Finite element analysis of plate and beam models

by

Anneke Labuschagne

Submitted in partial fulfillment of the requirements for the degree

Philosophiae Doctor

in the Faculty of Natural and Agricultural Sciences

University of Pretoria

Pretoria

July 2006
DECLARATION

I, the undersigned, hereby declare that the thesis submitted herewith for the degree Philosophiae Doctor to the University of Pretoria contains my own, independent work and has not been submitted for any degree at any other university.

Name: Anneke Labuschagne

Date: July 2006
Title: Finite element analysis of plate and beam models
Name: Anneke Labuschagne
Supervisor: Prof N F J van Rensburg
Co-supervisor: Dr A J van der Merwe
Department: Mathematics and Applied Mathematics
Degree: Philosophiae Doctor

Summary

We consider linear mathematical models for elastic plates and beams. To be specific, we consider the Euler-Bernoulli, Rayleigh and Timoshenko theories for beams and the Kirchhoff and Reissner-Mindlin theories for plates.

The theories mentioned above refer to the partial differential equations that model a beam or plate. The contact with other objects also need to be modelled. The equations that result are referred to as “interface conditions”.

We consider three problems concerning interface conditions for plates and beams: A vertical slender structure on a resilient seating, the built in end of a beam and a plate-beam system.

The vertical structure may be modelled as a vertically mounted beam. However, the dynamics of the seating must be included in the model and this increases the complexity of a finite element analysis considerably. We show that the interface conditions and additional equations can be accommodated in the variational form and that the finite element method yields excellent results.

Although the Timoshenko model is considered to be better than the Euler-Bernoulli model, some authors do not agree that it is an improvement for the case of a cantilever beam. In a modal analysis of a two-dimensional beam model, we show that the Timoshenko model is not only better, but it provides good results when the beam is so short that one is reluctant to use beam theory at all.

In applications, structures consisting of linked systems of beams and plates are encountered. We consider a rectangular plate connected to two beams. Combining the Reissner-Mindlin plate model and the Timoshenko beam model can be seen as a first step towards a better model while still avoiding the complexity of a fully three-dimensional model. However, the modelling of
the plate-beam system is more complex than in the case of the classical theory and the mathematical analysis and numerical analysis present additional difficulties.

A weak variational form is derived for all the model problems. This is necessary to apply general existence and uniqueness results. It is also necessary to apply general convergence results and derive error bounds. The setting for the weak variational forms are product spaces. This is due to the complex nature of the model problems.
Contents

1 Modelling interface conditions 1

1.1 Introduction .................................................... 1
  1.1.1 A vertical slender structure on a resilient seating .... 2
  1.1.2 Boundary conditions for the clamped end of a beam ... 3
  1.1.3 Plate-beam systems ........................................... 3

1.2 Beam theory .................................................... 4
  1.2.1 Equations of motion ......................................... 4
  1.2.2 The Timoshenko model ....................................... 5
  1.2.3 The Euler-Bernoulli and Rayleigh models ................. 6
  1.2.4 Dimensionless form .......................................... 7

1.3 Plate theory .................................................... 9
  1.3.1 Equations of motion ......................................... 9
  1.3.2 The Reissner-Mindlin and Kirchhoff models ............... 10
  1.3.3 Dimensionless forms ......................................... 11

1.4 Two-dimensional model for a beam .......................... 12
  1.4.1 Equation of motion .......................................... 12
  1.4.2 Dimensionless form .......................................... 14
# CONTENTS

1.5 Interface conditions ........................................ 15
  1.5.1 Vertical slender structure ............................... 15
  1.5.2 Boundary conditions for the clamped end of a beam . . 16
  1.5.3 Plate-beam system ..................................... 17

2 Model problems .............................................. 19
  2.1 Vertical slender structure ................................ 19
    2.1.1 Simplistic Models .................................. 20
    2.1.2 The dynamics of the foundation block and resilient seating .................. 20
    2.1.3 Rayleigh models .................................... 23
    2.1.4 Timoshenko models ................................ 26
  2.2 The cantilever beam .................................... 28
  2.3 Two-dimensional model for a cantilever beam .......... 29
  2.4 A plate-beam system ................................... 32
    2.4.1 The Reissner-Mindlin-Timoshenko model ............ 33
    2.4.2 Other models ..................................... 36

3 Variational forms .................................. 39
  3.1 Introduction ........................................... 39
  3.2 Vertical slender structure: Rayleigh models ............ 44
    3.2.1 Variational forms ................................ 44
    3.2.2 Weak variational forms ............................ 47
  3.3 Vertical slender structure: Timoshenko models ......... 55
CONTENTS

