let $\boldsymbol{S}_1 \sim W_p(n_1, \boldsymbol{\Sigma})$, $\boldsymbol{S}_2 \sim W_p(n_2, \boldsymbol{\Sigma})$ *and* $\boldsymbol{B} \sim W_p(m, \boldsymbol{\Sigma})$. *The ratios in* (5.1),

$$\boldsymbol{W}_i = (\boldsymbol{S}_1 + \boldsymbol{S}_2 + c\boldsymbol{B})^{-\frac{1}{2}} \boldsymbol{S}_i (\boldsymbol{S}_1 + \boldsymbol{S}_2 + c\boldsymbol{B})^{-\frac{1}{2}}, \quad i = 1, 2,$$

give $(\boldsymbol{W}_1, \boldsymbol{W}_2) \sim BB_p^{III}(n_1, n_2, m, c)$. *Let* $\Lambda_3 = |\boldsymbol{W}_1|^{\frac{1}{2}n_1} |\boldsymbol{W}_2|^{\frac{1}{2}n_2}$.

The pdf of Λ_3 *is given by*

$$\frac{\pi^{\frac{1}{4}p(p-1)}}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)\Gamma_p\left(\frac{m}{2}\right)} c^{-\frac{1}{2}mp} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{1}{k!} \Gamma_p\left(\frac{m}{2}, \kappa\right) \Gamma_p\left(\frac{n_1+n_2+m}{2}, \kappa\right) C_\kappa\left(\frac{c-1}{c} \boldsymbol{I}_p\right) H_{p,2p}^{2p,0}\left(\lambda_3 \Big|_{(b_1,\beta_1),\ldots,(b_{2p},\beta_{2p})}^{(a_1,\alpha_1),\ldots,(a_p,\alpha_p)}\right), \quad (5.24)$$

$0 < \lambda_3 < 1$, *where*

$a_j = \frac{m}{2} + k_j - \frac{1}{2}(j-1)$ *for* $j = 1, 2, 3, \ldots, p$,

$\alpha_j = \frac{n_1+n_2}{2}$ *for* $j = 1, 2, 3, \ldots, p$,

$$b_j = \begin{cases} -\frac{1}{4}(j-1) & \text{for } j = 1, 3, 5, \ldots, 2p-1 \\ -\frac{1}{4}(j-2) & \text{for } j = 2, 4, 6, \ldots, 2p, \end{cases}$$

$$\beta_j = \begin{cases} \frac{n_1}{2} & \text{for } j = 1, 3, 5, \ldots, 2p-1 \\ \frac{n_2}{2} & \text{for } j = 2, 4, 6, \ldots, 2p. \end{cases}$$

Proof:

From (5.21) the Mellin transform (see [2.8.1]) of $f(\lambda_3)$ is

$$\begin{aligned} & M_f(h) \\ \equiv\ & E\left(\Lambda_3^{h-1}\right) \\ =\ & E\left[\left(|\boldsymbol{W}_1|^{\frac{1}{2}n_1} |\boldsymbol{W}_2|^{\frac{1}{2}n_2}\right)^{h-1}\right] \\ =\ & \frac{\Gamma_p\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)} \frac{\Gamma_p\left[\frac{n_1}{2} + \frac{n_1}{2}(h-1)\right]\Gamma_p\left[\frac{n_2}{2} + \frac{n_2}{2}(h-1)\right]}{\Gamma_p\left[\frac{n_1+n_2+m}{2} + \frac{n_1+n_2}{2}(h-1)\right]} c^{-\frac{1}{2}mp} \\ & \cdot {}_2F_1\left(\frac{m}{2}, \frac{n_1+n_2+m}{2}; \frac{n_1+n_2+m}{2} + \frac{n_1+n_2}{2}(h-1); \frac{c-1}{c}\boldsymbol{I}_p\right) \\ =\ & \frac{\Gamma_p\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)} \frac{\Gamma_p\left(\frac{n_1}{2}h\right)\Gamma_p\left(\frac{n_2}{2}h\right)}{\Gamma_p\left(\frac{m}{2} + \frac{n_1+n_2}{2}h\right)} c^{-\frac{1}{2}mp}\ {}_2F_1\left(\frac{m}{2}, \frac{n_1+n_2+m}{2}; \frac{m}{2} + \frac{n_1+n_2}{2}h; \frac{c-1}{c}\boldsymbol{I}_p\right). \end{aligned} \quad (5.25)$$

From [2.6.1] and [2.3.3] the Gauss hypergeometric function of matrix argument in (5.25) can be written as

$$
\begin{aligned}
&{}_2F_1\left(\tfrac{m}{2}, \tfrac{n_1+n_2+m}{2}; \tfrac{m}{2} + \tfrac{n_1+n_2}{2}h; \tfrac{c-1}{c}\boldsymbol{I}_p\right) \\
&= \sum_{k=0}^{\infty}\sum_{\kappa}\frac{1}{k!}C_\kappa\left(\tfrac{c-1}{c}\boldsymbol{I}_p\right)\frac{\left(\frac{m}{2}\right)_\kappa\left(\frac{n_1+n_2+m}{2}\right)_\kappa}{\left(\frac{m}{2}+\frac{n_1+n_2}{2}h\right)_\kappa} \\
&= \sum_{k=0}^{\infty}\sum_{\kappa}\frac{1}{k!}C_\kappa\left(\tfrac{c-1}{c}\boldsymbol{I}_p\right)\frac{\Gamma_p\left(\frac{m}{2},\kappa\right)}{\Gamma_p\left(\frac{m}{2}\right)}\frac{\Gamma_p\left(\frac{n_1+n_2+m}{2},\kappa\right)}{\Gamma_p\left(\frac{n_1+n_2+m}{2}\right)}\frac{\Gamma_p\left(\frac{m}{2}+\frac{n_1+n_2}{2}h\right)}{\Gamma_p\left(\frac{m}{2}+\frac{n_1+n_2}{2}h,\kappa\right)}.
\end{aligned}
$$

This gives

$$
M_f(h) \equiv \frac{\Gamma_p\left(\frac{n_1}{2}h\right)\Gamma_p\left(\frac{n_2}{2}h\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)\Gamma_p\left(\frac{m}{2}\right)}c^{-\frac{1}{2}mp}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{1}{k!}\frac{\Gamma_p\left(\frac{m}{2},\kappa\right)\Gamma_p\left(\frac{n_1+n_2+m}{2},\kappa\right)}{\Gamma_p\left(\frac{m}{2}+\frac{n_1+n_2}{2}h,\kappa\right)}C_\kappa\left(\tfrac{c-1}{c}\boldsymbol{I}_p\right). \tag{5.26}
$$

Using [2.3.3] the generalised gamma function of weight κ in (5.26) can be written as

$$
\begin{aligned}
&\Gamma_p\left(\tfrac{m}{2} + \tfrac{n_1+n_2}{2}h, \kappa\right) \\
&= \pi^{\frac{1}{4}p(p-1)}\prod_{j=1}^{p}\Gamma\left[\tfrac{m}{2} + \tfrac{n_1+n_2}{2}h + k_j - \tfrac{1}{2}(j-1)\right] \\
&= \pi^{\frac{1}{4}p(p-1)}\prod_{j=1}^{p}\Gamma\left(a_j + \alpha_j h\right),
\end{aligned} \tag{5.27}
$$

where $a_j = \frac{m}{2} + k_j - \frac{1}{2}(j-1)$ for $j = 1, 2, 3, \ldots, p$,

and $\alpha_j = \frac{n_1+n_2}{2}$ for $j = 1, 2, 3, \ldots, p$.

From [2.2.2] the multivariate gamma functions in (5.26) can be written as

$$
\begin{aligned}
&\Gamma_p\left(\tfrac{n_1}{2}h\right)\Gamma_p\left(\tfrac{n_2}{2}h\right) \\
&= \pi^{\frac{1}{2}p(p-1)}\prod_{j=1}^{p}\Gamma\left[\tfrac{n_1}{2}h - \tfrac{1}{2}(j-1)\right]\prod_{j=1}^{p}\Gamma\left[\tfrac{n_2}{2}h - \tfrac{1}{2}(j-1)\right] \\
&= \pi^{\frac{1}{2}p(p-1)}\prod_{j=1}^{2p}\Gamma\left(b_j + \beta_j h\right)
\end{aligned} \tag{5.28}
$$

where

$$
b_j = \begin{cases} -\frac{1}{4}(j-1) & \text{for } j = 1, 3, 5, \ldots, 2p-1 \\ -\frac{1}{4}(j-2) & \text{for } j = 2, 4, 6, \ldots, 2p \end{cases}
$$

and

$$
\beta_j = \begin{cases} \frac{n_1}{2} & \text{for } j = 1, 3, 5, \ldots, 2p-1 \\ \frac{n_2}{2} & \text{for } j = 2, 4, 6, \ldots, 2p. \end{cases}
$$

Substituting (5.27) and (5.28) in (5.26) gives

$$M_f(h) \equiv \frac{\pi^{\frac{1}{4}p(p-1)}}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)\Gamma_p\left(\frac{m}{2}\right)} c^{-\frac{1}{2}mp} \sum_{k=0}^{\infty}\sum_{\kappa} \frac{1}{k!}\Gamma_p\left(\frac{m}{2},\kappa\right)\Gamma_p\left(\frac{n_1+n_2+m}{2},\kappa\right) C_\kappa\left(\frac{c-1}{c}\boldsymbol{I}_p\right) \frac{\prod_{j=1}^{2p}\Gamma\left(b_j+\beta_j h\right)}{\prod_{j=1}^{p}\Gamma\left(a_j+\alpha_j h\right)}. \tag{5.29}$$

The pdf of Λ_3 is obtained from the inverse Mellin transform of (5.29) (see [2.8.1]) and from the definition of Fox's H-function (see [2.8.3]) as

$$\begin{aligned}
& f(\lambda_3) \\
= & \frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty} M_f(h)\,\lambda_3^{-h}\mathrm{d}h \\
= & \frac{\pi^{\frac{1}{4}p(p-1)}}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)\Gamma_p\left(\frac{m}{2}\right)} c^{-\frac{1}{2}mp} \sum_{k=0}^{\infty}\sum_{\kappa} \frac{1}{k!}\Gamma_p\left(\frac{m}{2},\kappa\right)\Gamma_p\left(\frac{n_1+n_2+m}{2},\kappa\right) C_\kappa\left(\frac{c-1}{c}\boldsymbol{I}_p\right)\left[\frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty}\frac{\prod_{j=1}^{2p}\Gamma\left(b_j+\beta_j h\right)}{\prod_{j=1}^{p}\Gamma\left(a_j+\alpha_j h\right)}\lambda_3^{-h}\mathrm{d}h\right] \\
= & \frac{\pi^{\frac{1}{4}p(p-1)}}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)\Gamma_p\left(\frac{m}{2}\right)} c^{-\frac{1}{2}mp} \sum_{k=0}^{\infty}\sum_{\kappa} \frac{1}{k!}\Gamma_p\left(\frac{m}{2},\kappa\right)\Gamma_p\left(\frac{n_1+n_2+m}{2},\kappa\right) C_\kappa\left(\frac{c-1}{c}\boldsymbol{I}_p\right) H_{p,2p}^{2p,0}\left(\lambda_3\middle|_{(b_1,\beta_1),\ldots,(b_{2p},\beta_{2p})}^{(a_1,\alpha_1),\ldots,(a_p,\alpha_p)}\right). \quad \blacksquare
\end{aligned}$$

Remark 5.13
In (3.20) , substituting $\alpha = 1$ gives the pdf of $\Lambda_1 = |\boldsymbol{W}|$ where $\boldsymbol{W} \sim B_p^{III}(n, m, c)$ as

$$\frac{1}{\Gamma_p\left(\frac{n}{2}\right)} c^{\frac{1}{2}np} \sum_{k=0}^{\infty}\sum_{\kappa} \frac{1}{k!} C_\kappa\left((1-c)\boldsymbol{I}_p\right)\Gamma_p\left(\frac{n+m}{2},\kappa\right) G_{p,p}^{p,0}\left(\lambda_1\middle|_{b_1,\ldots,b_p}^{a_1,\ldots,a_p}\right), \quad 0 < \lambda_1 < 1,$$

where $a_j = \frac{n+m}{2} + k_j - \frac{1}{2}(j+1)$ for $j = 1, 2, \ldots, p,$

$b_j = \frac{n}{2} + k_j - \frac{1}{2}(j+1)$ for $j = 1, 2, \ldots, p.$

5.5 Role of the parameters

In this section we study the effect of the parameter c. The effect of the parameters n_1, n_2 and m was studied in Section 4.5.

Firstly, we consider the bivariate case, $p = 1$, to illustrate the effect of the parameter c on

(i) the form of the pdf of (W_1, W_2);
(ii) the correlation between W_1 and W_2;
(iii) the graphs of $E(W_2|w_1)$ and $var(W_2|w_1)$ plotted against w_1;
(iv) the form of the pdf of Λ_3.

From the result in (5.2) the joint pdf of W_1 and W_2 simplifies to

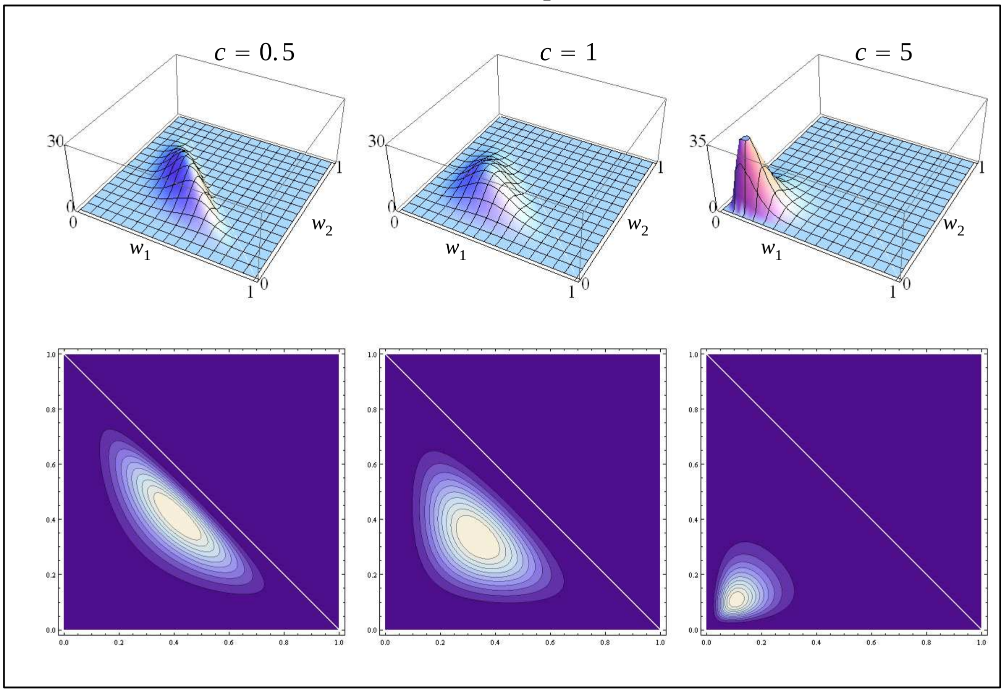
$$\begin{aligned}
f(w_1, w_2) &= \frac{\Gamma\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{m}{2}\right)} w_1^{\frac{1}{2}n_1-1} w_2^{\frac{1}{2}n_2-1}(1-w_1-w_2)^{\frac{1}{2}m-1} \\
&\quad \cdot c^{\frac{1}{2}(n_1+n_2)}\left[1+(c-1)w_1+(c-1)w_2\right]^{-\frac{1}{2}(n_1+n_2+m)},
\end{aligned} \tag{5.30}$$

$0 < w_i < 1$, $i = 1, 2$, $0 < w_1 + w_2 < 1$.

For $c = 2$, (5.30) simplifies to the pdf given by Cardeño et al. (2005). They also derived expressions for the marginal pdf and the $(h_1, h_2)^{th}$ product moment associated with (5.30).

Figure 5.1 shows graphs of the pdf of the $BB_1^{III}(10, 10, 10, c)$ distribution. An increase in the value of the parameter c with all the other parameters held constant, causes $f(w_1, w_2)$ to shift towards smaller values of both W_1 and W_2. Note that when $c = 1$, $(W_1, W_2) \sim BB_1^{I}(10, 10, 10)$ as discussed in Section 4.5.

Figure 5.1: Effect of c on $f(w_1, w_2)$, $(W_1, W_2) \sim BB_1^{III}(10, 10, 10, c)$

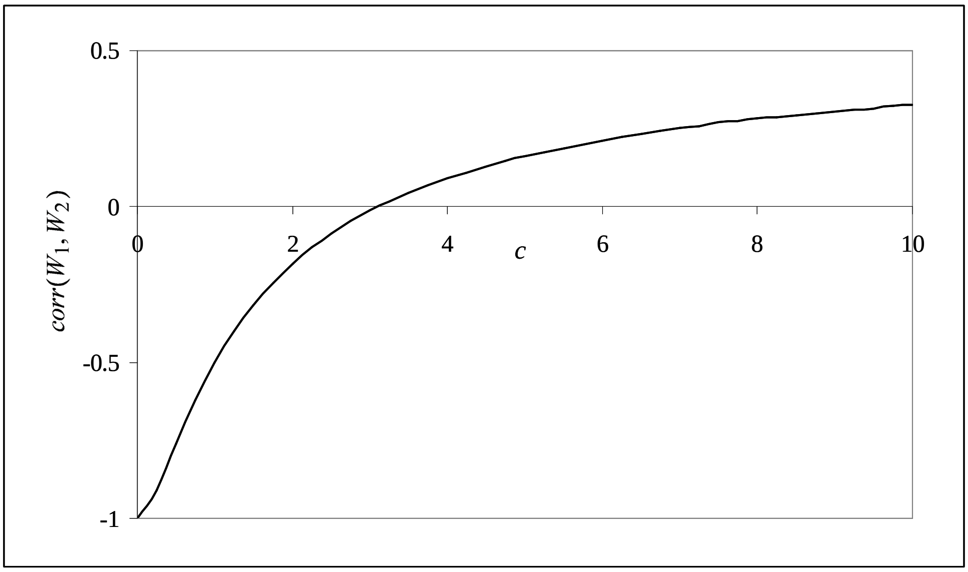


The $(h_1, h_2)^{th}$ product moment, $E\left(W_1^{h_1} W_2^{h_2}\right)$, associated with (5.30) is given by (5.21), that is

$$\begin{aligned} E\left(W_1^{h_1} W_2^{h_2}\right) &= \frac{\Gamma\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} \frac{\Gamma\left(\frac{n_1}{2}+h_1\right)\Gamma\left(\frac{n_2}{2}+h_2\right)}{\Gamma\left(\frac{n_1+n_2+m}{2}+h_1+h_2\right)} \\ &\quad \cdot c^{-\frac{1}{2}m} \, {}_2F_1\left(\frac{m}{2}, \frac{n_1+n_2+m}{2}; \frac{n_1+n_2+m}{2} + h_1 + h_2; \frac{c-1}{c}\right). \end{aligned} \tag{5.31}$$

The result in (5.31) was used to calculate the correlation coefficient, $corr(W_1, W_2)$. Figure 5.2 shows the graph of $corr(W_1, W_2)$ for increasing values of c. For small values of c the correlation coefficient between the variables is negative and increases as c increases. For c large enough, in this case $c > 3$, $corr(W_1, W_2)$ is positive. The bivariate beta type I distribution is used in Bayesian analysis as the natural conjugate prior for the multinomial distribution when the variables are negatively correlated. In some practical cases random variables may be positively correlated, hence the bivariate beta type I distribution will not be a reasonable choice to be a prior distribution. However, the bivariate beta type III accommodates positive correlation for specific choices of the parameter c. Bodvin (2010) illustrated the use of the bivariate beta type III distribution in the Bayes context. She proposed the use of Shannon entropy when determining the parameters of prior bivariate beta distributions as part of a Bayesian calibration methodology and illustrated the appropriateness of this bivariate beta distribution on Moody's default rate data because of its ability to deal with positive correlation in the underlying data.

Figure 5.2: Effect of c on $corr(W_1, W_2)$, $(W_1, W_2) \sim BB_1^{III}(10, 10, 10, c)$



From (5.12), the marginal pdf of W_1 for $p = 1$ simplifies to

$$\begin{aligned} f(w_1) &= \frac{\Gamma\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2+m}{2}\right)} c^{-\frac{1}{2}m} w_1^{\frac{1}{2}n_1-1} (1-w_1)^{\frac{1}{2}(n_2+m)-1} \\ &\quad \cdot {}_2F_1\left(\tfrac{m}{2}, \tfrac{n_1+n_2+m}{2}; \tfrac{n_2+m}{2}; \tfrac{c-1}{c}(1-w_1)\right), \quad 0 < w_1 < 1. \end{aligned}$$

Also, for the bivariate case, the conditional density of $(W_2|W_1 = w_1)$ given by (5.13) simplifies to

$$\begin{aligned} & f(w_2|w_1) \\ &= \frac{\Gamma\left(\frac{n_2+m}{2}\right)}{\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{m}{2}\right)} c^{\frac{1}{2}(n_1+n_2+m)} w_2^{\frac{1}{2}n_2-1} (1-w_1-w_2)^{\frac{1}{2}m-1} \left[1+(c-1)w_1+(c-1)w_2\right]^{-\frac{1}{2}(n_1+n_2+m)} \\ &\quad \cdot (1-w_1)^{-\frac{1}{2}(n_2+m)+1} \left[{}_2F_1\left(\tfrac{m}{2}, \tfrac{n_1+n_2+m}{2}; \tfrac{n_2+m}{2}; \tfrac{c-1}{c}(1-w_1)\right)\right]^{-1}, \quad 0 < w_2 < 1-w_1. \end{aligned} \tag{5.32}$$

The h^{th} moment, $E\left(W_2^h|W_1 = w_1\right)$, associated with (5.32) is derived in the next theorem.

Theorem 5.5
If $(W_1, W_2) \sim BB_1^{III}(n_1, n_2, m, c)$ *then*

$$\begin{aligned} & E\left(W_2^h|W_1 = w_1\right) \\ &= \frac{\Gamma\left(\frac{n_2+m}{2}\right)}{\Gamma\left(\frac{n_2}{2}\right)} \frac{\Gamma\left(\frac{n_2}{2}+h\right)}{\Gamma\left(\frac{n_2+m}{2}+h\right)} c^{\frac{1}{2}(n_1+n_2+m)} \left[1+(c-1)w_1\right]^{-\frac{1}{2}(n_1+n_2+m)} (1-w_1)^h \\ &\quad \cdot \left[{}_2F_1\left(\tfrac{m}{2}, \tfrac{n_1+n_2+m}{2}; \tfrac{n_2+m}{2}; \tfrac{c-1}{c}(1-w_1)\right)\right]^{-1} {}_2F_1\left(\frac{n_2}{2}+h, \frac{n_1+n_2+m}{2}; \frac{n_2+m}{2}+h; -\frac{(c-1)(1-w_1)}{1+(c-1)w_1}\right). \end{aligned} \tag{5.33}$$

Proof:

Using (5.32) and setting $z = \frac{w_2}{1-w_1}$ it follows that

$$
\begin{aligned}
& E\left(W_2^h | W_1 = w_1\right) \\
= & \frac{\Gamma\left(\frac{n_2+m}{2}\right)}{\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{m}{2}\right)} c^{\frac{1}{2}(n_1+n_2+m)} \left(1-w_1\right)^{-\frac{1}{2}(n_2+m)+1} \left[{}_2F_1\left(\frac{m}{2}, \frac{n_1+n_2+m}{2}; \frac{n_2+m}{2}; \frac{c-1}{c}\left(1-w_1\right)\right)\right]^{-1} \\
& \cdot \int_0^{1-w_1} w_2^{\frac{1}{2}n_2+h-1} \left(1-w_1-w_2\right)^{\frac{1}{2}m-1} \left[1+(c-1)w_1+(c-1)w_2\right]^{-\frac{1}{2}(n_1+n_2+m)} \mathrm{d}w_2 \\
= & \frac{\Gamma\left(\frac{n_2+m}{2}\right)}{\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{m}{2}\right)} c^{\frac{1}{2}(n_1+n_2+m)} \left(1-w_1\right)^{-\frac{1}{2}(n_2+m)+1} \left[{}_2F_1\left(\frac{m}{2}, \frac{n_1+n_2+m}{2}; \frac{n_2+m}{2}; \frac{c-1}{c}\left(1-w_1\right)\right)\right]^{-1} \\
& \cdot \left(1-w_1\right)^{\frac{1}{2}n_2+h-1} \left(1-w_1\right)^{\frac{1}{2}m-1} \left[1+(c-1)w_1\right]^{-\frac{1}{2}(n_1+n_2+m)} \left(1-w_1\right) \\
& \cdot \int_0^1 z^{\frac{1}{2}n_2+h-1} (1-z)^{\frac{1}{2}m-1} \left(1+\frac{(c-1)(1-w_1)}{1+(c-1)w_1} z\right)^{-\frac{1}{2}(n_1+n_2+m)} \mathrm{d}z.
\end{aligned}
\tag{5.34}
$$

Now, using [2.5.3] to solve the integral in (5.34) gives (5.33). ■

Figure 5.3 shows graphs of $E(W_2|w_1)$ given by (5.33) plotted against w_1 for different values of c where $(W_1, W_2) \sim BB_1^{III}(10, 10, 10, c)$. Consider the graph $E(W_2|w_1)$ against w_1 where $c = 1$ as a base, that is where $(W_1, W_2) \sim BB_1^{I}(10, 10, 10)$. For $c < 1$ the graph of $E(W_2|w_1)$ against w_1 lies above the base. For $c > 1$ the graph lies below the base and as c increases the graph becomes more curved. For a given value w_1 of W_1, $E(W_2|w_1)$ decreases as c increases. This corresponds to what is observed in Figure 5.1. Figure 5.4 shows graphs of $var(W_2|w_1)$ against w_1 where $(W_1, W_2) \sim BB_1^{III}(10, 10, 10, c)$. Again consider the case where $c = 1$ as the base. For both $c < 1$ and $c > 1$ the graph of $var(W_2|w_1)$ against w_1 lies below the base and is curved. For $c < 1$ the graph is strictly decreasing and for $c > 1$ the graph first increases and then decreases.

Figure 5.3: Effect of c on $E(W_2|w_1)$, $(W_1, W_2) \sim BB_1^{III}(10, 10, 10, c)$

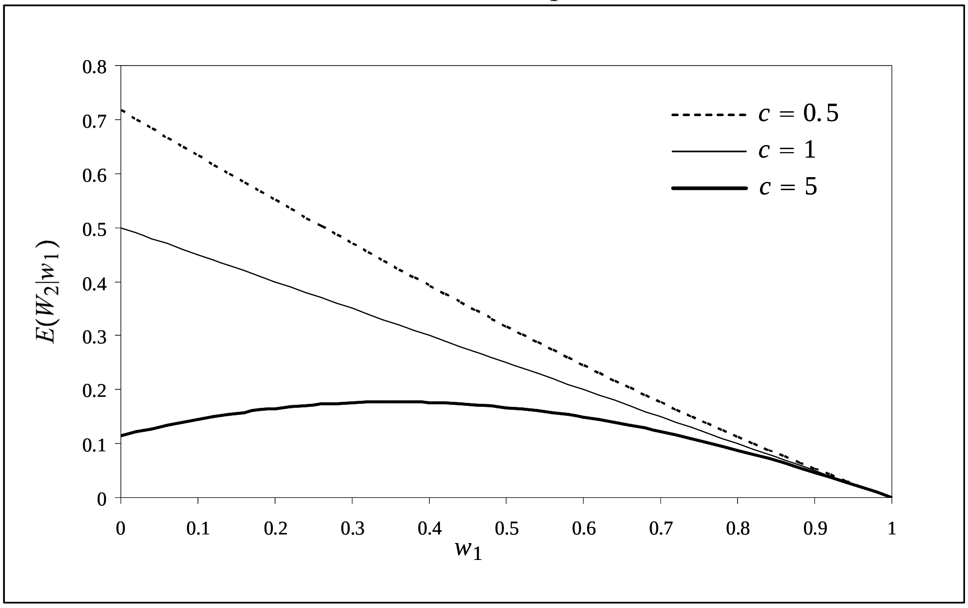
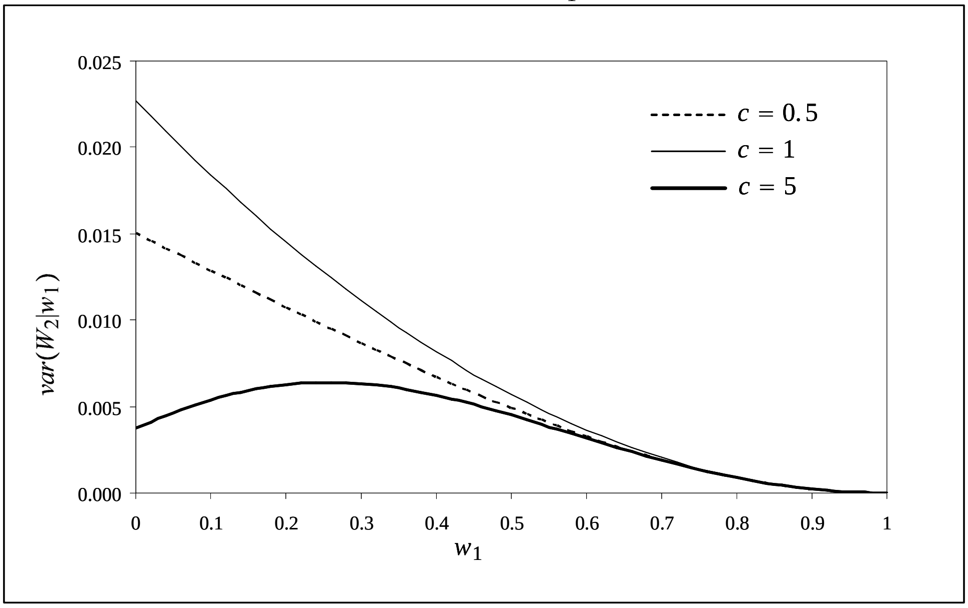


Figure 5.4: Effect of c on $var(W_2|w_1)$, $(W_1, W_2) \sim BB_1^{III}(10, 10, 10, c)$



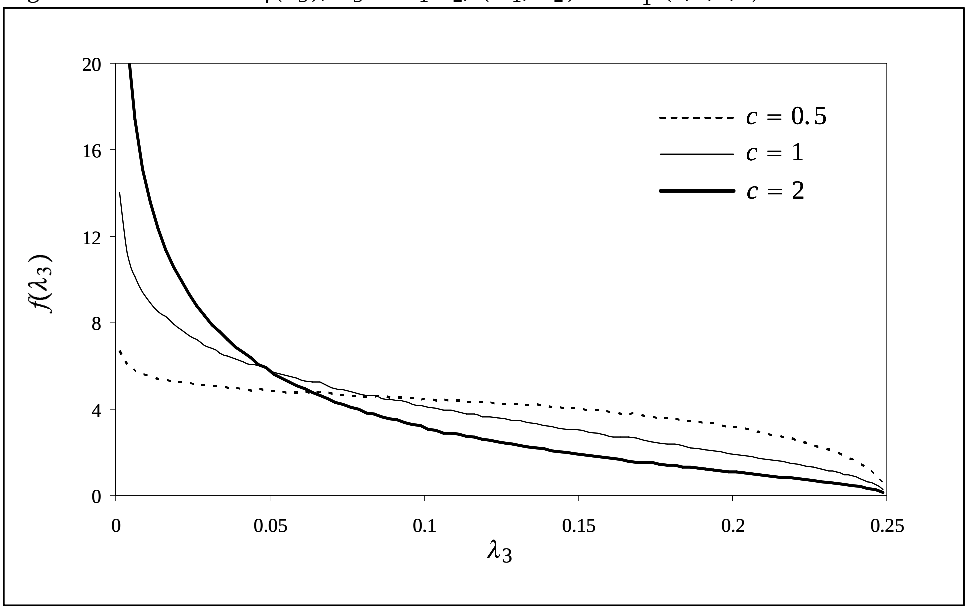
Finally, for the bivariate case, the effect of the additional parameter c on $f(\lambda_3)$, the pdf of $\Lambda_3 = W_1^{\frac{1}{2}n_1} W_2^{\frac{1}{2}n_2}$, was studied in Figure 5.5 where $(W_1, W_2) \sim BB_1^{III}(2, 2, 2, c)$. Considering the result in (5.24) for $p = 1$, it follows from [2.3.3] that $\Gamma_p\left(\frac{m}{2}, \kappa\right) = \Gamma\left(\frac{n_1+n_2+m}{2} + k\right)$, $\Gamma_p\left(\frac{n_1+n_2+m}{2}, \kappa\right) = \Gamma\left(\frac{n_1+n_2+m}{2} + k\right)$ and from [2.3.1] it follows that $\sum_{\kappa} C_\kappa\left(\frac{c-1}{c}\boldsymbol{I}_p\right) = \left(\frac{c-1}{c}\right)^k$. Therefore

$$f(\lambda_3) = \frac{1}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{m}{2}\right)} c^{-\frac{1}{2}m} \sum_{k=0}^{\infty} \frac{1}{k!} \Gamma\left(\frac{m}{2} + k\right) \Gamma\left(\frac{n_1+n_2+m}{2} + k\right) \left(\frac{c-1}{c}\right)^k H_{1,2}^{2,0}\left(\lambda_3 \middle| \begin{matrix} \left(\frac{m}{2}+k, \frac{n_1+n_2}{2}\right) \\ \left(0, \frac{n_1}{2}\right), \left(0, \frac{n_2}{2}\right) \end{matrix}\right), \tag{5.35}$$

$0 < \lambda_3 < 1$.

At smaller values of Λ_3 the pdf, $f(\lambda_3)$ (see (5.35)), increases as c increases.

Figure 5.5: Effect of c on $f(\lambda_3)$, $\Lambda_3 = W_1 W_2$, $(W_1, W_2) \sim BB_1^{III}(2, 2, 2, c)$



Secondly, we consider the bimatrix case, $p = 2$, to illustrate the effect of the parameter c on the pdf of Λ_3. From (5.24), the pdf of Λ_3 for $p = 2$ simplifies to

$$f(\lambda_3) = \frac{\sqrt{\pi}}{\Gamma_2\left(\frac{n_1}{2}\right)\Gamma_2\left(\frac{n_2}{2}\right)\Gamma_2\left(\frac{m}{2}\right)} c^{-m} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{1}{k!} \Gamma_2\left(\frac{m}{2}, \kappa\right) \Gamma_2\left(\frac{n_1+n_2+m}{2}, \kappa\right) C_\kappa\left(\frac{c-1}{c} \boldsymbol{I}_2\right) H_{2,4}^{4,0}\left(\lambda_3 \middle| {}^{(a_1,\alpha_1),(a_2,\alpha_2)}_{(b_1,\beta_1),\ldots,(b_4,\beta_4)}\right), \tag{5.36}$$

$0 < \lambda_3 < 1$, where

$a_j = \frac{m}{2} + k_j - \frac{1}{2}(j-1)$ for $j = 1, 2$,

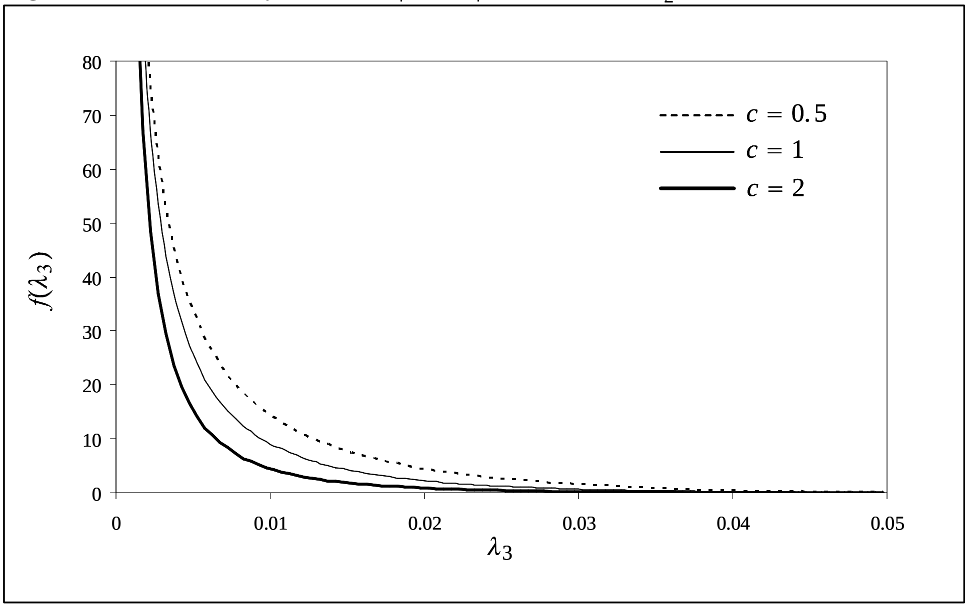
$\alpha_j = \frac{n_1+n_2}{2}$ for $j = 1, 2$,

$$b_j = \begin{cases} -\frac{1}{4}(j-1) & \text{for } j = 1, 3 \\ -\frac{1}{4}(j-2) & \text{for } j = 2, 4, \end{cases}$$

$$\beta_j = \begin{cases} \frac{n_1}{2} & \text{for } j = 1, 3 \\ \frac{n_2}{2} & \text{for } j = 2, 4. \end{cases}$$

Figure 5.6 illustrates the shape of $f(\lambda_3)$ (see (5.36)) for increasing values of c where $(\boldsymbol{W}_1, \boldsymbol{W}_2) \sim BB_2^{III}(2, 2, 2, c)$. We note that as c increases the pdf shifts towards smaller values of Λ_3.

Figure 5.6: Effect of c on $f(\lambda_3)$, $\Lambda_3 = |\boldsymbol{W}_1\boldsymbol{W}_2|$, $(\boldsymbol{W}_1, \boldsymbol{W}_2) \sim BB_2^{III}(2,2,2,c)$



6 Bimatrix variate beta type IV distribution

In this section we derive the joint pdf of two dependent matrix variate beta type I variates (see (3.3)) given by the Wishart ratios (1.4) and call it the bimatrix variate beta type IV distribution. The marginal and conditional pdfs are derived in Theorem 6.2 and the product moment of the determinants is derived in Theorem 6.3. The latter is used in Theorem 6.4 to derive an exact expression for the pdf of $\Lambda_4 = |\boldsymbol{X}_1\boldsymbol{X}_2|$ (see (1.10)), the product of two dependent Wilks' statistics (see Bekker, Roux, Ehlers and Arashi, 2010). The role of the parameters are studied in Section 6.5. This distribution is also known in the literature as the bimatrix variate generalised beta type I distribution and its pdf and some properties were derived independently from this study by Gupta and Nagar (2009b) and Díaz-García and Gutiérrez-Jáimez (2010).

6.1 Probability density function

The pdf of the bimatrix variate beta type IV distribution is derived from Wishart ratios in Theorem 6.1.

Theorem 6.1
Let $\boldsymbol{S}_1 \sim W_p(n_1, \boldsymbol{\Sigma})$, $\boldsymbol{S}_2 \sim W_p(n_2, \boldsymbol{\Sigma})$ *and* $\boldsymbol{B} \sim W_p(m, \boldsymbol{\Sigma})$ *be independently distributed. Define*

$$\boldsymbol{X}_i = \left(\boldsymbol{I}_p + \boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{S}_i\boldsymbol{B}^{-\frac{1}{2}}\right)^{-\frac{1}{2}} \boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{S}_i\boldsymbol{B}^{-\frac{1}{2}} \left(\boldsymbol{I}_p + \boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{S}_i\boldsymbol{B}^{-\frac{1}{2}}\right)^{-\frac{1}{2}}, \ i = 1, 2, \tag{6.1}$$

where $\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{B}^{\frac{1}{2}} = \boldsymbol{B}$ *(see* (1.4)*).*

The pdf of $(\boldsymbol{X}_1, \boldsymbol{X}_2) \sim BB_p^{IV}(n_1, n_2, m)$ *is*

$$\begin{aligned} f(\boldsymbol{X}_1, \boldsymbol{X}_2) = \quad & \left\{\beta_p\left(\tfrac{n_1}{2}, \tfrac{n_2}{2}; \tfrac{m}{2}\right)\right\}^{-1} \prod_{i=1}^{2} |\boldsymbol{X}_i|^{\frac{1}{2}n_i - \frac{1}{2}(p+1)} \\ & \cdot \prod_{i=1}^{2} |\boldsymbol{I}_p - \boldsymbol{X}_i|^{-\frac{1}{2}n_i - \frac{1}{2}(p+1)} \left|\boldsymbol{I}_p + \sum_{i=1}^{2} \boldsymbol{X}_i (\boldsymbol{I}_p - \boldsymbol{X}_i)^{-1}\right|^{-\frac{1}{2}(n_1+n_2+m)}, \end{aligned}$$

or alternatively

$$\begin{aligned} & f(\boldsymbol{X}_1, \boldsymbol{X}_2) \\ = \quad & \left\{\beta_p\left(\tfrac{n_1}{2}, \tfrac{n_2}{2}; \tfrac{m}{2}\right)\right\}^{-1} \prod_{i=1}^{2} |\boldsymbol{X}_i|^{\frac{1}{2}n_i - \frac{1}{2}(p+1)} \\ & \cdot |\boldsymbol{I}_p - \boldsymbol{X}_1|^{\frac{1}{2}(n_2+m) - \frac{1}{2}(p+1)} |\boldsymbol{I}_p - \boldsymbol{X}_2|^{\frac{1}{2}(n_1+m) - \frac{1}{2}(p+1)} |\boldsymbol{I}_p - \boldsymbol{X}_1\boldsymbol{X}_2|^{-\frac{1}{2}(n_1+n_2+m)}, \end{aligned} \tag{6.2}$$

$\boldsymbol{0} < \boldsymbol{X}_i < \boldsymbol{I}_p$, $i = 1, 2$, *where* $n_i > (p-1)$, $i = 1, 2$, *and* $m > (p-1)$.

Proof:
The pdf of $(\boldsymbol{S}_1, \boldsymbol{S}_2, \boldsymbol{B})$ is given by

$$K \prod_{i=1}^{2} \left[\mathrm{etr}\left(-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}_i\right) |\boldsymbol{S}_i|^{\frac{1}{2}(n_i - p - 1)}\right] \left[\mathrm{etr}\left(-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\right) |\boldsymbol{B}|^{\frac{1}{2}(m-p-1)}\right] \tag{6.3}$$

where $K^{-1} = \Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)\Gamma_p\left(\frac{m}{2}\right)|2\boldsymbol{\Sigma}|^{\frac{1}{2}(n_1+n_2+m)}$ (see [2.10.1]).

Considering the ratios given by (6.1) and letting $\boldsymbol{Z}_i = \boldsymbol{B}^{-\frac{1}{2}} \boldsymbol{S}_i \boldsymbol{B}^{-\frac{1}{2}}$ we get $\boldsymbol{S}_i = \boldsymbol{B}^{\frac{1}{2}} \boldsymbol{Z}_i \boldsymbol{B}^{\frac{1}{2}}$ and $\boldsymbol{X}_i = (\boldsymbol{I}_p + \boldsymbol{Z}_i)^{-\frac{1}{2}} \boldsymbol{Z}_i (\boldsymbol{I}_p + \boldsymbol{Z}_i)^{-\frac{1}{2}}$. Since $\boldsymbol{Z}_i$ commutes with any rational function of $\boldsymbol{Z}_i$, it follows that $\boldsymbol{X}_i = (\boldsymbol{I}_p + \boldsymbol{Z}_i)^{-1} \boldsymbol{Z}_i$, $i = 1, 2$. Thus, $\boldsymbol{Z}_i = \boldsymbol{X}_i (\boldsymbol{I}_p - \boldsymbol{X}_i)^{-1}$, $i = 1, 2$. From [2.1.3], [2.1.4], [2.1.6] and [2.1.9] the Jacobian of the transformations in (6.1) is

$$
\begin{aligned}
& J(\boldsymbol{S}_1, \boldsymbol{S}_2 \to \boldsymbol{X}_1, \boldsymbol{X}_2) \\
&= J(\boldsymbol{S}_1 \to \boldsymbol{X}_1) J(\boldsymbol{S}_2 \to \boldsymbol{X}_2) \\
&= \prod_{i=1}^{2} J(\boldsymbol{S}_i \to \boldsymbol{Z}_i) J(\boldsymbol{Z}_i \to \boldsymbol{X}_i) \\
&= |\boldsymbol{B}|^{(p+1)} \prod_{i=1}^{2} |\boldsymbol{I}_p - \boldsymbol{X}_i|^{-(p+1)}.
\end{aligned}
\tag{6.4}
$$

Substituting $\boldsymbol{S}_i = \boldsymbol{B}^{\frac{1}{2}} \boldsymbol{X}_i (\boldsymbol{I}_p - \boldsymbol{X}_i)^{-1} \boldsymbol{B}^{\frac{1}{2}}$ and (6.4) in (6.3) gives the pdf of $(\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{B})$ as

$$
\begin{aligned}
& f(\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{B}) \\
=\;& K \prod_{i=1}^{2} \left\{ \operatorname{etr}\left[-\tfrac{1}{2} \boldsymbol{\Sigma}^{-1} \boldsymbol{B}^{\frac{1}{2}} \boldsymbol{X}_i (\boldsymbol{I}_p - \boldsymbol{X}_i)^{-1} \boldsymbol{B}^{\frac{1}{2}} \right] \left| \boldsymbol{B}^{\frac{1}{2}} \boldsymbol{X}_i (\boldsymbol{I}_p - \boldsymbol{X}_i)^{-1} \boldsymbol{B}^{\frac{1}{2}} \right|^{\frac{1}{2}(n_i - p - 1)} \right\} \\
& \cdot \operatorname{etr}\left(-\tfrac{1}{2} \boldsymbol{\Sigma}^{-1} \boldsymbol{B} \right) |\boldsymbol{B}|^{\frac{1}{2}(m-p-1)} |\boldsymbol{B}|^{(p+1)} \prod_{i=1}^{2} |\boldsymbol{I}_p - \boldsymbol{X}_i|^{-(p+1)} \\
=\;& K \prod_{i=1}^{2} |\boldsymbol{X}_i|^{\frac{1}{2} n_i - \frac{1}{2}(p+1)} \prod_{i=1}^{2} |\boldsymbol{I}_p - \boldsymbol{X}_i|^{-\frac{1}{2} n_i - \frac{1}{2}(p+1)} \\
& \cdot |\boldsymbol{B}|^{\frac{1}{2}(n_1 + n_2 + m) - \frac{1}{2}(p+1)} \operatorname{etr}\left\{ -\tfrac{1}{2} \boldsymbol{\Sigma}^{-1} \boldsymbol{B}^{\frac{1}{2}} \left[\boldsymbol{I}_p + \sum_{i=1}^{2} \boldsymbol{X}_i (\boldsymbol{I}_p - \boldsymbol{X}_i)^{-1} \right] \boldsymbol{B}^{\frac{1}{2}} \right\}.
\end{aligned}
\tag{6.5}
$$

We consider the symmetrised density function of $(\boldsymbol{X}_1, \boldsymbol{X}_2)$ (see [2.9.1]), that is $f_s(\boldsymbol{X}_1, \boldsymbol{X}_2) \equiv \int_{\boldsymbol{B} > \boldsymbol{0}} \int_{O(p)} f(\boldsymbol{H}\boldsymbol{X}_1\boldsymbol{H}', \boldsymbol{H}\boldsymbol{X}_2\boldsymbol{H}', \boldsymbol{H}\boldsymbol{B}\boldsymbol{H}') \mathrm{d}\boldsymbol{H} \mathrm{d}\boldsymbol{B}$ where $\boldsymbol{H}$ $(p \times p)$ is orthogonal and $\mathrm{d}\boldsymbol{H}$ is the normalised Haar invariant measure on $O(p)$. Note that $\mathrm{d}\boldsymbol{B} = \mathrm{d}\boldsymbol{H}\boldsymbol{B}\boldsymbol{H}'$ (Díaz-García and Gutiérrez-Jáimez, 2006b). From (6.5)

$$
\begin{aligned}
& f(\boldsymbol{H}\boldsymbol{X}_1\boldsymbol{H}', \boldsymbol{H}\boldsymbol{X}_2\boldsymbol{H}', \boldsymbol{H}\boldsymbol{B}\boldsymbol{H}') \\
=\;& K \prod_{i=1}^{2} \left| \boldsymbol{H}\boldsymbol{X}_i\boldsymbol{H}' \right|^{\frac{1}{2} n_i - \frac{1}{2}(p+1)} \prod_{i=1}^{2} \left| \boldsymbol{I}_p - \boldsymbol{H}\boldsymbol{X}_i\boldsymbol{H}' \right|^{-\frac{1}{2} n_i - \frac{1}{2}(p+1)} \\
& \cdot \left| \boldsymbol{H}\boldsymbol{B}\boldsymbol{H}' \right|^{\frac{1}{2}(n_1 + n_2 + m) - \frac{1}{2}(p+1)} \operatorname{etr}\left\{ -\tfrac{1}{2} \boldsymbol{\Sigma}^{-1} (\boldsymbol{H}\boldsymbol{B}\boldsymbol{H}')^{\frac{1}{2}} \left[\boldsymbol{I}_p + \sum_{i=1}^{2} \boldsymbol{H}\boldsymbol{X}_i\boldsymbol{H}' (\boldsymbol{I}_p - \boldsymbol{H}\boldsymbol{X}_i\boldsymbol{H}')^{-1} \right] (\boldsymbol{H}\boldsymbol{B}\boldsymbol{H}')^{\frac{1}{2}} \right\} \\
=\;& K \prod_{i=1}^{2} |\boldsymbol{X}_i|^{\frac{1}{2} n_i - \frac{1}{2}(p+1)} \prod_{i=1}^{2} |\boldsymbol{I}_p - \boldsymbol{X}_i|^{-\frac{1}{2} n_i - \frac{1}{2}(p+1)} \\
& \cdot |\boldsymbol{B}|^{\frac{1}{2}(n_1 + n_2 + m) - \frac{1}{2}(p+1)} \operatorname{etr}\left\{ -\tfrac{1}{2} \boldsymbol{\Sigma}^{-1} \left(\boldsymbol{H}\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{H}' \right) \left[\boldsymbol{I}_p + \sum_{i=1}^{2} \boldsymbol{H}\boldsymbol{X}_i (\boldsymbol{I}_p - \boldsymbol{X}_i)^{-1} \boldsymbol{H}' \right] \left(\boldsymbol{H}\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{H}' \right) \right\}.
\end{aligned}
\tag{6.6}
$$

Then, from (6.6) and [2.3.6],

$$
\begin{aligned}
& f_s(\boldsymbol{X}_1, \boldsymbol{X}_2) \\
= \quad & K \prod_{i=1}^{2} |\boldsymbol{X}_i|^{\frac{1}{2}n_i - \frac{1}{2}(p+1)} \prod_{i=1}^{2} |\boldsymbol{I}_p - \boldsymbol{X}_i|^{-\frac{1}{2}n_i - \frac{1}{2}(p+1)} \\
& \cdot \int_{\boldsymbol{B}>\boldsymbol{0}} |\boldsymbol{B}|^{\frac{1}{2}(n_1+n_2+m) - \frac{1}{2}(p+1)} \\
& \qquad \cdot \int_{O(p)} \operatorname{etr}\left\{ -\tfrac{1}{2} \boldsymbol{H}' \boldsymbol{\Sigma}^{-1} \boldsymbol{H} \boldsymbol{B}^{\frac{1}{2}} \left[\boldsymbol{I}_p + \sum_{i=1}^{2} \boldsymbol{X}_i (\boldsymbol{I}_p - \boldsymbol{X}_i)^{-1} \right] \boldsymbol{B}^{\frac{1}{2}} \right\} \mathrm{d}\boldsymbol{H} \mathrm{d}\boldsymbol{B} \\
= \quad & K \prod_{i=1}^{2} |\boldsymbol{X}_i|^{\frac{1}{2}n_i - \frac{1}{2}(p+1)} \prod_{i=1}^{2} |\boldsymbol{I}_p - \boldsymbol{X}_i|^{-\frac{1}{2}n_i - \frac{1}{2}(p+1)} \\
& \cdot \int_{\boldsymbol{B}>\boldsymbol{0}} |\boldsymbol{B}|^{\frac{1}{2}(n_1+n_2+m) - \frac{1}{2}(p+1)} \\
& \qquad \cdot \int_{O(p)} \operatorname{etr}\left\{ -\tfrac{1}{2} \boldsymbol{H}' \boldsymbol{\Sigma}^{-1} \boldsymbol{H} \left[\boldsymbol{I}_p + \sum_{i=1}^{2} \boldsymbol{X}_i (\boldsymbol{I}_p - \boldsymbol{X}_i)^{-1} \right]^{\frac{1}{2}} \boldsymbol{B} \left[\boldsymbol{I}_p + \sum_{i=1}^{2} \boldsymbol{X}_i (\boldsymbol{I}_p - \boldsymbol{X}_i)^{-1} \right]^{\frac{1}{2}} \right\} \mathrm{d}\boldsymbol{H} \mathrm{d}\boldsymbol{B}.
\end{aligned} \tag{6.7}
$$

Interchanging the order of integration in (6.7) and integrating with respect to $\boldsymbol{B}$ by using [2.2.3] gives

$$
\begin{aligned}
& f_s(\boldsymbol{X}_1, \boldsymbol{X}_2) \\
= \quad & K \prod_{i=1}^{2} |\boldsymbol{X}_i|^{\frac{1}{2}n_i - \frac{1}{2}(p+1)} \prod_{i=1}^{2} |\boldsymbol{I}_p - \boldsymbol{X}_i|^{-\frac{1}{2}n_i - \frac{1}{2}(p+1)} \\
& \cdot \int_{O(p)} \int_{\boldsymbol{B}>\boldsymbol{0}} |\boldsymbol{B}|^{\frac{1}{2}(n_1+n_2+m) - \frac{1}{2}(p+1)} \\
& \qquad \cdot \operatorname{etr}\left\{ -\tfrac{1}{2} \left[\boldsymbol{I}_p + \sum_{i=1}^{2} \boldsymbol{X}_i (\boldsymbol{I}_p - \boldsymbol{X}_i)^{-1} \right]^{\frac{1}{2}} \boldsymbol{H}' \boldsymbol{\Sigma}^{-1} \boldsymbol{H} \left[\boldsymbol{I}_p + \sum_{i=1}^{2} \boldsymbol{X}_i (\boldsymbol{I}_p - \boldsymbol{X}_i)^{-1} \right]^{\frac{1}{2}} \boldsymbol{B} \right\} \mathrm{d}\boldsymbol{B} \mathrm{d}\boldsymbol{H} \\
= \quad & K \Gamma_p\left(\tfrac{n_1+n_2+m}{2}\right) \prod_{i=1}^{2} |\boldsymbol{X}_i|^{\frac{1}{2}n_i - \frac{1}{2}(p+1)} \prod_{i=1}^{2} |\boldsymbol{I}_p - \boldsymbol{X}_i|^{-\frac{1}{2}n_i - \frac{1}{2}(p+1)} \\
& \cdot \int_{O(p)} \left| \tfrac{1}{2} \left[\boldsymbol{I}_p + \sum_{i=1}^{2} \boldsymbol{X}_i (\boldsymbol{I}_p - \boldsymbol{X}_i)^{-1} \right]^{\frac{1}{2}} \boldsymbol{H}' \boldsymbol{\Sigma}^{-1} \boldsymbol{H} \left[\boldsymbol{I}_p + \sum_{i=1}^{2} \boldsymbol{X}_i (\boldsymbol{I}_p - \boldsymbol{X}_i)^{-1} \right]^{\frac{1}{2}} \right|^{-\frac{1}{2}(n_1+n_2+m)} \mathrm{d}\boldsymbol{H} \\
= \quad & \left\{ \beta_p\left(\tfrac{n_1}{2}, \tfrac{n_2}{2}; \tfrac{m}{2}\right) \right\}^{-1} \int_{O(p)} \prod_{i=1}^{2} |\boldsymbol{H}\boldsymbol{X}_i\boldsymbol{H}'|^{\frac{1}{2}n_i - \frac{1}{2}(p+1)} \prod_{i=1}^{2} |\boldsymbol{I}_p - \boldsymbol{H}\boldsymbol{X}_i\boldsymbol{H}'|^{-\frac{1}{2}n_i - \frac{1}{2}(p+1)} \\
& \qquad \cdot \left| \boldsymbol{I}_p + \boldsymbol{H} \sum_{i=1}^{2} \boldsymbol{X}_i (\boldsymbol{I}_p - \boldsymbol{X}_i)^{-1} \boldsymbol{H}' \right|^{-\frac{1}{2}(n_1+n_2+m)} \mathrm{d}\boldsymbol{H}.
\end{aligned} \tag{6.8}
$$

From (6.8) and [2.9.3] follows that

$$
\begin{aligned}
f(\boldsymbol{X}_1, \boldsymbol{X}_2) = \quad & \left\{ \beta_p\left(\tfrac{n_1}{2}, \tfrac{n_2}{2}; \tfrac{m}{2}\right) \right\}^{-1} \prod_{i=1}^{2} |\boldsymbol{X}_i|^{\frac{1}{2}n_i - \frac{1}{2}(p+1)} \prod_{i=1}^{2} |\boldsymbol{I}_p - \boldsymbol{X}_i|^{-\frac{1}{2}n_i - \frac{1}{2}(p+1)} \\
& \cdot \left| \boldsymbol{I}_p + \sum_{i=1}^{2} \boldsymbol{X}_i (\boldsymbol{I}_p - \boldsymbol{X}_i)^{-1} \right|^{-\frac{1}{2}(n_1+n_2+m)}.
\end{aligned} \tag{6.9}
$$

In (6.9) rewrite

$$\begin{aligned}
&\left|\boldsymbol{I}_p + \boldsymbol{X}_1\left(\boldsymbol{I}_p - \boldsymbol{X}_1\right)^{-1} + \boldsymbol{X}_2\left(\boldsymbol{I}_p - \boldsymbol{X}_2\right)^{-1}\right| \\
&= \left|\left(\boldsymbol{I}_p - \boldsymbol{X}_1\right) + \boldsymbol{X}_1 + \boldsymbol{X}_2\left(\boldsymbol{I}_p - \boldsymbol{X}_2\right)^{-1}\left(\boldsymbol{I}_p - \boldsymbol{X}_1\right)\right|\left|\boldsymbol{I}_p - \boldsymbol{X}_1\right|^{-1} \\
&= \left|\left(\boldsymbol{I}_p - \boldsymbol{X}_2\right) + \boldsymbol{X}_2 - \boldsymbol{X}_2\left(\boldsymbol{I}_p - \boldsymbol{X}_2\right)^{-1}\boldsymbol{X}_1\left(\boldsymbol{I}_p - \boldsymbol{X}_2\right)\right|\left|\boldsymbol{I}_p - \boldsymbol{X}_1\right|^{-1}\left|\boldsymbol{I}_p - \boldsymbol{X}_2\right|^{-1} \\
&= \left|\boldsymbol{X}_2\left(\boldsymbol{I}_p - \boldsymbol{X}_2\right)^{-1}\left(\boldsymbol{X}_2^{-1} - \boldsymbol{X}_1\right)\left(\boldsymbol{I}_p - \boldsymbol{X}_2\right)\right|\left|\boldsymbol{I}_p - \boldsymbol{X}_1\right|^{-1}\left|\boldsymbol{I}_p - \boldsymbol{X}_2\right|^{-1} \\
&= \left|\boldsymbol{I}_p - \boldsymbol{X}_1\boldsymbol{X}_2\right|\left|\boldsymbol{I}_p - \boldsymbol{X}_1\right|^{-1}\left|\boldsymbol{I}_p - \boldsymbol{X}_2\right|^{-1}.
\end{aligned} \tag{6.10}$$

From (6.10), the pdf of $(\boldsymbol{X}_1, \boldsymbol{X}_2)$ given by (6.9) can be written as

$$\begin{aligned}
&f\left(\boldsymbol{X}_1, \boldsymbol{X}_2\right) \\
&= \left\{\beta_p\left(\tfrac{n_1}{2}, \tfrac{n_2}{2}; \tfrac{m}{2}\right)\right\}^{-1}\prod_{i=1}^{2}\left|\boldsymbol{X}_i\right|^{\frac{1}{2}n_i - \frac{1}{2}(p+1)} \\
&\quad \cdot\left|\boldsymbol{I}_p - \boldsymbol{X}_1\right|^{\frac{1}{2}(n_2+m)-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p - \boldsymbol{X}_2\right|^{\frac{1}{2}(n_1+m)-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p - \boldsymbol{X}_1\boldsymbol{X}_2\right|^{-\frac{1}{2}(n_1+n_2+m)}. \quad \blacksquare
\end{aligned}$$

Remark 6.1

The following definition of Wishart ratios will also give matrix variates having the bimatrix variate beta type IV distribution with pdf given by (6.2) (see Díaz-García and Gutiérrez-Jáimez, 2010):

$$\boldsymbol{X}_i = \left(\boldsymbol{S}_i + \boldsymbol{B}\right)^{-\frac{1}{2}}\boldsymbol{S}_i\left(\boldsymbol{S}_i + \boldsymbol{B}\right)^{-\frac{1}{2}}, \ i = 1, 2.$$

Remark 6.2

The matrix variate Dirichlet IV distribution, denoted by $(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_r) \sim D_p^{IV}(n_1, \ldots, n_r, m)$, results by extending (6.1) to r independent Wishart matrix variates, $\boldsymbol{S}_i \sim W_p(n_i, \boldsymbol{\Sigma})$, $i = 1, \ldots, r$, all independent of $\boldsymbol{B} \sim W_p(m, \boldsymbol{\Sigma})$. The pdf of $(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_r)$ is given by

$$\begin{aligned}
&\left\{\beta_p\left(\tfrac{n_1}{2}, \ldots, \tfrac{n_r}{2}; \tfrac{m}{2}\right)\right\}^{-1}\prod_{i=1}^{r}\left|\boldsymbol{X}_i\right|^{\frac{1}{2}n_i - \frac{1}{2}(p+1)} \\
&\cdot\prod_{i=1}^{r}\left|\boldsymbol{I}_p - \boldsymbol{X}_i\right|^{-\frac{1}{2}n_i - \frac{1}{2}(p+1)}\left|\boldsymbol{I}_p + \sum_{i=1}^{r}\boldsymbol{X}_i\left(\boldsymbol{I}_p - \boldsymbol{X}_i\right)^{-1}\right|^{-\frac{1}{2}(n_1+\cdots+n_r+m)},
\end{aligned} \tag{6.11}$$

$\boldsymbol{0} < \boldsymbol{X}_i < \boldsymbol{I}_p$, $i = 1, \ldots, r$.

Remark 6.3

The pdfs of the bimatrix variate beta type IV distribution in (6.2) and the matrix variate Dirichlet type IV distribution in (6.11) are not members of the Liouville family of distributions (see [2.2.1]).

6.2 Marginal property and conditional density

In this section the marginal and conditional pdfs of the bimatrix variate beta type IV distribution are derived.

Theorem 6.2
If $(\boldsymbol{X}_1, \boldsymbol{X}_2) \sim BB_p^{IV}(n_1, n_2, m)$, *then* $\boldsymbol{X}_i \sim B_p^I(n_i, m)$, $i = 1, 2$, *and the pdf of* $\boldsymbol{X}_2 | \boldsymbol{X}_1$ *is given by*

$$\left\{\beta_p\left(\tfrac{n_2}{2}, \tfrac{n_1+m}{2}\right)\right\}^{-1} |\boldsymbol{I}_p - \boldsymbol{X}_1|^{\frac{1}{2}n_2} |\boldsymbol{X}_2|^{\frac{1}{2}n_2 - \frac{1}{2}(p+1)} |\boldsymbol{I}_p - \boldsymbol{X}_2|^{\frac{1}{2}(n_1+m)-\frac{1}{2}(p+1)} |\boldsymbol{I}_p - \boldsymbol{X}_1\boldsymbol{X}_2|^{-\frac{1}{2}(n_1+n_2+m)}, \quad (6.12)$$

$\mathbf{0} < \boldsymbol{X}_2 < \boldsymbol{I}_p$.

Proof:
The marginal pdf of $\boldsymbol{X}_1$ is the matrix variate beta type I given by (3.1). That is

$$f(\boldsymbol{X}_1) = \left\{\beta_p\left(\tfrac{n_1}{2}, \tfrac{m}{2}\right)\right\}^{-1} |\boldsymbol{X}_1|^{\frac{1}{2}n_1 - \frac{1}{2}(p+1)} |\boldsymbol{I}_p - \boldsymbol{X}_1|^{\frac{1}{2}m - \frac{1}{2}(p+1)}, \ \mathbf{0} < \boldsymbol{X}_1 < \boldsymbol{I}_p. \quad (6.13)$$

From (6.2) and (6.13) the pdf of $\boldsymbol{X}_2 | \boldsymbol{X}_1$ is given as

$$\begin{aligned}
& f(\boldsymbol{X}_2|\boldsymbol{X}_1) \\
= \ & \left\{\beta_p\left(\tfrac{n_1}{2}, \tfrac{n_2}{2}; \tfrac{m}{2}\right)\right\}^{-1} \prod_{i=1}^{2} |\boldsymbol{X}_i|^{\frac{1}{2}n_i - \frac{1}{2}(p+1)} |\boldsymbol{I}_p - \boldsymbol{X}_1|^{\frac{1}{2}(n_2+m)-\frac{1}{2}(p+1)} |\boldsymbol{I}_p - \boldsymbol{X}_2|^{\frac{1}{2}(n_1+m)-\frac{1}{2}(p+1)} \\
& \cdot |\boldsymbol{I}_p - \boldsymbol{X}_1\boldsymbol{X}_2|^{-\frac{1}{2}(n_1+n_2+m)} \left\{\beta_p\left(\tfrac{n_1}{2}, \tfrac{m}{2}\right)\right\} |\boldsymbol{X}_1|^{-\frac{1}{2}n_1 + \frac{1}{2}(p+1)} |\boldsymbol{I}_p - \boldsymbol{X}_1|^{-\frac{1}{2}m + \frac{1}{2}(p+1)} \\
= \ & \frac{\Gamma_p\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)\Gamma_p\left(\frac{m}{2}\right)} \frac{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{m}{2}\right)}{\Gamma_p\left(\frac{n_1+m}{2}\right)} |\boldsymbol{X}_2|^{\frac{1}{2}n_2 - \frac{1}{2}(p+1)} \\
& \cdot |\boldsymbol{I}_p - \boldsymbol{X}_1|^{\frac{1}{2}n_2} |\boldsymbol{I}_p - \boldsymbol{X}_2|^{\frac{1}{2}(n_1+m)-\frac{1}{2}(p+1)} |\boldsymbol{I}_p - \boldsymbol{X}_1\boldsymbol{X}_2|^{-\frac{1}{2}(n_1+n_2+m)} \\
= \ & \left\{\beta_p\left(\tfrac{n_2}{2}, \tfrac{n_1+m}{2}\right)\right\}^{-1} |\boldsymbol{X}_2|^{\frac{1}{2}n_2 - \frac{1}{2}(p+1)} |\boldsymbol{I}_p - \boldsymbol{X}_1|^{\frac{1}{2}n_2} |\boldsymbol{I}_p - \boldsymbol{X}_2|^{\frac{1}{2}(n_1+m)-\frac{1}{2}(p+1)} |\boldsymbol{I}_p - \boldsymbol{X}_1\boldsymbol{X}_2|^{-\frac{1}{2}(n_1+n_2+m)}. \quad \blacksquare
\end{aligned}$$

Remark 6.4
If $(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_r) \sim D_p^{IV}(n_1, \ldots, n_r, m)$ given by (6.11), then for $s \leq r$ the marginal pdf is $(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_s) \sim D_p^{IV}(n_1, \ldots, n_s, m)$ and the pdf of $(\boldsymbol{X}_{s+1}, \ldots, \boldsymbol{X}_r) \,|\, (\boldsymbol{X}_1, \ldots, \boldsymbol{X}_s)$ is given by

$$\begin{aligned}
& \left\{\beta_p\left(\tfrac{n_{s+1}}{2}, \ldots, \tfrac{n_r}{2}; \tfrac{n_1+\cdots+n_s+m}{2}\right)\right\}^{-1} \prod_{i=s+1}^{r} |\boldsymbol{X}_i|^{\frac{1}{2}n_i - \frac{1}{2}(p+1)} \prod_{i=s+1}^{r} |\boldsymbol{I}_p - \boldsymbol{X}_i|^{-\frac{1}{2}n_i - \frac{1}{2}(p+1)} \\
& \cdot \left|\boldsymbol{I}_p + \sum_{i=1}^{r} \boldsymbol{X}_i (\boldsymbol{I}_p - \boldsymbol{X}_i)^{-1}\right|^{-\frac{1}{2}(n_1+\cdots+n_r+m)} \left|\boldsymbol{I}_p + \sum_{i=1}^{s} \boldsymbol{X}_i (\boldsymbol{I}_p - \boldsymbol{X}_i)^{-1}\right|^{\frac{1}{2}(n_1+\cdots+n_s+m)}, \quad (6.14)
\end{aligned}$$

$\mathbf{0} < \boldsymbol{X}_i < \boldsymbol{I}_p$, $i = s+1, \ldots, r$.

