Chapter 8

Plate-beam system

8.1 Introduction

The differences between the Euler-Bernoulli, Rayleigh and Timoshenko beam models can be investigated by comparing the natural frequencies predicted by the different models. It is well known that in general, the shear corrections introduced by the Timoshenko model are larger than the rotary inertia corrections of the Rayleigh model. For the first (smallest) eigenvalue these corrections are small, but for the higher eigenvalues they are of significance. See Section 8.1.1 for a numerical example.

The same tendency is seen when we compare the eigenvalues for the classical plate models, i.e. the Kirchhoff model with and without rotary inertia, with those of the Reissner-Mindlin plate model. See Section 8.1.2 for a numerical example.

8.1.1 Pinned-pinned beam

For a pinned-pinned beam the eigenvalues and eigenfunctions for the Euler-Bernoulli model, the Rayleigh model and the Timoshenko model can be obtained in closed form.

Euler-Bernoulli model

The eigenvalues are

$$\lambda = \frac{k^4\pi^4}{\beta_b}, \quad k = 1,\, 2,\, \ldots,$$

with associated eigenfunctions

$$w(x) = \sin k\pi x.$$

Rayleigh model

The eigenvalues are

$$\lambda = \frac{k^4\pi^4}{\beta_b(1+\alpha_b^{-1}k^2\pi^2)}, \quad k = 1,\, 2,\, \ldots,$$

with associated eigenfunctions

$$w(x) = \sin k\pi x.$$

Timoshenko model

The eigenvalues are the roots of

$$\lambda^2 - \left(\alpha_b + \left(1 + \frac{\alpha_b}{\beta_b}\right) k^2\pi^2\right)\lambda + \frac{\alpha_b}{\beta_b}\, k^4\pi^4 = 0 \quad \text{for } k = 1,\, 2,\, \ldots$$

For each k, two eigenvalues λ_k and λ_k^* are obtained. If λ_k^* denotes the larger one of the two, it is known that $\lambda_k^* > \alpha_b$ for all k. In the numerical examples the first few eigenvalues are considered and the λ_k^* will not feature. The associated eigenfunction pairs are

$$w_k(x) = \sin k\pi x, \quad \phi_k(x) = \frac{k^2\pi^2 - \lambda_k}{k\pi}\, \cos k\pi x \quad \text{and}$$

$$w_k^*(x) = \sin k\pi x, \quad \phi_k^*(x) = \frac{k^2\pi^2 - \lambda_k^*}{k\pi}\, \cos k\pi x.$$

Comparison of eigenvalues

As an example we present some numerical results for a pinned-pinned beam with a length to depth ratio of 20:1 and a square profile, i.e. $\alpha_b = 4800$. We choose $\beta_b = 0.25$. The percentage differences for the first five eigenvalues are shown in Table 1, where $\lambda_i^{(EB)}$, $\lambda_i^{(R)}$ and $\lambda_i^{(T)}$ denote the i-th eigenvalue for the Euler-Bernoulli, Rayleigh and Timoshenko models respectively. The percentage differences are calculated with respect to the Euler-Bernoulli eigenvalues. Clearly, the shear corrections are larger than the corrections due to rotary inertia and the corrections for larger eigenvalues are significant. For a "shorter" beam (smaller α_b) these corrections are even larger.

Table 1: Corrections for a pinned-pinned beam

i	Rotary inertia $\frac{\lambda_i^{(EB)}-\lambda_i^{(R)}}{\lambda_i^{(EB)}}$	Shear $\frac{\lambda_i^{(EB)}-\lambda_i^{(T)}}{\lambda_i^{(EB)}}$
1	0.21 %	1.02%
2	0.82 %	3.93%
3	1.82 %	8.36%
4	3.19 %	13.85%
5	4.89 %	19.01%

8.1.2 Rigidly supported plate

For a plate supported rigidly on all four sides, the eigenvalues and eigenfunctions can be determined in closed form for all the different plate models, i.e. the Kirchhoff model with and without rotary inertia and the Reissner-Mindlin model.

