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Chapter 1

Modelling interface conditions

1.1 Introduction

In this thesis our concern is mathematical models for elastic plates and beams. In real life all objects are three-dimensional. Due to the proportions of a body, it is sometimes justifiable to consider a one-dimensional or two-dimensional model. These models are referred to as beam and plate models respectively.

We restrict our attention to linear models or linear theories for plates and beams. To be specific, we consider the Euler-Bernoulli, Rayleigh and Timoshenko theories for beams and the Kirchhoff and Reissner-Mindlin theories for plates.

The theories mentioned above refer to the partial differential equations that model a beam or plate. The contact − or lack thereof − with other objects also need to be modelled. The equations that result are usually referred to as boundary conditions, but we prefer the more inclusive term "interface conditions".

We consider three problems concerning interface conditions for plates and beams. In this section we present a brief introduction. A detailed discussion will be given in Section 1.5 after a review of the general theory.

The relevant aspects of beam theory and plate theory are presented in Sections 1.2 and 1.3 and a two-dimensional beam model in Section 1.4. We write all the problems in dimensionless form to facilitate numerical experiments.

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The model problems to be investigated are presented in Chapter 2.

1.1.1 A vertical slender structure on a resilient seating

Unwanted vibrations often occur in mechanical structures. The following design problem is described in [N1]:

> *"Because of their inherent low damping, free-standing welded steel structures are prone to oscillate in the wind. This may cause the chimney to fail due to metal fatigue. One method of artificially increasing the damping is to mount the chimney on a resilient foundation incorporating bearing pads made of a high-damping material."*

The structure may be modelled as a vertically mounted beam, i.e. a continuum model is used. Engineers often refer to continuum models as distributed parameter system (DPS) models.

In [N2], Newland discusses efforts to compute natural frequencies using DPS models. The results compared poorly with experimental results. Newland pointed out that the models needed to be improved to include the influence of the resilient seating. According to Newland, this increases the complexity of a finite element analysis considerably. As an alternative he proposed lumped parameter system models (LPS).

LPS models are useful for the analysis of vibrating systems when one is primarily interested in the lower order modes (see [CZ]). However, the accuracy is questionable and the theoretical tools for error estimation are not available. We considered it worthwhile to investigate beam models and to compare results.

Our initial objective was to match Newland's results using beam models. In doing so, we demonstrated the flexibility of DPS models in conjunction with the finite element method. We used the Euler-Bernoulli and Rayleigh models for the slender structure since they correspond to Newland's models.

Modelling the behaviour of the resilient seating and foundation leads to a hybrid system. We constructed four mathematical models to match those of Newland and showed that the interface conditions and additional equations can be accommodated in the variational form. Consequently the finite element method can be used. Using a small number of elements, our results

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compared well with those of Newland (see [N1], [N2] and [LVV]). The numerical results published in [LVV] show clearly the advantage of the finite element method.

In this thesis we investigate aspects not considered in [LVV]. First we use the Timoshenko theory to construct mathematical models and compare the results. We also consider theoretical aspects such as existence and uniqueness of solutions and convergence of finite element method approximations.

1.1.2 Boundary conditions for the clamped end of a beam

The Euler-Bernoulli beam is a popular model for the transverse vibration of a beam which is still used. Although the Timoshenko model is considered to be better (see e.g. [Fu], [I], [N1], [T] and [Wa]), some authors, for instance Duva and Simmonds ([DS]), do not agree that the Timoshenko model is an unqualified improvement. According to [DS], the corrections predicted by the Timoshenko model are in some cases erroneous. The authors claim that for the first eigenfrequency of the cantilever beam, the Timoshenko model provides a correction in the wrong direction and that this is due to "effects at the built in end".

Careful consideration of a clamped end of a beam leads to the conclusion that the boundary conditions for the two models are not compatible. This fact was pointed out in [V3] and an alternative boundary condition was proposed. However, the modified boundary condition worsened the disparities between the two models, i.e. the differences between the natural frequencies were larger. It became clear that further investigation was necessary and in this investigation two-dimensional effects must be taken into account. In order to do this, we consider two-dimensional models for a cantilever beam.

1.1.3 Plate-beam systems

In applications, structures consisting of linked systems of beams and plates are encountered. The reader is referred to [LLS] where a large variety of applications can be found.

We consider a rectangular plate connected to two beams. This problem was also considered in [ZVGV1], [ZVGV3] and [Ziet] using classical plate theory

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and the Euler-Bernoulli beam theory.

Combining the Reissner-Mindlin plate model and the Timoshenko beam model can be seen as a first step towards a better model while still avoiding the "complications" of a fully three-dimensional model.

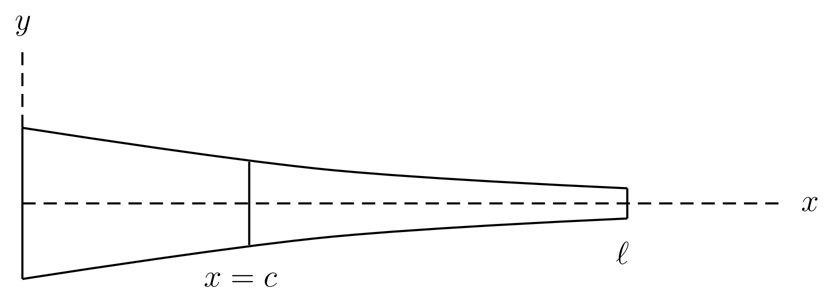
1.2 Beam theory

In this section we consider the transverse motion of a beam. We restrict our attention to a beam that is straight in its undeformed state. We assume that it has a well defined axis of symmetry and that all the cross sections are similar and have their centroids on the axis of symmetry.

The Euler-Bernoulli theory for a beam originated in the 18-th century. An improvement was introduced by Rayleigh in the 19-th century. In 1921, Timoshenko proposed his theory where shear is taken into account.

1.2.1 Equations of motion

Consider a beam as illustrated below. The x-axis is taken to coincide with the line of centroids of the cross sections. We assume that the cross sections and applied loads are symmetric with respect to the xy-plane and consequently the motion of the beam is parallel to the xy-plane.



Consider a cross section at $x = c$. Denote the axial force, shear force and moment by $S(c,t)$, $V(c,t)$ and $M(c,t)$ respectively. We follow the convention

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that S, V and M denote the forces and moment exerted by the part of the body for which $x > c$ on the rest.

Suppose the beam has constant density ρ, length ℓ and cross sectional area A. We consider a one-dimensional model and the reference configuration is the interval $[0, \ell]$. The transverse displacement (deflection) of the cross section at $x \in [0, \ell]$ at time t is denoted by $w(x, t)$. Assuming that plane cross sections remain plane, the rotation of a cross section is denoted by $\phi(x, t)$. Assume that the load P is in the transverse direction. The equations of motion are then given by

$$\rho A \partial_t^2 w = \partial_x V + P, \tag{1.2.1}$$
$$\rho I \partial_t^2 \phi = V + \partial_x M + L, \tag{1.2.2}$$

where I is the area moment of inertia (see [T, p 331-337] and [I, p 337]).

Remarks

1. The term $\rho I \partial_t^2 \phi$ in Equation (1.2.2) is usually referred to as the rotary inertia term.

2. Note the unusual term L present in Equation (1.2.2). This term represents a moment density term that will be used in some of the mathematical models (see Sections 2.1 and 2.4).

1.2.2 The Timoshenko model

To determine the forces S and V and the moment M, the stresses are integrated over a cross section. For more detail, see [Fu, Sec 7.7], [Co] and [I, p 337-338].

In the linear theory, it is assumed that $\partial_x w$ is small. The following constitutive equations for the moment M and the shear force V are used.

$$M = EI \partial_x \phi, \tag{1.2.3}$$
$$V = AG\kappa^2 \big(\partial_x w - \phi\big). \tag{1.2.4}$$

In these equations, E and G are elastic constants (see Section 1.4) and κ^2 the shear coefficient or shear correction factor. We refer the reader to [T, p 337-338], [Fu, p 323-324], [I, p 337-338] and [N1, p 392-395].

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Substituting the constitutive equations (1.2.3) and (1.2.4) into the equations of motion (1.2.1) and (1.2.2), yield the well known Timoshenko model for the free vibration of a beam.

$$\begin{aligned} \rho A \partial_t^2 w &= \partial_x \big(AG\kappa^2 (\partial_x w - \phi) \big), \\ \rho I \partial_t^2 \phi &= AG\kappa^2 \big(\partial_x w - \phi \big) + \partial_x \big(EI \partial_x \phi \big) + L. \end{aligned}$$

The partial differential equations above can be derived in different ways (see [Fu, p 322-323] and [Co]).