Remark 6.5
In the case of the bimatrix variate beta type I distribution the marginal pdf is that of the matrix variate beta type I (see (4.5)) and for the matrix variate Dirichlet type I distribution the marginal pdf of any subset of matrix variates is again matrix variate Dirichlet type I (see Remark 4.4). From Theorem 6.2 the marginal pdf of the bimatrix variate beta type IV distribution is the matrix variate beta type I and from Remark 6.4 the marginal pdf of the matrix variate Dirichlet type IV distribution is again matrix variate Dirichlet type IV.

Remark 6.6

The marginal and conditional pdfs of the matrix variate Dirichlet IV distribution (see Remark 6.4 and (6.14)) are not members of the Liouville family of distributions (see [2.2.1]).

6.3 Product moment of the determinants

The $(h_1, h_2)^{th}$ product moment, $E\left(|\boldsymbol{X}_1|^{h_1}|\boldsymbol{X}_2|^{h_2}\right)$, where $(\boldsymbol{X}_1, \boldsymbol{X}_2) \sim BB_p^{IV}(n_1, n_2, m)$ is derived in Theorem 6.3.

Theorem 6.3

If $(\boldsymbol{X}_1, \boldsymbol{X}_2) \sim BB_p^{IV}(n_1, n_2, m)$ *as given by* (6.2) *then*

$$E\left(|\boldsymbol{X}_1|^{h_1}|\boldsymbol{X}_2|^{h_2}\right) = \frac{\Gamma_p\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)\Gamma_p\left(\frac{m}{2}\right)}\frac{\Gamma_p\left(\frac{n_2}{2}+h_2\right)\Gamma_p\left(\frac{n_1+m}{2}\right)}{\Gamma_p\left(\frac{n_1+n_2+m}{2}+h_2\right)}\frac{\Gamma_p\left(\frac{n_1}{2}+h_1\right)\Gamma_p\left(\frac{n_2+m}{2}\right)}{\Gamma_p\left(\frac{n_1+n_2+m}{2}+h_1\right)}$$

$$\cdot\, {}_3F_2\left(\tfrac{n_2}{2}+h_2, \tfrac{n_1+n_2+m}{2}, \tfrac{n_1}{2}+h_1; \tfrac{n_1+n_2+m}{2}+h_2, \tfrac{n_1+n_2+m}{2}+h_1; \boldsymbol{I}_p\right), \tag{6.15}$$

where ${}_3F_2(\cdot)$ *is the hypergeometric function of matrix argument given in [2.6.1].*

Proof:

From (6.2) and by using [2.6.5] and [2.6.8] we get

$$E\left(|\boldsymbol{X}_1|^{h_1}|\boldsymbol{X}_2|^{h_2}\right)$$

$$= \left\{\beta_p\left(\tfrac{n_1}{2}, \tfrac{n_2}{2}; \tfrac{m}{2}\right)\right\}^{-1}\int_{\boldsymbol{0}<\boldsymbol{X}_1<\boldsymbol{I}_p}|\boldsymbol{X}_1|^{\frac{1}{2}n_1+h_1-\frac{1}{2}(p+1)}|\boldsymbol{I}_p-\boldsymbol{X}_1|^{\frac{1}{2}(n_2+m)-\frac{1}{2}(p+1)}$$

$$\cdot\left[\int_{\boldsymbol{0}<\boldsymbol{X}_2<\boldsymbol{I}_p}|\boldsymbol{X}_2|^{\frac{1}{2}n_2+h_2-\frac{1}{2}(p+1)}|\boldsymbol{I}_p-\boldsymbol{X}_2|^{\frac{1}{2}(n_1+m)-\frac{1}{2}(p+1)}|\boldsymbol{I}_p-\boldsymbol{X}_1\boldsymbol{X}_2|^{-\frac{1}{2}(n_1+n_2+m)}\mathrm{d}\boldsymbol{X}_2\right]\mathrm{d}\boldsymbol{X}_1$$

$$= \left\{\beta_p\left(\tfrac{n_1}{2}, \tfrac{n_2}{2}; \tfrac{m}{2}\right)\right\}^{-1}\frac{\Gamma_p\left(\frac{n_2}{2}+h_2\right)\Gamma_p\left(\frac{n_1+m}{2}\right)}{\Gamma_p\left(\frac{n_1+n_2+m}{2}+h_2\right)}\int_{\boldsymbol{0}<\boldsymbol{X}_1<\boldsymbol{I}_p}|\boldsymbol{X}_1|^{\frac{1}{2}n_1+h_1-\frac{1}{2}(p+1)}|\boldsymbol{I}_p-\boldsymbol{X}_1|^{\frac{1}{2}(n_2+m)-\frac{1}{2}(p+1)}$$

$$\cdot\, {}_2F_1\left(\tfrac{n_2}{2}+h_2, \tfrac{n_1+n_2+m}{2}; \tfrac{n_1+n_2+m}{2}+h_2; \boldsymbol{X}_1\right)\mathrm{d}\boldsymbol{X}_1$$

$$= \left\{\beta_p\left(\tfrac{n_1}{2}, \tfrac{n_2}{2}; \tfrac{m}{2}\right)\right\}^{-1}\frac{\Gamma_p\left(\frac{n_2}{2}+h_2\right)\Gamma_p\left(\frac{n_1+m}{2}\right)}{\Gamma_p\left(\frac{n_1+n_2+m}{2}+h_2\right)}\frac{\Gamma_p\left(\frac{n_1}{2}+h_1\right)\Gamma_p\left(\frac{n_2+m}{2}\right)}{\Gamma_p\left(\frac{n_1+n_2+m}{2}+h_1\right)}$$

$$\cdot\, {}_3F_2\left(\tfrac{n_2}{2}+h_2, \tfrac{n_1+n_2+m}{2}, \tfrac{n_1}{2}+h_1; \tfrac{n_1+n_2+m}{2}+h_2, \tfrac{n_1+n_2+m}{2}+h_1; \boldsymbol{I}_p\right). \qquad \blacksquare$$

6.4 Distribution of the product of determinants

An exact expression for the pdf of Λ_4 in (1.10) is derived in Theorem 6.4.

Theorem 6.4

Let $\boldsymbol{S}_1 \sim W_p(n_1, \boldsymbol{\Sigma})$, $\boldsymbol{S}_2 \sim W_p(n_2, \boldsymbol{\Sigma})$ *and* $\boldsymbol{B} \sim W_p(m, \boldsymbol{\Sigma})$. *The ratios in* (6.1),

$$\boldsymbol{X}_i = \left(\boldsymbol{I}_p + \boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{S}_i\boldsymbol{B}^{-\frac{1}{2}}\right)^{-\frac{1}{2}}\boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{S}_i\boldsymbol{B}^{-\frac{1}{2}}\left(\boldsymbol{I}_p + \boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{S}_i\boldsymbol{B}^{-\frac{1}{2}}\right)^{-\frac{1}{2}}, \quad i = 1, 2,$$

give $(\boldsymbol{X}_1, \boldsymbol{X}_2) \sim BB_p^{IV}(n_1, n_2, m)$. *Let* $\Lambda_4 = |\boldsymbol{X}_1\boldsymbol{X}_2| = \left|\frac{\boldsymbol{S}_1}{\boldsymbol{S}_1+\boldsymbol{B}}\right|\left|\frac{\boldsymbol{S}_2}{\boldsymbol{S}_2+\boldsymbol{B}}\right|$.

The pdf of Λ_4, *the product of two dependent Wilks' statistics, is given by*

$$\frac{\Gamma_p\left(\frac{n_1+m}{2}\right)\Gamma_p\left(\frac{n_2+m}{2}\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)\Gamma_p\left(\frac{m}{2}\right)}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{1}{k!}\Gamma_p\left(\frac{n_1+n_2+m}{2},\kappa\right)C_\kappa\left(\boldsymbol{I}_p\right)G_{2p,2p}^{2p,0}\left(\lambda_4\Big|_{b_1,\ldots,b_{2p}}^{a_1,\ldots,a_{2p}}\right), \tag{6.16}$$

$0<\lambda_4<1$, *where*

$$a_j=\begin{cases}\frac{n_1+n_2+m}{2}-1+k_{(j+1)/2}-\frac{1}{4}(j-1) & \textit{for}\ \ j=1,3,5,\ldots,2p-1\\ \frac{n_1+n_2+m}{2}-1+k_{j/2}-\frac{1}{4}(j-2) & \textit{for}\ \ j=2,4,6,\ldots,2p,\end{cases}$$

$$\textit{and}\ \ b_j=\begin{cases}\frac{n_1}{2}-1+k_{(j+1)/2}-\frac{1}{4}(j-1) & \textit{for}\ \ j=1,3,5,\ldots,2p-1\\ \frac{n_2}{2}-1+k_{j/2}-\frac{1}{4}(j-2) & \textit{for}\ \ j=2,4,6,\ldots,2p.\end{cases}$$

Proof:

From (6.15) the Mellin transform (see [2.8.1]) of $f(\lambda_4)$ is

$$\begin{aligned}M_f(h) &\equiv E\left(\Lambda_4^{h-1}\right)\\ &= E\left(\left|\boldsymbol{X}_1\boldsymbol{X}_2\right|^{h-1}\right)\\ &= \frac{\Gamma_p\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)\Gamma_p\left(\frac{m}{2}\right)}\frac{\Gamma_p\left(\frac{n_2}{2}+h-1\right)\Gamma_p\left(\frac{n_1+m}{2}\right)}{\Gamma_p\left(\frac{n_1+n_2+m}{2}+h-1\right)}\frac{\Gamma_p\left(\frac{n_1}{2}+h-1\right)\Gamma_p\left(\frac{n_2+m}{2}\right)}{\Gamma_p\left(\frac{n_1+n_2+m}{2}+h-1\right)}\\ &\quad\cdot{}_3F_2\left(\frac{n_2}{2}+h-1,\frac{n_1+n_2+m}{2},\frac{n_1}{2}+h-1;\frac{n_1+n_2+m}{2}+h-1,\frac{n_1+n_2+m}{2}+h-1;\boldsymbol{I}_p\right).\end{aligned} \tag{6.17}$$

From [2.6.1] and [2.3.3] the hypergeometric function of matrix argument in (6.17) can be written as

$$\begin{aligned}&{}_3F_2\left(\frac{n_2}{2}+h-1,\frac{n_1+n_2+m}{2},\frac{n_1}{2}+h-1;\frac{n_1+n_2+m}{2}+h-1,\frac{n_1+n_2+m}{2}+h-1;\boldsymbol{I}_p\right)\\ &=\sum_{k=0}^{\infty}\sum_{\kappa}\frac{1}{k!}C_\kappa\left(\boldsymbol{I}_p\right)\frac{\Gamma_p\left(\frac{n_2}{2}+h-1,\kappa\right)}{\Gamma_p\left(\frac{n_2}{2}+h-1\right)}\frac{\Gamma_p\left(\frac{n_1+n_2+m}{2},\kappa\right)}{\Gamma_p\left(\frac{n_1+n_2+m}{2}\right)}\frac{\Gamma_p\left(\frac{n_1}{2}+h-1,\kappa\right)}{\Gamma_p\left(\frac{n_1}{2}+h-1\right)}\frac{\Gamma_p\left(\frac{n_1+n_2+m}{2}+h-1\right)}{\Gamma_p\left(\frac{n_1+n_2+m}{2}+h-1,\kappa\right)}\frac{\Gamma_p\left(\frac{n_1+n_2+m}{2}+h-1\right)}{\Gamma_p\left(\frac{n_1+n_2+m}{2}+h-1,\kappa\right)}.\end{aligned}$$

Substituting this in (6.17) gives

$$M_f(h)\equiv\frac{\Gamma_p\left(\frac{n_1+m}{2}\right)\Gamma_p\left(\frac{n_2+m}{2}\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)\Gamma_p\left(\frac{m}{2}\right)}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{1}{k!}\frac{\Gamma_p\left(\frac{n_2}{2}+h-1,\kappa\right)\Gamma_p\left(\frac{n_1+n_2+m}{2},\kappa\right)\Gamma_p\left(\frac{n_1}{2}+h-1,\kappa\right)}{\Gamma_p\left(\frac{n_1+n_2+m}{2}+h-1,\kappa\right)\Gamma_p\left(\frac{n_1+n_2+m}{2}+h-1,\kappa\right)}C_\kappa\left(\boldsymbol{I}_p\right). \tag{6.18}$$

From [2.3.3] the generalised gamma functions of weight κ in (6.18) can be written as

$$\begin{aligned}&\Gamma_p\left(\frac{n_1+n_2+m}{2}+h-1,\kappa\right)\Gamma_p\left(\frac{n_1+n_2+m}{2}+h-1,\kappa\right)\\ &=\pi^{\frac{1}{4}p(p-1)}\prod_{j=1}^{p}\Gamma\left[\frac{n_1+n_2+m}{2}+h-1+k_j-\frac{1}{2}(j-1)\right]\pi^{\frac{1}{4}p(p-1)}\prod_{j=1}^{p}\Gamma\left[\frac{n_1+n_2+m}{2}+h-1+k_j-\frac{1}{2}(j-1)\right]\\ &=\pi^{\frac{1}{2}p(p-1)}\prod_{j=1}^{2p}\Gamma\left(a_j+h\right),\end{aligned} \tag{6.19}$$

$$\text{where}\ \ a_j=\begin{cases}\frac{n_1+n_2+m}{2}-1+k_{(j+1)/2}-\frac{1}{4}(j-1) & \text{for}\ \ j=1,3,5,\ldots,2p-1\\ \frac{n_1+n_2+m}{2}-1+k_{j/2}-\frac{1}{4}(j-2) & \text{for}\ \ j=2,4,6,\ldots,2p.\end{cases}$$

Also from [2.3.3],

$$\Gamma_p\left(\tfrac{n_1}{2}+h-1,\kappa\right)\Gamma_p\left(\tfrac{n_2}{2}+h-1,\kappa\right)$$
$$=\pi^{\frac{1}{4}p(p-1)}\prod_{j=1}^{p}\Gamma\left[\tfrac{n_1}{2}+h-1+k_j-\tfrac{1}{2}(j-1)\right]\pi^{\frac{1}{4}p(p-1)}\prod_{j=1}^{p}\Gamma\left[\tfrac{n_2}{2}+h-1+k_j-\tfrac{1}{2}(j-1)\right]$$
$$=\pi^{\frac{1}{2}p(p-1)}\prod_{j=1}^{2p}\Gamma\left(b_j+h\right) \tag{6.20}$$

where $b_j=\begin{cases}\frac{n_1}{2}-1+k_{(j+1)/2}-\frac{1}{4}(j-1) & \text{for } j=1,3,5,\ldots,2p-1\\ \frac{n_2}{2}-1+k_{j/2}-\frac{1}{4}(j-2) & \text{for } j=2,4,6,\ldots,2p.\end{cases}$

Substituting (6.19) and (6.20) in (6.18) gives

$$M_f\left(h\right)\equiv\frac{\Gamma_p\left(\frac{n_1+m}{2}\right)\Gamma_p\left(\frac{n_2+m}{2}\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)\Gamma_p\left(\frac{m}{2}\right)}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{1}{k!}C_\kappa\left(\boldsymbol{I}_p\right)\Gamma_p\left(\frac{n_1+n_2+m}{2},\kappa\right)\frac{\prod_{j=1}^{2p}\Gamma(b_j+h)}{\prod_{j=1}^{2p}\Gamma(a_j+h)}. \tag{6.21}$$

The pdf of Λ_4 is obtained from the inverse Mellin transform of (6.21) (see [2.8.1]) as

$$f\left(\lambda_4\right)=\frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty}M_f\left(h\right)\lambda_4^{-h}\mathrm{d}h$$
$$=\frac{\Gamma_p\left(\frac{n_1+m}{2}\right)\Gamma_p\left(\frac{n_2+m}{2}\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)\Gamma_p\left(\frac{m}{2}\right)}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{1}{k!}\Gamma_p\left(\frac{n_1+n_2+m}{2},\kappa\right)C_\kappa\left(\boldsymbol{I}_p\right)\left[\frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty}\frac{\prod_{j=1}^{2p}\Gamma(b_j+h)}{\prod_{j=1}^{2p}\Gamma(a_j+h)}\lambda_4^{-h}\mathrm{d}h\right]. \tag{6.22}$$

From the definition of Meijer's G-function (see [2.8.2]), (6.22) can be written as

$$f\left(\lambda_4\right)=\frac{\Gamma_p\left(\frac{n_1+m}{2}\right)\Gamma_p\left(\frac{n_2+m}{2}\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)\Gamma_p\left(\frac{m}{2}\right)}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{1}{k!}\Gamma_p\left(\frac{n_1+n_2+m}{2},\kappa\right)C_\kappa\left(\boldsymbol{I}_p\right)G_{2p,2p}^{2p,0}\left(\lambda_4\middle|_{b_1,\ldots,b_{2p}}^{a_1,\ldots,a_{2p}}\right). \qquad \blacksquare$$

6.5 Role of the parameters

In this section we study the effect of the parameters n_1, n_2 and m.

Firstly, we consider the bivariate case, $p=1$, to illustrate the effect of the parameters n_1, n_2 and m on

- (i) the form of the pdf of (X_1, X_2);
- (ii) the correlation between X_1 and X_2;
- (iii) the graphs of $E(X_2|x_1)$ and $var(X_2|x_1)$ plotted against x_1;
- (iv) the form of the pdf of Λ_4.

From the result in (6.2) the joint pdf of X_1 and X_2 is

$$f\left(x_1,x_2\right)=\frac{\Gamma\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{m}{2}\right)}x_1^{\frac{1}{2}n_1-1}x_2^{\frac{1}{2}n_2-1}\cdot\left(1-x_1\right)^{\frac{1}{2}(n_2+m)-1}\left(1-x_2\right)^{\frac{1}{2}(n_1+m)-1}\left(1-x_1x_2\right)^{-\frac{1}{2}(n_1+n_2+m)},\quad 0<x_1,x_2<1. \tag{6.23}$$

Libby and Novick (1982) and Chen and Novick (1984) proposed the multivariate case of the distribution given by (6.23) and called it the multivariate generalised beta distribution. They also presented applications in utility modelling and Bayesian analysis respectively. Libby and Novick (1982) considered ratios of $r+1$ independent gamma random variates with parameters α_i and β_i, $i = 0, \dots, r$, respectively. The bivariate distribution given by (6.23) is obtained from ratios of three independent chi-square or standard gamma random variates (see (6.1) for $p = 1$) and was also derived and studied by Jones (2001), Olkin and Liu (2003) and Nagar et al. (2009).

Figures 6.1a and 6.1b show graphs of the pdfs of $BB_1^{IV}(8, n_2, 8)$ and $BB_1^{IV}(8, 8, m)$ distributions respectively (see (6.23)). As n_2 increases, $f(x_1, x_2)$ shifts towards larger values of X_2. The effect of the parameter n_1 will be similar to that of n_2 but with $f(x_1, x_2)$ shifting towards larger values of X_1 as n_1 increases. As m increases (see Figure 6.1b) the joint pdf shifts towards smaller values of both X_1 and X_2.

Figure 6.1a: Effect of n_2 on $f(x_1, x_2)$, $(X_1, X_2) \sim BB_1^{IV}(8, n_2, 8)$

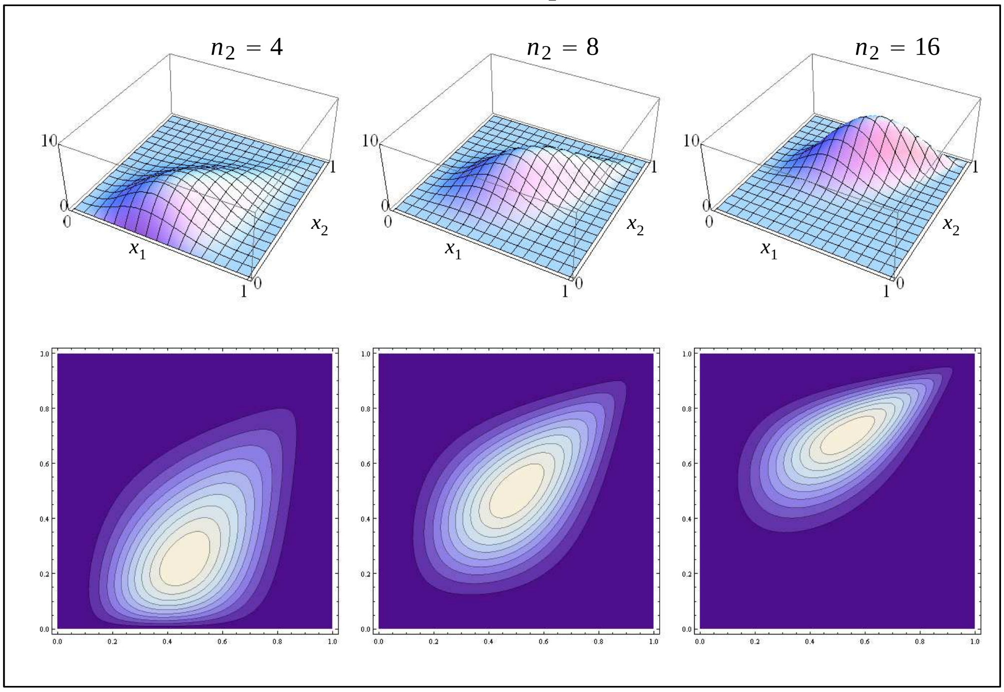
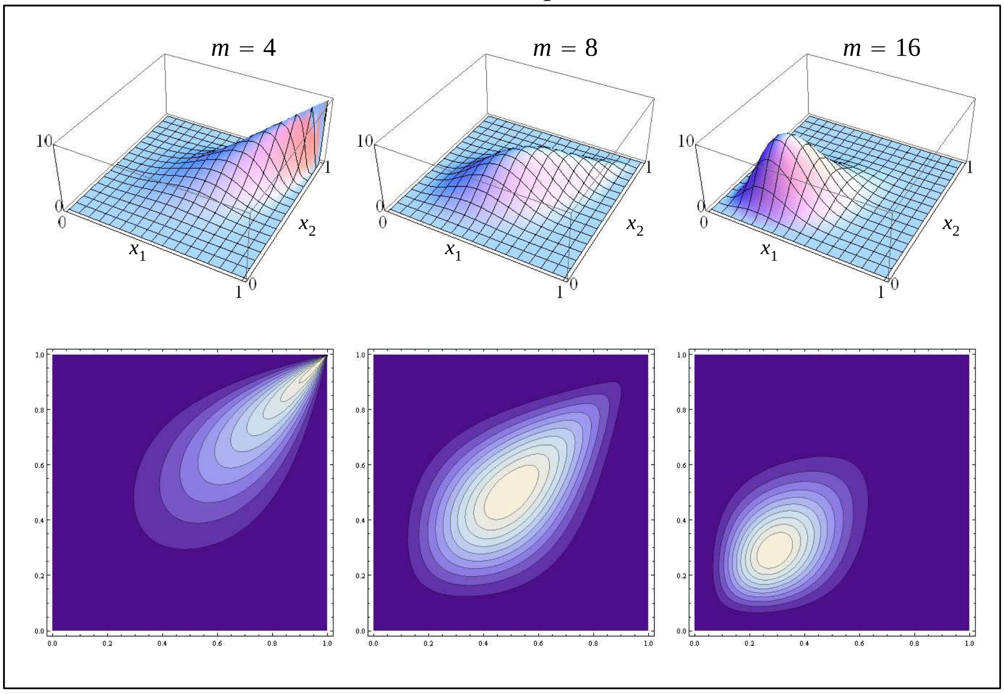


Figure 6.1b: Effect of m on $f(x_1, x_2)$, $(X_1, X_2) \sim BB_1^{IV}(8, 8, m)$



From (6.15) the $(h_1, h_2)^{th}$ product moment, $E\left(X_1^{h_1} X_2^{h_2}\right)$, associated with (6.23) is given by

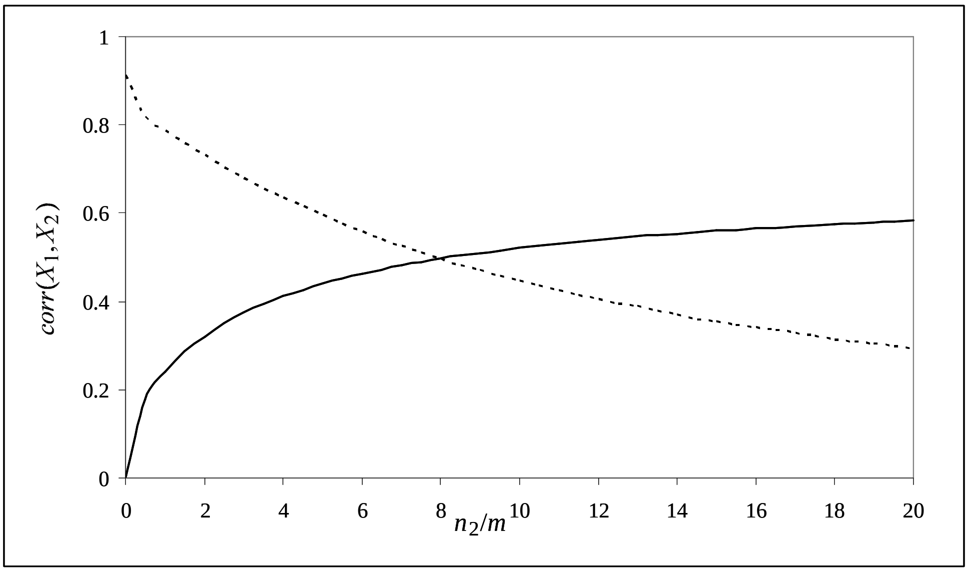
$$\begin{aligned} E\left(X_1^{h_1} X_2^{h_2}\right) &= \frac{\Gamma\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{m}{2}\right)} \frac{\Gamma\left(\frac{n_2}{2}+h_2\right)\Gamma\left(\frac{n_1+m}{2}\right)}{\Gamma\left(\frac{n_1+n_2+m}{2}+h_2\right)} \frac{\Gamma\left(\frac{n_1}{2}+h_1\right)\Gamma\left(\frac{n_2+m}{2}\right)}{\Gamma\left(\frac{n_1+n_2+m}{2}+h_1\right)} \\ &\quad \cdot {}_3F_2\left(\frac{n_2}{2}+h_2, \frac{n_1+n_2+m}{2}, \frac{n_1}{2}+h_1; \frac{n_1+n_2+m}{2}+h_2, \frac{n_1+n_2+m}{2}+h_1; 1\right). \end{aligned} \tag{6.24}$$

The result in (6.24) was used to calculate the correlation between X_1 and X_2 and Figure 6.2 shows graphs of $corr\left(X_1, X_2\right)$ for increasing values of n_2 and m. The correlation is positive and increases for increasing values of n_2 or decreasing values of m.

Figure 6.2: Effect of n_2 and m on $corr(X_1, X_2)$

(i) $(X_1, X_2) \sim BB_1^{IV}(8, n_2, 8)$ ———

(ii) $(X_1, X_2) \sim BB_1^{IV}(8, 8, m)$ - - - - - -



From (6.13), the marginal pdf of X_1 for $p = 1$ is

$$f\left(x_1\right) = \frac{\Gamma\left(\frac{n_1+m}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{m}{2}\right)} x_1^{\frac{1}{2}n_1-1}\left(1-x_1\right)^{\frac{1}{2}m-1}, \quad 0 < x_1 < 1,$$

that is $X_1 \sim B_1^I\left(n_1, m\right)$.

For the bivariate case, the conditional pdf of $\left(X_2 | X_1 = x_1\right)$ given by (6.12) simplifies to

$$f\left(x_2 | x_1\right) = \frac{\Gamma\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{n_1+m}{2}\right)}\left(1-x_1\right)^{\frac{1}{2}n_2} x_2^{\frac{1}{2}n_2-1}\left(1-x_2\right)^{\frac{1}{2}\left(n_1+m\right)-1}\left(1-x_1 x_2\right)^{-\frac{1}{2}\left(n_1+n_2+m\right)}, \quad 0 < x_2 < 1. \tag{6.25}$$

The h^{th} moment, $E\left(X_2^h | X_1 = x_1\right)$, associated with (6.25) is derived in the next theorem.

Theorem 6.5

If $(X_1, X_2) \sim BB_1^{IV}(n_1, n_2, m)$ *then*

$$E\left(X_2^h | X_1 = x_1\right) = \frac{\Gamma\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma\left(\frac{n_2}{2}\right)} \frac{\Gamma\left(\frac{n_2}{2}+h\right)}{\Gamma\left(\frac{n_1+n_2+m}{2}+h\right)} (1-x_1)^{\frac{1}{2}n_2} \;{}_2F_1\left(\frac{n_2}{2}+h, \frac{n_1+n_2+m}{2}; \frac{n_1+n_2+m}{2}+h; x_1\right). \quad (6.26)$$

Proof:

From (6.25) and by using [2.5.3] it follows that

$$\begin{aligned} & E\left(X_2^h | X_1 = x_1\right) \\ = \; & \frac{\Gamma\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{n_1+m}{2}\right)} (1-x_1)^{\frac{1}{2}n_2} \int_0^1 x_2^{\frac{1}{2}n_2+h-1} (1-x_2)^{\frac{1}{2}(n_1+m)-1} (1-x_1x_2)^{-\frac{1}{2}(n_1+n_2+m)} \mathrm{d}x_2 \\ = \; & \frac{\Gamma\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{n_1+m}{2}\right)} \frac{\Gamma\left(\frac{n_2}{2}+h\right)\Gamma\left(\frac{n_1+m}{2}\right)}{\Gamma\left(\frac{n_1+n_2+m}{2}+h\right)} (1-x_1)^{\frac{1}{2}n_2} \;{}_2F_1\left(\frac{n_2}{2}+h, \frac{n_1+n_2+m}{2}; \frac{n_1+n_2+m}{2}+h; x_1\right). \quad \blacksquare \end{aligned}$$

In Figures 6.3a and 6.3b graphs of $E(X_2|x_1)$ (see (6.26)) show the underlying structure of (6.25) for increasing values of n_2 and m respectively. For given values of the parameters, the relationship between $E(X_2|x_1)$ and x_1 is curved and increases as x_1 increases. For a given value x_1 of X_1, $E(X_2|x_1)$ increases as n_2 increases or as m decreases. This corresponds to what is observed in Figures 6.1a and 6.1b. From (6.26) the parameters n_1 and m have the same effect on $E(X_2|x_1)$.

Figure 6.3a: Effect of n_2 on $E(X_2|x_1)$, $(X_1, X_2) \sim BB_1^{IV}(8, n_2, 8)$

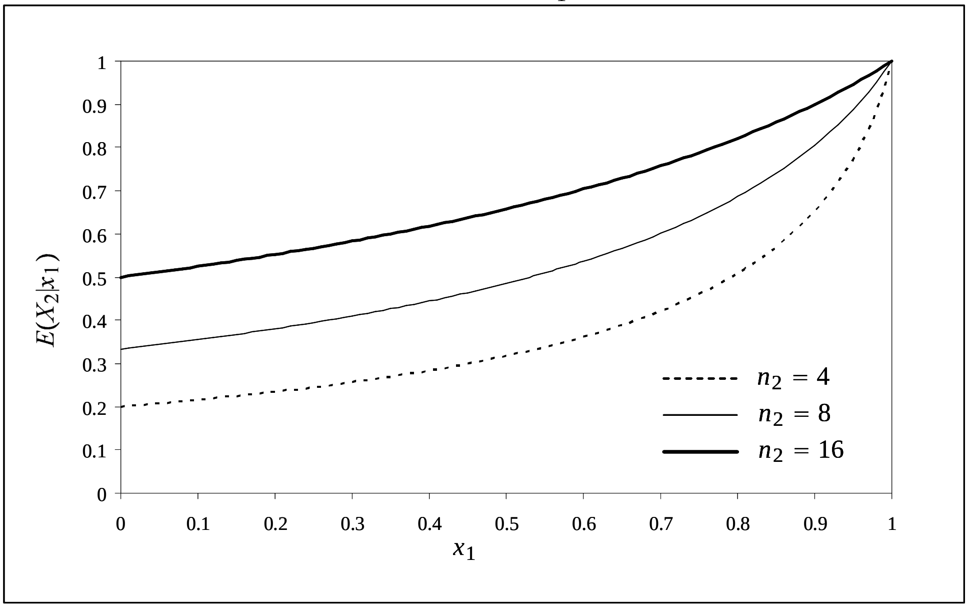
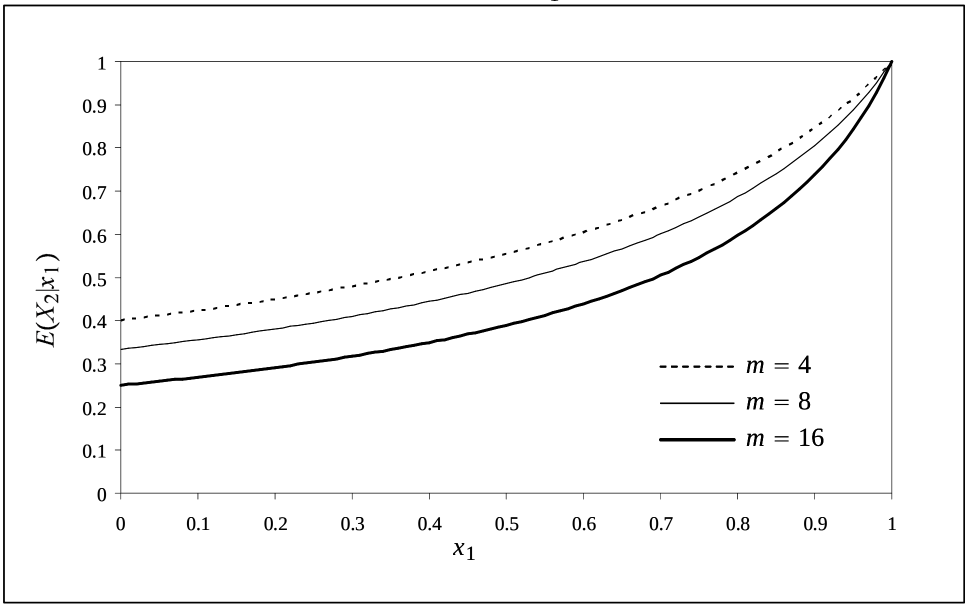


Figure 6.3b: Effect of m on $E(X_2|x_1)$, $(X_1, X_2) \sim BB_1^{IV}(8, 8, m)$



Figures 6.4a and 6.4b show graphs of $var\,(X_2|x_1)$ plotted against x_1 for different values of n_2 and m. At larger values of x_1, $var\,(X_2|x_1)$ increases as n_2 decreases. This is also observed at smaller values of x_1 as m decreases.

Figure 6.4a: Effect of n_2 on $var(X_2|x_1)$, $(X_1, X_2) \sim BB_1^{IV}(8, n_2, 8)$

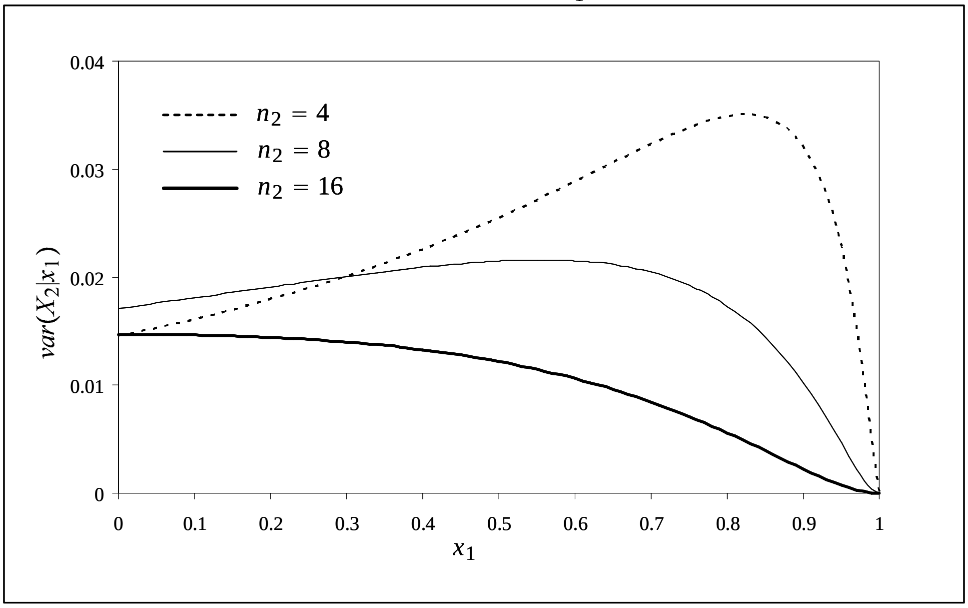
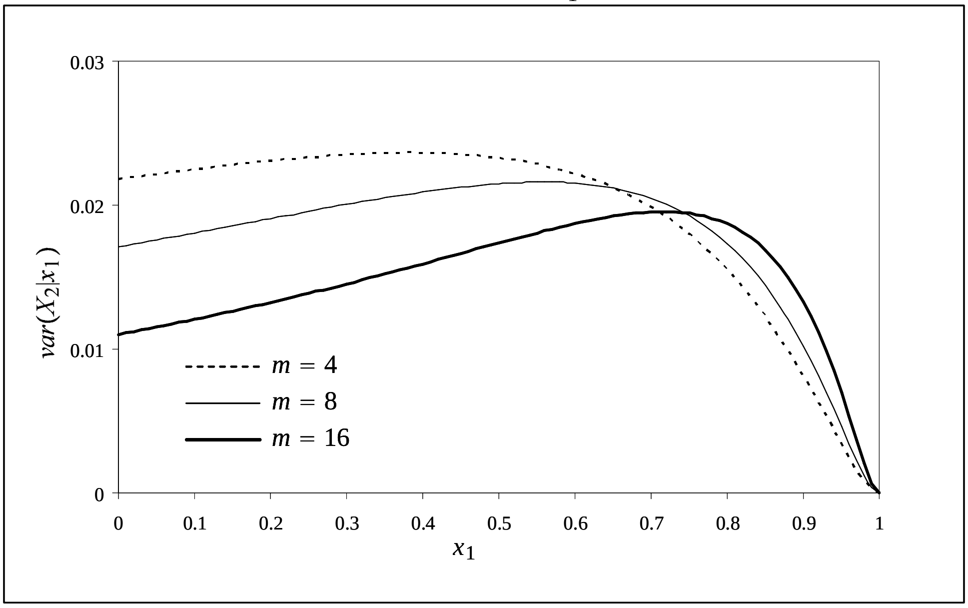


Figure 6.4b: Effect of m on $var(X_2|x_1)$, $(X_1,X_2) \sim BB_1^{IV}(8,8,m)$



Finally, for the bivariate case, the effect of the parameters n_1, n_2 and m on the pdf of $\Lambda_4 = X_1X_2$ was studied where $(X_1,X_2) \sim BB_1^{IV}(n_1,n_2,m)$. Considering the result in (6.16) for $p=1$, it follows from [2.3.3] that $\Gamma_p\left(\frac{n_1+n_2+m}{2},\kappa\right) = \Gamma\left(\frac{n_1+n_2+m}{2}+k\right)$ and from [2.3.1] that $\sum\limits_{\kappa} C_\kappa\left(\boldsymbol{I}_p\right) = 1$. Therefore

$$f\left(\lambda_4\right) = \frac{\Gamma\left(\frac{n_1+m}{2}\right)\Gamma\left(\frac{n_2+m}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{m}{2}\right)} \sum_{k=0}^{\infty} \frac{1}{k!}\Gamma\left(\frac{n_1+n_2+m}{2}+k\right) G_{2,2}^{2,0}\left(\lambda_4 \middle| \begin{matrix} \frac{n_1+n_2+m}{2}-1+k, \frac{n_1+n_2+m}{2}-1+k \\ \frac{n_1}{2}-1+k, \frac{n_2}{2}-1+k \end{matrix}\right), \tag{6.27}$$

$0 < \lambda_4 < 1$.

From [2.8.4] and [2.8.5] the G-function in (6.27) can be written as,

$$\begin{aligned} G_{2,2}^{2,0}\left(\lambda_4 \middle| \begin{matrix} \frac{n_1+n_2+m}{2}-1+k, \frac{n_1+n_2+m}{2}-1+k \\ \frac{n_1}{2}-1+k, \frac{n_2}{2}-1+k \end{matrix}\right) &= \lambda_4^k G_{2,2}^{2,0}\left(\lambda_4 \middle| \begin{matrix} \frac{n_1+n_2+m}{2}-1, \frac{n_1+n_2+m}{2}-1 \\ \frac{n_1}{2}-1, \frac{n_2}{2}-1 \end{matrix}\right) \\ &= \lambda_4^k \frac{\lambda_4^{\frac{1}{2}n_2-1}\left(1-\lambda_4\right)^{\frac{1}{2}\left(n_1+n_2+2m\right)-1}}{\Gamma\left(\frac{n_1+n_2+2m}{2}\right)} {}_2F_1\left(\frac{n_2+m}{2}, \frac{n_2+m}{2}; \frac{n_1+n_2+2m}{2}; 1-\lambda_4\right) . \end{aligned}$$

Substituting this in (6.27) and using [2.5.2] gives

$$\begin{aligned} & f\left(\lambda_4\right) \\ = \; & \frac{\beta\left(\frac{n_1+m}{2},\frac{n_2+m}{2}\right)}{\beta\left(\frac{n_1}{2},\frac{n_2}{2},\frac{m}{2}\right)} \lambda_4^{\frac{1}{2}n_2-1}\left(1-\lambda_4\right)^{\frac{1}{2}\left(n_1+n_2+2m\right)-1} \sum_{k=0}^{\infty} \frac{\lambda_4^k}{k!} \frac{\Gamma\left(\frac{n_1+n_2+m}{2}+k\right)}{\Gamma\left(\frac{n_1+n_2+m}{2}\right)} {}_2F_1\left(\frac{n_2+m}{2}, \frac{n_2+m}{2}; \frac{n_1+n_2+2m}{2}; 1-\lambda_4\right) \\ = \; & \frac{\beta\left(\frac{n_1+m}{2},\frac{n_2+m}{2}\right)}{\beta\left(\frac{n_1}{2},\frac{n_2}{2},\frac{m}{2}\right)} \lambda_4^{\frac{1}{2}n_2-1}\left(1-\lambda_4\right)^{\frac{1}{2}\left(n_1+n_2+2m\right)-1} {}_2F_1\left(\frac{n_2+m}{2}, \frac{n_2+m}{2}; \frac{n_1+n_2+2m}{2}; 1-\lambda_4\right) {}_1F_0\left(\frac{n_1+n_2+m}{2}; \lambda_4\right) \\ = \; & \frac{\beta\left(\frac{n_1+m}{2},\frac{n_2+m}{2}\right)}{\beta\left(\frac{n_1}{2},\frac{n_2}{2},\frac{m}{2}\right)} \lambda_4^{\frac{1}{2}n_2-1}\left(1-\lambda_4\right)^{\frac{1}{2}\left(n_1+n_2+2m\right)-1} {}_2F_1\left(\frac{n_2+m}{2}, \frac{n_2+m}{2}; \frac{n_1+n_2+2m}{2}; 1-\lambda_4\right) \left(1-\lambda_4\right)^{-\frac{1}{2}\left(n_1+n_2+m\right)} \\ = \; & \frac{\beta\left(\frac{n_1+m}{2},\frac{n_2+m}{2}\right)}{\beta\left(\frac{n_1}{2},\frac{n_2}{2},\frac{m}{2}\right)} \lambda_4^{\frac{1}{2}n_2-1}\left(1-\lambda_4\right)^{\frac{1}{2}m-1} {}_2F_1\left(\frac{n_2+m}{2}, \frac{n_2+m}{2}; \frac{n_1+n_2+2m}{2}; 1-\lambda_4\right). \end{aligned}$$

This is the pdf of Λ_4 written in terms of the Gauss hypergeometric function (see Nagar et al., 2009).

Figures 6.5a and 6.5b show the effect of n_2 and m on $f(\lambda_4)$ given by (6.27) where $(X_1, X_2) \sim BB_1^{IV}(8, n_2, 8)$ and $(X_1, X_2) \sim BB_1^{I}(8, 8, m)$ respectively. As n_2 increases the pdf shifts towards larger values of Λ_4. The opposite is observed as m increases.

Figure 6.5a: Effect of n_2 on $f(\lambda_4)$, $\Lambda_4 = X_1X_2$, $(X_1,X_2) \sim BB_1^{IV}(8,n_2,8)$

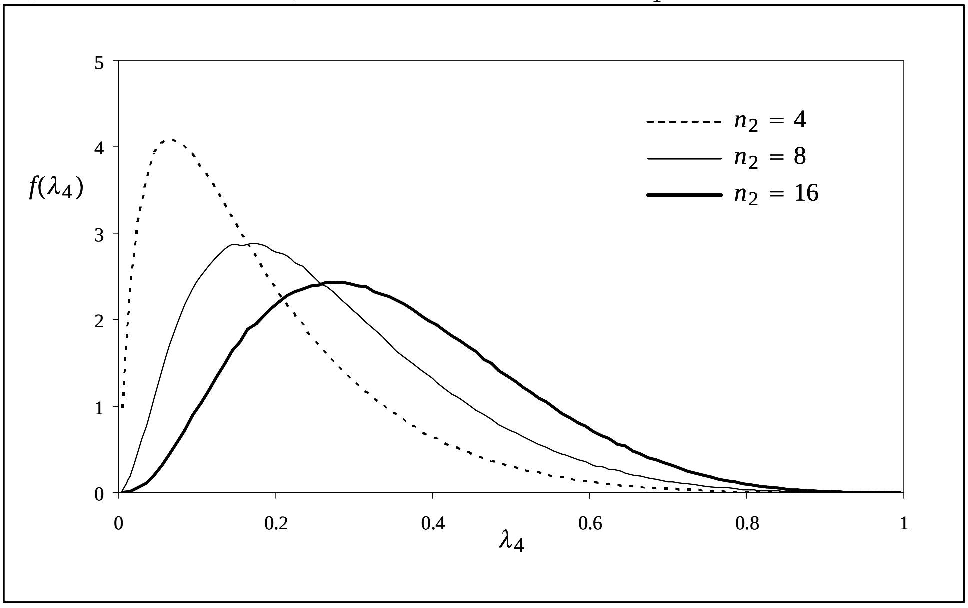
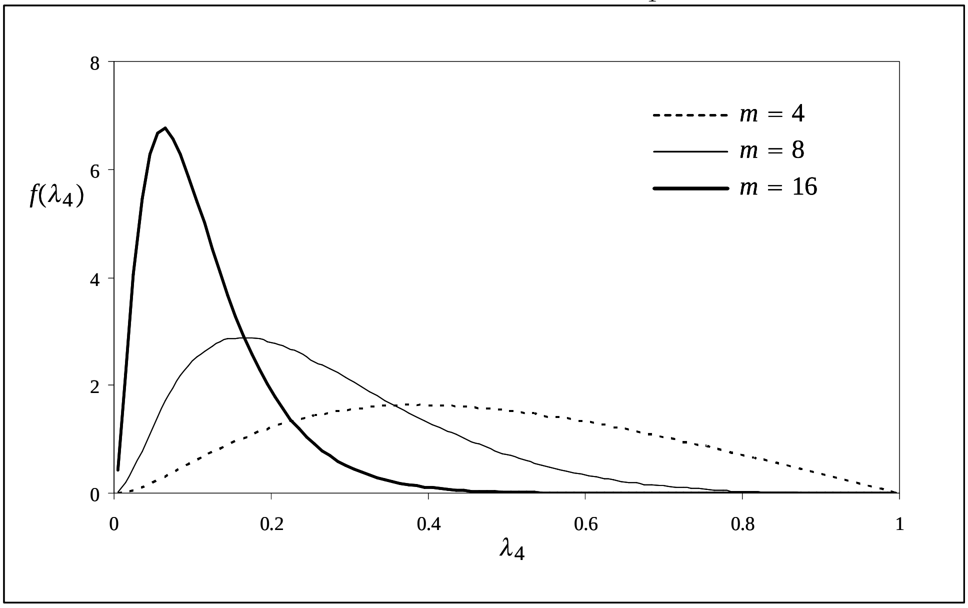


Figure 6.5b: Effect of m on $f(\lambda_4)$, $\Lambda_4 = X_1X_2$, $(X_1,X_2) \sim BB_1^{IV}(8,8,m)$



Secondly, we consider the bimatrix case where $p = 2$ to illustrate the effect of n_1, n_2 and m on the form of the pdf of Λ_4. From (6.16), the pdf of Λ_4 for $p = 2$ simplifies to

$$f(\lambda_4) = \frac{\Gamma_2\left(\frac{n_1+m}{2}\right)\Gamma_2\left(\frac{n_2+m}{2}\right)}{\Gamma_2\left(\frac{n_1}{2}\right)\Gamma_2\left(\frac{n_2}{2}\right)\Gamma_2\left(\frac{m}{2}\right)} \sum_{k=0}^{\infty}\sum_{\kappa}\frac{1}{k!}\Gamma_2\left(\frac{n_1+n_2+m}{2},\kappa\right)C_\kappa(\boldsymbol{I}_2)\, G_{4,4}^{4,0}\left(\lambda_4\middle|_{b_1,b_2,b_3,b_4}^{a_1,a_2,a_3,a_4}\right), \quad 0 < \lambda_4 < 1,$$

where

$a_1 = \frac{n_1+n_2+m}{2} - 1 + k_1$, $a_2 = \frac{n_1+n_2+m}{2} - 1 + k_1$, $a_3 = \frac{n_1+n_2+m}{2} + k_2 - \frac{3}{2}$ and $a_4 = \frac{n_1+n_2+m}{2} + k_2 - \frac{3}{2}$;
$b_1 = \frac{n_1}{2} - 1 + k_1$, $b_2 = \frac{n_2}{2} - 1 + k_1$, $b_3 = \frac{n_1}{2} + k_2 - \frac{3}{2}$ and $b_4 = \frac{n_2}{2} + k_2 - \frac{3}{2}$.

Figures 6.6a and Figure 6.6b illustrate the shape of the pdf, $f(\lambda_4)$, for increasing values of n_2 and m. We note that as n_2 increases the pdf shifts towards larger values of Λ_4. The opposite is observed as m increases.

Figure 6.6a: Effect of n_2 on $f(\lambda_4)$, $\Lambda_4 = |\boldsymbol{X}_1 \boldsymbol{X}_2|$, $(\boldsymbol{X}_1, \boldsymbol{X}_2) \sim BB_2^{IV}(8, n_2, 8)$

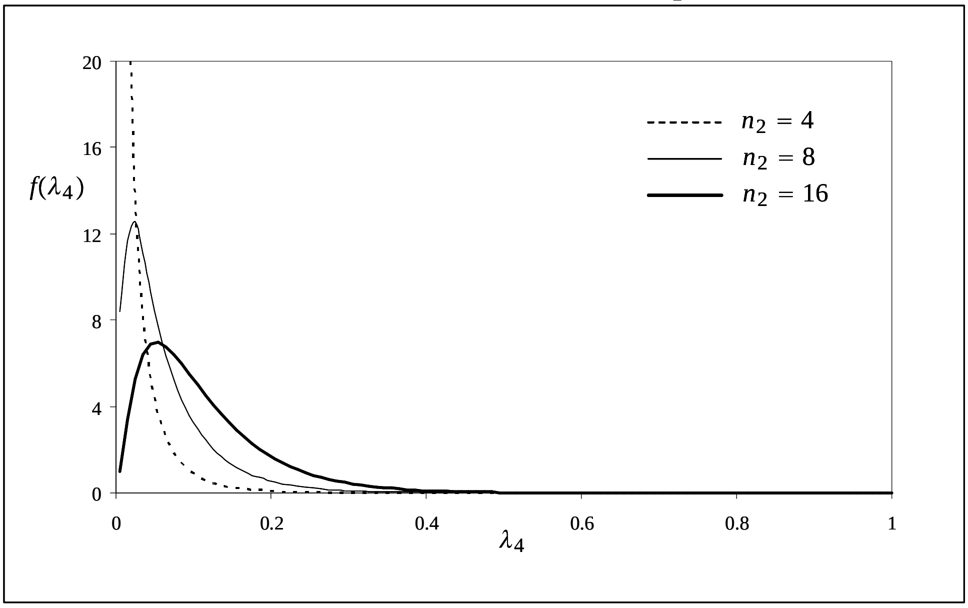
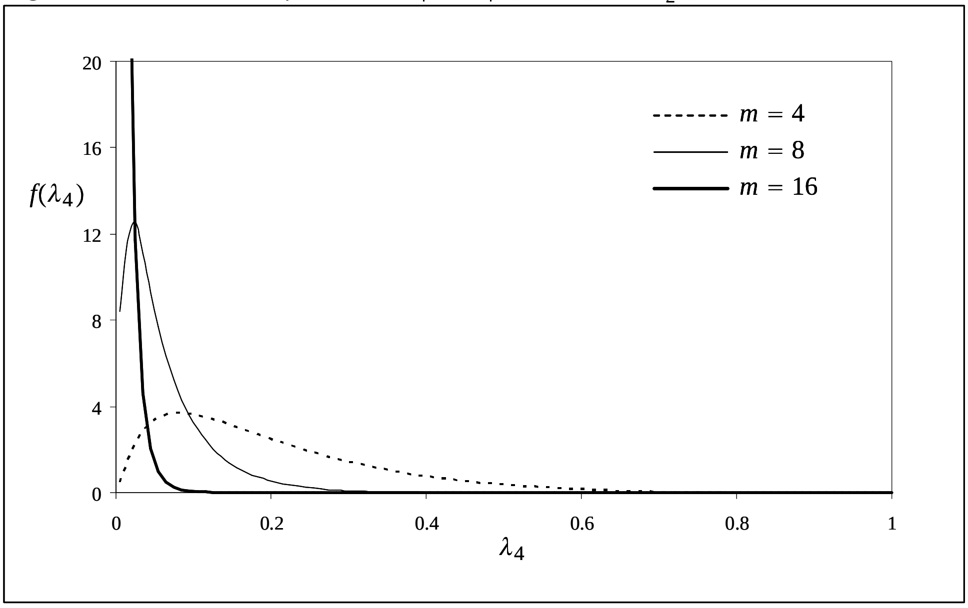


Figure 6.6b: Effect of m on $f(\lambda_4)$, $\Lambda_4 = |\boldsymbol{X}_1 \boldsymbol{X}_2|$, $(\boldsymbol{X}_1, \boldsymbol{X}_2) \sim BB_2^{IV}(8, 8, m)$



7 Bimatrix variate beta type V distribution

By introducing the additional parameters α_1 and α_2 in the ratios (1.3) we derive the pdf of (1.5) and call it the bimatrix variate beta type V distribution. The marginal and conditional pdfs are derived in Theorem 7.2. Theorem 7.3 derives the product moment of the determinants and the latter is then used in Theorem 7.4 to derive an exact expression for the pdf of $\Lambda_5 = |\boldsymbol{Q}_1|^{\frac{1}{2}n_1} |\boldsymbol{Q}_2|^{\frac{1}{2}n_2}$ (see (1.9)). The role of the additional parameters α_1 and α_2 is studied in Section 7.5.

7.1 Probability density function

The pdf of the bimatrix variate beta type V distribution is derived from Wishart ratios in Theorem 7.1.

Theorem 7.1
Let $\boldsymbol{S}_1 \sim W_p(n_1, \boldsymbol{\Sigma})$, $\boldsymbol{S}_2 \sim W_p(n_2, \boldsymbol{\Sigma})$ *and* $\boldsymbol{B} \sim W_p(m, \boldsymbol{\Sigma})$ *be independently distributed. Consider the definition in* (1.5), *that is*

$$\boldsymbol{Q}_i = \boldsymbol{S}^{-\frac{1}{2}} (\alpha_i \boldsymbol{S}_i) \boldsymbol{S}^{-\frac{1}{2}},\ i = 1, 2, \tag{7.1}$$

where $\boldsymbol{S} = \alpha_1 \boldsymbol{S}_1 + \alpha_2 \boldsymbol{S}_2 + c\boldsymbol{B}$, $\alpha_i > 0$, $c > 0$ *and* $\boldsymbol{S} = \boldsymbol{S}^{\frac{1}{2}} \boldsymbol{S}^{\frac{1}{2}}$.

The pdf of $(\boldsymbol{Q}_1, \boldsymbol{Q}_2) \sim BB_p^V(n_1, n_2, m, \alpha_1, \alpha_2, c)$ *is*

$$\begin{aligned} & f(\boldsymbol{Q}_1, \boldsymbol{Q}_2) \\ = \;& \left\{\beta_p\left(\tfrac{n_1}{2}, \tfrac{n_2}{2}; \tfrac{m}{2}\right)\right\}^{-1} \prod_{i=1}^{2} |\boldsymbol{Q}_i|^{\frac{1}{2}n_i - \frac{1}{2}(p+1)} \left| \boldsymbol{I}_p - \sum_{i=1}^{2} \boldsymbol{Q}_i \right|^{\frac{1}{2}m - \frac{1}{2}(p+1)} \prod_{i=1}^{2} \left(\tfrac{c}{\alpha_i}\right)^{\frac{1}{2}n_i p} \left| \boldsymbol{I}_p + \sum_{i=1}^{2} \tfrac{c - \alpha_i}{\alpha_i} \boldsymbol{Q}_i \right|^{-\frac{1}{2}(n_1+n_2+m)} \\ = \;& g(\boldsymbol{Q}_1, \boldsymbol{Q}_2) \prod_{i=1}^{2} \left(\tfrac{c}{\alpha_i}\right)^{\frac{1}{2}n_i p} \left| \boldsymbol{I}_p + \sum_{i=1}^{2} \tfrac{c - \alpha_i}{\alpha_i} \boldsymbol{Q}_i \right|^{-\frac{1}{2}(n_1+n_2+m)}, \end{aligned} \tag{7.2}$$

$\boldsymbol{0} < \boldsymbol{Q}_i < \boldsymbol{I}_p$, $i = 1, 2$, $\boldsymbol{0} < \sum_{i=1}^{2} \boldsymbol{Q}_i < \boldsymbol{I}_p$, *where* $n_i > (p-1)$, $i = 1, 2$, $m > (p-1)$ *and* $g(\cdot)$ *is the pdf of* $BB_p^I(n_1, n_2, m)$ *given by* (4.2).

Proof:
The pdf of $(\boldsymbol{S}_1, \boldsymbol{S}_2, \boldsymbol{B})$ is given by

$$K \prod_{i=1}^{2} \left[\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}_i\right) |\boldsymbol{S}_i|^{\frac{1}{2}(n_i-p-1)}\right] \left[\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\right) |\boldsymbol{B}|^{\frac{1}{2}(m-p-1)}\right] \tag{7.3}$$

where $K^{-1} = \Gamma_p\left(\frac{n_1}{2}\right) \Gamma_p\left(\frac{n_2}{2}\right) \Gamma_p\left(\frac{m}{2}\right) |2\boldsymbol{\Sigma}|^{\frac{1}{2}(n_1+n_2+m)}$ (see [2.10.1]).

