Chapter 3

Derivation and properties of the quasi-elastic equations in terrain-following coordinates based on the full pressure field

3.1 Introduction

As mentioned in Chapter 2, the first nonhydrostatic atmospheric flow model in pressure-based coordinates was formulated by Miller (1974) and Miller and Pearce (1974). This model employed the full pressure field as vertical coordinate. The approximations introduced by Miller (1974) to the fully elastic equations in p coordinates ensured the absence of vertically propagating sound waves from the equation set. Buoyancy modes remain undistorted by the simplifications, and horizontally propagating sound waves (Lamb waves) may be removed by applying the lower boundary condition $\omega = 0$ at $p = p_0$ where p_0 is a constant. The complete removal of sound waves yields the equation set to be anelastic. A sound theoretical basis for these equations (hereafter called the Miller-Pearce equations, MP equations or MP model) is provided by Miller and White (1984), and they also transformed the equations to σ coordinates. Lamb waves (Lamb, 1932) are present in the σ coordinate formulation, where the lower boundary condition is $\dot{\sigma} = 0$ at $\sigma = 1$ and the upper boundary is defined as $p = 0$. The presence of Lamb waves implies a computational penalty compared to the pressure coordinate formulation (Miller and White, 1984; Room et al., 2001). Because the MP equations in σ coordinates are filtered of vertically propagating sound waves, but do allow the propagation of sound waves in the horizontal, the equation set may be called quasi-elastic (or pseudo-anelastic (Room et al., 2001) .

The MP model has been applied successfully in pressure coordinates in numerous studies of systems where the initial state is fairly uniform. These include studies of cumulonimbus and other convective systems (Miller and Pearce, 1974; Moncrieff and Miller, 1976; Miller and Betts, 1977; Thorpe and Miller, 1978; Thorpe et al., 1982; Brugge and Moncrief, 1985). The MP model in σ coordinates was used in two spatial dimensions to simulate mountain waves (Xue and Thorpe, 1991) and for studying gravity wave drag and critical level resonance in three-spatial dimensions (Miranda and James, 1992; Miranda and Valente, 1997). Room et al. (2001) modified the MP model in σ coordinates by filtering out the Lamb waves. The filtering was achieved by choosing the lower boundary of the model domain to have a fixed pressure distribution in space and time, and deducing proper vertical boundary conditions for the nonhydrostatic geopotential height-equation (Room et al., 2001; also see Chapter 2). The anelastic σ coordinate equations allow the use of significantly larger time-steps during numerical integration compared to the corresponding MP equations (Room et al., 2001). A potential disadvantage of the anelastic equations derived by Room et al. (2001) is that the surface pressure fluctuation is calculated by assuming hydrostatic balance at the lowest model level, and by approximating the latter with the hydrostatic balance in terms of a reference state temperature profile. If the actual surface pressure is lower then the fixed background pressure, calculations in the model are performed below the actual terrain geometric height (see Chapter 2). Still, in a set of numerical experiments involving airflow over orographic obstacles, the surface pressure fields obtained from using the MP equations applied by Miranda and James (1992) and from applying the anelastic equations showed close correspondence (Room et al., 2001). The anelastic σ coordinate equations are called the NHAD model (nonhydrostatic adjusted dynamics, Room et al. (2001)).

Both the MP and NHAD models are restrictive in the sense that they are formulated in terms of a reference state that is in hydrostatic equilibrium and is a function of pressure only. A single reference state may not be typical everywhere in a given computational domain. In particular, this may be true if the model is applied in large-scale modelling or in mesoscale simulations of features such as frontal zones (White, 1989). White (1989) extended the MP model to be able to represent large temperature variations on pressure surfaces, by avoiding the explicit use of a reference state. The extended equations imply a p coordinate analogue of Ertel's potential vorticity conservation law and conserves energy (White, 1989). Salmon and Smith (1994) showed that White's equations have a Hamiltonian structure and that the conservation properties follow from the symmetry properties of the relevant Hamiltonian.

This Chapter commences with an overview of the MP model and White's equations in pressure coordinates, whereafter the σ coordinate analogue of the pressure coordinate equations of White (1989) is derived. In fact, the work in this chapter represents the first formulation of White's extended pressure coordinate equations in σ coordinates. The σ coordinate used is based on the full (nonhydrostatic) pressure field, just as the pressure and σ coordinates used in the MP-model and by White (1989) are based on the full pressure field. Recently formulated terrain-following nonhydrostatic models mostly make use (as discussed in Chapter 2) of vertical coordinates based on the hydrostatic pressure (Laprise, 1992; Juang, 1992; Gallus and Rancic, 1996; Janjic et al., 2001) or on a hydrostatic reference-pressure field (Dudhia, 1993; Hsu and Sun, 2001; Dudhia and Bresch, 2002). The σ coordinate used by Room et al. (2001) is a type of hybrid, since it is defined in terms of the full pressure field and a reference surface pressure distribution. The σ coordinate used in the present chapter makes use of the actual surface pressure, such as in the MP model and the equations of White (1989), and employs a prognostic equation for the surface pressure. Lamb waves may therefore be expected to form part of the solution set of these equations, and they are indeed shown to be quasi-elastic.

Numerical solutions of the NHAD and MP equations in p or σ coordinates all make use of an elliptic equation in the geopotential. A similar elliptic equation is derived for the σ coordinate quasi-elastic equations presented in this chapter. It is shown that the equation may be obtained from a coordinate transformation of the corresponding pressure coordinate equation derived by White (1989), or alternatively, directly from the quasi-elastic equations in σ coordinates. An energy equation for the quasi-elastic equations is also derived. Finally, the characteristics of the gravity and sound waves in the quasi-elastic σ coordinate equation system are discussed. It is shown how the phase speed of these fast travelling waves depends on the choice of the model top.

3.2 The Miller-Pearce model

3.2.1 Basic concepts

In pressure coordinates, the vertical velocity w may be expressed as:

$$\frac{Dw}{Dt} = \frac{\partial z}{\partial t} + u\frac{\partial z}{\partial x} + v\frac{\partial z}{\partial y} + \omega\frac{\partial z}{\partial p}. \tag{3.1}$$

Here

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial p} \tag{3.2}$$

is the material derivative; all partial derivatives with respect to t, x and y are carried out at constant p. The horizontal components of the wind are denoted by u and v. Note that p represents the full pressure field, and $\omega \equiv Dp/Dt$.

In the free atmosphere, that is, away from boundaries which may impose constraints on w, the last term on the right-hand side of (3.1) is dominant (Miller, 1974). It is common practice to ignore the contributions of the first two terms in

(3.1), and to use the hydrostatic aproximation (1.6) to approximate w as $-\omega/g\rho$ (e.g. Miller, 1974; Holton, 1992). It is less obvious that this approximation is valid when differentiated (Miller, 1974), that is, as

$$\frac{Dw}{Dt} \simeq \frac{D}{Dt}\left(\frac{\omega}{g\rho}\right). \tag{3.3}$$

It is also not immediately clear to what extent the properties of a given equation set are changed by the use of approximation (3.3). However, by scale analysis of the fully-elastic equations based on the full pressure field as vertical coordinate, it may be shown that (3.3) may be applied to find an approximated form of the vertical momentum equation that is consistent with other approximations made to the horizontal momentum and continuity equations (Miller, 1974, also see Chapter 2). Indeed, the first nonhydrostatic flow model in pressure coordinates as formulated by Miller (1974) and Miller and Pearce (1974) was based on approximation (3.3). In the MP-model the true vertical acceleration Dw/Dt is replaced in the vertical momentum equation by $D\tilde{w}_{ref}/Dt$, where

$$\tilde{w}_{ref} = \frac{-\omega}{g\rho_{ref}} = -\frac{R}{g}\frac{\omega T_{ref}}{p}. \tag{3.4}$$

Here g is the gravitational acceleration and R is the gas constant for a unit mass of air. $T_{ref} = T_{ref}(p)$ and $\rho = \rho_{ref}(p)$ are reference profiles of temperature T and density ρ. In conjunction with approximations made in the continuity and horizontal momentum equations, this replacement of Dw/Dt by $D\tilde{w}_{ref}/Dt$ ensures the absence of vertically propagating acoustic modes from the solution set of the MP equations, whilst buoyancy modes remain undistorted (and no spurious modes are introduced) (Miller, 1974; Miller and White, 1984). Horizontally propagating acoustic modes (the Lamb modes) may be removed from the pressure coordinate MP equations by applying the lower boundary condition $\omega = 0$ at $p = p_0$, where p_0 is a constant (Miller, 1974; Miller and White, 1984).