The boundary conditions depend on the configuration and a number of variations are possible (see [I, p 335, 338] and [Fu, p 323-324]).

Note that we will not use the partial differential equations above. When confronted by complex interface conditions, it is advisable to use the equations of motion and constitutive equations (Equations (1.2.1) − (1.2.4)), rather than the partial differential equations.

1.2.3 The Euler-Bernoulli and Rayleigh models

We consider first the **Rayleigh model**. It can be derived formally from the Timoshenko model. Combining Equations (1.2.1) and (1.2.2), we find that

$$\rho A \partial_t^2 w = \rho I \partial_t^2 \partial_x \phi - \partial_x^2 M + P - \partial_x L.$$

For this model, it is assumed that a cross section remains perpendicular to the neutral plane. This implies that $\partial_x w = \phi$, and the equation reduces to

$$\rho A \partial_t^2 w = \rho I \partial_t^2 \partial_x^2 w - \partial_x^2 M + P - \partial_x L.$$

This is the equation of motion for the Rayleigh model. The constitutive equation for the shear force V is now redundant and the constitutive equation for the bending moment is

$$M = EI \partial_x^2 w.$$

As mentioned before, we do not use the partial differential equations, but we present them for the purpose of comparison. The partial differential equation for the Rayleigh model is

$$\rho A \partial_t^2 w - \rho I \partial_t^2 \partial_x^2 w = -EI \partial_x^4 w + P - \partial_x L.$$

The **Euler-Bernoulli model** is a special case of the Rayleigh model where rotary inertia is ignored and the result is

$$\rho A \partial_t^2 w = -EI \partial_x^4 w + P - \partial_x L.$$

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1.2.4 Dimensionless form

In this subsection we write the equations of motion and constitutive equations in dimensionless form. Set

$$\tau = \frac{t}{t_0}\,, \quad \xi = \frac{x}{\ell}\,, \quad w^*(\xi,\tau) = \frac{w(x,t)}{\ell} \quad \text{and} \quad \phi^*(\xi,\tau) = \phi(x,t).$$

We introduce the dimensionless constants

$$\alpha = \frac{A\ell^2}{I}\,, \quad \beta = \frac{AG\kappa^2\ell^2}{EI} \quad \text{and} \quad \gamma = \frac{\beta}{\alpha} = \frac{G\kappa^2}{E}\,.$$

The constant γ depends on the elastic constants and the shear correction factor κ^2 that is determined by the shape of the cross section. The values of κ^2 range between $\frac{1}{2}$ and 1 (see [Co] or [BSSS, p 173]). On the other hand, for isotropic materials we assume that $\frac{G}{E} = \frac{1}{2(1+\nu)}$ (see [My, p 174] or [Fu, Sec 7.2]). Realistic values for γ range between $\frac{1}{6}$ and $\frac{1}{2}$. Timoshenko ([T, p 342]) used $\frac{2}{3}$ for κ^2 and $\frac{G}{E} = \frac{3}{8}$.

The constant α is subject to significant variation. With r^2 the radius of gyration we have $\alpha = \frac{A\ell^2}{I} = \frac{\ell^2}{r^2}$.

The forces and moments in dimensionless form are

$$L^*(\xi,\tau) = \frac{L(x,t)}{G\kappa^2 A}\,, \quad P^*(\xi,\tau) = \frac{\ell P(x,t)}{G\kappa^2 A}\,,$$

$$V^*(\xi,\tau) = \frac{V(x,t)}{G\kappa^2 A} \quad \text{and} \quad M^*(\xi,\tau) = \frac{M(x,t)}{\ell G\kappa^2 A}\,.$$

A convenient choice for t_0 is

$$t_0 = \ell\sqrt{\frac{\rho}{G\kappa^2}}\,.$$

Returning to the original notation we present the equations of motion and constitutive equations in dimensionless form.

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Timoshenko model

$$\partial_t^2 w = \partial_x V + P, \tag{1.2.5}$$
$$\frac{1}{\alpha}\,\partial_t^2 \phi = V + \partial_x M + L, \tag{1.2.6}$$
$$M = \frac{1}{\beta}\,\partial_x \phi, \tag{1.2.7}$$
$$V = \partial_x w - \phi. \tag{1.2.8}$$

Rayleigh model

$$\partial_t^2 w = \partial_x V + P, \tag{1.2.9}$$
$$\frac{1}{\alpha}\,\partial_t^2 \partial_x w = V + \partial_x M + L, \tag{1.2.10}$$
$$M = \frac{1}{\beta}\,\partial_x^2 w. \tag{1.2.11}$$

Euler-Bernoulli model

The Euler-Bernoulli model is obtained from the Rayleigh model by omitting the rotary inertia term $\frac{1}{\alpha}\,\partial_t^2 \partial_x w$.

Remark

Note that the rotary inertia term is simply omitted. It is not correct to reason that $\frac{1}{\alpha} \approx 0$, since that would imply that $\frac{1}{\beta} \approx 0$.

Since the Euler-Bernoulli model is a special case of the Rayleigh model, we will not refer to this model again in the theoretical discussions that follow. To obtain results for the Euler-Bernoulli model, one uses the relevant equations for the Rayleigh model with the modification mentioned above.

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1.3 Plate theory

In his book *Elastic Plates: Theory and Applications*, Reissman presents an interesting historical note (see [Rei]):

> *"The theory of plates has a colorful history. Classical plate theory was initiated by Mlle. Sophie Germain (1776 – 1831) in direct response to a prize offered by the French Academy (1811) for the explanation of the nodal curves of a vibrating plate, as demonstrated (experimentally) by E. Chladni (1756 – 1829) of Saxony. After two attempts, Mlle. Germain received the prize in 1816 but only after Lagrange, a member of the examination committee, corrected her initially submitted paper. Subsequently, a controversy ensued about the appropriate, associated boundary conditions, and this was settled approximately 34 years after the correct partial differential equations were discovered. No less than the authorities G. R. Kirchhoff (1824 – 1887) and Lord Kelvin (William Thompson) (1824 – 1907) were responsible for this part of the theory."*

From 1945 to 1950 improvements to classical plate theory were made by E. Reissner, H. Hencky, Y. S. Uflyand and R. D. Mindlin (see [Mi] for references).

1.3.1 Equations of motion

We consider small transverse vibration of a thin plate with thickness h and density ρ. The reference configuration for the plate is a domain Ω in the plane.

The transverse displacement of $\boldsymbol{x}$ at time t is denoted by $w(\boldsymbol{x}, t)$. The angle between a "material line" and a perpendicular to the plane is $\psi(\boldsymbol{x}, t)$ and the angle between the projection of the material line in the plane and the unit vector $\boldsymbol{e}_1$ is $\phi(\boldsymbol{x}, t)$ (see [Rei, Sec 3.2, Sec 3.5]). For a linear model $\boldsymbol{\psi}$ is approximated by

$$\boldsymbol{\psi} = [\psi_1 \ \psi_2]^T = [\psi \cos \phi \quad \psi \sin \phi]^T.$$

Then the equations of motion (see [Mi] and [Rei, p 152]) are given by

$$\rho h \partial_t^2 w = \operatorname{div} \boldsymbol{Q} + q, \tag{1.3.1}$$

$$\rho I \partial_t^2 \boldsymbol{\psi} = \operatorname{div} M - \boldsymbol{Q}, \tag{1.3.2}$$

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where $I = \dfrac{h^3}{12}$ is the length moment of inertia.

$\boldsymbol{Q}$ represents a force density, $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$ a moment density and q an external load on the plate.

1.3.2 The Reissner-Mindlin and Kirchhoff models

Constitutive equations

We restrict our attention to the linear theory. The following assumptions are made for small curvature and small partial derivatives (see [Rei, p 61] and [Mi]).

$$\boldsymbol{Q} = \kappa^2 G h (\nabla w + \boldsymbol{\psi}), \tag{1.3.3}$$

where G is the shear modulus and κ^2 a correction factor.

$$M = \frac{1}{2} D \begin{bmatrix} 2\big(\partial_1\psi_1 + \nu\partial_2\psi_2\big) & (1-\nu)\big(\partial_1\psi_2 + \partial_2\psi_1\big) \\ (1-\nu)\big(\partial_1\psi_2 + \partial_2\psi_1\big) & 2\big(\partial_2\psi_2 + \nu\partial_1\psi_1\big) \end{bmatrix}. \tag{1.3.4}$$

D is a measure of stiffness for the plate and is given by

$$D = \frac{EI}{1-\nu^2},$$

where E is Young's modulus and ν Poisson's ratio.