Kirchhoff model without rotary inertia

The eigenvalues are

$$\lambda = \frac{\pi^4(n^2+m^2)^2}{\beta_p(1-\nu_p^2)h_p}, \quad n = 1,\ 2,\ \ldots \quad \text{and} \quad m = 1,\ 2,\ \ldots$$

with associated eigenfunctions

$$w(x_1, x_2) = \sin(n\pi x_1)\,\sin(m\pi x_2).$$

Kirchhoff model with rotary inertia

The eigenvalues are

$$\lambda = \frac{\pi^4(n^2+m^2)^2}{\beta_p(1-\nu_p^2)(h_p + I_p\pi^2(n^2+m^2))}, \quad n = 1,\ 2,\ \ldots \quad \text{and} \quad m = 1,\ 2,\ \ldots$$

with associated eigenfunctions

$$w(x_1, x_2) = \sin(n\pi x_1)\,\sin(m\pi x_2).$$

Reissner-Mindlin model

The eigenvalues are the solutions of the quadratic equation

$$r\lambda^2 - (1 + (r+\gamma)f)\lambda + \gamma f^2 = 0.$$

In this equation

$$r = \frac{h_p^2}{12} \quad \text{and} \quad \gamma = \frac{1}{2\beta_p(1-\nu_p^2)h_p}.$$

A sequence of values for f are used, each yielding two eigenvalues.

$$f = \pi^2(n^2+m^2) \quad \text{for} \quad n = 1,\ 2,\ \ldots \quad \text{and} \quad m = 1,\ 2,\ \ldots.$$

The associated eigenfunction pairs are of the form

$$\begin{aligned} w(x_1, x_2) &= \sin(n\pi x_1)\,\sin(m\pi x_2), \\ \psi_1(x_1, x_2) &= A_{nm}\cos(n\pi x_1)\,\sin(m\pi x_2), \\ \psi_2(x_1, x_2) &= B_{nm}\sin(n\pi x_1)\,\cos(m\pi x_2). \end{aligned}$$

Since these formulae will not be used in our calculations, we do not display the closed form expressions for A_{nm} and B_{nm}.

Comparison of eigenvalues

For the numerical calculations we use a square plate with dimensionless thickness $h_p = 0.05$, Poisson's ratio $\nu_p = 0.3$ and shear correction factor $\kappa_p^2 = 5/6$. The first six eigenvalues for the three different models are given in Table 2. Note that due to the spatial symmetry of the problem, repeated eigenvalues occur, e.g. Eigenvalues 2 and 3, and also, 5 and 6.

Table 2: Eigenvalues for rigidly supported plate

i	Kirchhoff without rotary inertia	Kirchhoff with rotary inertia	Reissner-Mindlin
1	0.2783	0.2772	0.2733
2	1.7394	1.7217	1.6643
3	1.7394	1.7217	1.6643
4	4.4530	4.3809	4.1540
5	6.9578	6.8176	6.3849
6	6.9578	6.8176	6.3849

The percentage differences for the first six eigenvalues are shown in Table 3, where $\lambda_i^{(K)}$, $\lambda_i^{(KR)}$ and $\lambda_i^{(RM)}$ denote the i-th eigenvalue for the Kirchhoff model without rotary inertia, the Kirchhoff model with rotary inertia and the Reissner-Mindlin model respectively. The percentage differences are calculated with respect to the Kirchhoff eigenvalues. It is clear that the corrections due to shear are larger than the corrections due to rotary inertia.

Table 3: Corrections for rigidly supported plate

i	Rotary inertia $\frac{\lambda_i^{(K)}-\lambda_i^{(KR)}}{\lambda_i^{(K)}}$	Shear $\frac{\lambda_i^{(K)}-\lambda_i^{(RM)}}{\lambda_i^{(K)}}$
1	0.41 %	1.78 %
2	1.02 %	4.32 %
3	1.02 %	4.32 %
4	1.62 %	6.71 %
5	2.01 %	8.23 %
6	2.01 %	8.23 %

8.1.3 Plate-beam system

In [ZVGV3] a plate-beam system consisting of the classical plate model and the Euler-Bernoulli beam model is investigated. It is shown that introducing rotary inertia into the model does not causes a significant change in the eigenvalues. It is also shown that when the ratio d_b/h_p is increased, the eigenvalues of the plate-beam system tend to those of the rigidly supported plate.