3.2.2 The Miller-Pearce model in pressure coordinates

For an adiabatic, frictionless atmosphere the three-dimensional MP model with pressure as the vertical coordinate consists of the following approximate forms of the horizontal and vertical momentum equations (3.5) to (3.7), continuity equation (3.8) and the exact thermodynamic equation for a perfect gas (3.9):

$$\frac{Du}{Dt} - fv + \frac{\partial\phi'}{\partial x} = 0, \tag{3.5}$$

$$\frac{Dv}{Dt} + fu + \frac{\partial\phi'}{\partial y} = 0, \tag{3.6}$$

$$\frac{R}{g}\frac{D}{Dt}\left(\frac{\omega T_{ref}}{p}\right) + g\frac{T'}{T_{ref}} + \frac{gp}{RT_{ref}}\frac{\partial\phi'}{\partial p}, = 0 \tag{3.7}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p}, = 0 \tag{3.8}$$

$$\frac{DT}{Dt} - \kappa \frac{\omega T}{p} = 0. \tag{3.9}$$

Note that Coriolis terms are included in the horizontal component equations; f is the Coriolis parameter. The material derivative is defined by (3.2). In equations (3.5) to (3.9) all differentiations with respect to time and the horizontal coordinates are carried out at constant pressure; u and v are the velocity components in the x and y directions; $\kappa = R/c_p$ with c_p the specific heat at constant pressure and ϕ is the geopotential, gz, z being geometric height. $\phi^{'}$ and $T^{'}$ are the departures of ϕ and T from the reference profiles $\phi_{ref}(p)$ and $T_{ref}(p)$:

$$\phi = \phi_{ref}(p) + \phi^{'};\ T = T_{ref}(p) + T^{'}. \tag{3.10}$$

The reference state is chosen to be in hydrostatic balance, so that:

$$\frac{d\phi_{ref}}{dp} + \frac{RT_{ref}}{p} = 0. \tag{3.11}$$

Time integration of the MP equations makes use of an elliptic diagnostic equation for $\phi^{'}$ which results when (3.8) is applied to differentiated forms of equations (3.5) to (3.7) (Johnson, 1978; Miller and White (1984); Brugge and Moncrieff (1985):

$$\frac{\partial^2 \phi^{'}}{\partial x^2} + \frac{\partial^2 \phi^{'}}{\partial y^2} + \frac{\partial}{\partial p}\left(r_s^2 \frac{\partial \phi^{'}}{\partial p} \right) = 2J_3 - \frac{\partial}{\partial p}\left(g r_{ref} \frac{T^{'}}{T_{ref}} - \frac{\omega^2}{r_{ref}} \frac{dr_{ref}}{dp} \right) \tag{3.12}$$

in which

$$J_3 = \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} \right) + \left(\frac{\partial v}{\partial y}\frac{\partial \omega}{\partial p} - \frac{\partial \omega}{\partial y}\frac{\partial v}{\partial p} \right) + \left(\frac{\partial \omega}{\partial p}\frac{\partial u}{\partial x} - \frac{\partial u}{\partial p}\frac{\partial \omega}{\partial x} \right) \tag{3.13}$$

and

$$r_{ref} = \frac{gp}{RT_{ref}}. \tag{3.14}$$

Boundary conditions on the solution of (3.12) are obtained using equations (3.5) to (3.7) and (3.9) together with appropriate specifications of u, v, ω and $T^{'}$ at the boundaries (Miller and White, 1984).

Miller (1974) derived the MP equations for flow independent of one horizontal coordinate by transforming the geometric-height coordinate equations to pressure coordinates and then neglecting various small terms (also see Chapter 2). By using a scaling and power series expansion method, Miller and White (1984) gave a formal justification of the MP equations for flow independent of one horizontal coordinate. Their derivation assumed that the departures of $\phi^{'}$ and $T^{'}$ from the reference state profiles were due solely to processes occurring on the convective scale (White, 1989). They identified the quantity

$$\alpha^* \equiv -\frac{p_0}{\theta_{ref}}\frac{d\theta_{ref}}{dp} \tag{3.15}$$

as the key parameter whose smallness ($\alpha \ll 1$) validates the approximations made by Miller (1974). In (3.15), $\theta_{ref} = \theta_{ref}(p)$ is the potential temperature profile corresponding to $T_{ref}(p)$, that is, $\theta_{ref} = T_{ref}\,(p_0/p)^{\kappa}$ with p_0 a reference pressure level. Condition (3.15) is satisfied everywhere in the troposphere. Even in the extreme case of the Antarctica surface layer $\alpha \leq 0.3$ (Miranda and James, 1992).

Equations (3.5) - (3.9) imply an energy equation of the form

$$\frac{D}{Dt}(\frac{1}{2}u^2 + \frac{1}{2}v^2 + \frac{1}{2}\tilde{w}_{ref}^2 + c_pT) = -\bar{\nabla}_p \cdot (\phi\mathbf{v}) - \frac{\partial}{\partial p}(\omega\phi) \tag{3.16}$$

in which $\bar{\nabla} = (\partial/\partial x,\, \partial/\partial y)$ and $\mathbf{v} = (u\,,v)$ is the horizontal velocity (White, 1989). Equation (3.16) is an extension of the familiar energy equation for the hydrostatic equations, to include the contribution of the approximate vertical velocity $\tilde{w}_{ref}$ to the specific kinetic energy (White, 1989). Miller and White (1984) noted a 2-dimensional case of (3.16). Johnson (1978) carried out a detailed study of the vorticity properties of the MP model, and established the existence of an anologue of Ertel's potential vorticity theorem.

The MP model has been applied successfully in pressure coordinates in numerous studies of systems where the initial state is fairly uniform. These were mostly studies of cumulonimbus and other convective systems (Moncrieff and Miller, 1976; Miller and Bets, 1977; Thorpe and Miller, 1978; Thorpe et al., 1982; Brugge and Moncrieff, 1985). The appearence of the reference temperature $T_{ref}(p)$ in the definition of $\tilde{w}$ (and elsewhere in the vertical momentum equation) was not a hampering feature in these studies, since each was concerned with situations in which the initial state is horizontally homogeneous. Variations of the temperature on pressure surfaces is thus due solely to convection itself, and these turn out to be small. Should the MP model be used in mesoscale simulations - or, as speculatively suggested by Miller and White (1984), in larger scale modelling - the use of $T_{ref}(p)$ instead of the true local temperature, T, would be less sound. In frontal zones, for example, considerable variations of temperature may occur on pressure surfaces, and a single reference state might not be typical of different parts of the computational domain (White, 1989). However, it should be noted that similar height-based equation sets (e.g. Ogura and Phillips, 1962; Gal-Chen and Somerville, 1975) have been used to successfully simulate fronts. The tropopause may be a more problematic region to apply the MP model to. The tropopause slopes in the horizontal, so that there will be relatively larger departures in temperature from a reference state that depends on pressure, on a pressure surface that intersects the tropopause. A modification of the MP model to represent large temperature variations on pressure surfaces, and that is independent of the use of a thermodynamic reference profile, was stated by White (1989). This equation set is described in section 3.2.4. A normal-mode analysis of the MP and White (1989) equations (in pressure and σ coordinates), following recent work performed in height-based coordinates (Thuburn et al., 2002a; Thuburn et al., 2002b; Davies et al., 2003), may provide more clearity on the validity of these equation sets as a function of different flow regimes.

3.2.3 The Miller-Pearce model in sigma coordinates

The σ coordinate version of the MP equations was derived by Miller and White (1984) by means of a coordinate transformation of the equations in pressure coordinates. Here the σ coordinate form of the equations is defined by (2.6) in terms of the full pressure field. The σ coordinate equations, here stated in three spatial dimensions, are:

$$\frac{Du}{DT} - fv + \left(\frac{\partial \phi^{'}}{\partial x}\right) - \sigma \left(\frac{\partial \phi^{'}}{\partial \sigma}\right) \frac{\partial \ln p_{surf}}{\partial x} = 0, \qquad (3.17)$$

$$\frac{Dv}{DT} + fu + \left(\frac{\partial \phi^{'}}{\partial y}\right) - \sigma \left(\frac{\partial \phi^{'}}{\partial \sigma}\right) \frac{\partial \ln p_{surf}}{\partial y} = 0, \qquad (3.18)$$

$$\frac{R}{g}\frac{D}{Dt}\left[T_{ref}\left(\frac{D \ln p_s}{Dt} + \frac{\dot{\sigma}}{\sigma}\right)\right] + g\frac{T^{'}}{T_{ref}} + \frac{\sigma g}{RT_{ref}}\frac{\partial \phi^{'}}{\partial \sigma} = 0, \qquad (3.19)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \dot{\sigma}}{\partial \sigma} + \frac{D \ln p_{surf}}{Dt} = 0, \qquad (3.20)$$

$$\frac{DT^{'}}{Dt} - \left(\frac{D \ln p_{surf}}{Dt} + \frac{\dot{\sigma}}{\sigma}\right)\left(p_{surf}\sigma s_{ref} + \kappa T^{'}\right) = 0. \qquad (3.21)$$

In these equations partial differentiations with respect to x and y are carried out at constant σ. Note that s_{ref} denotes $-\left(T_{ref}/\theta_{ref}\right)\left(d\theta_{ref}/dp\right)$, the mean static stability function (Miller and White, 1984). As stated before in (3.10), $T^{'}$ and $\phi^{'}$ denotes departures from a hydrostatic reference state (2.25) that is a function of pressure only. The surface pressure is denoted by p_{surf} as in Chapter 2, and all the other variables and constants are defined as before.

