Appendix A

Derivation of the elliptic equation directly from the $\sigma$ coordinate quasi-elastic equations

In this appendix, the elliptic equation (3.81) is derived directly from the $\sigma$ coordinate equations (3.64) to (3.68). Multiplying (3.66) by $s$ gives:

$$s\frac{R}{g} \frac{D}{Dt} \left( \frac{\omega T}{p} \right) + gs + s^2 \frac{\partial \phi}{\partial \sigma} = 0. \quad (A.1)$$

Taking $\frac{\partial}{\partial x}$ of (3.64), $\frac{\partial}{\partial y}$ of (3.65) and $\frac{\partial}{\partial \sigma}$ of (A.1), and adding the three resulting equations gives:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial}{\partial \sigma} \left( \frac{s^2 \frac{\partial \phi}{\partial \sigma}}{\partial x} + \frac{\sigma}{\partial \sigma} \left( \frac{\partial^2 \ln p_s}{\partial x^2} + \frac{\partial^2 \ln p_s}{\partial y^2} \right) \right)
= - \frac{\partial}{\partial x} \left( \frac{Du}{Dt} \right) + \frac{\partial}{\partial x} \left( \frac{Dv}{Dt} \right)
$$

$$- \frac{\partial}{\partial \sigma} \left( \frac{p}{\rho_s T} \frac{D}{Dt} \left( \frac{\omega T}{p} \right) \right) + f \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) - u \frac{df}{dy} - \frac{\partial}{\partial \sigma} (sg). \quad (A.2)$$

In order to simplify (A.2), it is necessary to expand the terms $-\frac{\partial (Du/Dt)}{\partial x}$, $-\frac{\partial (Dv/Dt)}{\partial y}$ and $-\frac{\partial [(p/p_s T) D (\omega T/p)]/Dt}{\partial \sigma}$. First note that from expanding the total derivatives of the two components of the horizontal wind, it follows that:

$$- \frac{\partial}{\partial x} \left( \frac{Du}{Dt} \right) = - \frac{D}{Dt} \left( \frac{\partial u}{\partial x} \right) - \left( \frac{\partial u}{\partial x} \frac{\partial u}{\partial x} + \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + \frac{\partial \sigma}{\partial \sigma} \frac{\partial u}{\partial x} \right) \quad (A.3)$$
and

\[-\frac{\partial}{\partial y} \left( \frac{Dv}{Dt} \right) = -\frac{D}{Dt} \left( \frac{Dv}{\partial y} \right) - \left( \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} \frac{\partial v}{\partial y} + \frac{\partial v}{\partial y} \frac{\partial v}{\partial \sigma} \right). \quad (A.4)\]

Finding an appropriate expansion for \(-\partial [(p/p_sT) D(\omega T/p)/Dt] \partial \sigma\) requires a more extensive procedure. From the use of relationship (3.69), it follows that

\[
\frac{p}{p_s T} \frac{D}{Dt} \left( \frac{\omega T}{p} \right) = \frac{p}{p_s T} \frac{D}{Dt} \left[ \frac{T p_s}{p} \left( \sigma \frac{D \ln p_s}{Dt} + \dot{\sigma} \right) \right]
\]

\[
= \frac{p}{p_s T} \left[ \frac{D}{Dt} \left( \frac{T p_s \sigma}{p} \frac{D \ln p_s}{Dt} \right) + \frac{D}{Dt} \left( \frac{T p_s}{p} \sigma \right) \right]
\]

\[
= \frac{\sigma}{D^2 \ln p_s} \frac{D^2 \ln p_s}{Dt^2} \left[ A - B - C \right] + \left( \frac{p}{p_s T} \frac{D}{Dt} \left( \frac{T p_s}{p} \right) \right) \frac{D \ln p_s}{Dt}.
\quad (A.5)
\]