An initial aim is to compare the eigenvalues of the Reissner-Mindlin-Timoshenko (RMT) plate-beam system with those of the Kirchhoff-Euler-Bernoulli (KEB) plate-beam system. We will also consider the asymptotic behaviour of the eigenvalues of the RMT system when ratio d_b/h_p is increased. Some interesting phenomena present themselves and will be discussed in Section 8.5.5.

8.2 The eigenvalue problems

In this section the variational forms in Chapter 3 for the different plate-beam systems are used to derive the associated eigenvalue problems.

8.2.1 Reissner-Mindlin-Timoshenko plate-beam system

As explained in Section 3.9, if $\tilde{w}(\boldsymbol{x},t) = T(t)w(\boldsymbol{x})$ and $\tilde{\boldsymbol{\psi}}(\boldsymbol{x},t) = T(t)\boldsymbol{\psi}(\boldsymbol{x})$ is considered as a possible solution for Equations (3.6.6) and (3.6.12), the following eigenvalue problem is obtained.

Problem RMT

$$\begin{aligned} & \lambda \left\{ h_p \iint_\Omega wv\, dA + \eta_1 \left[\int_0^1 wv\, dx_1 \right]_{x_2=0} + \eta_1 \left[\int_0^1 wv\, dx_1 \right]_{x_2=a} \right\} \\ = \; & h_p \iint_\Omega (\nabla w + \boldsymbol{\psi}) \cdot \nabla v\, dA + \eta_2 \left[\int_0^1 (\partial_1 w + \psi_1)\partial_1 v\, dx_1 \right]_{x_2=0} \\ & + \eta_2 \left[\int_0^1 (\partial_1 w + \psi_1)\partial_1 v\, dx_1 \right]_{x_2=a} \qquad (8.2.1) \end{aligned}$$

for all v in $T_1(\Omega)$ and

$$\begin{aligned}
&\lambda\left\{I_p \iint_\Omega \boldsymbol{\psi}\cdot\boldsymbol{\phi}\,dA + \frac{\eta_1}{\alpha_b}\left[\int_0^1 \psi_1\phi_1\,dx_1\right]_{x_2=0} + \frac{\eta_1}{\alpha_b}\left[\int_0^1 \psi_1\phi_1\,dx_1\right]_{x_2=a}\right\}\\
= \;& b_B(\boldsymbol{\psi},\boldsymbol{\phi}) + h_p \iint_\Omega (\nabla w + \boldsymbol{\psi})\cdot\boldsymbol{\phi}\,dA\\
&+\frac{\eta_2}{\beta_b}\left[\int_0^1 \partial_1\psi_1\partial_1\phi_1\,dx_1\right]_{x_2=0} + \frac{\eta_2}{\beta_b}\left[\int_0^1 \partial_1\psi_1\partial_1\phi_1\,dx_1\right]_{x_2=a}\\
&+\eta_2\left[\int_0^1 (\partial_1 w + \psi_1)\phi_1\,dx_1\right]_{x_2=0} + \eta_2\left[\int_0^1 (\partial_1 w + \psi_1)\phi_1\,dx_1\right]_{x_2=a} \qquad (8.2.2)
\end{aligned}$$

for all $\boldsymbol{\phi}$ in $T_2(\Omega)$.

($T_1(\Omega)$, $T_2(\Omega)$ and b_B are defined in Section 3.6.1.)

8.2.2 Kirchhoff-Rayleigh plate-beam system

If $\tilde{w}(\boldsymbol{x},t) = T(t)w(\boldsymbol{x})$ is considered as a possible solution for Equation (3.6.15), the following eigenvalue problem is obtained.