3.3.1 Variational forms ........................................ 55
3.3.2 Weak variational forms .................................... 57
3.4 The cantilever beam ........................................ 63
3.5 Two-dimensional model for the cantilever beam .............. 64
  3.5.1 Variational forms ........................................ 64
  3.5.2 Weak variational forms .................................... 66
3.6 Plate-beam system ........................................ 69
  3.6.1 Variational form of problem RMT ...................... 69
  3.6.2 Variational form of Problems KR and KEB .......... 74
  3.6.3 Weak variational form of Problem RMT .............. 76
3.7 Equilibrium problems ........................................ 79
3.8 Vibration problems .......................................... 81
3.9 Modal analysis .............................................. 82
3.10 Nonmodal damping .......................................... 85

4 Interpolation ................................................... 87
  4.1 Hermite cubics .............................................. 87
  4.2 Hermite bicubic functions .................................. 89
  4.3 Standard estimates for the interpolation error ............. 90
    4.3.1 One-dimensional domain ................................ 91
    4.3.2 Two-dimensional domain ................................ 92
    4.3.3 Vector-valued functions ............................... 93
  4.4 Interpolation estimates for the one-dimensional hybrid models ............................................... 94
  4.5 Interpolation estimates for the plate-beam system .......... 96
5 Approximation

5.1 Projections ........................................... 99
  5.1.1 One-dimensional models ......................... 100
  5.1.2 Two-dimensional models ................. 101

5.2 Equilibrium problems .................................. 103

5.3 Symmetrical eigenvalue problems .............. 103

5.4 Non selfadjoint eigenvalue problem ............. 105
  5.4.1 Abstract eigenvalue problem ................. 105
  5.4.2 Galerkin approximation ..................... 107
  5.4.3 Operator approximations ................. 108
  5.4.4 Convergence ........................................ 110
  5.4.5 Application ........................................ 111

6 Vertical slender structure .............................. 115

6.1 Introduction ........................................... 115

6.2 The eigenvalue problem ................................ 116
  6.2.1 The Rayleigh model ............................. 116
  6.2.2 The Timoshenko model ...................... 117

6.3 Galerkin approximations for the eigenvalue problem ... 119
  6.3.1 Rayleigh models ................................. 119
  6.3.2 Timoshenko models ...................... 120

6.4 Matrix form of the semi-discrete problem ........... 123
  6.4.1 The Rayleigh models ...................... 123
  6.4.2 The Timoshenko models ................... 124
CONTENTS

6.5 Numerical results .............................................. 127
   6.5.1 Physical constants ............................... 128
   6.5.2 Convergence .................................. 129
   6.5.3 Effect of gravity, rotary inertia and shear ...... 129
   6.5.4 Conclusion .................................. 130

7 Cantilever beam ................................................... 133
   7.1 Scope of the investigation ......................... 133
   7.2 Boundary conditions and test functions .......... 135
      7.2.1 Boundary conditions ......................... 135
      7.2.2 Test functions ................................ 138
   7.3 Galerkin approximation ............................ 139
      7.3.1 Equilibrium problem ........................ 140
      7.3.2 The eigenvalue problem ...................... 141
   7.4 Matrix formulation .................................. 141
      7.4.1 Construction of the matrices $K$ and $M$ ...... 142
   7.5 Shear strain distribution ............................ 143
   7.6 Deflection ........................................ 146
   7.7 Eigenvalues and eigenfunctions ..................... 147

8 Plate-beam system ............................................... 151
   8.1 Introduction ........................................ 151
      8.1.1 Pinned-pinned beam .......................... 151
      8.1.2 Rigidly supported plate ....................... 153
      8.1.3 Plate-beam system ............................ 156
CONTENTS

8.2 The eigenvalue problems ........................................ 156
  8.2.1 Reissner-Mindlin-Timoshenko plate-beam system . . . 156
  8.2.2 Kirchhoff-Rayleigh plate-beam system ................. 157
  8.2.3 Kirchhoff-Euler-Bernoulli plate-beam system .......... 158

8.3 Galerkin approximations for the
  eigenvalue problems ........................................... 158
  8.3.1 Galerkin approximation for Problem RMT ............... 158
  8.3.2 Galerkin approximation for Problem KEB ............... 158

8.4 Matrix formulation of Galerkin
  approximations ................................................. 159
  8.4.1 Construction of $K$ and $M$ for Problem RMT .......... 160
  8.4.2 Construction of $K$ and $M$ for Problem KEB .......... 161

8.5 Numerical results ............................................ 161
  8.5.1 Parameters ................................................ 161
  8.5.2 Convergence .............................................. 162
  8.5.3 Comparison of Reissner-Mindlin-Timoshenko
    system with Kirchhoff-Euler-Bernoulli system .......... 163
  8.5.4 Comparison of Kirchhoff-Euler-Bernoulli system with
    a rigidly supported Kirchhoff plate ...................... 164
  8.5.5 Comparison of Reissner-Mindlin-Timoshenko
    system with a rigidly supported Reissner-Mindlin plate 165
  8.5.6 Pure shear modes ........................................ 166