Noting that

\[
B = \left[ \dot{\sigma} + \frac{\sigma p}{p_s T} \frac{D}{Dt} \left( \frac{T p_s}{p} \right) \right] \frac{D \ln p_s}{Dt}
\quad (A.6)
\]

and

\[
C = \frac{D}{Dt} \left( \frac{\sigma p}{p_s T} \frac{D}{Dt} \left( \frac{T p_s}{p} \right) \right)
\quad (A.7)
\]

it follows that

\[
\frac{\partial A}{\partial \sigma} = \frac{\sigma D^2 \ln p_s}{Dt^2} + \frac{\partial}{\partial \sigma} \left( \frac{\sigma D^2 \ln p_s}{Dt^2} \right),
\quad (A.8)
\]

\[
\frac{\partial B}{\partial \sigma} = \left[ \dot{\sigma} + \frac{\sigma p}{p_s T} \frac{D}{Dt} \left( \frac{T p_s}{p} \right) \right] \frac{D \ln p_s}{Dt}
\quad (A.9)
\]

and

\[
\frac{\partial C}{\partial \sigma} = \frac{D}{Dt} \left( \frac{\sigma p}{p_s T} \frac{D}{Dt} \left( \frac{T p_s}{p} \right) \right)
\quad (A.10)
\]
Equation (A.10) is derived from expanding the total derivative of \( \dot{\sigma} \) in (A.7). From combining (A.8) to (A.10) it is obtained that:

\[
- \frac{\partial}{\partial \sigma} \left[ \frac{p}{p_s T} \left( \frac{D}{Dt} \left( \frac{\omega}{p} \right) \right) \right] = - \frac{D}{Dt} \left( \frac{\partial \dot{\sigma}}{\partial \sigma} \right) - \left[ \frac{\partial u}{\partial \sigma} \frac{\partial \dot{\sigma}}{\partial x} + \frac{\partial v}{\partial \sigma} \frac{\partial \dot{\sigma}}{\partial y} + \frac{\partial \dot{\sigma}}{\partial \sigma} \frac{\partial \dot{\sigma}}{\partial \sigma} \right]
\]

\[
- \frac{\partial}{\partial \sigma} \left[ \sigma \left( \frac{p}{p_s T} \right) \frac{D}{Dt} \left( \frac{T_p}{p} \right) \right] = - \frac{D}{Dt} \left( \frac{D \ln p_s}{Dt} \right) - \sigma \frac{\partial}{\partial \sigma} \left( \frac{D^2 \ln p_s}{Dt^2} \right)
\]

\[
- \frac{\partial}{\partial \sigma} \left\{ \left[ \sigma + \frac{\sigma p}{p_s T} \frac{D}{Dt} \left( \frac{T_p}{p} \right) \right] \frac{D \ln p_s}{Dt} \right\}.
\]  (A.11)

From substituting (A.3), (A.4) and (A.11) into (A.2), and applying the continuity equation (3.67), it follows that:

\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial \phi}{\partial \sigma} \left( \frac{\partial^2 \ln p_s}{\partial x^2} + \frac{\partial^2 \ln p_s}{\partial y^2} \right)
\]

\[
- \sigma \left[ \frac{\partial \ln p_s}{\partial x} \left( \frac{\partial^2 \phi}{\partial x \partial \sigma} \right) + \frac{\partial \ln p_s}{\partial y} \left( \frac{\partial^2 \phi}{\partial y \partial \sigma} \right) \right]
\]

\[
+ \left( \frac{\partial u}{\partial x} \frac{\partial u}{\partial \sigma} - \frac{\partial v}{\partial x} \frac{\partial u}{\partial \sigma} \right)_{B_{2u}} + \left( \frac{\partial u}{\partial y} \frac{\partial v}{\partial \sigma} - \frac{\partial v}{\partial y} \frac{\partial v}{\partial \sigma} \right)_{B_{4v}}
\]

\[
+ \left[ \frac{\partial u}{\partial \sigma} \frac{\partial \dot{\sigma}}{\partial x} - \frac{\partial v}{\partial \sigma} \frac{\partial \dot{\sigma}}{\partial y} \right]_{B_{4u}} - \frac{\partial}{\partial \sigma} \left\{ \left[ \sigma + \frac{\sigma p}{p_s T} \frac{D}{Dt} \left( \frac{T_p}{p} \right) \right] \frac{D \ln p_s}{Dt} \right\}
\]

\[
- \frac{\partial}{\partial \sigma} \left\{ \sigma + \frac{\sigma p}{p_s T} \frac{D}{Dt} \left( \frac{T_p}{p} \right) \right\} \frac{D \ln p_s}{Dt} + f \left( \frac{\partial v}{\partial x} \frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} \frac{\partial u}{\partial \sigma} \right) - u \frac{df}{dy} - \frac{\partial}{\partial \sigma} (sg).
\]  (A.12)