Problem KR

$$\begin{aligned}
&\lambda\left\{h_p \iint_\Omega wv\,dA + I_p \iint_\Omega (\nabla w)\cdot\nabla v\,dA\right\}\\
&+\lambda\left\{\eta_1\left[\int_0^1 wv\,dx_1\right]_{x_2=0} + \eta_1\left[\int_0^1 wv\,dx_1\right]_{x_2=a}\right\}\\
&+\lambda\left\{\frac{\eta_1}{\alpha_b}\left[\int_0^1 (\partial_1 w)v\,dx_1\right]_{x_2=0} + \frac{\eta_1}{\alpha_b}\left[\int_0^1 (\partial_1 w)v\,dx_1\right]_{x_2=a}\right\}\\
= \;& b_B(w,v) + \frac{\eta_2}{\beta_b}\left[\int_0^1 \partial_1^2 w\partial_1^2 v\,dx_1\right]_{x_2=0} + \frac{\eta_2}{\beta_b}\left[\int_0^1 \partial_1^2 w\partial_1^2 v\,dx_1\right]_{x_2=a} \qquad (8.2.3)
\end{aligned}$$

for all $v \in T(\Omega)$.

($T(\Omega)$ and b_B are defined in Section 3.6.2.)

8.2.3 Kirchhoff-Euler-Bernoulli plate-beam system

The eigenvalue problem for the case where rotary inertia is ignored, is obtained by ignoring the terms containing I_p and η_1/α_b in (8.2.3). We refer to the corresponding problem as **Problem KEB**.

8.3 Galerkin approximations for the eigenvalue problems

For all three eigenvalue problems we consider an approximate solution

$$w^h(\boldsymbol{x}) = \sum_{i=1}^{N} w_i \gamma_i(\boldsymbol{x}), \quad \psi_1^h(\boldsymbol{x}) = \sum_{i=1}^{N} \psi_{1i}\gamma_i(\boldsymbol{x}) \quad \text{and} \quad \psi_2^h(\boldsymbol{x}) = \sum_{i=1}^{N} \psi_{2i}\gamma_i(\boldsymbol{x})$$

in terms of the bicubic basis functions

$$\gamma_i, \quad i = 1, 2, \ldots N,$$

where the functions ψ_1^h and ψ_2^h are only applicable for Problem RMT.

As $w^h \in T_1(\Omega)$ and $\boldsymbol{\psi}^h = [\psi_1^h \; \psi_2^h]^T \in T_2(\Omega)$, some of these coefficients will be equal to zero.

8.3.1 Galerkin approximation for Problem RMT

If $\boldsymbol{w} = [w_1 \, w_2 \, \ldots \, w_N]^T$, $\boldsymbol{\psi}_1 = [\psi_{11} \, \psi_{12} \, \ldots \, \psi_{1N}]^T$ and $\boldsymbol{\psi}_2 = [\psi_{21} \, \psi_{22} \, \ldots \, \psi_{2N}]^T$, the Galerkin approximation for Problem RMT is given by three matrix equations. These equations are obtained by choosing $v = \gamma_j$ in (8.2.1) and $\boldsymbol{\phi} = [\gamma_j \; 0]^T$ and $\boldsymbol{\phi} = [0 \; \gamma_j]^T$ in (8.2.2). Recall that $v \in T_1(\Omega)$ and $\boldsymbol{\phi} \in T_2(\Omega)$ and that only admissible basis functions should be used.

8.3.2 Galerkin approximation for Problem KEB

If $\boldsymbol{w} = [w_1 \; w_2 \; \ldots \; w_N]^T$, the Galerkin approximation for Problem KEB is given by a matrix equation. This equation is obtained by choosing $v = \gamma_j$ in (8.2.3). Recall that $v \in T(\Omega)$ and that only admissible basis functions should be used.

8.4 Matrix formulation of Galerkin approximations

The eigenvalue problem for both Problem RMT and KEB can be represented in matrix notation as

$$\mathcal{K}\boldsymbol{z} = \lambda\mathcal{M}\boldsymbol{z}.$$

The following matrices are required for defining the matrices $\mathcal{K}$ and $\mathcal{M}$ for the different eigenvalue problems.