The MP model in σ coordinates was used in two spatial dimensions to simulate mountain waves (Xue and Thorpe, 1991) and for studying gravity wave drag and critical level resonance in three spatial dimensions (Miranda and James, 1992; Miranda and Valente, 1997). The MP model in σ coordinates was also successfully applied in the "cold bubble test" (see Straka, 1993 and Chapter 5) to simulate a descending cold bubble in an isentropic environment (Xue, 1989; see also Gallus and Rancic, 1994). Note the use of a reference profile that depends on pressure in the equations, which may restrict the application of the model to situations where the atmospheric state is fairly uniform on isobaric levels (White, 1989).

3.3 White's extension of the MP model in pressure coordinates

3.3.1 The momentum, continuity and thermodynamic energy equations

Choosing to avoid the formulation in terms of a reference state used in the MP model, White (1989) stated the following set of pressure coordinate equations:

$$\frac{Du}{Dt} - fv + \frac{\partial \phi^{'}}{\partial x} = 0, \tag{3.22}$$

$$\frac{Dv}{Dt} + fu + \frac{\partial \phi^{'}}{\partial y} = 0, \tag{3.23}$$

$$\frac{R}{g}\frac{D}{Dt}(\frac{\omega T}{p}) + g\frac{T^{'}}{T} + \frac{gp}{RT}\frac{\partial \phi^{'}}{\partial p} = 0, \tag{3.24}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0, \tag{3.25}$$

$$\frac{DT}{Dt} - \kappa\frac{\omega T}{p} = 0. \tag{3.26}$$

Here (3.22) and (3.23) are the horizontal momentum equations, (3.24) is the vertical momentum equation, (3.25) is the continuity equation and (3.26) the thermodynamic energy equation. These equations are independent of the choice of hydrostatic reference state (3.16) (White, 1989). They could be written with $\phi^{'}$ and $T^{'}$ replaced by ϕ and T, but White (1989) argued that the subtraction of reference profiles $\phi_{ref}(p)$ and $T_{ref}(p)$ is desirable in computational practice. Thus, equations (3.22) to (3.26) differs from the MP model in pressure cooridinates (3.5) to (3.9) only in the replacement of the reference state temperature $T_{ref}(p)$ by the true temperature T. This appears not to be a trivial extension: The vorticity and potential vorticity dynamics of the extended set (3.22) to (3.26) are algebraicly different from the MP equations (3.5) to (3.9) (White, 1989). Equations (3.22) to (3.26) imply the energy equation

$$\frac{D}{Dt}(\frac{1}{2}u^2 + \frac{1}{2}v^2 + \frac{1}{2}\hat{w^2} + c_pT) = -\bar{\nabla}_p \cdot (\phi\mathbf{v}) - \frac{\partial}{\partial p}(\omega\phi), \tag{3.27}$$

where

$$\hat{w} = \frac{-\omega RT}{gp}. \tag{3.28}$$

Equation (3.27) is a generalization of (3.16) of the MP equations. White (1989) also showed that (3.22) to (3.26), as the MP equations, imply a pressure coordinate analogue of Ertel's potential vorticity conservation law.

In this study it is preferred not to make use of the reference profile (3.16) in White's equations (3.22) to (3.24). This decision is based on the belief that the use of perturbation quantities in (3.22) to (3.24) instead of the full fields does not imply significant advantages in computational practice (McGregor, personnel communicaton). In fact, the elliptic equations that are eventually derived for White's equations (see sections 3.3.2 and 3.4.7), can be most conveniently solved numerically if they are formulated in terms of the full geopotential field (see Chapter 4). Formulating the elliptic equation in terms of perturbation quantities of the hydrostatic reference profile (3.16) requires a more complicated specification of the lower boundary conditions needed during the numercial solution of the elliptic equation (see Xue, 1989; also see Chapter 4). Using (3.10) to write the perturbation quantities $\phi^{'}$ and $T^{'}$ in terms of the full fields ϕ and T, and substituting into (3.22) to (3.24), a statement of the momentum equations in terms of the full ϕ and T fields is obtained:

$$\frac{Du}{Dt} - fv + \frac{\partial\phi}{\partial x} = 0, \tag{3.29}$$

$$\frac{Dv}{Dt} + fu + \frac{\partial\phi}{\partial y} = 0, \tag{3.30}$$

$$\frac{R}{g}\frac{D}{Dt}(\frac{\omega T}{p}) + g + \frac{gp}{RT}\frac{\partial\phi}{\partial p} = 0. \tag{3.31}$$

Equations (3.29) to (3.31) is the form of the momentum equations of White (1989) that is used in the present study. It may be noted that White's equations (3.29) to (3.31), (3.25) and (3.26) were derived from a Hamiltonian perspective by Salmon and Smith (1994).

3.3.2 A diagnostic equation for ϕ in pressure coordinates

Following the procedure suggested by Miller and White (1984) for the two-dimensional MP-equations, a diagnostic equation for ϕ may be derived from equations (3.22) to (3.26).

Differentiating (3.22) with respect to x and (3.23) with respect to y, and forming $D\left(\partial u/\partial x\right)/Dt$ and $D\left(\partial v/\partial y\right)/Dt$ from the resulting two equations, gives

$$\frac{D}{Dt}\left(\frac{\partial u}{\partial x}\right)+\left(\frac{\partial u}{\partial x}\right)^2+\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}+\frac{\partial \omega}{\partial x}\frac{\partial u}{\partial p}-f\frac{\partial v}{\partial x}+\frac{\partial^2\phi}{\partial x^2}=0 \tag{3.32}$$

and

$$\frac{D}{Dt}\left(\frac{\partial v}{\partial y}\right)+\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}+\left(\frac{\partial v}{\partial y}\right)^2+\frac{\partial \omega}{\partial y}\frac{\partial v}{\partial p}+f\frac{\partial u}{\partial y}+u\frac{df}{dy}+\frac{\partial^2\phi}{\partial y^2}=0. \tag{3.33}$$

It may be noted that

$$\frac{D}{Dt}\left(\frac{\omega T}{p}\right)=\frac{T}{p}\frac{D\omega}{Dt}+\frac{\omega}{p}\frac{DT}{Dt}-\frac{\omega^2 T}{p^2}=\frac{T}{p}\frac{D\omega}{Dt}-\frac{1}{\gamma}\frac{\omega^2 T}{p^2}, \tag{3.34}$$

by the use of (3.26). Here $\gamma=c_p/c_v$. After substituting (3.34) in (3.24) and differentiating to p it follows, by forming $D\left(\partial\omega/\partial p\right)/Dt$ from the resulting equation, that

$$\frac{D}{Dt}\left(\frac{\partial \omega}{\partial p}\right)+\frac{\partial u}{\partial p}\frac{\partial \omega}{\partial x}+\frac{\partial v}{\partial p}\frac{\partial \omega}{\partial y}+\left(\frac{\partial \omega}{\partial p}\right)^2$$

$$+\frac{\partial}{\partial p}\left(rg-\frac{\omega^2}{\gamma p}+r^2\frac{\partial\phi}{\partial p}\right)=0. \tag{3.35}$$

Here $r=gp/RT$. Adding the terms $D\left(\partial u/\partial x\right)/Dt$, $D\left(\partial v/\partial y\right)/Dt$, and $D\left(\partial\omega/\partial p\right)/Dt$, and applying (3.25) leads to the required diagnostic equation for ϕ:

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial}{\partial p}\left(r^2\frac{\partial\phi}{\partial p}\right)=$$

$$2J_3+f\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)-u\frac{df}{dy}-\frac{\partial}{\partial p}\left(rg-\frac{\omega^2}{\gamma p}\right) \tag{3.36}$$

in which J_3 is given by (3.13). Equation (3.36) is mathematically equivalent to the elliptic equation stated by White (1989) in terms of perturbation quantities $\phi^{'}$ and $T^{'}$.