In order to simplify (A.12), the labeled terms need to be written in alternative form. Firstly, note that

\[
B_1 = - \sigma \frac{\partial}{\partial \sigma} \left( \frac{D^2 \ln p_s}{Dt^2} \right) = - \sigma \left[ \frac{\partial \ln p_s}{\partial x} \frac{\partial (Du)}{\partial \sigma} \frac{D}{Dt} + \frac{\partial \ln p_s}{\partial y} \frac{\partial (Dv)}{\partial \sigma} \frac{D}{Dt} \right]
\]

\[
- 2 \sigma \left[ \frac{\partial u}{\partial \sigma} \frac{D}{Dt} \left( \frac{\partial \ln p_s}{\partial x} \right) + \frac{\partial v}{\partial \sigma} \frac{D}{Dt} \left( \frac{\partial \ln p_s}{\partial y} \right) \right].
\]  (A.13)
By expanding the total derivatives of $u$ and $v$ in (A.13), it is obtained that:

$$
B_1 = -\sigma \frac{\partial}{\partial \sigma} \left( \frac{D^2 \ln p_s}{Dt^2} \right) = \sigma f \left( \frac{\partial u}{\partial \sigma} \frac{\partial \ln p_s}{\partial y} - \frac{\partial v}{\partial \sigma} \frac{\partial \ln p_s}{\partial x} \right) 
$$

$$
+ \sigma \left( \frac{\partial^2 \phi}{\partial x \partial \phi} \frac{\partial \ln p_s}{\partial x} + \frac{\partial^2 \phi}{\partial y \partial \phi} \frac{\partial \ln p_s}{\partial y} \right) - \sigma \frac{\partial}{\partial \sigma} \left( \frac{\partial \phi}{\partial \sigma} \right) \left[ \left( \frac{\partial \ln p_s}{\partial x} \right)^2 + \left( \frac{\partial \ln p_s}{\partial y} \right)^2 \right] 
$$

$$
- 2\sigma \left[ \frac{\partial u}{\partial \sigma} \frac{\partial \ln p_s}{\partial x} + \frac{\partial v}{\partial \sigma} \frac{\partial \ln p_s}{\partial y} \right]. \tag{A.14}
$$

From making use of the continuity equation (3.67) it follows that:

$$
B_{2a} + B_{2b} = 2 \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \left( \frac{D \ln p_s}{Dt} + \frac{\partial \sigma}{\partial \sigma} \right). \tag{A.15}
$$

From differentiating relationship (3.69) with respect to $\sigma$ it follows that

$$
\frac{\partial \sigma}{\partial \sigma} = \frac{\partial}{\partial \sigma} \left( \frac{p}{p_s} \right) - \sigma \frac{D \ln p_s}{Dt} - \sigma \left( \frac{\partial u}{\partial \sigma} \frac{\partial \ln p_s}{\partial x} + \frac{\partial v}{\partial \sigma} \frac{\partial \ln p_s}{\partial y} \right). \tag{A.16}
$$

It follows, from applying the continuity equation, that

$$
B_3 = - \left( \frac{\partial \sigma}{\partial \sigma} \right)^2 = \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \frac{\partial \sigma}{\partial \sigma} + \frac{D \ln p_s}{Dt} \frac{\partial \sigma}{\partial \sigma}. \tag{A.17}
$$