Making the transformations $\boldsymbol{Y}_i = \alpha_i \boldsymbol{S}_i$, $i = 1, 2$, and $\boldsymbol{Z} = c\boldsymbol{B}$ with Jacobian $J(\boldsymbol{S}_1, \boldsymbol{S}_2, \boldsymbol{B} \to \boldsymbol{Y}_1, \boldsymbol{Y}_2, \boldsymbol{Z}) = \prod_{i=1}^{2} J(\boldsymbol{S}_i \to \boldsymbol{Y}_i) J(\boldsymbol{B} \to \boldsymbol{Z}) = \left(\prod_{i=1}^{2} \alpha_i \cdot c\right)^{-\frac{1}{2}p(p+1)}$ (see [2.1.4] and [2.1.5]) and substituting in (7.3) gives the pdf of $(\boldsymbol{Y}_1, \boldsymbol{Y}_2, \boldsymbol{Z})$ as

$$\begin{aligned} & f(\boldsymbol{Y}_1, \boldsymbol{Y}_2, \boldsymbol{Z}) \\ = \;& K\mathrm{etr}\left(-\sum_{i=1}^{2}\tfrac{1}{2\alpha_i}\boldsymbol{\Sigma}^{-1}\boldsymbol{Y}_i\right)\prod_{i=1}^{2}\left|\tfrac{1}{\alpha_i}\boldsymbol{Y}_i\right|^{\frac{1}{2}n_i-\frac{1}{2}(p+1)}\mathrm{etr}\left(-\tfrac{1}{2c}\boldsymbol{\Sigma}^{-1}\boldsymbol{Z}\right)\left|\tfrac{1}{c}\boldsymbol{Z}\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)}\left(c\prod_{i=1}^{2}\alpha_i\right)^{-\frac{1}{2}p(p+1)} \\ = \;& K\prod_{i=1}^{2}\alpha_i^{-\frac{1}{2}n_ip}c^{-\frac{1}{2}mp}\prod_{i=1}^{2}|\boldsymbol{Y}_i|^{\frac{1}{2}n_i-\frac{1}{2}(p+1)}|\boldsymbol{Z}|^{\frac{1}{2}m-\frac{1}{2}(p+1)}\mathrm{etr}\left[-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\left(\sum_{i=1}^{2}\tfrac{1}{\alpha_i}\boldsymbol{Y}_i+\tfrac{1}{c}\boldsymbol{Z}\right)\right]. \end{aligned} \tag{7.4}$$

Next, consider the transformations $\boldsymbol{S} = \boldsymbol{Y}_1 + \boldsymbol{Y}_2 + \boldsymbol{Z}$ and $\boldsymbol{Q}_i = \boldsymbol{S}^{-\frac{1}{2}}\boldsymbol{Y}_i\boldsymbol{S}^{-\frac{1}{2}}$, $i = 1, 2$, which give $\boldsymbol{Y}_i = \boldsymbol{S}^{\frac{1}{2}}\boldsymbol{Q}_i\boldsymbol{S}^{\frac{1}{2}}$, $i = 1, 2$, and $\boldsymbol{Z} = \boldsymbol{S} - \sum_{i=1}^{2}\boldsymbol{Y}_i = \boldsymbol{S} - \sum_{i=1}^{2}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{Q}_i\boldsymbol{S}^{\frac{1}{2}} = \boldsymbol{S}^{\frac{1}{2}}\left(\boldsymbol{I}_p - \sum_{i=1}^{2}\boldsymbol{Q}_i\right)\boldsymbol{S}^{\frac{1}{2}}$ with Jacobian $J(\boldsymbol{Y}_1, \boldsymbol{Y}_2, \boldsymbol{Z} \to \boldsymbol{Q}_1, \boldsymbol{Q}_2, \boldsymbol{S}) = |\boldsymbol{S}|^{(p+1)}$ (see [2.1.4] and [2.1.8]). Substituting in (7.4) gives

$$\begin{aligned} & f(\boldsymbol{Q}_1, \boldsymbol{Q}_2, \boldsymbol{S}) \\ = \;& K\prod_{i=1}^{2}\alpha_i^{-\frac{1}{2}n_ip}c^{-\frac{1}{2}mp}\prod_{i=1}^{2}\left|\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{Q}_i\boldsymbol{S}^{\frac{1}{2}}\right|^{\frac{1}{2}n_i-\frac{1}{2}(p+1)}\left|\boldsymbol{S}^{\frac{1}{2}}\left(\boldsymbol{I}_p-\sum_{i=1}^{2}\boldsymbol{Q}_i\right)\boldsymbol{S}^{\frac{1}{2}}\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)} \\ & \cdot\mathrm{etr}\left\{-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\left[\sum_{i=1}^{2}\tfrac{1}{\alpha_i}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{Q}_i\boldsymbol{S}^{\frac{1}{2}}+\tfrac{1}{c}\boldsymbol{S}^{\frac{1}{2}}\left(\boldsymbol{I}_p-\sum_{i=1}^{2}\boldsymbol{Q}_i\right)\boldsymbol{S}^{\frac{1}{2}}\right]\right\}|\boldsymbol{S}|^{(p+1)} \\ = \;& K\prod_{i=1}^{2}\alpha_i^{-\frac{1}{2}n_ip}c^{-\frac{1}{2}mp}\prod_{i=1}^{2}|\boldsymbol{Q}_i|^{\frac{1}{2}n_i-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\sum_{i=1}^{2}\boldsymbol{Q}_i\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)} \\ & \cdot|\boldsymbol{S}|^{\frac{1}{2}(n_1+n_2+m)-\frac{1}{2}(p+1)}\mathrm{etr}\left\{-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}^{\frac{1}{2}}\left[\sum_{i=1}^{2}\tfrac{1}{\alpha_i}\boldsymbol{Q}_i+\tfrac{1}{c}\left(\boldsymbol{I}_p-\sum_{i=1}^{2}\boldsymbol{Q}_i\right)\right]\boldsymbol{S}^{\frac{1}{2}}\right\} \\ = \;& K\prod_{i=1}^{2}\alpha_i^{-\frac{1}{2}n_ip}c^{-\frac{1}{2}mp}\prod_{i=1}^{2}|\boldsymbol{Q}_i|^{\frac{1}{2}n_i-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\sum_{i=1}^{2}\boldsymbol{Q}_i\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)} \\ & \cdot|\boldsymbol{S}|^{\frac{1}{2}(n_1+n_2+m)-\frac{1}{2}(p+1)}\mathrm{etr}\left[-\tfrac{1}{2c}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}^{\frac{1}{2}}\left(\boldsymbol{I}_p+\sum_{i=1}^{2}\tfrac{c-\alpha_i}{\alpha_i}\boldsymbol{Q}_i\right)\right]. \end{aligned} \tag{7.5}$$

We consider the symmetrised density function of $(\boldsymbol{Q}_1, \boldsymbol{Q}_2)$ (see [2.9.1]), that is $f_s(\boldsymbol{Q}_1, \boldsymbol{Q}_2) \equiv \int_{\boldsymbol{S}>\boldsymbol{0}}\int_{O(p)} f(\boldsymbol{H}\boldsymbol{Q}_1\boldsymbol{H}', \boldsymbol{H}\boldsymbol{Q}_2\boldsymbol{H}', \boldsymbol{H}\boldsymbol{S}\boldsymbol{H}')\mathrm{d}\boldsymbol{H}\mathrm{d}\boldsymbol{S}$ where $\boldsymbol{H}$ $(p \times p)$ is orthogonal and $\mathrm{d}\boldsymbol{H}$ is the normalised Haar invariant measure on $O(p)$. Note that $\mathrm{d}\boldsymbol{S} = \mathrm{d}\boldsymbol{H}\boldsymbol{S}\boldsymbol{H}'$ (Díaz-García and Gutiérrez-Jáimez, 2006b). From (7.5)

$$\begin{aligned} & f(\boldsymbol{H}\boldsymbol{Q}_1\boldsymbol{H}', \boldsymbol{H}\boldsymbol{Q}_2\boldsymbol{H}', \boldsymbol{H}\boldsymbol{S}\boldsymbol{H}') \\ = \;& K\prod_{i=1}^{2}\alpha_i^{-\frac{1}{2}n_ip}c^{-\frac{1}{2}mp}\prod_{i=1}^{2}|\boldsymbol{H}\boldsymbol{Q}_i\boldsymbol{H}'|^{\frac{1}{2}n_i-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\sum_{i=1}^{2}\boldsymbol{H}\boldsymbol{Q}_i\boldsymbol{H}'\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)} \\ & \cdot|\boldsymbol{H}\boldsymbol{S}\boldsymbol{H}'|^{\frac{1}{2}(n_1+n_2+m)-\frac{1}{2}(p+1)}\mathrm{etr}\left[-\tfrac{1}{2c}\left(\boldsymbol{H}\boldsymbol{S}\boldsymbol{H}'\right)^{\frac{1}{2}}\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{H}\boldsymbol{S}\boldsymbol{H}'\right)^{\frac{1}{2}}\left(\boldsymbol{I}_p+\sum_{i=1}^{2}\tfrac{c-\alpha_i}{\alpha_i}\boldsymbol{H}\boldsymbol{Q}_i\boldsymbol{H}'\right)\right] \\ = \;& K\prod_{i=1}^{2}\alpha_i^{-\frac{1}{2}n_ip}c^{-\frac{1}{2}mp}\prod_{i=1}^{2}|\boldsymbol{Q}_i|^{\frac{1}{2}n_i-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\sum_{i=1}^{2}\boldsymbol{Q}_i\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)} \\ & \cdot|\boldsymbol{S}|^{\frac{1}{2}(n_1+n_2+m)-\frac{1}{2}(p+1)}\mathrm{etr}\left[-\tfrac{1}{2c}\boldsymbol{H}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{H}'\boldsymbol{\Sigma}^{-1}\boldsymbol{H}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{H}'\boldsymbol{H}\left(\boldsymbol{I}_p+\sum_{i=1}^{2}\tfrac{c-\alpha_i}{\alpha_i}\boldsymbol{Q}_i\right)\boldsymbol{H}'\right]. \end{aligned} \tag{7.6}$$

Then from (7.6)

$$\begin{aligned}
&f_s(\boldsymbol{Q}_1,\boldsymbol{Q}_2)\\
=\;& K\prod_{i=1}^{2}\alpha_i^{-\frac{1}{2}n_ip}c^{-\frac{1}{2}mp}\prod_{i=1}^{2}|\boldsymbol{Q}_i|^{\frac{1}{2}n_i-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\sum_{i=1}^{2}\boldsymbol{Q}_i\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)}\\
&\cdot\int_{\boldsymbol{S}>\boldsymbol{0}}|\boldsymbol{S}|^{\frac{1}{2}(n_1+n_2+m)-\frac{1}{2}(p+1)}\int_{O(p)}\operatorname{etr}\left[-\tfrac{1}{2c}\boldsymbol{H}'\boldsymbol{\Sigma}^{-1}\boldsymbol{H}\boldsymbol{S}^{\frac{1}{2}}\left(\boldsymbol{I}_p+\sum_{i=1}^{2}\tfrac{c-\alpha_i}{\alpha_i}\boldsymbol{Q}_i\right)\boldsymbol{S}^{\frac{1}{2}}\right]\mathrm{d}\boldsymbol{H}\,\mathrm{d}\boldsymbol{S}.
\end{aligned}\tag{7.7}$$

From [2.3.6] it follows that the integral in (7.7) over the orthogonal group can be written as

$$\int_{O(p)}\operatorname{etr}\left[-\tfrac{1}{2c}\boldsymbol{H}'\boldsymbol{\Sigma}^{-1}\boldsymbol{H}\left(\boldsymbol{I}_p+\sum_{i=1}^{2}\tfrac{c-\alpha_i}{\alpha_i}\boldsymbol{Q}_i\right)^{\frac{1}{2}}\boldsymbol{S}\left(\boldsymbol{I}_p+\sum_{i=1}^{2}\tfrac{c-\alpha_i}{\alpha_i}\boldsymbol{Q}_i\right)^{\frac{1}{2}}\right]d\boldsymbol{H}.\tag{7.8}$$

Using (7.8) in (7.7), changing the order of integration and integrating with respect to $\boldsymbol{S}$ using [2.2.3] gives

$$\begin{aligned}
&f_s(\boldsymbol{Q}_1,\boldsymbol{Q}_2)\\
=\;& K\prod_{i=1}^{2}\alpha_i^{-\frac{1}{2}n_ip}c^{-\frac{1}{2}mp}\prod_{i=1}^{2}|\boldsymbol{Q}_i|^{\frac{1}{2}n_i-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\sum_{i=1}^{2}\boldsymbol{Q}_i\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)}\\
&\cdot\int_{O(p)}\int_{\boldsymbol{S}>\boldsymbol{0}}|\boldsymbol{S}|^{\frac{1}{2}(n_1+n_2+m)-\frac{1}{2}(p+1)}\operatorname{etr}\left[-\tfrac{1}{2c}\left(\boldsymbol{I}_p+\sum_{i=1}^{2}\tfrac{c-\alpha_i}{\alpha_i}\boldsymbol{Q}_i\right)^{\frac{1}{2}}\boldsymbol{H}'\boldsymbol{\Sigma}^{-1}\boldsymbol{H}\left(\boldsymbol{I}_p+\sum_{i=1}^{2}\tfrac{c-\alpha_i}{\alpha_i}\boldsymbol{Q}_i\right)^{\frac{1}{2}}\boldsymbol{S}\right]\mathrm{d}\boldsymbol{S}\mathrm{d}\boldsymbol{H}\\
=\;& K\Gamma_p\left(\tfrac{n_1+n_2+m}{2}\right)\prod_{i=1}^{2}\alpha_i^{-\frac{1}{2}n_ip}c^{-\frac{1}{2}mp}\prod_{i=1}^{2}|\boldsymbol{Q}_i|^{\frac{1}{2}n_i-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\sum_{i=1}^{2}\boldsymbol{Q}_i\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)}\\
&\cdot\int_{O(p)}\left|\tfrac{1}{2c}\left(\boldsymbol{I}_p+\sum_{i=1}^{2}\tfrac{c-\alpha_i}{\alpha_i}\boldsymbol{Q}_i\right)^{\frac{1}{2}}\boldsymbol{H}'\boldsymbol{\Sigma}^{-1}\boldsymbol{H}\left(\boldsymbol{I}_p+\sum_{i=1}^{2}\tfrac{c-\alpha_i}{\alpha_i}\boldsymbol{Q}_i\right)^{\frac{1}{2}}\right|^{-\frac{1}{2}(n_1+n_2+m)}\mathrm{d}\boldsymbol{H}\\
=\;& \left\{\beta_p\left(\tfrac{n_1}{2},\tfrac{n_2}{2};\tfrac{m}{2}\right)\right\}^{-1}\prod_{i=1}^{2}\alpha_i^{-\frac{1}{2}n_ip}c^{\frac{1}{2}(n_1+n_2)p}\\
&\cdot\int_{O(p)}\prod_{i=1}^{2}|\boldsymbol{H}\boldsymbol{Q}_i\boldsymbol{H}'|^{\frac{1}{2}n_i-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\sum_{i=1}^{2}\boldsymbol{H}\boldsymbol{Q}_i\boldsymbol{H}'\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p+\sum_{i=1}^{2}\tfrac{c-\alpha_i}{\alpha_i}\boldsymbol{H}\boldsymbol{Q}_i\boldsymbol{H}'\right|^{-\frac{1}{2}(n_1+n_2+m)}\mathrm{d}\boldsymbol{H}.
\end{aligned}\tag{7.9}$$

The result follows from (7.9) and [2.9.3]. ■

Remark 7.1
The following definition of Wishart ratios will also give matrix variates having the bimatrix variate beta type V distribution with pdf given by (7.2):

$$\boldsymbol{Q}_i = \alpha_i \boldsymbol{S}_i^{\frac{1}{2}} \left(\alpha_1 \boldsymbol{S}_1 + \alpha_2 \boldsymbol{S}_2 + c\boldsymbol{B}\right)^{-1} \boldsymbol{S}_i^{\frac{1}{2}}, \ i = 1, 2,$$

and

$$\boldsymbol{Q}_i = \left(c\boldsymbol{I}_p + \sum_{i=1}^{2} \alpha_i \boldsymbol{B}^{-\frac{1}{2}} \boldsymbol{S}_i \boldsymbol{B}^{-\frac{1}{2}}\right)^{-\frac{1}{2}} \alpha_i \boldsymbol{B}^{-\frac{1}{2}} \boldsymbol{S}_i \boldsymbol{B}^{-\frac{1}{2}} \left(c\boldsymbol{I}_p + \sum_{i=1}^{2} \alpha_i \boldsymbol{B}^{-\frac{1}{2}} \boldsymbol{S}_i \boldsymbol{B}^{-\frac{1}{2}}\right)^{-\frac{1}{2}} \quad i = 1, 2. \tag{7.10}$$

Remark 7.2
The ratios (1.5) (or (7.1)) can also be written as

$$\boldsymbol{Q}_i = \left(\boldsymbol{S}_1^* + \boldsymbol{S}_2^* + \boldsymbol{B}^*\right)^{-\frac{1}{2}} \boldsymbol{S}_i^* \left(\boldsymbol{S}_1^* + \boldsymbol{S}_2^* + \boldsymbol{B}^*\right)^{-\frac{1}{2}}, \quad i = 1, 2,$$

where $\boldsymbol{S}_1^* = \alpha_1 \boldsymbol{S} \sim W_p\left(n_1, \alpha_1 \boldsymbol{\Sigma}\right)$, $\boldsymbol{S}_2^* = \alpha_2 \boldsymbol{S} \sim W_p\left(n_2, \alpha_2 \boldsymbol{\Sigma}\right)$ and $\boldsymbol{B}^* = c\boldsymbol{B} \sim W_p\left(m, c\boldsymbol{\Sigma}\right)$, all independent (see Theorem 3.4 and Remark 3.2).

Remark 7.3
The pdf of $\boldsymbol{Q} \sim B_p^V\left(n, m, \alpha, c\right)$ is given by (3.5) (see also Remark 3.2).

Remark 7.4
The matrix variate Dirichlet type V distribution, denoted as $\left(\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_r\right) \sim D_p^V\left(n_1, \ldots, n_r, m, \alpha_1, \ldots, \alpha_r, c\right)$, results by extending (7.1) to r independent Wishart matrix variates, $\boldsymbol{S}_i \sim W_p\left(n_i, \boldsymbol{\Sigma}\right)$, $i = 1, \ldots, r$, all independent of $\boldsymbol{B} \sim W_p\left(m, \boldsymbol{\Sigma}\right)$. The pdf of $\left(\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_r\right)$ is given by

$$\begin{aligned} &\left\{\beta_p\left(\tfrac{n_1}{2}, \ldots, \tfrac{n_r}{2}; \tfrac{m}{2}\right)\right\}^{-1} \prod_{i=1}^{r} \left(\frac{c}{\alpha_i}\right)^{\frac{1}{2} n_i p} \\ &\cdot \prod_{i=1}^{r} \left|\boldsymbol{Q}_i\right|^{\frac{1}{2} n_i - \frac{1}{2}(p+1)} \left|\boldsymbol{I}_p - \sum_{i=1}^{r} \boldsymbol{Q}_i\right|^{\frac{1}{2} m - \frac{1}{2}(p+1)} \left|\boldsymbol{I}_p + \sum_{i=1}^{r} \frac{c - \alpha_i}{\alpha_i} \boldsymbol{Q}_i\right|^{-\frac{1}{2}(n_1 + \cdots + n_r + m)}, \end{aligned} \tag{7.11}$$

$\boldsymbol{0} < \boldsymbol{Q}_i < \boldsymbol{I}_p$, $i = 1, \ldots, r$, $\boldsymbol{0} < \sum_{i=1}^{r} \boldsymbol{Q}_i < \boldsymbol{I}_p$.

Remark 7.5
If $\alpha_1 = \alpha_2 = \cdots = \alpha_r = \alpha$ the pdfs of the bimatrix variate beta type V distribution in (7.2) and the matrix variate Dirichlet type V distribution in (7.11) are members of the Liouville family of distributions of the second kind (see [2.2.1]).

7.2 Marginal property and conditional density

In this section the marginal and conditional pdfs of the bimatrix variate beta type V distribution are derived.

Theorem 7.2
If $(\boldsymbol{Q}_1, \boldsymbol{Q}_2) \sim BB_p^V(n_1, n_2, m, \alpha_1, \alpha_2, c)$, *then the pdf of* $\boldsymbol{Q}_1$ *is given by*

$$\begin{aligned} f(\boldsymbol{Q}_1) &= \left\{\beta_p\left(\tfrac{n_1}{2}, \tfrac{n_2+m}{2}\right)\right\}^{-1} \left(\tfrac{\alpha_2}{\alpha_1}\right)^{\frac{1}{2}n_1 p} \left(\tfrac{\alpha_2}{c}\right)^{\frac{1}{2}mp} \\ &\quad \cdot |\boldsymbol{Q}_1|^{\frac{1}{2}n_1-\frac{1}{2}(p+1)} |\boldsymbol{I}_p - \boldsymbol{Q}_1|^{\frac{1}{2}(n_2+m)-\frac{1}{2}(p+1)} \left|\boldsymbol{I}_p + \tfrac{\alpha_2-\alpha_1}{\alpha_1}\boldsymbol{Q}_1\right|^{-\frac{1}{2}(n_1+n_2+m)} \\ &\quad \cdot {}_2F_1\left(\tfrac{m}{2}, \tfrac{n_1+n_2+m}{2}; \tfrac{n_2+m}{2}; \tfrac{c-\alpha_2}{c}(\boldsymbol{I}_p - \boldsymbol{Q}_1)\left(\boldsymbol{I}_p + \tfrac{\alpha_2-\alpha_1}{\alpha_1}\boldsymbol{Q}_1\right)^{-1}\right) \\ &= g(\boldsymbol{Q}_1)\left(\tfrac{\alpha_2}{\alpha_1}\right)^{\frac{1}{2}n_1 p} \left(\tfrac{\alpha_2}{c}\right)^{\frac{1}{2}mp} \left|\boldsymbol{I}_p + \tfrac{\alpha_2-\alpha_1}{\alpha_1}\boldsymbol{Q}_1\right|^{-\frac{1}{2}(n_1+n_2+m)} \\ &\quad \cdot {}_2F_1\left(\tfrac{m}{2}, \tfrac{n_1+n_2+m}{2}; \tfrac{n_2+m}{2}; \tfrac{c-\alpha_2}{c}(\boldsymbol{I}_p - \boldsymbol{Q}_1)\left(\boldsymbol{I}_p + \tfrac{\alpha_2-\alpha_1}{\alpha_1}\boldsymbol{Q}_1\right)^{-1}\right), \end{aligned} \tag{7.12}$$

$\boldsymbol{0} < \boldsymbol{Q}_1 < \boldsymbol{I}_p$, *where* $g(\cdot)$ *is the pdf of* $B_p^I\left(\frac{n_1}{2}, \frac{n_2+m}{2}\right)$ *given by* (3.1).

Furthermore the pdf of $(\boldsymbol{Q}_2|\boldsymbol{Q}_1)$ *is*

$$\begin{aligned} &\left\{\beta_p\left(\tfrac{n_2}{2}, \tfrac{m}{2}\right)\right\}^{-1}\left(\tfrac{c}{\alpha_2}\right)^{\frac{1}{2}(n_1+n_2+m)p} |\boldsymbol{Q}_2|^{\frac{1}{2}n_2-\frac{1}{2}(p+1)} \left|\boldsymbol{I}_p - \sum_{i=1}^{2}\boldsymbol{Q}_i\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)} |\boldsymbol{I}_p - \boldsymbol{Q}_1|^{-\frac{1}{2}(n_2+m)+\frac{1}{2}(p+1)} \\ &\cdot \left|\boldsymbol{I}_p + \sum_{i=1}^{2}\tfrac{c-\alpha_i}{\alpha_i}\boldsymbol{Q}_i\right|^{-\frac{1}{2}(n_1+n_2+m)} \left|\boldsymbol{I}_p + \tfrac{\alpha_2-\alpha_1}{\alpha_1}\boldsymbol{Q}_1\right|^{\frac{1}{2}(n_1+n_2+m)} \\ &\cdot \left[{}_2F_1\left(\tfrac{m}{2}, \tfrac{n_1+n_2+m}{2}; \tfrac{n_2+m}{2}; \tfrac{c-\alpha_2}{c}(\boldsymbol{I}_p - \boldsymbol{Q}_1)\left(\boldsymbol{I}_p + \tfrac{\alpha_2-\alpha_1}{\alpha_1}\boldsymbol{Q}_1\right)^{-1}\right)\right]^{-1}, \end{aligned} \tag{7.13}$$

$\boldsymbol{0} < \boldsymbol{Q}_2 < \boldsymbol{I}_p - \boldsymbol{Q}_1$.

Proof:
The pdf of $(\boldsymbol{Q}_1, \boldsymbol{Q}_2)$ given by (7.2) can be rewritten as

$$\begin{aligned} f(\boldsymbol{Q}_1, \boldsymbol{Q}_2) &= K|\boldsymbol{Q}_1|^{\frac{1}{2}n_1-\frac{1}{2}(p+1)}|\boldsymbol{Q}_2|^{\frac{1}{2}n_2-\frac{1}{2}(p+1)} \\ &\quad \cdot |\boldsymbol{I}_p - \boldsymbol{Q}_1|^{\frac{1}{2}m-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p - (\boldsymbol{I}_p - \boldsymbol{Q}_1)^{-\frac{1}{2}}\boldsymbol{Q}_2(\boldsymbol{I}_p - \boldsymbol{Q}_1)^{-\frac{1}{2}}\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)} \\ &\quad \cdot \left|\boldsymbol{I}_p + \tfrac{c-\alpha_1}{\alpha_1}\boldsymbol{Q}_1 + \tfrac{c-\alpha_2}{\alpha_2}\boldsymbol{Q}_2\right|^{-\frac{1}{2}(n_1+n_2+m)}, \end{aligned} \tag{7.14}$$

where $K = \left\{\beta_p\left(\frac{n_1}{2}, \frac{n_2}{2}; \frac{m}{2}\right)\right\}^{-1}\prod_{i=1}^{2}\left(\frac{c}{\alpha_i}\right)^{\frac{1}{2}n_i p}$.

Consider the transformation where $\boldsymbol{Z}_2 = (\boldsymbol{I}_p - \boldsymbol{Q}_1)^{-\frac{1}{2}}\boldsymbol{Q}_2(\boldsymbol{I}_p - \boldsymbol{Q}_1)^{-\frac{1}{2}}$. Then $\boldsymbol{Q}_2 = (\boldsymbol{I}_p - \boldsymbol{Q}_1)^{\frac{1}{2}}\boldsymbol{Z}_2(\boldsymbol{I}_p - \boldsymbol{Q}_1)^{\frac{1}{2}}$ and from [2.1.6] the Jacobian is $J(\boldsymbol{Q}_2 \to \boldsymbol{Z}_2) = |\boldsymbol{I}_p - \boldsymbol{Q}_1|^{\frac{1}{2}(p+1)}$.

Substituting this in (7.14) gives

$$\begin{aligned}
&f\left(\boldsymbol{Q}_1,\boldsymbol{Z}_2\right)\\
=\;& K\left|\boldsymbol{Q}_1\right|^{\frac{1}{2}n_1-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\boldsymbol{Q}_1\right|^{\frac{1}{2}(n_2+m)-\frac{1}{2}(p+1)}\left|\boldsymbol{Z}_2\right|^{\frac{1}{2}n_2-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\boldsymbol{Z}_2\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)}\\
&\cdot\left|\boldsymbol{I}_p-\boldsymbol{Q}_1+\tfrac{c}{\alpha_1}\boldsymbol{Q}_1+\tfrac{c-\alpha_2}{\alpha_2}\left(\boldsymbol{I}_p-\boldsymbol{Q}_1\right)^{\frac{1}{2}}\boldsymbol{Z}_2\left(\boldsymbol{I}_p-\boldsymbol{Q}_1\right)^{\frac{1}{2}}\right|^{-\frac{1}{2}(n_1+n_2+m)}\\
=\;& K\left|\boldsymbol{Q}_1\right|^{\frac{1}{2}n_1-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\boldsymbol{Q}_1\right|^{\frac{1}{2}(n_2+m)-\frac{1}{2}(p+1)}\left|\boldsymbol{Z}_2\right|^{\frac{1}{2}n_2-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\boldsymbol{Z}_2\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)}\\
&\cdot\left|\boldsymbol{I}_p-\boldsymbol{Q}_1\right|^{-\frac{1}{2}(n_1+n_2+m)}\left|\boldsymbol{I}_p+\tfrac{c}{\alpha_1}\left(\boldsymbol{I}_p-\boldsymbol{Q}_1\right)^{-\frac{1}{2}}\boldsymbol{Q}_1\left(\boldsymbol{I}_p-\boldsymbol{Q}_1\right)^{-\frac{1}{2}}+\tfrac{c-\alpha_2}{\alpha_2}\boldsymbol{Z}_2\right|^{-\frac{1}{2}(n_1+n_2+m)}\\
=\;& K\left|\boldsymbol{Q}_1\right|^{\frac{1}{2}n_1-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\boldsymbol{Q}_1\right|^{\frac{1}{2}(n_2+m)-\frac{1}{2}(p+1)}\left|\boldsymbol{Z}_2\right|^{\frac{1}{2}n_2-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\boldsymbol{Z}_2\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)}\\
&\cdot\left|\boldsymbol{I}_p-\boldsymbol{Q}_1\right|^{-\frac{1}{2}(n_1+n_2+m)}\left|\boldsymbol{I}_p+\tfrac{c}{\alpha_1}\left(\boldsymbol{I}_p-\boldsymbol{Q}_1\right)^{-1}\boldsymbol{Q}_1+\tfrac{c-\alpha_2}{\alpha_2}\boldsymbol{Z}_2\right|^{-\frac{1}{2}(n_1+n_2+m)}\\
=\;& K\left|\boldsymbol{Q}_1\right|^{\frac{1}{2}n_1-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\boldsymbol{Q}_1\right|^{\frac{1}{2}(n_2+m)-\frac{1}{2}(p+1)}\left|\boldsymbol{Z}_2\right|^{\frac{1}{2}n_2-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\boldsymbol{Z}_2\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)}\\
&\cdot\left|\boldsymbol{I}_p+\tfrac{c-\alpha_1}{\alpha_1}\boldsymbol{Q}_1+\tfrac{c-\alpha_2}{\alpha_2}\left(\boldsymbol{I}_p-\boldsymbol{Q}_1\right)\boldsymbol{Z}_2\right|^{-\frac{1}{2}(n_1+n_2+m)}\\
=\;& K\left|\boldsymbol{Q}_1\right|^{\frac{1}{2}n_1-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\boldsymbol{Q}_1\right|^{\frac{1}{2}(n_2+m)-\frac{1}{2}(p+1)}\left|\boldsymbol{Z}_2\right|^{\frac{1}{2}n_2-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\boldsymbol{Z}_2\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)}\\
&\cdot\left|\boldsymbol{I}_p+\tfrac{c-\alpha_1}{\alpha_1}\boldsymbol{Q}_1\right|^{-\frac{1}{2}(n_1+n_2+m)}\left|\boldsymbol{I}_p+\tfrac{c-\alpha_2}{\alpha_2}\left(\boldsymbol{I}_p+\tfrac{c-\alpha_1}{\alpha_1}\boldsymbol{Q}_1\right)^{-1}\left(\boldsymbol{I}_p-\boldsymbol{Q}_1\right)\boldsymbol{Z}_2\right|^{-\frac{1}{2}(n_1+n_2+m)}.
\end{aligned} \tag{7.15}$$

Integrating (7.15) with respect to $\boldsymbol{Z}_2$ by using [2.6.5] gives

$$\begin{aligned}
&f\left(\boldsymbol{Q}_1\right)\\
=\;& K\left|\boldsymbol{Q}_1\right|^{\frac{1}{2}n_1-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\boldsymbol{Q}_1\right|^{\frac{1}{2}(n_2+m)-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p+\tfrac{c-\alpha_1}{\alpha_1}\boldsymbol{Q}_1\right|^{-\frac{1}{2}(n_1+n_2+m)}\\
&\cdot\int_{\boldsymbol{0}<\boldsymbol{Z}_2<\boldsymbol{I}_p}\left|\boldsymbol{Z}_2\right|^{\frac{1}{2}n_2-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\boldsymbol{Z}_2\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p+\tfrac{c-\alpha_2}{\alpha_2}\left(\boldsymbol{I}_p+\tfrac{c-\alpha_1}{\alpha_1}\boldsymbol{Q}_1\right)^{-1}\left(\boldsymbol{I}_p-\boldsymbol{Q}_1\right)\boldsymbol{Z}_2\right|^{-\frac{1}{2}(n_1+n_2+m)}\mathrm{d}\boldsymbol{Z}_2\\
=\;& \frac{\Gamma_p\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)\Gamma_p\left(\frac{m}{2}\right)}\prod_{i=1}^{2}\left(\frac{c}{\alpha_i}\right)^{\frac{1}{2}n_ip}\frac{\Gamma_p\left(\frac{n_2}{2}\right)\Gamma_p\left(\frac{m}{2}\right)}{\Gamma_p\left(\frac{n_2+m}{2}\right)}\left|\boldsymbol{Q}_1\right|^{\frac{1}{2}n_1-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\boldsymbol{Q}_1\right|^{\frac{1}{2}(n_2+m)-\frac{1}{2}(p+1)}\\
&\cdot\left|\boldsymbol{I}_p+\tfrac{c-\alpha_1}{\alpha_1}\boldsymbol{Q}_1\right|^{-\frac{1}{2}(n_1+n_2+m)}{}_2F_1\left(\tfrac{n_2}{2},\tfrac{n_1+n_2+m}{2};\tfrac{n_2+m}{2};-\tfrac{c-\alpha_2}{\alpha_2}\left(\boldsymbol{I}_p+\tfrac{c-\alpha_1}{\alpha_1}\boldsymbol{Q}_1\right)^{-1}\left(\boldsymbol{I}_p-\boldsymbol{Q}_1\right)\right).
\end{aligned} \tag{7.16}$$

Letting $\boldsymbol{S}=-\frac{c-\alpha_2}{\alpha_2}\left(\boldsymbol{I}_p+\frac{c-\alpha_1}{\alpha_1}\boldsymbol{Q}_1\right)^{-1}\left(\boldsymbol{I}_p-\boldsymbol{Q}_1\right)$ and using [2.6.7], the hypergeometric function of matrix argument in (7.16) can be written as

$${}_2F_1\left(\tfrac{n_2}{2},\tfrac{n_1+n_2+m}{2};\tfrac{n_2+m}{2};\boldsymbol{S}\right)=\left|\boldsymbol{I}_p-\boldsymbol{S}\right|^{-\frac{1}{2}(n_1+n_2+m)}{}_2F_1\left(\tfrac{m}{2},\tfrac{n_1+n_2+m}{2};\tfrac{n_2+m}{2};-\boldsymbol{S}\left(\boldsymbol{I}_p-\boldsymbol{S}\right)^{-1}\right). \tag{7.17}$$

Since

$$\begin{aligned}
&\boldsymbol{I}_p - \boldsymbol{S} \\
&= \boldsymbol{I}_p + \tfrac{c-\alpha_2}{\alpha_2}\left(\boldsymbol{I}_p + \tfrac{c-\alpha_1}{\alpha_1}\boldsymbol{Q}_1\right)^{-1}(\boldsymbol{I}_p - \boldsymbol{Q}_1) \\
&= \left(\boldsymbol{I}_p + \tfrac{c-\alpha_1}{\alpha_1}\boldsymbol{Q}_1\right)^{-1}\left[\boldsymbol{I}_p + \tfrac{c-\alpha_1}{\alpha_1}\boldsymbol{Q}_1 + \tfrac{c-\alpha_2}{\alpha_2}(\boldsymbol{I}_p - \boldsymbol{Q}_1)\right] \\
&= \left(\boldsymbol{I}_p + \tfrac{c-\alpha_1}{\alpha_1}\boldsymbol{Q}_1\right)^{-1}\left[\tfrac{c}{\alpha_1}\boldsymbol{Q}_1 + \tfrac{c}{\alpha_2}(\boldsymbol{I}_p - \boldsymbol{Q}_1)\right] \\
&= \tfrac{c}{\alpha_2}\left(\boldsymbol{I}_p + \tfrac{c-\alpha_1}{\alpha_1}\boldsymbol{Q}_1\right)^{-1}\left(\boldsymbol{I}_p + \tfrac{\alpha_2-\alpha_1}{\alpha_1}\boldsymbol{Q}_1\right)
\end{aligned}$$

it follows from (7.16) and (7.17) that

$$\begin{aligned}
&f(\boldsymbol{Q}_1) \\
= \;& \left\{\beta_p\left(\tfrac{n_1}{2}, \tfrac{n_2+m}{2}\right)\right\}^{-1}\prod_{i=1}^{2}\left(\tfrac{c}{\alpha_i}\right)^{\frac{1}{2}n_i p}|\boldsymbol{Q}_1|^{\frac{1}{2}n_1-\frac{1}{2}(p+1)}|\boldsymbol{I}_p - \boldsymbol{Q}_1|^{\frac{1}{2}(n_2+m)-\frac{1}{2}(p+1)} \\
&\cdot\left|\boldsymbol{I}_p + \tfrac{c-\alpha_1}{\alpha_1}\boldsymbol{Q}_1\right|^{-\frac{1}{2}(n_1+n_2+m)}\left|\tfrac{c}{\alpha_2}\left(\boldsymbol{I}_p + \tfrac{c-\alpha_1}{\alpha_1}\boldsymbol{Q}_1\right)^{-1}\left(\boldsymbol{I}_p + \tfrac{\alpha_2-\alpha_1}{\alpha_1}\boldsymbol{Q}_1\right)\right|^{-\frac{1}{2}(n_1+n_2+m)} \\
&\cdot{}_2F_1\left(\tfrac{m}{2}, \tfrac{n_1+n_2+m}{2}; \tfrac{n_2+m}{2}; \tfrac{c-\alpha_2}{\alpha_2}\left(\boldsymbol{I}_p + \tfrac{c-\alpha_1}{\alpha_1}\boldsymbol{Q}_1\right)^{-1}(\boldsymbol{I}_p - \boldsymbol{Q}_1)\tfrac{\alpha_2}{c}\left(\boldsymbol{I}_p + \tfrac{\alpha_2-\alpha_1}{\alpha_1}\boldsymbol{Q}_1\right)^{-1}\left(\boldsymbol{I}_p + \tfrac{c-\alpha_1}{\alpha_1}\boldsymbol{Q}_1\right)\right) \\
= \;& \left\{\beta_p\left(\tfrac{n_1}{2}, \tfrac{n_2+m}{2}\right)\right\}^{-1}\prod_{i=1}^{2}\alpha_i^{-\frac{1}{2}n_i p}\alpha_2^{\frac{1}{2}(n_1+n_2+m)p}c^{-\frac{1}{2}mp}|\boldsymbol{Q}_1|^{\frac{1}{2}n_1-\frac{1}{2}(p+1)}|\boldsymbol{I}_p - \boldsymbol{Q}_1|^{\frac{1}{2}(n_2+m)-\frac{1}{2}(p+1)} \\
&\cdot\left|\boldsymbol{I}_p + \tfrac{\alpha_2-\alpha_1}{\alpha_1}\boldsymbol{Q}_1\right|^{-\frac{1}{2}(n_1+n_2+m)}{}_2F_1\left(\tfrac{m}{2}, \tfrac{n_1+n_2+m}{2}; \tfrac{n_2+m}{2}; \tfrac{c-\alpha_2}{c}(\boldsymbol{I}_p - \boldsymbol{Q}_1)\left(\boldsymbol{I}_p + \tfrac{\alpha_2-\alpha_1}{\alpha_1}\boldsymbol{Q}_1\right)^{-1}\right).
\end{aligned}$$

This is the result in (7.12).

The conditional pdf of $(\boldsymbol{Q}_2|\boldsymbol{Q}_1)$ in (7.13) follows from $f(\boldsymbol{Q}_1, \boldsymbol{Q}_2)$ and $f(\boldsymbol{Q}_1)$ given by (7.2) and (7.12) respectively. ■

Remark 7.6
If $(\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_r) \sim D_p^V(n_1, \ldots, n_r, m, \alpha_1, \ldots, \alpha_r, c)$, as given by (7.11), then the pdf of $(\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_{r-1})$, is

$$\begin{aligned}
&f(\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_{r-1}) \\
= \;& \left\{\beta_p\left(\tfrac{n_1}{2}, \ldots, \tfrac{n_{r-1}}{2}; \tfrac{n_r+m}{2}\right)\right\}^{-1}\prod_{i=1}^{r}\left(\tfrac{\alpha_r}{\alpha_i}\right)^{\frac{1}{2}n_i p}\left(\tfrac{\alpha_r}{c}\right)^{\frac{1}{2}mp} \\
&\cdot\prod_{i=1}^{r-1}|\boldsymbol{Q}_i|^{\frac{1}{2}n_i-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p - \sum_{i=1}^{r-1}\boldsymbol{Q}_i\right|^{\frac{1}{2}(n_r+m)-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p + \sum_{i=1}^{r-1}\tfrac{\alpha_r-\alpha_i}{\alpha_i}\boldsymbol{Q}_i\right|^{-\frac{1}{2}(n_1+\cdots+n_r+m)} \\
&\cdot{}_2F_1\left(\tfrac{m}{2}, \tfrac{n_1+\cdots+n_r+m}{2}; \tfrac{n_r+m}{2}; \tfrac{c-\alpha_r}{c}\left(\boldsymbol{I}_p - \sum_{i=1}^{r-1}\boldsymbol{Q}_i\right)\left(\boldsymbol{I}_p + \sum_{i=1}^{r-1}\tfrac{\alpha_r-\alpha_i}{\alpha_i}\boldsymbol{Q}_i\right)^{-1}\right),
\end{aligned}$$

$\boldsymbol{0} < \boldsymbol{Q}_i < \boldsymbol{I}_p$, $i = 1, \ldots, r-1$, $\boldsymbol{0} < \sum_{i=1}^{r-1}\boldsymbol{Q}_i < \boldsymbol{I}_p$.

In the case where $\alpha_i = \alpha$, $i = 1, \ldots, r$, the pdf of $(\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_s)$, $s \leq r$ is

$$\begin{aligned} & f(\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_s) \\ = & \left\{\beta_p\left(\tfrac{n_1}{2}, \ldots, \tfrac{n_s}{2}; \tfrac{n_{s+1}+\cdots+n_r+m}{2}\right)\right\}^{-1} \left(\tfrac{\alpha}{c}\right)^{\frac{1}{2}mp} \prod_{i=1}^{s} |\boldsymbol{Q}_i|^{\frac{1}{2}n_i - \frac{1}{2}(p+1)} \left|\boldsymbol{I}_p - \sum_{i=1}^{s} \boldsymbol{Q}_i\right|^{\frac{1}{2}(n_{s+1}+\cdots+n_r+m)-\frac{1}{2}(p+1)} \\ & \cdot {}_2F_1\left(\tfrac{m}{2}, \tfrac{n_1+\cdots+n_r+m}{2}; \tfrac{n_{s+1}+\cdots+n_r+m}{2}; \tfrac{c-\alpha}{c}\left(\boldsymbol{I}_p - \sum_{i=1}^{s} \boldsymbol{Q}_i\right)\right), \end{aligned} \tag{7.18}$$

$\boldsymbol{0} < \boldsymbol{Q}_i < \boldsymbol{I}_p$, $i = 1, \ldots, s$, $\boldsymbol{0} < \sum_{i=1}^{s} \boldsymbol{Q}_i < \boldsymbol{I}_p$.

The pdf of $(\boldsymbol{Q}_{s+1}, \ldots, \boldsymbol{Q}_r) \mid (\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_s)$ is given by

$$\begin{aligned} & \left\{\beta_p\left(\tfrac{n_{s+1}}{2}, \ldots, \tfrac{n_r}{2}; \tfrac{m}{2}\right)\right\}^{-1} \left(\tfrac{c}{\alpha}\right)^{\frac{1}{2}(n_1+\cdots+n_r+m)p} \prod_{i=s+1}^{r} |\boldsymbol{Q}_i|^{\frac{1}{2}n_i-\frac{1}{2}(p+1)} \left|\boldsymbol{I}_p - \sum_{i=1}^{r} \boldsymbol{Q}_i\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)} \\ & \cdot \left|\boldsymbol{I}_p - \sum_{i=1}^{s} \boldsymbol{Q}_i\right|^{-\frac{1}{2}(n_{s+1}+\cdots+n_r+m)+\frac{1}{2}(p+1)} \left|\boldsymbol{I}_p + \tfrac{c-\alpha}{\alpha} \sum_{i=1}^{r} \boldsymbol{Q}_i\right|^{\frac{1}{2}(n_1+\cdots+n_r+m)} \\ & \cdot \left[{}_2F_1\left(\tfrac{m}{2}, \tfrac{n_1+\cdots+n_r+m}{2}; \tfrac{n_{s+1}+\cdots+n_r+m}{2}; \tfrac{c-\alpha}{c}\left(\boldsymbol{I}_p - \sum_{i=1}^{s} \boldsymbol{Q}_i\right)\right)\right]^{-1}, \end{aligned} \tag{7.19}$$

$\boldsymbol{0} < \boldsymbol{Q}_i < \boldsymbol{I}_p - \sum_{i=1}^{s} \boldsymbol{Q}_i$, $i = s+1, \ldots, r$, $\boldsymbol{0} < \sum_{i=s+1}^{r} \boldsymbol{Q}_i < \boldsymbol{I}_p - \sum_{i=1}^{s} \boldsymbol{Q}_i$.

Remark 7.7

In the case of the bimatrix variate beta type I distribution the marginal density is that of the matrix variate beta type I (see (4.5)) and for the matrix variate Dirichlet type I distribution the marginal density of any subset of matrix variates is again matrix variate Dirichlet type I (see Remark 4.4). From (7.12) and (7.18) this property does not hold for the bimatrix variate beta type V and matrix variate Dirichlet type V distributions.

Remark 7.8

The marginal and conditional pdfs of the matrix variate Dirichlet type V distribution given by (7.18) and (7.19) are members of the Liouville family of distributions of the second and first kind respectively (see [2.2.1]).

7.3 Product moment of the determinants, $(\alpha_1 = \alpha_2 = \alpha)$

The $(h_1, h_2)^{th}$ product moment, $E\left(|\boldsymbol{Q}_1|^{h_1} |\boldsymbol{Q}_2|^{h_2}\right)$, where $(\boldsymbol{Q}_1, \boldsymbol{Q}_2) \sim BB_p^V(n_1, n_2, m, \alpha, \alpha, c)$ is derived in Theorem 7.3.

Theorem 7.3
If $(\boldsymbol{Q}_1, \boldsymbol{Q}_2) \sim BB_p^V(n_1, n_2, m, \alpha, \alpha, c)$, *then the* $(h_1, h_2)^{th}$ *product moment is*

$$\begin{aligned} E\left(|\boldsymbol{Q}_1|^{h_1}|\boldsymbol{Q}_2|^{h_2}\right) &= \frac{\Gamma_p\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)}\frac{\Gamma_p\left(\frac{n_1}{2}+h_1\right)\Gamma_p\left(\frac{n_2}{2}+h_2\right)}{\Gamma_p\left(\frac{n_1+n_2}{2}+h_1+h_2\right)} \\ &\quad \cdot\left(\tfrac{c}{\alpha}\right)^{-\frac{1}{2}mp}{}_2F_1\left(\tfrac{m}{2}, \tfrac{n_1+n_2+m}{2}; \tfrac{n_1+n_2+m}{2}+h_1+h_2; \tfrac{c-\alpha}{c}\boldsymbol{I}_p\right) \\ &= E\left(|\boldsymbol{Q}_1^*|^{h_1}|\boldsymbol{Q}_2^*|^{h_2}\right) \\ &\quad \cdot\left(\tfrac{c}{\alpha}\right)^{-\frac{1}{2}mp}{}_2F_1\left(\tfrac{m}{2}, \tfrac{n_1+n_2+m}{2}; \tfrac{n_1+n_2+m}{2}+h_1+h_2; \tfrac{c-\alpha}{c}\boldsymbol{I}_p\right), \end{aligned} \tag{7.20}$$

where $\operatorname{Re}\left(\frac{1}{2}n_i+h_i\right) > \frac{1}{2}(p-1)$, $i=1,2$, $(\boldsymbol{Q}_1^*, \boldsymbol{Q}_2^*) \sim BB_p^I(n_1, n_2, m)$ *and* $E\left(|\boldsymbol{Q}_1^*|^{h_1}|\boldsymbol{Q}_2^*|^{h_2}\right)$ *is given by* (4.8).

Proof:
From the pdf of $(\boldsymbol{Q}_1, \boldsymbol{Q}_2)$ given by (7.2), with $\alpha_1 = \alpha_2 = \alpha$, we have

$$\begin{aligned} &E\left(|\boldsymbol{Q}_1|^{h_1}|\boldsymbol{Q}_2|^{h_2}\right) \\ &= \left\{\beta_p\left(\tfrac{n_1}{2}, \tfrac{n_2}{2}; \tfrac{m}{2}\right)\right\}^{-1}\left(\tfrac{c}{\alpha}\right)^{\frac{1}{2}(n_1+n_2)p} \\ &\quad \cdot \int\limits_{\substack{\boldsymbol{0}<\Sigma_{i=1}^2\boldsymbol{Q}_i<\boldsymbol{I}_p \\ \boldsymbol{0}<\boldsymbol{Q}_i<\boldsymbol{I}_p}}\int \prod_{i=1}^2|\boldsymbol{Q}_i|^{\frac{1}{2}n_i+h_i-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\sum_{i=1}^2\boldsymbol{Q}_i\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p+\tfrac{c-\alpha}{\alpha}\sum_{i=1}^2\boldsymbol{Q}_i\right|^{-\frac{1}{2}(n_1+n_2+m)}\prod_{i=1}^2 \mathrm{d}\boldsymbol{Q}_i. \end{aligned} \tag{7.21}$$

Letting $f\left(\sum_{i=1}^2\boldsymbol{Q}_i\right) = \left|\boldsymbol{I}_p-\sum_{i=1}^2\boldsymbol{Q}_i\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p+\frac{c-\alpha}{\alpha}\sum_{i=1}^2\boldsymbol{Q}_i\right|^{-\frac{1}{2}(n_1+n_2+m)}$ in (7.21), using [2.2.6], [2.6.5] and then [2.6.7] gives

$$\begin{aligned} &E\left(|\boldsymbol{Q}_1|^{h_1}|\boldsymbol{Q}_2|^{h_2}\right) \\ &= \left\{\beta_p\left(\tfrac{n_1}{2}, \tfrac{n_2}{2}; \tfrac{m}{2}\right)\right\}^{-1}\left(\tfrac{c}{\alpha}\right)^{\frac{1}{2}(n_1+n_2)p}\beta_p\left(\tfrac{n_1}{2}+h_1, \tfrac{n_2}{2}+h_2\right) \\ &\quad \cdot \int\limits_{\boldsymbol{0}<\boldsymbol{Z}<\boldsymbol{I}_p}|\boldsymbol{Z}|^{\frac{1}{2}(n_1+n_2)+h_1+h_2-\frac{1}{2}(p+1)}|\boldsymbol{I}_p-\boldsymbol{Z}|^{\frac{1}{2}m-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p+\tfrac{c-\alpha}{\alpha}\boldsymbol{Z}\right|^{-\frac{1}{2}(n_1+n_2+m)}\mathrm{d}\boldsymbol{Z} \\ &= \frac{\Gamma_p\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)\Gamma_p\left(\frac{m}{2}\right)}\frac{\Gamma_p\left(\frac{n_1}{2}+h_1\right)\Gamma_p\left(\frac{n_2}{2}+h_2\right)}{\Gamma_p\left(\frac{n_1+n_2}{2}+h_1+h_2\right)}\frac{\Gamma_p\left(\frac{n_1+n_2}{2}+h_1+h_2\right)\Gamma_p\left(\frac{m}{2}\right)}{\Gamma_p\left(\frac{n_1+n_2+m}{2}+h_1+h_2\right)}\left(\tfrac{c}{\alpha}\right)^{\frac{1}{2}(n_1+n_2)p} \\ &\quad \cdot {}_2F_1\left(\tfrac{n_1+n_2}{2}+h_1+h_2, \tfrac{n_1+n_2+m}{2}; \tfrac{n_1+n_2+m}{2}+h_1+h_2; \tfrac{\alpha-c}{\alpha}\boldsymbol{I}_p\right) \\ &= \frac{\Gamma_p\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)}\frac{\Gamma_p\left(\frac{n_1}{2}+h_1\right)\Gamma_p\left(\frac{n_2}{2}+h_2\right)}{\Gamma_p\left(\frac{n_1+n_2+m}{2}+h_1+h_2\right)} \\ &\quad \cdot\left(\tfrac{c}{\alpha}\right)^{-\frac{1}{2}mp}{}_2F_1\left(\tfrac{m}{2}, \tfrac{n_1+n_2+m}{2}; \tfrac{n_1+n_2+m}{2}+h_1+h_2; \tfrac{c-\alpha}{c}\boldsymbol{I}_p\right). \quad \blacksquare \end{aligned}$$

Remark 7.9
The h^{th} moment associated with $\boldsymbol{Q} \sim B_p^V(n, m, \alpha, c)$ is given by (3.18).

7.4 Distribution of the product of determinants, $(\alpha_1 = \alpha_2 = \alpha)$

An exact expression for the pdf of Λ_5 in (1.9) is derived in Theorem 7.4.

Theorem 7.4
Let $\boldsymbol{S}_1 \sim W_p(n_1, \boldsymbol{\Sigma})$, $\boldsymbol{S}_2 \sim W_p(n_2, \boldsymbol{\Sigma})$ *and* $\boldsymbol{B} \sim W_p(m, \boldsymbol{\Sigma})$. *The ratios in* (7.1),

$$\boldsymbol{Q}_i = (\alpha \boldsymbol{S}_1 + \alpha \boldsymbol{S}_2 + c\boldsymbol{B})^{-\frac{1}{2}} \alpha \boldsymbol{S}_i (\alpha \boldsymbol{S}_1 + \alpha \boldsymbol{S}_2 + c\boldsymbol{B})^{-\frac{1}{2}}, \quad i = 1, 2,$$

give $(\boldsymbol{Q}_1, \boldsymbol{Q}_2) \sim BB_p^V(n_1, n_2, m, \alpha, \alpha, c)$. *Let* $\Lambda_5 = |\boldsymbol{Q}_1|^{\frac{1}{2}n_1} |\boldsymbol{Q}_2|^{\frac{1}{2}n_2}$.
The pdf of Λ_5 *is given by*

$$\frac{\pi^{\frac{1}{4}p(p-1)}}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)\Gamma_p\left(\frac{m}{2}\right)} \left(\frac{c}{\alpha}\right)^{-\frac{1}{2}mp} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{1}{k!} \Gamma_p\left(\frac{m}{2}, \kappa\right) \Gamma_p\left(\frac{n_1+n_2+m}{2}, \kappa\right) C_\kappa\left(\frac{c-\alpha}{c} \boldsymbol{I}_p\right) H_{p,2p}^{2p,0}\left(\lambda_5 \middle| {}^{(a_1,\alpha_1),\ldots,(a_p,\alpha_p)}_{(b_1,\beta_1),\ldots,(b_{2p},\beta_{2p})}\right), \tag{7.22}$$

$0 < \lambda_5 < 1$, *where*

$a_j = \frac{m}{2} + k_j - \frac{1}{2}(j-1)$, *for* $j = 1, 2, 3, \ldots, p$,

$\alpha_j = \frac{n_1+n_2}{2}$ *for* $j = 1, 2, 3, \ldots, p$,

$$b_j = \begin{cases} -\frac{1}{4}(j-1) & \text{for } j = 1, 3, 5, \ldots, 2p-1 \\ -\frac{1}{4}(j-2) & \text{for } j = 2, 4, 6, \ldots, 2p, \end{cases}$$

$$\beta_j = \begin{cases} \frac{n_1}{2} & \text{for } j = 1, 3, 5, \ldots, 2p-1 \\ \frac{n_2}{2} & \text{for } j = 2, 4, 6, \ldots, 2p. \end{cases}$$

Proof:
Using (7.20) the Mellin transform (see [2.8.1]) of $f(\lambda_5)$ is

$$\begin{aligned} M_f(h) &\equiv E\left(\Lambda_5^{h-1}\right) \\ &= E\left[\left(|\boldsymbol{Q}_1|^{\frac{1}{2}n_1} |\boldsymbol{Q}_2|^{\frac{1}{2}n_2}\right)^{h-1}\right] \\ &= \frac{\Gamma_p\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)} \frac{\Gamma_p\left[\frac{n_1}{2} + \frac{n_1}{2}(h-1)\right]\Gamma_p\left[\frac{n_2}{2} + \frac{n_2}{2}(h-1)\right]}{\Gamma_p\left[\frac{n_1+n_2+m}{2} + \frac{n_1+n_2}{2}(h-1)\right]} \left(\frac{c}{\alpha}\right)^{-\frac{1}{2}mp} \\ &\quad \cdot {}_2F_1\left(\frac{m}{2}, \frac{n_1+n_2+m}{2}; \frac{n_1+n_2+m}{2} + \frac{n_1+n_2}{2}(h-1); \frac{c-\alpha}{c} \boldsymbol{I}_p\right) \\ &= \frac{\Gamma_p\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)} \frac{\Gamma_p\left(\frac{n_1}{2}h\right)\Gamma_p\left(\frac{n_2}{2}h\right)}{\Gamma_p\left(\frac{m}{2} + \frac{n_1+n_2}{2}h\right)} \left(\frac{c}{\alpha}\right)^{-\frac{1}{2}mp} \\ &\quad \cdot {}_2F_1\left(\frac{m}{2}, \frac{n_1+n_2+m}{2}; \frac{m}{2} + \frac{n_1+n_2}{2}h; \frac{c-\alpha}{c} \boldsymbol{I}_p\right). \end{aligned} \tag{7.23}$$

From [2.6.1] and [2.3.3] the Gauss hypergeometric function of matrix argument in (7.23) can be written as

$$
\begin{aligned}
&{}_2F_1\left(\tfrac{m}{2}, \tfrac{n_1+n_2+m}{2}; \tfrac{m}{2}+\tfrac{n_1+n_2}{2}h; \tfrac{c-\alpha}{c}\boldsymbol{I}_p\right)\\
&= \sum_{k=0}^{\infty}\sum_{\kappa}\tfrac{1}{k!}C_\kappa\left(\tfrac{c-\alpha}{c}\boldsymbol{I}_p\right)\frac{\left(\frac{m}{2}\right)_\kappa\left(\frac{n_1+n_2+m}{2}\right)_\kappa}{\left(\frac{m}{2}+\frac{n_1+n_2}{2}h\right)_\kappa}\\
&= \sum_{k=0}^{\infty}\sum_{\kappa}\tfrac{1}{k!}C_\kappa\left(\tfrac{c-\alpha}{c}\boldsymbol{I}_p\right)\frac{\Gamma_p\left(\frac{m}{2},\kappa\right)}{\Gamma_p\left(\frac{m}{2}\right)}\frac{\Gamma_p\left(\frac{n_1+n_2+m}{2},\kappa\right)}{\Gamma_p\left(\frac{n_1+n_2+m}{2}\right)}\frac{\Gamma_p\left(\frac{m}{2}+\frac{n_1+n_2}{2}h\right)}{\Gamma_p\left(\frac{m}{2}+\frac{n_1+n_2}{2}h,\kappa\right)}.
\end{aligned}
$$

Substituting in (7.23) gives

$$
M_f(h) \equiv \frac{\Gamma_p\left(\frac{n_1}{2}h\right)\Gamma_p\left(\frac{n_2}{2}h\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)\Gamma_p\left(\frac{m}{2}\right)}\left(\tfrac{c}{\alpha}\right)^{-\frac{1}{2}mp}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{1}{k!}\frac{\Gamma_p\left(\frac{m}{2},\kappa\right)\Gamma_p\left(\frac{n_1+n_2+m}{2},\kappa\right)}{\Gamma_p\left(\frac{m}{2}+\frac{n_1+n_2}{2}h,\kappa\right)}C_\kappa\left(\tfrac{c-\alpha}{c}\boldsymbol{I}_p\right). \tag{7.24}
$$

Using [2.3.3] the generalised gamma function of weight κ in (7.24) can be written as

$$
\begin{aligned}
&\Gamma_p\left(\tfrac{m}{2}+\tfrac{n_1+n_2}{2}h,\kappa\right)\\
&= \pi^{\frac{1}{4}p(p-1)}\prod_{j=1}^{p}\Gamma\left[\tfrac{m}{2}+\tfrac{n_1+n_2}{2}h+k_j-\tfrac{1}{2}(j-1)\right]\\
&= \pi^{\frac{1}{4}p(p-1)}\prod_{j=1}^{p}\Gamma\left(a_j+\alpha_j h\right),
\end{aligned}
\tag{7.25}
$$

where $a_j = \frac{m}{2}+k_j-\frac{1}{2}(j-1)$, for $j=1,2,3,\ldots,p$,

and $\alpha_j = \frac{n_1+n_2}{2}$ for $j=1,2,3,\ldots,p$.

Similarly, using [2.2.2] the multivariate gamma functions in (7.24) can be written as

$$
\begin{aligned}
&\Gamma_p\left(\tfrac{n_1}{2}h\right)\Gamma_p\left(\tfrac{n_2}{2}h\right)\\
&= \pi^{\frac{1}{2}p(p-1)}\prod_{j=1}^{p}\Gamma\left[\tfrac{n_1}{2}h-\tfrac{1}{2}(j-1)\right]\prod_{j=1}^{p}\Gamma\left[\tfrac{n_2}{2}h-\tfrac{1}{2}(j-1)\right]\\
&= \pi^{\frac{1}{2}p(p-1)}\prod_{j=1}^{2p}\Gamma\left(b_j+\beta_j h\right)
\end{aligned}
\tag{7.26}
$$

where $b_j = \begin{cases} -\frac{1}{4}(j-1) & \text{for } j=1,3,5,\ldots,2p-1 \\ -\frac{1}{4}(j-2) & \text{for } j=2,4,6,\ldots,2p, \end{cases}$

and $\beta_j = \begin{cases} \frac{n_1}{2} & \text{for } j=1,3,5,\ldots,2p-1 \\ \frac{n_2}{2} & \text{for } j=2,4,6,\ldots,2p. \end{cases}$

Now, substituting (7.25) and (7.26) in (7.24) gives

$$M_f(h) \equiv \frac{\pi^{\frac{1}{4}p(p-1)}}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)\Gamma_p\left(\frac{m}{2}\right)}\left(\frac{c}{\alpha}\right)^{-\frac{1}{2}mp}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{1}{k!}\Gamma_p\left(\frac{m}{2},\kappa\right)\Gamma_p\left(\frac{n_1+n_2+m}{2},\kappa\right)C_\kappa\left(\frac{c-\alpha}{c}\boldsymbol{I}_p\right)\frac{\prod_{j=1}^{2p}\Gamma\left(b_j+\beta_j h\right)}{\prod_{j=1}^{p}\Gamma\left(a_j+\alpha_j h\right)}. \quad (7.27)$$

The inverse Mellin transform (see [2.8.1]) of (7.27) is given by

$$\begin{aligned} f(\lambda_5) &= \frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty} M_f(h)\,\lambda_5^{-h}\mathrm{d}h \\ &= \frac{\pi^{\frac{1}{4}p(p-1)}}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)\Gamma_p\left(\frac{m}{2}\right)}\left(\frac{c}{\alpha}\right)^{-\frac{1}{2}mp}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{1}{k!}\Gamma_p\left(\frac{m}{2},\kappa\right)\Gamma_p\left(\frac{n_1+n_2+m}{2},\kappa\right)C_\kappa\left(\frac{c-\alpha}{c}\boldsymbol{I}_p\right) \\ &\quad\cdot\left[\frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty}\frac{\prod_{j=1}^{2p}\Gamma\left(b_j+\beta_j h\right)}{\prod_{j=1}^{p}\Gamma\left(a_j+\alpha_j h\right)}\lambda_5^{-h}\mathrm{d}h\right] \\ &= \frac{\pi^{\frac{1}{4}p(p-1)}}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)\Gamma_p\left(\frac{m}{2}\right)}\left(\frac{c}{\alpha}\right)^{-\frac{1}{2}mp}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{1}{k!}\Gamma_p\left(\frac{m}{2},\kappa\right)\Gamma_p\left(\frac{n_1+n_2+m}{2},\kappa\right)C_\kappa\left(\frac{c-\alpha}{c}\boldsymbol{I}_p\right) \\ &\quad\cdot H_{p,2p}^{2p,0}\left(\lambda_5\Big|_{(b_1,\beta_1),\ldots,(b_{2p},\beta_{2p})}^{(a_1,\alpha_1),\ldots,(a_p,\alpha_p)}\right). \end{aligned}$$

The last step follows from the definition of Fox's H-function (see [2.8.3]). ■

Remark 7.10
The pdf of $\Lambda_1 = |\boldsymbol{Q}|$ where $\boldsymbol{Q} \sim B_p^V(n, m, \alpha, c)$ is given by (3.20).

7.5 Role of the parameters

In this section we study the effect of the parameters α_1 and α_2. The effect of the parameters n_1, n_2 and m was studied in Section 4.5 and the effect of the parameter c was studied in Section 5.5.

Firstly, we consider the bivariate case, $p = 1$, to illustrate the effect of the additional parameters α_1 and α_2 on

- (i) the form of the pdf of (Q_1, Q_2);
- (ii) the correlation between Q_1 and Q_2;
- (iii) the graphs of $E(Q_2|q_1)$ and $var(Q_2|q_1)$ plotted against q_1;
- (iv) the form of the pdf of Λ_5.

The effect of the parameter c is discussed in Section 5.5.

From the result in (7.2) the joint pdf of Q_1 and Q_2 is given by

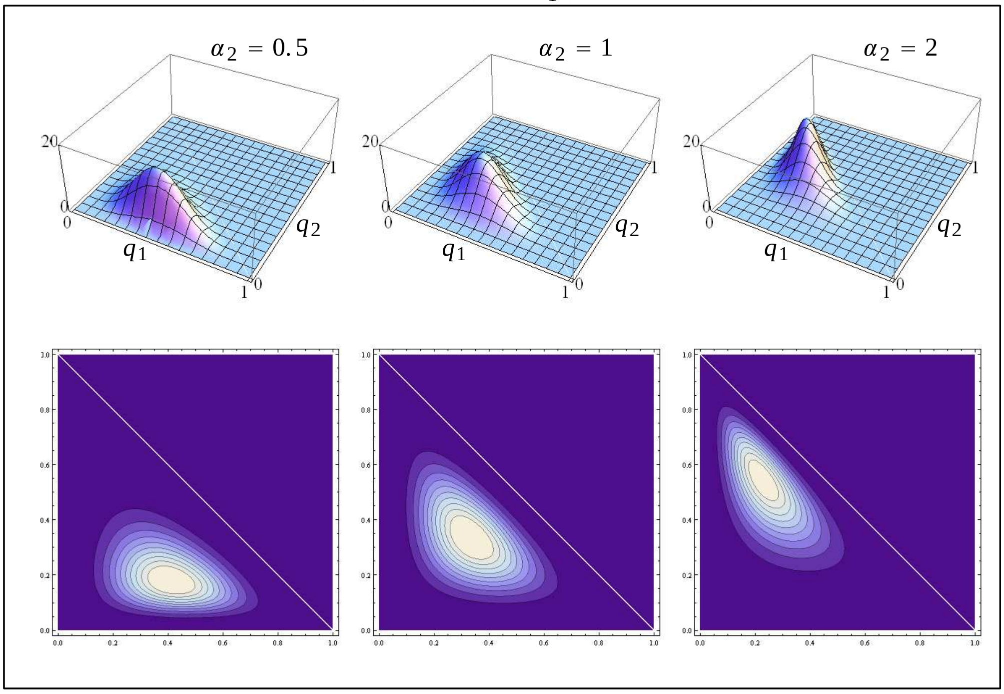
$$\begin{aligned} f(q_1, q_2) &= \frac{\Gamma\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{m}{2}\right)} q_1^{\frac{1}{2}n_1-1} q_2^{\frac{1}{2}n_2-1}(1-q_1-q_2)^{\frac{1}{2}m-1} \\ &\quad\cdot\left(\frac{c}{\alpha_1}\right)^{\frac{1}{2}n_1}\left(\frac{c}{\alpha_2}\right)^{\frac{1}{2}n_2}\left(1+\frac{c-\alpha_1}{\alpha_1}q_1+\frac{c-\alpha_2}{\alpha_2}q_2\right)^{-\frac{1}{2}(n_1+n_2+m)}, \end{aligned} \quad (7.28)$$

$0 < q_i < 1$, $i = 1, 2$, $0 < q_1 + q_2 < 1$.

The pdf in (7.28) is a special case of the generalised Dirichlet distribution derived by Craiu and Craiu (1969). It is also a special case of the F_1-beta distribution defined and studied by Nadarajah and Kotz (2005).

Figure 7.1 shows graphs of the pdf of the $BB_1^V(10, 10, 10, 1, \alpha_2, 1)$ distribution (see (7.28)). As α_2 increases, with all the other parameters constant, the pdf shifts towards smaller values of Q_1 and larger values of Q_2. The opposite will be observed for increasing values of α_1. Note that when $\alpha_1 = \alpha_2 = c = 1$, $(Q_1, Q_2) \sim BB_1^I(10, 10, 10)$ as discussed in Section 4.5 and when $\alpha_1 = \alpha_2 = 1$, $(Q_1, Q_2) \sim BB_1^{III}(10, 10, 10, c)$ as discussed in Section 5.5.

Figure 7.1: Effect of α_2 on $f(q_1, q_2)$, $(Q_1, Q_2) \sim BB_1^V(10, 10, 10, 1, \alpha_2, 1)$



The $(h_1, h_2)^{th}$ product moment, $E\left(Q_1^{h_1} Q_2^{h_2}\right)$, associated with (7.28) is derived in Theorem 7.5.

Theorem 7.5

If $(Q_1, Q_2) \sim BB_1^V(n_1, n_2, m, \alpha_1, \alpha_2, c)$ *with pdf given by* (7.28) *then*

$$\begin{aligned} & E\left(Q_1^{h_1} Q_2^{h_2}\right) \\ = \; & \frac{\Gamma\left(\frac{n_1}{2}+h_1\right)\Gamma\left(\frac{n_2}{2}+h_2\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} \frac{\Gamma\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma\left(\frac{n_1+n_2+m}{2}+h_1+h_2\right)} \\ & \cdot \left(\frac{c}{\alpha_1}\right)^{\frac{1}{2}n_1} \left(\frac{c}{\alpha_2}\right)^{\frac{1}{2}n_2} F_1\left(\frac{n_1+n_2+m}{2}, \frac{n_1}{2}+h_1, \frac{n_2}{2}+h_2, \frac{n_1+n_2+m}{2}+h_1+h_2; \frac{\alpha_1-c}{\alpha_1}, \frac{\alpha_2-c}{\alpha_2}\right), \end{aligned} \tag{7.29}$$

where F_1 *is the Appell function of the first kind (see [2.5.6]).*

Proof:

From (7.28) and [2.5.7] it follows that

$$\begin{aligned} E\left(Q_1^{h_1}Q_2^{h_2}\right) &= \frac{\Gamma\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{m}{2}\right)}\left(\frac{c}{\alpha_1}\right)^{\frac{1}{2}n_1}\left(\frac{c}{\alpha_2}\right)^{\frac{1}{2}n_2}\int\int_{\substack{0<q_1+q_2<1\\ 0<q_i<1,\ i=1,2}} q_1^{\frac{1}{2}n_1+h_1-1}q_2^{\frac{1}{2}n_2+h_2-1} \\ &\quad \cdot(1-q_1-q_2)^{\frac{1}{2}m-1}\left(1+\frac{c-\alpha_1}{\alpha_1}q_1+\frac{c-\alpha_2}{\alpha_2}q_2\right)^{-\frac{1}{2}(n_1+n_2+m)}\mathrm{d}q_1\mathrm{d}q_2 \\ &= \frac{\Gamma\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)}\frac{\Gamma\left(\frac{n_1}{2}+h_1\right)\Gamma\left(\frac{n_2}{2}+h_2\right)}{\Gamma\left(\frac{n_1+n_2+m}{2}+h_1+h_2\right)}\left(\frac{c}{\alpha_1}\right)^{\frac{1}{2}n_1}\left(\frac{c}{\alpha_2}\right)^{\frac{1}{2}n_2} \\ &\quad \cdot F_1\left(\frac{n_1+n_2+m}{2},\frac{n_1}{2}+h_1,\frac{n_2}{2}+h_2,\frac{n_1+n_2+m}{2}+h_1+h_2;\frac{\alpha_1-c}{\alpha_1},\frac{\alpha_2-c}{\alpha_2}\right). \quad \blacksquare \end{aligned}$$

Remark 7.11

If $\alpha_1=\alpha_2=\alpha$ it follows from [2.5.8] and [2.5.4] that (7.29) simplifies to

$$\begin{aligned} E\left(Q_1^{h_1}Q_2^{h_2}\right) &= \frac{\Gamma\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)}\frac{\Gamma\left(\frac{n_1}{2}+h_1\right)\Gamma\left(\frac{n_2}{2}+h_2\right)}{\Gamma\left(\frac{n_1+n_2+m}{2}+h_1+h_2\right)} \\ &\quad \cdot\left(\frac{c}{\alpha}\right)^{\frac{1}{2}(n_1+n_2)}\ {}_2F_1\left(\frac{n_1+n_2}{2}+h_1+h_2,\frac{n_1+n_2+m}{2};\frac{n_1+n_2+m}{2}+h_1+h_2;\frac{\alpha-c}{\alpha}\right) \\ &= \frac{\Gamma\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)}\frac{\Gamma\left(\frac{n_1}{2}+h_1\right)\Gamma\left(\frac{n_2}{2}+h_2\right)}{\Gamma\left(\frac{n_1+n_2+m}{2}+h_1+h_2\right)} \\ &\quad \cdot\left(\frac{c}{\alpha}\right)^{-\frac{1}{2}m}\ {}_2F_1\left(\frac{m}{2},\frac{n_1+n_2+m}{2};\frac{n_1+n_2+m}{2}+h_1+h_2;\frac{c-\alpha}{c}\right). \end{aligned}$$

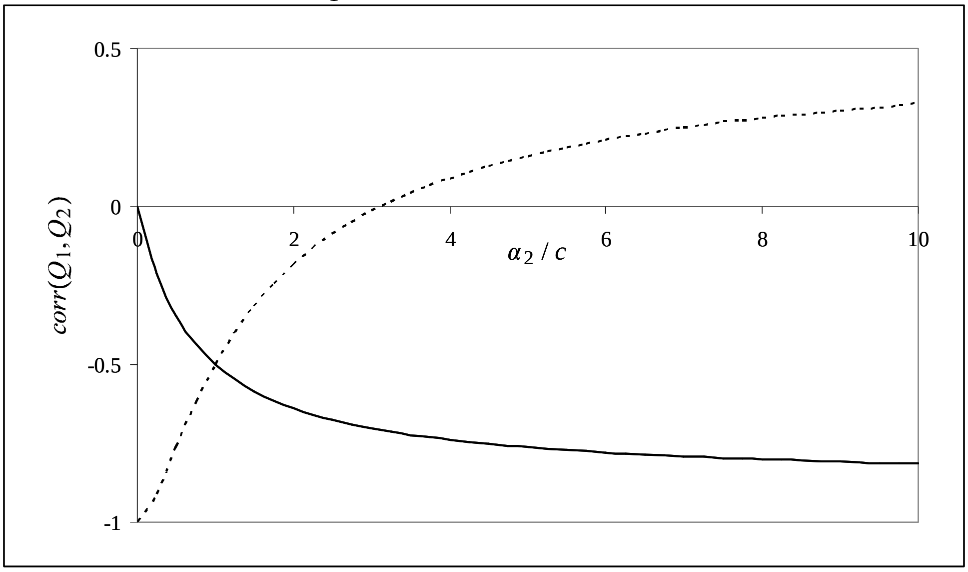
The result also follows from (7.20) with $p=1$.

