Appendix 1. Sobolev Spaces

The space $\mathcal{L}^2(\Omega)$

Consider an open subset Ω of $I\!R^n$. The space $\mathcal{L}^2(\Omega)$ consists of functions f such that f^2 is Lebesgue integrable on Ω. The first result is well known.

Theorem 1

The space $\mathcal{L}^2(\Omega)$ is a Hilbert space with **inner product**

$$(f,g) = \int_\Omega fg = \int_\Omega fg\ d\mu$$

where μ is the n-dimensional Lebesgue measure.

Theorem 2

The space $\mathcal{L}^2(\Omega)$ is separable (See [Ad, Th 2.15, p 28]).

Theorem 3

$C_0^\infty(\Omega)$ is dense in $\mathcal{L}^2(\Omega)$ (See [Ad, Th 2.13, p 28]).

The one-dimensional case

Suppose Ω is a **bounded open** interval. The **Sobolev spaces** $H^m(\Omega)$ are subspaces of functions in $\mathcal{L}^2(\Omega)$ with weak derivatives up to order m in $\mathcal{L}^2(\Omega)$.

Definition

For f and g in $H^m(\Omega)$,

$$[f,g]_m = (f^{(m)}, g^{(m)}) \quad \text{for} \quad m = 0,\ 1,\ \dots$$

For $m \geq 1$, the bilinear form $[\cdot,\cdot]_m$ has all the properties of an inner product except that there exist functions $f \neq 0$ such that $[f,f]_m = 0$.

Definition

For f in $H^m(\Omega)$,

$$|f|_m = \sqrt{[f,f]_m} \quad \text{for} \quad m = 0,\ 1,\ \dots$$

The function $|\cdot|_m$ is a semi-norm for $m \geq 1$.

The two-dimensional case

Suppose Ω is a **bounded open convex** subset of $I\!R^2$. The **Sobolev spaces** $H^m(\Omega)$ are subspaces of functions in $\mathcal{L}^2(\Omega)$ with weak partial derivatives up to order m in $\mathcal{L}^2(\Omega)$.

Remark

It is not necessary to require that Ω be convex, but it is sufficient for our purpose. In the theory it is usually assumed that Ω is star shaped or has the cone property.

Definition

For f and g in $H^m(\Omega)$,

$$[f,g]_m = \sum_{i+j=m} (\partial_1^i \partial_2^j f, \partial_1^i \partial_2^j g) \quad \text{for} \quad m = 0,\, 1,\, \ldots$$

For $m \geq 1$ the bilinear form $[\cdot,\cdot]_m$ has all the properties of an inner product except that there exist functions $f \neq 0$ such that $[f,f]_m = 0$.

Definition

For f in $H^m(\Omega)$,

$$|f|_m = \sqrt{[f,f]_m} \quad \text{for} \quad m = 0,\, 1,\, \ldots$$

The function $|\cdot|_m$ is a semi-norm for $m \geq 1$.

The boundary

Recall that a curve is called **smooth** if its parametrization has a continuous derivative. The boundary of Ω is called **piecewise smooth** if it consists of a finite number of smooth curves.

For a vector valued function r such that $r_i \in C^1[a,b]$ for $i = 1,\, 2$, the range $\boldsymbol{C}$ of r defines a smooth curve in the plane.

Suppose that $\boldsymbol{C}$ is a part of the boundary of Ω. A function f is Lebesgue integrable on $\boldsymbol{C}$ if $f \circ r \sqrt{(r_1')^2 + (r_2')^2}$ is Lebesgue integrable on the interval $[a,b]$.

A function f is in $\mathcal{L}^2(\boldsymbol{C})$ if f^2 is Lebesgue integrable over $\boldsymbol{C}$. The inner product for $\mathcal{L}^2(\boldsymbol{C})$ is defined by

$$(f,g)_{\boldsymbol{C}} = \int_{\boldsymbol{C}} fg \; ds = \int_a^b (f \circ r)\,(g \circ r)\,\sqrt{(r_1')^2 + (r_2')^2}\;.$$

When necessary, we use the **notation** $(f,g)_\Omega$ and $(f,g)_\Gamma$ to avoid confusion.

Sobolev spaces of vector valued functions

Definition

$u \in \mathcal{L}^2(\Omega)^2$ if $u_i \in \mathcal{L}^2(\Omega)$ for $i = 1, 2$.