The correction factor κ^2 is chosen in such a way that the solution of the plate model compares well with the solution of the three-dimensional model. The value of κ^2 depends on Poisson's ratio ν and ranges almost linearly from 0.76 to 0.91 if ν increases from 0 to 0.5 (see [Mi]). Also mentioned in this reference is that Reissner used $\kappa^2 = \dfrac{5}{6}$.

The equations of motion and the constitutive equations above are known as the **Reissner-Mindlin plate model**.

The constitutive equations may be substituted into the equations of motion, leading to a system of three partial differential equations (see [Rei, p 152] and [Mi]). In our approach these partial differential equations are not used.

For classical plate theory, $\boldsymbol{\psi}$ is replaced by $-\nabla w$ and the constitutive equation for $\boldsymbol{Q}$ is no longer necessary. This is sometimes referred to as the **Kirchhoff plate model**.

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1.3.3 Dimensionless forms

We introduce the dimensionless variables

$$\tau = \frac{t}{t_0}, \quad \xi_1 = \frac{x_1}{\ell} \quad \text{and} \quad \xi_2 = \frac{x_2}{\ell}\,,$$

where ℓ is a suitable length and t_0 must still be specified.

The dimensionless variables, with $\boldsymbol{x} = (x_1, x_2)$ and $\boldsymbol{\xi} = (\xi_1, \xi_2)$, are

$$w^*(\boldsymbol{\xi}, \tau) = \left(\frac{1}{\ell}\right) w(\boldsymbol{x}, t), \quad \boldsymbol{\psi}^*(\boldsymbol{\xi}, \tau) = \boldsymbol{\psi}(\boldsymbol{x}, t),$$

$$\boldsymbol{Q}^*(\boldsymbol{\xi}, \tau) = \left(\frac{1}{\ell G\kappa^2}\right) \boldsymbol{Q}(\boldsymbol{x}, t), \quad M^*(\boldsymbol{\xi}, \tau) = \left(\frac{1}{\ell^2 G\kappa^2}\right) M(\boldsymbol{x}, t)$$

$$\text{and} \quad q^*(\boldsymbol{\xi}, \tau) = \left(\frac{1}{G\kappa^2}\right) q(\boldsymbol{x}, t).$$

The dimensionless constants that are used are given by

$$h_p = \frac{h}{\ell}\,, \quad I_p = \frac{h_p^3}{12} \quad \text{and} \quad \beta_p = \frac{\ell^3 G\kappa^2}{EI}\,.$$

The constant h_p denotes the dimensionless thickness of the plate and I_p the dimensionless length moment of inertia.

We choose $t_0 = \ell\sqrt{\dfrac{\rho}{G\kappa^2}}$ (for convenience) and use the original notation for the corresponding dimensionless quantities. The equations of motion and constitutive equations in dimensionless form are presented below.

Reissner-Mindlin plate model

$$h_p\,\partial_t^2 w = \operatorname{div}\boldsymbol{Q} + q, \tag{1.3.5}$$

$$I_p\,\partial_t^2 \boldsymbol{\psi} = \operatorname{div} M - \boldsymbol{Q}, \tag{1.3.6}$$

$$\boldsymbol{Q} = h_p\big(\nabla w + \boldsymbol{\psi}\big), \tag{1.3.7}$$

$$M = \frac{1}{2\beta_p(1-\nu^2)} \begin{bmatrix} 2\big(\partial_1\psi_1 + \nu\partial_2\psi_2\big) & (1-\nu)\big(\partial_1\psi_2 + \partial_2\psi_1\big) \\ (1-\nu)\big(\partial_1\psi_2 + \partial_2\psi_1\big) & 2\big(\partial_2\psi_2 + \nu\partial_1\psi_1\big) \end{bmatrix} \tag{1.3.8}$$

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Classical plate model

$$h_p\,\partial_t^2 w = \operatorname{div}\boldsymbol{Q} + q, \tag{1.3.9}$$
$$I_p\,\partial_t^2\left(\boldsymbol{\nabla} w\right) = \boldsymbol{Q} - \operatorname{div} M, \tag{1.3.10}$$

$$M = -\frac{1}{\beta_p(1-\nu^2)}\begin{bmatrix} \left(\partial_1^2 w + \nu\partial_2^2\right)w & (1-\nu)\partial_1\partial_2 w \\ (1-\nu)\partial_1\partial_2 w & \left(\partial_2^2 w + \nu\partial_1^2\right)w \end{bmatrix}. \tag{1.3.11}$$

Generally the rotary inertia term $I_p\partial_t^2\left(\nabla w\right)$ in Equation (1.3.10) is ignored.

1.4 Two-dimensional model for a beam

As mentioned in the introduction, we also consider a two-dimensional model for a beam. To facilitate the discussion, we include a brief review of linear elasticity.

1.4.1 Equation of motion

Consider an elastic body with density ρ. The displacement of a point $\boldsymbol{x}$ in the reference configuration at time t is $\boldsymbol{u}(\boldsymbol{x},t)$ and the velocity is $\boldsymbol{v} = \partial_t\boldsymbol{u}$.

From the conservation law for momentum, we have the **equation of motion** (see [Fu, Sec 5.5, 5.7]) or [AF, p 125])

$$\rho\partial_t^2\boldsymbol{u} = \operatorname{div} T + \boldsymbol{Q},$$

where T is the first Piola stress tensor and $\boldsymbol{Q}$ an external body force (density force).

In the case of small local displacements, the **infinitesimal theory of elasticity** or **linear elasticity** may be used. In this case the first Piola stress tensor is approximated by the Cauchy stress tensor (which is symmetric). For an explanation, see [AF, p 45-46, 122, 125]. Another explanation is given in [Fu, Sec 7.1].

In the matrix representation of T the stress components are denoted by σ_{ij} and $\operatorname{div} T$ is a vector with components

$$[\operatorname{div} T]_i = \partial_1\sigma_{i1} + \partial_2\sigma_{i2} + \partial_3\sigma_{i3} \quad \text{for } i = 1,\ 2,\ 3.$$

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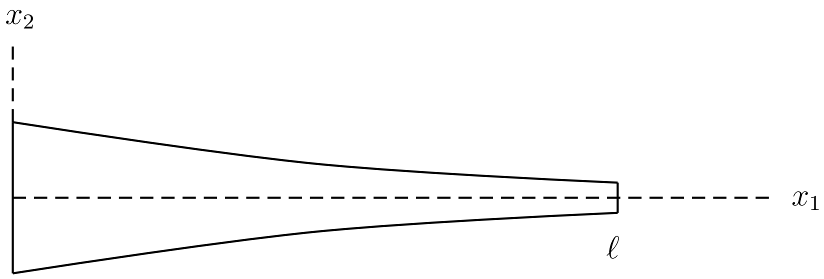
Simplifying assumptions

Now consider a beam as illustrated below. The x_1-axis is taken to coincide with the line of centroids of the cross sections. We assume that the cross sections and applied loads are symmetric with respect to the x_1x_2-plane and consequently the motion of the beam is parallel to the x_1x_2-plane.



For beam problems it is reasonable to assume that the body or beam is in a state of plane stress. To be specific, we assume that $\sigma_{3i} = \sigma_{i3} = 0$. However, this does not imply that the problem is two-dimensional since $\partial_3\sigma_{ij}$ need not be zero. This is an assumption that we make. The interpretation is that the stresses are averages across the width of the beam. This approach is in line with Cowper's ([Co]) derivation of the Timoshenko model. It is reasonable to assume that the two-dimensional model is more accurate than beam models (but obviously less accurate than three-dimensional models).

Constitutive equations

The **infinitesimal strain** $\mathcal{E}$ is given by

$$e_{ij} = \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right).$$

(See [AF, p 25] or [Fu, p 155].)

Constitutive equations are required to express the relationship between the stress T and the strain $\mathcal{E}$. These depend on the elastic properties of the material under consideration. An **isotropic** material exhibits no preferred direction in its response to a given state of stress. For a **homogeneous**

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material the elastic properties are the same at all points of the reference configuration.