The result in (7.29) was used to calculate the correlation coefficient, $corr\,(Q_1,Q_2)$, and Figure 7.2 shows graphs of the correlation between Q_1 and Q_2 for increasing values of α_2 and c. The correlation shifts towards -1 as α_2 increases or c decreases. The effect of α_1 on $corr\,(Q_1,Q_2)$ will be the same as that of α_2. The bivariate beta type I distribution is used in Bayesian analysis as the natural conjugate prior for the multinomial distribution when the variables are negatively correlated. In some practical cases random variables may be positively correlated, hence the bivariate beta type I distribution will not be a reasonable choice to be a prior distribution. However, the bivariate beta type V accommodates positive correlation for specific choices of the parameters α_1, α_2 and c. See Bodvin (2010) (also Section 5.5).

Figure 7.2: Effect of α_2 and c on $corr(Q_1, Q_2)$

(i) $(Q_1, Q_2) \sim BB_1^V(10, 10, 10, 1, \alpha_2, 1)$ ———

(ii) $(Q_1, Q_2) \sim BB_1^V(10, 10, 10, 1, 1, c)$ - - - - - -



From (7.12), the marginal pdf of Q_1 for the bivariate case simplifies to

$$f(q_1) = \frac{\Gamma\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2+m}{2}\right)} \left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{1}{2}n_1} \left(\frac{\alpha_2}{c}\right)^{\frac{1}{2}m} q_1^{\frac{1}{2}n_1-1} (1-q_1)^{\frac{1}{2}(n_2+m)-1}$$
$$\cdot \left(1 + \frac{\alpha_2-\alpha_1}{\alpha_1} q_1\right)^{-\frac{1}{2}(n_1+n_2+m)} {}_2F_1\left(\frac{m}{2}, \frac{n_1+n_2+m}{2}; \frac{n_2+m}{2}; \frac{c-\alpha_2}{c} \frac{1-q_1}{1+\frac{\alpha_2-\alpha_1}{\alpha_1}q_1}\right),$$

$0 < q_1 < 1$.

Also, for the bivariate case, the conditional pdf of $(Q_2|Q_1 = q_1)$ given by (7.13) simplifies to

$$f(q_2|q_1)$$
$$= \frac{\Gamma\left(\frac{n_2+m}{2}\right)}{\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{m}{2}\right)} \left(\frac{c}{\alpha_2}\right)^{\frac{1}{2}(n_1+n_2+m)} q_2^{\frac{1}{2}n_2-1} (1-q_1-q_2)^{\frac{1}{2}m-1} \left(1 + \frac{c-\alpha_1}{\alpha_1}q_1 + \frac{c-\alpha_2}{\alpha_2}q_2\right)^{-\frac{1}{2}(n_1+n_2+m)}$$
$$\cdot \left(1 + \frac{\alpha_2-\alpha_1}{\alpha_1}q_1\right)^{\frac{1}{2}(n_1+n_2+m)} (1-q_1)^{-\frac{1}{2}(n_2+m)+1} \left[{}_2F_1\left(\frac{m}{2}, \frac{n_1+n_2+m}{2}; \frac{n_2+m}{2}; \frac{c-\alpha_2}{c} \frac{1-q_1}{1+\frac{\alpha_2-\alpha_1}{\alpha_1}q_1}\right)\right]^{-1}, \tag{7.30}$$

$0 < q_2 < 1 - q_1$.

The h^{th} moment associated with (7.30) is derived in Theorem 7.6.

Theorem 7.6
If $(Q_1, Q_2) \sim BB_1^V(n_1, n_2, m, \alpha_1, \alpha_2, c)$ *then*

$$\begin{aligned} & E\left(Q_2^h | Q_1 = q_1\right) \\ = \; & \frac{\Gamma\left(\frac{n_2+m}{2}\right)}{\Gamma\left(\frac{n_2}{2}\right)} \frac{\Gamma\left(\frac{n_2}{2}+h\right)}{\Gamma\left(\frac{n_2+m}{2}+h\right)} \left(\frac{c}{\alpha_2}\right)^{\frac{1}{2}(n_1+n_2+m)} \\ & \left(1+\frac{\alpha_2-\alpha_1}{\alpha_1}q_1\right)^{\frac{1}{2}(n_1+n_2+m)} \left(1+\frac{c-\alpha_1}{\alpha_1}q_1\right)^{-\frac{1}{2}(n_1+n_2+m)} (1-q_1)^h \\ & \cdot \left[{}_2F_1\left(\frac{m}{2}, \frac{n_1+n_2+m}{2}; \frac{n_2+m}{2}; \frac{c-\alpha_2}{c} \frac{1-q_1}{1+\frac{\alpha_2-\alpha_1}{\alpha_1}q_1}\right)\right]^{-1} {}_2F_1\left(\frac{n_2}{2}+h, \frac{n_1+n_2+m}{2}; \frac{n_2+m}{2}+h; -\frac{\frac{c-\alpha_2}{\alpha_2}(1-q_1)}{1+\frac{c-\alpha_1}{\alpha_1}q_1}\right). \end{aligned} \tag{7.31}$$

Proof:
Let $z = \frac{q_2}{1-q_1}$. From (7.30)

$$\begin{aligned} & E\left(Q_2^h | Q_1 = q_1\right) \\ = \; & \frac{\Gamma\left(\frac{n_2+m}{2}\right)}{\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{m}{2}\right)} \left(\frac{c}{\alpha_2}\right)^{\frac{1}{2}(n_1+n_2+m)} \left(1+\frac{\alpha_2-\alpha_1}{\alpha_1}q_1\right)^{\frac{1}{2}(n_1+n_2+m)} (1-q_1)^{-\frac{1}{2}(n_2+m)+1} \\ & \cdot \left[{}_2F_1\left(\frac{m}{2}, \frac{n_1+n_2+m}{2}; \frac{n_2+m}{2}; \frac{c-\alpha_2}{c} \frac{1-q_1}{1+\frac{\alpha_2-\alpha_1}{\alpha_1}q_1}\right)\right]^{-1} \\ & \cdot (1-q_1)^{\frac{1}{2}n_2+h-1} (1-q_1)^{\frac{1}{2}m-1} \left(1+\frac{c-\alpha_1}{\alpha_1}q_1\right)^{-\frac{1}{2}(n_1+n_2+m)} (1-q_1) \\ & \cdot \int_0^1 z^{\frac{1}{2}n_2+h-1} (1-z)^{\frac{1}{2}m-1} \left(1+\frac{\frac{c-\alpha_2}{\alpha_2}(1-q_1)}{1+\frac{c-\alpha_1}{\alpha_1}q_1} z\right)^{-\frac{1}{2}(n_1+n_2+m)} \mathrm{d}z. \end{aligned} \tag{7.32}$$

Using [2.5.3] to solve the integral in (7.32) gives (7.31). ■

Figure 7.3 shows graphs of $E(Q_2|q_1)$ given by (7.31) plotted against q_1 for different values of the parameters α_1 and α_2 where $(Q_1, Q_2) \sim BB_1^V(10, 10, 10, \alpha_1, \alpha_2, 1)$. Note from (7.31) that when $\alpha_2 = c$, the graph of $E(Q_2|q_1)$ against q_1 is a straight line and is the same as the case where $(Q_1, Q_2) \sim BB_1^I(10, 10, 10)$ discussed in Section 4.5. Consider the graph of $E(Q_2|q_1)$ against q_1 where $\alpha_2 = c = 1$ as a base line. If $\alpha_2 < c$ the graph lies below the base and as α_1 increases (with α_2 and c constant) the graph shifts downwards, further away from the base. If $\alpha_2 > c$ the graph is above the base and as α_1 increases (with α_2 and c constant) the graph shifts upwards, also further away from the base. For fixed α_1 and c the graph of $E(Q_2|q_1)$ against q_1 shifts upwards as α_2 increases. This corresponds to what is observed in Figure 7.1. Figure 7.4 shows graphs of $var(Q_2|q_1)$ plotted against q_1 for different values of α_1 and α_2 where $(Q_1, Q_2) \sim BB_1^V(10, 10, 10, \alpha_1, \alpha_2, 1)$. As α_1 increases (with α_2 and c constant), $var(Q_2|q_1)$ decreases. The opposite is observed for increasing values of α_2.

Figure 7.3: Effect of α_1 and α_2 on $E(Q_2|q_1)$, $(Q_1,Q_2) \sim BB_1^V(10,10,10,\alpha_1,\alpha_2,1)$

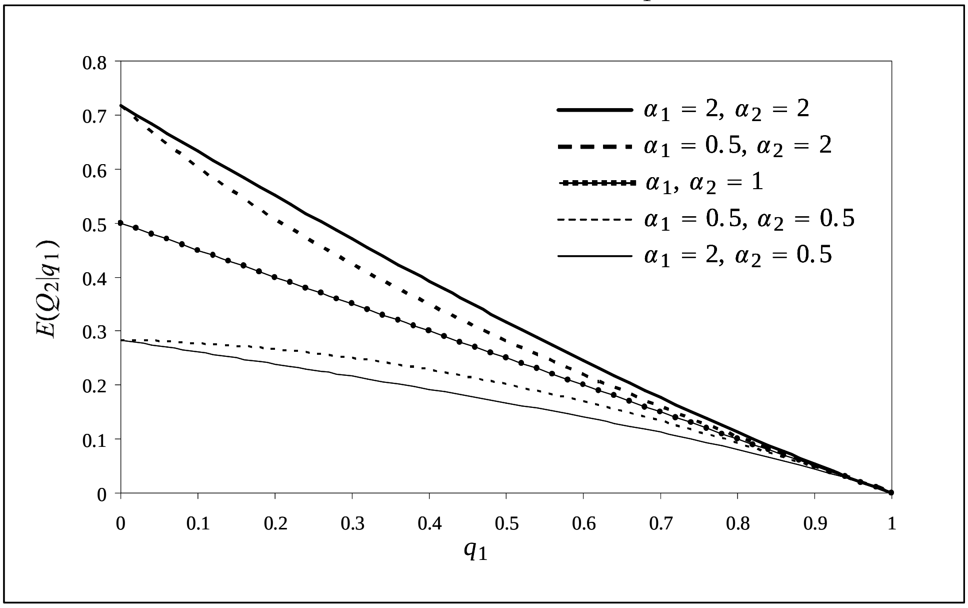
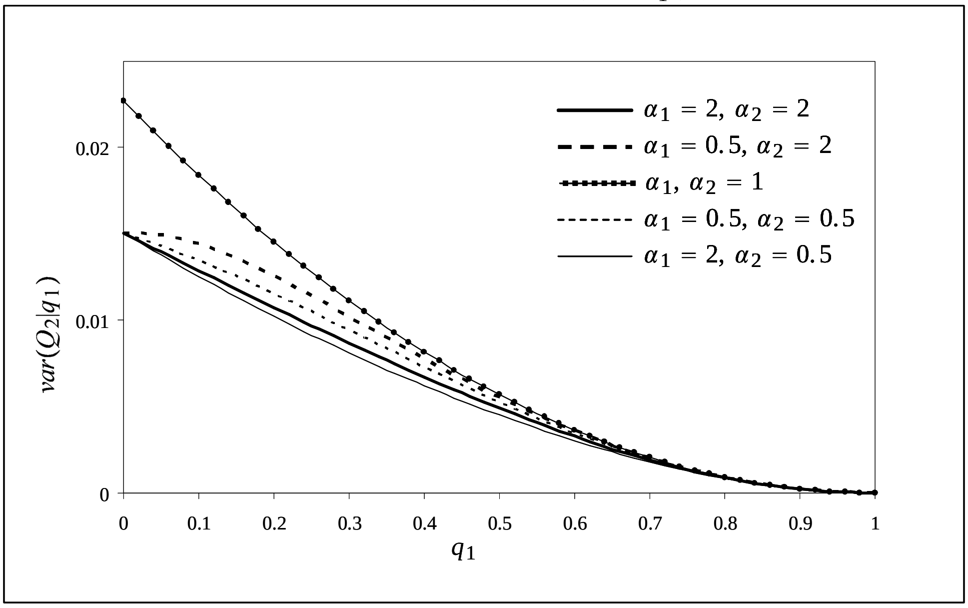


Figure 7.4: Effect of α_1 and α_2 on $var(Q_2|q_1)$, $(Q_1,Q_2) \sim BB_1^V(10,10,10,\alpha_1,\alpha_2,1)$



Finally, for the bivariate case, the effect of the parameter α_1 on the pdf of $\Lambda_5 = Q_1^{\frac{1}{2}n_1} Q_2^{\frac{1}{2}n_2}$ was studied where $(Q_1, Q_2) \sim BB_1^V(n_1, n_2, m, \alpha_1, \alpha_2, c)$ and Λ_5 is given by (1.9). The pdf of Λ_5 is derived in Theorem 7.7 for any α_1 and α_2 (the result in (7.22) gives the pdf when $\alpha_1 = \alpha_2 = \alpha$). Rogers and Young (1973) derived the pdf of Λ_5 for $\alpha_1 = \alpha_2 = c$.

Theorem 7.7
Let $S_1 \sim \chi^2(n_1)$, $S_2 \sim \chi^2(n_2)$ *and* $B \sim \chi^2(m)$. *The ratios*

$$(Q_1, Q_2) = \left(\tfrac{\alpha_1 S_1}{\alpha_1 S_1 + \alpha_2 S_2 + cB}, \tfrac{\alpha_2 S_2}{\alpha_1 S_1 + \alpha_2 S_2 + cB}\right),$$

give $(Q_1, Q_2) \sim BB_1^V(n_1, n_2, m, \alpha_1, \alpha_2, c)$. *Let* $\Lambda_5 = Q_1^{\frac{1}{2}n_1} Q_2^{\frac{1}{2}n_2}$.

The pdf of Λ_5 *is given by*

$$\frac{1}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)}\left(\frac{c}{\alpha_1}\right)^{\frac{1}{2}n_1}\left(\frac{c}{\alpha_2}\right)^{\frac{1}{2}n_2}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{1}{k!l!}\Gamma\left(\frac{n_1+n_2+m}{2}+k+l\right)\left(\frac{\alpha_1-c}{\alpha_1}\right)^k\left(\frac{\alpha_2-c}{\alpha_2}\right)^l H_{1,2}^{2,0}\left(\lambda_5\middle|^{(a_1,\gamma_1)}_{(b_1,\beta_1),(b_2,\beta_2)}\right), \tag{7.33}$$

$0 < \lambda_5 < 1$, *where* $a_1 = \frac{m}{2} + k + l$, $\gamma_1 = \frac{n_1+n_2}{2}$, $b_1 = k$, $b_2 = l$, $\beta_1 = \frac{n_1}{2}$, $\beta_2 = \frac{n_2}{2}$ *and where* $H(\cdot)$ *is Fox's H-function (see [2.8.3]).*

Proof:
Using (7.29) the Mellin transform (see [2.8.1]) of $f(\lambda_5)$ is

$$\begin{aligned}
&M_f(h) \\
\equiv\ & E\left(\Lambda_5^{h-1}\right) \\
=\ & E\left[\left(Q_1^{\frac{1}{2}n_1} Q_2^{\frac{1}{2}n_2}\right)^{h-1}\right] \\
=\ & \frac{\Gamma\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)}\frac{\Gamma\left(\frac{n_1}{2}h\right)\Gamma\left(\frac{n_2}{2}h\right)}{\Gamma\left(\frac{m}{2}+\frac{n_1+n_2}{2}h\right)}\left(\frac{c}{\alpha_1}\right)^{\frac{1}{2}n_1}\left(\frac{c}{\alpha_2}\right)^{\frac{1}{2}n_2} F_1\left(\frac{n_1+n_2+m}{2}, \frac{n_1}{2}h, \frac{n_2}{2}h, \frac{m}{2}+\frac{n_1+n_2}{2}h; \frac{\alpha_1-c}{\alpha_1}, \frac{\alpha_2-c}{\alpha_2}\right).
\end{aligned} \tag{7.34}$$

From [2.5.6] and [2.3.2] the Appell function of the first kind in (7.34) can be written as

$$\begin{aligned}
&F_1\left(\frac{n_1+n_2+m}{2}, \frac{n_1}{2}h, \frac{n_2}{2}h, \frac{m}{2}+\frac{n_1+n_2}{2}h; \frac{\alpha_1-c}{\alpha_1}, \frac{\alpha_2-c}{\alpha_2}\right) \\
=\ & \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{1}{k!l!}\frac{\left(\frac{n_1+n_2+m}{2}\right)_{k+l}\left(\frac{n_1}{2}h\right)_k\left(\frac{n_2}{2}h\right)_l}{\left(\frac{m}{2}+\frac{n_1+n_2}{2}h\right)_{k+l}}\left(\frac{\alpha_1-c}{\alpha_1}\right)^k\left(\frac{\alpha_2-c}{\alpha_2}\right)^l \\
=\ & \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{1}{k!l!}\frac{\Gamma\left(\frac{n_1+n_2+m}{2}+k+l\right)\Gamma\left(\frac{n_1}{2}h+k\right)\Gamma\left(\frac{n_2}{2}h+l\right)\Gamma\left(\frac{m}{2}+\frac{n_1+n_2}{2}h\right)}{\Gamma\left(\frac{n_1+n_2+m}{2}\right)\Gamma\left(\frac{n_1}{2}h\right)\Gamma\left(\frac{n_2}{2}h\right)\Gamma\left(\frac{m}{2}+\frac{n_1+n_2}{2}h+k+l\right)}.
\end{aligned}$$

Substituting in (7.34) gives

$$\begin{aligned}
M_f(h) \equiv\ & \left(\frac{c}{\alpha_1}\right)^{\frac{1}{2}n_1}\left(\frac{c}{\alpha_2}\right)^{\frac{1}{2}n_2}\frac{1}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} \\
& \cdot \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{1}{k!l!}\frac{\Gamma\left(\frac{n_1+n_2+m}{2}+k+l\right)\Gamma\left(\frac{n_1}{2}h+k\right)\Gamma\left(\frac{n_2}{2}h+l\right)}{\Gamma\left(\frac{m}{2}+\frac{n_1+n_2}{2}h+k+l\right)}\left(\frac{\alpha_1-c}{\alpha_1}\right)^k\left(\frac{\alpha_2-c}{\alpha_2}\right)^l.
\end{aligned} \tag{7.35}$$

The gamma functions in (7.35) can be written as

$$\prod_{i=1}^{1}\Gamma\left(a_i+\gamma_i h\right) \tag{7.36}$$

where $a_1 = \frac{m}{2} + k + l$ and $\gamma_1 = \frac{n_1+n_2}{2}$,

and

$$\prod_{i=1}^{2} \Gamma(b_i + \beta_i h) \tag{7.37}$$

where $b_1 = k$, $b_2 = l$ and $\beta_1 = \frac{n_1}{2}$, $\beta_2 = \frac{n_2}{2}$.

Substituting (7.36) and (7.37) in (7.35) gives

$$\begin{aligned} M_f(h) \equiv & \frac{1}{\Gamma(\frac{n_1}{2})\Gamma(\frac{n_2}{2})} \left(\frac{c}{\alpha_1}\right)^{\frac{1}{2}n_1} \left(\frac{c}{\alpha_2}\right)^{\frac{1}{2}n_2} \\ & \cdot \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{1}{k!l!} \Gamma\left(\frac{n_1+n_2+m}{2} + k + l\right) \left(\frac{\alpha_1 - c}{\alpha_1}\right)^k \left(\frac{\alpha_2 - c}{\alpha_2}\right)^l \frac{\prod_{i=1}^{2} \Gamma(b_i + \beta_i h)}{\prod_{i=1}^{1} \Gamma(a_i + \gamma_i h)}. \end{aligned} \tag{7.38}$$

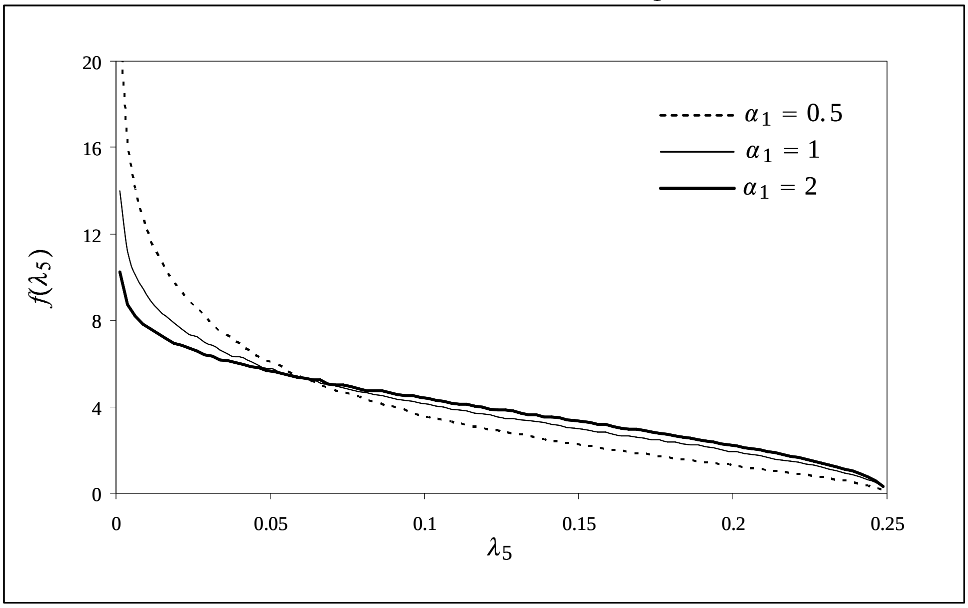
The inverse Mellin transform (see [2.8.1]) of (7.38) gives the pdf of Λ_5 as

$$\begin{aligned} & f(\lambda_5) \\ = & \frac{1}{2\pi i} \int_{\omega - i\infty}^{\omega + i\infty} M_f(h)\, \lambda_5^{-h} \mathrm{d}h \\ = & \frac{1}{\Gamma(\frac{n_1}{2})\Gamma(\frac{n_2}{2})} \left(\frac{c}{\alpha_1}\right)^{\frac{1}{2}n_1} \left(\frac{c}{\alpha_2}\right)^{\frac{1}{2}n_2} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{1}{k!l!} \Gamma\left(\frac{n_1+n_2+m}{2} + k + l\right) \left(\frac{\alpha_1 - c}{\alpha_1}\right)^k \left(\frac{\alpha_2 - c}{\alpha_2}\right)^l H_{1,2}^{2,0}\left(\lambda_5 \middle|_{(b_1,\beta_1),(b_2,\beta_2)}^{(a_1,\gamma_1)}\right), \end{aligned}$$

where $H(\cdot)$ is Fox's H-function (see [2.8.3]). ■

Figure 7.5 shows the effect of α_1 on $f(\lambda_5)$ (see (7.33)) where $(Q_1, Q_2) \sim BB_1^V(2, 2, 2, \alpha_1, 1, 1)$. At smaller values of Λ_5 the pdf, $f(\lambda_5)$, increases as α_1 decreases.

Figure 7.5: Effect of α_1 on $f(\lambda_5)$, $\Lambda_5 = Q_1 Q_2$, $(Q_1, Q_2) \sim BB_1^V(2,2,2,\alpha_1,1,1)$



Secondly, we consider the bimatrix case, $p = 2$, to illustrate the effect of the parameter $\alpha_1 = \alpha_2 = \alpha$ on Λ_5. From (7.22), the pdf of Λ_5 for $p = 2$ simplifies to

$$f(\lambda_5) = \frac{\sqrt{\pi}}{\Gamma_2\left(\frac{n_1}{2}\right)\Gamma_2\left(\frac{n_2}{2}\right)\Gamma_2\left(\frac{m}{2}\right)} \left(\frac{c}{\alpha}\right)^{-m} \sum_{k=0}^{\infty}\sum_{\kappa} \frac{1}{k!}\Gamma_2\left(\frac{m}{2},\kappa\right)\Gamma_2\left(\frac{n_1+n_2+m}{2},\kappa\right) C_\kappa\left(\frac{c-\alpha}{c}\boldsymbol{I}_2\right) H_{2,4}^{4,0}\left(\lambda_5 \middle|_{(b_1,\beta_1),\ldots,(b_4,\beta_4)}^{(a_1,\alpha_1),(a_2,\alpha_2)}\right), \tag{7.39}$$

$0 < \lambda_5 < 1$, where

$a_j = \frac{m}{2} + k_j - \frac{1}{2}(j-1)$, for $j = 1, 2$,

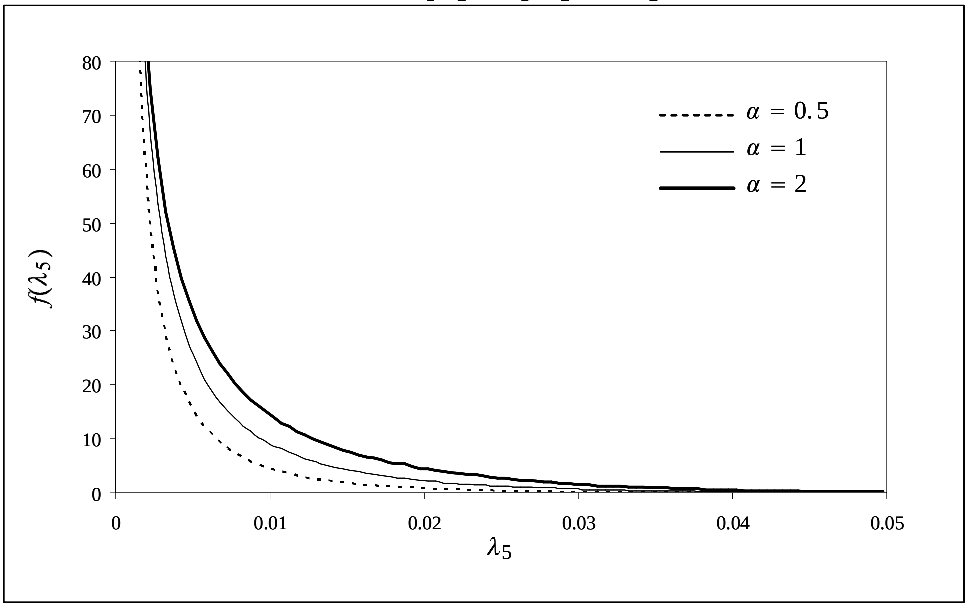
$\alpha_j = \frac{n_1+n_2}{2}$ for $j = 1, 2$,

$$b_j = \begin{cases} -\frac{1}{4}(j-1) & \text{for } j = 1, 3 \\ -\frac{1}{4}(j-2) & \text{for } j = 2, 4, \end{cases}$$

$$\beta_j = \begin{cases} \frac{n_1}{2} & \text{for } j = 1, 3 \\ \frac{n_2}{2} & \text{for } j = 2, 4. \end{cases}$$

Figure 7.6 illustrates the shape of the pdf $f(\lambda_5)$ given by (7.39), for increasing values of α where $(\boldsymbol{Q}_1, \boldsymbol{Q}_2) \sim BB_2^V(2, 2, 2, \alpha, \alpha, 1)$. We note that as α increases the pdf shifts towards larger values of Λ_5.

Figure 7.6: Effect of α on $f(\lambda_5)$, $\Lambda_5 = |\boldsymbol{Q}_1\boldsymbol{Q}_2|$, $(\boldsymbol{Q}_1, \boldsymbol{Q}_2) \sim BB_2^V(2,2,2,\alpha,\alpha,1)$



II

Noncentral distributions

8 Noncentral matrix variate beta type I distribution

In this section the pdf of the noncentral matrix variate beta type I distribution with $\boldsymbol{U}$ defined as in (3.3), where $\boldsymbol{B}$ has a noncentral Wishart distribution, is derived and the corresponding moment of the determinant and the pdf of the Wilks' statistic Λ_1 are also given.

8.1 Probability density function

De Waal (1968) derived the pdf of $\boldsymbol{U} \sim B_p^I(n, m; \boldsymbol{\Theta})$ defined as in (1.1) when $\boldsymbol{B}$ is noncentral, with the result given in integral form as

$$\left[\Gamma_p\left(\tfrac{n}{2}\right)\Gamma_p\left(\tfrac{m}{2}\right)|2\boldsymbol{\Sigma}|^{\frac{1}{2}(n+m)}\right]^{-1}\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)|\boldsymbol{U}|^{\frac{1}{2}n-\frac{1}{2}(p+1)}|\boldsymbol{I}_p-\boldsymbol{U}|^{\frac{1}{2}m-\frac{1}{2}(p+1)}$$
$$\cdot\int_{\boldsymbol{S}>\boldsymbol{0}}|\boldsymbol{S}|^{\frac{1}{2}(n+m)-\frac{1}{2}(p+1)}\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}\right)\,{}_0F_1\left(\tfrac{m}{2};\tfrac{1}{4}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}^{\frac{1}{2}}\left(\boldsymbol{I}_p-\boldsymbol{U}\right)\boldsymbol{S}^{\frac{1}{2}}\right)\mathrm{d}\boldsymbol{S},$$

$\boldsymbol{0} < \boldsymbol{U} < \boldsymbol{I}_p$. Kshirsagar (1961) derived the pdf of $\boldsymbol{U}$ for the linear case, that is when $\boldsymbol{B} \sim W_p(m, \boldsymbol{\Sigma}; \boldsymbol{\Theta})$ and $\boldsymbol{\Theta} = \operatorname{diag}(\theta, 0, \ldots, 0)$. In this section the pdf of $\boldsymbol{U}$ (see (3.3)) is derived when the covariance matrices of the Wishart distributions are $\boldsymbol{\Sigma}$.

Theorem 8.1
Let $\boldsymbol{S} \sim W_p(n, \boldsymbol{\Sigma})$ *and* $\boldsymbol{B} \sim W_p(m, \boldsymbol{\Sigma}; \boldsymbol{\Theta})$ *be independently distributed. Consider the definition of* $\boldsymbol{U}$ *given by* (3.3)*,*

$$\boldsymbol{U} = \left(\boldsymbol{I}_p + \boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{S}\boldsymbol{B}^{-\frac{1}{2}}\right)^{-\frac{1}{2}}\boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{S}\boldsymbol{B}^{-\frac{1}{2}}\left(\boldsymbol{I}_p + \boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{S}\boldsymbol{B}^{-\frac{1}{2}}\right)^{-\frac{1}{2}}. \tag{8.1}$$

The pdf of $\boldsymbol{U} \sim B_p^I(n, m; \boldsymbol{\Theta})$ *is given by*

$$\begin{aligned} & f(\boldsymbol{U}) \\ = \; & \left\{\beta_p\left(\tfrac{n}{2}, \tfrac{m}{2}\right)\right\}^{-1}\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)|\boldsymbol{U}|^{\frac{1}{2}n-\frac{1}{2}(p+1)} \\ & \cdot|\boldsymbol{I}_p-\boldsymbol{U}|^{\frac{1}{2}m-\frac{1}{2}(p+1)}\;{}_1F_1\left(\tfrac{n+m}{2};\tfrac{m}{2};\tfrac{1}{2}\left(\boldsymbol{I}_p-\boldsymbol{U}\right)\boldsymbol{\Theta}\right) \\ = \; & g(\boldsymbol{U})\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\;{}_1F_1\left(\tfrac{n+m}{2};\tfrac{m}{2};\tfrac{1}{2}\left(\boldsymbol{I}_p-\boldsymbol{U}\right)\boldsymbol{\Theta}\right), \quad \boldsymbol{0} < \boldsymbol{U} < \boldsymbol{I}_p, \end{aligned} \tag{8.2}$$

where $\beta_p\left(\frac{n}{2}, \frac{m}{2}\right)$ *is the multivariate beta function given in [2.2.4],* $n > (p-1)$, $m > (p-1)$, ${}_1F_1(\cdot)$ *is the confluent hypergeometric function of matrix argument given in [2.6.4] and* $g(\cdot)$ *is the pdf of* $B_p^I(n, m)$ *given by* (3.1).

Proof:
The pdf of $(\boldsymbol{S}, \boldsymbol{B})$ is given by

$$K\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}\right)|\boldsymbol{S}|^{\frac{1}{2}(n-p-1)}\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\right)|\boldsymbol{B}|^{\frac{1}{2}(m-p-1)}\;{}_0F_1\left(\tfrac{m}{2};\tfrac{1}{4}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\right) \tag{8.3}$$

where $K^{-1} = \Gamma_p\left(\frac{n}{2}\right)\Gamma_p\left(\frac{m}{2}\right)|2\boldsymbol{\Sigma}|^{\frac{1}{2}(n+m)}$ (see [2.10.2]) and ${}_0F_1(\cdot)$ is the hypergeometric function of the matrix argument given in [2.6.1].

Making the transformation $\boldsymbol{V} = \boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{S}\boldsymbol{B}^{-\frac{1}{2}}$ with Jacobian $J(\boldsymbol{S},\boldsymbol{B} \to \boldsymbol{V},\boldsymbol{B}) = |\boldsymbol{B}|^{\frac{1}{2}(p+1)}$ (see [2.1.6]), substituting in (8.3) and integrating with respect to $\boldsymbol{B}$ gives the pdf of $\boldsymbol{V}$ as

$$\begin{aligned} f(\boldsymbol{V}) = \ & K\mathrm{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)|\boldsymbol{V}|^{\frac{1}{2}n-\frac{1}{2}(p+1)} \\ & \cdot\int_{\boldsymbol{B}>\boldsymbol{0}}\mathrm{etr}\left[-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}^{\frac{1}{2}}(\boldsymbol{I}_p+\boldsymbol{V})\boldsymbol{B}^{\frac{1}{2}}\right]|\boldsymbol{B}|^{\frac{1}{2}(n+m)-\frac{1}{2}(p+1)}\ {}_0F_1\left(\tfrac{m}{2};\tfrac{1}{4}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\right)\mathrm{d}\boldsymbol{B}. \end{aligned} \tag{8.4}$$

We consider the symmetrised density function of $\boldsymbol{V}$ (see [2.9.1]), that is $f_s(\boldsymbol{V}) \equiv \int_{O(p)} f\left(\boldsymbol{H}\boldsymbol{V}\boldsymbol{H}'\right)\mathrm{d}\boldsymbol{H}$ where $\boldsymbol{H}$ $(p\times p)$ is orthogonal and $\mathrm{d}\boldsymbol{H}$ is the normalised Haar invariant measure on $O(p)$. From (8.4)

$$\begin{aligned} f(\boldsymbol{H}\boldsymbol{V}\boldsymbol{H}) = \ & K\mathrm{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\left|\boldsymbol{H}\boldsymbol{V}\boldsymbol{H}'\right|^{\frac{1}{2}n-\frac{1}{2}(p+1)} \\ & \cdot\int_{\boldsymbol{B}>\boldsymbol{0}}|\boldsymbol{B}|^{\frac{1}{2}(n+m)-\frac{1}{2}(p+1)}\mathrm{etr}\left[-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}^{\frac{1}{2}}\left(\boldsymbol{I}_p+\boldsymbol{H}\boldsymbol{V}\boldsymbol{H}'\right)\boldsymbol{B}^{\frac{1}{2}}\right]\ {}_0F_1\left(\tfrac{m}{2};\tfrac{1}{4}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\right)\mathrm{d}\boldsymbol{B}. \end{aligned} \tag{8.5}$$

Hence, from (8.5), [2.3.6] and [2.6.6], we get

$$\begin{aligned} & f_s(\boldsymbol{V}) \\ = \ & K\mathrm{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)|\boldsymbol{V}|^{\frac{1}{2}n-\frac{1}{2}(p+1)} \\ & \cdot\int_{\boldsymbol{B}>\boldsymbol{0}}|\boldsymbol{B}|^{\frac{1}{2}(n+m)-\frac{1}{2}(p+1)}\ {}_0F_1\left(\tfrac{m}{2};\tfrac{1}{4}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\right)\int_{O(p)}\mathrm{etr}\left[-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{H}(\boldsymbol{I}_p+\boldsymbol{V})\boldsymbol{H}'\boldsymbol{B}^{\frac{1}{2}}\right]\mathrm{d}\boldsymbol{H}\mathrm{d}\boldsymbol{B} \\ = \ & K\mathrm{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)|\boldsymbol{V}|^{\frac{1}{2}n-\frac{1}{2}(p+1)} \\ & \cdot\int_{\boldsymbol{S}>\boldsymbol{0}}|\boldsymbol{B}|^{\frac{1}{2}(n+m)-\frac{1}{2}(p+1)}\ {}_0F_1\left(\tfrac{m}{2};\tfrac{1}{4}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\right)\int_{O(p)}\mathrm{etr}\left[-\tfrac{1}{2}\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{B}\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{H}(\boldsymbol{I}_p+\boldsymbol{V})\boldsymbol{H}'\right]\mathrm{d}\boldsymbol{H}\mathrm{d}\boldsymbol{B} \\ = \ & K\mathrm{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)|\boldsymbol{V}|^{\frac{1}{2}n-\frac{1}{2}(p+1)} \\ & \cdot\int_{O(p)}\int_{\boldsymbol{B}>\boldsymbol{0}}|\boldsymbol{B}|^{\frac{1}{2}(n+m)-\frac{1}{2}(p+1)}\ {}_0F_1\left(\tfrac{m}{2};\tfrac{1}{4}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\right)\mathrm{etr}\left[-\tfrac{1}{2}\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{H}(\boldsymbol{I}_p+\boldsymbol{V})\boldsymbol{H}'\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{B}\right]\mathrm{d}\boldsymbol{B}\mathrm{d}\boldsymbol{H} \\ = \ & K\mathrm{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)|\boldsymbol{V}|^{\frac{1}{2}n-\frac{1}{2}(p+1)}\ \Gamma_p\left(\tfrac{n+m}{2}\right) \\ & \cdot\int_{O(p)}\left|\tfrac{1}{2}\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{H}(\boldsymbol{I}_p+\boldsymbol{V})\boldsymbol{H}'\boldsymbol{\Sigma}^{-\frac{1}{2}}\right|^{-\frac{1}{2}(n+m)}\ {}_1F_1\left(\tfrac{n+m}{2};\tfrac{m}{2};\tfrac{2}{4}\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{H}(\boldsymbol{I}_p+\boldsymbol{V})^{-1}\boldsymbol{H}'\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\right)\mathrm{d}\boldsymbol{H} \\ = \ & \left\{\beta_p\left(\tfrac{n}{2},\tfrac{m}{2}\right)\right\}^{-1}\mathrm{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)|\boldsymbol{V}|^{\frac{1}{2}n-\frac{1}{2}(p+1)}|\boldsymbol{I}_p+\boldsymbol{V}|^{-\frac{1}{2}(n+m)} \\ & \cdot\int_{O(p)}{}_1F_1\left(\tfrac{n+m}{2};\tfrac{m}{2};\tfrac{1}{2}\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{H}(\boldsymbol{I}_p+\boldsymbol{V})^{-1}\boldsymbol{H}'\right)\mathrm{d}\boldsymbol{H}. \end{aligned} \tag{8.6}$$

Since $\boldsymbol{\Theta} = \boldsymbol{\Sigma}^{-1}\boldsymbol{M}\boldsymbol{M}'$ (see [2.10.4]) it follows from [2.6.1] and [2.3.4] that the integral in (8.6) over the orthogonal group can be rewritten as

$$\begin{aligned}
&\sum_{k=0}^{\infty}\sum_{\kappa}\frac{\left(\frac{n+m}{2}\right)_{\kappa}}{\left(\frac{m}{2}\right)_{\kappa}}\frac{1}{k!}\int_{O(p)}C_{\kappa}\left(\tfrac{1}{2}\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{\Sigma}^{-1}\boldsymbol{M}\boldsymbol{M}'\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{H}\left(\boldsymbol{I}_p+\boldsymbol{V}\right)^{-1}\boldsymbol{H}'\right)\mathrm{d}\boldsymbol{H}\\
&=\sum_{k=0}^{\infty}\sum_{\kappa}\frac{\left(\frac{n+m}{2}\right)_{\kappa}}{\left(\frac{m}{2}\right)_{\kappa}}\frac{1}{k!}\frac{C_{\kappa}\left(\frac{1}{2}\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{M}\boldsymbol{M}'\boldsymbol{\Sigma}^{-\frac{1}{2}}\right)C_{\kappa}\left(\left(\boldsymbol{I}_p+\boldsymbol{V}\right)^{-1}\right)}{C_{\kappa}\left(\boldsymbol{I}_p\right)}\\
&=\sum_{k=0}^{\infty}\sum_{\kappa}\frac{\left(\frac{n+m}{2}\right)_{\kappa}}{\left(\frac{m}{2}\right)_{\kappa}}\frac{1}{k!}\int_{O(p)}C_{\kappa}\left(\tfrac{1}{2}\boldsymbol{\Theta}\boldsymbol{H}\left(\boldsymbol{I}_p+\boldsymbol{V}\right)^{-1}\boldsymbol{H}'\right)\mathrm{d}\boldsymbol{H}\\
&=\int_{O(p)}{}_1F_1\left(\tfrac{n+m}{2};\tfrac{m}{2};\tfrac{1}{2}\boldsymbol{\Theta}\boldsymbol{H}\left(\boldsymbol{I}_p+\boldsymbol{V}\right)^{-1}\boldsymbol{H}'\right)\mathrm{d}\boldsymbol{H}.
\end{aligned}\tag{8.7}$$

Substituting (8.7) in (8.6) gives

$$\begin{aligned}
f_s\left(\boldsymbol{V}\right) = {} & \left\{\beta_p\left(\tfrac{n}{2},\tfrac{m}{2}\right)\right\}^{-1}\mathrm{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\left|\boldsymbol{V}\right|^{\frac{1}{2}n-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p+\boldsymbol{V}\right|^{-\frac{1}{2}(n+m)}\\
&\cdot\int_{O(p)}{}_1F_1\left(\tfrac{n+m}{2};\tfrac{m}{2};\tfrac{1}{2}\boldsymbol{\Theta}\boldsymbol{H}\left(\boldsymbol{I}_p+\boldsymbol{V}\right)^{-1}\boldsymbol{H}'\right)\mathrm{d}\boldsymbol{H}.
\end{aligned}\tag{8.8}$$

Since $f_s\left(\boldsymbol{V}\right)\equiv\int_{O(p)}f\left(\boldsymbol{H}\boldsymbol{V}\boldsymbol{H}'\right)\mathrm{d}\boldsymbol{H}$ it follows from (8.8) and [2.9.2] that

$$\begin{aligned}
&f\left(\boldsymbol{H}\boldsymbol{V}\boldsymbol{H}'\right)\\
&= \left\{\beta_p\left(\tfrac{n}{2},\tfrac{m}{2}\right)\right\}^{-1}\mathrm{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\left|\boldsymbol{H}\boldsymbol{V}\boldsymbol{H}'\right|^{\frac{1}{2}n-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p+\boldsymbol{H}\boldsymbol{V}\boldsymbol{H}'\right|^{-\frac{1}{2}(n+m)}\\
&\quad\cdot{}_1F_1\left(\tfrac{n+m}{2};\tfrac{m}{2};\tfrac{1}{2}\boldsymbol{\Theta}\left(\boldsymbol{I}_p+\boldsymbol{H}\boldsymbol{V}\boldsymbol{H}'\right)^{-1}\right).
\end{aligned}$$

From [2.9.2] the pdf of $\boldsymbol{V}$ is identified as

$$\left\{\beta_p\left(\tfrac{n}{2},\tfrac{m}{2}\right)\right\}^{-1}\mathrm{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\left|\boldsymbol{V}\right|^{\frac{1}{2}n-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p+\boldsymbol{V}\right|^{-\frac{1}{2}(n+m)}\;{}_1F_1\left(\tfrac{n+m}{2};\tfrac{m}{2};\tfrac{1}{2}\left(\boldsymbol{I}_p+\boldsymbol{V}\right)^{-1}\boldsymbol{\Theta}\right).\tag{8.9}$$

Next, consider the transformation in (8.1) written in terms of $\boldsymbol{V}$, that is

$$\begin{aligned}
\boldsymbol{U} &= \left(\boldsymbol{I}_p+\boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{S}\boldsymbol{B}^{-\frac{1}{2}}\right)^{-\frac{1}{2}}\boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{S}\boldsymbol{B}^{-\frac{1}{2}}\left(\boldsymbol{I}_p+\boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{S}\boldsymbol{B}^{-\frac{1}{2}}\right)^{-\frac{1}{2}}\\
&= \left(\boldsymbol{I}_p+\boldsymbol{V}\right)^{-\frac{1}{2}}\boldsymbol{V}\left(\boldsymbol{I}_p+\boldsymbol{V}\right)^{-\frac{1}{2}}.
\end{aligned}$$

The Jacobian of the transformation is $J\left(\boldsymbol{V}\rightarrow\boldsymbol{U}\right)=\left|\boldsymbol{I}_p-\boldsymbol{U}\right|^{-(p+1)}$ (see [2.1.9]). Since $\boldsymbol{V}$ commutes with any rational function, $\boldsymbol{U}=\left(\boldsymbol{I}_p+\boldsymbol{V}\right)^{-\frac{1}{2}}\boldsymbol{V}\left(\boldsymbol{I}_p+\boldsymbol{V}\right)^{-\frac{1}{2}}=\left(\boldsymbol{I}_p+\boldsymbol{V}\right)^{-1}\boldsymbol{V}$, that is $\boldsymbol{V}=\boldsymbol{U}\left(\boldsymbol{I}_p-\boldsymbol{U}\right)^{-1}$. Substituting in (8.9) gives

$$\begin{aligned}
&f\left(\boldsymbol{U}\right)\\
&= \left\{\beta_p\left(\tfrac{n}{2},\tfrac{m}{2}\right)\right\}^{-1}\mathrm{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\left|\boldsymbol{I}_p-\boldsymbol{U}\right|^{-(p+1)}\left|\boldsymbol{U}\left(\boldsymbol{I}_p-\boldsymbol{U}\right)^{-1}\right|^{\frac{1}{2}n-\frac{1}{2}(p+1)}\\
&\quad\cdot\left|\boldsymbol{I}_p+\boldsymbol{U}\left(\boldsymbol{I}_p-\boldsymbol{U}\right)^{-1}\right|^{-\frac{1}{2}(n+m)}{}_1F_1\left(\tfrac{n+m}{2};\tfrac{m}{2};\tfrac{1}{2}\left[\boldsymbol{I}_p+\boldsymbol{U}\left(\boldsymbol{I}_p-\boldsymbol{U}\right)^{-1}\right]^{-1}\boldsymbol{\Theta}\right)\\
&= \left\{\beta_p\left(\tfrac{n}{2},\tfrac{m}{2}\right)\right\}^{-1}\mathrm{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\left|\boldsymbol{U}\right|^{\frac{1}{2}n-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\boldsymbol{U}\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)}\;{}_1F_1\left(\tfrac{n+m}{2};\tfrac{m}{2};\tfrac{1}{2}\left(\boldsymbol{I}_p-\boldsymbol{U}\right)\boldsymbol{\Theta}\right).\quad\blacksquare
\end{aligned}$$

Remark 8.1
The result in (8.2) is the same as obtained by Díaz-García and Gutiérrez-Jáimez (2006a) when using the ratio given by (1.1) where $\boldsymbol{S} \sim W_p\left(n, \boldsymbol{I}_p\right)$ and $\boldsymbol{B} \sim W_p\left(m, \boldsymbol{I}_p; \boldsymbol{\Theta}\right)$.

Remark 8.2
The ratio in (8.1) was used to correspond with the rest of the study (see (9.1), (10.1), (11.1) and (12.1)). The following definition of Wishart ratio (see (1.1)) will also give a matrix variate having the noncentral matrix variate beta type I distribution with pdf given by (8.2):

$$\boldsymbol{U} = (\boldsymbol{S}+\boldsymbol{B})^{-\frac{1}{2}} \boldsymbol{S} (\boldsymbol{S}+\boldsymbol{B})^{-\frac{1}{2}}.$$

8.2 Moment of the determinant

The h^{th} moment of the determinant of $\boldsymbol{U}$ where $\boldsymbol{U} \sim B_p^I(n, m; \boldsymbol{\Theta})$ is given in Theorem 8.2.

Theorem 8.2 *(De Waal, 1968, Equation 2.2)*
If $\boldsymbol{U} \sim B_p^I(n, m; \boldsymbol{\Theta})$ *then*

$$\begin{aligned} E\left(|\boldsymbol{U}|^h\right) &= \frac{\Gamma_p\left(\frac{n+m}{2}\right)}{\Gamma_p\left(\frac{n}{2}\right)} \frac{\Gamma_p\left(\frac{n}{2}+h\right)}{\Gamma_p\left(\frac{n+m}{2}+h\right)} \operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) {}_1F_1\left(\tfrac{n+m}{2}; \tfrac{n+m}{2}+h; \tfrac{1}{2}\boldsymbol{\Theta}\right) \\ &= E\left(|\boldsymbol{U}^*|^h\right) \operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) {}_1F_1\left(\tfrac{n+m}{2}; \tfrac{n+m}{2}+h; \tfrac{1}{2}\boldsymbol{\Theta}\right), \end{aligned}$$

where $\operatorname{Re}\left(\frac{n}{2}+h\right) > \frac{1}{2}(p-1)$, $\boldsymbol{U}^* \sim B_p^I(n, m)$ *and* $E\left(|\boldsymbol{U}^*|^h\right)$ *is given by* (3.17).

Remark 8.3
Note that the product moment of $\boldsymbol{U}$ is invariant with respect to symmetrisation of the pdf of $\boldsymbol{U}$ (see Greenacre, 1972, Proposition 3.1.1, page 20).

8.3 Probability density function of the Wilks' statistic

De Waal (1968) derived an asymptotic distribution of a suitable function of the Wilks' statistic Λ_1 in (1.6) where $\boldsymbol{U} \sim B_p^I(n, m; \boldsymbol{\Theta})$. He also considered the linear case, that is when $\boldsymbol{\Theta}$ is of rank one. This is the distribution under some nonnull hypothesis and can be used to calculate the power of the test based on Λ_1. Gupta and Javier (1986) derived the exact distribution of this function of Λ_1 in terms of incomplete gamma functions and then used the distribution to do power computations of the test statistic in the linear case. An exact expression for the pdf of Λ_1 was derived by Bekker, Roux and Arashi (2010) and the result is given in this section.

Theorem 8.3 *(Bekker, Roux and Arashi, 2010)*
Let $\boldsymbol{U} \sim B_p^I(n, m; \boldsymbol{\Theta})$ *with pdf given by* (8.2) *and* $\Lambda_1 = |\boldsymbol{U}|$*, then the pdf of* Λ_1 *is given by*

$$f(\lambda_1) = \frac{1}{\Gamma_p\left(\frac{n}{2}\right)} \operatorname{etr}\left(-\frac{1}{2}\boldsymbol{\Theta}\right) \sum_{k=0}^{\infty} \sum_{\kappa} \frac{1}{k!} \Gamma_p\left(\frac{n+m}{2}, \kappa\right) C_\kappa\left(\frac{1}{2}\boldsymbol{\Theta}\right) G_{p,p}^{p,0}\left(\lambda_1 \Big|_{b_1, \ldots, b_p}^{a_1, \ldots, a_p}\right), \quad 0 < \lambda_1 < 1,$$

where

$a_j = \frac{n+m}{2} + k_j - \frac{1}{2}(j+1)$ *for* $j = 1, 2, \ldots, p,$

$\beta_j = \frac{n}{2} - \frac{1}{2}(j+1)$ *for* $j = 1, 2, \ldots, p.$

9 Noncentral bimatrix variate beta type I distribution

In this section an exact expression for the pdf of the noncentral bimatrix variate beta type I distribution with $(\boldsymbol{U}_1, \boldsymbol{U}_2)$ defined as in (4.3) is derived, as well as the corresponding product moment of the determinants and the pdf of $\Lambda_2 = |\boldsymbol{U}_1|^{\frac{1}{2}n_1} |\boldsymbol{U}_2|^{\frac{1}{2}n_2}$ (see (1.7)).

9.1 Probability density function

De Waal (1972) derived the pdf of $(\boldsymbol{U}_1, \boldsymbol{U}_2) \sim BB_p^I(n_1, n_2, m; \boldsymbol{\Theta})$ defined as in (1.2) when $\boldsymbol{B}$ is noncentral, with the result given in integral form as

$$\begin{aligned}
&\left[\Gamma_p\left(\tfrac{n_1}{2}\right)\Gamma_p\left(\tfrac{n_2}{2}\right)\Gamma_p\left(\tfrac{m}{2}\right)|2\boldsymbol{\Sigma}|^{\frac{1}{2}(n_1+n_2+m)}\right]^{-1}\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\\
&\cdot\prod_{i=1}^{2}|\boldsymbol{U}_i|^{\frac{1}{2}n_i-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\sum_{i=1}^{2}\boldsymbol{U}_i\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)}\\
&\cdot\int_{\boldsymbol{S}>\boldsymbol{0}}|\boldsymbol{S}|^{\frac{1}{2}(n_1+n_2+m)-\frac{1}{2}(p+1)}\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}\right)\,{}_0F_1\left(\tfrac{m}{2};\tfrac{1}{4}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}^{\frac{1}{2}}\left(\boldsymbol{I}_p-\sum_{i=1}^{2}\boldsymbol{U}_i\right)\boldsymbol{S}^{\frac{1}{2}}\right)\mathrm{d}\boldsymbol{S},
\end{aligned}$$

$\boldsymbol{0} < \boldsymbol{U}_i < \boldsymbol{I}_p$, $i = 1, 2$, $\boldsymbol{0} < \sum_{i=1}^{2}\boldsymbol{U}_i < \boldsymbol{I}_p$. Troskie (1967) derived the pdf of $(\boldsymbol{U}_1, \boldsymbol{U}_2)$ for the linear case, that is when $\boldsymbol{B} \sim W_p(m, \boldsymbol{\Sigma}, \boldsymbol{\Theta})$ and $\boldsymbol{\Theta} = \operatorname{diag}(\theta, 0, \ldots, 0)$. In Theorem 9.1 an exact expression for the pdf of $(\boldsymbol{U}_1, \boldsymbol{U}_2) \sim BB_p^I(n_1, n_2, m; \boldsymbol{\Theta})$ is derived.

Theorem 9.1
Let $\boldsymbol{S}_1 \sim W_p(n_1, \boldsymbol{\Sigma})$, $\boldsymbol{S}_2 \sim W_p(n_2, \boldsymbol{\Sigma})$ *and* $\boldsymbol{B} \sim W_p(m, \boldsymbol{\Sigma}, \boldsymbol{\Theta})$ *be independently distributed. As in* (4.3) *define*

$$\boldsymbol{U}_i = \left(\boldsymbol{I}_p + \sum_{i=1}^{2}\boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{S}_i\boldsymbol{B}^{-\frac{1}{2}}\right)^{-\frac{1}{2}}\boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{S}_i\boldsymbol{B}^{-\frac{1}{2}}\left(\boldsymbol{I}_p + \sum_{i=1}^{2}\boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{S}_i\boldsymbol{B}^{-\frac{1}{2}}\right)^{-\frac{1}{2}}, \quad i = 1, 2. \tag{9.1}$$

The pdf of $(\boldsymbol{U}_1, \boldsymbol{U}_2) \sim BB_p^I(n_1, n_2, m; \boldsymbol{\Theta})$ *is*

$$\begin{aligned}
&f(\boldsymbol{U}_1, \boldsymbol{U}_2)\\
=\;&\left\{\beta_p\left(\tfrac{n_1}{2}, \tfrac{n_2}{2}; \tfrac{m}{2}\right)\right\}^{-1}\prod_{i=1}^{2}|\boldsymbol{U}_i|^{\frac{1}{2}n_i-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\sum_{i=1}^{2}\boldsymbol{U}_i\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)}\\
&\cdot\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\,{}_1F_1\left(\tfrac{n_1+n_2+m}{2};\tfrac{m}{2};\tfrac{1}{2}\left(\boldsymbol{I}_p-\sum_{i=1}^{2}\boldsymbol{U}_i\right)\boldsymbol{\Theta}\right)\\
=\;&g(\boldsymbol{U}_1, \boldsymbol{U}_2)\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\,{}_1F_1\left(\tfrac{n_1+n_2+m}{2};\tfrac{m}{2};\tfrac{1}{2}\left(\boldsymbol{I}_p-\sum_{i=1}^{2}\boldsymbol{U}_i\right)\boldsymbol{\Theta}\right),
\end{aligned}\tag{9.2}$$

$\boldsymbol{0} < \boldsymbol{U}_i < \boldsymbol{I}_p$, $i = 1, 2$, $\boldsymbol{0} < \sum_{i=1}^{2}\boldsymbol{U}_i < \boldsymbol{I}_p$, *where* $n_i > (p-1)$, $i = 1, 2$, $m > (p-1)$ *and* $g(\cdot)$ *is the pdf of* $BB_p^I(n_1, n_2, m)$ *given by* (4.2).

Proof:
The pdf of $(\boldsymbol{S}_1, \boldsymbol{S}_2, \boldsymbol{B})$ is given by

$$K\prod_{i=1}^{2}\left[\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}_i\right)|\boldsymbol{S}_i|^{\frac{1}{2}(n_i-p-1)}\right]\left[\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\right)|\boldsymbol{B}|^{\frac{1}{2}(m-p-1)}\,{}_0F_1\left(\tfrac{m}{2};\tfrac{1}{4}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\right)\right] \tag{9.3}$$

where $K^{-1} = \Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)\Gamma_p\left(\frac{m}{2}\right)|2\boldsymbol{\Sigma}|^{\frac{1}{2}(n_1+n_2+m)}$ (see [2.10.2]).

Making the transformations $\boldsymbol{V}_i = \boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{S}_i\boldsymbol{B}^{-\frac{1}{2}}$, $i = 1, 2$, with Jacobian $J(\boldsymbol{S}_1, \boldsymbol{S}_2, \boldsymbol{B} \to \boldsymbol{V}_1, \boldsymbol{V}_2, \boldsymbol{B}) = |\boldsymbol{B}|^{p+1}$ (see [2.1.4] and [2.1.6]) substituted in (9.3) and integrated with respect to $\boldsymbol{B}$ gives the pdf of $(\boldsymbol{V}_1, \boldsymbol{V}_2)$ as

$$
\begin{aligned}
& f(\boldsymbol{V}_1, \boldsymbol{V}_2) \\
= \quad & K\mathrm{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)|\boldsymbol{V}_1|^{\frac{1}{2}n_1-\frac{1}{2}(p+1)}|\boldsymbol{V}_2|^{\frac{1}{2}n_2-\frac{1}{2}(p+1)} \\
& \cdot\int_{\boldsymbol{B}>\boldsymbol{0}} \mathrm{etr}\left[-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}^{\frac{1}{2}}(\boldsymbol{I}_p + \boldsymbol{V}_1 + \boldsymbol{V}_2)\boldsymbol{B}^{\frac{1}{2}}\right]|\boldsymbol{B}|^{\frac{1}{2}(n_1+n_2+m)-\frac{1}{2}(p+1)}\ {}_0F_1\left(\tfrac{m}{2}; \tfrac{1}{4}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\right)\mathrm{d}\boldsymbol{B}.
\end{aligned}
\tag{9.4}
$$

We consider the symmetrised density function of $(\boldsymbol{V}_1, \boldsymbol{V}_2)$ (see [2.9.1]), that is $f_s(\boldsymbol{V}_1, \boldsymbol{V}_2) \equiv \int_{O(p)} f(\boldsymbol{H}\boldsymbol{V}_1\boldsymbol{H}', \boldsymbol{H}\boldsymbol{V}_2\boldsymbol{H}')\mathrm{d}\boldsymbol{H}$ where $\boldsymbol{H}$ $(p\times p)$ is orthogonal and $\mathrm{d}\boldsymbol{H}$ is the normalised Haar invariant measure on $O(p)$. From (9.4)

$$
\begin{aligned}
& f(\boldsymbol{H}\boldsymbol{V}_1\boldsymbol{H}', \boldsymbol{H}\boldsymbol{V}_2\boldsymbol{H}') \\
= \quad & K\mathrm{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\prod_{i=1}^{2}|\boldsymbol{H}\boldsymbol{V}_i\boldsymbol{H}'|^{\frac{1}{2}n_i-\frac{1}{2}(p+1)} \\
& \cdot\int_{\boldsymbol{B}>\boldsymbol{0}}|\boldsymbol{B}|^{\frac{1}{2}(n_1+n_2+m)-\frac{1}{2}(p+1)}\mathrm{etr}\left[-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}^{\frac{1}{2}}\left(\boldsymbol{I}_p + \sum_{i=1}^{2}\boldsymbol{H}\boldsymbol{V}_i\boldsymbol{H}'\right)\boldsymbol{B}^{\frac{1}{2}}\right]{}_0F_1\left(\tfrac{m}{2}; \tfrac{1}{4}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\right)\mathrm{d}\boldsymbol{B}.
\end{aligned}
\tag{9.5}
$$

Hence, from (9.5), [2.3.6] and [2.6.6], we get

$$
\begin{aligned}
& f_s(\boldsymbol{V}_1, \boldsymbol{V}_2) \\
= \quad & K\mathrm{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\prod_{i=1}^{2}|\boldsymbol{V}_i|^{\frac{1}{2}n_i-\frac{1}{2}(p+1)} \\
& \cdot\int_{\boldsymbol{B}>\boldsymbol{0}}|\boldsymbol{B}|^{\frac{1}{2}(n_1+n_2+m)-\frac{1}{2}(p+1)}\ {}_0F_1\left(\tfrac{m}{2}; \tfrac{1}{4}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\right)\int_{O(p)}\mathrm{etr}\left[-\tfrac{1}{2}\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{B}\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{H}\left(\boldsymbol{I}_p + \sum_{i=1}^{2}\boldsymbol{V}_i\right)\boldsymbol{H}'\right]\mathrm{d}\boldsymbol{H}\mathrm{d}\boldsymbol{B} \\
= \quad & K\mathrm{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\prod_{i=1}^{2}|\boldsymbol{V}_i|^{\frac{1}{2}n_i-\frac{1}{2}(p+1)} \\
& \cdot\int_{O(p)}\int_{\boldsymbol{B}>\boldsymbol{0}}|\boldsymbol{B}|^{\frac{1}{2}(n_1+n_2+m)-\frac{1}{2}(p+1)}\ {}_0F_1\left(\tfrac{m}{2}; \tfrac{1}{4}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\right)\mathrm{etr}\left[-\tfrac{1}{2}\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{H}\left(\boldsymbol{I}_p + \sum_{i=1}^{2}\boldsymbol{V}_i\right)\boldsymbol{H}'\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{B}\right]\mathrm{d}\boldsymbol{B}\mathrm{d}\boldsymbol{H} \\
= \quad & K\mathrm{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\prod_{i=1}^{2}|\boldsymbol{V}_i|^{\frac{1}{2}n_i-\frac{1}{2}(p+1)}\Gamma_p\left(\tfrac{n_1+n_2+m}{2}\right)\int_{O(p)}\left|\tfrac{1}{2}\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{H}\left(\boldsymbol{I}_p + \sum_{i=1}^{2}\boldsymbol{V}_i\right)\boldsymbol{H}'\boldsymbol{\Sigma}^{-\frac{1}{2}}\right|^{-\frac{1}{2}(n_1+n_2+m)} \\
& \qquad\cdot{}_1F_1\left(\tfrac{n_1+n_2+m}{2}; \tfrac{m}{2}; \tfrac{2}{4}\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{H}\left(\boldsymbol{I}_p + \sum_{i=1}^{2}\boldsymbol{V}_i\right)^{-1}\boldsymbol{H}'\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\right)\mathrm{d}\boldsymbol{H} \\
= \quad & \left\{\beta_p\left(\tfrac{n_1}{2}, \tfrac{n_2}{2}; \tfrac{m}{2}\right)\right\}^{-1}\mathrm{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\prod_{i=1}^{2}|\boldsymbol{V}_i|^{\frac{1}{2}n_i-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p + \sum_{i=1}^{2}\boldsymbol{V}_i\right|^{-\frac{1}{2}(n_1+n_2+m)} \\
& \cdot\int_{O(p)}{}_1F_1\left(\tfrac{n_1+n_2+m}{2}; \tfrac{m}{2}; \tfrac{1}{2}\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{H}\left(\boldsymbol{I}_p + \sum_{i=1}^{2}\boldsymbol{V}_i\right)^{-1}\boldsymbol{H}'\right)\mathrm{d}\boldsymbol{H}.
\end{aligned}
\tag{9.6}
$$

Since $\boldsymbol{\Theta}=\boldsymbol{\Sigma}^{-1}\boldsymbol{M}\boldsymbol{M}'$ (see [2.10.4]) it follows from [2.6.1], [2.3.4] and [2.3.5] that the integral in (9.6) over the orthogonal group can be rewritten as

$$\begin{aligned}
&\sum_{k=0}^{\infty}\sum_{\kappa}\frac{\left(\frac{n_1+n_2+m}{2}\right)_\kappa}{\left(\frac{m}{2}\right)_\kappa}\frac{1}{k!}\int_{O(p)}C_\kappa\left(\tfrac{1}{2}\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{\Sigma}^{-1}\boldsymbol{M}\boldsymbol{M}'\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{H}\left(\boldsymbol{I}_p+\sum_{i=1}^{2}\boldsymbol{V}_i\right)^{-1}\boldsymbol{H}'\right)\mathrm{d}\boldsymbol{H}\\
&=\sum_{k=0}^{\infty}\sum_{\kappa}\frac{\left(\frac{n_1+n_2+m}{2}\right)_\kappa}{\left(\frac{m}{2}\right)_\kappa}\frac{1}{k!}\frac{C_\kappa\left(\frac{1}{2}\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{M}\boldsymbol{M}'\boldsymbol{\Sigma}^{-\frac{1}{2}}\right)C_\kappa\left(\left(\boldsymbol{I}_p+\sum_{i=1}^{2}\boldsymbol{V}_i\right)^{-1}\right)}{C_\kappa(\boldsymbol{I}_p)}\\
&=\sum_{k=0}^{\infty}\sum_{\kappa}\frac{\left(\frac{n_1+n_2+m}{2}\right)_\kappa}{\left(\frac{m}{2}\right)_\kappa}\frac{1}{k!}\int_{O(p)}C_\kappa\left(\tfrac{1}{2}\boldsymbol{\Theta}\boldsymbol{H}\left(\boldsymbol{I}_p+\sum_{i=1}^{2}\boldsymbol{V}_i\right)^{-1}\boldsymbol{H}'\right)\mathrm{d}\boldsymbol{H}\\
&=\int_{O(p)}{}_1F_1\left(\tfrac{n_1+n_2+m}{2};\tfrac{m}{2};\tfrac{1}{2}\boldsymbol{\Theta}\boldsymbol{H}\left(\boldsymbol{I}_p+\sum_{i=1}^{2}\boldsymbol{V}_i\right)^{-1}\boldsymbol{H}'\right)\mathrm{d}\boldsymbol{H}.
\end{aligned}\tag{9.7}$$

Substituting (9.7) in (9.6) gives

$$\begin{aligned}
f_s(\boldsymbol{V}_1,\boldsymbol{V}_2)=&\left\{\beta_p\left(\tfrac{n_1}{2},\tfrac{n_2}{2};\tfrac{m}{2}\right)\right\}^{-1}\mathrm{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\prod_{i=1}^{2}|\boldsymbol{V}_i|^{\frac{1}{2}n_i-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p+\sum_{i=1}^{2}\boldsymbol{V}_i\right|^{-\frac{1}{2}(n_1+n_2+m)}\\
&\cdot\int_{O(p)}{}_1F_1\left(\tfrac{n_1+n_2+m}{2};\tfrac{m}{2};\tfrac{1}{2}\boldsymbol{\Theta}\boldsymbol{H}\left(\boldsymbol{I}_p+\sum_{i=1}^{2}\boldsymbol{V}_i\right)^{-1}\boldsymbol{H}'\right)\mathrm{d}\boldsymbol{H}.
\end{aligned}\tag{9.8}$$

Since $f_s(\boldsymbol{V}_1,\boldsymbol{V}_2)\equiv\int_{O(p)}f\left(\boldsymbol{H}\boldsymbol{V}_1\boldsymbol{H}',\boldsymbol{H}\boldsymbol{V}_2\boldsymbol{H}'\right)\mathrm{d}\boldsymbol{H}$ it follows from [2.9.2] that the pdf of $(\boldsymbol{V}_1,\boldsymbol{V}_2)$ is

$$\begin{aligned}
f(\boldsymbol{V}_1,\boldsymbol{V}_2)\;=\;&\left\{\beta_p\left(\tfrac{n_1}{2},\tfrac{n_2}{2};\tfrac{m}{2}\right)\right\}^{-1}\mathrm{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\prod_{i=1}^{2}|\boldsymbol{V}_i|^{\frac{1}{2}n_i-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p+\sum_{i=1}^{2}\boldsymbol{V}_i\right|^{-\frac{1}{2}(n_1+n_2+m)}\\
&\cdot{}_1F_1\left(\tfrac{n_1+n_2+m}{2};\tfrac{m}{2};\tfrac{1}{2}\left(\boldsymbol{I}_p+\sum_{i=1}^{2}\boldsymbol{V}_i\right)^{-1}\boldsymbol{\Theta}\right),\quad\boldsymbol{V}_i>\boldsymbol{0},\ i=1,2.
\end{aligned}\tag{9.9}$$

Next, consider the transformations in (9.1) written in terms of $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$, that is

$$\begin{aligned}
\boldsymbol{U}_i\;&=\;\left(\boldsymbol{I}_p+\sum_{i=1}^{2}\boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{S}_i\boldsymbol{B}^{-\frac{1}{2}}\right)^{-\frac{1}{2}}\boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{S}_i\boldsymbol{B}^{-\frac{1}{2}}\left(\boldsymbol{I}_p+\sum_{i=1}^{2}\boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{S}_i\boldsymbol{B}^{-\frac{1}{2}}\right)^{-\frac{1}{2}}\\
&=\;\left(\boldsymbol{I}_p+\sum_{i=1}^{2}\boldsymbol{V}_i\right)^{-\frac{1}{2}}\boldsymbol{V}_i\left(\boldsymbol{I}_p+\sum_{i=1}^{2}\boldsymbol{V}_i\right)^{-\frac{1}{2}},\quad i=1,2.
\end{aligned}\tag{9.10}$$

Let $\boldsymbol{Z} = \boldsymbol{I}_p + \sum_{i=1}^{2} \boldsymbol{V}_i$, then $\boldsymbol{U}_1 = \boldsymbol{Z}^{-\frac{1}{2}} \boldsymbol{V}_1 \boldsymbol{Z}^{-\frac{1}{2}}$, $\boldsymbol{U}_2 = \boldsymbol{I}_p - \boldsymbol{Z}^{-1} - \boldsymbol{U}_1$ and $\boldsymbol{Z} = \left(\boldsymbol{I}_p - \sum_{i=1}^{2} \boldsymbol{U}_i\right)^{-1}$. From [2.1.2], [2.1.3], [2.1.7] and [2.1.8] the Jacobian of the transformations in (9.10) is

$$\begin{aligned}
&J\left(\boldsymbol{V}_1, \boldsymbol{V}_2 \rightarrow \boldsymbol{U}_1, \boldsymbol{U}_2\right) \\
&= J\left(\boldsymbol{V}_1, \boldsymbol{V}_2 \rightarrow \boldsymbol{U}_1, \boldsymbol{Z}\right) \cdot J\left(\boldsymbol{U}_1, \boldsymbol{Z} \rightarrow \boldsymbol{U}_1, \boldsymbol{U}_2\right) \\
&= |\boldsymbol{Z}|^{\frac{1}{2}(p+1)} |\boldsymbol{Z}|^{p+1} \\
&= \left|\boldsymbol{I}_p - \sum_{i=1}^{2} \boldsymbol{U}_i\right|^{-\frac{3}{2}(p+1)}
\end{aligned} \tag{9.11}$$

and (9.10) can be written as

$$\boldsymbol{V}_i = \boldsymbol{Z}^{\frac{1}{2}} \boldsymbol{U}_i \boldsymbol{Z}^{\frac{1}{2}} = \left(\boldsymbol{I}_p - \sum_{i=1}^{2} \boldsymbol{U}_i\right)^{-\frac{1}{2}} \boldsymbol{U}_i \left(\boldsymbol{I}_p - \sum_{i=1}^{2} \boldsymbol{U}_i\right)^{-\frac{1}{2}}, \quad i = 1, 2. \tag{9.12}$$

Substituting (9.11) and (9.12) in (9.9) gives

$$\begin{aligned}
&f\left(\boldsymbol{U}_1, \boldsymbol{U}_2\right) \\
&= \left\{\beta_p\left(\tfrac{n_1}{2}, \tfrac{n_2}{2}; \tfrac{m}{2}\right)\right\}^{-1} \operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) \left|\boldsymbol{I}_p - \sum_{i=1}^{2} \boldsymbol{U}_i\right|^{-\frac{3}{2}(p+1)} \prod_{i=1}^{2} \left|\left(\boldsymbol{I}_p - \sum_{i=1}^{2} \boldsymbol{U}_i\right)^{-\frac{1}{2}} \boldsymbol{U}_i \left(\boldsymbol{I}_p - \sum_{i=1}^{2} \boldsymbol{U}_i\right)^{-\frac{1}{2}}\right|^{\frac{1}{2}n_i - \frac{1}{2}(p+1)} \\
&\quad \cdot \left|\boldsymbol{I}_p + \sum_{i=1}^{2} \left(\boldsymbol{I}_p - \sum_{i=1}^{2} \boldsymbol{U}_i\right)^{-\frac{1}{2}} \boldsymbol{U}_i \left(\boldsymbol{I}_p - \sum_{i=1}^{2} \boldsymbol{U}_i\right)^{-\frac{1}{2}}\right|^{-\frac{1}{2}(n_1+n_2+m)} \\
&\quad \cdot {}_1F_1\left(\tfrac{n_1+n_2+m}{2}; \tfrac{m}{2}; \tfrac{1}{2}\left[\boldsymbol{I}_p + \sum_{i=1}^{2} \left(\boldsymbol{I}_p - \sum_{i=1}^{2} \boldsymbol{U}_i\right)^{-\frac{1}{2}} \boldsymbol{U}_i \left(\boldsymbol{I}_p - \sum_{i=1}^{2} \boldsymbol{U}_i\right)^{-\frac{1}{2}}\right]^{-1} \boldsymbol{\Theta}\right) \\
&= \left\{\beta_p\left(\tfrac{n_1}{2}, \tfrac{n_2}{2}; \tfrac{m}{2}\right)\right\}^{-1} \operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) \prod_{i=1}^{2} |\boldsymbol{U}_i|^{\frac{1}{2}n_i - \frac{1}{2}(p+1)} \left|\boldsymbol{I}_p - \sum_{i=1}^{2} \boldsymbol{U}_i\right|^{-\frac{1}{2}(n_1+n_2) - \frac{1}{2}(p+1)} \\
&\quad \cdot \left|\boldsymbol{I}_p - \sum_{i=1}^{2} \boldsymbol{U}_i\right|^{\frac{1}{2}(n_1+n_2+m)} \left|\boldsymbol{I}_p - \sum_{i=1}^{2} \boldsymbol{U}_i + \sum_{i=1}^{2} \boldsymbol{U}_i\right|^{-\frac{1}{2}(n_1+n_2+m)} \\
&\quad \cdot {}_1F_1\left(\tfrac{n_1+n_2+m}{2}; \tfrac{m}{2}; \tfrac{1}{2}\left(\boldsymbol{I}_p - \sum_{i=1}^{2} \boldsymbol{U}_i\right)^{\frac{1}{2}} \left(\boldsymbol{I}_p - \sum_{i=1}^{2} \boldsymbol{U}_i + \sum_{i=1}^{2} \boldsymbol{U}_i\right)^{-1} \left(\boldsymbol{I}_p - \sum_{i=1}^{2} \boldsymbol{U}_i\right)^{\frac{1}{2}} \boldsymbol{\Theta}\right) \\
&= \left\{\beta_p\left(\tfrac{n_1}{2}, \tfrac{n_2}{2}; \tfrac{m}{2}\right)\right\}^{-1} \operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) \prod_{i=1}^{2} |\boldsymbol{U}_i|^{\frac{1}{2}n_i - \frac{1}{2}(p+1)} \left|\boldsymbol{I}_p - \sum_{i=1}^{2} \boldsymbol{U}_i\right|^{\frac{1}{2}m - \frac{1}{2}(p+1)} \\
&\quad \cdot {}_1F_1\left(\tfrac{n_1+n_2+m}{2}; \tfrac{m}{2}; \tfrac{1}{2}\left(\boldsymbol{I}_p - \sum_{i=1}^{2} \boldsymbol{U}_i\right) \boldsymbol{\Theta}\right). \quad \blacksquare
\end{aligned}$$

Remark 9.1

The ratio in (9.1) was used to derive the exact expression for the pdf of $(\boldsymbol{U}_1, \boldsymbol{U}_2)$ given by (9.2). Using the ratio in (4.1) to derive the pdf of $(\boldsymbol{U}_1, \boldsymbol{U}_2)$ gives an expression for $f(\boldsymbol{U}_1, \boldsymbol{U}_2)$ in integral form (de Waal, 1972).