$$I_{ij}^{\Omega 12} = \iint_\Omega \partial_1\partial_2\gamma_j\partial_1\partial_2\gamma_i \, dA \,,$$

$$J_{ij}^{\Omega 11} = \iint_\Omega \partial_1^2\gamma_j\partial_1^2\gamma_i \, dA \,, \quad J_{ij}^{\Omega 22} = \iint_\Omega \partial_2^2\gamma_j\partial_2^2\gamma_i \, dA \,, \quad J_{ij}^{\Omega 12} = \iint_\Omega \partial_1^2\gamma_j\partial_2^2\gamma_i \, dA \,,$$

$$J_{ij}^{0} = \int_0^1 \partial_1^2\gamma_j(x_1,0)\partial_1^2\gamma_i(x_1,0) \, dx_1 \,, \quad J_{ij}^{1} = \int_0^1 \partial_1^2\gamma_j(x_1,a)\partial_1^2\gamma_i(x_1,a) \, dx_1 \,,$$

$$K_{ij}^{\Omega 11} = \iint_\Omega \partial_1\gamma_j\partial_1\gamma_i \, dA \,, \quad K_{ij}^{\Omega 22} = \iint_\Omega \partial_2\gamma_j\partial_2\gamma_i \, dA \,, \quad K_{ij}^{\Omega 12} = \iint_\Omega \partial_1\gamma_j\partial_2\gamma_i \, dA \,,$$

$$K_{ij}^{0} = \int_0^1 \partial_1\gamma_j(x_1,0)\partial_1\gamma_i(x_1,0) \, dx_1 \,, \quad K_{ij}^{1} = \int_0^1 \partial_1\gamma_j(x_1,a)\partial_1\gamma_i(x_1,a) \, dx_1 \,,$$

$$L_{ij}^{\Omega 1} = \iint_\Omega \gamma_j\partial_1\gamma_i \, dA \,, \quad L_{ij}^{\Omega 2} = \iint_\Omega \gamma_j\partial_2\gamma_i \, dA \,,$$

$$L_{ij}^{0} = \int_0^1 \gamma_j(x_1,0)\partial_1\gamma_i(x_1,0) \, dx_1 \,, \quad L_{ij}^{1} = \int_0^1 \gamma_j(x_1,a)\partial_1\gamma_i(x_1,a) \, dx_1 \,,$$

$$M_{ij}^{\Omega} = \iint_\Omega \gamma_j\gamma_i \, dA \,,$$

$$M_{ij}^{0} = \int_0^1 \gamma_j(x_1,0)\gamma_i(x_1,0) \, dx_1 \,, \quad M_{ij}^{1} = \int_0^1 \gamma_j(x_1,a)\gamma_i(x_1,a) \, dx_1 .$$

8.4.1 Construction of $\mathcal{K}$ and $\mathcal{M}$ for Problem RMT

We define the following matrices which are needed to construct $\mathcal{K}$ and $\mathcal{M}$.

$$
\begin{aligned}
K_w &= h_p \left(K^{\Omega 11} + K^{\Omega 22}\right) + \eta_2 \left(K^0 + K^1\right), \\
L_1 &= h_p \, L^{\Omega 1} + \eta_2 \left(L^0 + L^1\right), \\
L_2 &= h_p \, L^{\Omega 2}, \\
K_1 &= \frac{1}{\beta_p \left(1-\nu_p^2\right)} \left(K^{\Omega 11} + \frac{1-\nu_p}{2} K^{\Omega 22}\right) + \frac{\eta_2}{\beta_b} \left(K^0 + K^1\right) \\
&\quad + h_p \, M^{\Omega} + \eta_2 \left(M^0 + M^1\right), \\
K_\nu &= \frac{1}{\beta_p \left(1-\nu_p^2\right)} \left(\nu_p \left(K^{\Omega 12}\right)^T + \frac{1-\nu_p}{2} K^{\Omega 12}\right), \\
K_2 &= \frac{1}{\beta_p \left(1-\nu_p^2\right)} \left(\frac{1-\nu_p}{2} K^{\Omega 11} + K^{\Omega 22}\right) + h_p \, M^{\Omega}, \\
M_w &= h_p \, M^{\Omega} + \eta_1 \left(M^0 + M^1\right), \\
M_1 &= I_p \, M^{\Omega} + \frac{\eta_1}{\alpha_b} \left(M^0 + M^1\right), \\
M_2 &= I_p \, M^{\Omega}.
\end{aligned}
$$