We use **Hooke's law for homogeneous isotropic materials** ([Fu, Sec 9.1] or [My, p 173, 182]) for the special case of plane stress.

$$\begin{aligned} \sigma_{11} &= \frac{E}{1-\nu^2}\left(e_{11}+\nu e_{22}\right), \\ \sigma_{22} &= \frac{E}{1-\nu^2}\left(e_{22}+\nu e_{11}\right), \\ \sigma_{12} = \sigma_{21} &= \frac{E}{1+\nu}\; e_{12}, \end{aligned}$$

where E is Young's modulus and ν Poisson's ratio.

The constitutive equation in terms of the components of $\boldsymbol{u}$ follow as

$$\begin{aligned} \sigma_{11} &= \frac{E}{1-\nu^2}\left(\partial_1 u_1+\nu\partial_2 u_2\right), \\ \sigma_{22} &= \frac{E}{1-\nu^2}\left(\partial_2 u_2+\nu\partial_1 u_1\right), \\ \sigma_{12} = \sigma_{21} &= \frac{E}{2(1+\nu)}\left(\partial_1 u_2+\partial_2 u_1\right). \end{aligned}$$

Substitution of the constitutive equation into the equation of motion yields a system of partial differential equations for the components of the displacement. We will not make use of this system of partial differential equations.

1.4.2 Dimensionless form

The dimensionless variables and constants must be the same or compatible with those in Section 1.2. Set

$$\tau = \frac{t}{t_0}, \quad \xi_i = \frac{x_i}{\ell} \quad \boldsymbol{u}^*(\boldsymbol{\xi},\tau) = \frac{1}{\ell}\,\boldsymbol{u}(\boldsymbol{x},t),$$

$$\text{and} \quad \sigma^*_{ij}(\boldsymbol{\xi},\tau) = \left(\frac{1}{G\kappa^2}\right)\sigma_{ij}(\boldsymbol{x},t).$$

Recall that $t_0 = \ell\sqrt{\dfrac{\rho}{G\kappa^2}}$ and $\gamma = \dfrac{G\kappa^2}{E}$.

Returning to the original notation we present the equations of motion and constitutive equations in **dimensionless form**. In the problems under consideration $\boldsymbol{Q} = \boldsymbol{0}$.

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Equation of motion

$$\partial_t^2 \boldsymbol{u} = \operatorname{div} T, \quad \text{where} \tag{1.4.1}$$

$$\operatorname{div} T = \begin{bmatrix} \partial_1\sigma_{11} + \partial_2\sigma_{12} \\ \partial_1\sigma_{21} + \partial_2\sigma_{22} \\ 0 \end{bmatrix}.$$

Constitutive equations

$$\begin{aligned} \sigma_{11} &= \frac{1}{\gamma(1-\nu^2)} \big(\partial_1 u_1 + \nu\partial_2 u_2\big), \\ \sigma_{22} &= \frac{1}{\gamma(1-\nu^2)} \big(\partial_2 u_2 + \nu\partial_1 u_1\big), \\ \sigma_{12} = \sigma_{21} &= \frac{1}{2\gamma(1+\nu)} \big(\partial_1 u_2 + \partial_2 u_1\big). \end{aligned} \tag{1.4.2}$$

1.5 Interface conditions

It is now possible to provide more detail concerning the problems that we investigate.

1.5.1 Vertical slender structure

The vertical slender structure (for example a chimney), is modelled as a vertical beam with $x = 0$ at the ground level. The boundary conditions at the top present no problem and we have that

$$M(1,t) = V(1,t) = 0.$$

For a built in beam the conventional boundary conditions at the bottom are given by $w(0,t) = \partial_x w(0,t) = 0$. However, the conventional boundary conditions yielded poor results (as Newland mentioned in [N2]).

Modelling the behaviour of the resilient seating and foundation leads to a complex hybrid system with interface conditions and additional equations. This was done in [LVV] with satisfactory results – as mentioned before. In this thesis we adapt the interface conditions for the Timoshenko theory.

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1.5.2 Boundary conditions for the clamped end of a beam

First we show that the boundary conditions used for the Euler-Bernoulli and Timoshenko models are incompatible. Consider a beam in equilibrium clamped at $x = 0$ and an external vertical force F at the endpoint $x = 1$.

The usual boundary conditions at $x = 0$ for an Euler-Bernoulli beam are

$$w(0) = w'(0) = 0\,.$$

For a Timoshenko beam the boundary conditions at $x = 0$ are

$$w(0) = \phi(0) = 0\,.$$

When an external force F is applied at $x = 1$, the implication is that the shear force throughout the beam is constant and equal to F, hence $V(0) = F$. Since $\phi(0) = 0$, it follows from the constitutive equation (1.2.8) that

$$w'(0) = V(0) = F\,.$$

However, for the Euler-Bernoulli and the Rayleigh models it is assumed that $w'(0) = 0$. Clearly $\phi(0)$ and $w'(0)$ can not both be zero.

The boundary condition $w'(0) = 0$ is realistic from a modelling perspective. This suggests that the boundary conditions at a built in end for the Timoshenko theory deserves closer examination.

One possibility is the boundary condition proposed in [V3], which we consider in this thesis. However, as mentioned in Section 1.1.2, this boundary

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condition creates larger disparities and it is logical to consider other possibilities.

We consider the possibility that the constitutive equation (1.2.8) does not reflect reality at the built in end. The quantity $w' - \phi$ represents the average shear for a cross section. As x tends to zero, both w' and ϕ become small, but the shear force V remains constant. These facts suggests that $\dfrac{V}{w' - \phi}$ is not constant. The Timoshenko theory implies that a cross section remains plane and that the shearing strain $w' - \phi$ is constant on a cross section. In reality, the strain is zero at both the bottom and the top of a horizontal beam. In the Timoshenko theory, the quantity $w' - \phi$ represents the average strain of a cross section. It is possible that this is not realistic at the clamped end.

To investigate the difficulties mentioned, we consider a prismatic beam with the simplifying assumptions mentioned in Section 1.4. Usually the boundary condition for the "fixed end" is to set the displacement $\boldsymbol{u} = \mathbf{0}$. This will not do if the objective is to determine the strain at the clamped end, hence we also consider configurations where part of the beam is embedded (see Section 2.3).

Finally, there is another aspect that needs to be mentioned. The first two or three eigenfrequencies of the Euler-Bernoulli and the Timoshenko models for a cantilever beam differ very little, unless the beam is short (relative to its thickness) – to be precise, when the parameter α is small. The first eigenfrequency differs appreciably when the beam is so short that one is reluctant to use beam theory at all. Comparisons are given in Section 7.1.

1.5.3 Plate-beam system

When a plate and a beam are connected, numerous aspects need to be considered. These aspects may be classified under geometrical constraints and mechanical interaction. A Reissner-Mindlin-Timoshenko plate-beam system is extremely complex due to the presence of five equations of motion. One could say that the boundary conditions are partial differential equations themselves.

Another complication is the fact that the angles ψ (for the plate) and ϕ (for the beam) do not present a physically reality but convenient averages. Consequently it is not clear what the geometrical constraints should be.

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Not only is the modelling for a Reissner-Mindlin-Timoshenko plate-beam system more complex, but the mathematical analysis and numerical analysis present additional difficulties. Finally, the numerical algorithms also present nontrivial difficulties not present in the plate-beam system using classical plate and beam theory.

In this thesis we consider a Reissner-Mindlin plate supported by two Timoshenko beams. The case where the plate is connected rigidly to the beam, can be found in [LLS].

One expects that in some cases the Reissner-Mindlin-Timoshenko model will compare well with the Kirchhoff-Euler-Bernoulli model that is investigated in [ZVGV1], [ZVGV3] and [Ziet]. In Chapter 8 we present some results on this comparison.

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Chapter 2

Model problems

2.1 Vertical slender structure

In this section we present DPS models that correspond to Newland's LPS models [N1, p 129-132] and [N2]. The slender structure (e.g. a steel chimney) is modelled as a Euler-Bernoulli, a Rayleigh or a Timoshenko beam mounted vertically and gravity is taken into account. (The reason for including gravity in the model, is to match Newland's models.)

From Section 1.2 we have the relevant equations of motion for the Rayleigh and Timoshenko theories in dimensionless form. In this case we have free vibration and therefore $P = 0$.

The relevant constitutive equations are also given in Section 1.2. The term $L = -S\partial_x w$ is a moment density (measured in Newton) due to gravity. The axial force due to gravity is given by

$$S(x) = -\rho A g(\ell - x).$$

With $\mu = \dfrac{\rho g \ell}{G\kappa^2}$ and using the original notation, the dimensionless moment density (see Section 1.2) is given by

$$L(x,t) = \mu(1-x)\partial_x w(x,t).$$

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2.1.1 Simplistic Models

Initially we considered the Rayleigh theory as this corresponds to the Newland models.