Remark 9.2

The noncentral matrix variate Dirichlet I distribution, denoted by $(\boldsymbol{U}_1, \dots, \boldsymbol{U}_r) \sim D_p^I(n_1, \dots, n_r, m; \boldsymbol{\Theta})$, results by extending (9.1) to r independent Wishart matrix variates, $\boldsymbol{S}_i \sim W_p(n_i, \boldsymbol{\Sigma})$, $i = 1, \dots, r$, all independent of $\boldsymbol{B} \sim W_p(m, \boldsymbol{\Sigma}; \boldsymbol{\Theta})$. The pdf of $(\boldsymbol{U}_1, \dots, \boldsymbol{U}_r)$ is given by

$$\begin{aligned}
&\left\{\beta_p\left(\tfrac{n_1}{2}, \dots, \tfrac{n_r}{2}; \tfrac{m}{2}\right)\right\}^{-1} \operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) \prod_{i=1}^{r} |\boldsymbol{U}_i|^{\frac{1}{2}n_i - \frac{1}{2}(p+1)} \left|\boldsymbol{I}_p - \sum_{i=1}^{r} \boldsymbol{U}_i\right|^{\frac{1}{2}m - \frac{1}{2}(p+1)} \\
&\cdot {}_1F_1\left(\tfrac{n_1 + \dots + n_r + m}{2}; \tfrac{m}{2}; \tfrac{1}{2}\left(\boldsymbol{I}_p - \sum_{i=1}^{r} \boldsymbol{U}_i\right)\boldsymbol{\Theta}\right),
\end{aligned} \tag{9.13}$$

$\boldsymbol{0} < \boldsymbol{U}_i < \boldsymbol{I}_p$, $i = 1, \dots, r$, $\boldsymbol{0} < \sum_{i=1}^{r} \boldsymbol{U}_i < \boldsymbol{I}_p$.

Sánchez and Nagar (2002, 2003) extended (1.2) to r independent Wishart matrix variates, $\boldsymbol{S}_i \sim W_p(n_i, \boldsymbol{\Sigma}; \boldsymbol{\Theta}_i)$, $i = 1, \dots, r$, all independent of $\boldsymbol{B} \sim W_p(m, \boldsymbol{\Sigma}; \boldsymbol{\Theta}_{r+1})$, where $\boldsymbol{\Theta}_i = \operatorname{diag}(\theta_i^2, 0, \dots, 0)$ and derived the pdf of $(\boldsymbol{U}_1, \dots, \boldsymbol{U}_r)$.

Remark 9.3

The pdfs of the noncentral bimatrix variate beta type I distribution in (9.2) and the noncentral matrix variate Dirichlet type I distribution in (9.13) are members of the Liouville family of distributions of the second kind (see [2.2.1]).

9.2 Product moment of the determinants

The $(h_1, h_2)^{th}$ product moment, $E\left(|\boldsymbol{U}_1|^{h_1} |\boldsymbol{U}_2|^{h_2}\right)$, where $(\boldsymbol{U}_1, \boldsymbol{U}_2) \sim BB_p^I(n_1, n_2, m; \boldsymbol{\Theta})$, is given in Theorem 9.2.

Theorem 9.2 *(In analogy to de Waal, 1972)*
If $(\boldsymbol{U}_1, \boldsymbol{U}_2) \sim BB_p^I(n_1, n_2, m; \boldsymbol{\Theta})$ *as given by* (9.2) *then*

$$\begin{aligned}
E\left(|\boldsymbol{U}_1|^{h_1} |\boldsymbol{U}_2|^{h_2}\right) &= \frac{\Gamma_p\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)} \frac{\Gamma_p\left(\frac{n_1}{2}+h_1\right)\Gamma_p\left(\frac{n_2}{2}+h_2\right)}{\Gamma_p\left(\frac{n_1+n_2+m}{2}+h_1+h_2\right)} \\
&\quad \cdot \operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) {}_1F_1\left(\tfrac{n_1+n_2+m}{2}; \tfrac{n_1+n_2+m}{2} + h_1 + h_2; \tfrac{1}{2}\boldsymbol{\Theta}\right) \\
&= E\left(|\boldsymbol{U}_1^*|^{h_1} |\boldsymbol{U}_2^*|^{h_2}\right) \operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) {}_1F_1\left(\tfrac{n_1+n_2+m}{2}; \tfrac{n_1+n_2+m}{2} + h_1 + h_2; \tfrac{1}{2}\boldsymbol{\Theta}\right),
\end{aligned} \tag{9.14}$$

where $\operatorname{Re}\left(\frac{n_i}{2} + h_i\right) > \frac{1}{2}(p-1)$, $i = 1, 2$, $(\boldsymbol{U}_1^*, \boldsymbol{U}_2^*) \sim BB_p^I(n_1, n_2, m)$ *and* $E\left(|\boldsymbol{U}_1^*|^{h_1} |\boldsymbol{U}_2^*|^{h_2}\right)$ *is given by* (4.8).

9.3 Distribution of the product of determinants

De Waal (1972) derived an asymptotic distribution of a suitable function of the product of determinants of the bimatrix beta type I variates, Λ_2 in (1.7) where $(\boldsymbol{U}_1, \boldsymbol{U}_2) \sim BB_p^I(n_1, n_2, m; \boldsymbol{\Theta})$. Gupta and Nagar (1987) also derived asymptotic and exact distributions of this function of Λ_2. They also considered the linear case, that is when $\boldsymbol{\Theta}$ is of rank one. In Theorem 9.3 an exact expression is derived for the pdf of $\Lambda_2 = |\boldsymbol{U}_1|^{\frac{1}{2}n_1} |\boldsymbol{U}_2|^{\frac{1}{2}n_2}$ where $(\boldsymbol{U}_1, \boldsymbol{U}_2) \sim BB_p^I(n_1, n_2, m; \boldsymbol{\Theta})$ as given by (9.2).

Theorem 9.3
Let $(\boldsymbol{U}_1, \boldsymbol{U}_2) \sim BB_p^I(n_1, n_2, m; \boldsymbol{\Theta})$ *with pdf given by* (9.2) *and let* $\Lambda_2 = |\boldsymbol{U}_1|^{\frac{1}{2}n_1} |\boldsymbol{U}_2|^{\frac{1}{2}n_2}$, *then the pdf of* Λ_2 *is given by*

$$\frac{\pi^{\frac{1}{4}p(p-1)}}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)} \operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) \sum_{k=0}^{\infty}\sum_{\kappa} \frac{1}{k!}\Gamma_p\left(\tfrac{n_1+n_2+m}{2}, \kappa\right) C_\kappa\left(\tfrac{1}{2}\boldsymbol{\Theta}\right) H_{p,2p}^{2p,0}\left(\lambda_2 \Big|_{(b_1,\beta_1),\ldots,(b_{2p},\beta_{2p})}^{(a_1,\alpha_1),\ldots,(a_p,\alpha_p)}\right), \tag{9.15}$$

$0 < \lambda_2 < 1$, *where* $H(\cdot)$ *denotes Fox's H-function (see [2.8.3]) and*

$a_j = \frac{m}{2} + k_j - \frac{1}{2}(j-1)$ *for* $j = 1, 2, \ldots, p$,

$\alpha_j = \frac{n_1+n_2}{2}$ *for* $j = 1, 2, \ldots, p$,

$$b_j = \begin{cases} -\frac{1}{4}(j-1) & \text{for } j = 1, 3, 5, \ldots, 2p-1 \\ -\frac{1}{4}(j-2) & \text{for } j = 2, 4, 6, \ldots, 2p, \end{cases}$$

$$\beta_j = \begin{cases} \frac{n_1}{2} & \text{for } j = 1, 3, 5, \ldots, 2p-1 \\ \frac{n_2}{2} & \text{for } j = 2, 4, 6, \ldots, 2p. \end{cases}$$

Proof:
Using (9.14), [2.6.1] and [2.3.3] the Mellin transform (see [2.8.1]) of $f(\lambda_2)$ can be written as

$$\begin{aligned}
M_f(h) &\equiv E\left(\Lambda_2^{h-1}\right) \\
&= E\left[\left(|\boldsymbol{U}_1|^{\frac{1}{2}n_1} |\boldsymbol{U}_2|^{\frac{1}{2}n_2}\right)^{h-1}\right] \\
&= \frac{\Gamma_p\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)\Gamma_p\left(\frac{m}{2}\right)} \frac{\Gamma_p\left[\frac{n_1}{2}+\frac{n_1}{2}(h-1)\right]\Gamma_p\left[\frac{n_2}{2}+\frac{n_2}{2}(h-1)\right]\Gamma_p\left(\frac{m}{2}\right)}{\Gamma_p\left[\frac{n_1+n_2+m}{2}+\frac{n_1+n_2}{2}(h-1)\right]} \operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) \\
&\quad \cdot {}_1F_1\left(\tfrac{n_1+n_2+m}{2}; \tfrac{n_1+n_2+m}{2} + \tfrac{n_1+n_2}{2}(h-1); \tfrac{1}{2}\boldsymbol{\Theta}\right) \\
&= \frac{\Gamma_p\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)} \frac{\Gamma_p\left(\frac{n_1}{2}h\right)\Gamma_p\left(\frac{n_2}{2}h\right)}{\Gamma_p\left(\frac{m}{2}+\frac{n_1+n_2}{2}h\right)} \operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) \sum_{k=0}^{\infty}\sum_{\kappa} \frac{1}{k!} C_\kappa\left(\tfrac{1}{2}\boldsymbol{\Theta}\right) \frac{\left(\frac{n_1+n_2+m}{2}\right)_\kappa}{\left(\frac{m}{2}+\frac{n_1+n_2}{2}h\right)_\kappa} \\
&= \frac{\Gamma_p\left(\frac{n_1}{2}h\right)\Gamma_p\left(\frac{n_2}{2}h\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)} \operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) \sum_{k=0}^{\infty}\sum_{\kappa} \frac{1}{k!} \frac{\Gamma_p\left(\frac{n_1+n_2+m}{2},\kappa\right)}{\Gamma_p\left(\frac{m}{2}+\frac{n_1+n_2}{2}h,\kappa\right)} C_\kappa\left(\tfrac{1}{2}\boldsymbol{\Theta}\right).
\end{aligned} \tag{9.16}$$

From [2.3.3] the generalised gamma function of weight κ in (9.16) can be written as

$$\Gamma_p\left(\frac{m}{2}+\frac{n_1+n_2}{2}h,\kappa\right)$$
$$=\pi^{\frac{1}{4}p(p-1)}\prod_{j=1}^{p}\Gamma\left(a_j+\alpha_j h\right), \tag{9.17}$$

where $a_j=\frac{m}{2}+k_j-\frac{1}{2}(j-1)$ for $j=1,2,3,\ldots,p$

and $\alpha_j=\frac{n_1+n_2}{2}$ for $j=1,2,3,\ldots,p$.

Also, from [2.2.2], the multivariate gamma functions in (9.16) can be written as

$$\begin{aligned}
&\Gamma_p\left(\tfrac{n_1}{2}h\right)\Gamma_p\left(\tfrac{n_2}{2}h\right)\\
&=\pi^{\frac{1}{2}p(p-1)}\prod_{j=1}^{p}\Gamma\left[\tfrac{n_1}{2}h-\tfrac{1}{2}(j-1)\right]\prod_{j=1}^{p}\Gamma\left[\tfrac{n_2}{2}h-\tfrac{1}{2}(j-1)\right]\\
&=\pi^{\frac{1}{2}p(p-1)}\prod_{j=1}^{2p}\Gamma\left(b_j+\beta_j h\right)
\end{aligned} \tag{9.18}$$

where

$$b_j=\begin{cases}-\frac{1}{4}(j-1) & \text{for } j=1,3,5,\ldots,2p-1\\ -\frac{1}{4}(j-2) & \text{for } j=2,4,6,\ldots,2p\end{cases}$$

and

$$\beta_j=\begin{cases}\frac{n_1}{2} & \text{for } j=1,3,5,\ldots,2p-1\\ \frac{n_2}{2} & \text{for } j=2,4,6,\ldots,2p.\end{cases}$$

Now, substituting (9.17) and (9.18) in (9.16) gives

$$M_f(h)\equiv\frac{\pi^{\frac{1}{4}p(p-1)}}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)}\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\sum_{k=0}^{\infty}\sum_{\kappa}\frac{1}{k!}\Gamma_p\left(\tfrac{n_1+n_2+m}{2},\kappa\right)C_\kappa\left(\tfrac{1}{2}\boldsymbol{\Theta}\right)\frac{\prod_{j=1}^{2p}\Gamma\left(b_j+\beta_j h\right)}{\prod_{j=1}^{p}\Gamma\left(a_j+\alpha_j h\right)}. \tag{9.19}$$

Using (9.19) the inverse Mellin transform (see [2.8.1]) is given by

$$\begin{aligned}
f(\lambda_2) &= \frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty}M_f(h)\,\lambda_2^{-h}\mathrm{d}h\\
&=\frac{\pi^{\frac{1}{4}p(p-1)}}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)}\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\sum_{k=0}^{\infty}\sum_{\kappa}\frac{1}{k!}\Gamma_p\left(\tfrac{n_1+n_2+m}{2},\kappa\right)C_\kappa\left(\tfrac{1}{2}\boldsymbol{\Theta}\right)\left[\frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty}\frac{\prod_{j=1}^{2p}\Gamma\left(b_j+\beta_j h\right)}{\prod_{j=1}^{p}\Gamma\left(a_j+\alpha_j h\right)}\lambda_2^{-h}\mathrm{d}h\right]\\
&=\frac{\pi^{\frac{1}{4}p(p-1)}}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)}\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\sum_{k=0}^{\infty}\sum_{\kappa}\frac{1}{k!}\Gamma_p\left(\tfrac{n_1+n_2+m}{2},\kappa\right)C_\kappa\left(\tfrac{1}{2}\boldsymbol{\Theta}\right)H_{p,2p}^{2p,0}\left(\lambda_2\middle|_{(b_1,\beta_1),\ldots,(b_{2p},\beta_{2p})}^{(a_1,\alpha_1),\ldots,(a_p,\alpha_p)}\right).
\end{aligned}$$

The last step follows from the definition of Fox's H-function (see [2.8.3]) and gives (9.15). ∎

9.4 Role of the parameters

In this section we study the effect of the noncentrality parameter. The effect of the parameters n_1, n_2 and m was studied in Section 4.5.

Firstly, we consider the bivariate case, $p = 1$, to illustrate the effect of the parameter θ on

(i) the form of the pdf of (U_1, U_2);
(ii) the correlation between U_1 and U_2;
(iii) the form of the pdf of Λ_2.

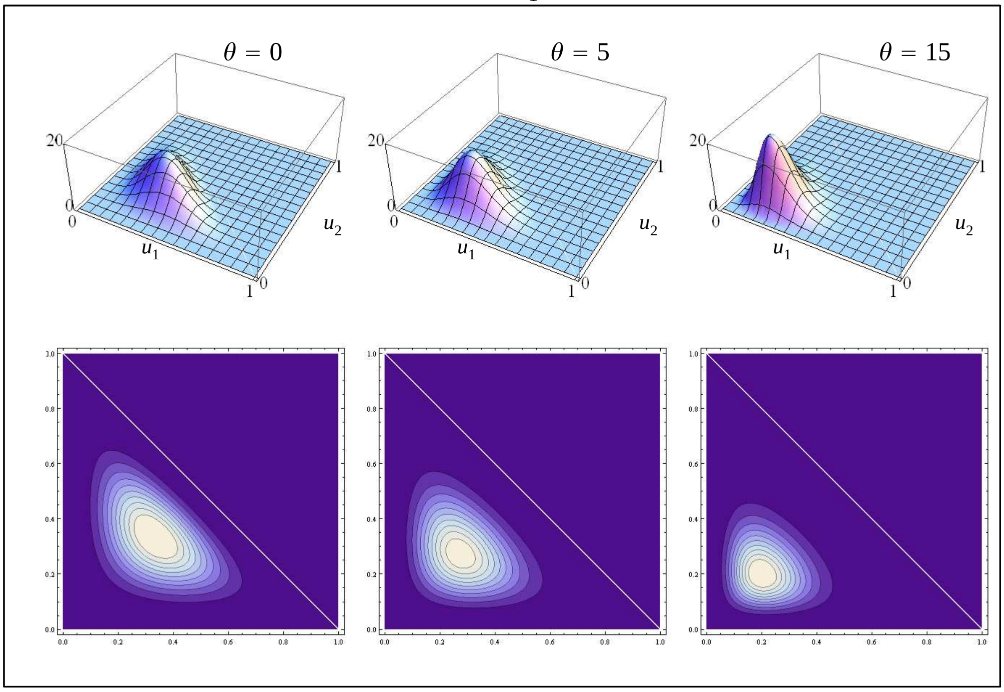
From (9.2) the joint pdf of U_1 and U_2 simplifies to

$$\begin{aligned} f(u_1, u_2) &= \frac{\Gamma\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{m}{2}\right)} u_1^{\frac{1}{2}n_1-1} u_2^{\frac{1}{2}n_2-1} (1-u_1-u_2)^{\frac{1}{2}m-1} \\ &\quad \cdot e^{-\frac{1}{2}\theta}\ {}_1F_1\left(\tfrac{n_1+n_2+m}{2}; \tfrac{m}{2}; \tfrac{\theta}{2}(1-u_1-u_2)\right), \end{aligned} \tag{9.20}$$

$0 < u_i < 1,\ i = 1, 2,\ 0 < u_1 + u_2 < 1$ (see Balakrishnan and Lai, 2009).

Figure 9.1 shows graphs of the pdf of a $BB_1^I(10, 10, 10; \theta)$ distribution for increasing values of θ. As θ increases with all the other parameters constant, the pdf shifts towards smaller values of both U_1 and U_2.

Figure 9.1: Effect of θ on $f(u_1, u_2)$, $(U_1, U_2) \sim BB_1^I(10, 10, 10; \theta)$

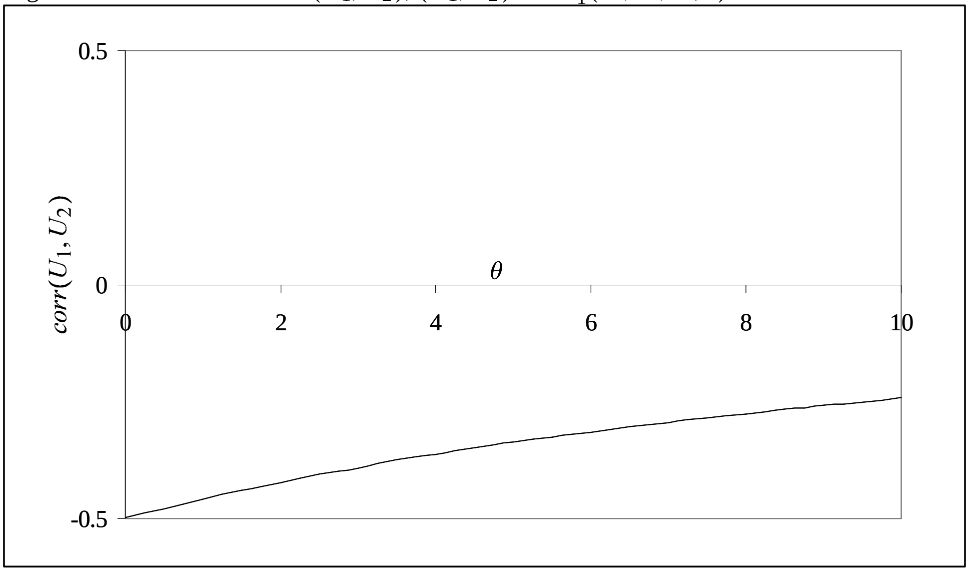


From (9.14) the $(h_1, h_2)^{th}$ product moment, $E\left(U_1^{h_1} U_2^{h_2}\right)$, associated with (9.20) is given by

$$\begin{aligned} E\left(U_1^{h_1} U_2^{h_2}\right) &= \frac{\Gamma\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} \frac{\Gamma\left(\frac{n_1}{2}+h_1\right)\Gamma\left(\frac{n_2}{2}+h_2\right)}{\Gamma\left(\frac{n_1+n_2+m}{2}+h_1+h_2\right)} \\ &\quad \cdot e^{-\frac{1}{2}\theta}\ {}_1F_1\left(\tfrac{n_1+n_2+m}{2}; \tfrac{n_1+n_2+m}{2} + h_1 + h_2; \tfrac{1}{2}\theta\right). \end{aligned} \tag{9.21}$$

The correlation coefficient, $corr\,(U_1, U_2)$, was calculated by using (9.21). Figure 9.2 shows the graph of $corr\,(U_1, U_2)$ for increasing values of θ. The correlation is negative and as θ increases, $corr\,(U_1, U_2)$ shifts towards 0.

Figure 9.2: Effect of θ on $corr(U_1, U_2)$, $(U_1, U_2) \sim BB_1^I(10, 10, 10; \theta)$

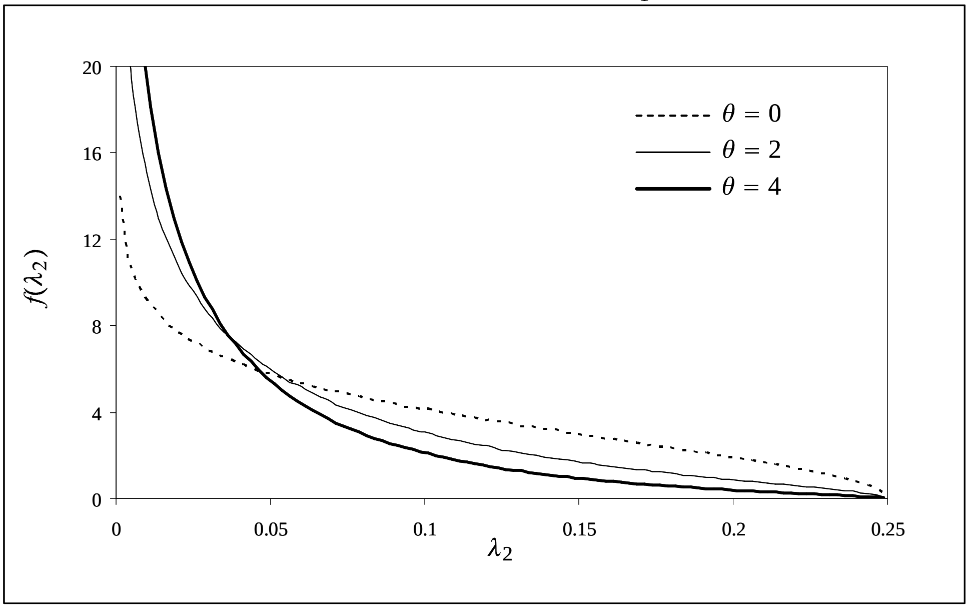


The effect of the noncentrality parameter θ on $f(\lambda_2)$, the pdf of $\Lambda_2 = U_1^{\frac{1}{2}n_1} U_2^{\frac{1}{2}n_2}$, was studied where $(U_1, U_2) \sim BB_1^I(n_1, n_2, m; \theta)$. Considering the result in (9.15) for $p = 1$, it follows from [2.3.3] that $\Gamma_p\left(\frac{n_1+n_2+m}{2}, \kappa\right) = \Gamma\left(\frac{n_1+n_2+m}{2} + k\right)$ and from [2.3.1] that $\sum_{\kappa} C_\kappa\left(\frac{1}{2}\mathbf{\Theta}\right) = \left(\frac{\theta}{2}\right)^k$. Therefore

$$f(\lambda_2) = \frac{1}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} e^{-\frac{1}{2}\theta} \sum_{k=0}^{\infty} \frac{1}{k!} \Gamma\left(\frac{n_1+n_2+m}{2} + k\right) \left(\frac{\theta}{2}\right)^k H_{1,2}^{2,0}\left(\lambda_2 \middle| \begin{smallmatrix} \left(\frac{m}{2}+k, \frac{n_1+n_2}{2}\right) \\ \left(0, \frac{n_1}{2}\right), \left(0, \frac{n_2}{2}\right) \end{smallmatrix}\right), \; 0 < \lambda_2 < 1. \qquad (9.22)$$

Figure 9.3 shows the effect of θ on $f(\lambda_2)$ in (9.22) where $(U_1, U_2) \sim BB_1^I(2, 2, 2; \theta)$. At smaller values of Λ_2 the pdf increases as θ increases.

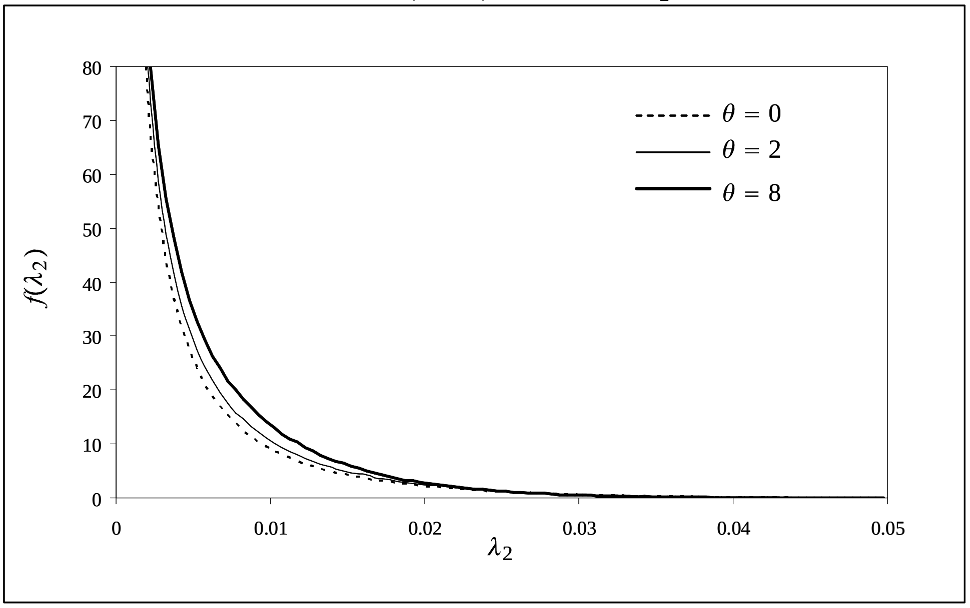
Figure 9.3: Effect of θ on $f(\lambda_2)$, $\Lambda_2 = U_1 U_2$, $(U_1, U_2) \sim BB_1^I(2, 2, 2; \theta)$



Secondly, we consider the bimatrix case, $p = 2$, to illustrate the effect of the noncentrality parameter $\mathbf{\Theta}$ on the pdf of Λ_2 given by (9.15). The case is considered where $(\boldsymbol{U}_1, \boldsymbol{U}_2) \sim BB_2^I(2, 2, 2; \mathbf{\Theta})$, $\mathbf{\Theta} = \theta \boldsymbol{I}_2$.

Figure 9.4 illustrates the shape of the pdf, $f(\lambda_2)$, for increasing values of θ. As θ increases the pdf shifts towards larger values of Λ_2.

Figure 9.4: Effect of $\mathbf{\Theta}$ on $f(\lambda_2)$, $\Lambda_2 = |\boldsymbol{U}_1 \boldsymbol{U}_2|$, $(\boldsymbol{U}_1, \boldsymbol{U}_2) \sim BB_2^I(2, 2, 2; \mathbf{\Theta})$, $\mathbf{\Theta} = \theta \boldsymbol{I}_2$



10 Noncentral bimatrix variate beta type III distribution

In this section an exact expression for the pdf of the noncentral bimatrix variate beta type III distribution with $(\boldsymbol{W}_1, \boldsymbol{W}_2)$ defined in (5.8) is derived as well as the corresponding product moment of the determinants and the pdf of $\Lambda_3 = |\boldsymbol{W}_1|^{\frac{1}{2}n_1} |\boldsymbol{W}_2|^{\frac{1}{2}n_2}$ (see (1.8)).

10.1 Probability density function

In Theorem 10.1 an exact expression for the pdf of the noncentral bimatrix variate beta type III distribution is derived.

Theorem 10.1
Let $\boldsymbol{S}_1 \sim W_p(n_1, \boldsymbol{\Sigma})$, $\boldsymbol{S}_2 \sim W_p(n_2, \boldsymbol{\Sigma})$ *and* $\boldsymbol{B} \sim W_p(m, \boldsymbol{\Sigma}, \boldsymbol{\Theta})$ *be independently distributed. Define*

$$\boldsymbol{W}_i = \left(c\boldsymbol{I}_p + \sum_{i=1}^{2} \boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{S}_i\boldsymbol{B}^{-\frac{1}{2}}\right)^{-\frac{1}{2}} \boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{S}_i\boldsymbol{B}^{-\frac{1}{2}} \left(c\boldsymbol{I}_p + \sum_{i=1}^{2} \boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{S}_i\boldsymbol{B}^{-\frac{1}{2}}\right)^{-\frac{1}{2}}, \quad i = 1, 2, \tag{10.1}$$

(see (5.8)*). The pdf of* $(\boldsymbol{W}_1, \boldsymbol{W}_2) \sim BB_p^{III}(n_1, n_2, m, c; \boldsymbol{\Theta})$ *is*

$$\begin{aligned}
& f(\boldsymbol{W}_1, \boldsymbol{W}_2) \\
= \quad & \left\{\beta_p\left(\tfrac{n_1}{2}, \tfrac{n_2}{2}; \tfrac{m}{2}\right)\right\}^{-1} \prod_{i=1}^{2} |\boldsymbol{W}_i|^{\frac{1}{2}n_i - \frac{1}{2}(p+1)} \left|\boldsymbol{I}_p - \sum_{i=1}^{2} \boldsymbol{W}_i\right|^{\frac{1}{2}m - \frac{1}{2}(p+1)} \\
& \cdot c^{\frac{1}{2}(n_1+n_2)p} \left|\boldsymbol{I}_p + (c-1)\sum_{i=1}^{2} \boldsymbol{W}_i\right|^{-\frac{1}{2}(n_1+n_2+m)} \\
& \cdot \operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) {}_1F_1\left(\tfrac{n_1+n_2+m}{2}; \tfrac{m}{2}; \tfrac{1}{2}\left(\boldsymbol{I}_p - \sum_{i=1}^{2} \boldsymbol{W}_i\right)^{\frac{1}{2}} \left[\boldsymbol{I}_p + (c-1)\sum_{i=1}^{2} \boldsymbol{W}_i\right]^{-1} \left(\boldsymbol{I}_p - \sum_{i=1}^{2} \boldsymbol{W}_i\right)^{\frac{1}{2}} \boldsymbol{\Theta}\right) \\
= \quad & g(\boldsymbol{W}_1, \boldsymbol{W}_2) \\
& \cdot \operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) {}_1F_1\left(\tfrac{n_1+n_2+m}{2}; \tfrac{m}{2}; \tfrac{1}{2}\left(\boldsymbol{I}_p - \sum_{i=1}^{2} \boldsymbol{W}_i\right)^{\frac{1}{2}} \left[\boldsymbol{I}_p + (c-1)\sum_{i=1}^{2} \boldsymbol{W}_i\right]^{-1} \left(\boldsymbol{I}_p - \sum_{i=1}^{2} \boldsymbol{W}_i\right)^{\frac{1}{2}} \boldsymbol{\Theta}\right),
\end{aligned} \tag{10.2}$$

$\boldsymbol{0} < \boldsymbol{W}_i < \boldsymbol{I}_p$, $i = 1, 2$, $\boldsymbol{0} < \sum_{i=1}^{2} \boldsymbol{W}_i < \boldsymbol{I}_p$, *where* $n_i > (p-1)$, $i = 1, 2$, $m > (p-1)$ *and* $g(\cdot)$ *is the pdf of* $BB_p^{III}(n_1, n_2, m, c)$ *given by* (5.2).

Proof:
The pdf of $(\boldsymbol{S}_1, \boldsymbol{S}_2, \boldsymbol{B})$ is given by

$$K \prod_{i=1}^{2} \left[\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}_i\right) |\boldsymbol{S}_i|^{\frac{1}{2}(n_i-p-1)}\right] \left[\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) \operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\right) |\boldsymbol{B}|^{\frac{1}{2}(m-p-1)} {}_0F_1\left(\tfrac{m}{2}; \tfrac{1}{4}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\right)\right]$$

where $K^{-1} = \Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)\Gamma_p\left(\frac{m}{2}\right)|2\boldsymbol{\Sigma}|^{\frac{1}{2}(n_1+n_2+m)}$ (see [2.10.2]).

It was shown in the first part of Theorem 9.1 that if $\boldsymbol{V}_i = \boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{S}_i\boldsymbol{B}^{-\frac{1}{2}}$, $i=1,2$, the pdf of $(\boldsymbol{V}_1, \boldsymbol{V}_2)$ is given by (9.9), as

$$\begin{aligned} f(\boldsymbol{V}_1, \boldsymbol{V}_2) &= \left\{\beta_p\left(\tfrac{n_1}{2}, \tfrac{n_2}{2}; \tfrac{m}{2}\right)\right\}^{-1} \operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) \prod_{i=1}^{2} |\boldsymbol{V}_i|^{\frac{1}{2}n_i - \frac{1}{2}(p+1)} \left|\boldsymbol{I}_p + \sum_{i=1}^{2}\boldsymbol{V}_i\right|^{-\frac{1}{2}(n_1+n_2+m)} \\ &\quad \cdot {}_1F_1\left(\tfrac{n_1+n_2+m}{2}; \tfrac{m}{2}; \tfrac{1}{2}\left(\boldsymbol{I}_p + \sum_{i=1}^{2}\boldsymbol{V}_i\right)^{-1}\boldsymbol{\Theta}\right), \quad \boldsymbol{V}_i > \boldsymbol{0},\ i=1,2. \end{aligned} \tag{10.3}$$

Consider the transformations in (10.1) written in terms of $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$, that is

$$\begin{aligned} \boldsymbol{W}_i &= \left(c\boldsymbol{I}_p + \sum_{i=1}^{2}\boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{S}_i\boldsymbol{B}^{-\frac{1}{2}}\right)^{-\frac{1}{2}} \boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{S}_i\boldsymbol{B}^{-\frac{1}{2}} \left(c\boldsymbol{I}_p + \sum_{i=1}^{2}\boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{S}_i\boldsymbol{B}^{-\frac{1}{2}}\right)^{-\frac{1}{2}} \\ &= \left(\boldsymbol{I}_p + \sum_{i=1}^{2}\tfrac{1}{c}\boldsymbol{V}_i\right)^{-\frac{1}{2}} \tfrac{1}{c}\boldsymbol{V}_i \left(\boldsymbol{I}_p + \sum_{i=1}^{2}\tfrac{1}{c}\boldsymbol{V}_i\right)^{-\frac{1}{2}}, \quad i=1,2. \end{aligned} \tag{10.4}$$

Let $\boldsymbol{Y}_i = \frac{1}{c}\boldsymbol{V}_i$, $i=1,2$, $\boldsymbol{Z} = \boldsymbol{I}_p + \sum_{i=1}^{2}\boldsymbol{Y}_i$ and $\boldsymbol{W}_1 = \boldsymbol{Z}^{-\frac{1}{2}}\boldsymbol{Y}_1\boldsymbol{Z}^{-\frac{1}{2}}$. Then $\boldsymbol{W}_2 = \boldsymbol{I}_p - \boldsymbol{Z}^{-1} - \boldsymbol{W}_1$ and $\boldsymbol{Z} = \left(\boldsymbol{I}_p - \sum_{i=1}^{2}\boldsymbol{W}_i\right)^{-1}$. From [2.1.2], [2.1.3], [2.1.4], [2.1.5], [2.1.7] and [2.1.8] the Jacobian of the transformations in (10.4) is

$$\begin{aligned} & J(\boldsymbol{V}_1, \boldsymbol{V}_2 \to \boldsymbol{W}_1, \boldsymbol{W}_2) \\ &= J(\boldsymbol{V}_1, \boldsymbol{V}_2 \to \boldsymbol{Y}_1, \boldsymbol{Y}_2) \cdot J(\boldsymbol{Y}_1, \boldsymbol{Y}_2 \to \boldsymbol{W}_1, \boldsymbol{Z}) \cdot J(\boldsymbol{W}_1, \boldsymbol{Z} \to \boldsymbol{W}_1, \boldsymbol{W}_2) \\ &= c^{p(p+1)} \cdot |\boldsymbol{Z}|^{\frac{1}{2}(p+1)} \cdot |\boldsymbol{Z}|^{p+1} \\ &= c^{p(p+1)} \left|\boldsymbol{I}_p - \sum_{i=1}^{2}\boldsymbol{W}_i\right|^{-\frac{3}{2}(p+1)}. \end{aligned} \tag{10.5}$$

Rewriting (10.4) as

$$\boldsymbol{V}_i = c\left(\boldsymbol{I}_p - \sum_{i=1}^{2}\boldsymbol{W}_i\right)^{-\frac{1}{2}} \boldsymbol{W}_i \left(\boldsymbol{I}_p - \sum_{i=1}^{2}\boldsymbol{W}_i\right)^{-\frac{1}{2}}, \quad i=1,2, \tag{10.6}$$

and substituting (10.5) and (10.6) in (10.3) gives

$$
\begin{aligned}
& f\left(\boldsymbol{W}_1, \boldsymbol{W}_2\right) \\
= & \left\{\beta_p\left(\tfrac{n_1}{2}, \tfrac{n_2}{2}; \tfrac{m}{2}\right)\right\}^{-1} \operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) c^{p(p+1)} \prod_{i=1}^{2}\left| c\left(\boldsymbol{I}_p - \sum_{i=1}^{2}\boldsymbol{W}_i\right)^{-\frac{1}{2}} \boldsymbol{W}_i \left(\boldsymbol{I}_p - \sum_{i=1}^{2}\boldsymbol{W}_i\right)^{-\frac{1}{2}}\right|^{\frac{1}{2}n_i - \frac{1}{2}(p+1)} \\
& \cdot \left| \boldsymbol{I}_p + c\sum_{i=1}^{2}\left(\boldsymbol{I}_p - \sum_{i=1}^{2}\boldsymbol{W}_i\right)^{-\frac{1}{2}} \boldsymbol{W}_i \left(\boldsymbol{I}_p - \sum_{i=1}^{2}\boldsymbol{W}_i\right)^{-\frac{1}{2}}\right|^{-\frac{1}{2}(n_1+n_2+m)} \left|\boldsymbol{I}_p - \sum_{i=1}^{2}\boldsymbol{W}_i\right|^{-\frac{3}{2}(p+1)} \\
& \cdot {}_1F_1\left(\tfrac{n_1+n_2+m}{2}; \tfrac{m}{2}; \tfrac{1}{2}\left[\boldsymbol{I}_p + c\sum_{i=1}^{2}\left(\boldsymbol{I}_p - \sum_{i=1}^{2}\boldsymbol{W}_i\right)^{-\frac{1}{2}} \boldsymbol{W}_i \left(\boldsymbol{I}_p - \sum_{i=1}^{2}\boldsymbol{W}_i\right)^{-\frac{1}{2}}\right]^{-1}\boldsymbol{\Theta}\right) \\
= & \left\{\beta_p\left(\tfrac{n_1}{2}, \tfrac{n_2}{2}; \tfrac{m}{2}\right)\right\}^{-1} \operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) c^{\frac{1}{2}(n_1+n_2)p} \prod_{i=1}^{2}\left|\boldsymbol{W}_i\right|^{\frac{1}{2}n_i - \frac{1}{2}(p+1)} \left|\boldsymbol{I}_p - \sum_{i=1}^{2}\boldsymbol{W}_i\right|^{-\frac{1}{2}(n_1+n_2)-\frac{1}{2}(p+1)} \\
& \cdot \left|\boldsymbol{I}_p - \sum_{i=1}^{2}\boldsymbol{W}_i\right|^{\frac{1}{2}(n_1+n_2+m)} \left|\boldsymbol{I}_p - \sum_{i=1}^{2}\boldsymbol{W}_i + c\sum_{i=1}^{2}\boldsymbol{W}_i\right|^{-\frac{1}{2}(n_1+n_2+m)} \\
& \cdot {}_1F_1\left(\tfrac{n_1+n_2+m}{2}; \tfrac{m}{2}; \tfrac{1}{2}\left(\boldsymbol{I}_p - \sum_{i=1}^{2}\boldsymbol{W}_i\right)^{\frac{1}{2}} \left(\boldsymbol{I}_p - \sum_{i=1}^{2}\boldsymbol{W}_i + c\sum_{i=1}^{2}\boldsymbol{W}_i\right)^{-1} \left(\boldsymbol{I}_p - \sum_{i=1}^{2}\boldsymbol{W}_i\right)^{\frac{1}{2}}\boldsymbol{\Theta}\right) \\
= & \left\{\beta_p\left(\tfrac{n_1}{2}, \tfrac{n_2}{2}; \tfrac{m}{2}\right)\right\}^{-1} \operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) c^{\frac{1}{2}(n_1+n_2)p} \\
& \cdot \prod_{i=1}^{2}\left|\boldsymbol{W}_i\right|^{\frac{1}{2}n_i - \frac{1}{2}(p+1)} \left|\boldsymbol{I}_p - \sum_{i=1}^{2}\boldsymbol{W}_i\right|^{\frac{1}{2}m - \frac{1}{2}(p+1)} \left|\boldsymbol{I}_p + (c-1)\sum_{i=1}^{2}\boldsymbol{W}_i\right|^{-\frac{1}{2}(n_1+n_2+m)} \\
& \cdot {}_1F_1\left(\tfrac{n_1+n_2+m}{2}; \tfrac{m}{2}; \tfrac{1}{2}\left(\boldsymbol{I}_p - \sum_{i=1}^{2}\boldsymbol{W}_i\right)^{\frac{1}{2}} \left[\boldsymbol{I}_p + (c-1)\sum_{i=1}^{2}\boldsymbol{W}_i\right]^{-1} \left(\boldsymbol{I}_p - \sum_{i=1}^{2}\boldsymbol{W}_i\right)^{\frac{1}{2}}\boldsymbol{\Theta}\right). \qquad \blacksquare
\end{aligned}
$$

Remark 10.1
The ratio in (10.1) was used (in stead of (5.1)) to derive the exact expression for the pdf of $\left(\boldsymbol{W}_1, \boldsymbol{W}_2\right)$ given by (10.2) (see Remark 9.1).

Remark 10.2
Let $\boldsymbol{S} \sim W_p\left(n, \boldsymbol{\Sigma}\right)$ independent of $\boldsymbol{B} \sim W_p\left(m, \boldsymbol{\Sigma}; \boldsymbol{\Theta}\right)$. Define

$$
\boldsymbol{W} = \left(c\boldsymbol{I}_p + \boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{S}\boldsymbol{B}^{-\frac{1}{2}}\right)^{-\frac{1}{2}} \boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{S}\boldsymbol{B}^{-\frac{1}{2}} \left(c\boldsymbol{I}_p + \boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{S}\boldsymbol{B}^{-\frac{1}{2}}\right)^{-\frac{1}{2}}.
$$

The pdf of $\boldsymbol{W}$ is

$$
\begin{aligned}
f\left(\boldsymbol{W}\right) &= \left\{\beta_p\left(\tfrac{n}{2}, \tfrac{m}{2}\right)\right\}^{-1} c^{\frac{1}{2}np} \operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) \left|\boldsymbol{W}\right|^{\frac{1}{2}n - \frac{1}{2}(p+1)} \left|\boldsymbol{I}_p - \boldsymbol{W}\right|^{\frac{1}{2}m - \frac{1}{2}(p+1)} \left|\boldsymbol{I}_p + (c-1)\boldsymbol{W}\right|^{-\frac{1}{2}(n+m)} \\
&\quad \cdot {}_1F_1\left(\tfrac{n+m}{2}; \tfrac{m}{2}; \tfrac{1}{2}\left(\boldsymbol{I}_p - \boldsymbol{W}\right)^{\frac{1}{2}} \left[\boldsymbol{I}_p + (c-1)\boldsymbol{W}\right]^{-1} \left(\boldsymbol{I}_p - \boldsymbol{W}\right)^{\frac{1}{2}}\boldsymbol{\Theta}\right) \\
&= g\left(\boldsymbol{W}\right) \operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) {}_1F_1\left(\tfrac{n+m}{2}; \tfrac{m}{2}; \tfrac{1}{2}\left(\boldsymbol{I}_p - \boldsymbol{W}\right)^{\frac{1}{2}} \left[\boldsymbol{I}_p + (c-1)\boldsymbol{W}\right]^{-1} \left(\boldsymbol{I}_p - \boldsymbol{W}\right)^{\frac{1}{2}}\boldsymbol{\Theta}\right),
\end{aligned}
$$

$\boldsymbol{0} < \boldsymbol{W} < \boldsymbol{I}_p$, where $n > (p-1)$, $m > (p-1)$ and $g(\cdot)$ is the pdf of $B_p^{III}\left(n, m, c\right)$ given by (5.10). This is the noncentral matrix variate beta type III distribution and is denoted by $\boldsymbol{W} \sim B_p^{III}\left(n, m, c; \boldsymbol{\Theta}\right)$.

Remark 10.3

The noncentral matrix variate Dirichlet type III distribution, denoted by $(\boldsymbol{W}_1, \ldots, \boldsymbol{W}_r) \sim D_p^{III}(n_1, \ldots, n_r, m, c; \boldsymbol{\Theta})$, results by extending (10.1) to r independent Wishart matrix variates, $\boldsymbol{S}_i \sim W_p(n_i, \boldsymbol{\Sigma})$, $i = 1, \ldots, r$, all independent of $\boldsymbol{B} \sim W_p(m, \boldsymbol{\Sigma}; \boldsymbol{\Theta})$. The pdf of $(\boldsymbol{W}_1, \ldots, \boldsymbol{W}_r)$ is given by

$$\begin{aligned}
&\left\{\beta_p\left(\tfrac{n_1}{2}, \ldots, \tfrac{n_r}{2}; \tfrac{m}{2}\right)\right\}^{-1} c^{\frac{1}{2}(n_1+\cdots+n_r)p} \operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) \\
&\cdot \prod_{i=1}^{r} |\boldsymbol{W}_i|^{\frac{1}{2}n_i - \frac{1}{2}(p+1)} \left|\boldsymbol{I}_p - \sum_{i=1}^{r} \boldsymbol{W}_i\right|^{\frac{1}{2}m - \frac{1}{2}(p+1)} \left|\boldsymbol{I}_p + (c-1)\sum_{i=1}^{r} \boldsymbol{W}_i\right|^{-\frac{1}{2}(n_1+\cdots+n_r+m)} \\
&\cdot {}_1F_1\left(\tfrac{n_1+\cdots+n_r+m}{2}; \tfrac{m}{2}; \tfrac{1}{2}\left(\boldsymbol{I}_p - \sum_{i=1}^{r} \boldsymbol{W}_i\right)^{\frac{1}{2}} \left[\boldsymbol{I}_p + (c-1)\sum_{i=1}^{r} \boldsymbol{W}_i\right]^{-1} \left(\boldsymbol{I}_p - \sum_{i=1}^{r} \boldsymbol{W}_i\right)^{\frac{1}{2}} \boldsymbol{\Theta}\right), \qquad (10.7)
\end{aligned}$$

$\boldsymbol{0} < \boldsymbol{W}_i < \boldsymbol{I}_p$, $i = 1, \ldots, r$, $\boldsymbol{0} < \sum_{i=1}^{r} \boldsymbol{W}_i < \boldsymbol{I}_p$.

Remark 10.4

The pdfs of the noncentral bimatrix variate beta type III distribution in (10.2) and the noncentral matrix variate Dirichlet type III distribution in (10.7) are members of the Liouville family of distributions of the second kind (see [2.2.1]).

10.2 Product moment of the determinants

The $(h_1, h_2)^{th}$ product moment, $E\left(|\boldsymbol{W}_1|^{h_1} |\boldsymbol{W}_2|^{h_2}\right)$, where $(\boldsymbol{W}_1, \boldsymbol{W}_2) \sim BB_p^{III}(n_1, n_2, m, c; \boldsymbol{\Theta})$, is derived in Theorem 10.2.

Theorem 10.2

If $(\boldsymbol{W}_1, \boldsymbol{W}_2) \sim BB_p^{III}(n_1, n_2, m, c; \boldsymbol{\Theta})$ *as given by* (10.2) *then* $E\left(|\boldsymbol{W}_1|^{h_1} |\boldsymbol{W}_2|^{h_2}\right)$ *is given by*

$$\frac{\Gamma_p\left(\frac{n_1}{2}+h_1\right)\Gamma_p\left(\frac{n_2}{2}+h_2\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)} c^{-\frac{1}{2}mp} \operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) \sum_{\kappa,\tau;\phi} \theta_\phi^{\kappa,\tau} \frac{1}{k!t!} \frac{\Gamma_p\left(\frac{m}{2},\phi\right)\Gamma_p\left(\frac{n_1+n_2+m}{2},\phi\right)}{\Gamma_p\left(\frac{m}{2},\kappa\right)\Gamma_p\left(\frac{n_1+n_2+m}{2}+h_1+h_2,\phi\right)} C_\phi^{\kappa,\tau}\left(\tfrac{1}{2c}\boldsymbol{\Theta}, \tfrac{c-1}{c}\boldsymbol{I}_p\right), \qquad (10.8)$$

where $\sum_{\kappa,\tau;\phi} = \sum_{k=0}^{\infty}\sum_{\kappa}\sum_{t=0}^{\infty}\sum_{\tau}\sum_{\phi\in\kappa\cdot\tau}$, $\phi \in \kappa\cdot\tau$ *is explained in [2.4.1],* $C_\phi^{\kappa,\tau}\left(\frac{1}{2c}\boldsymbol{\Theta}, \frac{c-1}{c}\boldsymbol{I}_p\right)$ *is a homogeneous invariant polynomial of degrees k and t in the elements of the symmetric matrices* $\frac{1}{2c}\boldsymbol{\Theta}$ *and* $\frac{c-1}{c}\boldsymbol{I}_p$ *(see [2.4.1]) and* $\theta_\phi^{\kappa,\tau} = \frac{C_\phi^{\kappa,\tau}(\boldsymbol{I}_p, \boldsymbol{I}_p)}{C_\phi(\boldsymbol{I}_p)}$.

Proof:

Let $\boldsymbol{S}_1 \sim W_p(n_1, \boldsymbol{\Sigma})$, $\boldsymbol{S}_2 \sim W_p(n_2, \boldsymbol{\Sigma})$ and $\boldsymbol{B} \sim W_p(m, \boldsymbol{\Sigma}, \boldsymbol{\Theta})$ be independently distributed. The pdf of $(\boldsymbol{S}_1, \boldsymbol{S}_2, \boldsymbol{B})$ is given by

$$K\left\{\prod_{i=1}^{2}\left[\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}_i\right)|\boldsymbol{S}_i|^{\frac{1}{2}(n_i-p-1)}\right]\right\}\left[\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\right)|\boldsymbol{B}|^{\frac{1}{2}(m-p-1)}\ {}_0F_1\left(\tfrac{m}{2}; \tfrac{1}{4}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\right)\right] \qquad (10.9)$$

where $K^{-1} = \Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)\Gamma_p\left(\frac{m}{2}\right)|2\boldsymbol{\Sigma}|^{\frac{1}{2}(n_1+n_2+m)}$ (see [2.10.2]).