We define the matrices $\mathcal{K}^{RMT}$ and $\mathcal{M}^{RMT}$ by

$$
\mathcal{K}^{RMT} = \begin{bmatrix} K_w & L_1 & L_2 \\ L_1^T & K_1 & K_\nu \\ L_2^T & K_\nu^T & K_2 \end{bmatrix} \quad \text{and} \quad \mathcal{M}^{RMT} = \begin{bmatrix} M_w & 0 & 0 \\ 0 & M_1 & 0 \\ 0 & 0 & M_2 \end{bmatrix}.
$$

The matrices $\mathcal{K}$ and $\mathcal{M}$ that are needed for Problem RMT are found from the matrices above by omitting rows and columns according to the restrictions on the test functions.

8.4.2 Construction of $\mathcal{K}$ and $\mathcal{M}$ for Problem KEB

We define the matrices

$$\begin{aligned}
\mathcal{K}^{KEB} &= \frac{1}{\beta_p(1-\nu_p^2)}\left(J^{\Omega 11} + J^{\Omega 22} + \nu_p\,(\,J^{\Omega 12} + J^{\Omega 21}\,)\right) \\
&\quad + \frac{2}{\beta_p(1+\nu_p)}\,I^{\Omega 12} + \frac{\eta_2}{\beta_b}\left(J^0 + J^1\right), \\
\mathcal{M}^{KEB} &= h_p M^{\Omega} + \eta_1(M^0 + M^1).
\end{aligned}$$

The matrices $\mathcal{K}$ and $\mathcal{M}$ that are needed for Problem KEB are constructed from the matrices above by omitting rows and columns in accordance to the restrictions on the test functions.

8.5 Numerical results

8.5.1 Parameters

For the numerical results we consider consider a square plate and beams with a rectangular profile of thickness d and height $5d$. The dimensionless thickness d_b of the beams is denoted by $d_b = d/\ell$. For both the plate and the beams, we choose Poisson's ratio $\nu_p = \nu_b = 0.3$ and the shear correction factors $\kappa_p^2 = \kappa_b^2 = 5/6$. We also assume that the plate and the beams are made of the same isotropic material and therefore we use $G = \dfrac{E}{2(1+\nu)}$.

For this special case the dimensionless constants reduce to

$$\begin{aligned}
\eta_1 &= 5d_b^2\,, \\
\eta_2 &= 5\left(\frac{\kappa_b^2}{\kappa_p^2}\right) d_b^2\,, \\
I_p &= \frac{h_p^3}{12}\,, \\
\frac{1}{\alpha_b} &= \frac{25d_b^2}{12}\,, \\
\frac{1}{\beta_p} &= \frac{(1+\nu_p)h_p^3}{6\kappa_p^2}\,, \\
\frac{1}{\beta_b} &= \frac{25(1+\nu_b)d_b^2}{6\kappa_b^2}\,.
\end{aligned}$$

In all the numerical experiments a square plate is considered (i.e. $a = 1$) and the value of h_p is fixed at $h_p = 0.05$, while the value of d_b is varied to allow for different values of the ratio d_b/h_p.

8.5.2 Convergence

MATLAB programs have been written for calculating the eigenvalues of the RMT and KEB plate-beam systems, using the finite element method. The results of convergence tests are discussed briefly for Problem RMT. In this case $h_p = d_b = 0.05$.

In Table 4 the first ten eigenvalues of the RMT plate-beam system are listed for a 2×2, 4×4, 8×8 and a 16×16 grid. The value $\lambda_i^{(k)}$ denotes the approximation for eigenvalue i when using a $k \times k$ grid for the finite element calculations. When the grid is refined, the eigenvalues form a decreasing sequence, which is in line with the theory.