3.3.3 Two-dimensional equations in pressure coordinates

The two-dimensional version of White's equations in pressure coordinates (neglecting the Coriolis effect) is:

$$\frac{Du}{Dt}+\frac{\partial\phi}{\partial x}=0, \tag{3.37}$$

$$\frac{R}{g}\frac{D}{Dt}\left(\frac{\omega T}{p}\right)+g+\frac{gp}{RT}\frac{\partial\phi}{\partial p}=0, \tag{3.38}$$

$$\frac{\partial u}{\partial x}+\frac{\partial\omega}{\partial p}=0, \tag{3.39}$$

$$\frac{DT}{Dt}-\kappa\frac{\omega T}{p}=0. \tag{3.40}$$

These equations may be compared to (2.35) to (2.38), the two-dimensional MP equations in presure coordinates. In two spatial dimensions, the elliptic equation assumes the form:

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial}{\partial p}\left(r^2\frac{\partial\phi}{\partial p}\right)=$$

$$2\left(\frac{\partial\omega}{\partial p}\frac{\partial u}{\partial x}-\frac{\partial u}{\partial p}\frac{\partial\omega}{\partial x}\right)-\frac{\partial}{\partial p}\left(rg-\frac{\omega^2}{\gamma p}\right). \tag{3.41}$$

3.4 Derivation of White's equations in σ coordinates by a coordinate transformation

3.4.1 Transformation relations

In the following sections the equations derived by White (1989) in pressure coordinates are transformed to σ coordinates, with

$$\sigma=\frac{p-p_T}{p_{surf}-p_T}=\frac{p-p_T}{p_s} \tag{3.42}$$

Here p represents the full (nonhydrostatic) pressure field, p_T is the prescribed pressure at the model top (a constant), p_{surf} is the actual surface pressure and $p_s=p_{surf}-p_T$. Note that $p_{surf}=p_{surf}(x,y,t)$ and $p_s=p_s(x,y,t)$.

Let A be any scalar (or vector) dependent variable. The required transformation relationships btween σ and pressure coordinates (Phillips, 1957; Kasahara, 1974; Haltiner and Williams, 1980) are:

$$\left(\frac{\partial A}{\partial q}\right)_\sigma=\left(\frac{\partial A}{\partial q}\right)_p+\frac{\partial A}{\partial p}\left(\frac{\partial p}{\partial q}\right)_\sigma \tag{3.43}$$

with $q=x$, y or t, and

$$\frac{\partial A}{\partial\sigma}=\frac{\partial A}{\partial p}\frac{\partial p}{\partial\sigma}. \tag{3.44}$$

Alternatively,

$$\frac{\partial A}{\partial p} = \frac{\partial A}{\partial \sigma}\frac{\partial \sigma}{\partial p}. \tag{3.45}$$

If equation (3.45) is substituted into equation (3.43), the result is

$$(\frac{\partial A}{\partial q})_\sigma = (\frac{\partial A}{\partial q})_p + \frac{\partial A}{\partial \sigma}\frac{\partial \sigma}{\partial p}(\frac{\partial p}{\partial q})_\sigma. \tag{3.46}$$

It is worthwhile to note that

$$\frac{\partial \sigma}{\partial p} = \frac{\partial}{\partial p}(\frac{p - p_T}{p_{surf} - p_T}) = \frac{1}{p_{surf - }p_T}\frac{\partial p}{\partial p} = \frac{1}{p_s} \tag{3.47}$$

or, alternatively,

$$\frac{\partial p}{\partial \sigma} = p_s \tag{3.48}$$

and

$$(\frac{\partial p}{\partial q})_\sigma = \frac{\partial}{\partial q}(\sigma p_s + p_T)_\sigma = \sigma(\frac{\partial p_s}{\partial q})_\sigma. \tag{3.49}$$

Equations (3.47) and (3.49) can now be substituted into equation (3.46) to obtain the transformation relationship

$$\left(\frac{\partial A}{\partial q}\right)_\sigma = \left(\frac{\partial A}{\partial q}\right)_p + \frac{\partial A}{\partial \sigma}\frac{\sigma}{p_s}\left(\frac{\partial p_s}{\partial q}\right)_\sigma = \left(\frac{\partial A}{\partial q}\right)_p + \frac{\partial A}{\partial \sigma}\sigma\left(\frac{\partial \ln p_s}{\partial q}\right)_\sigma, \tag{3.50}$$

whilst substituting equation (3.47) in (3.45) gives

$$\frac{\partial A}{\partial p} = \frac{1}{p_s}\frac{\partial A}{\partial \sigma}. \tag{3.51}$$

3.4.2 The horizontal momentum equations

From equations (3.50) it follows that

$$\left(\frac{\partial \phi}{\partial x}\right)_p = \left(\frac{\partial \phi}{\partial x}\right)_\sigma - \sigma\frac{\partial \phi}{\partial \sigma}\left(\frac{\partial \ln p_s}{\partial x}\right)_\sigma. \tag{3.52}$$

The equation for momentum conservation in the x direction in pressure coordinates (3.29) therefore transforms to σ coordinates as

$$\frac{Du}{Dt} - fv + \frac{\partial \phi}{\partial x} - \sigma\frac{\partial \phi}{\partial \sigma}\frac{\partial \ln p_s}{\partial x} = 0. \tag{3.53}$$

Similarly, the pressure coordinate equation for momentum conservation in the y-direction (3.30) may be tranformed to σcoordinates to obtain

$$\frac{Dv}{Dt} + fu + \frac{\partial \phi}{\partial y} - \sigma \frac{\partial \phi}{\partial \sigma} \frac{\partial \ln p_s}{\partial y} = 0. \tag{3.54}$$

3.4.3 The vertical momentum equation

From equations (3.51) it follows that

$$\frac{\partial \phi}{\partial p} = \frac{\partial \phi}{\partial \sigma} \frac{\partial \sigma}{\partial p} = \frac{1}{p_s} \frac{\partial \phi}{\partial \sigma}. \tag{3.55}$$

From this, the vertical momentum equation in pressure coordinates (3.31) transforms to

$$\frac{R}{g} \frac{D}{Dt} \left(\frac{\omega T}{p} \right) + g + \frac{p}{p_s} \frac{g}{RT} \frac{\partial \phi}{\partial \sigma} = 0. \tag{3.56}$$

3.4.4 The continuity equation

From the transformation relationship (3.50) it follows that

$$\left(\frac{\partial u}{\partial x} \right)_p = \left(\frac{\partial u}{\partial x} \right)_\sigma - \sigma \frac{\partial u}{\partial \sigma} \left(\frac{\partial \ln p_s}{\partial x} \right)_\sigma \tag{3.57}$$

and

$$\left(\frac{\partial v}{\partial y} \right)_p = \left(\frac{\partial v}{\partial y} \right)_\sigma - \sigma \frac{\partial v}{\partial \sigma} \left(\frac{\partial \ln p_s}{\partial y} \right)_\sigma . \tag{3.58}$$

Noting the relationship between $\dot{\sigma}$ and ω

$$\begin{aligned} \omega \equiv \frac{Dp}{Dt} &= \frac{\partial p}{\partial t} + u \frac{\partial p}{\partial x} + v \frac{\partial p}{\partial y} + \dot{\sigma} \frac{\partial p}{\partial \sigma} \\ &= \sigma \frac{\partial p_s}{\partial t} + u\sigma \frac{\partial p_s}{\partial x} + v\sigma \frac{\partial p_s}{\partial y} + \dot{\sigma} p_s, \end{aligned} \tag{3.59}$$

from the use of equations (3.48) and (3.49), it follows from equation (3.51) that

$$\frac{\partial \omega}{\partial p} = \frac{1}{p_s} \frac{\partial \omega}{\partial \sigma}$$

$$= \frac{1}{p_s} \left[\frac{\partial p_s}{\partial t} + u \frac{\partial p_s}{\partial x} + v \frac{\partial p_s}{\partial y} + \sigma \left(\frac{\partial p_s}{\partial x} \frac{\partial u}{\partial \sigma} + \frac{\partial p_s}{\partial y} \frac{\partial v}{\partial \sigma} \right) + \frac{\partial \dot{\sigma}}{\partial \sigma} p_s \right]. \tag{3.60}$$