$u \in \mathcal{L}^2(\Gamma)^2$ if $u_i \in \mathcal{L}^2(\Gamma)$ for $i = 1, 2$.

$u \in H^k\Omega)^2$ if $u_i \in H^k(\Omega)$ for $i = 1, 2$.

$[u, v]_{m,2} = [u_1, v_1]_m + [u_2, v_2]_m$ for $u \in \mathcal{L}^2(\Omega)^2$ and $v \in \mathcal{L}^2(\Omega)^2$.

$|u|_{m,2} = \sqrt{[u, u]_{m,2}}$ for $u \in \mathcal{L}^2(\Omega)^2$.

The function $|\cdot|_{m,2}$ is a semi-norm for $m \geq 1$.

When we need to distinguish between domains, we will use superscripts Ω and Γ in the cases of a double subscript, e.g. $\|\cdot\|^{\Omega}_{m,2}$ and $\|\cdot\|^{\Gamma}_{m,2}$.

General definitions and results

Suppose Ω is a **bounded open interval** or a **bounded open convex** subset of $I\!R^2$.

Notation

$H^0(\Omega) = \mathcal{L}^2(\Omega)$ and $H^0(\Omega)^2 = \mathcal{L}^2(\Omega)^2$.

Definition

The inner product for $H^m(\Omega)$ is defined by

$$(f, g)_m = \sum_{k=0}^{m} [f, g]_k \quad \text{for} \quad m = 0, 1, \ldots$$

Definition

The norm for $H^m(\Omega)$ is defined by

$$\|f\|_m = \sqrt{(f,g)_m} \quad \text{for} \quad m = 0,\, 1,\, \ldots$$

Definition

The inner product for $H^m(\Omega)^2$ is defined by

$$(f,g)_{m,2} = \sum_{k=0}^{m} [f,g]_{k,2} \quad \text{for} \quad m = 0,\, 1,\, \ldots$$

Definition

The norm for $H^m(\Omega)^2$ is defined by

$$\|f\|_{m,2} = \sqrt{(f,g)_{m,2}} \quad \text{for} \quad m = 0,\, 1,\, \ldots$$

Theorem 4

The space $H^m(\Omega)$ is complete (See [Ad, Th 3.2, p 45]).

Theorem 5

$C^m(\bar{\Omega})$ is dense in $H^m(\Omega)$ with respect to the norm of $H^m(\Omega)$.
(See [OR, Th 2.10, p 53].)

Theorem 6

The space $H^m(\Omega)$ is separable (See [Ad, Th 3.5, p 47]).

Theorem 7 (Rellich)

For m any nonnegative integer, the embedding of $H^{m+1}(\Omega)$ into $H^m(\Omega)$ is compact (See [Ad, Th 6.2, p 144]).

Notation

$$\begin{aligned}\partial^\alpha &= \partial_1^{\alpha_1}\partial_2^{\alpha_2}\dots\partial_n^{\alpha_n}\,, \quad \text{where}\\ |\alpha| &= \alpha_1 + \alpha_2 \cdots \alpha_n\end{aligned}$$

and $|\alpha|$ denotes the order of the derivative.

Theorem 8 (Sobolev's lemma)

Let m be any nonnegative integer. If $u \in H^p(\Omega)$ where $p > n/2$, then $u \in C^m(\bar{\Omega})$ and

$$\|\partial^\alpha u\|_{\sup} \le \|u\|_p \quad \text{for} \quad |\alpha| \le m.$$

(See [OR, Th 3.10, p 80].)

Remarks

1. Theorems 4 to 8 are also true for vector valued functions. The proofs are all trivial.
2. When we need to distinguish between different domains, say Ω and Γ, they will appear as superscripts, for instance $\|\cdot\|_k^\Omega$ and $(f,g)_{m,2}^\Gamma$.

Appendix 2. Inequalities

The one-dimensional case

Proposition 1

Consider any $u \in C^1[0,1]$. For any two points x and y in $[0,1]$,

$$|u(x)| \le \|u'\| + |u(y)|.$$

Proof

Assuming that $x > y$ (without loss of generality), we have

$$u(x) = \int_y^x u' + u(y).$$

But $|\int_y^x f| \le \|f\|$ for any $f \in \mathcal{L}^2(0,1)$. This follows from the Cauchy-Schwartz inequality

$$\left(\int_y^x fg\right)^2 \le \left(\int_y^x f^2\right)\left(\int_y^x g^2\right)$$

by choosing $g = 1$. The rest is obvious.