Table 4: Convergence

i	$\lambda_i^{(2)}$	$\lambda_i^{(4)}$	$\lambda_i^{(8)}$	$\lambda_i^{(16)}$
1	2.3517×10^{-1}	2.3412×10^{-1}	2.3401×10^{-1}	2.3400×10^{-1}
2	7.9829×10^{-1}	7.7665×10^{-1}	7.7474×10^{-1}	7.7443×10^{-1}
3	1.1934×10^{0}	1.1822×10^{0}	1.1790×10^{0}	1.1785×10^{0}
4	1.9352×10^{0}	1.6459×10^{0}	1.6408×10^{0}	1.6406×10^{0}
5	2.8759×10^{0}	2.4348×10^{0}	2.4271×10^{0}	2.4266×10^{0}
6	4.4995×10^{0}	3.9430×10^{0}	3.9317×10^{0}	3.9311×10^{0}
7	8.6576×10^{0}	6.4642×10^{0}	6.3653×10^{0}	6.3615×10^{0}
8	9.8681×10^{0}	7.4870×10^{0}	7.3860×10^{0}	7.3816×10^{0}
9	1.0396×10^{1}	8.7615×10^{0}	8.6805×10^{0}	8.6743×10^{0}
10	1.3119×10^{1}	1.0499×10^{1}	1.0391×10^{1}	1.0386×10^{1}

The relative errors for the first 10 eigenvalues are displayed in Table 5.

Table 5: Convergence

i	$\dfrac{\lambda_i^{(4)} - \lambda_i^{(2)}}{\lambda_i^{(4)}}$	$\dfrac{\lambda_i^{(8)} - \lambda_i^{(4)}}{\lambda_i^{(8)}}$	$\dfrac{\lambda_i^{(16)} - \lambda_i^{(8)}}{\lambda_i^{(16)}}$
1	4.4676×10^{-3}	4.5354×10^{-4}	6.5675×10^{-5}
2	2.7862×10^{-1}	2.4688×10^{-3}	3.9173×10^{-4}
3	9.4811×10^{-3}	2.6543×10^{-3}	4.4116×10^{-4}
4	1.7577×10^{-1}	3.1193×10^{-3}	1.1919×10^{-4}
5	1.8115×10^{-1}	3.1752×10^{-3}	2.2742×10^{-4}
6	1.4113×10^{-1}	2.8553×10^{-3}	1.6457×10^{-4}
7	3.3931×10^{-1}	1.5547×10^{-1}	5.9373×10^{-4}
8	3.1804×10^{-1}	1.3668×10^{-1}	6.0509×10^{-4}
9	1.8660×10^{-1}	9.3339×10^{-3}	7.1587×10^{-4}
10	2.4949×10^{-1}	1.0448×10^{-1}	4.1081×10^{-4}

We find that the first six eigenvalues, which we will consider in the following experiments, are accurate to three significant digits for a 16×16 grid.

8.5.3 Comparison of Reissner-Mindlin-Timoshenko system with Kirchhoff-Euler-Bernoulli system

In [ZVGV3], a numerical investigation of a similar plate-beam system is done for a combination of the classical plate model and the Euler-Bernoulli beam model. It was found that the inclusion of rotary inertia in the plate and beam models had little effect on the eigenvalues.

We now compare the eigenvalues for the RMT system to those of the KEB system for $d_b/h_p = 1$ and show that the shear corrections on the higher eigenvalues are of more significance than the corrections due to rotary inertia.

Table 6: Eigenvalues for plate-beam system

i	KEB	RMT	Shear correction
1	0.2413	0.2340	3.03 %
2	0.8765	0.7744	11.65 %
3	1.3715	1.1785	14.07 %
4	1.7197	1.6406	4.60 %
5	2.6642	2.4266	8.92 %
6	4.2835	3.9311	8.23 %

8.5.4 Comparison of Kirchhoff-Euler-Bernoulli system with a rigidly supported Kirchhoff plate

[ZVGV3] contains a numerical experiment where the eigenvalues of the KEB plate-beam system is compared to the eigenvalues of a rigidly supported Kirchhoff plate, for different values of the ratio d_b/h_p. The experiment is repeated for the sake of completeness, as well as the fact that a different time scaling is used in [ZVGV3]. The exact eigenvalues for the supported Kirchhoff plate appear in the last column. From Table 7 it is clear that the eigenvalues of the KEB plate-beam system tend to the eigenvalues of the rigidly supported plate as the ratio d_b/h_p is increased.