Making use of (3.69) to write $\dot{\sigma}$ in terms of $\omega$ and $D \ln p_s / Dt$ implies that

$$
B_{4a} + B_{4b} + B_{4c} + B_{4d} = -2 \left( \frac{\partial u}{\partial \sigma} \frac{\partial \sigma}{\partial x} + \frac{\partial v}{\partial \sigma} \frac{\partial \sigma}{\partial y} \right) = -2 \left[ \frac{\partial u}{\partial \sigma} \frac{\partial \sigma}{\partial x} \left( \frac{p}{p_s} \right) + \frac{\partial v}{\partial \sigma} \frac{\partial \sigma}{\partial y} \left( \frac{p}{p_s} \right) \right]
$$

$$
+ 2\sigma \left[ \frac{\partial u}{\partial \sigma} \frac{D \ln p_s}{Dt} \frac{\partial \ln p_s}{\partial x} + \frac{\partial v}{\partial \sigma} \frac{D \ln p_s}{Dt} \frac{\partial \ln p_s}{\partial y} \right]
$$

$$
+ 2\sigma \left( \frac{\partial u}{\partial \sigma} \frac{\partial \ln p_s}{\partial x} + \frac{\partial v}{\partial \sigma} \frac{\partial \ln p_s}{\partial y} \right) + 2\sigma \left( \frac{\partial u}{\partial \sigma} \frac{\partial \ln p_s}{\partial x} + \frac{\partial v}{\partial \sigma} \frac{\partial \ln p_s}{\partial y} \right). \tag{A.18}
$$
By substituting (A.14), (A.15), (A.17) and (A.18) into (A.12), it follows, after some cancellations between terms, that:

\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} - \sigma \frac{\partial \phi}{\partial \sigma} \left( \frac{\partial^2 \ln p_s}{\partial x^2} + \frac{\partial^2 \ln p_s}{\partial y^2} \right)
\]

\[
+ \sigma \frac{\partial}{\partial \sigma} \left[ \left( \frac{\partial \ln p_s}{\partial x} \right)^2 + \left( \frac{\partial \ln p_s}{\partial y} \right)^2 \right] - 2\sigma \left[ \frac{\partial \ln p_s}{\partial x} \left( \frac{\partial^2 \phi}{\partial x \partial \sigma} \right) + \frac{\partial \ln p_s}{\partial y} \left( \frac{\partial^2 \phi}{\partial y \partial \sigma} \right) \right]
\]

\[
= 2 \left( \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial v}{\partial x} \frac{\partial u}{\partial y} \right) + 2 \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \left( \frac{\partial \phi}{\partial \sigma} \right) + \frac{D \ln p_s}{\partial x} \frac{\partial \phi}{\partial \sigma} + \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \left( \frac{D \ln p_s}{\partial x} \right)
\]

\[
\sigma \int \frac{\partial u}{\partial \sigma} \left( \frac{\partial \ln p_s}{\partial x} \right) + \frac{\partial v}{\partial \sigma} \left( \frac{\partial \ln p_s}{\partial y} \right) - 2 \sigma \left[ \int \frac{\partial u}{\partial \sigma} \left( \frac{\partial \ln p_s}{\partial x} \right) + \frac{\partial v}{\partial \sigma} \left( \frac{\partial \ln p_s}{\partial y} \right) \right]
\]

\[
\sigma f \left[ \frac{\partial u}{\partial \sigma} \frac{\partial \ln p_s}{\partial y} - \frac{\partial v}{\partial \sigma} \frac{\partial \ln p_s}{\partial x} \right] + f \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) - u \frac{df}{dy} - \partial (sg)
\]

\[
- \frac{\partial}{\partial \sigma} \left[ \left( \frac{\partial}{\partial \sigma} \left( \frac{\partial \ln p_s}{\partial x} \right) \right) \left( \frac{\partial}{\partial \sigma} \left( \frac{\partial \ln p_s}{\partial x} \right) \right) \right] - \frac{\partial}{\partial \sigma} \left\{ \left( \frac{\partial}{\partial \sigma} \left( \frac{\partial \ln p_s}{\partial x} \right) \right) \left( \frac{\partial}{\partial \sigma} \left( \frac{\partial \ln p_s}{\partial x} \right) \right) \right\}.
\]

(A.19)