From the use of equations (3.57), (3.58) and (3.60) the continuity equation (3.25) transforms to σ coordinates as follows:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial \dot{\sigma}}{\partial \sigma}+\frac{\partial \ln p_s}{\partial t}+u\frac{\partial \ln p_s}{\partial x}+v\frac{\partial \ln p_s}{\partial y} = 0, \tag{3.61}$$

or, in a more compact form:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial \dot{\sigma}}{\partial \sigma}+\frac{D \ln p_s}{Dt}=0. \tag{3.62}$$

3.4.5 The thermodynamic energy equation

The form of the thermodynamic energy equation in σ coordinates is the same as for the pressure coordinate equation (3.40):

$$\frac{DT}{Dt}-\kappa\frac{\omega T}{p}=0. \tag{3.63}$$

3.4.6 The extended nonhydrostatic equation set

For the sake of convenience and future reference, the set of equations derived in σ coordinates based on the full pressure field is restated and renumbered:

$$\frac{Du}{Dt}-fv+\frac{\partial \phi}{\partial x}-\sigma\frac{\partial \phi}{\partial \sigma}\frac{\partial \ln p_s}{\partial x}=0, \tag{3.64}$$

$$\frac{Dv}{Dt}+fu+\frac{\partial \phi}{\partial y}-\sigma\frac{\partial \phi}{\partial \sigma}\frac{\partial \ln p_s}{\partial y}=0, \tag{3.65}$$

$$\frac{R}{g}\frac{D}{Dt}\left(\frac{\omega T}{p}\right)+g+\frac{p}{p_s}\frac{g}{RT}\frac{\partial \phi}{\partial \sigma}=0, \tag{3.66}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial \dot{\sigma}}{\partial \sigma}+\frac{D \ln p_s}{Dt}=0, \tag{3.67}$$

$$\frac{DT}{Dt}-\kappa\frac{\omega T}{p}=0. \tag{3.68}$$

It is convenient to introduce the quantity Ω by using relation (3.59) between the variables $\dot{\sigma}$ and ω:

$$\Omega=\frac{\omega}{p}=\frac{p_s}{\sigma p_s+p_T}\left(\sigma\frac{D \ln p_s}{Dt}+\dot{\sigma}\right). \tag{3.69}$$

3.4.7 A diagnostic equation for ϕ in σ coordinates

Note that from this point in the text and onwards, partial derivatives with respect to x, y and t are taken on constant σ levels, unless when stated otherwise. From the transformation relationships (3.50) and (3.51) it follows that:

$$\left(\frac{\partial \phi^2}{\partial x^2}\right)_p = \left(\frac{\partial \phi^2}{\partial x^2}\right) - \sigma \frac{\partial \ln p_s}{\partial x}\frac{\partial^2 \phi}{\partial x \partial \sigma} - \sigma \frac{\partial \phi}{\partial \sigma}\frac{\partial}{\partial x}\left(\frac{\partial \ln p_s}{\partial x}\right) - \sigma \frac{\partial \ln p_s}{\partial x}\frac{\partial^2 \phi}{\partial x \partial \sigma}$$

$$+\frac{\partial}{\partial \sigma}\left(\frac{\partial \phi}{\partial \sigma}\frac{\sigma}{p_s}\frac{\partial p_s}{\partial x}\right)\frac{\sigma}{p_s}\frac{\partial p_s}{\partial x}$$

$$= \left(\frac{\partial^2 \phi}{\partial x^2}\right)_\sigma - 2\sigma \frac{\partial \ln p_s}{\partial x}\frac{\partial^2 \phi}{\partial x \partial \sigma} + \sigma \frac{\partial}{\partial \sigma}\left(\sigma \frac{\partial \phi}{\partial \sigma}\right)\left(\frac{\partial \ln p_s}{\partial x}\right)^2 - \sigma \frac{\partial \phi}{\partial \sigma}\frac{\partial^2 \ln p_s}{\partial x^2}. \quad (3.70)$$

The last two terms in equation (3.70) can be written in alternative forms:

$$\sigma \frac{\partial}{\partial \sigma}\left(\sigma \frac{\partial \phi}{\partial \sigma}\right)\left(\frac{\partial \ln p_s}{\partial x}\right)^2 = \frac{\partial}{\partial \sigma}\left(\sigma^2 \frac{\partial \phi}{\partial \sigma}\right)\left(\frac{\partial \ln p_s}{\partial x}\right)^2 - \sigma \frac{\partial \phi}{\partial \sigma}\left(\frac{\partial \ln p_s}{\partial x}\right)^2 \quad (3.71)$$

and

$$-\sigma \frac{\partial \phi}{\partial \sigma}\frac{\partial^2 \ln p_s}{\partial x^2} = -\sigma \frac{\partial \phi}{\partial \sigma}\frac{1}{p_s}\frac{\partial^2 p_s}{\partial x^2} + \sigma \frac{\partial \phi}{\partial \sigma}(\frac{\partial \ln p_s}{\partial})^2. \quad (3.72)$$

Substituting (3.71) and (3.72) into (3.70) gives the required coordinate transformation for the term $\left(\partial^2 \phi / \partial x^2\right)_p$:

$$(\frac{\partial^2 \phi}{\partial x^2})_p = (\frac{\partial^2 \phi}{\partial x^2})_\sigma - 2\sigma \frac{\partial \ln p_s}{\partial x}\frac{\partial^2 \phi}{\partial x \partial \sigma} + \frac{\partial}{\partial \sigma}(\sigma^2 \frac{\partial \phi}{\partial \sigma})(\frac{\partial \ln p_s}{\partial x})^2 - \sigma \frac{\partial \phi}{\partial \sigma}\frac{1}{p_s}\frac{\partial^2 p_s}{\partial x^2}. \quad (3.73)$$

Similarly, it can be shown that

$$(\frac{\partial^2 \phi}{\partial y^2})_p = (\frac{\partial^2 \phi}{\partial y^2})_\sigma - 2\sigma \frac{\partial \ln p_s}{\partial y}\frac{\partial^2 \phi}{\partial y \partial \sigma} + \frac{\partial}{\partial \sigma}(\sigma^2 \frac{\partial \phi}{\partial \sigma})(\frac{\partial \ln p_s}{\partial y})^2 - \sigma \frac{\partial \phi}{\partial \sigma}\frac{1}{p_s}\frac{\partial^2 p_s}{\partial y^2}. \quad (3.74)$$

To complete the tranformation of terms on the left-hand side of (3.36), transformation relationship (3.51) can be used to show that

$$\frac{\partial}{\partial p}(r^2 \frac{\partial \phi}{\partial p}) = \frac{\partial}{\partial \sigma}(s^2 \frac{\partial \phi}{\partial \sigma}). \quad (3.75)$$

Here $r = gp/RT$ as before and $s = (\sigma + p_T/p_s)\,(g/RT) = (p/p_s)\,(g/RT)$.

The Jacobian terms (3.13) on the right-hand side of (3.36) transforms to σ coordinates using transformation relationships (3.50) and (3.51) as follows:

$$\frac{\partial \omega}{\partial p}\frac{\partial u}{\partial x} - \frac{\partial u}{\partial p}\frac{\partial \omega}{\partial x} = \frac{1}{p_s}\frac{\partial \omega}{\partial \sigma}\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial \sigma}\sigma\frac{\partial \ln p_s}{\partial x}\right) - \frac{1}{p_s}\frac{\partial u}{\partial \sigma}\left(\frac{\partial \omega}{\partial x} - \frac{\partial \omega}{\partial \sigma}\sigma\frac{\partial \ln p_s}{\partial x}\right)$$

$$= \frac{\partial u}{\partial x}\frac{\partial}{\partial \sigma}\left(\Omega\frac{p}{p_s}\right) - \frac{1}{p_s}\frac{\partial u}{\partial \sigma}\frac{\partial}{\partial x}\left(\Omega p\right) \tag{3.76}$$

and similarly,

$$\frac{\partial \omega}{\partial p}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial p}\frac{\partial \omega}{\partial y} = \frac{\partial v}{\partial y}\frac{\partial}{\partial \sigma}\left(\Omega\frac{p}{p_s}\right) - \frac{1}{p_s}\frac{\partial v}{\partial \sigma}\frac{\partial}{\partial y}\left(\Omega p\right). \tag{3.77}$$

The remaining terms in the Jacobian transform to:

$$\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y}\right)_p = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y}$$

$$+\sigma\left[\frac{\partial \ln p_s}{\partial x}\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial \sigma} - \frac{\partial v}{\partial y}\frac{\partial u}{\partial \sigma}\right) + \frac{\partial \ln p_s}{\partial y}\left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial \sigma} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial \sigma}\right)\right]. \tag{3.78}$$

The Coriolis terms transform as follows:

$$f\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)_p = f\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + f\sigma\left[\frac{\partial u}{\partial \sigma}\left(\frac{\partial \ln p_s}{\partial y}\right) - \frac{\partial v}{\partial \sigma}\left(\frac{\partial \ln p_s}{\partial x}\right)\right], \tag{3.79}$$

whilst the term $-udf/dy$ remains unchanged by the coordinate transformation.