Making the transformations $\boldsymbol{W}_i = \boldsymbol{S}^{-\frac{1}{2}}\boldsymbol{S}_i\boldsymbol{S}^{-\frac{1}{2}}$, $i = 1, 2$, where $\boldsymbol{S} = \boldsymbol{S}_1 + \boldsymbol{S}_2 + c\boldsymbol{B}$ gives $\boldsymbol{S}_i = \boldsymbol{S}^{\frac{1}{2}}\boldsymbol{W}_i\boldsymbol{S}^{\frac{1}{2}}$, $i = 1, 2$, $\boldsymbol{B} = \frac{1}{c}\left(\boldsymbol{S} - \boldsymbol{S}_1 - \boldsymbol{S}_2\right) = \frac{1}{c}\boldsymbol{S}^{\frac{1}{2}}\left(\boldsymbol{I}_p - \sum_{i=1}^{2}\boldsymbol{W}_i\right)\boldsymbol{S}^{\frac{1}{2}}$. The Jacobian of the transformations is $J\left(\boldsymbol{S}_1, \boldsymbol{S}_2, \boldsymbol{B} \rightarrow \boldsymbol{W}_1, \boldsymbol{W}_2, \boldsymbol{S}\right) = c^{-\frac{1}{2}p(p+1)}\left|\boldsymbol{S}\right|^{(p+1)}$ (see [2.1.4] and [2.1.7]). Substituting in (10.9) gives

$$
\begin{aligned}
& f\left(\boldsymbol{W}_1, \boldsymbol{W}_2, \boldsymbol{S}\right) \\
= \quad & K\mathrm{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\left\{\prod_{i=1}^{2}\left[\mathrm{etr}\left(-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{W}_i\boldsymbol{S}^{\frac{1}{2}}\right)\left|\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{W}_i\boldsymbol{S}^{\frac{1}{2}}\right|^{\frac{1}{2}n_i-\frac{1}{2}(p+1)}\right]\right\}c^{-\frac{1}{2}p(p+1)}\left|\boldsymbol{S}\right|^{(p+1)} \\
& \cdot\mathrm{etr}\left[-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\tfrac{1}{c}\boldsymbol{S}^{\frac{1}{2}}\left(\boldsymbol{I}_p - \sum_{i=1}^{2}\boldsymbol{W}_i\right)\boldsymbol{S}^{\frac{1}{2}}\right]\left|\tfrac{1}{c}\boldsymbol{S}^{\frac{1}{2}}\left(\boldsymbol{I}_p - \sum_{i=1}^{2}\boldsymbol{W}_i\right)\boldsymbol{S}^{\frac{1}{2}}\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)} \\
& \cdot {}_0F_1\left(\tfrac{m}{2}; \tfrac{1}{4}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\tfrac{1}{c}\boldsymbol{S}^{\frac{1}{2}}\left(\boldsymbol{I}_p - \sum_{i=1}^{2}\boldsymbol{W}_i\right)\boldsymbol{S}^{\frac{1}{2}}\right) \\
= \quad & K\mathrm{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)c^{-\frac{1}{2}mp}\prod_{i=1}^{2}\left|\boldsymbol{W}_i\right|^{\frac{1}{2}n_i-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p - \sum_{i=1}^{2}\boldsymbol{W}_i\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)}\left|\boldsymbol{S}\right|^{\frac{1}{2}(n_1+n_2+m)-\frac{1}{2}(p+1)} \\
& \cdot\mathrm{etr}\left\{-\tfrac{1}{2c}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}^{\frac{1}{2}}\left[\boldsymbol{I}_p + (c-1)\sum_{i=1}^{2}\boldsymbol{W}_i\right]\boldsymbol{S}^{\frac{1}{2}}\right\}{}_0F_1\left(\tfrac{m}{2}; \tfrac{1}{4c}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}^{\frac{1}{2}}\left(\boldsymbol{I}_p - \sum_{i=1}^{2}\boldsymbol{W}_i\right)\boldsymbol{S}^{\frac{1}{2}}\right).
\end{aligned}
$$

From this, the pdf of $\left(\boldsymbol{W}_1, \boldsymbol{W}_2\right)$ is

$$
\begin{aligned}
& Kc^{-\frac{1}{2}mp}\mathrm{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\prod_{i=1}^{2}\left|\boldsymbol{W}_i\right|^{\frac{1}{2}n_i-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p - \sum_{i=1}^{2}\boldsymbol{W}_i\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)} \\
& \cdot\int_{\boldsymbol{S}>\boldsymbol{0}}\left|\boldsymbol{S}\right|^{\frac{1}{2}(n_1+n_2+m)-\frac{1}{2}(p+1)}\mathrm{etr}\left\{-\tfrac{1}{2c}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}^{\frac{1}{2}}\left[\boldsymbol{I}_p + (c-1)\sum_{i=1}^{2}\boldsymbol{W}_i\right]\boldsymbol{S}^{\frac{1}{2}}\right\} \\
& \quad \cdot {}_0F_1\left(\tfrac{m}{2}; \tfrac{1}{4c}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}^{\frac{1}{2}}\left(\boldsymbol{I}_p - \sum_{i=1}^{2}\boldsymbol{W}_i\right)\boldsymbol{S}^{\frac{1}{2}}\right)\mathrm{d}\boldsymbol{S}.
\end{aligned} \tag{10.10}
$$

From (10.10)

$$
\begin{aligned}
& E\left(\left|\boldsymbol{W}_1\right|^{h_1}\left|\boldsymbol{W}_2\right|^{h_2}\right) \\
= \quad & Kc^{-\frac{1}{2}mp}\mathrm{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\int_{\boldsymbol{S}>\boldsymbol{0}}\left|\boldsymbol{S}\right|^{\frac{1}{2}(n_1+n_2+m)-\frac{1}{2}(p+1)}\int\int_{\substack{\boldsymbol{0}<\boldsymbol{W}_1+\boldsymbol{W}_2<\boldsymbol{I}_p \\ \boldsymbol{W}_i>\boldsymbol{0}}}\prod_{i=1}^{2}\left|\boldsymbol{W}_i\right|^{\frac{1}{2}n_i+h_i-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p - \sum_{i=1}^{2}\boldsymbol{W}_i\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)} \\
& \cdot\mathrm{etr}\left\{-\tfrac{1}{2c}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}^{\frac{1}{2}}\left[\boldsymbol{I}_p + (c-1)\sum_{i=1}^{2}\boldsymbol{W}_i\right]\boldsymbol{S}^{\frac{1}{2}}\right\}{}_0F_1\left(\tfrac{m}{2}; \tfrac{1}{4c}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}^{\frac{1}{2}}\left(\boldsymbol{I}_p - \sum_{i=1}^{2}\boldsymbol{W}_i\right)\boldsymbol{S}^{\frac{1}{2}}\right)\mathrm{d}\boldsymbol{W}_1\mathrm{d}\boldsymbol{W}_2\mathrm{d}\boldsymbol{S}.
\end{aligned} \tag{10.11}
$$

Let

$$
\begin{aligned}
f\left(\sum_{i=1}^{2}\boldsymbol{W}_i\right) = \quad & \left|\boldsymbol{I}_p - \sum_{i=1}^{2}\boldsymbol{W}_i\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)}\mathrm{etr}\left\{-\tfrac{1}{2c}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}^{\frac{1}{2}}\left[\boldsymbol{I}_p + (c-1)\sum_{i=1}^{2}\boldsymbol{W}_i\right]\boldsymbol{S}^{\frac{1}{2}}\right\} \\
& \cdot {}_0F_1\left(\tfrac{m}{2}; \tfrac{1}{4c}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}^{\frac{1}{2}}\left(\boldsymbol{I}_p - \sum_{i=1}^{2}\boldsymbol{W}_i\right)\boldsymbol{S}^{\frac{1}{2}}\right)
\end{aligned}
$$

and use [2.2.6] to write (10.11) as

$$\begin{aligned}
&E\left(\left|\boldsymbol{W}_1\right|^{h_1}\left|\boldsymbol{W}_2\right|^{h_2}\right)\\
=\;& Kc^{-\frac{1}{2}mp}\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\int_{\boldsymbol{S}>\boldsymbol{0}}\left|\boldsymbol{S}\right|^{\frac{1}{2}(n_1+n_2+m)-\frac{1}{2}(p+1)}\beta_p\left(\tfrac{n_1}{2}+h_1,\tfrac{n_2}{2}+h_2\right)\\
&\cdot\int_{\boldsymbol{0}<\boldsymbol{Z}<\boldsymbol{I}_p}\left|\boldsymbol{Z}\right|^{\frac{1}{2}(n_1+n_2)+h_1+h_2-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\boldsymbol{Z}\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)}\\
&\cdot\operatorname{etr}\left\{-\tfrac{1}{2c}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}^{\frac{1}{2}}\left[\boldsymbol{I}_p+(c-1)\boldsymbol{Z}\right]\boldsymbol{S}^{\frac{1}{2}}\right\}{}_0F_1\left(\tfrac{m}{2};\tfrac{1}{4c}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}^{\frac{1}{2}}\left(\boldsymbol{I}_p-\boldsymbol{Z}\right)\right)\mathrm{d}\boldsymbol{Z}\mathrm{d}\boldsymbol{S}.
\end{aligned}\tag{10.12}$$

In (10.12), consider the integral with respect to $\boldsymbol{Z}$ and let $\boldsymbol{X}=\boldsymbol{I}_p-\boldsymbol{Z}$. Using [2.6.1], [2.6.2], [2.4.2] and [2.4.5] gives

$$\begin{aligned}
&\int_{\boldsymbol{0}<\boldsymbol{Z}<\boldsymbol{I}_p}\left|\boldsymbol{Z}\right|^{\frac{1}{2}(n_1+n_2)+h_1+h_2-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\boldsymbol{Z}\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)}\\
&\cdot\operatorname{etr}\left\{-\tfrac{1}{2c}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}^{\frac{1}{2}}\left[\boldsymbol{I}_p+(c-1)\boldsymbol{Z}\right]\boldsymbol{S}^{\frac{1}{2}}\right\}{}_0F_1\left(\tfrac{m}{2};\tfrac{1}{4c}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}^{\frac{1}{2}}\left(\boldsymbol{I}_p-\boldsymbol{Z}\right)\right)\mathrm{d}\boldsymbol{Z}\\
=\;&\int_{\boldsymbol{0}<\boldsymbol{X}<\boldsymbol{I}_p}\left|\boldsymbol{I}_p-\boldsymbol{X}\right|^{\frac{1}{2}(n_1+n_2)+h_1+h_2-\frac{1}{2}(p+1)}\left|\boldsymbol{X}\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)}\\
&\cdot\operatorname{etr}\left\{-\tfrac{1}{2c}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}^{\frac{1}{2}}\left[\boldsymbol{I}_p+(c-1)\left(\boldsymbol{I}_p-\boldsymbol{X}\right)\right]\right\}{}_0F_1\left(\tfrac{m}{2};\tfrac{1}{4c}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{X}\right)\mathrm{d}\boldsymbol{X}\\
=\;&\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}^{\frac{1}{2}}\right)\int_{\boldsymbol{0}<\boldsymbol{X}<\boldsymbol{I}_p}\left|\boldsymbol{X}\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\boldsymbol{X}\right|^{\frac{1}{2}(n_1+n_2)+h_1+h_2-\frac{1}{2}(p+1)}\\
&\cdot\operatorname{etr}\left\{\tfrac{c-1}{2c}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{X}\right\}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{1}{\left(\frac{m}{2}\right)_\kappa}\frac{1}{k!}C_\kappa\left(\tfrac{1}{4c}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{X}\right)\mathrm{d}\boldsymbol{X}\\
=\;&\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}\right)\int_{\boldsymbol{0}<\boldsymbol{X}<\boldsymbol{I}_p}\left|\boldsymbol{X}\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\boldsymbol{X}\right|^{\frac{1}{2}(n_1+n_2)+h_1+h_2-\frac{1}{2}(p+1)}\\
&\cdot\sum_{t=0}^{\infty}\sum_{\tau}\frac{1}{t!}C_\tau\left(\tfrac{c-1}{2c}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{X}\right)\sum_{k=0}^{\infty}\sum_{\kappa}\frac{1}{\left(\frac{m}{2}\right)_\kappa}\frac{1}{k!}C_\kappa\left(\tfrac{1}{4c}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{X}\right)\mathrm{d}\boldsymbol{X}\\
=\;&\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}\right)\sum_{k=0}^{\infty}\sum_{\kappa}\sum_{t=0}^{\infty}\sum_{\tau}\sum_{\phi\in\kappa\cdot\tau}\theta_\phi^{\kappa,\tau}\frac{1}{k!t!}\frac{1}{\left(\frac{m}{2}\right)_\kappa}\\
&\int_{\boldsymbol{0}<\boldsymbol{X}<\boldsymbol{I}_p}\left|\boldsymbol{X}\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\boldsymbol{X}\right|^{\frac{1}{2}(n_1+n_2)+h_1+h_2-\frac{1}{2}(p+1)}C_\phi^{\kappa,\tau}\left(\tfrac{1}{4c}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{X},\tfrac{c-1}{2c}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{X}\right)\mathrm{d}\boldsymbol{X}\\
=\;&\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}\right)\sum_{k=0}^{\infty}\sum_{\kappa}\sum_{t=0}^{\infty}\sum_{\tau}\sum_{\phi\in\kappa\cdot\tau}\theta_\phi^{\kappa,\tau}\frac{1}{k!t!}\frac{1}{\left(\frac{m}{2}\right)_\kappa}\frac{\Gamma_p\left(\frac{m}{2},\phi\right)\Gamma_p\left(\frac{n_1+n_2}{2}+h_1+h_2\right)}{\Gamma_p\left(\frac{n_1+n_2+m}{2}+h_1+h_2,\phi\right)}C_\phi^{\kappa,\tau}\left(\tfrac{1}{4c}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}^{\frac{1}{2}},\tfrac{c-1}{2c}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}^{\frac{1}{2}}\right).
\end{aligned}\tag{10.13}$$

Substituting (10.13) in (10.12) and using [2.4.4] to solve the integral with respect to $\boldsymbol{S}$ gives

$$
\begin{aligned}
&E\left(|\boldsymbol{W}_1|^{h_1}|\boldsymbol{W}_2|^{h_2}\right)\\
=\;& K\frac{\Gamma_p\left(\frac{n_1}{2}+h_1\right)\Gamma_p\left(\frac{n_2}{2}+h_2\right)}{\Gamma_p\left(\frac{n_1+n_2}{2}+h_1+h_2\right)}c^{-\frac{1}{2}mp}\mathrm{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\sum_{k=0}^{\infty}\sum_{\kappa}\sum_{t=0}^{\infty}\sum_{\tau}\sum_{\phi\in\kappa\cdot\tau}\theta_{\phi}^{\kappa,\tau}\frac{1}{k!t!}\frac{1}{\left(\frac{m}{2}\right)_{\kappa}}\frac{\Gamma_p\left(\frac{m}{2},\phi\right)\Gamma_p\left(\frac{n_1+n_2}{2}+h_1+h_2\right)}{\Gamma_p\left(\frac{n_1+n_2+m}{2}+h_1+h_2,\phi\right)}\\
&\cdot\int_{\boldsymbol{S}>\boldsymbol{0}}|\boldsymbol{S}|^{\frac{1}{2}(n_1+n_2+m)-\frac{1}{2}(p+1)}\mathrm{etr}\left(-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}\right)C_{\phi}^{\kappa,\tau}\left(\tfrac{1}{4c}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{S},\tfrac{c-1}{2c}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}\right)\mathrm{d}\boldsymbol{S}\\
=\;& K\Gamma_p\left(\tfrac{n_1}{2}+h_1\right)\Gamma_p\left(\tfrac{n_2}{2}+h_2\right)\Gamma_p\left(\tfrac{m}{2}\right)c^{-\frac{1}{2}mp}\mathrm{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\sum_{k=0}^{\infty}\sum_{\kappa}\sum_{t=0}^{\infty}\sum_{\tau}\sum_{\phi\in\kappa\cdot\tau}\theta_{\phi}^{\kappa,\tau}\frac{1}{k!t!}\frac{\Gamma_p\left(\frac{m}{2},\phi\right)}{\Gamma_p\left(\frac{m}{2},\kappa\right)\Gamma_p\left(\frac{n_1+n_2+m}{2}+h_1+h_2,\phi\right)}\\
&\cdot\Gamma_p\left(\tfrac{n_1+n_2+m}{2},\phi\right)\left|\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\right|^{-\frac{1}{2}(n_1+n_2+m)}C_{\phi}^{\kappa,\tau}\left(\tfrac{1}{4c}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}2\boldsymbol{\Sigma},\tfrac{c-1}{2c}\boldsymbol{\Sigma}^{-1}2\boldsymbol{\Sigma}\right)\\
=\;& \frac{\Gamma_p\left(\frac{n_1}{2}+h_1\right)\Gamma_p\left(\frac{n_2}{2}+h_2\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)}c^{-\frac{1}{2}mp}\mathrm{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\sum_{k=0}^{\infty}\sum_{\kappa}\sum_{t=0}^{\infty}\sum_{\tau}\sum_{\phi\in\kappa\cdot\tau}\theta_{\phi}^{\kappa,\tau}\frac{1}{k!t!}\frac{\Gamma_p\left(\frac{m}{2},\phi\right)\Gamma_p\left(\frac{n_1+n_2+m}{2},\phi\right)}{\Gamma_p\left(\frac{m}{2},\kappa\right)\Gamma_p\left(\frac{n_1+n_2+m}{2}+h_1+h_2,\phi\right)}C_{\phi}^{\kappa,\tau}\left(\tfrac{1}{2c}\boldsymbol{\Theta},\tfrac{c-1}{c}\boldsymbol{I}_p\right).
\end{aligned}
$$

■

10.3 Distribution of the product of determinants

In Theorem 10.3 an exact expression is derived for the pdf of $\Lambda_3=|\boldsymbol{W}_1|^{\frac{1}{2}n_1}|\boldsymbol{W}_2|^{\frac{1}{2}n_2}$ where $(\boldsymbol{W}_1,\boldsymbol{W}_2)\sim BB_p^{III}(n_1,n_2,m,c;\boldsymbol{\Theta})$ as given by (10.2).

Theorem 10.3
Let $(\boldsymbol{W}_1,\boldsymbol{W}_2)\sim BB_p^{III}(n_1,n_2,m,c;\boldsymbol{\Theta})$ with pdf given by (10.2) and let $\Lambda_3=|\boldsymbol{W}_1|^{\frac{1}{2}n_1}|\boldsymbol{W}_2|^{\frac{1}{2}n_2}$. Then the pdf of Λ_3 is given by

$$
\frac{\pi^{\frac{1}{4}p(p-1)}}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)}c^{-\frac{1}{2}mp}\mathrm{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\sum_{\kappa,\tau;\phi}\theta_{\phi}^{\kappa,\tau}\frac{1}{k!t!}\frac{\Gamma_p\left(\frac{m}{2},\phi\right)\Gamma_p\left(\frac{n_1+n_2+m}{2},\phi\right)}{\Gamma_p\left(\frac{m}{2},\kappa\right)}C_{\phi}^{\kappa,\tau}\left(\tfrac{1}{2c}\boldsymbol{\Theta},\tfrac{c-1}{c}\boldsymbol{I}_p\right)H_{p,2p}^{2p,0}\left(\lambda_3\middle|_{(b_1,\beta_1),\ldots,(b_{2p},\beta_{2p})}^{(a_1,\alpha_1),\ldots,(a_p,\alpha_p)}\right),\tag{10.14}
$$

$0<\lambda_3<1$, where $\sum_{\kappa,\tau;\phi}=\sum_{k=0}^{\infty}\sum_{\kappa}\sum_{t=0}^{\infty}\sum_{\tau}\sum_{\phi\in\kappa\cdot\tau}$ and

$a_j=\frac{m}{2}+(k_j+t_j)-\frac{1}{2}(j-1)$ *for* $j=1,2,3,\ldots,p$,

$\alpha_j=\frac{n_1+n_2}{2}$ *for* $j=1,2,3,\ldots,p$,

$$
b_j=\begin{cases}-\frac{1}{4}(j-1) & \textit{for } j=1,3,5,\ldots,2p-1\\ -\frac{1}{4}(j-2) & \textit{for } j=2,4,6,\ldots,2p,\end{cases}
$$

$$
\beta_j=\begin{cases}\frac{n_1}{2} & \textit{for } j=1,3,5,\ldots,2p-1\\ \frac{n_2}{2} & \textit{for } j=2,4,6,\ldots,2p.\end{cases}
$$

Proof:

Using (10.8), the Mellin transform (see [2.8.1]) of $f(\lambda_3)$ is

$$\begin{aligned}
M_f(h) &\equiv E\left(\Lambda_3^{h-1}\right) \\
&= E\left[\left(|\boldsymbol{W}_1|^{\frac{1}{2}n_1}|\boldsymbol{W}_2|^{\frac{1}{2}n_2}\right)^{h-1}\right] \\
&= c^{-\frac{1}{2}mp}\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\frac{\Gamma_p\left[\frac{n_1}{2}+\frac{n_1}{2}(h-1)\right]\Gamma_p\left[\frac{n_2}{2}+\frac{n_2}{2}(h-1)\right]}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)} \\
&\quad \cdot\sum_{\kappa,\tau;\phi}\theta_\phi^{\kappa,\tau}\frac{1}{k!t!}\frac{\Gamma_p\left(\frac{m}{2},\phi\right)\Gamma_p\left(\frac{n_1+n_2+m}{2},\phi\right)}{\Gamma_p\left[\frac{n_1+n_2+m}{2}+\frac{n_1+n_2}{2}(h-1),\phi\right]\Gamma_p\left(\frac{m}{2},\kappa\right)}C_\phi^{\kappa,\tau}\left(\tfrac{1}{2c}\boldsymbol{\Theta},\tfrac{c-1}{c}\boldsymbol{I}_p\right) \\
&= \frac{\Gamma_p\left(\frac{n_1}{2}h\right)\Gamma_p\left(\frac{n_2}{2}h\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)}c^{-\frac{1}{2}mp}\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\sum_{\kappa,\tau;\phi}\theta_\phi^{\kappa,\tau}\frac{1}{k!t!}\frac{\Gamma_p\left(\frac{m}{2},\phi\right)\Gamma_p\left(\frac{n_1+n_2+m}{2},\phi\right)}{\Gamma_p\left(\frac{m}{2},\kappa\right)\Gamma_p\left(\frac{m}{2}+\frac{n_1+n_2}{2}h,\phi\right)}C_\phi^{\kappa,\tau}\left(\tfrac{1}{2c}\boldsymbol{\Theta},\tfrac{c-1}{c}\boldsymbol{I}_p\right).
\end{aligned} \tag{10.15}$$

From [2.3.3] the generalised gamma function of weight ϕ in (10.15) can be written as

$$\begin{aligned}
&\Gamma_p\left(\tfrac{m}{2}+\tfrac{n_1+n_2}{2}h,\phi\right) \\
&= \pi^{\frac{1}{4}p(p-1)}\prod_{j=1}^{p}\Gamma\left(a_j+\alpha_j h\right),
\end{aligned} \tag{10.16}$$

where $a_j = \frac{m}{2}+(k_j+t_j)-\frac{1}{2}(j-1)$ for $j=1,2,3,\ldots,p$

and $\alpha_j = \frac{n_1+n_2}{2}$ for $j=1,2,3,\ldots,p$.

From [2.2.2] the multivariate gamma functions in (10.15) can be written as

$$\begin{aligned}
&\Gamma_p\left(\tfrac{n_1}{2}h\right)\Gamma_p\left(\tfrac{n_2}{2}h\right) \\
&\quad = \pi^{\frac{1}{2}p(p-1)}\prod_{j=1}^{p}\Gamma\left[\tfrac{n_1}{2}h-\tfrac{1}{2}(j-1)\right]\prod_{j=1}^{p}\Gamma\left[\tfrac{n_2}{2}h-\tfrac{1}{2}(j-1)\right] \\
&\quad = \pi^{\frac{1}{2}p(p-1)}\prod_{j=1}^{2p}\Gamma\left(b_j+\beta_j h\right)
\end{aligned} \tag{10.17}$$

where

$$b_j = \begin{cases} -\frac{1}{4}(j-1) & \text{for } j=1,3,5,\ldots,2p-1 \\ -\frac{1}{4}(j-2) & \text{for } j=2,4,6,\ldots,2p, \end{cases}$$

and

$$\beta_j = \begin{cases} \frac{n_1}{2} & \text{for } j=1,3,5,\ldots,2p-1 \\ \frac{n_2}{2} & \text{for } j=2,4,6,\ldots,2p. \end{cases}$$

Now, substituting (10.16) and (10.17) in (10.15) gives

$$M_f(h) \equiv \frac{\pi^{\frac{1}{4}p(p-1)}}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)} c^{-\frac{1}{2}mp} \operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) \sum_{\kappa,\tau;\phi} \theta_\phi^{\kappa,\tau} \frac{1}{k!t!} \frac{\Gamma_p\left(\frac{m}{2},\phi\right)\Gamma_p\left(\frac{n_1+n_2+m}{2},\phi\right)}{\Gamma_p\left(\frac{m}{2},\kappa\right)} C_\phi^{\kappa,\tau}\left(\tfrac{1}{2c}\boldsymbol{\Theta}, \tfrac{c-1}{c}\boldsymbol{I}_p\right) \frac{\prod_{j=1}^{2p}\Gamma\left(b_j+\beta_j h\right)}{\prod_{j=1}^{p}\Gamma\left(a_j+\alpha_j h\right)}. \tag{10.18}$$

Using (10.18) the inverse Mellin transform (see [2.8.1]) is given by

$$\begin{aligned} f(\lambda_3) &= \frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty} M_f(h)\,\lambda_3^{-h}\mathrm{d}h \\ &= \frac{\pi^{\frac{1}{4}p(p-1)}}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)} c^{-\frac{1}{2}mp}\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) \\ &\quad \cdot \sum_{\kappa,\tau;\phi} \theta_\phi^{\kappa,\tau}\frac{1}{k!t!}\frac{\Gamma_p\left(\frac{m}{2},\phi\right)\Gamma_p\left(\frac{n_1+n_2+m}{2},\phi\right)}{\Gamma_p\left(\frac{m}{2},\kappa\right)} C_\phi^{\kappa,\tau}\left(\tfrac{1}{2c}\boldsymbol{\Theta}, \tfrac{c-1}{c}\boldsymbol{I}_p\right)\left[\frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty}\frac{\prod_{j=1}^{2p}\Gamma\left(b_j+\beta_j h\right)}{\prod_{j=1}^{p}\Gamma\left(a_j+\alpha_j h\right)}\lambda_3^{-h}\mathrm{d}h\right] \\ &= \frac{\pi^{\frac{1}{4}p(p-1)}}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)} c^{-\frac{1}{2}mp}\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\sum_{\kappa,\tau;\phi} \theta_\phi^{\kappa,\tau}\frac{1}{k!t!}\frac{\Gamma_p\left(\frac{m}{2},\phi\right)\Gamma_p\left(\frac{n_1+n_2+m}{2},\phi\right)}{\Gamma_p\left(\frac{m}{2},\kappa\right)} C_\phi^{\kappa,\tau}\left(\tfrac{1}{2c}\boldsymbol{\Theta}, \tfrac{c-1}{c}\boldsymbol{I}_p\right) \\ &\quad \cdot H_{p,2p}^{2p,0}\left(\lambda_3\Big|_{(b_1,\beta_1),\ldots,(b_{2p},\beta_{2p})}^{(a_1,\alpha_1),\ldots,(a_p,\alpha_p)}\right). \end{aligned}$$

The last step follows from the definition of Fox's H-function (see [2.8.3]) and gives (10.14). ■

10.4 Bivariate distribution

The bivariate case is considered in this section, that is where $(W_1, W_2) \sim BB_1^{III}\left(\frac{n_1}{2}, \frac{n_2}{2}, \frac{m}{2}, c; \theta\right)$. The pdf $f(W_1, W_2)$ is given and $E\left(W_1^{h_1}W_2^{h_2}\right)$ and the pdf of $\Lambda_3 = W_1^{\frac{1}{2}n_1}W_2^{\frac{1}{2}n_2}$ are derived in Theorems 10.4 and 10.5 respectively.

From (10.2) the pdf of $(W_1, W_2) \sim BB_1^{III}\left(\frac{n_1}{2}, \frac{n_2}{2}, \frac{m}{2}, c; \theta\right)$ is given by

$$\begin{aligned} f(w_1, w_2) = \;& \frac{\Gamma\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{m}{2}\right)} w_1^{\frac{1}{2}n_1-1}w_2^{\frac{1}{2}n_2-1}(1-w_1-w_2)^{\frac{1}{2}m-1} \\ & \cdot c^{\frac{1}{2}(n_1+n_2)}\left[1+(c-1)w_1+(c-1)w_2\right]^{-\frac{1}{2}(n_1+n_2+m)} \\ & \cdot e^{-\frac{1}{2}\theta}\,{}_1F_1\left(\frac{n_1+n_2+m}{2}; \frac{m}{2}; \frac{\theta}{2}\frac{1-w_1-w_2}{1+(c-1)w_1+(c-1)w_2}\right), \end{aligned} \tag{10.19}$$

$0 < w_i < 1,\ i = 1, 2,\ \ 0 < w_1 + w_2 < 1.$

Theorem 10.4
If $(W_1, W_2) \sim BB_1^{III}\left(\frac{n_1}{2}, \frac{n_2}{2}, \frac{m}{2}, c; \theta\right)$ *as given by* (10.19) *then*

$$\begin{aligned} E\left(W_1^{h_1}W_2^{h_2}\right) &= \frac{\Gamma\left(\frac{n_1}{2}+h_1\right)\Gamma\left(\frac{n_2}{2}+h_2\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} c^{-\frac{1}{2}m}e^{-\frac{1}{2}\theta}\sum_{l=0}^{\infty}\frac{1}{l!}\frac{\Gamma\left(\frac{n_1+n_2+m}{2}+l\right)}{\Gamma\left(\frac{n_1+n_2+m}{2}+l+h_1+h_2\right)}\left(\frac{\theta}{2c}\right)^l \\ &\quad \cdot {}_2F_1\left(\frac{m}{2}+l, \frac{n_1+n_2+m}{2}+l; \frac{n_1+n_2+m}{2}+l+h_1+h_2; \frac{c-1}{c}\right). \end{aligned} \tag{10.20}$$

Proof:
From (10.19), [2.5.1] and (5.30)

$$
\begin{aligned}
& E\left(W_1^{h_1}W_2^{h_2}\right) \\
= & \frac{\Gamma\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{m}{2}\right)} c^{\frac{1}{2}(n_1+n_2)} e^{-\frac{1}{2}\theta} \int\limits_{\substack{0<w_1+w_2<1 \\ 0<w_i<1,\ i=1,2}}\int w_1^{\frac{1}{2}n_1+h_1-1} w_2^{\frac{1}{2}n_2+h_2-1} (1-w_1-w_2)^{\frac{1}{2}m-1} \\
& \cdot [1+(c-1)w_1+(c-1)w_2]^{-\frac{1}{2}(n_1+n_2+m)} \; {}_1F_1\left(\frac{n_1+n_2+m}{2}; \frac{m}{2}; \frac{\theta}{2}\frac{1-w_1-w_2}{1+(c-1)w_1+(c-1)w_2}\right) \mathrm{d}w_1 \mathrm{d}w_2 \\
= & \frac{\Gamma\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{m}{2}\right)} c^{\frac{1}{2}(n_1+n_2)} e^{-\frac{1}{2}\theta} \sum_{l=0}^{\infty} \frac{1}{l!}\left(\frac{\theta}{2}\right)^l \frac{\Gamma\left(\frac{n_1+n_2+m}{2}+l\right)}{\Gamma\left(\frac{n_1+n_2+m}{2}\right)} \frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m}{2}+l\right)} \\
& \cdot \int\limits_{\substack{0<w_1+w_2<1 \\ 0<w_i<1,\ i=1,2}}\int w_1^{\frac{1}{2}n_1+h_1-1} w_2^{\frac{1}{2}n_2+h_2-1} (1-w_1-w_2)^{\frac{1}{2}m+l-1} [1+(c-1)w_1+(c-1)w_2]^{-\frac{1}{2}(n_1+n_2+m)-l} \mathrm{d}w_1 \mathrm{d}w_2 \\
= & e^{-\frac{1}{2}\theta} \sum_{l=0}^{\infty} \frac{1}{l!}\left(\frac{\theta}{2}\right)^l E_l\left(W_1^{h_1}W_2^{h_2}\right),
\end{aligned}
\tag{10.21}
$$

where $E_l\left(W_1^{h_1}W_2^{h_2}\right)$ is the $(h_1,h_2)^{th}$ product moment of $(W_1,W_2) \sim BB_1^{III}\left(\frac{n_1}{2},\frac{n_2}{2},\frac{m}{2}+l,c\right)$ given by (5.31). Using (5.31) in (10.21) gives

$$
\begin{aligned}
E\left(W_1^{h_1}W_2^{h_2}\right) = & \frac{\Gamma\left(\frac{n_1}{2}+h_1\right)\Gamma\left(\frac{n_2}{2}+h_2\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} e^{-\frac{1}{2}\theta} \sum_{l=0}^{\infty} \frac{1}{l!} \frac{\Gamma\left(\frac{n_1+n_2+m}{2}+l\right)}{\Gamma\left(\frac{n_1+n_2+m}{2}+l+h_1+h_2\right)} c^{-\frac{1}{2}m-l}\left(\frac{\theta}{2}\right)^l \\
& \cdot {}_2F_1\left(\frac{m}{2}+l, \frac{n_1+n_2+m}{2}+l; \frac{n_1+n_2+m}{2}+l+h_1+h_2; \frac{c-1}{c}\right). \quad \blacksquare
\end{aligned}
$$

Theorem 10.5
If $(W_1,W_2) \sim BB_1^{III}\left(\frac{n_1}{2},\frac{n_2}{2},\frac{m}{2},c;\theta\right)$ *then the pdf of* $\Lambda_3 = W_1^{\frac{1}{2}n_1}W_2^{\frac{1}{2}n_2}$ *is*

$$
\begin{aligned}
f(\lambda_3) = & \frac{1}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} c^{-\frac{1}{2}m} e^{-\frac{1}{2}\theta} \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{1}{k!l!}\Gamma\left(\frac{n_1+n_2+m}{2}+k+l\right) \frac{\Gamma\left(\frac{m}{2}+l+k\right)}{\Gamma\left(\frac{m}{2}+l\right)} \\
& \cdot \left(\frac{\theta}{2c}\right)^l \left(\frac{c-1}{c}\right)^k H_{1,2}^{2,0}\left(\lambda_3\Big|_{\left(0,\frac{n_1}{2}\right),\left(0,\frac{n_2}{2}\right)}^{\left(\frac{m}{2}+k+l,\frac{n_1+n_2}{2}\right)}\right), \quad 0<\lambda_3<1.
\end{aligned}
\tag{10.22}
$$

Proof:
From (10.20) and [2.5.1] the Mellin transform of $f(\lambda_3)$ (see [2.8.1]) is

$$\begin{aligned}
M_f(h) &\equiv E\left(\Lambda_3^{h-1}\right) \\
&= E\left[\left(W_1^{\frac{1}{2}n_1} W_2^{\frac{1}{2}n_2}\right)^{h-1}\right] \\
&= \frac{\Gamma\left(\frac{n_1}{2}h\right)\Gamma\left(\frac{n_2}{2}h\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} c^{-\frac{1}{2}m} e^{-\frac{1}{2}\theta} \sum_{l=0}^{\infty} \frac{1}{l!} \frac{\Gamma\left(\frac{n_1+n_2+m}{2}+l\right)}{\Gamma\left(\frac{m}{2}+l+\frac{n_1+n_2}{2}h\right)} \left(\frac{\theta}{2c}\right)^l \\
&\quad \cdot {}_2F_1\left(\frac{m}{2}+l, \frac{n_1+n_2+m}{2}+l; \frac{m}{2}+l+\frac{n_1+n_2}{2}h; \frac{c-1}{c}\right) \\
&= \frac{\Gamma\left(\frac{n_1}{2}h\right)\Gamma\left(\frac{n_2}{2}h\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} c^{-\frac{1}{2}m} e^{-\frac{1}{2}\theta} \sum_{l=0}^{\infty} \frac{1}{l!} \frac{\Gamma\left(\frac{n_1+n_2+m}{2}+l\right)}{\Gamma\left(\frac{m}{2}+l+\frac{n_1+n_2}{2}h\right)} \left(\frac{\theta}{2c}\right)^l \\
&\quad \cdot \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\Gamma\left(\frac{m}{2}+l+k\right)}{\Gamma\left(\frac{m}{2}+l\right)} \frac{\Gamma\left(\frac{n_1+n_2+m}{2}+l+k\right)}{\Gamma\left(\frac{n_1+n_2+m}{2}+l\right)} \frac{\Gamma\left(\frac{m}{2}+l+\frac{n_1+n_2}{2}h\right)}{\Gamma\left(\frac{m}{2}+l+\frac{n_1+n_2}{2}h+k\right)} \left(\frac{c-1}{c}\right)^k \\
&= \frac{\Gamma\left(\frac{n_1}{2}h\right)\Gamma\left(\frac{n_2}{2}h\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} c^{-\frac{1}{2}m} e^{-\frac{1}{2}\theta} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{1}{k!l!} \frac{\Gamma\left(\frac{m}{2}+l+k\right)}{\Gamma\left(\frac{m}{2}+l\right)} \frac{\Gamma\left(\frac{n_1+n_2+m}{2}+l+k\right)}{\Gamma\left(\frac{m}{2}+l+\frac{n_1+n_2}{2}h+k\right)} \left(\frac{\theta}{2c}\right)^l \left(\frac{c-1}{c}\right)^k .
\end{aligned} \tag{10.23}$$

The gamma function in (10.23) can be written as

$$\begin{aligned}
&\Gamma\left(\frac{m}{2}+l+\frac{n_1+n_2}{2}h+k\right) \\
&= \prod_{j=1}^{1} \Gamma\left(a_j+\alpha_j h\right),
\end{aligned} \tag{10.24}$$

where $a_1 = \frac{m}{2}+k+l$ and $\alpha_1 = \frac{n_1+n_2}{2}$.

Similarly

$$\begin{aligned}
&\Gamma\left(\frac{n_1}{2}h\right)\Gamma\left(\frac{n_2}{2}h\right) \\
&= \prod_{j=1}^{2} \Gamma\left(b_j+\beta_j h\right),
\end{aligned} \tag{10.25}$$

where $b_1 = b_2 = 0$, $\beta_1 = \frac{n_1}{2}$ and $\beta_2 = \frac{n_2}{2}$.

Now, substituting (10.24) and (10.25) in (10.23) gives

$$M_f(h) \equiv \frac{1}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} c^{-\frac{1}{2}m} e^{-\frac{1}{2}\theta} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{1}{k!l!} \frac{\Gamma\left(\frac{m}{2}+l+k\right)}{\Gamma\left(\frac{m}{2}+l\right)} \Gamma\left(\frac{n_1+n_2+m}{2}+l+k\right) \left(\frac{\theta}{2c}\right)^l \left(\frac{c-1}{c}\right)^k \frac{\prod_{j=1}^{2}\Gamma\left(b_j+\beta_j h\right)}{\prod_{j=1}^{1}\Gamma\left(a_j+\alpha_j h\right)}. \tag{10.26}$$

The pdf of Λ_3, given by (10.22), is obtained from the inverse Mellin transform of (10.26) (see [2.8.1]) and the definition of Fox's H-function (see [2.8.3]). ■

Remark 10.5
The effect of the parameters was studied in Sections 4.5, 5.5 and 9.4.

11 Noncentral bimatrix variate beta type IV distribution

In this section an exact expression for the pdf of the noncentral bimatrix variate beta type IV distribution with $(\boldsymbol{X}_1, \boldsymbol{X}_2)$ defined in (1.4) is derived as well as the corresponding product moment of the determinants and the pdf of $\Lambda_4 = |\boldsymbol{X}_1\boldsymbol{X}_2|$ (see (1.10)). This distribution is also known in the literature as the noncentral bimatrix variate generalised beta type I distribution and its pdf and some properties were derived independently from this study by Díaz-García and Gutiérrez-Jáimez (2009).

11.1 Probability density function

In Theorem 11.1 an exact expression for the pdf of the noncentral bimatrix variate beta type IV distribution is derived.

Theorem 11.1
Let $\boldsymbol{S}_1 \sim W_p(n_1, \boldsymbol{\Sigma})$, $\boldsymbol{S}_2 \sim W_p(n_2, \boldsymbol{\Sigma})$ *and* $\boldsymbol{B} \sim W_p(m, \boldsymbol{\Sigma}, \boldsymbol{\Theta})$ *be independently distributed. Define*

$$\boldsymbol{X}_i = \left(\boldsymbol{I}_p + \boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{S}_i\boldsymbol{B}^{-\frac{1}{2}}\right)^{-\frac{1}{2}} \boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{S}_i\boldsymbol{B}^{-\frac{1}{2}} \left(\boldsymbol{I}_p + \boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{S}_i\boldsymbol{B}^{-\frac{1}{2}}\right)^{-\frac{1}{2}}, \ i = 1, 2, \tag{11.1}$$

where $\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{B}^{\frac{1}{2}} = \boldsymbol{B}$ *(see* (1.4)*).*

The pdf of $(\boldsymbol{X}_1, \boldsymbol{X}_2) \sim BB_p^{IV}(n_1, n_2, m; \boldsymbol{\Theta})$ *is*

$$\begin{aligned}
& f(\boldsymbol{X}_1, \boldsymbol{X}_2) \\
= \; & \left\{\beta_p\left(\tfrac{n_1}{2}, \tfrac{n_2}{2}; \tfrac{m}{2}\right)\right\}^{-1} \prod_{i=1}^{2} |\boldsymbol{X}_i|^{\frac{1}{2}n_i - \frac{1}{2}(p+1)} \prod_{i=1}^{2} |\boldsymbol{I}_p - \boldsymbol{X}_i|^{-\frac{1}{2}n_i - \frac{1}{2}(p+1)} \left|\boldsymbol{I}_p + \sum_{i=1}^{2} \boldsymbol{X}_i (\boldsymbol{I}_p - \boldsymbol{X}_i)^{-1}\right|^{-\frac{1}{2}(n_1+n_2+m)} \\
& \cdot \operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) {}_1F_1\left(\tfrac{n_1+n_2+m}{2}; \tfrac{m}{2}; \tfrac{1}{2}\left[\boldsymbol{I}_p + \sum_{i=1}^{2} \boldsymbol{X}_i (\boldsymbol{I}_p - \boldsymbol{X}_i)^{-1}\right]^{-1} \boldsymbol{\Theta}\right) \\
= \; & \left\{\beta_p\left(\tfrac{n_1}{2}, \tfrac{n_2}{2}; \tfrac{m}{2}\right)\right\}^{-1} \prod_{i=1}^{2} |\boldsymbol{X}_i|^{\frac{1}{2}n_i - \frac{1}{2}(p+1)} \\
& \cdot |\boldsymbol{I}_p - \boldsymbol{X}_1|^{\frac{1}{2}(n_2+m) - \frac{1}{2}(p+1)} |\boldsymbol{I}_p - \boldsymbol{X}_2|^{\frac{1}{2}(n_1+m) - \frac{1}{2}(p+1)} |\boldsymbol{I}_p - \boldsymbol{X}_1\boldsymbol{X}_2|^{-\frac{1}{2}(n_1+n_2+m)} \\
& \cdot \operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) {}_1F_1\left(\tfrac{n_1+n_2+m}{2}; \tfrac{m}{2}; \tfrac{1}{2}\left[\boldsymbol{I}_p + \sum_{i=1}^{2} \boldsymbol{X}_i (\boldsymbol{I}_p - \boldsymbol{X}_i)^{-1}\right]^{-1} \boldsymbol{\Theta}\right) \\
= \; & g(\boldsymbol{X}_1, \boldsymbol{X}_2) \operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) {}_1F_1\left(\tfrac{n_1+n_2+m}{2}; \tfrac{m}{2}; \tfrac{1}{2}\left[\boldsymbol{I}_p + \sum_{i=1}^{2} \boldsymbol{X}_i (\boldsymbol{I}_p - \boldsymbol{X}_i)^{-1}\right]^{-1} \boldsymbol{\Theta}\right),
\end{aligned} \tag{11.2}$$

$\boldsymbol{0} < \boldsymbol{X}_i < \boldsymbol{I}_p$, $i = 1, 2$, *where* $n_i > (p-1)$, $i = 1, 2$, $m > (p-1)$ *and* $g(\cdot)$ *is the pdf of* $BB_p^{IV}(n_1, n_2, m)$ *given by* (6.2).

Proof:
The pdf of $(\boldsymbol{S}_1, \boldsymbol{S}_2, \boldsymbol{B})$ is given by

$$K \prod_{i=1}^{2} \left[\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}_i\right) |\boldsymbol{S}_i|^{\frac{1}{2}(n_i - p - 1)}\right] \left[\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) \operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\right) |\boldsymbol{B}|^{\frac{1}{2}(m-p-1)} \; {}_0F_1\left(\tfrac{m}{2}; \tfrac{1}{4}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\right)\right] \tag{11.3}$$

where $K^{-1} = \Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)\Gamma_p\left(\frac{m}{2}\right)|2\boldsymbol{\Sigma}|^{\frac{1}{2}(n_1+n_2+m)}$ (see [2.10.2]).

Making the transformations given by (11.1) with Jacobian $J\left(\boldsymbol{S}_1,\boldsymbol{S}_2 \to \boldsymbol{X}_1,\boldsymbol{X}_2\right) = |\boldsymbol{B}|^{(p+1)}\prod_{i=1}^{2}|\boldsymbol{I}_p-\boldsymbol{X}_i|^{-(p+1)}$ (see (6.4)) substituted in (11.3) gives

$$
\begin{aligned}
&f\left(\boldsymbol{X}_1,\boldsymbol{X}_2,\boldsymbol{B}\right)\\
&= K\mathrm{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\prod_{i=1}^{2}\left\{\mathrm{etr}\left[-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{X}_i\left(\boldsymbol{I}_p-\boldsymbol{X}_i\right)^{-1}\boldsymbol{B}^{\frac{1}{2}}\right]\left|\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{X}_i\left(\boldsymbol{I}_p-\boldsymbol{X}_i\right)^{-1}\boldsymbol{B}^{\frac{1}{2}}\right|^{\frac{1}{2}(n_i-p-1)}\right\}\\
&\quad\cdot\mathrm{etr}\left(-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\right)|\boldsymbol{B}|^{\frac{1}{2}(m-p-1)}\ {}_0F_1\left(\tfrac{m}{2};\tfrac{1}{4}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\right)|\boldsymbol{B}|^{(p+1)}\prod_{i=1}^{2}|\boldsymbol{I}_p-\boldsymbol{X}_i|^{-(p+1)}\\
&= K\mathrm{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\prod_{i=1}^{2}|\boldsymbol{X}_i|^{\frac{1}{2}n_i-\frac{1}{2}(p+1)}\prod_{i=1}^{2}|\boldsymbol{I}_p-\boldsymbol{X}_i|^{-\frac{1}{2}n_i-\frac{1}{2}(p+1)}\\
&\quad\cdot|\boldsymbol{B}|^{\frac{1}{2}(n_1+n_2+m)-\frac{1}{2}(p+1)}\mathrm{etr}\left\{-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}^{\frac{1}{2}}\left[\boldsymbol{I}_p+\sum_{i=1}^{2}\boldsymbol{X}_i\left(\boldsymbol{I}_p-\boldsymbol{X}_i\right)^{-1}\right]\boldsymbol{B}^{\frac{1}{2}}\right\}\ {}_0F_1\left(\tfrac{m}{2};\tfrac{1}{4}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\right).
\end{aligned}
\tag{11.4}
$$

The pdf of $(\boldsymbol{X}_1,\boldsymbol{X}_2)$ is obtained by integrating (11.4) with respect to $\boldsymbol{B}$,

$$
\begin{aligned}
&f\left(\boldsymbol{X}_1,\boldsymbol{X}_2\right)\\
&= K\mathrm{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\prod_{i=1}^{2}|\boldsymbol{X}_i|^{\frac{1}{2}n_i-\frac{1}{2}(p+1)}\prod_{i=1}^{2}|\boldsymbol{I}_p-\boldsymbol{X}_i|^{-\frac{1}{2}n_i-\frac{1}{2}(p+1)}\\
&\quad\cdot\int_{\boldsymbol{B}>\boldsymbol{0}}|\boldsymbol{B}|^{\frac{1}{2}(n_1+n_2+m)-\frac{1}{2}(p+1)}\mathrm{etr}\left\{-\tfrac{1}{2}\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}^{\frac{1}{2}}\left[\boldsymbol{I}_p+\sum_{i=1}^{2}\boldsymbol{X}_i\left(\boldsymbol{I}_p-\boldsymbol{X}_i\right)^{-1}\right]\right\}\ {}_0F_1\left(\tfrac{m}{2};\tfrac{1}{4}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\right)\mathrm{d}\boldsymbol{B}.
\end{aligned}
\tag{11.5}
$$

Next, we consider the symmetrised density function of $(\boldsymbol{X}_1,\boldsymbol{X}_2)$ (see [2.9.1]), that is $f_s\left(\boldsymbol{X}_1,\boldsymbol{X}_2\right) \equiv \int_{O(p)} f\left(\boldsymbol{H}\boldsymbol{X}_1\boldsymbol{H}',\boldsymbol{H}\boldsymbol{X}_2\boldsymbol{H}'\right)\mathrm{d}\boldsymbol{H}$ where $\boldsymbol{H}$ $(p\times p)$ is orthogonal and $\mathrm{d}\boldsymbol{H}$ is the normalised Haar invariant measure on $O(p)$. From (11.5)

$$
\begin{aligned}
&f\left(\boldsymbol{H}\boldsymbol{X}_1\boldsymbol{H}',\boldsymbol{H}\boldsymbol{X}_2\boldsymbol{H}'\right)\\
&= K\mathrm{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\prod_{i=1}^{2}\left|\boldsymbol{H}\boldsymbol{X}_i\boldsymbol{H}'\right|^{\frac{1}{2}n_i-\frac{1}{2}(p+1)}\prod_{i=1}^{2}\left|\boldsymbol{I}_p-\boldsymbol{H}\boldsymbol{X}_i\boldsymbol{H}'\right|^{-\frac{1}{2}n_i-\frac{1}{2}(p+1)}\int_{\boldsymbol{B}>\boldsymbol{0}}|\boldsymbol{B}|^{\frac{1}{2}(n_1+n_2+m)-\frac{1}{2}(p+1)}\\
&\qquad\cdot\mathrm{etr}\left\{-\tfrac{1}{2}\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}^{\frac{1}{2}}\left[\boldsymbol{I}_p+\sum_{i=1}^{2}\boldsymbol{H}\boldsymbol{X}_i\boldsymbol{H}'\left(\boldsymbol{I}_p-\boldsymbol{H}\boldsymbol{X}_i\boldsymbol{H}'\right)^{-1}\right]\right\}\ {}_0F_1\left(\tfrac{m}{2};\tfrac{1}{4}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\right)\mathrm{d}\boldsymbol{B}.
\end{aligned}
\tag{11.6}
$$

Then from (11.6), [2.3.6] and [2.6.6],

$$
\begin{aligned}
& f_s(\boldsymbol{X}_1, \boldsymbol{X}_2) \\
= \;& K \operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) \prod_{i=1}^{2} |\boldsymbol{X}_i|^{\frac{1}{2}n_i - \frac{1}{2}(p+1)} \prod_{i=1}^{2} |\boldsymbol{I}_p - \boldsymbol{X}_i|^{-\frac{1}{2}n_i - \frac{1}{2}(p+1)} \\
& \cdot \int_{\boldsymbol{B}>\boldsymbol{0}} |\boldsymbol{B}|^{\frac{1}{2}(n_1+n_2+m) - \frac{1}{2}(p+1)} \; {}_0F_1\left(\tfrac{m}{2}; \tfrac{1}{4}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\right) \\
& \cdot \int_{O(p)} \operatorname{etr}\left\{-\tfrac{1}{2}\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{H}\left[\boldsymbol{I}_p + \sum_{i=1}^{2}\boldsymbol{X}_i(\boldsymbol{I}_p - \boldsymbol{X}_i)^{-1}\right]\boldsymbol{H}'\right\} \mathrm{d}\boldsymbol{H}\,\mathrm{d}\boldsymbol{B} \\
= \;& K \operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) \prod_{i=1}^{2} |\boldsymbol{X}_i|^{\frac{1}{2}n_i - \frac{1}{2}(p+1)} \prod_{i=1}^{2} |\boldsymbol{I}_p - \boldsymbol{X}_i|^{-\frac{1}{2}n_i - \frac{1}{2}(p+1)} \\
& \cdot \int_{O(p)} \int_{\boldsymbol{B}>\boldsymbol{0}} |\boldsymbol{B}|^{\frac{1}{2}(n_1+n_2+m) - \frac{1}{2}(p+1)} \; {}_0F_1\left(\tfrac{m}{2}; \tfrac{1}{4}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\right) \\
& \qquad \cdot \operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{H}\left[\boldsymbol{I}_p + \sum_{i=1}^{2}\boldsymbol{X}_i(\boldsymbol{I}_p - \boldsymbol{X}_i)^{-1}\right]\boldsymbol{H}'\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{B}\right)\mathrm{d}\boldsymbol{B}\,\mathrm{d}\boldsymbol{H} \\
= \;& K\Gamma_p\left(\tfrac{n_1+n_2+m}{2}\right) \operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) \prod_{i=1}^{2} |\boldsymbol{X}_i|^{\frac{1}{2}n_i - \frac{1}{2}(p+1)} \prod_{i=1}^{2} |\boldsymbol{I}_p - \boldsymbol{X}_i|^{-\frac{1}{2}n_i - \frac{1}{2}(p+1)} \\
& \cdot \int_{O(p)} \left|\tfrac{1}{2}\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{H}\left[\boldsymbol{I}_p + \sum_{i=1}^{2}\boldsymbol{X}_i(\boldsymbol{I}_p - \boldsymbol{X}_i)^{-1}\right]\boldsymbol{H}'\boldsymbol{\Sigma}^{-\frac{1}{2}}\right|^{-\frac{1}{2}(n_1+n_2+m)} \\
& \qquad \cdot {}_1F_1\left(\tfrac{n_1+n_2+m}{2}; \tfrac{m}{2}; \tfrac{2}{4}\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{H}\left[\boldsymbol{I}_p + \sum_{i=1}^{2}\boldsymbol{X}_i(\boldsymbol{I}_p - \boldsymbol{X}_i)^{-1}\right]^{-1}\boldsymbol{H}'\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\right)\mathrm{d}\boldsymbol{H}.
\end{aligned}
\tag{11.7}
$$

Since $\boldsymbol{\Theta} = \boldsymbol{\Sigma}^{-1}\boldsymbol{M}\boldsymbol{M}'$ (see [2.10.4]) it follows from [2.6.1], [2.3.4] and [2.3.5] that the integral in (11.7) over the orthogonal group can be written as

$$
\begin{aligned}
& \int_{O(p)} {}_1F_1\left(\tfrac{n_1+n_2+m}{2}; \tfrac{m}{2}; \tfrac{1}{2}\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{H}\left[\boldsymbol{I}_p + \sum_{i=1}^{2}\boldsymbol{X}_i(\boldsymbol{I}_p - \boldsymbol{X}_i)^{-1}\right]^{-1}\boldsymbol{H}'\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\right)\mathrm{d}\boldsymbol{H} \\
&= \sum_{k=0}^{\infty}\sum_{\kappa} \frac{\left(\frac{n_1+n_2+m}{2}\right)_\kappa}{\left(\frac{m}{2}\right)_\kappa}\frac{1}{k!}\int_{O(p)} C_\kappa\left(\tfrac{1}{2}\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{H}\left[\boldsymbol{I}_p + \sum_{i=1}^{2}\boldsymbol{X}_i(\boldsymbol{I}_p - \boldsymbol{X}_i)^{-1}\right]^{-1}\boldsymbol{H}'\right)\mathrm{d}\boldsymbol{H} \\
&= \sum_{k=0}^{\infty}\sum_{\kappa} \frac{\left(\frac{n_1+n_2+m}{2}\right)_\kappa}{\left(\frac{m}{2}\right)_\kappa}\frac{1}{k!}\int_{O(p)} C_\kappa\left(\tfrac{1}{2}\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{\Sigma}^{-1}\boldsymbol{M}\boldsymbol{M}'\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{H}\left[\boldsymbol{I}_p + \sum_{i=1}^{2}\boldsymbol{X}_i(\boldsymbol{I}_p - \boldsymbol{X}_i)^{-1}\right]^{-1}\boldsymbol{H}'\right)\mathrm{d}\boldsymbol{H} \\
&= \sum_{k=0}^{\infty}\sum_{\kappa} \frac{\left(\frac{n_1+n_2+m}{2}\right)_\kappa}{\left(\frac{m}{2}\right)_\kappa}\frac{1}{k!}\frac{C_\kappa\left(\frac{1}{2}\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{M}\boldsymbol{M}'\boldsymbol{\Sigma}^{-\frac{1}{2}}\right)C_\kappa\left(\left[\boldsymbol{I}_p + \sum_{i=1}^{2}\boldsymbol{X}_i(\boldsymbol{I}_p - \boldsymbol{X}_i)^{-1}\right]^{-1}\right)}{C_\kappa(\boldsymbol{I}_p)} \\
&= \sum_{k=0}^{\infty}\sum_{\kappa} \frac{\left(\frac{n_1+n_2+m}{2}\right)_\kappa}{\left(\frac{m}{2}\right)_\kappa}\frac{1}{k!}\int_{O(p)} C_\kappa\left(\tfrac{1}{2}\boldsymbol{\Theta}\boldsymbol{H}\left[\boldsymbol{I}_p + \sum_{i=1}^{2}\boldsymbol{X}_i(\boldsymbol{I}_p - \boldsymbol{X}_i)^{-1}\right]^{-1}\boldsymbol{H}'\right)\mathrm{d}\boldsymbol{H} \\
&= \int_{O(p)} {}_1F_1\left(\tfrac{n_1+n_2+m}{2}; \tfrac{m}{2}; \tfrac{1}{2}\boldsymbol{\Theta}\boldsymbol{H}\left[\boldsymbol{I}_p + \sum_{i=1}^{2}\boldsymbol{X}_i(\boldsymbol{I}_p - \boldsymbol{X}_i)^{-1}\right]^{-1}\boldsymbol{H}'\right)\mathrm{d}\boldsymbol{H}.
\end{aligned}
\tag{11.8}
$$

Substituting (11.8) in (11.7) gives

$$\begin{aligned}
&f_s\left(\boldsymbol{X}_1, \boldsymbol{X}_2\right) \\
&= \left\{\beta_p\left(\tfrac{n_1}{2}, \tfrac{n_2}{2}; \tfrac{m}{2}\right)\right\}^{-1} \operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) \prod_{i=1}^{2} \left|\boldsymbol{X}_i\right|^{\frac{1}{2}n_i-\frac{1}{2}(p+1)} \prod_{i=1}^{2} \left|\boldsymbol{I}_p - \boldsymbol{X}_i\right|^{-\frac{1}{2}n_i-\frac{1}{2}(p+1)} \\
&\quad \cdot \left|\boldsymbol{I}_p + \sum_{i=1}^{2} \boldsymbol{X}_i\left(\boldsymbol{I}_p - \boldsymbol{X}_i\right)^{-1}\right|^{-\frac{1}{2}(n_1+n_2+m)} \int_{O(p)} {}_1F_1\left(\tfrac{n_1+n_2+m}{2}; \tfrac{m}{2}; \tfrac{1}{2}\boldsymbol{\Theta}\boldsymbol{H}\left[\boldsymbol{I}_p + \sum_{i=1}^{2} \boldsymbol{X}_i\left(\boldsymbol{I}_p - \boldsymbol{X}_i\right)^{-1}\right]^{-1} \boldsymbol{H}'\right) \mathrm{d}\boldsymbol{H}.
\end{aligned} \tag{11.9}$$

Since $f_s\left(\boldsymbol{X}_1, \boldsymbol{X}_2\right) \equiv \int_{O(p)} f\left(\boldsymbol{H}\boldsymbol{X}_1\boldsymbol{H}', \boldsymbol{H}\boldsymbol{X}_2\boldsymbol{H}'\right)\mathrm{d}\boldsymbol{H}$ it follows from (11.9) that

$$\begin{aligned}
&f\left(\boldsymbol{H}\boldsymbol{X}_1\boldsymbol{H}', \boldsymbol{H}\boldsymbol{X}_2\boldsymbol{H}'\right) \\
&= \left\{\beta_p\left(\tfrac{n_1}{2}, \tfrac{n_2}{2}; \tfrac{m}{2}\right)\right\}^{-1} \operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) \prod_{i=1}^{2} \left|\boldsymbol{H}\boldsymbol{X}_i\boldsymbol{H}'\right|^{\frac{1}{2}n_i-\frac{1}{2}(p+1)} \prod_{i=1}^{2} \left|\boldsymbol{I}_p - \boldsymbol{H}\boldsymbol{X}_i\boldsymbol{H}'\right|^{-\frac{1}{2}n_i-\frac{1}{2}(p+1)} \\
&\quad \cdot \left|\boldsymbol{I}_p + \sum_{i=1}^{2} \boldsymbol{H}\boldsymbol{X}_i\boldsymbol{H}'\left(\boldsymbol{I}_p - \boldsymbol{H}\boldsymbol{X}_i\boldsymbol{H}'\right)^{-1}\right|^{-\frac{1}{2}(n_1+n_2+m)} {}_1F_1\left(\tfrac{n_1+n_2+m}{2}; \tfrac{m}{2}; \tfrac{1}{2}\boldsymbol{\Theta}\boldsymbol{H}\left[\boldsymbol{I}_p + \sum_{i=1}^{2} \boldsymbol{X}_i\left(\boldsymbol{I}_p - \boldsymbol{X}_i\right)^{-1}\right]^{-1} \boldsymbol{H}'\right).
\end{aligned} \tag{11.10}$$

From (11.10) and [2.9.2] the pdf of $\left(\boldsymbol{X}_1, \boldsymbol{X}_2\right)$ is

$$\begin{aligned}
&f\left(\boldsymbol{X}_1, \boldsymbol{X}_2\right) \\
&= \left\{\beta_p\left(\tfrac{n_1}{2}, \tfrac{n_2}{2}; \tfrac{m}{2}\right)\right\}^{-1} \operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) \prod_{i=1}^{2} \left|\boldsymbol{X}_i\right|^{\frac{1}{2}n_i-\frac{1}{2}(p+1)} \prod_{i=1}^{2} \left|\boldsymbol{I}_p - \boldsymbol{X}_i\right|^{-\frac{1}{2}n_i-\frac{1}{2}(p+1)} \\
&\quad \cdot \left|\boldsymbol{I}_p + \sum_{i=1}^{2} \boldsymbol{X}_i\left(\boldsymbol{I}_p - \boldsymbol{X}_i\right)^{-1}\right|^{-\frac{1}{2}(n_1+n_2+m)} {}_1F_1\left(\tfrac{n_1+n_2+m}{2}; \tfrac{m}{2}; \tfrac{1}{2}\left[\boldsymbol{I}_p + \sum_{i=1}^{2} \boldsymbol{X}_i\left(\boldsymbol{I}_p - \boldsymbol{X}_i\right)^{-1}\right]^{-1} \boldsymbol{\Theta}\right).
\end{aligned} \tag{11.11}$$

Since $\left|\boldsymbol{I}_p + \sum_{i=1}^{2} \boldsymbol{X}_i\left(\boldsymbol{I}_p - \boldsymbol{X}_i\right)^{-1}\right| = \left|\boldsymbol{I}_p - \boldsymbol{X}_1\boldsymbol{X}_2\right| \left|\boldsymbol{I}_p - \boldsymbol{X}_1\right|^{-1} \left|\boldsymbol{I}_p - \boldsymbol{X}_2\right|^{-1}$, we can write (11.11) as

$$\begin{aligned}
&f\left(\boldsymbol{X}_1, \boldsymbol{X}_2\right) \\
&= \left\{\beta_p\left(\tfrac{n_1}{2}, \tfrac{n_2}{2}; \tfrac{m}{2}\right)\right\}^{-1} \operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) \prod_{i=1}^{2} \left|\boldsymbol{X}_i\right|^{\frac{1}{2}n_i-\frac{1}{2}(p+1)} \left|\boldsymbol{I}_p - \boldsymbol{X}_1\right|^{\frac{1}{2}(n_2+m)-\frac{1}{2}(p+1)} \left|\boldsymbol{I}_p - \boldsymbol{X}_2\right|^{\frac{1}{2}(n_1+m)-\frac{1}{2}(p+1)} \\
&\quad \cdot \left|\boldsymbol{I}_p - \boldsymbol{X}_1\boldsymbol{X}_2\right|^{-\frac{1}{2}(n_1+n_2+m)} {}_1F_1\left(\tfrac{n_1+n_2+m}{2}; \tfrac{m}{2}; \tfrac{1}{2}\left[\boldsymbol{I}_p + \sum_{i=1}^{2} \boldsymbol{X}_i\left(\boldsymbol{I}_p - \boldsymbol{X}_i\right)^{-1}\right]^{-1} \boldsymbol{\Theta}\right).
\end{aligned}$$

This is the result given by (11.2). ■

Remark 11.1

The ratio in (11.1) was used to correspond with the rest of the study (see (8.1), (9.1), (10.1) and (12.1)). The following definition of Wishart ratios (see Remark 6.1) will also give matrix variates having the noncentral bimatrix variate beta type IV distribution with pdf given by (11.2) (see Díaz-García and Gutiérrez-Jáimez, 2009):

$$\boldsymbol{X}_i = \left(\boldsymbol{S}_i + \boldsymbol{B}\right)^{-\frac{1}{2}} \boldsymbol{S}_i \left(\boldsymbol{S}_i + \boldsymbol{B}\right)^{-\frac{1}{2}}, \ i = 1, 2.$$

Remark 11.2

The noncentral matrix variate Dirichlet type IV distribution denoted by $(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_r) \sim D_p^{IV}(n_1, \ldots, n_r, m; \boldsymbol{\Theta})$, results by extending (11.1) to r independent Wishart matrix variates, $\boldsymbol{S}_i \sim W_p(n_i, \boldsymbol{\Sigma})$, $i = 1, \ldots, r$, all independent of $\boldsymbol{B} \sim W_p(m, \boldsymbol{\Sigma}; \boldsymbol{\Theta})$. The pdf of $(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_r)$ is given by

$$\begin{aligned}&\left\{\beta_p\left(\tfrac{n_1}{2}, \ldots, \tfrac{n_r}{2}, \tfrac{m}{2}\right)\right\}^{-1} \operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) \prod_{i=1}^{r} |\boldsymbol{X}_i|^{\frac{1}{2}n_i - \frac{1}{2}(p+1)} \prod_{i=1}^{r} |\boldsymbol{I}_p - \boldsymbol{X}_i|^{-\frac{1}{2}n_i - \frac{1}{2}(p+1)} \\ &\cdot \left| I_p + \sum_{i=1}^{r} \boldsymbol{X}_i (\boldsymbol{I}_p - \boldsymbol{X}_i)^{-1} \right|^{-\frac{1}{2}(n+m)} {}_1F_1\left(\tfrac{n+m}{2}; \tfrac{m}{2}; \tfrac{1}{2}\boldsymbol{\Theta}\left[I_p + \sum_{i=1}^{r} \boldsymbol{X}_i (\boldsymbol{I}_p - \boldsymbol{X}_i)^{-1}\right]^{-1}\right), \end{aligned} \tag{11.12}$$

$\boldsymbol{0} < \boldsymbol{X}_i < \boldsymbol{I}_p$, $i = 1, \ldots, r$, where $n = n_1 + \cdots + n_r$.

Remark 11.3

The pdfs of the noncentral bimatrix variate beta type IV distribution in (11.2) and the noncentral matrix variate Dirichlet type IV distribution in (11.12) are not members of the Liouville family of distributions (see [2.2.1]).

11.2 Product moment of the determinants

The $(h_1, h_2)^{th}$ product moment, $E\left(|\boldsymbol{X}_1|^{h_1} |\boldsymbol{X}_2|^{h_2}\right)$, where $(\boldsymbol{X}_1, \boldsymbol{X}_2) \sim BB_p^{IV}(n_1, n_2, m; \boldsymbol{\Theta})$ is derived in Theorem 11.2.