It may finally be noted that

\[
C_1 = \frac{2}{\rho_s} \left[ \frac{\partial u}{\partial \sigma} \frac{\partial p}{\partial \sigma} \left( \rho \Omega \right) + \frac{\partial v}{\partial \sigma} \frac{\partial p}{\partial \sigma} \left( \rho \Omega \right) \right] + \frac{2\rho \omega}{\rho_s} \left( \frac{\partial \ln p_s}{\partial x} \frac{\partial u}{\partial \sigma} + \frac{\partial \ln p_s}{\partial y} \frac{\partial v}{\partial \sigma} \right),
\]

(A.20)

\[
C_2 = 2 \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \left( \frac{\partial \phi}{\partial \sigma} \right) + \frac{D \ln p_s}{\partial x} \frac{\partial \phi}{\partial \sigma} + \left( \frac{\partial \phi}{\partial \sigma} \right) \left( \frac{D \ln p_s}{\partial x} \right) \left( \frac{D \ln p_s}{\partial x} \right)
\]

\[
= 2 \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \left( \frac{\partial \sigma}{\partial \sigma} \right) - \left( \frac{D \ln p_s}{\partial x} \right)^2 = 2 \frac{D \ln p_s}{\partial x} \frac{\partial \phi}{\partial \sigma} + \left( \frac{D \ln p_s}{\partial x} \right)^2
\]

(A.21)
\[ C_3 + C_4 = -\frac{\partial}{\partial \sigma} \left( \frac{\mathcal{D}}{\sigma} \left( \frac{D \ln p_s}{D t} \right) \right) - \frac{\partial}{\partial \sigma} \left[ \frac{p}{p_s T} \frac{D T p_s}{D t} \frac{p \omega}{p_s} \right] \]

\[ = -\frac{\partial}{\partial \sigma} \left( \mathcal{D} \frac{D \ln p_s}{D t} \right) + \frac{1}{\gamma} \left[ \frac{p \omega}{p_s} \frac{\partial \omega}{\partial \sigma} + \frac{\partial (p \omega)}{\partial \sigma} \frac{\omega}{p_s} \right] - \frac{p \omega}{p_s} \frac{\partial}{\partial \sigma} \left( \frac{D \ln p_s}{D t} \right) \]

\[ - \frac{\partial}{\partial \sigma} \left( p \Omega \right) \frac{1}{p_s} \frac{D \ln p_s}{D t} \]

\[ = -2 \frac{\partial \mathcal{D}}{\partial \sigma} \frac{D \ln p_s}{D t} - 2 \frac{p \omega}{p_s} \left( \frac{\partial u}{\partial \sigma} \frac{\partial \ln p_s}{\partial x} + \frac{\partial v}{\partial \sigma} \frac{\partial \ln p_s}{\partial y} \right) - \left( \frac{D \ln p_s}{D t} \right)^2 \]

\[ + \frac{1}{\gamma} \left[ \frac{p \omega}{p_s} \frac{\partial \omega}{\partial \sigma} + \frac{\partial (p \omega)}{\partial \sigma} \frac{\omega}{p_s} \right]. \quad \text{(A.22)} \]

Equations (A.19) to (A.22) may now be substituted in (A.18). After some cancellations and reorganization of the terms, the elliptic equation (3.81) is obtained.
Appendix B

Alternative derivation of the linearized elliptic equation

Linearizing equation (3.81) gives

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial}{\partial \sigma} \left( s_0^2 \frac{\partial \phi}{\partial \sigma} \right) + \frac{\partial}{\partial \sigma} \left( s_0 \frac{T'}{T_0} g \right) = 0. \quad (B.1)$$

Substituting wave-like solutions of the form (3.106) into (B.1) leads to

$$\frac{d}{d\sigma} \left( \frac{\sigma p_0 + p_T}{p_0} \right)^2 \frac{d\tilde{\phi}'}{d\sigma} - H_0^2 k^2 \tilde{\phi}' + H_0^2 \frac{d}{d\sigma} \left( s_0 \frac{T'}{T_0} g \right) = 0. \quad (B.2)$$