Theorem 1

For any $u \in C^1[0,1]$ with a zero in $[0,1]$ we have

$$\|u\| \le \|u'\|.$$

Proof

Suppose $u(y) = 0$, then $|u(x)| \leq \|u'\|$ by Proposition 1.

Hence $\|u\|_{sup} \leq \|u'\|$. The rest is obvious since $\|u\| \leq \|u\|_{sup}$.

Proposition 2

For any $u \in C^1[0,1]$, $\quad |u(0)| \leq \sqrt{2}\ \|u\|_1$.

Proof

Let $g(x) = 1 - x$ and $v = gu$ and consider the fact that

$$u(0) = v(0) = -\int_0^1 v' + v(1)\,.$$

Since $v(1) = 0$,

$$|u(0)| = \left|\int_0^1 (u'g + ug')\right| \leq \|u'\|\|g\| + \|u\|\|g'\| \leq \|u'\| + \|u\|.$$

Using the inequality $(a+b)^2 \leq 2a^2 + 2b^2$, it follows that

$$|u(0)|^2 \leq 2\|u'\|^2 + 2\|u\|^2\,.$$

The two-dimensional case

Suppose Ω is a bounded open convex subset of $I\!R^2$ with a piecewise smooth boundary. The following result is referred to as the Poincare-Friedrichs inequality or Friedrichs's inequality or Poincare's inequality.

Theorem 2

Suppose Σ is a part of the boundary of Ω with nonzero length. Denote the set

$$\{u \in C^1(\bar{\Omega}) \,\big|\, u = 0 \text{ on } \Sigma\}$$

by $F(\Omega)$. There exists a constant c_F such that, for each $u \in F(\Omega)$,

$$\|u\| \leq c_F |u|_1.$$

Proof

See e.g. [Br, p 30].

Corollary

Suppose Σ_1 and Σ_2 are parts of the boundary of Ω with nonzero length. Denote the set

$$\{u \in C^1(\bar{\Omega})^2 \,\big|\, u_1 = 0 \text{ on } \Sigma_1 \text{ and } u_2 = 0 \text{ on } \Sigma_2\}$$

by $F(\Omega)^2$. There exists a constant c_F such that for each $u \in F(\Omega)^2$,

$$\|u\|_{0,2} \leq c_F |u|_{1,2}\,.$$

Note that Σ_1 and Σ_2 may overlap and even be equal.

Theorem 3 (Korn's inequality)

Suppose b_B is the bilinear form for the Reissner-Mindlin plate. There exists a constant c_Ω such that

$$|u|_{1,2}^2 \leq c_\Omega\, b_B(u,u)$$

for each $u \in V$.

Proof

See e.g. [Br, p 288-289].

Appendix 3. Trace

The one-dimensional case

Recall that for each $u \in H^1(0,1)$, there exists a sequence $\{u_n\} \subset C^1[0,1]$ such that $\|u_n - u\|_1 \to 0$ as $n \to \infty$.

Theorem 1

For each $u \in H^1(0,1)$, there exists a unique real number γu with the following property: For each sequence $\{u_n\} \subset C^1[0,1]$ such that $\|u_n - u\|_1 \to 0$ as $n \to \infty$, we have

$$\lim_{n\to\infty} u_n(0) = \gamma u.$$

Proof

Due to Proposition 2 in Appendix 2, $\lim\limits_{n\to\infty} u_n(0)$ exists for each sequence $\{u_n\} \subset C^1[0,1]$ such that $\|u_n - u\|_1 \to 0$ as $n \to \infty$. Also due to this proposition, the limit is independent of the choice of the sequence $\{u_n\}$.

Theorem 2

The mapping γ is linear and bounded. In fact,

$$|\gamma u| \leq \sqrt{2}\, \|u\|_1 .$$

Proof

The linearity follows from the properties of limits and the estimate from Proposition 2 in Appendix 2 by considering the limits.

Remark

The mapping γ is a bounded linear functional.