Theorem 11.2

If $(\boldsymbol{X}_1, \boldsymbol{X}_2) \sim BB_p^{IV}(n_1, n_2, m; \boldsymbol{\Theta})$ *then* $E\left(|\boldsymbol{X}_1|^{h_1} |\boldsymbol{X}_2|^{h_2}\right)$ *is given by*

$$\begin{aligned}&E\left(|\boldsymbol{X}_1|^{h_1} |\boldsymbol{X}_2|^{h_2}\right) \\ &= \frac{\left[\Gamma_p\left(\frac{p+1}{2}\right)\right]^2}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)} \operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) \sum_{\kappa,\lambda,\tau,\rho,J;\phi,\phi^*} \frac{1}{k!t!j!} \binom{\kappa}{\lambda}\binom{\tau}{\rho} g_{\lambda,\rho}^{\phi} \theta_{\phi^*}^{J,\phi} \\ &\cdot \frac{C_\kappa(\boldsymbol{I}_p)}{C_\lambda(\boldsymbol{I}_p)} \frac{C_\tau(\boldsymbol{I}_p)}{C_\rho(\boldsymbol{I}_p)} \frac{\Gamma_p\left(\frac{n_1}{2}+\frac{p+1}{2},\kappa\right)}{\Gamma_p\left(\frac{n_1}{2}+\frac{p+1}{2},\lambda\right)} \frac{\Gamma_p\left(\frac{n_1}{2}+h_1,\kappa\right)}{\Gamma_p\left(\frac{n_1}{2}+h_1+\frac{p+1}{2},\kappa\right)} \frac{\Gamma_p\left(\frac{n_2}{2}+\frac{p+1}{2},\tau\right)}{\Gamma_p\left(\frac{n_2}{2}+\frac{p+1}{2},\rho\right)} \frac{\Gamma_p\left(\frac{n_2}{2}+h_2,\tau\right)}{\Gamma_p\left(\frac{n_2}{2}+h_2+\frac{p+1}{2},\tau\right)} \frac{\Gamma_p\left(\frac{n_1+n_2+m}{2},\phi^*\right)}{\Gamma_p\left(\frac{m}{2},J\right)} C_{\phi^*}^{J,\phi}\left(\tfrac{1}{2}\boldsymbol{\Theta}, -\boldsymbol{I}_p\right),\end{aligned} \tag{11.13}$$

where $\sum_{\kappa,\lambda,\tau,\rho,J;\phi,\phi^*} = \sum_{k=0}^{\infty}\sum_{\kappa}\sum_{l=0}^{k}\sum_{\lambda}\sum_{t=0}^{\infty}\sum_{\tau}\sum_{r=0}^{t}\sum_{\rho}\sum_{\phi\in\lambda\cdot\rho}\sum_{j=0}^{\infty}\sum_{J}\sum_{\phi^*\in J\cdot\phi}$, $\phi \in \lambda \cdot \rho$ *and* $\phi^* \in J \cdot \phi$ *are explained in [2.4.1],* $C_{\phi^*}^{J,\phi}\left(\frac{1}{2}\boldsymbol{\Theta}, -\boldsymbol{I}_p\right)$ *is a homogeneous invariant polynomial of degrees* j *and* $(l+r)$ *in the elements of the symmetric matrices* $\frac{1}{2}\boldsymbol{\Theta}$ *and* $-\boldsymbol{I}_p$ *(see [2.4.1]),* $\theta_{\phi^*}^{J,\phi}$ *is explained in [2.4.2] and* $g_{\lambda,\rho}^{\phi}$ *is explained in [2.4.3]).*

Proof:

From (11.5)

$$
\begin{aligned}
& E\left(\left|\boldsymbol{X}_1\right|^{h_1}\left|\boldsymbol{X}_2\right|^{h_2}\right) \\
= \quad & K\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\int_{\boldsymbol{B}>\boldsymbol{0}}\int_{\boldsymbol{0}<\boldsymbol{X}_1<\boldsymbol{I}_p}\int_{\boldsymbol{0}<\boldsymbol{X}_2<\boldsymbol{I}_p}\prod_{i=1}^{2}\left|\boldsymbol{X}_i\right|^{\frac{1}{2}n_i+h_i-\frac{1}{2}(p+1)}\prod_{i=1}^{2}\left|\boldsymbol{I}_p-\boldsymbol{X}_i\right|^{-\frac{1}{2}n_i-\frac{1}{2}(p+1)}\left|\boldsymbol{B}\right|^{\frac{1}{2}(n_1+n_2+m)-\frac{1}{2}(p+1)} \\
& \cdot\operatorname{etr}\left\{-\tfrac{1}{2}\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}^{\frac{1}{2}}\left[\boldsymbol{I}_p+\sum_{i=1}^{2}\boldsymbol{X}_i\left(\boldsymbol{I}_p-\boldsymbol{X}_i\right)^{-1}\right]\right\}\ {}_0F_1\left(\tfrac{m}{2};\tfrac{1}{4}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\right)\mathrm{d}\boldsymbol{X}_2\mathrm{d}\boldsymbol{X}_1\mathrm{d}\boldsymbol{B} \\
= \quad & K\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\int_{\boldsymbol{B}>\boldsymbol{0}}\left|\boldsymbol{B}\right|^{\frac{1}{2}(n_1+n_2+m)-\frac{1}{2}(p+1)}\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\right)\ {}_0F_1\left(\tfrac{m}{2};\tfrac{1}{4}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\right) \\
& \cdot\int_{\boldsymbol{0}<\boldsymbol{X}_1<\boldsymbol{I}_p}\left|\boldsymbol{X}_1\right|^{\frac{1}{2}n_1+h_1-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\boldsymbol{X}_1\right|^{-\frac{1}{2}n_1-\frac{1}{2}(p+1)}\operatorname{etr}\left[-\tfrac{1}{2}\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{X}_1\left(\boldsymbol{I}_p-\boldsymbol{X}_1\right)^{-1}\right]\mathrm{d}\boldsymbol{X}_1 \\
& \cdot\int_{\boldsymbol{0}<\boldsymbol{X}_2<\boldsymbol{I}_p}\left|\boldsymbol{X}_2\right|^{\frac{1}{2}n_2+h_2-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\boldsymbol{X}_2\right|^{-\frac{1}{2}n_2-\frac{1}{2}(p+1)}\operatorname{etr}\left[-\tfrac{1}{2}\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{X}_2\left(\boldsymbol{I}_p-\boldsymbol{X}_2\right)^{-1}\right]\mathrm{d}\boldsymbol{X}_2\mathrm{d}\boldsymbol{B}
\end{aligned}
\tag{11.14}
$$

where $K^{-1}=\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)\Gamma_p\left(\frac{m}{2}\right)\left|2\boldsymbol{\Sigma}\right|^{\frac{1}{2}(n_1+n_2+m)}$. Consider the integral with respect to $\boldsymbol{X}_1$. That is,

$$I_1=\int_{\boldsymbol{0}<\boldsymbol{X}_1<\boldsymbol{I}_p}\left|\boldsymbol{X}_1\right|^{\frac{1}{2}n_1+h_1-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\boldsymbol{X}_1\right|^{-\frac{1}{2}n_1-\frac{1}{2}(p+1)}\operatorname{etr}\left[-\tfrac{1}{2}\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{X}_1\left(\boldsymbol{I}_p-\boldsymbol{X}_1\right)^{-1}\right]\mathrm{d}\boldsymbol{X}_1.$$

Making the transformation $\boldsymbol{X}_1\to\boldsymbol{H}\boldsymbol{X}_1\boldsymbol{H}'$, where $\boldsymbol{H}\in O(p)$, and using [2.3.4], [2.6.9], [2.7.1], [2.7.2], [2.3.8] and [2.7.3] gives

$$
\begin{aligned}
& I_1 \\
= \quad & \sum_{k=0}^{\infty}\sum_{\kappa}\frac{1}{k!}\int_{\boldsymbol{0}<\boldsymbol{X}_1<\boldsymbol{I}_p}\left|\boldsymbol{X}_1\right|^{\frac{1}{2}n_1+h_1-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\boldsymbol{X}_1\right|^{-\frac{1}{2}n_1-\frac{1}{2}(p+1)} \\
& \cdot\int_{O(p)}C_\kappa\left[-\tfrac{1}{2}\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{H}\boldsymbol{X}_1\left(\boldsymbol{I}_p-\boldsymbol{X}_1\right)^{-1}\boldsymbol{H}'\right]\mathrm{d}\boldsymbol{H}\mathrm{d}\boldsymbol{X}_1 \\
= \quad & \int_{\boldsymbol{0}<\boldsymbol{X}_1<\boldsymbol{I}_p}\left|\boldsymbol{X}_1\right|^{\frac{1}{2}n_1+h_1-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\boldsymbol{X}_1\right|^{-\frac{1}{2}n_1-\frac{1}{2}(p+1)}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{1}{k!}\frac{C_\kappa\left(-\frac{1}{2}\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}^{\frac{1}{2}}\right)C_\kappa\left[\boldsymbol{X}_1(\boldsymbol{I}_p-\boldsymbol{X}_1)^{-1}\right]}{C_\kappa(\boldsymbol{I}_p)}\mathrm{d}\boldsymbol{X}_1 \\
= \quad & \int_{\boldsymbol{0}<\boldsymbol{X}_1<\boldsymbol{I}_p}\left|\boldsymbol{X}_1\right|^{\frac{1}{2}n_1+h_1-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\boldsymbol{X}_1\right|^{-\frac{1}{2}n_1-\frac{1}{2}(p+1)}\ {}_0F_0^{(p)}\left(-\tfrac{1}{2}\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}^{\frac{1}{2}},\boldsymbol{X}_1\left(\boldsymbol{I}_p-\boldsymbol{X}_1\right)^{-1}\right)\mathrm{d}\boldsymbol{X}_1 \\
= \quad & \int_{\boldsymbol{0}<\boldsymbol{X}_1<\boldsymbol{I}_p}\left|\boldsymbol{X}_1\right|^{\frac{1}{2}n_1+h_1-\frac{1}{2}(p+1)}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{1}{k!}\frac{L_\kappa^{\frac{1}{2}n_1}\left(\frac{1}{2}\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}^{\frac{1}{2}}\right)C_\kappa(\boldsymbol{X}_1)}{C_\kappa(\boldsymbol{I}_p)}\mathrm{d}\boldsymbol{X}_1 \\
= \quad & \sum_{k=0}^{\infty}\sum_{\kappa}\frac{1}{k!}L_\kappa^{\frac{1}{2}n_1}\left(\tfrac{1}{2}\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}^{\frac{1}{2}}\right)\int_{\boldsymbol{0}<\boldsymbol{X}_1<\boldsymbol{I}_p}\left|\boldsymbol{X}_1\right|^{\frac{1}{2}n_1+h_1-\frac{1}{2}(p+1)}\frac{C_\kappa(\boldsymbol{X}_1)}{C_\kappa(\boldsymbol{I}_p)}\mathrm{d}\boldsymbol{X}_1 \\
= \quad & \sum_{k=0}^{\infty}\sum_{\kappa}\frac{1}{k!}L_\kappa^{\frac{1}{2}n_1}\left(\tfrac{1}{2}\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}^{\frac{1}{2}}\right)\frac{\Gamma_p\left(\frac{n_1}{2}+h_1,\kappa\right)\Gamma_p\left(\frac{p+1}{2}\right)}{\Gamma_p\left(\frac{n_1}{2}+h_1+\frac{p+1}{2},\kappa\right)} \\
= \quad & \sum_{k=0}^{\infty}\sum_{\kappa}\sum_{l=0}^{k}\sum_{\lambda}\frac{1}{k!}\left(\tfrac{n_1}{2}+\tfrac{p+1}{2}\right)_\kappa C_\kappa(\boldsymbol{I}_p)\binom{\kappa}{\lambda}\frac{C_\lambda(-\frac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{B})}{\left(\frac{n_1}{2}+\frac{p+1}{2}\right)_\lambda C_\lambda(\boldsymbol{I}_p)}\frac{\Gamma_p\left(\frac{n_1}{2}+h_1,\kappa\right)\Gamma_p\left(\frac{p+1}{2}\right)}{\Gamma_p\left(\frac{n_1}{2}+h_1+\frac{p+1}{2},\kappa\right)} \\
= \quad & \sum_{k=0}^{\infty}\sum_{\kappa}\sum_{l=0}^{k}\sum_{\lambda}\frac{1}{k!}\binom{\kappa}{\lambda}\frac{C_\kappa(\boldsymbol{I}_p)}{C_\lambda(\boldsymbol{I}_p)}\frac{\Gamma_p\left(\frac{n_1}{2}+\frac{p+1}{2},\kappa\right)}{\Gamma_p\left(\frac{n_1}{2}+\frac{p+1}{2},\lambda\right)}\frac{\Gamma_p\left(\frac{n_1}{2}+h_1,\kappa\right)\Gamma_p\left(\frac{p+1}{2}\right)}{\Gamma_p\left(\frac{n_1}{2}+h_1+\frac{p+1}{2},\kappa\right)}C_\lambda(-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}).
\end{aligned}
\tag{11.15}
$$

Similarly the integral with respect to $\boldsymbol{X}_2$ can be written as

$$\begin{aligned} I_2 &= \int_{\mathbf{0}<\boldsymbol{X}_2<\boldsymbol{I}_p} |\boldsymbol{X}_2|^{\frac{1}{2}n_2+h_2-\frac{1}{2}(p+1)} |\boldsymbol{I}_p-\boldsymbol{X}_2|^{-\frac{1}{2}n_2-\frac{1}{2}(p+1)} \operatorname{etr}\left[-\tfrac{1}{2}\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{X}_2(\boldsymbol{I}_p-\boldsymbol{X}_2)^{-1}\right] \mathrm{d}\boldsymbol{X}_2 \\ &= \sum_{t=0}^{\infty}\sum_{\tau}\sum_{r=0}^{t}\sum_{\rho} \frac{1}{t!}\binom{\tau}{\rho} \frac{C_\tau(\boldsymbol{I}_p)}{C_\rho(\boldsymbol{I}_p)} \frac{\Gamma_p\left(\frac{n_2}{2}+\frac{p+1}{2},\tau\right)}{\Gamma_p\left(\frac{n_2}{2}+\frac{p+1}{2},\rho\right)} \frac{\Gamma_p\left(\frac{n_2}{2}+h_2,\tau\right)\Gamma_p\left(\frac{p+1}{2}\right)}{\Gamma_p\left(\frac{n_2}{2}+h_2+\frac{p+1}{2},\tau\right)} C_\rho(-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}). \end{aligned} \tag{11.16}$$

Substituting (11.15) and (11.16) in (11.14) gives

$$\begin{aligned} &E\left(|\boldsymbol{X}_1|^{h_1}|\boldsymbol{X}_2|^{h_2}\right) \\ =\; & K\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\left[\Gamma_p\left(\tfrac{p+1}{2}\right)\right]^2 \sum_{k=0}^{\infty}\sum_{\kappa}\sum_{l=0}^{k}\sum_{\lambda}\frac{1}{k!}\binom{\kappa}{\lambda}\frac{C_\kappa(\boldsymbol{I}_p)}{C_\lambda(\boldsymbol{I}_p)}\frac{\Gamma_p\left(\frac{n_1}{2}+\frac{p+1}{2},\kappa\right)}{\Gamma_p\left(\frac{n_1}{2}+\frac{p+1}{2},\lambda\right)}\frac{\Gamma_p\left(\frac{n_1}{2}+h_1,\kappa\right)}{\Gamma_p\left(\frac{n_1}{2}+h_1+\frac{p+1}{2},\kappa\right)} \\ & \cdot\sum_{t=0}^{\infty}\sum_{\tau}\sum_{r=0}^{t}\sum_{\rho}\frac{1}{t!}\binom{\tau}{\rho}\frac{C_\tau(\boldsymbol{I}_p)}{C_\rho(\boldsymbol{I}_p)}\frac{\Gamma_p\left(\frac{n_2}{2}+\frac{p+1}{2},\tau\right)}{\Gamma_p\left(\frac{n_2}{2}+\frac{p+1}{2},\rho\right)}\frac{\Gamma_p\left(\frac{n_2}{2}+h_2,\tau\right)}{\Gamma_p\left(\frac{n_2}{2}+h_2+\frac{p+1}{2},\tau\right)} \\ & \cdot\int_{\boldsymbol{B}>\mathbf{0}}|\boldsymbol{B}|^{\frac{1}{2}(n_1+n_2+m)-\frac{1}{2}(p+1)}\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\right)\,{}_0F_1\left(\tfrac{m}{2};\tfrac{1}{4}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\right)C_\lambda(-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{B})C_\rho(-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{B})\mathrm{d}\boldsymbol{B} \end{aligned} \tag{11.17}$$

Considering the integral with respect to B in (11.17) and using [2.4.3], [2.6.1], [2.4.2] and [2.4.4] gives

$$\begin{aligned} & I_3 \\ =\; & \int_{\boldsymbol{B}>\mathbf{0}}|\boldsymbol{B}|^{\frac{1}{2}(n_1+n_2+m)-\frac{1}{2}(p+1)}\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\right)\,{}_0F_1\left(\tfrac{m}{2};\tfrac{1}{4}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\right)C_\lambda\left(-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\right)C_\rho\left(-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\right)\mathrm{d}\boldsymbol{B} \\ =\; & \sum_{\phi\in\lambda\cdot\rho} g_{\lambda,\rho}^{\phi}\int_{\boldsymbol{B}>\mathbf{0}}|\boldsymbol{B}|^{\frac{1}{2}(n_1+n_2+m)-\frac{1}{2}(p+1)}\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\right)\,{}_0F_1\left(\tfrac{m}{2};\tfrac{1}{4}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\right)C_\phi\left(-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\right)\mathrm{d}\boldsymbol{B} \\ =\; & \sum_{\phi\in\lambda\cdot\rho} g_{\lambda,\rho}^{\phi}\sum_{j=0}^{\infty}\sum_{J}\frac{1}{j!}\frac{1}{\left(\frac{m}{2}\right)_J}\int_{\boldsymbol{B}>\mathbf{0}}|\boldsymbol{B}|^{\frac{1}{2}(n_1+n_2+m)-\frac{1}{2}(p+1)}\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\right)\,C_J\left(\tfrac{1}{4}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\right)C_\phi\left(-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\right)\mathrm{d}\boldsymbol{B} \\ =\; & \sum_{\phi\in\lambda\cdot\rho} g_{\lambda,\rho}^{\phi}\sum_{j=0}^{\infty}\sum_{J}\frac{1}{j!}\frac{1}{\left(\frac{m}{2}\right)_J}\sum_{\phi^*\in J\cdot\phi}\theta_{\phi^*}^{J,\phi}\int_{\boldsymbol{B}>\mathbf{0}}|\boldsymbol{B}|^{\frac{1}{2}(n_1+n_2+m)-\frac{1}{2}(p+1)}\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\right)\,C_{\phi^*}^{J,\phi}\left(\tfrac{1}{4}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{B},-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\right)\mathrm{d}\boldsymbol{B} \\ =\; & \sum_{\phi\in\lambda\cdot\rho} g_{\lambda,\rho}^{\phi}\sum_{j=0}^{\infty}\sum_{J}\frac{1}{j!}\frac{1}{\left(\frac{m}{2}\right)_J}\sum_{\phi^*\in J\cdot\phi}\theta_{\phi^*}^{J,\phi}\Gamma_p\left(\tfrac{n_1+n_2+m}{2},\phi^*\right)\left|\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\right|^{-\frac{1}{2}(n_1+n_2+m)}C_{\phi^*}^{J,\phi}\left(\tfrac{1}{4}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}2\boldsymbol{\Sigma},-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}2\boldsymbol{\Sigma}\right) \\ =\; & \sum_{\phi\in\lambda\cdot\rho} g_{\lambda,\rho}^{\phi}\sum_{j=0}^{\infty}\sum_{J}\frac{1}{j!}\frac{1}{\left(\frac{m}{2}\right)_J}\sum_{\phi^*\in J\cdot\phi}\theta_{\phi^*}^{J,\phi}\Gamma_p\left(\tfrac{n_1+n_2+m}{2},\phi^*\right)\left|\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\right|^{-\frac{1}{2}(n_1+n_2+m)}C_{\phi^*}^{J,\phi}\left(\tfrac{1}{2}\boldsymbol{\Theta},-\boldsymbol{I}_p\right). \end{aligned} \tag{11.18}$$

Substituting (11.18) in (11.17) gives

$$\begin{aligned} &E\left(|\boldsymbol{X}_1|^{h_1}|\boldsymbol{X}_2|^{h_2}\right) \\ =\; & \frac{\left[\Gamma_p\left(\frac{p+1}{2}\right)\right]^2}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)}\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\sum_{k=0}^{\infty}\sum_{\kappa}\sum_{l=0}^{k}\sum_{\lambda}\frac{1}{k!}\binom{\kappa}{\lambda}\frac{C_\kappa(\boldsymbol{I}_p)}{C_\lambda(\boldsymbol{I}_p)}\frac{\Gamma_p\left(\frac{n_1}{2}+\frac{p+1}{2},\kappa\right)}{\Gamma_p\left(\frac{n_1}{2}+\frac{p+1}{2},\lambda\right)}\frac{\Gamma_p\left(\frac{n_1}{2}+h_1,\kappa\right)}{\Gamma_p\left(\frac{n_1}{2}+h_1+\frac{p+1}{2},\kappa\right)} \\ & \cdot\sum_{t=0}^{\infty}\sum_{\tau}\sum_{r=0}^{t}\sum_{\rho}\frac{1}{t!}\binom{\tau}{\rho}\frac{C_\tau(\boldsymbol{I}_p)}{C_\rho(\boldsymbol{I}_p)}\frac{\Gamma_p\left(\frac{n_2}{2}+\frac{p+1}{2},\tau\right)}{\Gamma_p\left(\frac{n_2}{2}+\frac{p+1}{2},\rho\right)}\frac{\Gamma_p\left(\frac{n_2}{2}+h_2,\tau\right)}{\Gamma_p\left(\frac{n_2}{2}+h_2+\frac{p+1}{2},\tau\right)} \\ & \cdot\sum_{\phi\in\lambda\cdot\rho} g_{\lambda,\rho}^{\phi}\sum_{j=0}^{\infty}\sum_{J}\frac{1}{j!}\frac{1}{\Gamma_p\left(\frac{m}{2},J\right)}\sum_{\phi^*\in J\cdot\phi}\theta_{\phi^*}^{J,\phi}\Gamma_p\left(\tfrac{n_1+n_2+m}{2},\phi^*\right)C_{\phi^*}^{J,\phi}\left(\tfrac{1}{2}\boldsymbol{\Theta},-\boldsymbol{I}_p\right). \end{aligned}$$

■

11.3 Distribution of the product of determinants

In Theorem 11.3 an exact expression is derived for the pdf of $\Lambda_4 = |\boldsymbol{X}_1|^{\frac{1}{2}n_1} |\boldsymbol{X}_2|^{\frac{1}{2}n_2}$ where $(\boldsymbol{X}_1, \boldsymbol{X}_2) \sim BB_p^{IV}(n_1, n_2, m; \boldsymbol{\Theta})$ as given by (11.2).

Theorem 11.3

Let $\boldsymbol{S}_1 \sim W_p(n_1, \boldsymbol{\Sigma})$, $\boldsymbol{S}_2 \sim W_p(n_2, \boldsymbol{\Sigma})$ *and* $\boldsymbol{B} \sim W_p(m, \boldsymbol{\Sigma}, \boldsymbol{\Theta})$. *The ratios in* (11.1)

$$\boldsymbol{X}_i = \left(\boldsymbol{I}_p + \boldsymbol{B}^{-\frac{1}{2}} \boldsymbol{S}_i \boldsymbol{B}^{-\frac{1}{2}}\right)^{-\frac{1}{2}} \boldsymbol{B}^{-\frac{1}{2}} \boldsymbol{S}_i \boldsymbol{B}^{-\frac{1}{2}} \left(\boldsymbol{I}_p + \boldsymbol{B}^{-\frac{1}{2}} \boldsymbol{S}_i \boldsymbol{B}^{-\frac{1}{2}}\right)^{-\frac{1}{2}},\ i = 1, 2,$$

give $(\boldsymbol{X}_1, \boldsymbol{X}_2) \sim BB_p^{IV}(n_1, n_2, m; \boldsymbol{\Theta})$. *Let* $\Lambda_4 = |\boldsymbol{X}_1 \boldsymbol{X}_2|$.

The pdf of Λ_4 *is given by*

$$\frac{\left[\Gamma_p\left(\frac{p+1}{2}\right)\right]^2}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)} \operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) \sum_{\kappa,\lambda,\tau,\rho,J;\phi,\phi^*} \frac{1}{k!t!j!} \binom{\kappa}{\lambda}\binom{\tau}{\rho} g^{\phi}_{\lambda,\rho} \theta^{J,\phi}_{\phi^*} \frac{C_\kappa(\boldsymbol{I}_p)}{C_\lambda(\boldsymbol{I}_p)} \frac{C_\tau(\boldsymbol{I}_p)}{C_\rho(\boldsymbol{I}_p)} \tag{11.19}$$

$$\cdot \frac{\Gamma_p\left(\frac{n_1}{2}+\frac{p+1}{2},\kappa\right)}{\Gamma_p\left(\frac{n_1}{2}+\frac{p+1}{2},\lambda\right)} \frac{\Gamma_p\left(\frac{n_2}{2}+\frac{p+1}{2},\tau\right)}{\Gamma_p\left(\frac{n_2}{2}+\frac{p+1}{2},\rho\right)} \frac{\Gamma_p\left(\frac{n_1+n_2+m}{2},\phi^*\right)}{\Gamma_p\left(\frac{m}{2},J\right)} C^{J,\phi}_{\phi^*}\left(\tfrac{1}{2}\boldsymbol{\Theta}, -\boldsymbol{I}_p\right) G^{2p,0}_{2p,2p}\left(\lambda_4 \middle|^{a_1,\ldots,a_{2p}}_{b_1,\ldots,b_{2p}}\right),$$

$0 < \lambda_4 < 1$, *where* $\sum_{\kappa,\lambda,\tau,\rho,J;\phi,\phi^*} = \sum_{k=0}^{\infty}\sum_{\kappa}\sum_{l=0}^{k}\sum_{\lambda}\sum_{t=0}^{\infty}\sum_{\tau}\sum_{r=0}^{t}\sum_{\rho}\sum_{\phi\in\lambda\cdot\rho}\sum_{j=0}^{\infty}\sum_{J}\sum_{\phi^*\in J\cdot\phi}$ *and*

$$a_i = \begin{cases} \frac{n_1}{2} + \frac{p-1}{2} + k_{(i+1)/2} - \frac{1}{4}(i-1) & \text{for } i = 1,3,5,\ldots,2p-1 \\ \frac{n_2}{2} + \frac{p-1}{2} + t_{i/2} - \frac{1}{4}(i-2) & \text{for } i = 2,4,6,\ldots,2p, \end{cases}$$

$$b_i = \begin{cases} \frac{n_1}{2} - 1 + k_{(i+1)/2} - \frac{1}{4}(i-1) & \text{for } i = 1,3,5,\ldots,2p-1 \\ \frac{n_2}{2} - 1 + t_{i/2} - \frac{1}{4}(i-2) & \text{for } i = 2,6,10,\ldots,2p. \end{cases}$$

Proof:

Using (11.13) the Mellin transform (see [2.8.1]) of $f(\lambda_4)$ is

$$\begin{aligned} M_f(h) &\equiv E\left(\Lambda_4^{h-1}\right) \\ &= E\left[\left(|\boldsymbol{X}_1\boldsymbol{X}_2|\right)^{h-1}\right] \\ &= \frac{\left[\Gamma_p\left(\frac{p+1}{2}\right)\right]^2}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)} \operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) \sum_{\kappa,\lambda,\tau,\rho,J;\phi,\phi^*} \frac{1}{k!t!j!} \binom{\kappa}{\lambda}\binom{\tau}{\rho} g^{\phi}_{\lambda,\rho} \theta^{J,\phi}_{\phi^*} \frac{C_\kappa(\boldsymbol{I}_p)}{C_\lambda(\boldsymbol{I}_p)} \frac{C_\tau(\boldsymbol{I}_p)}{C_\rho(\boldsymbol{I}_p)} \\ &\quad \cdot \frac{\Gamma_p\left(\frac{n_1}{2}+\frac{p+1}{2},\kappa\right)}{\Gamma_p\left(\frac{n_1}{2}+\frac{p+1}{2},\lambda\right)} \frac{\Gamma_p\left(\frac{n_2}{2}+\frac{p+1}{2},\tau\right)}{\Gamma_p\left(\frac{n_2}{2}+\frac{p+1}{2},\rho\right)} \frac{\Gamma_p\left(\frac{n_1+n_2+m}{2},\phi^*\right)}{\Gamma_p\left(\frac{m}{2},J\right)} \frac{\Gamma_p\left(\frac{n_1}{2}+h-1,\kappa\right)}{\Gamma_p\left(\frac{n_1}{2}+h+\frac{p-1}{2},\kappa\right)} \frac{\Gamma_p\left(\frac{n_2}{2}+h-1,\tau\right)}{\Gamma_p\left(\frac{n_2}{2}+h+\frac{p-1}{2},\tau\right)} C^{J,\phi}_{\phi^*}\left(\tfrac{1}{2}\boldsymbol{\Theta}, -\boldsymbol{I}_p\right). \end{aligned} \tag{11.20}$$

From [2.3.3] the generalised gamma functions of weights κ and τ in (11.20) can be written as

$$\begin{aligned} &\Gamma_p\left(\tfrac{n_1}{2}+h+\tfrac{p-1}{2},\kappa\right)\Gamma_p\left(\tfrac{n_2}{2}+h+\tfrac{p-1}{2},\tau\right) \\ &= \pi^{\frac{1}{2}p(p-1)} \prod_{i=1}^{2p} \Gamma(a_i+h), \end{aligned} \tag{11.21}$$

$$\text{where} \quad a_i = \begin{cases} \frac{n_1}{2} + \frac{p-1}{2} + k_{(i+1)/2} - \frac{1}{4}(i-1) & \text{for} \quad i = 1,3,5,\ldots,2p-1 \\ \frac{n_2}{2} + \frac{p-1}{2} + t_{i/2} - \frac{1}{4}(i-2) & \text{for} \quad i = 2,4,6,\ldots,2p. \end{cases}$$

Also from [2.3.3] the generalised gamma functions of weights κ and τ in (11.20) can be written as

$$\begin{aligned} &\Gamma_p\left(\tfrac{n_1}{2}+h-1,\kappa\right)\Gamma_p\left(\tfrac{n_2}{2}+h-1,\tau\right) \\ &= \pi^{\frac{1}{2}p(p-1)}\prod_{i=1}^{2p}\Gamma(b_i+h) \\ &\text{where} \quad b_i = \begin{cases} \frac{n_1}{2} - 1 + k_{(i+1)/2} - \frac{1}{4}(i-1) & \text{for} \quad i = 1,3,5,\ldots,2p-1 \\ \frac{n_2}{2} - 1 + t_{i/2} - \frac{1}{4}(i-2) & \text{for} \quad i = 2,6,10,\ldots,2p. \end{cases} \end{aligned} \tag{11.22}$$

Now, substituting (11.21) and (11.22) in (11.20) gives

$$\begin{aligned} M_f(h) \equiv \; & \frac{\left[\Gamma_p\left(\frac{p+1}{2}\right)\right]^2}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)}\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\sum_{\kappa,\lambda,\tau,\rho,J;\phi,\phi^*}\frac{1}{k!t!j!}\binom{\kappa}{\lambda}\binom{\tau}{\rho}g^{\phi}_{\lambda,\rho}\theta^{J,\phi}_{\phi^*}\frac{C_\kappa(\boldsymbol{I}_p)}{C_\lambda(\boldsymbol{I}_p)}\frac{C_\tau(\boldsymbol{I}_p)}{C_\rho(\boldsymbol{I}_p)} \\ & \cdot\frac{\Gamma_p\left(\frac{n_1}{2}+\frac{p+1}{2},\kappa\right)}{\Gamma_p\left(\frac{n_1}{2}+\frac{p+1}{2},\lambda\right)}\frac{\Gamma_p\left(\frac{n_2}{2}+\frac{p+1}{2},\tau\right)}{\Gamma_p\left(\frac{n_2}{2}+\frac{p+1}{2},\rho\right)}\frac{\Gamma_p\left(\frac{n_1+n_2+m}{2},\phi^*\right)}{\Gamma_p\left(\frac{m}{2},J\right)}C^{J,\phi}_{\phi^*}\left(\tfrac{1}{2}\boldsymbol{\Theta},-\boldsymbol{I}_p\right)\frac{\prod_{i=1}^{2p}\Gamma(b_i+h)}{\prod_{i=1}^{2p}\Gamma(a_i+h)}. \end{aligned} \tag{11.23}$$

The pdf of Λ_4 is obtained from the inverse Mellin transform of (11.23) (see [2.8.1]) and is given by

$$\begin{aligned} f(\lambda_4) &= \frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty} M_f(h)\,\lambda_4^{-h}\mathrm{d}h \\ &= \frac{\left[\Gamma_p\left(\frac{p+1}{2}\right)\right]^2}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)}\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\sum_{\kappa,\lambda,\tau,\rho,J;\phi,\phi^*}\frac{1}{k!t!j!}\binom{\kappa}{\lambda}\binom{\tau}{\rho}g^{\phi}_{\lambda,\rho}\theta^{J,\phi}_{\phi^*}\frac{C_\kappa(\boldsymbol{I}_p)}{C_\lambda(\boldsymbol{I}_p)}\frac{C_\tau(\boldsymbol{I}_p)}{C_\rho(\boldsymbol{I}_p)} \\ &\quad \cdot\frac{\Gamma_p\left(\frac{n_1}{2}+\frac{p+1}{2},\kappa\right)}{\Gamma_p\left(\frac{n_1}{2}+\frac{p+1}{2},\lambda\right)}\frac{\Gamma_p\left(\frac{n_2}{2}+\frac{p+1}{2},\tau\right)}{\Gamma_p\left(\frac{n_2}{2}+\frac{p+1}{2},\rho\right)}\frac{\Gamma_p\left(\frac{n_1+n_2+m}{2},\phi^*\right)}{\Gamma_p\left(\frac{m}{2},J\right)}C^{J,\phi}_{\phi^*}\left(\tfrac{1}{2}\boldsymbol{\Theta},-\boldsymbol{I}_p\right)G^{2p,0}_{2p,2p}\left(\lambda_4\middle|^{a_1,\ldots,a_{2p}}_{b_1,\ldots,b_{2p}}\right). \end{aligned}$$

The last step follows from the definition of Meijer's G-function (see [2.8.2]) and gives (11.19). ■

11.4 Role of the parameters

In this section we study the effect of the noncentrality parameter. The effect of the parameters n_1, n_2 and m was studied in Section 6.5.

Firstly, we consider the bivariate case, $p = 1$, to illustrate the effect of the parameter θ on

(i) the form of the pdf of (X_1, X_2);
(ii) the correlation between X_1 and X_2;
(iii) the form of the pdf of Λ_4.

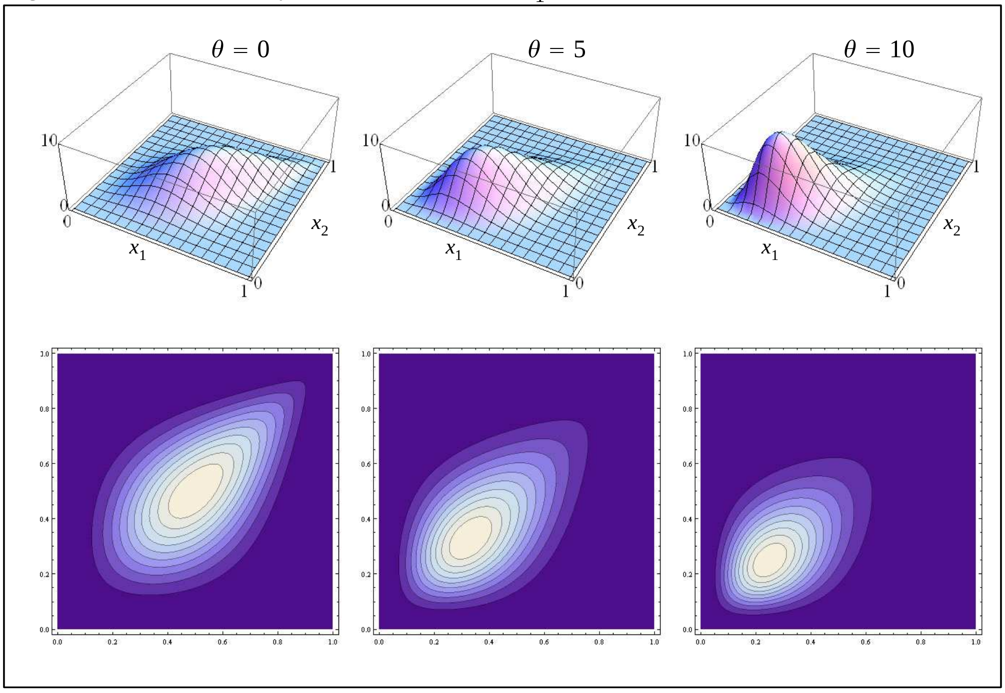
From the result in (11.2) the joint pdf of X_1 and X_2 is

$$\begin{aligned} f(x_1,x_2) &= \frac{\Gamma\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{m}{2}\right)} x_1^{\frac{1}{2}n_1-1} x_2^{\frac{1}{2}n_2-1} \\ &\cdot (1-x_1)^{\frac{1}{2}(n_2+m)-1}(1-x_2)^{\frac{1}{2}(n_1+m)-1}(1-x_1x_2)^{-\frac{1}{2}(n_1+n_2+m)} \\ &\cdot e^{-\frac{1}{2}\theta} \, {}_1F_1\left(\frac{n_1+n_2+m}{2}; \frac{m}{2}; \frac{\theta}{2}\frac{(1-x_1)(1-x_2)}{1-x_1x_2}\right), \quad 0 < x_1, x_2 < 1. \end{aligned} \tag{11.24}$$

The result in (11.24) was also derived by Gupta, Orozco-Castañeda and Nagar (2009).

Figure 11.1 shows graphs of the pdf of the $BB_1^{IV}(8,8,8;\theta)$ distribution (see (11.24)). The joint pdf of X_1 and X_2 shifts towards smaller values of both X_1 and X_2 as θ increases.

Figure 11.1: Effect of θ on $f(x_1,x_2)$, $(X_1,X_2) \sim BB_1^{IV}(8,8,8;\theta)$



Theorem 11.4 derives the $(h_1,h_2)^{th}$ product moment, $E\left(X_1^{h_1}X_2^{h_2}\right)$, associated with (11.24). Independent of this study, the result was also derived by Gupta, Orozco-Castañeda and Nagar (2009).

Theorem 11.4
If $(X_1,X_2) \sim BB_1^{IV}(n_1,n_2,m;\theta)$ *then*

$$\begin{aligned} &E\left(X_1^{h_1}X_2^{h_2}\right) \\ &= \frac{\Gamma\left(\frac{n_1}{2}+h_1\right)\Gamma\left(\frac{n_2}{2}+h_2\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} e^{-\frac{1}{2}\theta}\sum_{k=0}^{\infty}\frac{1}{k!}\frac{\Gamma\left(\frac{n_1+n_2+m}{2}+k\right)}{\Gamma\left(\frac{m}{2}+k\right)}\frac{\Gamma\left(\frac{n_2+m}{2}+k\right)}{\Gamma\left(\frac{n_1+n_2+m}{2}+h_1+k\right)}\frac{\Gamma\left(\frac{n_1+m}{2}+k\right)}{\Gamma\left(\frac{n_1+n_2+m}{2}+h_2+k\right)}\left(\frac{\theta}{2}\right)^k \\ &\cdot {}_3F_2\left(\frac{n_1}{2}+h_1, \frac{n_1+n_2+m}{2}+k, \frac{n_2}{2}+h_2; \frac{n_1+n_2+m}{2}+h_1+k, \frac{n_1+n_2+m}{2}+h_2+k; 1\right). \end{aligned} \tag{11.25}$$

Proof:
From (11.24) and [2.5.1] it follows that

$$
\begin{aligned}
&E\left(X_1^{h_1}X_2^{h_2}\right)\\
=\;& \frac{\Gamma\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{m}{2}\right)} e^{-\frac{1}{2}\theta}\int_0^1\int_0^1 x_1^{\frac{1}{2}n_1+h_1-1}x_2^{\frac{1}{2}n_2+h_2-1}\left(1-x_1\right)^{\frac{1}{2}(n_2+m)-1}\left(1-x_2\right)^{\frac{1}{2}(n_1+m)-1}\\
&\cdot\left(1-x_1x_2\right)^{-\frac{1}{2}(n_1+n_2+m)}\;{}_1F_1\left(\frac{n_1+n_2+m}{2};\frac{m}{2};\frac{\theta}{2}\frac{(1-x_1)(1-x_2)}{1-x_1x_2}\right)\mathrm{d}x_1\mathrm{d}x_2\\
=\;& \frac{\Gamma\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{m}{2}\right)} e^{-\frac{1}{2}\theta}\int_0^1\int_0^1 x_1^{\frac{1}{2}n_1+h_1-1}x_2^{\frac{1}{2}n_2+h_2-1}\left(1-x_1\right)^{\frac{1}{2}(n_2+m)-1}\left(1-x_2\right)^{\frac{1}{2}(n_1+m)-1}\\
&\cdot\left(1-x_1x_2\right)^{-\frac{1}{2}(n_1+n_2+m)}\sum_{k=0}^{\infty}\frac{1}{k!}\frac{\Gamma\left(\frac{n_1+n_2+m}{2}\right)_k}{\Gamma\left(\frac{m}{2}\right)_k}\left(\frac{\theta}{2}\right)^k\frac{(1-x_1)^k(1-x_2)^k}{(1-x_1x_2)^k}\mathrm{d}x_1\mathrm{d}x_2\\
=\;& \frac{\Gamma\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{m}{2}\right)} e^{-\frac{1}{2}\theta}\sum_{k=0}^{\infty}\frac{1}{k!}\frac{\Gamma\left(\frac{n_1+n_2+m}{2}\right)_k}{\Gamma\left(\frac{m}{2}\right)_k}\left(\frac{\theta}{2}\right)^k\\
&\int_0^1\left[\int_0^1 x_1^{\frac{1}{2}n_1+h_1-1}\left(1-x_1\right)^{\frac{1}{2}(n_2+m)+k-1}\left(1-x_1x_2\right)^{-\frac{1}{2}(n_1+n_2+m)-k}\mathrm{d}x_1\right]x_2^{\frac{1}{2}n_2+h_2-1}\left(1-x_2\right)^{\frac{1}{2}(n_1+m)+k-1}\mathrm{d}x_2.
\end{aligned}
\tag{11.26}
$$

Using [2.5.3], the solution of the integral in (11.26) with respect to x_1 is

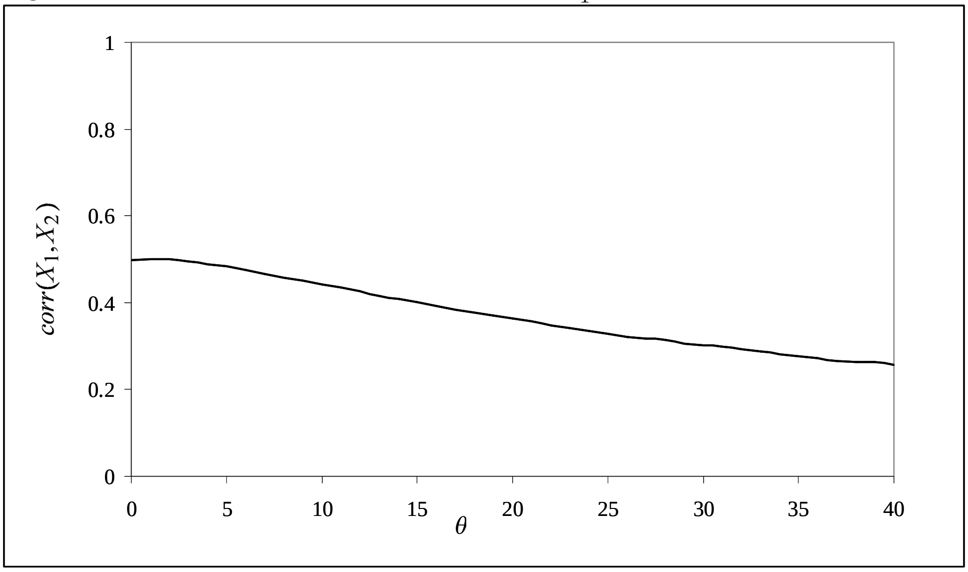
$$
\begin{aligned}
&\int_0^1 x_1^{\frac{1}{2}n_1+h_1-1}\left(1-x_1\right)^{\frac{1}{2}(n_2+m)+k-1}\left(1-x_1x_2\right)^{-\frac{1}{2}(n_1+n_2+m)-k}\mathrm{d}x_1\\
=\;&\frac{\Gamma\left(\frac{n_1}{2}+h_1\right)\Gamma\left(\frac{n_2+m}{2}+k\right)}{\Gamma\left(\frac{n_1+n_2+m}{2}+h_1+k\right)}\;{}_2F_1\left(\frac{n_1}{2}+h_1,\frac{n_1+n_2+m}{2}+k;\frac{n_1+n_2+m}{2}+h_1+k;x_2\right).
\end{aligned}
\tag{11.27}
$$

Substituting (11.27) in (11.26) and solving the integral with respect to x_2 by using [2.5.5] gives

$$
\begin{aligned}
&E\left(X_1^{h_1}X_2^{h_2}\right)\\
=\;& \frac{\Gamma\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{m}{2}\right)} e^{-\frac{1}{2}\theta}\sum_{k=0}^{\infty}\frac{1}{k!}\frac{\Gamma\left(\frac{n_1+n_2+m}{2}\right)_k}{\Gamma\left(\frac{m}{2}\right)_k}\left(\frac{\theta}{2}\right)^k\frac{\Gamma\left(\frac{n_1}{2}+h_1\right)\Gamma\left(\frac{n_2+m}{2}+k\right)}{\Gamma\left(\frac{n_1+n_2+m}{2}+h_1+k\right)}\\
&\int_0^1 x_2^{\frac{1}{2}n_2+h_2-1}\left(1-x_2\right)^{\frac{1}{2}(n_1+m)+k-1}\;{}_2F_1\left(\frac{n_1}{2}+h_1,\frac{n_1+n_2+m}{2}+k;\frac{n_1+n_2+m}{2}+h_1+k;x_2\right)\mathrm{d}x_2\\
=\;& \frac{\Gamma\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{m}{2}\right)} e^{-\frac{1}{2}\theta}\sum_{k=0}^{\infty}\frac{1}{k!}\frac{\Gamma\left(\frac{n_1+n_2+m}{2}+k\right)}{\Gamma\left(\frac{n_1+n_2+m}{2}\right)}\frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m}{2}+k\right)}\left(\frac{\theta}{2}\right)^k\frac{\Gamma\left(\frac{n_1}{2}+h_1\right)\Gamma\left(\frac{n_2+m}{2}+k\right)}{\Gamma\left(\frac{n_1+n_2+m}{2}+h_1+k\right)}\frac{\Gamma\left(\frac{n_2}{2}+h_2\right)\Gamma\left(\frac{n_1+m}{2}+k\right)}{\Gamma\left(\frac{n_1+n_2+m}{2}+h_2+k\right)}\\
&\cdot\;{}_3F_2\left(\frac{n_1}{2}+h_1,\frac{n_1+n_2+m}{2}+k,\frac{n_2}{2}+h_2;\frac{n_1+n_2+m}{2}+h_1+k,\frac{n_1+n_2+m}{2}+h_2+k;1\right)\\
=\;& \frac{1}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} e^{-\frac{1}{2}\theta}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\theta}{2}\right)^k\frac{\Gamma\left(\frac{n_1+n_2+m}{2}+k\right)}{\Gamma\left(\frac{m}{2}+k\right)}\frac{\Gamma\left(\frac{n_1}{2}+h_1\right)\Gamma\left(\frac{n_2+m}{2}+k\right)}{\Gamma\left(\frac{n_1+n_2+m}{2}+h_1+k\right)}\frac{\Gamma\left(\frac{n_2}{2}+h_2\right)\Gamma\left(\frac{n_1+m}{2}+k\right)}{\Gamma\left(\frac{n_1+n_2+m}{2}+h_2+k\right)}\\
&\cdot\;{}_3F_2\left(\frac{n_1}{2}+h_1,\frac{n_1+n_2+m}{2}+k,\frac{n_2}{2}+h_2;\frac{n_1+n_2+m}{2}+h_1+k,\frac{n_1+n_2+m}{2}+h_2+k;1\right). \quad \blacksquare
\end{aligned}
$$

Figure 11.2 shows the graph of the correlation between X_1 and X_2, $corr\left(X_1, X_2\right)$, for increasing values of θ. The result in (11.25) can be used to calculate $corr\left(X_1, X_2\right)$. The correlation is positive and decreases for increasing values of θ.

Figure 11.2: Effect of θ on $corr(X_1, X_2)$, $(X_1, X_2) \sim BB_1^{IV}(8,8,8;\theta)$



In Theorem 11.5 the pdf of $\Lambda_4 = X_1 X_2$ is derived where $\left(X_1, X_2\right) \sim BB_1^{IV}\left(n_1, n_2, m; \theta\right)$. Independent of this study, the result was also derived by Gupta, Orozco-Castañeda and Nagar (2009).

Theorem 11.5

If $\left(X_1, X_2\right) \sim BB_1^{IV}\left(n_1, n_2, m; \theta\right)$ *then the pdf of* $\Lambda_4 = X_1 X_2$ *is*

$$\begin{aligned} f\left(\lambda_4\right) = \quad & \frac{1}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} e^{-\frac{1}{2}\theta} \lambda_4^{\frac{1}{2}n_2 - 1} \sum_{k=0}^{\infty} \frac{1}{k!} \left(1 - \lambda_4\right)^{\frac{1}{2}m + k - 1} \frac{\Gamma\left(\frac{n_2+m}{2}+k\right)\Gamma\left(\frac{n_1+m}{2}+k\right)\Gamma\left(\frac{n_1+n_2+m}{2}+k\right)}{\Gamma\left(\frac{m}{2}+k\right)\Gamma\left(\frac{n_1+n_2+2m}{2}+2k\right)} \left(\frac{\theta}{2}\right)^k \\ & \cdot {}_2F_1\left(\frac{n_2+m}{2} + k, \frac{n_2+m}{2} + k; \frac{n_1+n_2+2m}{2} + 2k; 1 - \lambda_4\right), \quad 0 < \lambda_4 < 1. \end{aligned} \qquad (11.28)$$

Proof:

From (11.25) and [2.5.1] the Mellin transform of $f(\lambda_4)$ (see [2.8.1]) is

$$\begin{aligned} M_f(h) &\equiv E\left(\Lambda_4^{h-1}\right) \\ &= E\left[(X_1X_2)^{h-1}\right] \\ &= e^{-\frac{1}{2}\theta}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\theta}{2}\right)^k \frac{\Gamma\left(\frac{n_1+n_2+m}{2}+k\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{m}{2}+k\right)} \frac{\Gamma\left(\frac{n_1}{2}+h-1\right)\Gamma\left(\frac{n_2+m}{2}+k\right)}{\Gamma\left(\frac{n_1+n_2+m}{2}+h-1+k\right)} \frac{\Gamma\left(\frac{n_2}{2}+h-1\right)\Gamma\left(\frac{n_1+m}{2}+k\right)}{\Gamma\left(\frac{n_1+n_2+m}{2}+h-1+k\right)} \\ &\quad \cdot {}_3F_2\left(\frac{n_1}{2}+h-1, \frac{n_1+n_2+m}{2}+k, \frac{n_2}{2}+h-1; \frac{n_1+n_2+m}{2}+h-1+k, \frac{n_1+n_2+m}{2}+h-1+k; 1\right) \\ &= e^{-\frac{1}{2}\theta}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\theta}{2}\right)^k \frac{\Gamma\left(\frac{n_1+n_2+m}{2}+k\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{m}{2}+k\right)} \frac{\Gamma\left(\frac{n_1}{2}+h-1\right)\Gamma\left(\frac{n_2+m}{2}+k\right)}{\Gamma\left(\frac{n_1+n_2+m}{2}+h-1+k\right)} \frac{\Gamma\left(\frac{n_2}{2}+h-1\right)\Gamma\left(\frac{n_1+m}{2}+k\right)}{\Gamma\left(\frac{n_1+n_2+m}{2}+h-1+k\right)} \\ &\quad \cdot \sum_{t=0}^{\infty}\frac{1}{t!}\frac{\Gamma\left(\frac{n_1}{2}+h-1+t\right)}{\Gamma\left(\frac{n_1}{2}+h-1\right)} \frac{\Gamma\left(\frac{n_1+n_2+m}{2}+k+t\right)}{\Gamma\left(\frac{n_1+n_2+m}{2}+k\right)} \frac{\Gamma\left(\frac{n_2}{2}+h-1+t\right)}{\Gamma\left(\frac{n_2}{2}+h-1\right)} \frac{\Gamma\left(\frac{n_1+n_2+m}{2}+h-1+k\right)}{\Gamma\left(\frac{n_1+n_2+m}{2}+h-1+k+t\right)} \frac{\Gamma\left(\frac{n_1+n_2+m}{2}+h-1+k\right)}{\Gamma\left(\frac{n_1+n_2+m}{2}+h-1+k+t\right)} \\ &= \frac{1}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} e^{-\frac{1}{2}\theta}\sum_{k=0}^{\infty}\sum_{t=0}^{\infty}\frac{1}{k!t!}\frac{\Gamma\left(\frac{n_2+m}{2}+k\right)\Gamma\left(\frac{n_1+m}{2}+k\right)\Gamma\left(\frac{n_1+n_2+m}{2}+k+t\right)}{\Gamma\left(\frac{m}{2}+k\right)} \\ &\quad \cdot \frac{\Gamma\left(\frac{n_1}{2}+h-1+t\right)\Gamma\left(\frac{n_2}{2}+h-1+t\right)}{\Gamma\left(\frac{n_1+n_2+m}{2}+h-1+k+t\right)\Gamma\left(\frac{n_1+n_2+m}{2}+h-1+k+t\right)}\left(\frac{\theta}{2}\right)^k . \end{aligned} \tag{11.29}$$

The gamma functions in (11.29) can be written as

$$\begin{aligned} &\Gamma\left(\frac{n_1+n_2+m}{2}+h-1+k+t\right)\Gamma\left(\frac{n_1+n_2+m}{2}+h-1+k+t\right) \\ &= \prod_{j=1}^{2}\Gamma\left(a_j+h\right), \end{aligned} \tag{11.30}$$

where $a_1 = a_2 = \frac{n_1+n_2+m}{2}+k+t-1$,

and

$$\begin{aligned} &\Gamma\left(\frac{n_1}{2}+h-1+t\right)\Gamma\left(\frac{n_2}{2}+h-1+t\right) \\ &= \prod_{j=1}^{2}\Gamma\left(b_j+h\right), \end{aligned} \tag{11.31}$$

where $b_j = \frac{n_j}{2}+t-1,\ j=1,2.$

Now, substituting (11.30) and (11.31) in (11.29) gives

$$M_f(h) \equiv \frac{1}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} e^{-\frac{1}{2}\theta}\sum_{k=0}^{\infty}\sum_{t=0}^{\infty}\frac{1}{k!t!}\frac{\Gamma\left(\frac{n_2+m}{2}+k\right)\Gamma\left(\frac{n_1+m}{2}+k\right)\Gamma\left(\frac{n_1+n_2+m}{2}+k+t\right)}{\Gamma\left(\frac{m}{2}+k\right)}\left(\frac{\theta}{2}\right)^k \frac{\prod_{j=1}^{2}\Gamma(b_j+h)}{\prod_{j=1}^{2}\Gamma(a_j+h)}. \tag{11.32}$$

The pdf of Λ_4 is obtained from the inverse Mellin transform of (11.32) (see [2.8.1]) and is given by

$$f(\lambda_4) = \frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty} M_f(h)\,\lambda_4^{-h}\mathrm{d}h.$$

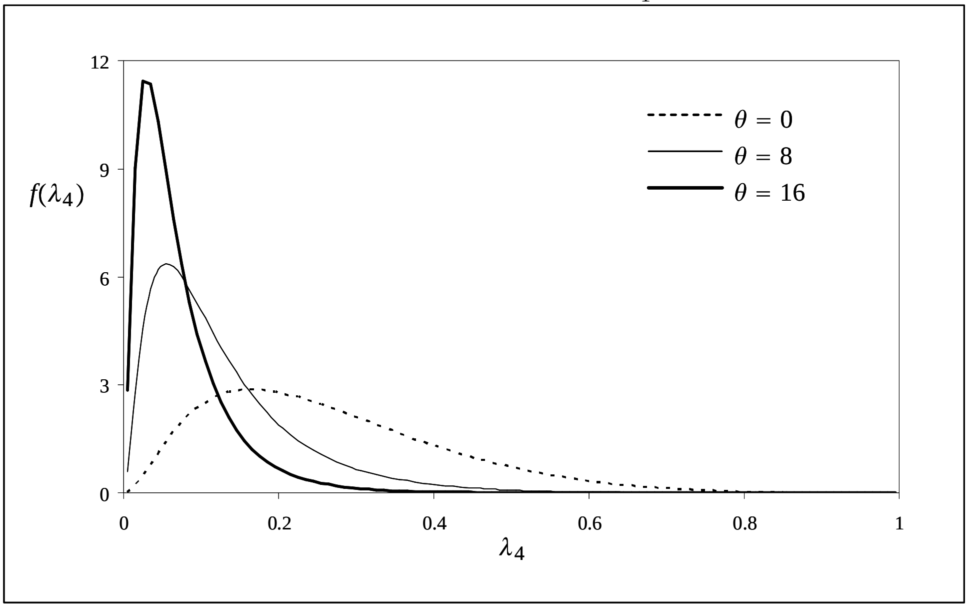
From the definition of the G-function (see [2.8.2]) and the results in [2.8.4], [2.8.5] and [2.5.2], it follows that

$$\begin{aligned}
f(\lambda_4) &= e^{-\frac{1}{2}\theta}\sum_{k=0}^{\infty}\sum_{t=0}^{\infty}\frac{1}{k!t!}\left(\frac{\theta}{2}\right)^k\frac{\Gamma\left(\frac{n_2+m}{2}+k\right)\Gamma\left(\frac{n_1+m}{2}+k\right)\Gamma\left(\frac{n_1+n_2+m}{2}+k+t\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{m}{2}+k\right)}G_{2,2}^{2,0}\left(\lambda_4|_{b_1,b_2}^{a_1,a_2}\right) \\
&= e^{-\frac{1}{2}\theta}\sum_{k=0}^{\infty}\sum_{t=0}^{\infty}\frac{1}{k!t!}\left(\frac{\theta}{2}\right)^k\frac{\Gamma\left(\frac{n_2+m}{2}+k\right)\Gamma\left(\frac{n_1+m}{2}+k\right)\Gamma\left(\frac{n_1+n_2+m}{2}+k+t\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{m}{2}+k\right)} \\
&\quad \cdot\frac{\lambda_4^{\frac{1}{2}n_2+t-1}(1-\lambda_4)^{\frac{1}{2}(n_1+n_2+2m)+2k-1}}{\Gamma\left(\frac{n_1+n_2+2m}{2}+2k\right)}\,{}_2F_1\left(\frac{n_2+m}{2}+k,\frac{n_2+m}{2}+k;\frac{n_1+n_2+2m}{2}+2k;1-\lambda_4\right) \\
&= e^{-\frac{1}{2}\theta}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\theta}{2}\right)^k\frac{\Gamma\left(\frac{n_2+m}{2}+k\right)\Gamma\left(\frac{n_1+m}{2}+k\right)\Gamma\left(\frac{n_1+n_2+m}{2}+k\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{m}{2}+k\right)}\frac{\lambda_4^{\frac{1}{2}n_2-1}(1-\lambda_4)^{\frac{1}{2}(n_1+n_2+2m)+2k-1}}{\Gamma\left(\frac{n_1+n_2+2m}{2}+2k\right)} \\
&\quad \cdot{}_2F_1\left(\frac{n_2+m}{2}+k,\frac{n_2+m}{2}+k;\frac{n_1+n_2+2m}{2}+2k;1-\lambda_4\right)\sum_{t=0}^{\infty}\frac{\lambda_4^t}{t!}\frac{\Gamma\left(\frac{n_1+n_2+m}{2}+k+t\right)}{\Gamma\left(\frac{n_1+n_2+m}{2}+k\right)} \\
&= e^{-\frac{1}{2}\theta}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\theta}{2}\right)^k\frac{\Gamma\left(\frac{n_2+m}{2}+k\right)\Gamma\left(\frac{n_1+m}{2}+k\right)\Gamma\left(\frac{n_1+n_2+m}{2}+k\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{m}{2}+k\right)}\frac{\lambda_4^{\frac{1}{2}n_2-1}(1-\lambda_4)^{\frac{1}{2}(n_1+n_2+2m)+2k-1}}{\Gamma\left(\frac{n_1+n_2+2m}{2}+2k\right)} \\
&\quad \cdot{}_2F_1\left(\frac{n_2+m}{2}+k,\frac{n_2+m}{2}+k;\frac{n_1+n_2+2m}{2}+2k;1-\lambda_4\right)\,{}_1F_0\left(\frac{n_1+n_2+m}{2}+k;\lambda_4\right) \\
&= \frac{1}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)}e^{-\frac{1}{2}\theta}\lambda_4^{\frac{1}{2}n_2-1}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\theta}{2}\right)^k\frac{\Gamma\left(\frac{n_2+m}{2}+k\right)\Gamma\left(\frac{n_1+m}{2}+k\right)\Gamma\left(\frac{n_1+n_2+m}{2}+k\right)}{\Gamma\left(\frac{m}{2}+k\right)}\frac{(1-\lambda_4)^{\frac{1}{2}(n_1+n_2+2m)+2k-1}}{\Gamma\left(\frac{n_1+n_2+2m}{2}+2k\right)} \\
&\quad \cdot{}_2F_1\left(\frac{n_2+m}{2}+k,\frac{n_2+m}{2}+k;\frac{n_1+n_2+2m}{2}+2k;1-\lambda_4\right)(1-\lambda_4)^{-\frac{1}{2}(n_1+n_2+m)-k}.
\end{aligned}$$

This is the result in (11.28). ■

Figure 11.3 shows the effect of θ on $f(\lambda_4)$ given by (11.28) where $(X_1, X_2) \sim BB_1^{IV}(8,8,8;\theta)$. As θ increases the pdf $f(\lambda_4)$ shifts towards smaller values of Λ_4.

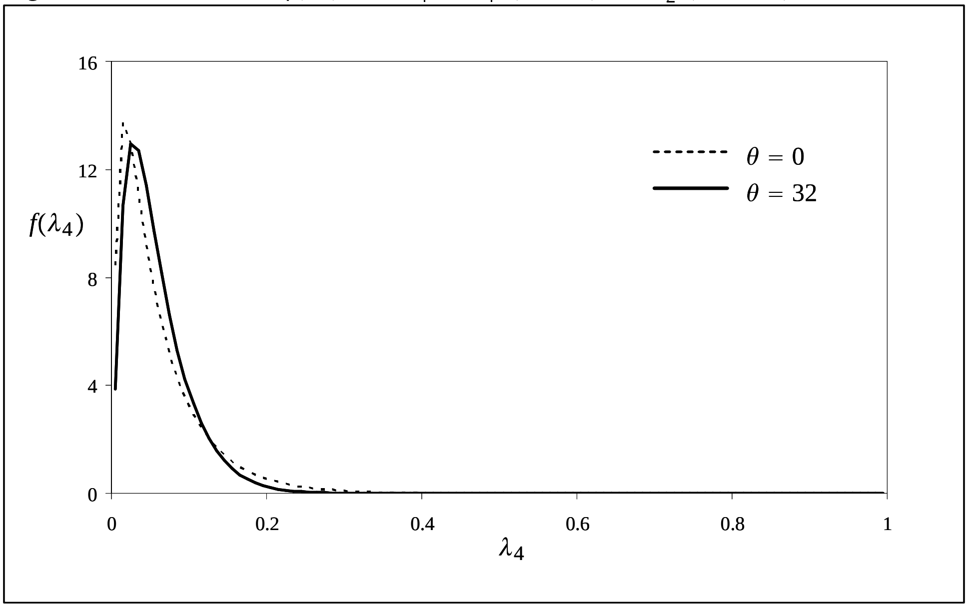
Figure 11.3: Effect of θ on $f(\lambda_4)$, $\Lambda_4 = X_1X_2$, $(X_1,X_2) \sim BB_1^{IV}(8,8,8;\theta)$



Secondly, we consider the bimatrix case, $p = 2$, to illustrate the effect of the noncentrality parameter $\boldsymbol{\Theta}$ on the pdf of Λ_4 given by (11.19). The case is considered where $(\boldsymbol{X}_1, \boldsymbol{X}_2) \sim BB_2^{IV}(8, 8, 8; \boldsymbol{\Theta})$, $\boldsymbol{\Theta} = \theta \boldsymbol{I}_2$.

Figure 11.4 illustrates the shape of the pdf, $f(\lambda_4)$, for increasing values of θ. We note that as θ increases the pdf shifts towards larger values of Λ_4.

Figure 11.4: Effect of $\boldsymbol{\Theta}$ on $f(\lambda_4)$, $\Lambda_4 = |\boldsymbol{X}_1 \boldsymbol{X}_2|$, $(\boldsymbol{X}_1, \boldsymbol{X}_2) \sim BB_2^{IV}(8, 8, 8; \boldsymbol{\Theta})$, $\boldsymbol{\Theta} = \theta \boldsymbol{I}_2$



12 Noncentral bimatrix variate beta type V distribution

In this section an exact expression for the pdf of the noncentral bimatrix variate beta type V distribution with $(\boldsymbol{Q}_1, \boldsymbol{Q}_2)$ defined as in (7.10) is derived as well as the corresponding product moment of the determinants and the pdf of $\Lambda_5 = |\boldsymbol{Q}_1|^{h_1} |\boldsymbol{Q}_2|^{h_2}$ (see (1.9).

12.1 Probability density function

In Theorem 12.1 an exact expression for the pdf of the noncentral bimatrix variate beta type V distribution is derived.

Theorem 12.1
Let $\boldsymbol{S}_1 \sim W_p(n_1, \boldsymbol{\Sigma})$, $\boldsymbol{S}_2 \sim W_p(n_2, \boldsymbol{\Sigma})$ *and* $\boldsymbol{B} \sim W_p(m, \boldsymbol{\Sigma}, \boldsymbol{\Theta})$ *be independently distributed. Consider the ratios given by* (7.10),

$$\boldsymbol{Q}_i = \left(c\boldsymbol{I}_p + \sum_{i=1}^{2} \alpha_i \boldsymbol{B}^{-\frac{1}{2}} \boldsymbol{S}_i \boldsymbol{B}^{-\frac{1}{2}}\right)^{-\frac{1}{2}} \alpha_i \boldsymbol{B}^{-\frac{1}{2}} \boldsymbol{S}_i \boldsymbol{B}^{-\frac{1}{2}} \left(c\boldsymbol{I}_p + \sum_{i=1}^{2} \alpha_i \boldsymbol{B}^{-\frac{1}{2}} \boldsymbol{S}_i \boldsymbol{B}^{-\frac{1}{2}}\right)^{-\frac{1}{2}}, \quad i = 1, 2. \tag{12.1}$$

The pdf of $(\boldsymbol{Q}_1, \boldsymbol{Q}_2) \sim BB_p^V(n_1, n_2, m, \alpha_1, \alpha_2, c; \boldsymbol{\Theta})$ *is*

$$\begin{aligned}
& f(\boldsymbol{Q}_1, \boldsymbol{Q}_2) \\
= \quad & \left\{\beta_p\left(\tfrac{n_1}{2}, \tfrac{n_2}{2}; \tfrac{m}{2}\right)\right\}^{-1} \prod_{i=1}^{2} |\boldsymbol{Q}_i|^{\frac{1}{2}n_i - \frac{1}{2}(p+1)} \left|\boldsymbol{I}_p - \sum_{i=1}^{2} \boldsymbol{Q}_i\right|^{\frac{1}{2}m - \frac{1}{2}(p+1)} \\
& \cdot \prod_{i=1}^{2} \left(\tfrac{c}{\alpha_i}\right)^{\frac{1}{2}n_i p} \left|\boldsymbol{I}_p + \sum_{i=1}^{2} \tfrac{c-\alpha_i}{\alpha_i} \boldsymbol{Q}_i\right|^{-\frac{1}{2}(n_1+n_2+m)} \\
& \cdot \operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) {}_1F_1\left(\tfrac{n_1+n_2+m}{2}; \tfrac{m}{2}; \tfrac{1}{2}\left(\boldsymbol{I}_p - \sum_{i=1}^{2} \boldsymbol{Q}_i\right)^{\frac{1}{2}} \left(\boldsymbol{I}_p + \sum_{i=1}^{2} \tfrac{c-\alpha_i}{\alpha_i} \boldsymbol{Q}_i\right)^{-1} \left(\boldsymbol{I}_p - \sum_{i=1}^{2} \boldsymbol{Q}_i\right)^{\frac{1}{2}} \boldsymbol{\Theta}\right) \\
= \quad & g(\boldsymbol{Q}_1, \boldsymbol{Q}_2) \\
& \cdot \operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) {}_1F_1\left(\tfrac{n_1+n_2+m}{2}; \tfrac{m}{2}; \tfrac{1}{2}\left(\boldsymbol{I}_p - \sum_{i=1}^{2} \boldsymbol{Q}_i\right)^{\frac{1}{2}} \left(\boldsymbol{I}_p + \sum_{i=1}^{2} \tfrac{c-\alpha_i}{\alpha_i} \boldsymbol{Q}_i\right)^{-1} \left(\boldsymbol{I}_p - \sum_{i=1}^{2} \boldsymbol{Q}_i\right)^{\frac{1}{2}} \boldsymbol{\Theta}\right),
\end{aligned} \tag{12.2}$$

$\boldsymbol{0} < \boldsymbol{Q}_i < \boldsymbol{I}_p$, $i = 1, 2$, $\boldsymbol{0} < \sum_{i=1}^{2} \boldsymbol{Q}_i < \boldsymbol{I}_p$, *where* $n_i > (p-1)$, $i = 1, 2$, $m > (p-1)$, *and* $g(\cdot)$ *is the pdf of* $BB_p^V(n_1, n_2, m, \alpha_1, \alpha_2, c)$ *given by* (7.2).

Proof:
The pdf of $(\boldsymbol{S}_1, \boldsymbol{S}_2, \boldsymbol{B})$ is given by

$$K \prod_{i=1}^{2} \left[\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}_i\right) |\boldsymbol{S}_i|^{\frac{1}{2}(n_i-p-1)}\right] \left[\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) \operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\right) |\boldsymbol{B}|^{\frac{1}{2}(m-p-1)} \; {}_0F_1\left(\tfrac{m}{2}; \tfrac{1}{4}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\right)\right]$$

where $K^{-1} = \Gamma_p\left(\frac{n_1}{2}\right) \Gamma_p\left(\frac{n_2}{2}\right) \Gamma_p\left(\frac{m}{2}\right) |2\boldsymbol{\Sigma}|^{\frac{1}{2}(n_1+n_2+m)}$ (see [2.10.2]).