Table 7: Kirchhoff-Euler-Bernoulli

	λ_i Plate-beam system $h_p = 0.05$				λ_i
i	$d_b/h_p = 1$	$d_b/h_p = 2$	$d_b/h_p = 4$	$d_b/h_p = 8$	Supported plate
1	0.2413	0.2760	0.2782	0.2783	0.2783
2	0.8765	1.6853	1.7368	1.7393	1.7394
3	1.3715	1.7383	1.7394	1.7395	1.7394
4	1.7197	4.4436	4.4525	4.4530	4.4530
5	2.6642	5.2472	6.9312	6.9574	6.9578
6	4.2835	6.1048	6.9587	6.9687	6.9578

8.5.5 Comparison of Reissner-Mindlin-Timoshenko system with a rigidly supported Reissner-Mindlin plate

In Table 8 the eigenvalues of the Reissner-Mindlin-Timoshenko plate-beam system are compared to the eigenvalues of a Reissner-Mindlin plate that is rigidly supported on all four sides. The exact eigenvalues for the rigidly supported plate is presented in the last column.

It is clear that, as expected, the eigenvalues of the Reissner-Mindlin-Timoshenko plate-beam system tend to the eigenvalues of the rigidly supported Reissner-Mindlin plate as the ratio d_b/h_p is increased.

Table 8: Reissner-Mindlin-Timoshenko

	λ_i Plate-beam system $h_p = 0.05$				λ_i
i	$d_b/h_p = 1$	$d_b/h_p = 2$	$d_b/h_p = 4$	$d_b/h_p = 8$	Supported plate
1	0.2340	0.2702	0.2730	0.2733	0.2733
2	0.7744	1.5695	1.6552	1.6627	1.6643
3	1.1785	1.6619	1.6639	1.6642	1.6643
				3.0030	
				3.0030	
4	1.6406	3.2510	4.1503	4.1532	4.1540
5	2.4266	3.5914	5.8931	6.3471	6.3849
6	3.9311	4.1320	6.3844	6.3849	6.3849

Two interesting phenomena in this table warrant some further comment. The first is that the eigenvalues of the plate-beam system corresponding to the the double eigenvalues of the supported plate remain further apart for the RMT system than for the KEB system.

Secondly, for large values of the ratio d_b/h_p, an "extra" pair of eigenvalues appear for the Reissner-Mindlin-Timoshenko system. For $d_b/h_p = 8$ in Table 8, the double eigenvalue $\lambda \approx 3$ does not correspond to an eigenvalue of the supported plate. These eigenvalues did not appear in numerical experimentation with the Kirchhoff-Euler-Bernoulli system. The explanation for these extra eigenvalues for the RMT system lies in the fact that "pure shear" modes exist for the RMT system under consideration. These modes are discussed in the next section.

8.5.6 Pure shear modes

A fact that is often overlooked is that for certain configurations, "pure shear" modes exist for the Timoshenko beam model and for the Reissner-Mindlin plate model.

Timoshenko model

For a pinned-pinned Timoshenko beam it is easy to see that $\lambda = \alpha_b$ is an eigenvalue with the associated pair of eigenfunctions

$$w(x) = 0, \quad \phi(x) = 1.$$

Reissner-Mindlin-Timoshenko plate-beam system

Returning to the numerical results for the RMT plate-beam system in Table 8, note that if $d_b/h_p = 8$ and $h_p = 0.05$, then $d_b = 0.4$ and hence $\alpha_b = 3$. It seems likely that the pair of "extra eigenvalues" in Table 8 is a consequence of the pure shear mode of the Timoshenko beam model. This conjecture is supported by the graphs of the eigenfunction pairs of the system in Figure 1.

Remark

Note that as the height of the beam is $5d_b$, it means that in this case, the length to height ratio for the beam is 1 : 2. One would not expect the Timoshenko beam model to yield realistic results and consequently the RMT plate-beam system will also not be a reasonable model to use. Hence this phenomenon is only of theoretical significance.

Figure 8: Eigenfunctions for the RMT plate-beam system

(Note the differences in scaling.)