Boundary conditions at $x = 0$

There are a number of possibilities for the boundary conditions at $x = 0$. Following Newland ([N1], [N2]), four models are considered in [LVV]. The first two are rather simplistic. In Model 1, the foundation is completely rigid and the boundary conditions at the base are given by

$$w(0,t) \;=\; \partial_x w(0,t) \;=\; 0.$$

In the second model that corresponds to the model in [N1, p 133], the effect of the resilient seating is taken into account. The foundation is modelled to be elastic with damping. Hence the moment $M(0,t)$ is determined by the elasticity and damping of the foundation. In this case the boundary conditions at the base for the Rayleigh model are given by

$$\begin{aligned} w(0,t) &= 0, \\ M(0,t) &= k\,\partial_x w(0,t) + c\,\partial_t\partial_x w(0,t), \end{aligned}$$

where the constants k and c are nonnegative.

Our results for these models were compared to Newland's results and was published in [LVV]. Models 1 and 2 are not considered in this thesis.

2.1.2 The dynamics of the foundation block and resilient seating

The mathematical models presented later in this Section as Problem VR 3 and Problem VR 4, were published in [LVV]. For these models the dynamics of the resilient seating and foundation block is taken into account.

As our point of departure, we consider the physical model in [N2]. Figure 1 corresponds to Figure 2 in [N2].

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Figure 1: Simplified sketch of the system

The springs and damping mechanisms in the sketch are schematic.

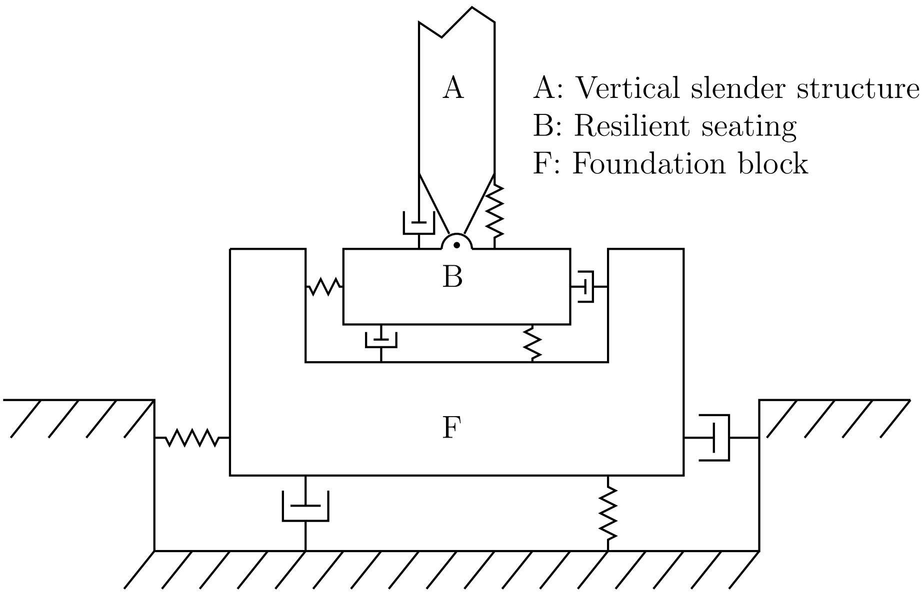
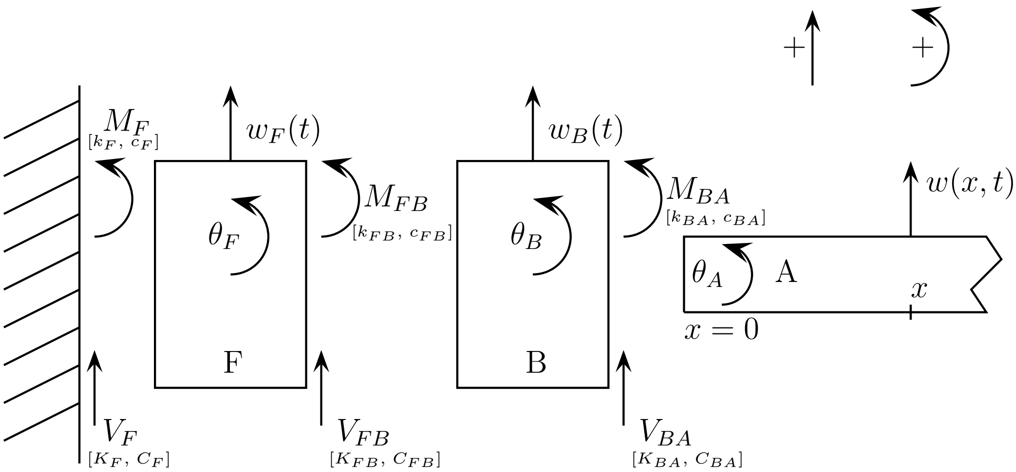


Figure 2: Displacements, angles of rotation, moments and forces

Convention: Moments and forces are denoted by the action of right on left. For instance, M_{FB} denotes the moment exerted by B on F.



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To formulate the boundary conditions at the base, it is necessary to consider the equations of motion for the resilient seating and foundation block. Both are modelled as rigid bodies connected to linear elastic springs and linear damping mechanisms.

Equations of motion

$$
\begin{aligned}
m_F \ddot{w}_F &= V_{FB} - V_F, \\
m_B \ddot{w}_B &= V_{BA} - V_{FB}, \\
I_F \ddot{\theta}_F &= M_{FB} - M_F, \\
I_B \ddot{\theta}_B &= M_{BA} - M_{FB}.
\end{aligned}
$$

Constitutive equations

$$
\begin{aligned}
V_F &= K_F w_F + C_F \dot{w}_F, \\
V_{FB} &= K_{FB}(w_B - w_F) + C_{FB}(\dot{w}_B - \dot{w}_F), \\
M_F &= k_F \theta_F + c_F \dot{\theta}_F, \\
M_{FB} &= k_{FB}(\theta_B - \theta_F) + c_{FB}(\dot{\theta}_B - \dot{\theta}_F).
\end{aligned}
$$

The reader must take note of the use of upper case and lower case letters for the constants.

Interface conditions

Let $\theta_A(t)$ denote the rotation of the end point of the vertical structure.

$$
\begin{aligned}
M_{BA}(t) &= k_{BA}\Big(\theta_A(t) - \theta_B(t)\Big) + c_{BA}\Big(\dot{\theta}_A(t) - \dot{\theta}_B(t)\Big), \\
M_{BA}(t) &= M(0,t), \\
V_{BA}(t) &= V(0,t), \\
w_B(t) &= w(0,t), \\
\theta_B(t) &\neq \theta_A(t) \quad \text{(in general)}.
\end{aligned}
$$

We make the following assumptions for $\theta_A(t)$:

- $\theta_A(t) = \partial_x w(0,t)$ for the Rayleigh models and
- $\theta_A(t) = \phi(0,t)$ for the Timoshenko models.

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Dimensionless constants

The dimensionless constants for the foundation block and the resilient seating are

$$m^* = \frac{m}{\ell \rho A}\,, \quad \text{and} \quad I^* = \frac{I}{\ell^3 \rho A}\,.$$

The different elastic and damping constants are

$$K^* = \frac{K\ell}{AG\kappa^2}\,, \quad k^* = \frac{k}{AG\kappa^2 \ell}\,, \quad C^* = \frac{C\ell}{AG\kappa^2 t_0} \quad \text{and} \quad c^* = \frac{c}{AG\kappa^2 t_0 \ell}\,.$$

The following equalities hold for the the scaling factors of m and I:

$$\ell \rho A = \frac{t_0^2 AG\kappa^2}{\ell} \quad \text{and} \quad \ell^3 \rho A = t_0^2 AG\kappa^2 \ell^2$$

All the constants in the equations of motion for the foundation block and resilient seating and the equations for the interface conditions, must be replaced by the corresponding dimensionless constants.

The diagrams and equations in this subsection are from [LVV].

2.1.3 Rayleigh models

The Rayleigh theory applied to Models 3 and 4 yields the same equations of motion, constitutive equations and boundary conditions at the top.