Theorem 3

For any $u \in H^1(0,1)$,

$$\|u\| \leq \|u'\| + |\gamma u|.$$

Proof

Consider any $u \in C^1[0,1]$. Proposition 1 in Appendix 2 implies that

$$\|u\|_{sup} \leq \|u'\| + |u(0)|.$$

Consequently,

$$\|u\| \leq \|u'\| + |\gamma u|.$$

The same inequality holds for each $u \in H^1(0,1)$ since $C^1[0,1]$ is dense in $H^1(0,1)$.

The two-dimensional case

Definition (Trace operator γ)

For $u \in C(\bar{\Omega})$, the function γu is the restriction of the function u to Γ.

Theorem 4

The trace operator γ can be extended to a bounded linear operator mapping $H^1(\Omega)$ onto $L^2(\Gamma)$ and $\|\gamma u\|_\Gamma \leq K\|u\|_1^\Omega$.

Proof

This result is a special case of results in [OR, p 141-142].

Definition

For $u \in H^1(\Omega)^2$, we define γu by

$$\gamma u = \langle \gamma u_1, \gamma u_2 \rangle.$$

Theorem 5

Suppose Ω is the open rectangle $0 < x_1 < 1,\ 0 < x_2 < a$. Γ is the side where $x_2 = 0$ and $\gamma_0 u = u(\cdot, 0)$. Then there exists a constant K such that

$$\|u\|_\Omega \leq K|u|_1^\Omega + K\|\gamma_0 u\|_\Gamma$$

for all $u \in H^1(\Omega)$.

Proof

Proposition 1 Appendix 2 implies that for each $x_2 \in [0, a]$,

$$|u(x)|^2 \leq 2a^2 \int_0^a \big[\partial_2 u(x_1, \cdot)\big]^2 + 2\big[u(x_1, 0)\big]^2 .$$

Therefore

$$\int_0^a \big[u(x_1, \cdot)\big]^2 \leq 2a^2 \int_0^a \big[\partial_2 u(x_1, \cdot)\big]^2 + 2\big[u(x_1, 0)\big]^2 .$$

Integration with respect to x_1 yields

$$\|u\|_\Omega^2 \leq 2a^2 \|\partial_2 u\|_\Omega^2 + 2\|\gamma_0 u\|_\Gamma^2 .$$

The result follows.

Appendix 4. The spaces $C^k(J;Y)$

Consider $J = (a,b)$ or $J = [a,b)$. Let Y be any Banach space and consider a function u with values in Y. Let t be any interior point of J.

Definition (Derivative)

Suppose there exists a $v \in Y$ such that

$$\lim_{h \to 0} \left\| h^{-1}\big(u(t+h) - u(t)\big) - v \right\|_Y = 0,$$

then v is the derivative of u at t. We write $u'(t)$ for the derivative.

It is obvious how to adapt the definition for the case $t = a$. The derivative (function) u' and the second order derivative u'' are defined in the usual way.

Notation

$C^k\big([0,\infty);Y\big)$ and $C^k\big((0,\infty);Y\big)$

Appendix 5. Proofs

All the results and proofs are from [V4] and presented here for completeness.

Plate-beam system

Proposition 1

There exists a constant K_T such that

$$\|w\|^2 + \|\phi\|^2 \leq K_T \left(\|\phi'\|^2 + \|w' - \phi\|^2\right)$$

for each $(w, \phi) \in T(0,1) \times C^1[0,1]$

Proof

Suppose it is not true. Then there exists a sequence $\{(w_n, \phi_n)\}$ such that

$$\|w_n\|^2 + \|\phi_n\|^2 = 1,$$

while

$$\|\phi_n'\|^2 + \|w_n' - \phi_n\|^2 \to 0 \quad \text{as } n \to \infty.$$

- We prove first that for n sufficiently large, ϕ_n does not have a zero.

Suppose ϕ_n does have a zero. Then $\|\phi_n\| \leq \|\phi_n'\|$. We also have $\|w_n\| \leq \|w_n'\|$ since $w_n(0) = 0$. Consequently

$$\|w_n\| \leq \|w_n'\| \leq \|w_n' - \phi_n\| + \|\phi_n\| \leq \|w_n' - \phi_n\| + \|\phi_n'\|.$$

This implies that

$$\|w_n\| + \|\phi_n\| \leq \|w_n' - \phi_n\| + \|\phi_n'\| \quad \text{for each } n.$$

This is a contradiction.

• We now show that $\|\phi_n\| > 1/2$ for n sufficiently large.