Here

$$H_0^2 \frac{d}{d\sigma} \left( s_0 \frac{T'}{T_0} g \right) = H_0^2 g \frac{d}{d\sigma} \left( s_0 \frac{T'}{T_0} + \frac{T'}{H_0} \right). \quad (B.3)$$

From (3.110) it follows that

$$\frac{dT'}{d\sigma} = -\kappa \left[ \left( \frac{p_0}{\sigma p_0 + p_T} \right) \left( \frac{1}{ikc} \right) \left( \frac{d\tilde{\sigma}}{d\sigma} - ikc\tilde{\pi} \right) \right] T_0 + \kappa \left[ \left( \frac{p_0}{\sigma p_0 + p_T} \right)^2 \left( \frac{1}{ikc} \right) \left( \tilde{\sigma} - ikc\tilde{\sigma} \right) \right] T_0 = 0. \quad (B.4)$$

Making use of (3.110) and (B.4) to substitute for $\tilde{T}'$ and $dT'/d\sigma$ in (B.3) gives:

$$H_0^2 \frac{d}{d\sigma} \left( s_0 \frac{T'}{T_0} g \right) = -H_0 g \kappa \left( \frac{1}{ikc} \right) \left( \frac{d\tilde{\sigma}}{d\sigma} - ikc\tilde{\pi} \right). \quad (B.5)$$
Substituting from (3.107) and (3.109) gives

\[ H_0^2 \frac{d}{da} \left( \frac{T'}{s_0 \Gamma_0 g} \right) = \frac{H_0 g k}{c^2} \frac{\dot{\phi}'}{\dot{\phi}} = H_0^2 \frac{N^2}{c^2} \frac{\dot{\phi}'}{\dot{\phi}}. \]  

(B.6)

From substituting (B.6) in (B.2) equation (3.113) is obtained.
Appendix C

The linearized elliptic equation under transformations $Z$ and $F$

The transformation relationship (3.114) implies that for any function $G = G(\sigma)$, it holds that

$$\frac{dG}{d\sigma} = \frac{dG}{dZ} \frac{dZ}{d\sigma} = \left(-H_0 \frac{p_0}{\sigma p_0 + p_T} \right) \frac{dG}{dZ}. \quad \text{(C.1)}$$

From (3.114) it may also be noted that

$$\frac{d\sigma}{dZ} = -\frac{1}{H_0} \left( \frac{\sigma p_0 + p_T}{p_0} \right). \quad \text{(C.2)}$$

Equation (3.113) may written alternatively as:

$$2 \left( \frac{\sigma p_0 + p_T}{p_0} \right) \frac{d\phi'}{d\sigma} + \left( \frac{\sigma p_0 + p_T}{p_0} \right)^2 \frac{d^2\phi'}{d\sigma^2} + H_0^2 \left( \frac{N^2}{c^2} - k^2 \right) \phi' = 0. \quad \text{(C.3)}$$

From applying (C.1), (C.3) transforms to:

$$-2H_0 \frac{d\phi'}{dZ} - H_0 \left( \frac{\sigma p_0 + p_T}{p_0} \right) \frac{d}{dZ} \left[-H_0 \left( \frac{p_0}{\sigma p_0 + p_T} \right) \frac{d\phi'}{dZ} \right] + H_0^2 \left( \frac{N^2}{c^2} - k^2 \right) \phi' = 0. \quad \text{(C.4)}$$

Applying transformation relationship (3.115) to (C.4) leads to:

$$-2H_0 \frac{dF}{dZ} \exp^{z/2H_0} - F \exp^{z/2H_0} + H_0^2 \left( \frac{1}{4H_0^2} F + \frac{1}{H_0} \frac{dF}{dZ} + \frac{d^2F}{dZ^2} \right) \exp^{z/2H_0}$$
\[ +H_0 \frac{dF}{dZ} \exp^{z/2H_0} + \frac{1}{2} F \exp^{z/2H_0} + H_0^2 \left( \frac{N^2}{c^2} - k^2 \right) F \exp^{z/2H_0} = 0, \quad (C.5) \]

which reduces to the required relationship (3.116).
Appendix D

Alternative derivation of the Lamb wave frequency equation

Relationship (3.118) may also be consistently obtained from considering the boundary conditions that apply to $\tilde{\sigma}$, $\tilde{\phi}'$ and $d\tilde{\phi}'/d\tilde{\sigma}$. The details are as follows.