The remaining two terms on the right-hand side of (3.36) transform as follows:

$$-\frac{\partial}{\partial p}\left(rg - \frac{\omega^2}{\gamma p}\right) = -\frac{\partial}{\partial \sigma}\left(sg - \frac{1}{\gamma}\Omega^2\frac{p}{p_s}\right). \tag{3.80}$$

Combining the above transformation relationships (3.73 to 3.80) yields an elliptic equation for ϕ in σ coordinates:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial}{\partial \sigma}\left(s^2\frac{\partial \phi}{\partial \sigma}\right) - 2\sigma\left(\frac{\partial \ln p_s}{\partial x}\frac{\partial^2 \phi}{\partial x \partial \sigma} + \frac{\partial \ln p_s}{\partial y}\frac{\partial^2 \phi}{\partial y \partial \sigma}\right) +$$

$$\left[\left(\frac{\partial \ln p_s}{\partial x}\right)^2 + \left(\frac{\partial \ln p_s}{\partial y}\right)^2\right]\left[\frac{\partial}{\partial \sigma}\left(\sigma^2\frac{\partial \phi}{\partial \sigma}\right)\right] - \frac{\sigma}{p_s}\left(\frac{\partial^2 p_s}{\partial x^2} + \frac{\partial^2 p_s}{\partial y^2}\right)\frac{\partial \phi}{\partial \sigma} =$$

$$2\{\left\{\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\frac{\partial}{\partial \sigma}\left(\Omega\frac{p}{p_s}\right)\right.$$

$$-\frac{1}{p_s}\left[\frac{\partial}{\partial x}(\Omega p)\frac{\partial u}{\partial \sigma}+\frac{\partial}{\partial y}(\Omega p)\frac{\partial v}{\partial \sigma}\right]+\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}+$$

$$\left.\sigma\left[\frac{\partial \ln p_s}{\partial x}\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial \sigma}-\frac{\partial v}{\partial y}\frac{\partial u}{\partial \sigma}\right)+\frac{\partial \ln p_s}{\partial y}\left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial \sigma}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial \sigma}\right)\right]\right\}$$

$$+f\left[\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}+\sigma\left(\frac{\partial u}{\partial \sigma}\frac{\partial \ln p_s}{\partial y}-\frac{\partial v}{\partial \sigma}\frac{\partial \ln p_s}{\partial x}\right)\right]-u\frac{df}{dy}-\frac{\partial}{\partial \sigma}\left(sg-\frac{p}{p_s}\Omega^2\frac{1}{\gamma}\right). \tag{3.81}$$

Equation (3.81) may also be derived directly from equations (3.64) to (3.69). This tedious derivation can be found in Appendix A.

3.4.8 Two-dimensional version of White's equations in σ coordinates

In two spatial dimensions, the pressure coordinate momentum, continuity and thermodynamic energy equations of White (1989) transform to the following σ coordinate equations:

$$\frac{Du}{Dt}+\left(\frac{\partial \phi}{\partial x}\right)-\sigma\left(\frac{\partial \phi}{\partial \sigma}\right)\frac{\partial \ln p_s}{\partial x}=0, \tag{3.82}$$

$$\frac{R}{g}\frac{D}{Dt}\left(\frac{\omega T}{p}\right)+g+\frac{p}{p_s}\frac{g}{RT}\frac{\partial \phi}{\partial \sigma}=0, \tag{3.83}$$

$$\frac{\partial u}{\partial x}+\frac{\partial \dot{\sigma}}{\partial \sigma}+\frac{D\ln p_s}{Dt}=0, \tag{3.84}$$

$$\frac{DT}{Dt}-\kappa\frac{\omega T}{p}=0. \tag{3.85}$$

The two-dimensional diagnostic equation for ϕ in σ coordinates is:

$$\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial}{\partial \sigma}\left(s^2\frac{\partial \phi}{\partial \sigma}\right)-2\sigma\frac{\partial \ln p_s}{\partial x}\frac{\partial^2 \phi}{\partial x \partial \sigma}+\left(\frac{\partial \ln p_s}{\partial x}\right)^2\frac{\partial}{\partial \sigma}\left(\sigma^2\frac{\partial \phi}{\partial \sigma}\right)-\frac{\sigma}{p_s}\left(\frac{\partial^2 p_s}{\partial x^2}\right)\frac{\partial \phi}{\partial \sigma}=$$

$$2\left[\frac{\partial u}{\partial x}\frac{\partial}{\partial \sigma}\left(\Omega\frac{p}{p_s}\right)-\frac{1}{p_s}\frac{\partial}{\partial x}(\Omega p)\frac{\partial u}{\partial \sigma}\right]-\frac{\partial}{\partial \sigma}\left[(sg)-\frac{1}{\gamma}\frac{p}{p_s}\Omega^2\right]. \tag{3.86}$$

3.5 Properties of the nonhydrostatic σ coordinate equations based on the full pressure field

In the following sections, the properties of the derived σ coordinate formulation of the equations of White (1989) are studied. Links and differences with respect to the MP equations in σ coordinates are pointed out.

3.5.1 Physical implications of the approximated vertical momentum equation

The unapproximated vertical momentum equation, for motion on an f plane, is

$$\frac{Dw}{Dt} + g + \frac{1}{\rho}\frac{\partial p}{\partial z} = 0. \tag{3.87}$$

Here w is the Cartesian vertical velocity Dz/Dt. Following the discussion of White (1989) for the extended pressure coordinate equations, (3.87) may be written as

$$\frac{1}{g}\frac{\partial \phi}{\partial \sigma}\frac{Dw}{Dt} + \frac{\partial \phi}{\partial \sigma} + RT\frac{p_s}{p} = 0. \tag{3.88}$$

Note that the reference state relation (3.11) transforms to σ coordinates as

$$\frac{\partial \phi_{ref}}{\partial \sigma} + \frac{p_s}{p}RT_{ref} = 0 \tag{3.89}$$

on the use of (3.51). Subtracting the reference state relation from equation (3.88), and rearranging the terms, gives

$$\frac{Dw}{Dt} + g\frac{\partial \sigma}{\partial \phi}\left(\frac{\partial \phi'}{\partial \sigma} + \frac{p_s}{p}RT'\right) = 0. \tag{3.90}$$

Substituting for $\partial\sigma/\partial\phi$ from equation (3.88) leads to

$$\frac{Dw}{Dt} + g\frac{\partial \sigma}{\partial \phi}\left(\frac{\partial \phi'}{\partial \sigma} + \frac{p_s}{p}RT'\right) = 0. \tag{3.91}$$

From considering (3.91) possible approximations become apparent. Upon introducing the approximations

$$Dw/Dt \ll g \tag{3.92}$$

and

$$w \simeq \hat{w} \equiv -\omega/\rho g = -\omega RT/gp, \tag{3.93}$$

(3.91) reduces to (3.67), the σ coordinate form of the extended MP vertical momentum equation. It does not contain the further replacement of T by T_{ref}, from which the MP vertical momentum equation is obtained.