If it is not true, then

$$\|w_n\| \leq \|w_n'\| \leq \|w_n' - \phi_n\| + \|\phi_n\| \leq 1/4 + 1/2 < 3/4$$

for n sufficiently large. Consequently

$$\|w_n\|^2 + \|\phi_n\|^2 < 9/16 + 1/4 = 13/16 < 1.$$

This is a contradiction.

• Next we show that $\int_0^1 \phi_n > 1/10$ for n sufficiently large.

We may assume without loss of generality that $\phi_n > 0$. Writing ϕ for ϕ_n, we have

$$\phi_{\max} - \phi_{\min} \leq \|\phi'\| \leq 1/20 \quad \text{and} \quad \phi_{\max} \leq 21/20$$

for n sufficiently large. Consequently

$$\phi_{\min}^2 = \int_0^1 \phi_{\min}^2 = \int_0^1 [\phi_{\min}^2 - \phi^2] + \int_0^1 \phi^2$$

$$= \int_0^1 \phi^2 - \int_0^1 [\phi^2 - \phi_{\min}^2] \geq 1/4 - 1/10 > 1/10.$$

Therefore

$$\int_0^1 \phi \geq \int_0^1 \phi_{\min} > 1/10.$$

• $\int_0^1 w_n' > 0$ for n sufficiently large.

$$\left| \int_0^1 w_n' - \int_0^1 \phi_n \right| \leq \int_0^1 |w_n' - \phi_n| \leq \|w' - \phi\| \leq 1/20.$$

• Finally we obtain a contradiction.

Since $\int_0^1 w_n' > 0$ and $w_n(0) = 0$, we have $w_n(1) > 0$ which is a contradiction. $\square$

Corollary

There exists a constant K such that

$$\|w\|_1^2 + \|\phi\|_1^2 \leq K\left(\|\phi'\|^2 + \|w' - \phi\|^2\right) .$$

Proposition 2

$$\|w\|_\Omega^2 + \left(\|\boldsymbol{\psi}\|_{0,2}^\Omega\right)^2 \leq K\left(\|\nabla w + \boldsymbol{\psi}\|_{0,2}^\Omega\right)^2 + K\left(|\boldsymbol{\psi}|_{1,2}^\Omega\right)^2 + K\|\gamma_0\psi_1\|_I^2$$

for each $(w, \boldsymbol{\psi}) \in T_1(\Omega) \times T_2(\Omega)$.

Proof

Since ψ_2 is zero on a part of the boundary, we may use the Friedrichs inequality $\|\psi_2\|_\Omega \leq c_F|\psi_2|_1^\Omega$. We also use

$$\|\psi_1\|_\Omega \leq c_1|\psi_1|_1^\Omega + c_1\|\gamma\psi_1\|_I$$

(see Theorem 5 Appendix 3). Combining the two inequalities, we have

$$\left(\|\boldsymbol{\psi}\|_{0,2}^\Omega\right)^2 \leq c_2\left(|\boldsymbol{\psi}|_{1,2}^\Omega\right)^2 + c_2\|\gamma\psi_1\|_I^2.$$

Since w is zero on a part of the boundary, $\|w\|_\Omega \leq c_F|w|_1^\Omega$, using the Friedrichs inequality. Therefore

$$\|w\|_\Omega \leq c_F|w|_1^\Omega = c_F\|\nabla w\|_{0,2}^\Omega \leq c_F\|\nabla w + \boldsymbol{\psi}\|_{0,2}^\Omega + c_F\|\boldsymbol{\psi}\|_{0,2}^\Omega .$$

Consequently

$$\begin{aligned}
\|w\|_\Omega^2 + \left(\|\boldsymbol{\psi}\|_{0,2}^\Omega\right)^2 &\leq K\left(\|\nabla w + \boldsymbol{\psi}\|_{0,2}^\Omega\right)^2 + K\left(\|\boldsymbol{\psi}\|_{0,2}^\Omega\right)^2 \\
&\leq K\left(\|\nabla w + \boldsymbol{\psi}\|_{0,2}^\Omega\right)^2 + K\left(|\boldsymbol{\psi}|_{1,2}^\Omega\right)^2 + K\|\gamma_0\psi_1\|_I^2 ,
\end{aligned}$$

where K is a generic constant depending on c_1 and c_F .