Equations of motion

$$\partial_t^2 w = \partial_x V, \tag{2.1.1}$$

$$\frac{1}{\alpha}\,\partial_t^2 \partial_x w = V + \partial_x M + L, \tag{2.1.2}$$

Constitutive equations

$$M = \frac{1}{\beta}\,\partial_x^2 w, \tag{2.1.3}$$

$$L(x,t) = \mu(1-x)\,\partial_x w(x,t). \tag{2.1.4}$$

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Boundary conditions at $x = 1$

$$M(1,t) = V(1,t) = 0.$$

The interface conditions for these two problems differ. We will refer to the two problems as Problem VR 3 and Problem VR 4 (corresponding to models 3 and 4 in [LVV]).

Problem VR 3

Equations of motion: (2.1.1) and (2.1.2).

Constitutive equations: (2.1.3) and (2.1.4).

Boundary conditions at $x = 1$: $M(1,t) = V(1,t) = 0$.

The motion of B is neglected and B is considered to be rigidly connected to the foundation block F and this case corresponds to Model 2 in [N2]. The conditions are

$$w_F(t) = w_B(t) = w(0,t), \quad \theta_F = \theta_B, \quad V_{FB}(t) = V_{BA}(t) = V(0,t)$$

$$\text{and} \quad M_{FB}(t) = M_{BA}(t) = M(0,t).$$

The constant k_{BA} is replaced by k and c_{BA} by c.

The interface conditions and the equations of motion of the foundation block and resilient seating reduce to the following three equations:

$$m_F \partial_t^2 w(0,t) = V(0,t) - K_F w(0,t) - C_F \partial_t w(0,t), \tag{2.1.5}$$

$$\begin{aligned} I_F \ddot{\theta}_F(t) &= k\Big(\partial_x w(0,t) - \theta_F(t)\Big) + c\Big(\partial_t \partial_x w(0,t) - \dot{\theta}_F(t)\Big) \\ &\quad - k_F \theta_F(t) - c_F \dot{\theta}_F(t), \end{aligned} \tag{2.1.6}$$

$$M(0,t) = k\Big(\partial_x w(0,t) - \theta_F(t)\Big) + c\Big(\partial_t \partial_x w(0,t) - \dot{\theta}_F(t)\Big). \tag{2.1.7}$$

Problem VR 4

Equations of motion: (2.1.1) and (2.1.2).

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Constitutive equations: (2.1.3) and (2.1.4).

Boundary conditions at $x = 1$: $M(1,t) = V(1,t) = 0$.

In Model 4, the interface conditions and the equations of motion of the foundation block and resilient seating follow directly from the discussion in Section 2.1.2.

The following five equations formulate the interface conditions.

$$\begin{aligned} m_B\partial_t^2 w(0,t) &= V(0,t) - K_{FB}\Big(w(0,t) - w_F(t)\Big) \\ &\quad -C_{FB}\Big(\partial_t w(0,t) - \dot{w}_F(t)\Big), \end{aligned} \tag{2.1.8}$$

$$\begin{aligned} I_B\ddot{\theta}_B(t) &= k_{BA}\Big(\partial_x w(0,t) - \theta_B(t)\Big) + c_{BA}\Big(\partial_t\partial_x w(0,t) - \dot{\theta}_B(t)\Big) \\ &\quad -k_{FB}\Big(\theta_B(t) - \theta_F(t)\Big) - c_{FB}\Big(\dot{\theta}_B(t) - \dot{\theta}_F(t)\Big), \end{aligned} \tag{2.1.9}$$

$$\begin{aligned} M(0,t) &= k_{BA}\Big(\partial_x w(0,t) - \theta_B(t)\Big) \\ &\quad +c_{BA}\Big(\partial_t\partial_x w(0,t) - \dot{\theta}_B(t)\Big), \end{aligned} \tag{2.1.10}$$

$$\begin{aligned} m_F\ddot{w}_F(t) &= K_{FB}\Big(w(0,t) - w_F(t)\Big) + C_{FB}\Big(\partial_t w(0,t) - \dot{w}_F(t)\Big) \\ &\quad -K_F w_F(t) - C_F\dot{w}_F(t), \end{aligned} \tag{2.1.11}$$

$$\begin{aligned} I_F\ddot{\theta}_F(t) &= k_{FB}\Big(\theta_B(t) - \theta_F(t)\Big) + c_{FB}\Big(\dot{\theta}_B(t) - \dot{\theta}_F(t)\Big) \\ &\quad -k_F\theta_F(t) - c_F\dot{\theta}_F(t). \end{aligned} \tag{2.1.12}$$

Remarks

1. The stiffness and damping in the mounting are modelled to be due to linear springs and linear dashpots. The limitations of these assumptions are discussed in [N2].

2. Problems VR 3 and VR 4 are from [LVV].

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2.1.4 Timoshenko models

As mentioned before, results for Model 3 and Model 4 using the Rayleigh and Euler-Bernoulli theory were published in [LVV]. In this thesis our main objective is to use the Timoshenko beam theory in the models and compare the results to the results where the Rayleigh theory is used. We refer to these problems as Problem VT 3 and Problem VT 4.

The equations of motion, constitutive equations and the boundary conditions at the top are the same for both problems.

Equations of motion

$$\partial_t^2 w = \partial_x V, \tag{2.1.13}$$

$$\frac{1}{\alpha}\,\partial_t^2 \phi = \partial_x M + V + L. \tag{2.1.14}$$

Constitutive equations

$$M = \frac{1}{\beta}\,\partial_x \phi, \tag{2.1.15}$$

$$V = \partial_x w - \phi, \tag{2.1.16}$$

$$L(x,t) = \mu(1-x)\partial_x w(x,t). \tag{2.1.17}$$

Boundary conditions at $x = 1$

$$M(1,t) = V(1,t) = 0.$$

Modifications on some of the interface conditions are necessary for the Timoshenko theory and we state the full set of interface conditions. Note that the first and last interface condition differ from those for the Rayleigh theory.

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Interface conditions

$$
\begin{aligned}
M_{BA}(t) &= k_{BA}\Big(\phi(0,t)-\theta_B(t)\Big)+c_{BA}\Big(\partial_t\phi(0,t)-\dot{\theta}_B(t)\Big),\\
M_{BA}(t) &= M(0,t),\\
V_{BA}(t) &= V(0,t),\\
w_B(t) &= w(0,t),\\
\theta_B(t) &\neq \phi(0,t) \quad \text{(in general)}.
\end{aligned}
$$

Problem VT 3

Equations of motion: (2.1.13) and (2.1.14).

Constitutive equations: (2.1.15), (2.1.16) and (2.1.17).

Boundary conditions at $x = 1$: $M(1,t) = V(1,t) = 0$.

As in the Rayleigh models, the motion of B is neglected and B is considered to be rigidly connected to F. Hence

$$
w_F(t) = w_B(t) = w(0,t), \quad \theta_F = \theta_B, \quad V_{FB}(t) = V_{BA}(t) = V(0,t)
$$

$$
\text{and} \quad M_{FB}(t) = M_{BA}(t) = M(0,t).
$$

The constant k_{BA} is replaced by k and c_{BA} by c.

The interface conditions and the equations of motion of the foundation block and resilient seating reduce to the following three equations:

$$
m_F\partial_t^2 w(0,t) = V(0,t) - K_F w(0,t) - C_F\partial_t w(0,t), \tag{2.1.18}
$$

$$
\begin{aligned}
I_F\ddot{\theta}_F(t) &= k\Big(\phi(0,t)-\theta_F(t)\Big)+c\Big(\partial_t\phi(0,t)-\dot{\theta}_F(t)\Big)\\
&\quad -k_F\theta_F(t)-c_F\dot{\theta}_F(t),
\end{aligned} \tag{2.1.19}
$$

$$
M(0,t) = k\Big(\phi(0,t)-\theta_F(t)\Big)+c\Big(\partial_t\phi(0,t)-\dot{\theta}_F(t)\Big). \tag{2.1.20}
$$

Problem VT 4

Equations of motion: (2.1.13) and (2.1.14).

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Constitutive equations: (2.1.15), (2.1.16) and (2.1.17).

Boundary conditions at $x = 1$: $M(1,t) = V(1,t) = 0$.