3.5.2 Energetics of the σ coordinate equations

In order to obtain an equation for the energy budget of the σ coordinate equations (3.64) to (3.68), multiplying (3.64) by u, (3.65) by v, (3.66) by $\hat{w}$, (3.68) by c_p and adding the equations give:

$$\frac{D}{Dt}\left(\frac{1}{2}u^2+\frac{1}{2}v^2+\frac{1}{2}\hat{w^2}+c_pT\right)=-u\frac{\partial\phi}{\partial x}-v\frac{\partial\phi}{\partial y}+\sigma\frac{\partial\phi}{\partial\sigma}\left(u\frac{\partial\ln p_s}{\partial x}+v\frac{\partial\ln p_s}{\partial y}\right)$$
$$+\hat{w}\left(g+\frac{p}{p_s}\frac{g}{RT}\frac{\partial\phi}{\partial\sigma}\right)+R\frac{\omega T}{p}. \tag{3.94}$$

From multiplying (3.94) by p_s, it follows that

$$p_s\frac{D}{Dt}\left(\frac{1}{2}u^2+\frac{1}{2}v^2+\frac{1}{2}\hat{w^2}+c_pT\right)=-p_s\left(u\frac{\partial\phi}{\partial x}+v\frac{\partial\phi}{\partial y}\right)+\sigma\frac{\partial\phi}{\partial\sigma}\left(u\frac{\partial p_s}{\partial x}+v\frac{\partial p_s}{\partial y}\right)$$
$$+\hat{w}p_s\left(g+\frac{p}{p_s}\frac{g}{RT}\frac{\partial\phi}{\partial\sigma}\right)+p_sR\frac{\omega T}{p}. \tag{3.95}$$

By using (3.93) it can be shown that (3.95) reduces to:

$$p_s\frac{D}{Dt}\left(\frac{1}{2}u^2+\frac{1}{2}v^2+\frac{1}{2}\hat{w^2}+c_pT\right)=\sigma\frac{\partial\phi}{\partial\sigma}\left(u\frac{\partial p_s}{\partial x}+v\frac{\partial p_s}{\partial y}\right)$$
$$-\left[\frac{\partial\left(p_su\phi\right)}{\partial x}-\phi\frac{\partial\left(up_s\right)}{\partial x}+\frac{\partial\left(p_sv\phi\right)}{\partial y}-\phi\frac{\partial\left(vp_s\right)}{\partial y}\right]-\omega\frac{\partial\phi}{\partial\sigma}$$
$$=\quad-\frac{\partial\left(p_su\phi\right)}{\partial x}-\frac{\partial\left(p_sv\phi\right)}{\partial y}+\phi\left(u\frac{\partial p_s}{\partial x}+v\frac{\partial p_s}{\partial y}\right)+\phi p_s\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$
$$-\omega\frac{\partial\phi}{\partial\sigma}+\sigma\frac{\partial\phi}{\partial\sigma}\left(u\frac{\partial p_s}{\partial x}+v\frac{\partial p_s}{\partial y}\right). \tag{3.96}$$

From the use of the continuity equation (3.67) and relationship (3.69) it is finally obtained that:

$$p_s\frac{D}{Dt}\left(\frac{1}{2}u^2+\frac{1}{2}v^2+\frac{1}{2}\hat{w^2}+c_pT\right)=-\frac{\partial\left(p_su\phi\right)}{\partial x}-\frac{\partial\left(p_sv\phi\right)}{\partial y}-\frac{\partial}{\partial\sigma}\left(p_s\dot{\sigma}\phi\right)$$
$$-\frac{\partial}{\partial\sigma}\left(\phi\sigma\right)\frac{\partial p_s}{\partial t}. \tag{3.97}$$

Equation (3.97) corresponds to the energy equation found by Miller and White (1984) for the two-dimensional MP equations in σ coordinates, with the difference that the aproximated "vertical" specific kinectic energy per unit mass is evaluated in terms of a reference temperature in the MP equations, by using (3.4). As pointed out by Miller and White (1984) for the energy equation corresponding to the MP equations, (3.97) is similar to the total energy equation implied by the usual hydrostatic σ coordinate equations (e.g. Haltiner, 1971). The main difference with respect to the energy equation of the hydrostatic σ coordinate equations, is that the contribution of vertical velocity to the specific kinetic energy is present in approximated form in (3.97), whilst it is neglected in the hydrostatic form of the equation.

3.5.3 Linearized equations

Knowledge of the types of motion described by a given equation set is essential not only for understanding the dynamical properties of the system, but also for practical purposes when numerical computations are carried out and a suitable algorithm is to be chosen (Miller, 1974).

The linear analysis presented in this section closely follows that of Miller and White (1984). However, the use of the modified σ coordinate (3.42) with $p_T > 0$ requires a more complicated treatment than the case of $p_T = 0$ (as used by Miller and White, 1984). The two-dimensional σ coordinate equations (3.82) to (3.85) are studied in order to keep the analysis compact, but the extension to three dimensions is obvious. Equations (3.82) to (3.85) are linearized around a reference state of no motion, that is $u_{ref} = 0$ and $\dot{\sigma}_{ref} = 0$. The reference state is chosen to be isothermal and in hydrostatic balance:

$$\frac{d\phi_{ref}}{dp} = -\frac{RT_0}{p}, \tag{3.98}$$

which by the use of transformation relation (3.51) translates to

$$\frac{\partial\phi_{ref}}{\partial\sigma} = \frac{-p_s}{p} RT_0. \tag{3.99}$$

Here T_0 is the temperature of the reference state and $\phi_{ref} = \phi_{ref}(p)$. The reference surface pressure p_{surf_ref} is constant. Note that the reference state is a function of pressure only, which implies that the reference state geopotential depends on the horizontal position at levels of constant σ. In fact it is useful to note that, from the transformation relation (3.50),

$$\begin{aligned}\left(\frac{\partial\phi_{ref}}{\partial x}\right)_\sigma &= \left(\frac{\partial\phi_{ref}}{\partial x}\right)_p + \frac{\partial\phi_{ref}}{\partial\sigma}\frac{\sigma}{p_s}\left(\frac{\partial p_s}{\partial x}\right)_\sigma \\ &= \sigma\frac{\partial\phi_{ref}}{\partial\sigma}\left(\frac{\partial \ln p_s}{\partial x}\right)_\sigma = -\sigma\frac{p_s}{p}RT_0\left(\frac{\partial \ln p_s}{\partial x}\right)_\sigma,\end{aligned} \tag{3.100}$$

since ϕ_{ref} is a function of pressure only.

The perturbation fields $\alpha^{'}$ are defined by

$$\alpha = \alpha_{ref} + \alpha^{'} \tag{3.101}$$

for a particular field variable α. Following the usual linearization procedure, all variables in (3.82) to (3.86) are written in terms of reference and perturbation parts. By neglecting the products of perturbations, and by making use of (3.99) and (3.100), the linearized equations may be shown to be:

$$\frac{\partial u}{\partial t} + \frac{\partial \phi^{'}}{\partial x} = 0, \tag{3.102}$$

$$H_0 \frac{\partial}{\partial t}\left[\frac{p_0}{\sigma p_0 + p_T}\left(\frac{\sigma}{p_0}\frac{\partial p_s}{\partial t} + \dot{\sigma}\right)\right] + \frac{T^{'}}{T_0} g + \left(\frac{\sigma p_0 + p_T}{p_0}\right)\frac{1}{H_0}\frac{\partial \phi^{'}}{\partial \sigma} = 0, \tag{3.103}$$

$$\frac{\partial u}{\partial x} + \frac{\partial \dot{\sigma}}{\partial \sigma} + \frac{1}{p_0}\frac{\partial p_s}{\partial t} = 0, \tag{3.104}$$

$$\frac{\partial T^{'}}{\partial t} - \kappa\left[\frac{p_0}{\sigma p_0 + p_T}\left(\frac{\sigma}{p_0}\frac{\partial p_s}{\partial t} + \dot{\sigma}\right)\right] T_0 = 0. \tag{3.105}$$

Here $H_0 = RT_0/g$ and $p_0 = p_{surf\_ref} - p_T$. Note that in deriving the linearized equations, the assumptions $\left|T^{'}\right| \ll T_0$ and $\left|p_s^{'}\right| \ll p_0$ are made explicitly.

3.5.4 Towards wave-like solutions of the linearized equations

In order to investigate the presence and characteristics of sound and gravity waves that form part of the solution set of (3.82) to (3.85), wave-like solutions of the form

$$Q\,(x,\,t) = \hat{Q}\exp^{ik(x-ct)} \tag{3.106}$$

are posed for (3.82) to (3.85), for each field variable Q (with the field variables being u, $\dot{\sigma}$, $\phi^{'}$, $T^{'}$ and p_s) . Here $\hat{Q}$ is a function of σ at most. It is convenient to write $\hat{p_s}$ in terms of p_0, as $\hat{p_s} = p_0\hat{\pi}$. Note that $kc = \nu$, where ν is the frequency of oscillation, $k = 2\pi/L_x$ is the wave number and L_x is the wave length. For propagating waves the phase speed is constant for an observer moving at the phase speed c. The following equations relating the amplitudes of the posed solutions are obtained from substituting (3.106) into (3.102) to (3.105):

$$-c\hat{u} + \hat{\phi}^{'} = 0, \tag{3.107}$$

$$H_0 \left\{ \frac{p_0}{\sigma p_0 + p_T} \left[-ikc\hat{\dot{\sigma}} + (ikc)^2 \sigma \hat{\pi} \right] \right\} + g\frac{\hat{T}'}{T_0} + \left(\frac{\sigma p_0 + p_T}{p_0} \right) \frac{1}{H_0} \frac{d\hat{\phi}'}{d\sigma} = 0, \tag{3.108}$$

$$ik\hat{u} + \frac{d\hat{\dot{\sigma}}}{d\sigma} - ikc\hat{\pi} = 0, \tag{3.109}$$

$$(-ikc)\,\hat{T}' - \kappa \left[\frac{p_0}{\sigma p_0 + p_T} \left(\hat{\dot{\sigma}} - ikc\sigma\hat{\pi} \right) \right] T_0 = 0. \tag{3.110}$$

Eliminating $\hat{u}$ and $\hat{T}'$ from (3.107) to (3.110) gives

$$\frac{d\hat{\dot{\sigma}}}{d\sigma} - ikc\hat{\pi} + i\frac{k}{c}\hat{\phi}' = 0, \tag{3.111}$$

$$icH_0^2 \left(k^2 - \frac{N^2}{c^2} \right) \left(\hat{\dot{\sigma}} - ikc\sigma\hat{\pi} \right) - \left(\frac{\sigma p_0 + p_T}{p_0} \right)^2 k \frac{d\hat{\phi}'}{d\sigma} = 0, \tag{3.112}$$

where $N^2 = g\kappa/H_0$.