Corollary

$$\left(\|w\|_1^\Omega\right)^2 + \left(\|\boldsymbol{\psi}\|_{1,2}^\Omega\right)^2 \leq K\left(\|\nabla w + \boldsymbol{\psi}\|_{0,2}^\Omega\right)^2 + K\left(|\boldsymbol{\psi}|_{1,2}^\Omega\right)^2 + K\|\gamma_0\psi_1\|_I^2 .$$

Theorem 1

The inertia space X is a separable Hilbert space and V is dense in X.

Proof

Since $C_0^\infty(\Omega)$ is dense in $\mathcal{L}^2(\Omega)$, we have that $T_1(\Omega)$ is dense in $\mathcal{L}^2(\Omega)$ and $T_2(\Omega)$ is dense in $\mathcal{L}^2(\Omega)^2$. Also $T(I)$ is dense in $\mathcal{L}^2(I)$. We conclude that the space T is dense in the space X equipped with the inner product

$$(u,v)_{\mathcal{L}^2} = (u_1,v_1)_\Omega + (u_2,v_2)_{0,2}^\Omega + \sum_{j=3}^{6}(u_j,v_j)_I\,.$$

The norms $\|u\|_X$ and $\|u\|_{\mathcal{L}^2}$, where

$$\|u\|_X^2 = c(u,u) \quad \text{and} \quad \|u\|_{\mathcal{L}^2}^2 = (u,u)_{\mathcal{L}^2}\,,$$

are equivalent. Therefor T is a dense subset of X with respect to the inertia norm and $T \subset V \subset X$.

Theorem 2 (Korn's inequality)

$$b_B(u_2,u_2) \geq K|u_2|_{1,2}^2 \quad \text{for each } u \in T.$$

Theorem 3

There exist constants c_1 and c_2 such that

$$\|u\|_X \leq c_1\|u\|_{H^1} \leq c_2\|u\|_V$$

for each $u \in T$.

Proof

From the corollary to Proposition 1, we have

$$\|u_3\|_1^2 + \|u_4\|_1^2 + \|u_5\|_1^2 + \|u_6\|_1^2 \leq C_\Gamma b_\Gamma(u,u).$$

Rewrite the corollary to Proposition 2.

$$\left(\|u_1\|_1^\Omega\right)^2 + \left(\|u_2\|_{1,2}^\Omega\right)^2 \leq K\left(\|\nabla u_1 + u_2\|_{0,2}^\Omega\right)^2 + K\left(|u_2|_{1,2}^\Omega\right)^2 + K\|\gamma_0 u_{21}\|_I^2 .$$

Combining the results, we have

$$\|u\|_{H^1}^2 \quad \leq \quad C\left(\|\nabla u_1 + u_2\|_{0,2}^\Omega\right)^2 + C\left(|u_2|_{1,2}^\Omega\right)^2 + Cb_\Gamma(u,u) ,$$

using Proposition 1 again. Now use Korn's inequality.

Nonmodal damping

Theorem 1

For each $y \in H$ there exists a unique $x \in H$ such that

$$\begin{aligned} x_2 &= y_1 \\ b(x_1, v) + a(x_2, v) &= -c(y_2, v) \quad \text{for each } v \in V. \end{aligned}$$

Proof

Let

$$g(v) = -a(y_1, v) - c(y_2, v) \quad \text{for each } v \in V,$$

then g is clearly a linear functional on V. Furthermore

$$|g(v)| \leq K\|y_1\|_V\|v\|_V + c\|y_2\|_X\|v\|_X \quad \text{for each } v \in V,$$

showing that g is bounded. The result follows from the well known theorem of Riesz.

Theorem 2

Λ is bounded.

Proof

Consider any $y \in H$ and suppose $x = \Lambda y$.

$$\begin{aligned} x_2 &= y_1 \\ b(x_1, v) + a(x_2, v) &= -c(y_2, v) \quad \text{for each } v \in V. \end{aligned}$$

It follows that

$$\|x_2\|_X \leq K\|x_2\|_V = \|y_1\|_V$$

and

$$\begin{aligned} \|x_1\|_V^2 &= b(x_1, x_1) \\ &\leq |a(x_2, x_1)| + |c(y_2, x_1)| \\ &\leq K\|x_2\|_V \, \|x_1\|_V + K\|y_2\|_X \, \|x_1\|_V \end{aligned}$$

Consequently

$$\|x_1\|_V \leq K\|y_1\|_V + K\|y_2\|_X \ \leq K\|y\|_H.$$

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