The interface conditions are given by

$$\begin{aligned} m_B\partial_t^2 w(0,t) &= V(0,t) - K_{FB}\Big(w(0,t) - w_F(t)\Big) \\ &\quad - C_{FB}\Big(\partial_t w(0,t) - \dot{w}_F(t)\Big), \end{aligned} \tag{2.1.21}$$

$$\begin{aligned} I_B\ddot{\theta}_B(t) &= k_{BA}\Big(\phi(0,t) - \theta_B(t)\Big) + c_{BA}\Big(\partial_t\phi(0,t) - \dot{\theta}_B(t)\Big) \\ &\quad - k_{FB}\Big(\theta_B(t) - \theta_F(t)\Big) - c_{FB}\Big(\dot{\theta}_B(t) - \dot{\theta}_F(t)\Big), \end{aligned} \tag{2.1.22}$$

$$M(0,t) = k_{BA}\Big(\phi(0,t) - \theta_B(t)\Big) + c_{BA}\Big(\partial_t\phi(0,t) - \dot{\theta}_B(t)\Big), \tag{2.1.23}$$

$$\begin{aligned} m_F\ddot{w}_F(t) &= K_{FB}\Big(w(0,t) - w_F(t)\Big) + C_{FB}\Big(\partial_t w(0,t) - \dot{w}_F(t)\Big) \\ &\quad - K_F w_F(t) - C_F\dot{w}_F(t), \end{aligned} \tag{2.1.24}$$

$$\begin{aligned} I_F\ddot{\theta}_F(t) &= k_{FB}\Big(\theta_B(t) - \theta_F(t)\Big) + c_{FB}\Big(\dot{\theta}_B(t) - \dot{\theta}_F(t)\Big) \\ &\quad - k_F\theta_F(t) - c_F\dot{\theta}_F(t). \end{aligned} \tag{2.1.25}$$

2.2 The cantilever beam

In Chapter 7 we compare the natural frequencies of the Euler-Bernoulli and Timoshenko models for the free vibration of a cantilever beam. For reference purposes we state the equations of motion, constitutive equations and the standard boundary conditions.

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Timoshenko theory

$$
\begin{aligned}
\partial_t^2 w &= \partial_x V, \\
\frac{1}{\alpha}\,\partial_t^2 \phi &= V + \partial_x M, \\
M &= \frac{1}{\beta}\,\partial_x \phi, \\
V &= \partial_x w - \phi, \\
M(1,t) &= V(1,t) \;=\; 0, \\
w(0,t) &= \phi(0,t) \;=\; 0.
\end{aligned}
$$

Euler-Bernoulli theory

$$
\begin{aligned}
\partial_t^2 w &= \partial_x V, \\
0 &= V + \partial_x M, \\
M &= \frac{1}{\beta}\,\partial_x^2 w, \\
M(1,t) &= V(1,t) \;=\; 0, \\
w(0,t) &= \partial_x w(0,t) \;=\; 0.
\end{aligned}
$$

A modification of the boundary conditions for the Timoshenko model (suggested in [V3]) is

$$
\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}
\begin{bmatrix} V(0,t) \\ M(0,t) \end{bmatrix}
=
\begin{bmatrix} w(0,t) \\ \phi(0,t) \end{bmatrix}.
$$

The standard boundary conditions for the Timoshenko model is a special case of the modified boundary conditions, where

$$
c_{11} = c_{12} = c_{21} = c_{22} = 0.
$$

2.3 Two-dimensional model for a cantilever beam

We consider a prismatic beam built in at one end. In Section 1.4 a two-dimensional model is proposed. The equation of motion and the constitutive equation are given by Equations (1.4.1) and (1.4.2).

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It is not obvious how to model the built in end of a cantilever beam. Therefore we consider different configurations and discuss them briefly. A detailed discussion is given in Section 7.2.1.

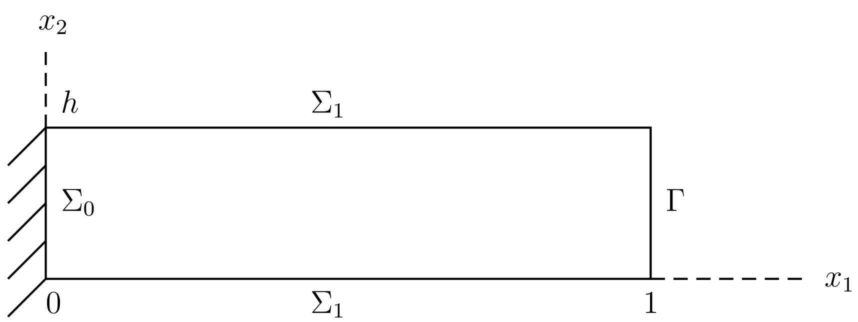
Rigidly attached beam

We consider a rigidly attached beam as in Figure 1. For this case the reference configuration Ω is the rectangle given by

$$0 \leq x_1 \leq 1, \quad 0 \leq x_2 \leq h$$

and the beam is attached at $x_1 = 0$.

Figure 1: Rigidly attached beam



Built in beam

In this case we consider a beam that is built in at $x_1 = 0$ as in Figure 2. The reference configuration Ω is the rectangle given by

$$-a \leq x_1 \leq 1, \quad 0 \leq x_2 \leq h.$$

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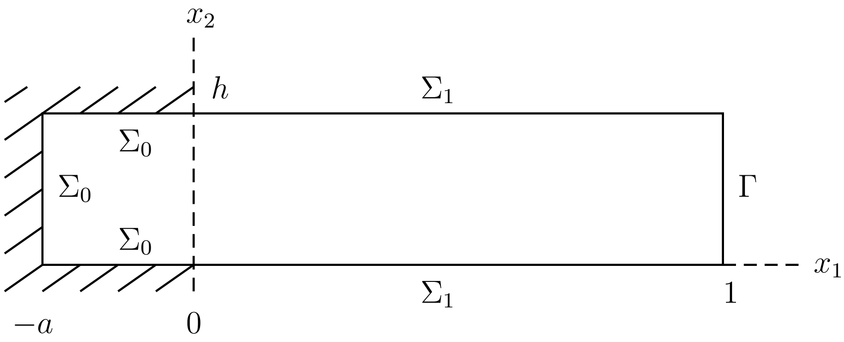
Figure 2: Built in beam



To apply the theory, it is preferable to formulate the model problems for a general domain. Let Ω be an open convex subset in the plane. The boundary of Ω consists of smooth curves, $\Sigma_1, \Sigma_2, \ldots, \Sigma_m$ and Γ.

Boundary conditions

The traction $\boldsymbol{t} = T\boldsymbol{n}$ is specified on Γ and on Σ_i we have $T\boldsymbol{n} \cdot \boldsymbol{u} = 0$ for each i, with the additional restriction that $u_1 = 0$ on at least one of the sets Σ_i and $u_2 = 0$ on at least one of the sets Σ_j.

Equilibrium problem

For the equilibrium problem a transverse force is applied at Γ. However, for the boundary value problem it is necessary to prescribe the traction on Γ.

Problem CTD 1

$$\begin{aligned} \operatorname{div} T &= \mathbf{0} \quad \text{in } \Omega, \\ T\boldsymbol{n} \cdot \boldsymbol{u} &= 0 \quad \text{on } \Sigma, \\ T\boldsymbol{n} &= \boldsymbol{t} \quad \text{on } \Gamma, \end{aligned}$$

with the constitutive equation given by Equation (1.4.2).

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Free vibration

Problem CTD 2

$$
\begin{aligned}
\partial_t^2 \boldsymbol{u} &= \operatorname{div} T \quad \text{in } \Omega, \\
T\boldsymbol{n} \cdot \boldsymbol{u} &= 0 \quad \text{on } \Sigma, \\
T\boldsymbol{n} &= \boldsymbol{0} \quad \text{on } \Gamma,
\end{aligned}
$$

with the constitutive equation given by Equation (1.4.2).

Remark

The condition $T\boldsymbol{n} \cdot \boldsymbol{u} = 0$ represents a number of possibilities, e. g. $\boldsymbol{u} = \boldsymbol{0}$ or $T\boldsymbol{n} = \boldsymbol{0}$ or various different combinations. The different configurations are given in Chapter 7.

2.4 A plate-beam system

Consider small transverse vibration of a thin rectangular plate supported by identical beams at two opposing sides and rigidly supported at the remaining sides. The beams are supported at their endpoints. Assume furthermore the case of free vibration, i.e. $q = 0$. The displacement for the system is measured with respect to the equilibrium state. (Due to gravity, the equilibrium state is not the same as the undeformed state.) It is assumed that the plate remains in contact with the beams and supporting structure at all times. This mathematical model is considered in [V4].