At the lower boundary, where $\sigma = 1$ and $\tilde{\sigma} = 0$ by definition, it follows from substituting (3.112) and (3.126) in (3.127) that:

$$c^2 H_0^2 \left[ k^2 - \frac{N^2}{c^2} \right] \tilde{\pi} = AH_0 \left( \mu - \frac{1}{2H_0} \right) \left( \frac{p_0 + p_T}{p_0} \right)^{H_0(\mu + \frac{1}{2H_0})} + BH_0 \left( -\mu - \frac{1}{2H_0} \right) \left( \frac{p_0 + p_T}{p_0} \right)^{H_0(-\mu + \frac{1}{2H_0})}.$$  \hspace{1cm} (D.1)

Substituting (3.126) into (D.1), and applying (3.129) gives:

$$\left( \mu^2 - \frac{1}{4H_0^2} \right) \left[ \left( \frac{p_0 + p_T}{p_0} \right)^{H_0(\mu + \frac{1}{2H_0})} - \left( \frac{p_T}{p_0} \right)^{2H_0\mu} \left( \frac{p_0 + p_T}{p_0} \right)^{H_0(-\mu + \frac{1}{2H_0})} \right] =$$

$$\left( k^2 - \frac{N^2}{c^2} \right) \left[ \left( \frac{p_0 + p_T}{p_0} \right)^{H_0(\mu + \frac{1}{2H_0})} - \left( \frac{p_T}{p_0} \right)^{2H_0\mu} \left( \frac{p_0 + p_T}{p_0} \right)^{H_0(-\mu + \frac{1}{2H_0})} \right] +$$

$$\left[ \left( \frac{p_T}{p_0} \right)^{2H_0\mu} \left( \frac{p_T}{p_0} \right)^{H_0(-\mu + \frac{1}{2H_0})} - \left( \frac{p_T}{p_0} \right)^{H_0(\mu + \frac{1}{2H_0})} \right]. \hspace{1cm} (D.2)
Noting that the last two terms in (D.2) cancel, it follows that:

\[
\left[ \left( \mu^2 - \frac{1}{4H_0^2} \right) - \left( k^2 - \frac{N^2}{c^2} \right) \right] \left[ \left( \frac{p_0 + p_T}{p_0} \right)^{H_0(\mu + \frac{1}{4H_0})} - \left( \frac{p_T}{p_0} \right)^{2H_0(\mu + \frac{1}{4H_0})} \left( \frac{p_0 + p_T}{p_0} \right)^{H_0(\mu + \frac{1}{4H_0})} \right] = 0
\]

(D.3)

which implies (3.118).
Appendix E

Applying the continuity equation for the case of solutions with sinusoidal variation in height

Substituting (3.143) in (3.117) gives:

\[ \hat{\rho} = \frac{1}{c^2} \left( A \int_0^1 \exp^{imH_0 \ln X} X^{-1/2} d\sigma + B \int_0^1 \exp^{-imH_0 \ln X} X^{-1/2} d\sigma \right). \]  (E.1)

Both integrals appearing in (E.1) have the form \( \int_0^1 \exp^{i a \ln X} X^{-1/2} d\sigma \), with \( a \) a real number and \( X = (\sigma p_0 + p_T)/p_0 \), and

\[ \int_0^1 \exp^{i a \ln X} X^{-1/2} d\sigma = \int_0^1 \cos \left( a \ln X \right) X^{-1/2} d\sigma + i \int_0^1 \sin \left( a \ln X \right) X^{-1/2} d\sigma. \]  (E.2)