It was shown in the first part of Theorem 9.1 that for the transformations $\boldsymbol{V}_i = \boldsymbol{B}^{-\frac{1}{2}} \boldsymbol{S}_i \boldsymbol{B}^{-\frac{1}{2}}$, $i = 1, 2$, the pdf of $(\boldsymbol{V}_1, \boldsymbol{V}_2)$ is given by (9.9), that is

$$\begin{aligned} f(\boldsymbol{V}_1, \boldsymbol{V}_2) = \quad & \left\{\beta_p\left(\tfrac{n_1}{2}, \tfrac{n_2}{2}; \tfrac{m}{2}\right)\right\}^{-1} \operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) \prod_{i=1}^{2} |\boldsymbol{V}_i|^{\frac{1}{2}n_i - \frac{1}{2}(p+1)} \left|\boldsymbol{I}_p + \sum_{i=1}^{2} \boldsymbol{V}_i\right|^{-\frac{1}{2}(n_1+n_2+m)} \\ & \cdot {}_1F_1\left(\tfrac{n_1+n_2+m}{2}; \tfrac{m}{2}; \tfrac{1}{2}\left(\boldsymbol{I}_p + \sum_{i=1}^{2} \boldsymbol{V}_i\right)^{-1} \boldsymbol{\Theta}\right), \quad \boldsymbol{V}_i > \boldsymbol{0},\ i = 1, 2. \end{aligned} \tag{12.3}$$

Consider the transformations in (12.1) written in terms of $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$, that is

$$\begin{aligned} \boldsymbol{Q}_i &= \left(c\boldsymbol{I}_p + \sum_{i=1}^{2} \alpha_i \boldsymbol{B}^{-\frac{1}{2}} \boldsymbol{S}_i \boldsymbol{B}^{-\frac{1}{2}}\right)^{-\frac{1}{2}} \alpha_i \boldsymbol{B}^{-\frac{1}{2}} \boldsymbol{S}_i \boldsymbol{B}^{-\frac{1}{2}} \left(c\boldsymbol{I}_p + \sum_{i=1}^{2} \alpha_i \boldsymbol{B}^{-\frac{1}{2}} \boldsymbol{S}_i \boldsymbol{B}^{-\frac{1}{2}}\right)^{-\frac{1}{2}} \\ &= \left(\boldsymbol{I}_p + \sum_{i=1}^{2} \tfrac{\alpha_i}{c} \boldsymbol{V}_i\right)^{-\frac{1}{2}} \tfrac{\alpha_i}{c} \boldsymbol{V}_i \left(\boldsymbol{I}_p + \sum_{i=1}^{2} \tfrac{\alpha_i}{c} \boldsymbol{V}_i\right)^{-\frac{1}{2}}, \quad i = 1, 2. \end{aligned} \tag{12.4}$$

Let $\boldsymbol{Y}_i = \frac{\alpha_i}{c} \boldsymbol{V}_i$, $i = 1, 2$, $\boldsymbol{Z} = \boldsymbol{I}_p + \sum_{i=1}^{2} \boldsymbol{Y}_i$ and $\boldsymbol{Q}_1 = \boldsymbol{Z}^{-\frac{1}{2}} \boldsymbol{Y}_1 \boldsymbol{Z}^{-\frac{1}{2}}$. Then $\boldsymbol{Q}_2 = \boldsymbol{I}_p - \boldsymbol{Z}^{-1} - \boldsymbol{Q}_1$ and $\boldsymbol{Z} = \left(\boldsymbol{I}_p - \sum_{i=1}^{2} \boldsymbol{Q}_i\right)^{-1}$. From [2.1.2], [2.1.3], [2.1.4], [2.1.5], [2.1.7] and [2.1.8] the Jacobian of the transformations in (12.4) is

$$\begin{aligned} & J(\boldsymbol{V}_1, \boldsymbol{V}_2 \to \boldsymbol{Q}_1, \boldsymbol{Q}_2) \\ &= J(\boldsymbol{V}_1, \boldsymbol{V}_2 \to \boldsymbol{Y}_1, \boldsymbol{Y}_2) \cdot J(\boldsymbol{Y}_1, \boldsymbol{Y}_2 \to \boldsymbol{Q}_1, \boldsymbol{Z}) \cdot J(\boldsymbol{Q}_1, \boldsymbol{Z} \to \boldsymbol{Q}_1, \boldsymbol{Q}_2) \\ &= c^{p(p+1)} \prod_{i=1}^{2} \alpha_i^{-\frac{1}{2}p(p+1)} |\boldsymbol{Z}|^{\frac{1}{2}(p+1)} |\boldsymbol{Z}|^{p+1} \\ &= c^{p(p+1)} \prod_{i=1}^{2} \alpha_i^{-\frac{1}{2}p(p+1)} \left|\boldsymbol{I}_p - \sum_{i=1}^{2} \boldsymbol{Q}_i\right|^{-\frac{3}{2}(p+1)}. \end{aligned} \tag{12.5}$$

Rewriting (12.4) as

$$\boldsymbol{V}_i = \tfrac{c}{\alpha_i}\left(\boldsymbol{I}_p - \sum_{i=1}^{2} \boldsymbol{Q}_i\right)^{-\frac{1}{2}} \boldsymbol{Q}_i \left(\boldsymbol{I}_p - \sum_{i=1}^{2} \boldsymbol{Q}_i\right)^{-\frac{1}{2}}, \quad i = 1, 2, \tag{12.6}$$

and substituting (12.5) and (12.6) in (12.3) gives

$$
\begin{aligned}
& f\left(\boldsymbol{Q}_1,\boldsymbol{Q}_2\right) \\
= \; & \left\{\beta_p\left(\tfrac{n_1}{2},\tfrac{n_2}{2};\tfrac{m}{2}\right)\right\}^{-1}\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)c^{p(p+1)}\prod_{i=1}^{2}\alpha_i^{-\frac{1}{2}p(p+1)}\left|\boldsymbol{I}_p-\sum_{i=1}^{2}\boldsymbol{Q}_i\right|^{-\frac{3}{2}(p+1)} \\
& \cdot\prod_{i=1}^{2}\left|\tfrac{c}{\alpha_i}\left(\boldsymbol{I}_p-\sum_{i=1}^{2}\boldsymbol{Q}_i\right)^{-\frac{1}{2}}\boldsymbol{Q}_i\left(\boldsymbol{I}_p-\sum_{i=1}^{2}\boldsymbol{Q}_i\right)^{-\frac{1}{2}}\right|^{\frac{1}{2}n_i-\frac{1}{2}(p+1)} \\
& \cdot\left|\boldsymbol{I}_p+\sum_{i=1}^{2}\tfrac{c}{\alpha_i}\left(\boldsymbol{I}_p-\sum_{i=1}^{2}\boldsymbol{Q}_i\right)^{-\frac{1}{2}}\boldsymbol{Q}_i\left(\boldsymbol{I}_p-\sum_{i=1}^{2}\boldsymbol{Q}_i\right)^{-\frac{1}{2}}\right|^{-\frac{1}{2}(n_1+n_2+m)} \\
& \cdot{}_1F_1\left(\tfrac{n_1+n_2+m}{2};\tfrac{m}{2};\tfrac{1}{2}\left[\boldsymbol{I}_p+\sum_{i=1}^{2}\tfrac{c}{\alpha_i}\left(\boldsymbol{I}_p-\sum_{i=1}^{2}\boldsymbol{Q}_i\right)^{-\frac{1}{2}}\boldsymbol{Q}_i\left(\boldsymbol{I}_p-\sum_{i=1}^{2}\boldsymbol{Q}_i\right)^{-\frac{1}{2}}\right]^{-1}\boldsymbol{\Theta}\right) \\
= \; & \left\{\beta_p\left(\tfrac{n_1}{2},\tfrac{n_2}{2};\tfrac{m}{2}\right)\right\}^{-1}\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\prod_{i=1}^{2}\alpha_i^{-\frac{1}{2}n_ip}c^{\frac{1}{2}(n_1+n_2)p}\prod_{i=1}^{2}\left|\boldsymbol{Q}_i\right|^{\frac{1}{2}n_i-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\sum_{i=1}^{2}\boldsymbol{Q}_i\right|^{-\frac{1}{2}(n_1+n_2)-\frac{1}{2}(p+1)} \\
& \cdot\left|\boldsymbol{I}_p-\sum_{i=1}^{2}\boldsymbol{Q}_i\right|^{\frac{1}{2}(n_1+n_2+m)}\left|\boldsymbol{I}_p-\sum_{i=1}^{2}\boldsymbol{Q}_i+\sum_{i=1}^{2}\tfrac{c}{\alpha_i}\boldsymbol{Q}_i\right|^{-\frac{1}{2}(n_1+n_2+m)} \\
& \cdot{}_1F_1\left(\tfrac{n_1+n_2+m}{2};\tfrac{m}{2};\tfrac{1}{2}\left(\boldsymbol{I}_p-\sum_{i=1}^{2}\boldsymbol{Q}_i\right)^{\frac{1}{2}}\left(\boldsymbol{I}_p-\sum_{i=1}^{2}\boldsymbol{Q}_i+\sum_{i=1}^{2}\tfrac{c}{\alpha_i}\boldsymbol{Q}_i\right)^{-1}\left(\boldsymbol{I}_p-\sum_{i=1}^{2}\boldsymbol{Q}_i\right)^{\frac{1}{2}}\boldsymbol{\Theta}\right) \\
= \; & \left\{\beta_p\left(\tfrac{n_1}{2},\tfrac{n_2}{2};\tfrac{m}{2}\right)\right\}^{-1}\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\prod_{i=1}^{2}\left(\tfrac{c}{\alpha_i}\right)^{\frac{1}{2}n_ip}\prod_{i=1}^{2}\left|\boldsymbol{Q}_i\right|^{\frac{1}{2}n_i-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p+\sum_{i=1}^{2}\tfrac{c-\alpha_i}{\alpha_i}\boldsymbol{Q}_i\right|^{-\frac{1}{2}(n_1+n_2+m)} \\
& \cdot\left|\boldsymbol{I}_p-\sum_{i=1}^{2}\boldsymbol{Q}_i\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)}{}_1F_1\left(\tfrac{n_1+n_2+m}{2};\tfrac{m}{2};\tfrac{1}{2}\left(\boldsymbol{I}_p-\sum_{i=1}^{2}\boldsymbol{Q}_i\right)^{\frac{1}{2}}\left(\boldsymbol{I}_p+\sum_{i=1}^{2}\tfrac{c-\alpha_i}{\alpha_i}\boldsymbol{Q}_i\right)^{-1}\left(\boldsymbol{I}_p-\sum_{i=1}^{2}\boldsymbol{Q}_i\right)^{\frac{1}{2}}\boldsymbol{\Theta}\right). \quad \blacksquare
\end{aligned}
$$

Remark 12.1
Ehlers, Bekker and Roux (2010) derived the result in (12.2) for the bivariate case, that is where $p = 1$. They also derived some properties of the noncentral bivariate beta type V distribution.

Remark 12.2
Let $\boldsymbol{S} \sim W_p(n, \boldsymbol{\Sigma})$ independent of $\boldsymbol{B} \sim W_p(m, \boldsymbol{\Sigma}; \boldsymbol{\Theta})$. Define

$$
\boldsymbol{Q}=\left(c\boldsymbol{I}_p+\alpha\boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{S}\boldsymbol{B}^{-\frac{1}{2}}\right)^{-\frac{1}{2}}\alpha\boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{S}\boldsymbol{B}^{-\frac{1}{2}}\left(c\boldsymbol{I}_p+\alpha\boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{S}\boldsymbol{B}^{-\frac{1}{2}}\right)^{-\frac{1}{2}}.
$$

The pdf of $\boldsymbol{Q}$ is

$$
\begin{aligned}
f(\boldsymbol{Q}) &= \left\{\beta_p\left(\tfrac{n}{2},\tfrac{m}{2}\right)\right\}^{-1}\left(\tfrac{c}{\alpha}\right)^{\frac{1}{2}np}\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\left|\boldsymbol{Q}\right|^{\frac{1}{2}n-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\boldsymbol{Q}\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p+\tfrac{c-\alpha}{\alpha}\boldsymbol{Q}\right|^{-\frac{1}{2}(n+m)} \\
&\quad\cdot{}_1F_1\left(\tfrac{n+m}{2};\tfrac{m}{2};\tfrac{1}{2}\left(\boldsymbol{I}_p-\boldsymbol{Q}\right)^{\frac{1}{2}}\left(\boldsymbol{I}_p+\tfrac{c-\alpha}{\alpha}\boldsymbol{Q}\right)^{-1}\left(\boldsymbol{I}_p-\boldsymbol{Q}\right)^{\frac{1}{2}}\boldsymbol{\Theta}\right) \\
&= g(\boldsymbol{Q})\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\;{}_1F_1\left(\tfrac{n+m}{2};\tfrac{m}{2};\tfrac{1}{2}\left(\boldsymbol{I}_p-\boldsymbol{Q}\right)^{\frac{1}{2}}\left(\boldsymbol{I}_p+\tfrac{c-\alpha}{\alpha}\boldsymbol{Q}\right)^{-1}\left(\boldsymbol{I}_p-\boldsymbol{Q}\right)^{\frac{1}{2}}\boldsymbol{\Theta}\right),
\end{aligned}
$$

$\boldsymbol{0} < \boldsymbol{Q} < \boldsymbol{I}_p$, where $n > (p-1)$, $m > (p-1)$ and where $g(\cdot)$ is the pdf of $B_p^V(n, m, \alpha, c)$ given by (3.5). This is the noncentral matrix variate beta type V distribution and is denoted by $\boldsymbol{Q} \sim B_p^V(n, m, \alpha, c; \boldsymbol{\Theta})$.

Remark 12.3

The noncentral matrix variate Dirichlet type V distribution denoted by $(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_r) \sim D_p^V(n_1,\ldots,n_r,m,\alpha_1,\ldots\alpha_r,c;\boldsymbol{\Theta})$, results by extending (12.1) to r independent Wishart matrix variates, $\boldsymbol{S}_i \sim W_p(n_i,\boldsymbol{\Sigma})$, $i=1,\ldots,r$, all independent of $\boldsymbol{B} \sim W_p(m,\boldsymbol{\Sigma};\boldsymbol{\Theta})$. The pdf of $(\boldsymbol{Q}_1,\ldots,\boldsymbol{Q}_r)$ is given by

$$\begin{aligned}&\left\{\beta_p\left(\tfrac{n_1}{2},\ldots,\tfrac{n_r}{2};\tfrac{m}{2}\right)\right\}^{-1}\left(\tfrac{c}{\alpha}\right)^{\frac{1}{2}(n_1+\cdots+n_r)p}\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\\ &\cdot\prod_{i=1}^{r}|\boldsymbol{Q}_i|^{\frac{1}{2}n_i-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\sum_{i=1}^{r}\boldsymbol{Q}_i\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p+\sum_{i=1}^{r}\tfrac{c-\alpha_i}{\alpha_i}\boldsymbol{Q}_i\right|^{-\frac{1}{2}(n_1+\cdots+n_r+m)}\\ &\cdot{}_1F_1\left(\tfrac{n_1+\cdots+n_r+m}{2};\tfrac{m}{2};\tfrac{1}{2}\left(\boldsymbol{I}_p-\sum_{i=1}^{r}\boldsymbol{Q}_i\right)^{\frac{1}{2}}\left(\boldsymbol{I}_p+\sum_{i=1}^{r}\tfrac{c-\alpha_i}{\alpha_i}\boldsymbol{Q}_i\right)^{-1}\left(\boldsymbol{I}_p-\sum_{i=1}^{r}\boldsymbol{Q}_i\right)^{\frac{1}{2}}\boldsymbol{\Theta}\right),\end{aligned}\tag{12.7}$$

$\boldsymbol{0}<\boldsymbol{Q}_i<\boldsymbol{I}_p$, $i=1,\ldots,r$, $\boldsymbol{0}<\sum_{i=1}^{r}\boldsymbol{Q}_i<\boldsymbol{I}_p$.

Remark 12.4

If $\alpha_1=\cdots=\alpha_r=\alpha$ then the pdfs of the noncentral bimatrix variate beta type V distribution in (12.2) and the noncentral matrix variate Dirichlet type V distribution in (12.7) are members of the Liouville family of distributions of the second kind (see [2.2.1]).

12.2 Product moment of the determinants

The $(h_1,h_2)^{th}$ product moment, $E\left(|\boldsymbol{Q}_1|^{h_1}|\boldsymbol{Q}_2|^{h_2}\right)$, where $(\boldsymbol{Q}_1,\boldsymbol{Q}_2) \sim BB_p^V(n_1,n_2,m,\alpha_1,\alpha_2,c;\boldsymbol{\Theta})$ is derived in Theorem 12.2.

Theorem 12.2

If $(\boldsymbol{Q}_1,\boldsymbol{Q}_2) \sim BB_p^V(n_1,n_2,m,\alpha_1,\alpha_2,c;\boldsymbol{\Theta})$ as given by (12.2) then for $\alpha_1=\alpha_2=\alpha$, $E\left(|\boldsymbol{Q}_1|^{h_1}|\boldsymbol{Q}_2|^{h_2}\right)$ is given by

$$\frac{\Gamma_p\left(\frac{n_1}{2}+h_1\right)\Gamma_p\left(\frac{n_2}{2}+h_2\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)}\left(\tfrac{c}{\alpha}\right)^{-\frac{1}{2}mp}\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\sum_{\kappa,\tau;\phi}\theta_\phi^{\kappa,\tau}\frac{1}{k!t!}\frac{\Gamma_p\left(\frac{m}{2},\phi\right)\Gamma_p\left(\frac{n_1+n_2+m}{2},\phi\right)}{\Gamma_p\left(\frac{m}{2},\kappa\right)\Gamma_p\left(\frac{n_1+n_2+m}{2}+h_1+h_2,\phi\right)}C_\phi^{\kappa,\tau}\left(\tfrac{\alpha}{2c}\boldsymbol{\Theta},\tfrac{c-\alpha}{c}\boldsymbol{I}_p\right),\tag{12.8}$$

where $\sum_{\kappa,\tau;\phi}=\sum_{k=0}^{\infty}\sum_{\kappa}\sum_{t=0}^{\infty}\sum_{\tau}\sum_{\phi\in\kappa\cdot\tau}$, $\phi\in\kappa\cdot\tau$ is explained in [2.4.1], $C_\phi^{\kappa,\tau}\left(\frac{\alpha}{2c}\boldsymbol{\Theta},\frac{c-\alpha}{c}\boldsymbol{I}_p\right)$ is a homogeneous invariant polynomial of degrees k and t in the elements of the symmetric matrices $\frac{\alpha}{2c}\boldsymbol{\Theta}$ and $\frac{c-\alpha}{c}\boldsymbol{I}_p$ (see [2.4.1]) and $\theta_\phi^{\kappa,\tau}=\frac{C_\phi^{\kappa,\tau}(\boldsymbol{I}_p,\boldsymbol{I}_p)}{C_\phi(\boldsymbol{I}_p)}$.

Proof:

Let $\boldsymbol{S}_1 \sim W_p(n_1,\boldsymbol{\Sigma})$, $\boldsymbol{S}_2 \sim W_p(n_2,\boldsymbol{\Sigma})$ and $\boldsymbol{B} \sim W_p(m,\boldsymbol{\Sigma},\boldsymbol{\Theta})$ be independently distributed and consider the ratios given by (7.1)

$$\boldsymbol{Q}_i=\boldsymbol{S}^{-\frac{1}{2}}\alpha_i\boldsymbol{S}_i\boldsymbol{S}^{-\frac{1}{2}},\ i=1,2,$$

where $\boldsymbol{S}=\alpha_1\boldsymbol{S}_1+\alpha_2\boldsymbol{S}_2+c\boldsymbol{B}$. This gives $(\boldsymbol{Q}_1,\boldsymbol{Q}_2) \sim BB_p^V(n_1,n_2,m,\alpha_1,\alpha_2,c;\boldsymbol{\Theta})$.

The pdf of $(\boldsymbol{S}_1,\boldsymbol{S}_2,\boldsymbol{B})$ is given by

$$K\left\{\prod_{i=1}^{2}\left[\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}_i\right)|\boldsymbol{S}_i|^{\frac{1}{2}(n_i-p-1)}\right]\right\}\left[\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\right)|\boldsymbol{B}|^{\frac{1}{2}(m-p-1)}\ {}_0F_1\left(\tfrac{m}{2};\tfrac{1}{4}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\right)\right].\tag{12.9}$$

where $K^{-1}=\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)\Gamma_p\left(\frac{m}{2}\right)|2\boldsymbol{\Sigma}|^{\frac{1}{2}(n_1+n_2+m)}$ (see [2.10.2]).

From Theorem 7.1, making the transformations $\boldsymbol{S}=\alpha_1\boldsymbol{S}_1+\alpha_2\boldsymbol{S}_2+c\boldsymbol{B}$ and $\boldsymbol{Q}_i=\boldsymbol{S}^{-\frac{1}{2}}\alpha_i\boldsymbol{S}_i\boldsymbol{S}^{-\frac{1}{2}}$, $i=1,2$, substituted in (12.9) gives

$$\begin{aligned}
& f\left(\boldsymbol{Q}_1,\boldsymbol{Q}_2\right) \\
= \quad & K\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\prod_{i=1}^{2}\alpha_i^{-\frac{1}{2}n_ip}c^{-\frac{1}{2}mp}\prod_{i=1}^{2}|\boldsymbol{Q}_i|^{\frac{1}{2}n_i-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\sum_{i=1}^{2}\boldsymbol{Q}_i\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)} \\
& \cdot\int_{\boldsymbol{S}>\boldsymbol{0}}|\boldsymbol{S}|^{\frac{1}{2}(n_1+n_2+m)-\frac{1}{2}(p+1)}\operatorname{etr}\left[-\tfrac{1}{2c}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}^{\frac{1}{2}}\left(\boldsymbol{I}_p+\sum_{i=1}^{2}\tfrac{c-\alpha_i}{\alpha_i}\boldsymbol{Q}_i\right)\right] \\
& \cdot{}_0F_1\left(\tfrac{m}{2};\tfrac{1}{4c}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}^{\frac{1}{2}}\left(\boldsymbol{I}_p-\sum_{i=1}^{2}\boldsymbol{Q}_i\right)\boldsymbol{S}^{\frac{1}{2}}\right).
\end{aligned} \tag{12.10}$$

From (12.10)

$$\begin{aligned}
& E\left(|\boldsymbol{Q}_1|^{h_1}|\boldsymbol{Q}_2|^{h_2}\right) \\
= \quad & K\prod_{i=1}^{2}\alpha_i^{-\frac{1}{2}n_ip}c^{-\frac{1}{2}mp}\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\int_{\boldsymbol{S}>\boldsymbol{0}}|\boldsymbol{S}|^{\frac{1}{2}(n_1+n_2+m)-\frac{1}{2}(p+1)}\int\int_{\substack{\boldsymbol{0}<\boldsymbol{Q}_1+\boldsymbol{Q}_2<\boldsymbol{I}_p\\ \boldsymbol{Q}_i>\boldsymbol{0}}}\prod_{i=1}^{2}|\boldsymbol{Q}_i|^{\frac{1}{2}n_i+h_i-\frac{1}{2}(p+1)} \\
& \cdot\left|\boldsymbol{I}_p-\sum_{i=1}^{2}\boldsymbol{Q}_i\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)}\operatorname{etr}\left[-\tfrac{1}{2c}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}^{\frac{1}{2}}\left(\boldsymbol{I}_p+\sum_{i=1}^{2}\tfrac{c-\alpha_i}{\alpha_i}\boldsymbol{Q}_i\right)\right] \\
& \cdot{}_0F_1\left(\tfrac{m}{2};\tfrac{1}{4c}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}^{\frac{1}{2}}\left(\boldsymbol{I}_p-\sum_{i=1}^{2}\boldsymbol{Q}_i\right)\boldsymbol{S}^{\frac{1}{2}}\right)\mathrm{d}\boldsymbol{Q}_1\mathrm{d}\boldsymbol{Q}_2\mathrm{d}\boldsymbol{S}.
\end{aligned} \tag{12.11}$$

Let $\alpha_1=\alpha_2=\alpha$ and

$$\begin{aligned}
f\left(\sum_{i=1}^{2}\boldsymbol{Q}_i\right)= \quad & \left|\boldsymbol{I}_p-\sum_{i=1}^{2}\boldsymbol{Q}_i\right|^{\frac{1}{2}m-\frac{1}{2}(p+1)}\operatorname{etr}\left\{-\tfrac{1}{2c}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}^{\frac{1}{2}}\left(\boldsymbol{I}_p+\tfrac{c-\alpha}{\alpha}\sum_{i=1}^{2}\boldsymbol{Q}_i\right)\right\} \\
& \cdot{}_0F_1\left(\tfrac{m}{2};\tfrac{1}{4c}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}^{\frac{1}{2}}\left(\boldsymbol{I}_p-\sum_{i=1}^{2}\boldsymbol{Q}_i\right)\boldsymbol{S}^{\frac{1}{2}}\right)
\end{aligned}$$

and use [2.2.6] to write (12.11) as

$$\begin{aligned}
& E\left(|\boldsymbol{Q}_1|^{h_1}|\boldsymbol{Q}_2|^{h_2}\right) \\
= \quad & K\alpha^{-\frac{1}{2}(n_1+n_2)p}c^{-\frac{1}{2}mp}\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\int_{\boldsymbol{S}>\boldsymbol{0}}|\boldsymbol{S}|^{\frac{1}{2}(n_1+n_2+m)-\frac{1}{2}(p+1)}\beta_p\left(\tfrac{n_1}{2}+h_1,\tfrac{n_2}{2}+h_2\right) \\
& \cdot\int_{\boldsymbol{0}<\boldsymbol{Z}<\boldsymbol{I}_p}|\boldsymbol{Z}|^{\frac{1}{2}(n_1+n_2)+h_1+h_2-\frac{1}{2}(p+1)}|\boldsymbol{I}_p-\boldsymbol{Z}|^{\frac{1}{2}m-\frac{1}{2}(p+1)}\operatorname{etr}\left\{-\tfrac{1}{2c}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}^{\frac{1}{2}}\left(\boldsymbol{I}_p+\tfrac{c-\alpha}{\alpha}\boldsymbol{Z}\right)\right\} \\
& \cdot{}_0F_1\left(\tfrac{m}{2};\tfrac{1}{4c}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}^{\frac{1}{2}}\left(\boldsymbol{I}_p-\boldsymbol{Z}\right)\right)\mathrm{d}\boldsymbol{Z}\mathrm{d}\boldsymbol{S}.
\end{aligned} \tag{12.12}$$

In (12.12), consider the integral with respect to $\boldsymbol{Z}$, that is

$$\begin{aligned}
I_1= \quad & \int_{\boldsymbol{0}<\boldsymbol{Z}<\boldsymbol{I}_p}|\boldsymbol{Z}|^{\frac{1}{2}(n_1+n_2)+h_1+h_2-\frac{1}{2}(p+1)}|\boldsymbol{I}_p-\boldsymbol{Z}|^{\frac{1}{2}m-\frac{1}{2}(p+1)}\operatorname{etr}\left[-\tfrac{1}{2c}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}^{\frac{1}{2}}\left(\boldsymbol{I}_p+\tfrac{c-\alpha}{\alpha}\boldsymbol{Z}\right)\right] \\
& \cdot{}_0F_1\left(\tfrac{m}{2};\tfrac{1}{4c}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}^{\frac{1}{2}}\left(\boldsymbol{I}_p-\boldsymbol{Z}\right)\right)\mathrm{d}\boldsymbol{Z}
\end{aligned}$$

and let $\boldsymbol{X}=\boldsymbol{I}_p-\boldsymbol{Z}$.

Then, using [2.6.1], [2.6.2], [2.4.2] and [2.4.5] we get

$$
\begin{aligned}
I_1 &= \int_{\mathbf{0}<\boldsymbol{X}<\boldsymbol{I}_p} |\boldsymbol{I}_p-\boldsymbol{X}|^{\frac{1}{2}(n_1+n_2)+h_1+h_2-\frac{1}{2}(p+1)} |\boldsymbol{X}|^{\frac{1}{2}m-\frac{1}{2}(p+1)} \\
&\quad \cdot \mathrm{etr}\left\{-\tfrac{1}{2c}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}^{\frac{1}{2}}\left[\boldsymbol{I}_p+\tfrac{c-\alpha}{\alpha}\left(\boldsymbol{I}_p-\boldsymbol{X}\right)\right]\right\} {}_0F_1\left(\tfrac{m}{2};\tfrac{1}{4c}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{X}\right)\mathrm{d}\boldsymbol{X} \\
&= \mathrm{etr}\left(-\tfrac{1}{2c}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}^{\frac{1}{2}}\right)\mathrm{etr}\left(-\tfrac{c-\alpha}{2c\alpha}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}^{\frac{1}{2}}\right) \\
&\quad \cdot \int_{\mathbf{0}<\boldsymbol{X}<\boldsymbol{I}_p} |\boldsymbol{X}|^{\frac{1}{2}m-\frac{1}{2}(p+1)} |\boldsymbol{I}_p-\boldsymbol{X}|^{\frac{1}{2}(n_1+n_2)+h_1+h_2-\frac{1}{2}(p+1)} \mathrm{etr}\left\{\tfrac{c-\alpha}{2c\alpha}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{X}\right\} \\
&\quad \cdot \sum_{k=0}^{\infty}\sum_{\kappa}\frac{1}{\left(\frac{m}{2}\right)_\kappa}\frac{1}{k!}C_\kappa\left(\tfrac{1}{4c}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{X}\right)\mathrm{d}\boldsymbol{X} \\
&= \mathrm{etr}\left(-\tfrac{1}{2\alpha}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}\right)\int_{\mathbf{0}<\boldsymbol{X}<\boldsymbol{I}_p} |\boldsymbol{X}|^{\frac{1}{2}m-\frac{1}{2}(p+1)} |\boldsymbol{I}_p-\boldsymbol{X}|^{\frac{1}{2}(n_1+n_2)+h_1+h_2-\frac{1}{2}(p+1)} \\
&\quad \cdot \sum_{t=0}^{\infty}\sum_{\tau}\frac{1}{t!}C_\tau\left(\tfrac{c-\alpha}{2c\alpha}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{X}\right)\sum_{k=0}^{\infty}\sum_{\kappa}\frac{1}{\left(\frac{m}{2}\right)_\kappa}\frac{1}{k!}C_\kappa\left(\tfrac{1}{4c}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{X}\right)\mathrm{d}\boldsymbol{X} \\
&= \mathrm{etr}\left(-\tfrac{1}{2\alpha}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}\right)\sum_{k=0}^{\infty}\sum_{\kappa}\sum_{t=0}^{\infty}\sum_{\tau}\sum_{\phi\in\kappa\cdot\tau}\theta_\phi^{\kappa,\tau}\frac{1}{k!t!}\frac{1}{\left(\frac{m}{2}\right)_\kappa} \\
&\quad \int_{\mathbf{0}<\boldsymbol{X}<\boldsymbol{I}_p} |\boldsymbol{X}|^{\frac{1}{2}m-\frac{1}{2}(p+1)} |\boldsymbol{I}_p-\boldsymbol{X}|^{\frac{1}{2}(n_1+n_2)+h_1+h_2-\frac{1}{2}(p+1)} C_\phi^{\kappa,\tau}(\tfrac{1}{4c}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{X},\tfrac{c-\alpha}{2c\alpha}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{X})\mathrm{d}\boldsymbol{X} \\
&= \mathrm{etr}\left(-\tfrac{1}{2\alpha}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}\right)\sum_{\kappa,\tau;\phi}\theta_\phi^{\kappa,\tau}\frac{1}{k!t!}\frac{\Gamma_p\left(\frac{m}{2}\right)}{\Gamma_p\left(\frac{m}{2},\kappa\right)}\frac{\Gamma_p\left(\frac{m}{2},\phi\right)\Gamma_p\left(\frac{n_1+n_2}{2}+h_1+h_2\right)}{\Gamma_p\left(\frac{n_1+n_2+m}{2}+h_1+h_2,\phi\right)}C_\phi^{\kappa,\tau}(\tfrac{1}{4c}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{S},\tfrac{c-\alpha}{2c\alpha}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}).
\end{aligned}
\tag{12.13}
$$

Substituting (12.13) in (12.12) and using [2.4.4] gives

$$
\begin{aligned}
&E\left(|\boldsymbol{Q}_1|^{h_1}|\boldsymbol{Q}_2|^{h_2}\right) \\
&= K\alpha^{-\frac{1}{2}(n_1+n_2)p}c^{-\frac{1}{2}mp}\mathrm{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\frac{\Gamma_p\left(\frac{n_1}{2}+h_1\right)\Gamma_p\left(\frac{n_2}{2}+h_2\right)}{\Gamma_p\left(\frac{n_1+n_2}{2}+h_1+h_2\right)}\sum_{\kappa,\tau;\phi}\theta_\phi^{\kappa,\tau}\frac{1}{k!t!}\frac{\Gamma_p\left(\frac{m}{2}\right)}{\Gamma_p\left(\frac{m}{2},\kappa\right)}\frac{\Gamma_p\left(\frac{m}{2},\phi\right)\Gamma_p\left(\frac{n_1+n_2}{2}+h_1+h_2\right)}{\Gamma_p\left(\frac{n_1+n_2+m}{2}+h_1+h_2,\phi\right)} \\
&\quad \cdot\int_{\boldsymbol{S}>\mathbf{0}}\mathrm{etr}\left(-\tfrac{1}{2\alpha}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}\right)|\boldsymbol{S}|^{\frac{1}{2}(n_1+n_2+m)-\frac{1}{2}(p+1)}C_\phi^{\kappa,\tau}(\tfrac{1}{4c}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}\boldsymbol{S},\tfrac{c-\alpha}{2c\alpha}\boldsymbol{\Sigma}^{-1}\boldsymbol{S})\mathrm{d}\boldsymbol{S} \\
&= |2\boldsymbol{\Sigma}|^{-\frac{1}{2}(n_1+n_2+m)}\alpha^{-\frac{1}{2}(n_1+n_2)p}c^{-\frac{1}{2}mp}\mathrm{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\frac{\Gamma_p\left(\frac{n_1}{2}+h_1\right)\Gamma_p\left(\frac{n_2}{2}+h_2\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)}\sum_{\kappa,\tau;\phi}\theta_\phi^{\kappa,\tau}\frac{1}{k!t!}\frac{\Gamma_p\left(\frac{m}{2},\phi\right)}{\Gamma_p\left(\frac{m}{2},\kappa\right)\Gamma_p\left(\frac{n_1+n_2+m}{2}+h_1+h_2,\phi\right)} \\
&\quad \cdot\Gamma_p\left(\tfrac{n_1+n_2+m}{2},\phi\right)\left|\tfrac{1}{2\alpha}\boldsymbol{\Sigma}^{-1}\right|^{-\frac{1}{2}(n_1+n_2+m)}C_\phi^{\kappa,\tau}(\tfrac{1}{4c}\boldsymbol{\Theta}\boldsymbol{\Sigma}^{-1}2\alpha\boldsymbol{\Sigma},\tfrac{c-\alpha}{2c\alpha}\boldsymbol{\Sigma}^{-1}2\alpha\boldsymbol{\Sigma}) \\
&= \left(\tfrac{c}{\alpha}\right)^{-\frac{1}{2}mp}\mathrm{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\frac{\Gamma_p\left(\frac{n_1}{2}+h_1\right)\Gamma_p\left(\frac{n_2}{2}+h_2\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)}\sum_{\kappa,\tau;\phi}\theta_\phi^{\kappa,\tau}\frac{1}{k!t!}\frac{\Gamma_p\left(\frac{m}{2},\phi\right)\Gamma_p\left(\frac{n_1+n_2+m}{2},\phi\right)}{\Gamma_p\left(\frac{m}{2},\kappa\right)\Gamma_p\left(\frac{n_1+n_2+m}{2}+h_1+h_2,\phi\right)}C_\phi^{\kappa,\tau}(\tfrac{\alpha}{2c}\boldsymbol{\Theta},\tfrac{c-\alpha}{c}\boldsymbol{I}_p). \quad \blacksquare
\end{aligned}
$$

12.3 Distribution of the product of determinants

In Theorem 12.3 an exact expression is derived for the pdf of $\Lambda_5 = |\boldsymbol{Q}_1|^{\frac{1}{2}n_1}|\boldsymbol{Q}_2|^{\frac{1}{2}n_2}$ where $(\boldsymbol{Q}_1,\boldsymbol{Q}_2) \sim BB_p^V(n_1,n_2,m,\alpha,\alpha,c;\boldsymbol{\Theta})$ as given by (12.2).

Theorem 12.3
Let $(\boldsymbol{Q}_1, \boldsymbol{Q}_2) \sim BB_p^V(n_1, n_2, m, \alpha, \alpha, c; \boldsymbol{\Theta})$ *with pdf given by* (12.2) *and let* $\Lambda_5 = |\boldsymbol{Q}_1|^{\frac{1}{2}n_1} |\boldsymbol{Q}_2|^{\frac{1}{2}n_2}$. *Then the pdf of* Λ_5 *is given by*

$$\frac{\pi^{\frac{1}{4}p(p-1)}}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)} \left(\frac{c}{\alpha}\right)^{-\frac{1}{2}mp} \operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) \sum_{\kappa,\tau;\phi} \theta_\phi^{\kappa,\tau} \frac{1}{k!t!} \frac{\Gamma_p\left(\frac{m}{2},\phi\right)\Gamma_p\left(\frac{n_1+n_2+m}{2},\phi\right)}{\Gamma_p\left(\frac{m}{2},\kappa\right)} C_\phi^{\kappa,\tau}\left(\tfrac{\alpha}{2c}\boldsymbol{\Theta}, \tfrac{c-\alpha}{c}\boldsymbol{I}_p\right) H_{p,2p}^{2p,0}\left(\lambda_5 \middle|^{(a_1,\alpha_1),\ldots,(a_p,\alpha_p)}_{(b_1,\beta_1),\ldots,(b_{2p},\beta_{2p})}\right), \tag{12.14}$$

$0 < \lambda_5 < 1$, *where* $\sum_{\kappa,\tau;\phi} = \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{t=0}^{\infty} \sum_{\tau} \sum_{\phi \in \kappa \cdot \tau}$ *and*

$a_j = \frac{m}{2} + (k_j + t_j) - \frac{1}{2}(j-1)$ *for* $j = 1, 2, 3, \ldots, p$,

$\alpha_j = \frac{n_1+n_2}{2}$ *for* $j = 1, 2, 3, \ldots, p$,

$$b_j = \begin{cases} -\frac{1}{4}(j-1) & \text{for } j = 1, 3, 5, \ldots, 2p-1 \\ -\frac{1}{4}(j-2) & \text{for } j = 2, 4, 6, \ldots, 2p, \end{cases}$$

$$\beta_j = \begin{cases} \frac{n_1}{2} & \text{for } j = 1, 3, 5, \ldots, 2p-1 \\ \frac{n_2}{2} & \text{for } j = 2, 4, 6, \ldots, 2p. \end{cases}$$

Proof:
Using (12.8) the Mellin transform (see [2.8.1]) of $f(\lambda_5)$ is

$$\begin{aligned} M_f(h) &\equiv E\left(\Lambda_5^{h-1}\right) \\ &= E\left[\left(|\boldsymbol{Q}_1|^{\frac{1}{2}n_1} |\boldsymbol{Q}_2|^{\frac{1}{2}n_2}\right)^{h-1}\right] \\ &= \left(\tfrac{\alpha}{c}\right)^{\frac{1}{2}mp} \operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) \frac{\Gamma_p\left[\frac{n_1}{2}+\frac{n_1}{2}(h-1)\right]\Gamma_p\left[\frac{n_2}{2}+\frac{n_2}{2}(h-1)\right]}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)} \\ &\quad \cdot \sum_{\kappa,\tau;\phi} \theta_\phi^{\kappa,\tau} \frac{1}{k!t!} \frac{\Gamma_p\left(\frac{m}{2},\phi\right)\Gamma_p\left(\frac{n_1+n_2+m}{2},\phi\right)}{\Gamma_p\left[\frac{n_1+n_2+m}{2}+\frac{n_1+n_2}{2}(h-1),\phi\right]\Gamma_p\left(\frac{m}{2},\kappa\right)} C_\phi^{\kappa,\tau}\left(\tfrac{\alpha}{2c}\boldsymbol{\Theta}, \tfrac{c-\alpha}{c}\boldsymbol{I}_p\right) \\ &= \frac{\Gamma_p\left(\frac{n_1}{2}h\right)\Gamma_p\left(\frac{n_2}{2}h\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)} \left(\tfrac{c}{\alpha}\right)^{-\frac{1}{2}mp} \operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) \sum_{\kappa,\tau;\phi} \theta_\phi^{\kappa,\tau} \frac{1}{k!t!} \frac{\Gamma_p\left(\frac{m}{2},\phi\right)\Gamma_p\left(\frac{n_1+n_2+m}{2},\phi\right)}{\Gamma_p\left(\frac{m}{2},\kappa\right)\Gamma_p\left(\frac{m}{2}+\frac{n_1+n_2}{2}h,\phi\right)} C_\phi^{\kappa,\tau}\left(\tfrac{\alpha}{2c}\boldsymbol{\Theta}, \tfrac{c-\alpha}{c}\boldsymbol{I}_p\right). \end{aligned} \tag{12.15}$$

From [2.3.3] the generalised gamma function of weight ϕ in (12.15) can be written as

$$\begin{aligned} &\Gamma_p\left(\tfrac{m}{2} + \tfrac{n_1+n_2}{2}h, \phi\right) \\ &= \pi^{\frac{1}{4}p(p-1)} \prod_{j=1}^{p} \Gamma\left(a_j + \alpha_j h\right), \end{aligned} \tag{12.16}$$

where $a_j = \frac{m}{2} + (k_j + t_j) - \frac{1}{2}(j-1)$ for $j = 1, 2, 3, \ldots, p$

and $\alpha_j = \frac{n_1+n_2}{2}$ for $j = 1, 2, 3, \ldots, p$.

From [2.2.2] the multivariate gamma functions in (12.15) can be written as

$$\begin{aligned}
&\Gamma_p\left(\tfrac{n_1}{2}h\right)\Gamma_p\left(\tfrac{n_2}{2}h\right) \\
&= \pi^{\frac{1}{2}p(p-1)} \prod_{j=1}^{p} \Gamma\left[\tfrac{n_1}{2}h - \tfrac{1}{2}(j-1)\right] \prod_{j=1}^{p} \Gamma\left[\tfrac{n_2}{2}h - \tfrac{1}{2}(j-1)\right] \\
&= \pi^{\frac{1}{2}p(p-1)} \prod_{j=1}^{2p} \Gamma\left(b_j + \beta_j h\right)
\end{aligned} \tag{12.17}$$

where $b_j = \begin{cases} -\frac{1}{4}(j-1) & \text{for } j = 1, 3, 5, \ldots, 2p-1 \\ -\frac{1}{4}(j-2) & \text{for } j = 2, 4, 6, \ldots, 2p, \end{cases}$

and $\beta_j = \begin{cases} \frac{n_1}{2} & \text{for } j = 1, 3, 5, \ldots, 2p-1 \\ \frac{n_2}{2} & \text{for } j = 2, 4, 6, \ldots, 2p. \end{cases}$

Now, substituting (12.16) and (12.17) in (12.15) gives

$$M_f(h) \equiv \left(\tfrac{\alpha}{c}\right)^{\frac{1}{2}mp} \operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) \frac{\pi^{\frac{1}{4}p(p-1)}}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)} \sum_{\kappa,\tau;\phi} \theta_\phi^{\kappa,\tau} \frac{1}{k!t!} \frac{\Gamma_p\left(\frac{m}{2},\phi\right)\Gamma_p\left(\frac{n_1+n_2+m}{2},\phi\right)}{\Gamma_p\left(\frac{m}{2},\kappa\right)} C_\phi^{\kappa,\tau}\left(\tfrac{\alpha}{2c}\boldsymbol{\Theta}, \tfrac{c-\alpha}{c}\boldsymbol{I}_p\right) \frac{\prod_{j=1}^{2p}\Gamma\left(b_j+\beta_j h\right)}{\prod_{j=1}^{p}\Gamma\left(a_j+\alpha_j h\right)}. \tag{12.18}$$

Using (12.18) the inverse Mellin transform (see [2.8.1]) is given by

$$\begin{aligned}
f(\lambda_5) &= \frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty} M_f(h)\,\lambda_5^{-h}\mathrm{d}h \\
&= \left(\tfrac{\alpha}{c}\right)^{\frac{1}{2}mp} \operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) \frac{\pi^{\frac{1}{4}p(p-1)}}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)} \\
&\quad \cdot \sum_{\kappa,\tau;\phi} \theta_\phi^{\kappa,\tau} \frac{1}{k!t!} \frac{\Gamma_p\left(\frac{m}{2},\phi\right)\Gamma_p\left(\frac{n_1+n_2+m}{2},\phi\right)}{\Gamma_p\left(\frac{m}{2},\kappa\right)} C_\phi^{\kappa,\tau}\left(\tfrac{\alpha}{2c}\boldsymbol{\Theta}, \tfrac{c-\alpha}{c}\boldsymbol{I}_p\right)\left[\frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty} \frac{\prod_{j=1}^{2p}\Gamma\left(b_j+\beta_j h\right)}{\prod_{j=1}^{p}\Gamma\left(a_j+\alpha_j h\right)}\lambda_5^{-h}\mathrm{d}h\right] \\
&= \left(\tfrac{\alpha}{c}\right)^{\frac{1}{2}mp} \operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) \frac{\pi^{\frac{1}{4}p(p-1)}}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)} \sum_{\kappa,\tau;\phi} \theta_\phi^{\kappa,\tau} \frac{1}{k!t!} \frac{\Gamma_p\left(\frac{m}{2},\phi\right)\Gamma_p\left(\frac{n_1+n_2+m}{2},\phi\right)}{\Gamma_p\left(\frac{m}{2},\kappa\right)} C_\phi^{\kappa,\tau}\left(\tfrac{\alpha}{2c}\boldsymbol{\Theta}, \tfrac{c-\alpha}{c}\boldsymbol{I}_p\right) \\
&\quad \cdot H_{p,2p}^{2p,0}\left(\lambda_5\middle|_{(b_1,\beta_1),\ldots,(b_{2p},\beta_{2p})}^{(a_1,\alpha_1),\ldots,(a_p,\alpha_p)}\right).
\end{aligned}$$

The last step follows from the definition of Fox's H-function (see [2.8.3]) and gives (12.14). ■

12.4 Bivariate distribution

The bivariate distribution is considered in this section, that is where $(Q_1, Q_2) \sim BB_1^V\left(\frac{n_1}{2}, \frac{n_2}{2}, \frac{m}{2}, \alpha_1, \alpha_2, c; \theta\right)$. The pdf $f(Q_1, Q_2)$ is given and $E\left(Q_1^{h_1}Q_2^{h_2}\right)$ and the pdf of $\Lambda_5 = Q_1^{\frac{1}{2}n_1}Q_2^{\frac{1}{2}n_2}$ are derived in Theorems 12.4 and 12.5 respectively.

From (12.2) the pdf of $(Q_1, Q_2) \sim BB_1^V\left(\frac{n_1}{2}, \frac{n_2}{2}, \frac{m}{2}, \alpha_1, \alpha_2, c; \theta\right)$ is given by

$$\begin{aligned} f(q_1, q_2) = {} & \frac{\Gamma\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{m}{2}\right)} q_1^{\frac{1}{2}n_1-1} q_2^{\frac{1}{2}n_2-1} (1-q_1-q_2)^{\frac{1}{2}m-1} \\ & \cdot \left(\frac{c}{\alpha_1}\right)^{\frac{1}{2}n_1 p} \left(\frac{c}{\alpha_2}\right)^{\frac{1}{2}n_2 p} \left(1 + \frac{c-\alpha_1}{\alpha_1} q_1 + \frac{c-\alpha_2}{\alpha_2} q_2\right)^{-\frac{1}{2}(n_1+n_2+m)} \\ & \cdot e^{-\frac{1}{2}\theta} {}_1F_1\left(\frac{n_1+n_2+m}{2}; \frac{m}{2}; \frac{\theta}{2} \frac{1-q_1-q_2}{1+\frac{c-\alpha_1}{\alpha_1}q_1 + \frac{c-\alpha_2}{\alpha_2}q_2}\right), \end{aligned} \quad (12.19)$$

$0 < q_i < 1,\ i = 1, 2,\ 0 < q_1 + q_2 < 1.$

Theorem 12.4
If $(Q_1, Q_2) \sim BB_1^V\left(\frac{n_1}{2}, \frac{n_2}{2}, \frac{m}{2}, \alpha_1, \alpha_2, c; \theta\right)$ *as given by* (12.19) *then*

$$\begin{aligned} E\left(Q_1^{h_1} Q_2^{h_2}\right) = {} & \frac{\Gamma\left(\frac{n_1}{2}+h_1\right)\Gamma\left(\frac{n_2}{2}+h_2\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} \left(\frac{c}{\alpha_1}\right)^{\frac{1}{2}n_1} \left(\frac{c}{\alpha_2}\right)^{\frac{1}{2}n_2} e^{-\frac{1}{2}\theta} \sum_{l=0}^{\infty} \frac{1}{l!} \frac{\Gamma\left(\frac{n_1+n_2+m}{2}+l\right)}{\Gamma\left(\frac{n_1+n_2+m}{2}+l+h_1+h_2\right)} \left(\frac{\theta}{2}\right)^l \\ & \cdot F_1\left(\frac{n_1+n_2+m}{2}+l, \frac{n_1}{2}+h_1, \frac{n_2}{2}+h_2, \frac{n_1+n_2+m}{2}+l+h_1+h_2; \frac{\alpha_1-c}{\alpha_1}, \frac{\alpha_2-c}{\alpha_2}\right). \end{aligned} \quad (12.20)$$

Proof:
From (12.19) and [2.5.1]

$$\begin{aligned} & E\left(Q_1^{h_1} Q_2^{h_2}\right) \\ = {} & \frac{\Gamma\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{m}{2}\right)} \left(\frac{c}{\alpha_1}\right)^{\frac{1}{2}n_1 p} \left(\frac{c}{\alpha_2}\right)^{\frac{1}{2}n_2 p} e^{-\frac{1}{2}\theta} \int\limits_{\substack{0<q_1+q_2<1 \\ 0<q_i<1,\ i=1,2}} \int q_1^{\frac{1}{2}n_1+h_1-1} q_2^{\frac{1}{2}n_2+h_2-1} (1-q_1-q_2)^{\frac{1}{2}m-1} \\ & \cdot \left(1 + \frac{c-\alpha_1}{\alpha_1} q_1 + \frac{c-\alpha_2}{\alpha_2} q_2\right)^{-\frac{1}{2}(n_1+n_2+m)} {}_1F_1\left(\frac{n_1+n_2+m}{2}; \frac{m}{2}; \frac{\theta}{2} \frac{1-q_1-q_2}{1+\frac{c-\alpha_1}{\alpha_1}q_1 + \frac{c-\alpha_2}{\alpha_2}q_2}\right) \mathrm{d}q_1 \mathrm{d}q_2 \\ = {} & \frac{\Gamma\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{m}{2}\right)} \left(\frac{c}{\alpha_1}\right)^{\frac{1}{2}n_1 p} \left(\frac{c}{\alpha_2}\right)^{\frac{1}{2}n_2 p} e^{-\frac{1}{2}\theta} \sum_{l=0}^{\infty} \frac{1}{l!} \frac{\Gamma\left(\frac{n_1+n_2+m}{2}+l\right)}{\Gamma\left(\frac{n_1+n_2+m}{2}\right)} \frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m}{2}+l\right)} \left(\frac{\theta}{2}\right)^l \\ & \cdot \int\limits_{\substack{0<q_1+q_2<1 \\ 0<q_i<1,\ i=1,2}} \int q_1^{\frac{1}{2}n_1+h_1-1} q_2^{\frac{1}{2}n_2+h_2-1} (1-q_1-q_2)^{\frac{1}{2}m+l-1} \left(1 + \frac{c-\alpha_1}{\alpha_1} q_1 + \frac{c-\alpha_2}{\alpha_2} q_2\right)^{-\frac{1}{2}(n_1+n_2+m)-l} \mathrm{d}q_1 \mathrm{d}q_2 \\ = {} & e^{-\frac{1}{2}\theta} \sum_{l=0}^{\infty} \frac{1}{l!} \left(\frac{\theta}{2}\right)^l E_l\left(Q_1^{h_1} Q_2^{h_2}\right), \end{aligned} \quad (12.21)$$

where $E_l\left(Q_1^{h_1} Q_2^{h_2}\right)$ is the $(h_1, h_2)^{th}$ product moment of $(Q_1, Q_2) \sim BB_1^V\left(\frac{n_1}{2}, \frac{n_2}{2}, \frac{m}{2}, \alpha_1, \alpha_2, c\right)$ given by (7.29). Using (7.29) in (12.21) gives the result. ■

Remark 12.5

If $\alpha_1 = \alpha_2 = \alpha$ it follows from [2.5.8] and [2.5.4] that (12.20) simplifies to

$$\begin{aligned}
E\left(Q_1^{h_1} Q_2^{h_2}\right) &= \frac{\Gamma\left(\frac{n_1}{2}+h_1\right)\Gamma\left(\frac{n_2}{2}+h_2\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)}\left(\frac{c}{\alpha}\right)^{\frac{1}{2}(n_1+n_2)} e^{-\frac{1}{2}\theta}\sum_{l=0}^{\infty}\frac{1}{l!}\frac{\Gamma\left(\frac{n_1+n_2+m}{2}+l\right)}{\Gamma\left(\frac{n_1+n_2+m}{2}+l+h_1+h_2\right)}\left(\frac{\theta}{2}\right)^l \\
&\quad \cdot {}_2F_1\left(\frac{n_1}{2}+\frac{n_2}{2}+h_1+h_2, \frac{n_1+n_2+m}{2}+l; \frac{n_1+n_2+m}{2}+l+h_1+h_2; \frac{\alpha-c}{\alpha}\right) \\
&= \frac{\Gamma\left(\frac{n_1}{2}+h_1\right)\Gamma\left(\frac{n_2}{2}+h_2\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)}\left(\frac{c}{\alpha}\right)^{-\frac{1}{2}m} e^{-\frac{1}{2}\theta}\sum_{l=0}^{\infty}\frac{1}{l!}\frac{\Gamma\left(\frac{n_1+n_2+m}{2}+l\right)}{\Gamma\left(\frac{n_1+n_2+m}{2}+l+h_1+h_2\right)}\left(\frac{\alpha\theta}{2c}\right)^l \\
&\quad \cdot {}_2F_1\left(\frac{m}{2}+l, \frac{n_1+n_2+m}{2}+l; \frac{n_1+n_2+m}{2}+l+h_1+h_2; \frac{c-\alpha}{c}\right).
\end{aligned}$$

Theorem 12.5

If $(Q_1, Q_2) \sim BB_1^V\left(\frac{n_1}{2}, \frac{n_2}{2}, \frac{m}{2}, \alpha_1, \alpha_2, c; \theta\right)$ *then the pdf of* $\Lambda_5 = Q_1^{\frac{1}{2}n_1} Q_2^{\frac{1}{2}n_2}$ *is*

$$\begin{aligned}
f(\lambda_5) = &\frac{1}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)}\left(\frac{c}{\alpha_1}\right)^{\frac{1}{2}n_1}\left(\frac{c}{\alpha_2}\right)^{\frac{1}{2}n_2} e^{-\frac{1}{2}\theta}\sum_{k=0}^{\infty}\sum_{t=0}^{\infty}\sum_{l=0}^{\infty}\frac{1}{k!t!l!}\Gamma\left(\frac{n_1+n_2+m}{2}+k+t+l\right)\left(\frac{\theta}{2}\right)^l\left(\frac{\alpha_1-c}{\alpha_1}\right)^k\left(\frac{\alpha_2-c}{\alpha_2}\right)^t \\
&\cdot H_{1,2}^{2,0}\left(\lambda_5\middle|_{(k,\frac{n_1}{2}),(t,\frac{n_2}{2})}^{(\frac{m}{2}+k+t+l,\frac{n_1+n_2}{2})}\right), \quad 0 < \lambda_5 < 1.
\end{aligned} \tag{12.22}$$

Proof:

From (12.20) and [2.5.6] the Mellin transform of $f(\lambda_5)$ (see [2.8.1]) is

$$\begin{aligned}
&M_f(h) \\
\equiv\ & E\left(\Lambda_5^{h-1}\right) \\
=\ & E\left[\left(Q_1^{\frac{1}{2}n_1} Q_2^{\frac{1}{2}n_2}\right)^{h-1}\right] \\
=\ & \frac{\Gamma\left(\frac{n_1}{2}h\right)\Gamma\left(\frac{n_2}{2}h\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)}\left(\frac{c}{\alpha_1}\right)^{\frac{1}{2}n_1}\left(\frac{c}{\alpha_2}\right)^{\frac{1}{2}n_2} e^{-\frac{1}{2}\theta}\sum_{l=0}^{\infty}\frac{1}{l!}\frac{\Gamma\left(\frac{n_1+n_2+m}{2}+l\right)}{\Gamma\left(\frac{m}{2}+l+\frac{n_1+n_2}{2}h\right)}\left(\frac{\theta}{2}\right)^l \\
& \cdot F_1\left(\frac{n_1+n_2+m}{2}+l, \frac{n_1}{2}h, \frac{n_2}{2}h, \frac{m}{2}+l+\frac{n_1+n_2}{2}h; \frac{\alpha_1-c}{\alpha_1}, \frac{\alpha_2-c}{\alpha_2}\right) \\
=\ & \frac{\Gamma\left(\frac{n_1}{2}h\right)\Gamma\left(\frac{n_2}{2}h\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)}\left(\frac{c}{\alpha_1}\right)^{\frac{1}{2}n_1}\left(\frac{c}{\alpha_2}\right)^{\frac{1}{2}n_2} e^{-\frac{1}{2}\theta}\sum_{l=0}^{\infty}\frac{1}{l!}\frac{\Gamma\left(\frac{n_1+n_2+m}{2}+l\right)}{\Gamma\left(\frac{m}{2}+l+\frac{n_1+n_2}{2}h\right)}\left(\frac{\theta}{2}\right)^l \\
& \cdot \sum_{k=0}^{\infty}\sum_{t=0}^{\infty}\frac{1}{k!t!}\frac{\left(\frac{n_1+n_2+m}{2}+l\right)_{k+t}\left(\frac{n_1}{2}h\right)_k\left(\frac{n_2}{2}h\right)_t}{\left(\frac{m}{2}+l+\frac{n_1+n_2}{2}h\right)_{k+t}}\left(\frac{\alpha_1-c}{\alpha_1}\right)^k\left(\frac{\alpha_2-c}{\alpha_2}\right)^t \\
=\ & \frac{\Gamma\left(\frac{n_1}{2}h\right)\Gamma\left(\frac{n_2}{2}h\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)}\left(\frac{c}{\alpha_1}\right)^{\frac{1}{2}n_1}\left(\frac{c}{\alpha_2}\right)^{\frac{1}{2}n_2} e^{-\frac{1}{2}\theta}\sum_{k=0}^{\infty}\sum_{t=0}^{\infty}\sum_{l=0}^{\infty}\frac{1}{k!t!l!}\frac{\Gamma\left(\frac{n_1+n_2+m}{2}+l\right)}{\Gamma\left(\frac{m}{2}+l+\frac{n_1+n_2}{2}h\right)} \\
& \cdot \frac{\Gamma\left(\frac{n_1+n_2+m}{2}+l+k+t\right)}{\Gamma\left(\frac{n_1+n_2+m}{2}+l\right)}\frac{\Gamma\left(\frac{n_1}{2}h+k\right)\Gamma\left(\frac{n_2}{2}h+t\right)}{\Gamma\left(\frac{n_1}{2}h\right)\Gamma\left(\frac{n_2}{2}h\right)}\frac{\Gamma\left(\frac{m}{2}+l+\frac{n_1+n_2}{2}h\right)}{\Gamma\left(\frac{m}{2}+l+\frac{n_1+n_2}{2}h+k+t\right)}\left(\frac{\theta}{2}\right)^l\left(\frac{\alpha_1-c}{\alpha_1}\right)^k\left(\frac{\alpha_2-c}{\alpha_2}\right)^t \\
=\ & \frac{1}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)}\left(\frac{c}{\alpha_1}\right)^{\frac{1}{2}n_1}\left(\frac{c}{\alpha_2}\right)^{\frac{1}{2}n_2} e^{-\frac{1}{2}\theta} \\
& \cdot \sum_{k=0}^{\infty}\sum_{t=0}^{\infty}\sum_{l=0}^{\infty}\frac{1}{k!t!l!}\Gamma\left(\frac{n_1+n_2+m}{2}+l+k+t\right)\frac{\Gamma\left(\frac{n_1}{2}h+k\right)\Gamma\left(\frac{n_2}{2}h+t\right)}{\Gamma\left(\frac{m}{2}+l+\frac{n_1+n_2}{2}h+k+t\right)}\left(\frac{\theta}{2}\right)^l\left(\frac{\alpha_1-c}{\alpha_1}\right)^k\left(\frac{\alpha_2-c}{\alpha_2}\right)^t.
\end{aligned} \tag{12.23}$$

The gamma functions in (12.23) can be written as

$$\prod_{j=1}^{1}\Gamma\left(a_j+\alpha_j h\right), \tag{12.24}$$

where $a_1=\frac{m}{2}+l+k+t$ and $\alpha_1=\frac{n_1+n_2}{2}$.

Similarly

$$\begin{aligned}&\Gamma\left(\tfrac{n_1}{2}h+k\right)\Gamma\left(\tfrac{n_2}{2}h+t\right)\\&=\prod_{j=1}^{2}\Gamma\left(b_j+\beta_j h\right),\end{aligned} \tag{12.25}$$

where $b_1=k$, $b_2=t$, $\beta_1=\frac{n_1}{2}$ and $\beta_2=\frac{n_2}{2}$.

Now, substituting (12.24) and (12.25) in (12.23) gives

$$\begin{aligned}M_f(h)\equiv\; &\frac{1}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)}\left(\frac{c}{\alpha_1}\right)^{\frac{1}{2}n_1}\left(\frac{c}{\alpha_2}\right)^{\frac{1}{2}n_2}e^{-\frac{1}{2}\theta}\sum_{k=0}^{\infty}\sum_{t=0}^{\infty}\sum_{l=0}^{\infty}\frac{1}{k!t!l!}\Gamma\left(\frac{n_1+n_2+m}{2}+l+k+t\right)\\&\cdot\left(\frac{\theta}{2}\right)^{l}\left(\frac{\alpha_1-c}{\alpha_1}\right)^{k}\left(\frac{\alpha_2-c}{\alpha_2}\right)^{t}\frac{\prod_{j=1}^{2}\Gamma\left(b_j+\beta_j h\right)}{\prod_{j=1}^{1}\Gamma\left(a_j+\alpha_j h\right)}.\end{aligned} \tag{12.26}$$

The pdf of Λ_5, given by (12.22), is obtained from the inverse Mellin transform of (12.26) (see [2.8.1]) and the definition of Fox's H-function (see [2.8.3]). ■

Remark 12.6
The effect of the parameters was studied in Sections 4.5, 5.5, 6.5 and 9.4.

13 Conclusion

In this thesis a bimatrix group of beta distributions with bounded domain were developed from different dependent Wishart ratios. (See the display in the following table.)

Wishart ratios $i=1,2$	Distribution	Application	Chapter $\boldsymbol{B} \sim W_p(m,\boldsymbol{\Sigma})$	Chapter $\boldsymbol{B} \sim W_p(m,\boldsymbol{\Sigma};\boldsymbol{\Theta})$
(1.2) $\boldsymbol{U}_i = (\boldsymbol{S}_1+\boldsymbol{S}_2+\boldsymbol{B})^{-\frac{1}{2}}\boldsymbol{S}_i(\boldsymbol{S}_1+\boldsymbol{S}_2+\boldsymbol{B})^{-\frac{1}{2}}$	Bimatrix beta type I	(1.7) $\Lambda_2 = \prod_{i=1}^2 \left\lvert \frac{\boldsymbol{S}_i}{\boldsymbol{S}_1+\boldsymbol{S}_2+\boldsymbol{B}} \right\rvert^{\frac{1}{2}n_i}$	4	9
(1.3) $\boldsymbol{W}_i = (\boldsymbol{S}_1+\boldsymbol{S}_2+c\boldsymbol{B})^{-\frac{1}{2}}\boldsymbol{S}_i(\boldsymbol{S}_1+\boldsymbol{S}_2+c\boldsymbol{B})^{-\frac{1}{2}}$	Bimatrix beta type III	(1.8) $\Lambda_3 = \prod_{i=1}^2 \left\lvert \frac{\boldsymbol{S}_i}{\boldsymbol{S}_1+\boldsymbol{S}_2+c\boldsymbol{B}} \right\rvert^{\frac{1}{2}n_i}$	5	10
(1.4) $\boldsymbol{X}_i = \left(\boldsymbol{I}_p+\boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{S}_i\boldsymbol{B}^{-\frac{1}{2}}\right)^{-\frac{1}{2}}\boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{S}_i\boldsymbol{B}^{-\frac{1}{2}}\left(\boldsymbol{I}_p+\boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{S}_i\boldsymbol{B}^{-\frac{1}{2}}\right)^{-\frac{1}{2}}$	Bimatrix beta type IV	(1.10) $\Lambda_4 = \prod_{i=1}^2 \left\lvert \frac{\boldsymbol{S}_i}{\boldsymbol{S}_i+\boldsymbol{B}} \right\rvert$	6	11
(1.5) $\boldsymbol{Q}_i = (\alpha_1\boldsymbol{S}_1+\alpha_2\boldsymbol{S}_2+c\boldsymbol{B})^{-\frac{1}{2}}\alpha_i\boldsymbol{S}_i(\alpha_1\boldsymbol{S}_1+\alpha_2\boldsymbol{S}_2+c\boldsymbol{B})^{-\frac{1}{2}}$	Bimatrix beta type V	(1.9) $\Lambda_5 = \prod_{i=1}^2 \left\lvert \frac{\alpha_i\boldsymbol{S}_i}{\alpha_1\boldsymbol{S}_1+\alpha_2\boldsymbol{S}_2+c\boldsymbol{B}} \right\rvert^{\frac{1}{2}n_i}$	7	12

In this study the goal was to derive the exact pdfs for each of the proposed Wishart ratios ((1.2) to (1.5)) for $\boldsymbol{B}$ having the central or the noncentral Wishart distribution. This was achieved by using symmetrised density functions defined by Greenacre (1973) followed by applying the symmetrised density functions in an inverse way. For each distribution some statistical properties were established; specifically the product moment. The marginal and conditional properties were also studied for the central distributions. In each chapter a direct application in multivariate statistics is presented by linking the results to statistics that are functions of the product of determinants of bimatrix beta variates. By making use of inverse Mellin transforms, exact expressions were derived for the pdfs of the product of the determinants for both the central and noncentral cases. These exact expressions for the pdfs were in terms of zonal polynomials, invariant polynomials, hypergeometric functions with matrix argument, Meijer's G-function and Fox's H-function. These functions have recently become more computable due to dynamic programming and the availability of packages and algorithms, therefore the theory is transformed into practice for the user.

14 List of Figures

The graphs in this study were drawn by using the computer software packages Mathematica, Excel and SAS. There are some algorithms available for calculating Meijer's G-function and Fox's H-function and facilitating the use of these distributions (see Gutierrez et al. (2000) and Koev and Edelman (2006)). There are also mathematical packages, such as Maple and Mathematica for computing and drawing densities in terms of Meijer's G-function. In this study we simulated the densities of Λ_2, Λ_3, Λ_4 and Λ_5 (see (1.7), (1.8), (1.10), and (1.9) respectively) for illustrative purposes.

6.1a Effect of n_2 on $f(x_1, x_2)$, $(X_1, X_2) \sim BB_1^{IV}(8, n_2, 8)$

6.1b Effect of m on $f(x_1, x_2)$, $(X_1, X_2) \sim BB_1^{IV}(8, 8, m)$

6.2 Effect of n_2 and m on $corr(X_1, X_2)$
(i) $(X_1, X_2) \sim BB_1^{IV}(8, n_2, 8)$
(ii) $(X_1, X_2) \sim BB_1^{IV}(8, 8, m)$

6.3a Effect of n_2 on $E(X_2|x_1)$, $(X_1, X_2) \sim BB_1^{IV}(8, n_2, 8)$

6.3b Effect of m on $E(X_2|x_1)$, $(X_1, X_2) \sim BB_1^{IV}(8, 8, m)$

6.4a Effect of n_2 on $var(X_2|x_1)$, $(X_1, X_2) \sim BB_1^{IV}(8, n_2, 8)$

6.4b Effect of m on $var(X_2|x_1)$, $(X_1, X_2) \sim BB_1^{IV}(8, 8, m)$

6.5a Effect of n_2 on $f(\lambda_4)$, $\Lambda_4 = X_1X_2$, $(X_1, X_2) \sim BB_1^{IV}(8, n_2, 8)$

6.5b Effect of m on $f(\lambda_4)$, $\Lambda_4 = X_1X_2$, $(X_1, X_2) \sim BB_1^{IV}(8, 8, m)$

6.6a Effect of n_2 on $f(\lambda_4)$, $\Lambda_4 = |\mathbf{X}_1\mathbf{X}_2|$, $(\mathbf{X}_1, \mathbf{X}_2) \sim BB_2^{IV}(8, n_2, 8)$

6.6b Effect of m on $f(\lambda_4)$, $\Lambda_4 = |\mathbf{X}_1\mathbf{X}_2|$, $(\mathbf{X}_1, \mathbf{X}_2) \sim BB_2^{IV}(8, 8, m)$

7.1 Effect of α_2 on $f(q_1, q_2)$, $(Q_1, Q_2) \sim BB_1^{V}(10, 10, 10, 1, \alpha_2, 1)$

7.2 Effect of α_2 and c on $corr(Q_1, Q_2)$
(i) $(Q_1, Q_2) \sim BB_1^{V}(10, 10, 10, 1, \alpha_2, 1)$
(ii) $(Q_1, Q_2) \sim BB_1^{V}(10, 10, 10, 1, 1, c)$

7.3 Effect of α_1 and α_2 on $E(Q_2|q_1)$, $(Q_1, Q_2) \sim BB_1^{V}(10, 10, 10, \alpha_1, \alpha_2, 1)$

7.4 Effect of α_1 and α_2 on $var(Q_2|q_1)$, $(Q_1, Q_2) \sim BB_1^{V}(10, 10, 10, \alpha_1, \alpha_2, 1)$

7.5 Effect of α_1 on $f(\lambda_5)$, $\Lambda_5 = Q_1Q_2$, $(Q_1, Q_2) \sim BB_1^{V}(2, 2, 2, \alpha_1, 1, 1)$

7.6 Effect of α on $f(\lambda_5)$, $\Lambda_5 = |\boldsymbol{Q}_1\boldsymbol{Q}_2|$, $(\boldsymbol{Q}_1, \boldsymbol{Q}_2) \sim BB_2^{V}(2, 2, 2, \alpha, \alpha, 1)$

9.1 Effect of θ on $f(u_1, u_2)$, $(U_1, U_2) \sim BB_1^{I}(10, 10, 10; \theta)$

9.2 Effect of θ on $corr(U_1, U_2)$, $(U_1, U_2) \sim BB_1^{I}(10, 10, 10; \theta)$

9.3 Effect of θ on $f(\lambda_2)$, $\Lambda_2 = U_1U_2$, $(U_1, U_2) \sim BB_1^{I}(2, 2, 2; \theta)$

9.4 Effect of $\boldsymbol{\Theta}$ on $f(\lambda_2)$, $\Lambda_2 = |\boldsymbol{U}_1\boldsymbol{U}_2|$, $(\boldsymbol{U}_1, \boldsymbol{U}_2) \sim BB_2^{I}(2, 2, 2; \boldsymbol{\Theta})$, $\boldsymbol{\Theta} = \theta\boldsymbol{I}_2$

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