The reference configuration for the plate is the rectangle Ω, where $0 \leq x_1 \leq 1$ and $0 \leq x_2 \leq a$. The plate is rigidly supported at $x_1 = 0$ and $x_1 = 1$. These sections of the boundary of Ω are denoted by Σ_0 and Σ_1 respectively. The plate is supported by beams at $x_2 = 0$ and $x_2 = a$ and these sections are denoted by Γ_0 and Γ_1 respectively. Figure 1 depicts the reference configuration. The shaded areas represent the beams.

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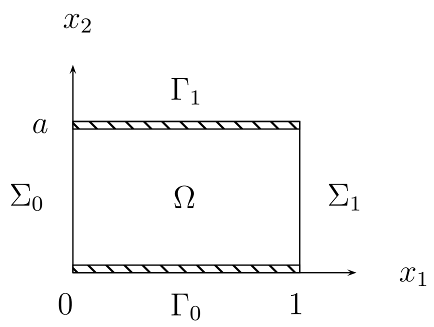
Figure 1: Reference configuration of the plate-beam system.



Notation

To avoid confusion, we adapt (if necessary) the symbols used for quantities related to the beams by using the subscript "b".

2.4.1 The Reissner-Mindlin-Timoshenko model

For the mathematical model we use the Reissner-Mindlin plate theory and the Timoshenko beam theory. On the rectangle Ω, the equations of motion (1.3.1) and (1.3.2) are satisfied and on Γ_0 and Γ_1, the two sets of equations of motion are given by (1.2.1) and (1.2.2). In Equation (1.2.2), L represents a moment density transmitted from the plate to the beam and P a force desity transmitted from the plate to the beam.

Boundary conditions on Σ_0 and Σ_1

On these sections of the boundary, the conventional homogeneous boundary conditions for a rigidly supported plate are used, i.e.

$$w = 0, \quad \psi_2 = 0 \quad \text{and} \quad M\boldsymbol{n} \cdot \boldsymbol{n} = 0, \tag{2.4.1}$$

where $\boldsymbol{n}$ is the unit exterior normal (see [Rei, p 66]). The third condition reduces to $M_{11} = 0$.

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Interface conditions on Γ_0 and Γ_1

On Γ_0 and Γ_1 the interaction between the plate and the beams is considered. The interface conditions are given in [V4] for a general case. For this special case they reduce to

$$w_b(x_1,t) = w(x_1,0,t) \text{ on } \Gamma_0\,, \quad w_b(x_1,t) = w(x_1,a,t) \text{ on } \Gamma_1\,, \tag{2.4.2}$$
$$\phi_b(x_1,t) = -\psi_1(x_1,0,t) \text{ on } \Gamma_0\,, \quad \phi_b(x_1,t) = -\psi_1(x_1,a,t) \text{ on } \Gamma_1\,. \tag{2.4.3}$$

The interface conditions for the force densities and moment densities on Γ_0 and Γ_1 are given by

$$\boldsymbol{Q}\cdot\boldsymbol{n} = -P, \tag{2.4.4}$$
$$M\boldsymbol{n}\cdot\boldsymbol{\tau} = L, \tag{2.4.5}$$
$$M\boldsymbol{n}\cdot\boldsymbol{n} = 0, \tag{2.4.6}$$

where $\boldsymbol{\tau}$ is the unit tangent oriented in such a way that Ω is on the left hand side of $\boldsymbol{\tau}$. For a detailed explanation of the moments $M\boldsymbol{n}\cdot\boldsymbol{n}$ and $M\boldsymbol{n}\cdot\boldsymbol{\tau}$, see [Rei, p 66].

Remarks

1. Note the difference in sign convention for measuring the angles $\boldsymbol{\psi}$ and ϕ_b in the plate and beam models.

2. Care should be taken to also incorporate the difference between sign conventions for moments in the plate and beam models. The beam equations for Γ_1 is derived for a beam oriented from left to right. When applying the interface condition (2.4.5) on Γ_1, the moment L has to be replaced by $-L$.

Conditions at the endpoints of Γ_0 and Γ_1

At the endpoints of Γ_0 and Γ_1 we have the obvious boundary conditions for the beams, namely

$$w_b = 0 \quad \text{and} \quad M_b = 0. \tag{2.4.7}$$

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Dimensionless form

The dimensionless form for the plate model has been derived in Section 1.3.3. For the beam equations it has to be recalculated using the scaling of the plate model and

$$\tau = \frac{t}{t_0} \quad \text{and} \quad \xi_1 = \frac{x_1}{\ell}\,.$$

Also set

$$w_b^* = \left(\frac{1}{\ell}\right) w_b\,, \quad \phi_b^* = \phi_b\,,$$

$$P^* = \left(\frac{1}{\ell G\kappa^2}\right) P, \quad V^* = \left(\frac{1}{\ell^2 G\kappa^2}\right) V,$$

$$M_b^* = \left(\frac{1}{\ell^3 G\kappa^2}\right) M_b \quad \text{and} \quad L^* = \left(\frac{1}{\ell^2 G\kappa^2}\right) L.$$

Note that the parameters of the plate are used for the scaling. Choosing $t_0 = \ell\sqrt{\dfrac{\rho}{G\kappa^2}}$ as in Section 1.3.3 and using the original notation for the corresponding dimensionless quantities, the dimensionless beam model is given by

$$\eta_1\,\partial_t^2 w_b = \partial_1 V + P, \tag{2.4.8}$$

$$\eta_1\,\partial_t^2 \phi_b = \alpha_b\,(\partial_1 M_b + V + L), \tag{2.4.9}$$

$$V = \eta_2\,(\partial_1 w_b - \phi_b), \tag{2.4.10}$$

$$\beta_b M_b = \eta_2\,\partial_1 \phi_b. \tag{2.4.11}$$

The dimensionless constants α_b and β_b are as in Section 1.2.4, i.e.

$$\alpha_b = \frac{A_b \ell^2}{I_b}, \quad \beta_b = \frac{A_b G_b \kappa_b^2 \ell^2}{E_b I_b}\,.$$

The two additional dimensionless constants η_1 and η_2 express ratios for the material properties and the geometrical properties of the plate and the beams:

$$\eta_1 = \left(\frac{\rho_b}{\rho}\right)\left(\frac{A_b}{\ell^2}\right) \quad \text{and} \quad \eta_2 = \left(\frac{G_b}{G}\right)\left(\frac{\kappa_b^2}{\kappa^2}\right)\left(\frac{A_b}{\ell^2}\right).$$

The interface conditions remain unchanged.

The mathematical model

The vibration problem for the plate-beam system is given by the following equations.

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Problem RMT

Equations of motion for the plate: (1.3.5) and (1.3.6) on Ω.

Constitutive equations for the plate: (1.3.7) and (1.3.8) on Ω.

Equations of motion for the beams: (2.4.8) and (2.4.9) on Γ_0 and Γ_1 .

Constitutive equations for the beams: (2.4.10) and (2.4.11) on Γ_0 and Γ_1 .

Interface conditions: (2.4.2) to (2.4.6) on Γ_0 and Γ_1 .

Boundary conditions: (2.4.1) on Σ_0 and Σ_1 .

Endpoint conditions: (2.4.7) at the endpoints of Γ_0 and Γ_1 .

2.4.2 Other models

A simplified model is obtained if the Kirchhoff plate model (with rotary inertia) and the Rayleigh beam model is used. Formally, this model problem can be derived from Problem RMT. We consider the model problems referred to for the purpose of comparison. It should be noted that the scaling for the dimensionless form differs from the scaling used in [ZVGV3] and [Ziet].

In this case the vibration problem for the plate-beam system is given by the following equations.

Problem KR

Equations of motion for the plate: (1.3.9) and (1.3.10) on Ω.

Constitutive equation for the plate: (1.3.11) on Ω.

Equations of motion for the beams:

$$\eta_1 \partial_t^2 w_b = \partial_1 V + P \text{ on } \Gamma_0 \text{ and } \Gamma_1\,,$$

$$\eta_1 \partial_t^2 \partial_x w_b = \alpha_b(\partial_1 M_b + V + L) \text{ on } \Gamma_0 \text{ and } \Gamma_1\,.$$

Constitutive equation for the beams:

$$\beta_b M_b = \eta_2 \partial_1^2 w_b \text{ on } \Gamma_0 \text{ and } \Gamma_1\,.$$

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Interface conditions: (2.4.2) to (2.4.6) on Γ_0 and Γ_1 .

Boundary conditions: (2.4.1) on Σ_0 and Σ_1 .

Endpoint conditions: (2.4.7) at the endpoints of Γ_0 and Γ_1 .

Problem KEB

An even simpler model is obtained if rotary inertia is ignored in the plate and the beams.