The two integrals implied by (E.2) may both be evaluated by using integration by parts, which gives:

\[ (1 + 4a^2) \int_0^1 \cos \left( a \ln X \right) X^{-1/2} d\sigma = \left[ 2X^{1/2} \cos \left( a \ln X \right) \right]_{\sigma=0}^{\sigma=1} + \left[ 4aX^{1/2} \sin \left( a \ln X \right) \right]_{\sigma=0}^{\sigma=1} \]

\[ = 2 \left\{ \left( \frac{p_0 + p_T}{p_0} \right)^{1/2} \cos \left[ a \ln \left( \frac{p_0 + p_T}{p_0} \right) \right] - \left( \frac{p_T}{p_0} \right)^{1/2} \cos \left[ a \ln \left( \frac{p_T}{p_0} \right) \right] \right\} + \]

\[ 4a \left\{ \left( \frac{p_0 + p_T}{p_0} \right)^{1/2} \sin \left[ a \ln \left( \frac{p_0 + p_T}{p_0} \right) \right] - \left( \frac{p_T}{p_0} \right)^{1/2} \sin \left[ a \ln \left( \frac{p_T}{p_0} \right) \right] \right\}. \]  (E.3)
and

\[
(1 + 4a^2) \int_0^1 \sin(a \ln X) X^{-1/2} d\sigma = \left[ 2X^{1/2} \sin(a \ln X) \right]_{\sigma=0}^{\sigma=1} - \left[ 4a X^{1/2} \cos(a \ln X) \right]_{\sigma=0}^{\sigma=1}
\]

\[
= 2 \left\{ \left( \frac{p_0 + pt}{p_0} \right)^{1/2} \sin \left[ a \ln \frac{p_0 + pt}{p_0} \right] - \left( \frac{pt}{p_0} \right)^{1/2} \sin \left[ a \ln \frac{pt}{p_0} \right] \right\} - \n \]

\[
4a \left\{ \left( \frac{p_0 + pt}{p_0} \right)^{1/2} \cos \left[ a \ln \frac{p_0 + pt}{p_0} \right] - \left( \frac{pt}{p_0} \right)^{1/2} \cos \left[ a \ln \frac{pt}{p_0} \right] \right\}. \tag{E.4}
\]

Substituting results (E.3) and (E.4) in (E.2) leads to:

\[
\int_0^1 \exp^{ia \ln X} X^{-1/2} d\sigma = \frac{1}{c^2 (1 + 4a^2)} \left[ (2 - 4ai) \left( \frac{p_0 + pt}{p_0} \right)^{1/2} \exp^{ia \ln[(p_0 + pt)/p_0]} \right. \\
+ \left. (-2 + 4ai) \left( \frac{pt}{p_0} \right)^{1/2} \exp^{ia \ln(pt/p_0)} \right]
\]

\[
\tag{E.5}
\]

Applying result (E.5) in (E.1) gives the required equation (3.145).
Bibliography


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[40] Engelbrecht FA (2000) Nested climate modelling over southern Africa with
a semi-Lagrangian limited area model. MSc-thesis, University of Pretoria,
134 pp.

[41] Engelbrecht FA (2005) Simulations of climate and climate change over
southern and tropical Africa with the conformal-cubic atmospheric model.
In Potential Impacts and Vulnerabilities of Climate Change on Hydrological
Responses in southern Africa, ed. R.E. Schulze, chap. 4. WRC Report
1430/1/05. Water Research Commission, Pretoria.

climate modelling over southern Africa. SA Tydskrif vir Natuurwetenskap
en Tegnologie 19 47-51.

(2002) January and July climate simulations over the SADC region using
the limited-area model DARLAM. Water SA, 28, 361-374.

primitive equations for a barotropic fluid with application to the boundary
current problem. Tellus 4 405-412.

SMRP Research paper No 98 University of Chicago, Illinois.


and cyclonic storms and their motion as determined from numerical model
experiments. J. Atmos. Sci. 35 1070-1096.

[48] Gal-Chen T and Somerville RCJ (1975) On the use of a coordinate transfor-
mation for the solution of the Navier-Stokes equations. ibid. 17 209-228.


of convective storms over the escarpment of northeastern South Africa.


rain-bearing synoptic systems. J. Climatol. 4 547-560.