Following Miller and White (1984) the best way to proceed is to eliminate $\hat{\dot{\sigma}}$ from equations (3.111 and 3.112) to obtain a second-order differential equation for $\hat{\phi}'$:

$$\frac{d}{d\sigma}[(\frac{\sigma p_0 + p_T}{p_0})^2 \frac{d\hat{\phi}'}{d\sigma}] + H_0^2(\frac{N^2}{c^2} - k^2)\hat{\phi}' = 0. \tag{3.113}$$

Equation (3.113) may alternatively be derived from the linearized version of the elliptic equation (3.81). This approach is useful in the eventual analysis of numerical schemes used to solve the system (3.64) to (3.68) (see Chapter 4), and is presented in Appendix B.

Under the transformations

$$Z = -H_0 \ln(\frac{\sigma p_0 + p_T}{p_0}), \tag{3.114}$$

$$F = \hat{\phi}' \exp^{-Z/2H_0}, \tag{3.115}$$

(3.113) becomes

$$\frac{D^2F}{DZ^2} + (\frac{N^2}{c^2} - k^2 - \frac{1}{4H_0^2})F = 0. \tag{3.116}$$

The algebra involved in deriving (3.116) is discussed in Appendix C.

For later application, it is useful to find a the relation between $\hat{\pi}$ and $\hat{\phi}'$. This is achieved by integrating (3.111) over $\sigma = [0, 1]$ and by applying $\hat{\dot{\sigma}} = 0$ (that is, $\dot{\sigma} = 0$) at $\sigma = 0$ and 1

$$\hat{\pi} = \frac{1}{c^2} \int_0^1 \hat{\phi}' d\sigma. \tag{3.117}$$

3.5.5 Solutions of (3.116) with exponential variation in height

3.5.5.1 Form of the sound wave solutions

Solutions of (3.116) with exponential height variation (if they exist) have

$$\mu^2 = \frac{1}{4H_0^2} - \left(\frac{N^2}{c^2} - k^2 \right) > 0, \tag{3.118}$$

so that

$$\hat{\phi}' = \left[A \exp^{-\mu Z} + B \exp^{\mu Z} \right] \exp^{\frac{Z}{2H_0}} \tag{3.119}$$

and one may define $\mu > 0$. Equation (3.118) is the frequency equation for Lamb-waves described by (3.82) to (3.85) (see the analysis by Miller (1974) and Miller and White (1984) for the MP equations, also see the following sections 3.5.5.5 and 3.5.5.6).

It may be noted that relationship (3.118) may consistently be obtained by considering the upper and lower boundary conditions that apply to $\dot{\sigma}$, $\hat{\phi}'$, and $d\hat{\phi}'/d\sigma$. The details of this derivation are given in Appendix D and make use of results obtained in sections 3.5.5.3 and 3.5.5.4. A second equation relating μ and c needs to be obtained, in order to find an expression for c in terms of the reference state parameters and physical constants. To this end, it is useful to linearize the term $D\phi/Dt$ at $\sigma = 1$. Note that

$$\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + u \frac{\partial \phi}{\partial x} \tag{3.120}$$

at $\sigma = 1$. Using (3.50) and (3.51), this may be written as:

$$\frac{D\phi}{Dt} = \frac{\partial \phi'}{\partial t} + \sigma \frac{d\phi_{ref}}{dp} \left(\frac{\partial p_s}{\partial t} + u \frac{\partial p_s}{\partial x} \right) + u \frac{\partial \phi'}{\partial x}$$

$$= \frac{\partial \phi'}{\partial t} + \sigma \left(\frac{-RT_0}{\sigma p_s + p_T} \right) \left(\frac{\partial p_s}{\partial t} + u \frac{\partial p_s}{\partial x} \right) + u \frac{\partial \phi'}{\partial x}. \tag{3.121}$$

Linearizing (3.121) and requiring that $w = 0$ at $\sigma = 1$ (where $\dot{\sigma} = 0$ by definition) gives (at $\sigma = 1$):

$$\frac{\partial \phi'}{\partial t} = \frac{RT_0}{p_0}\frac{\partial p_s}{\partial t} = \frac{gH_0}{p_0}\frac{\partial p_s}{\partial t}. \tag{3.122}$$

Note that by requiring $w = 0$ at $\sigma = 1$ (that is $Z = 0$), specifies $\sigma = 1$ as a level surface. Equation (3.122) reduces to

$$\hat{\phi}' = gH_0\hat{\pi} \tag{3.123}$$

at $\sigma = 1$, on the substitution of (3.106). The required second equation relating μ and c may be obtained by substituting appropriate expressions for $\hat{\phi}'$ and $\hat{\pi}$ in (3.123).

3.5.5.2 Finding an expression for $\hat{\phi}'$ at $\sigma = 1$

Recalling the definition $Z = -H_0 \ln[\frac{\sigma p_0 + p_T}{p_0}]$, it follows from (3.119) that

$$\hat{\phi}' = A(\frac{\sigma p_0 + p_T}{p_0})^{H_0(\mu - \frac{1}{2H_0})} + B(\frac{\sigma p_0 + p_T}{p_0})^{H_0(-\mu - \frac{1}{2H_0})}. \tag{3.124}$$

From (3.124) it follows that

$$\hat{\phi}'(\sigma = 1)) = A\left(\frac{p_0 + p_T}{p_0}\right)^{H_0(\mu - 1/2H_0)} + B\left(\frac{p_0 + p_T}{p_0}\right)^{H_0(-\mu - 1/2H_0)}. \tag{3.125}$$

3.5.5.3 Applying the linearized continuity equation

From substituting (3.124) in (3.117) it is obtained that:

$$\hat{\pi} = \frac{1}{c^2}\left\{A\left[\frac{1}{H_0(\mu + 1/2H_0)}\right]\left[\left(\frac{p_0 + p_T}{p_0}\right)^{H_0(\mu + 1/2H_0)} - \left(\frac{p_T}{p_0}\right)^{H_0(\mu + 1/2H_0)}\right]\right.$$

$$\left. + B\left[\frac{1}{H_0(-\mu + 1/2H_0)}\right]\left[\left(\frac{p_0 + p_T}{p_0}\right)^{H_0(-\mu + 1/2H_0)} - \left(\frac{p_T}{p_0}\right)^{H_0(-\mu + 1/2H_0)}\right]\right\}. \tag{3.126}$$

Equations (3.126) and (3.125) may now be substituted into (3.123), in order to find the required second relationship between c and μ. However, in order to eliminate constants A and B from the resulting equation, it is necessary to find the relationship between the constants. This relationship is derived in the next subsection.

3.5.5.4 Utilizing the upper boundary condition on $d\hat{\phi}'/d\sigma$

From differentiating equation (3.124) with respect to σ it follows that

$$\frac{d\hat{\phi}'}{d\sigma} = AH_0\left(\mu - 1/2H_0\right)\left(\frac{\sigma p_0 + p_T}{p_0}\right)^{H_0\left(\mu - \frac{3}{2H_0}\right)}$$
$$+BH_0\left(-\mu - 1/2H_0\right)\left(\frac{\sigma p_0 + p_T}{p_0}\right).^{H_0\left(-\mu - \frac{3}{2H_0}\right)} \tag{3.127}$$

Equation (3.112) yields, for the case $p_T > 0$, that $d\hat{\phi}'/d\sigma = 0$ at the upper boundary $\sigma = 0$ (from noting that $\dot{\sigma} = 0$ at $\sigma = 0$ per definition). From this result an equation relating A and B may be obtained by the use of (3.127):

$$A\left[H_0\left(\mu - \frac{1}{2H_0}\right)\left(\frac{p_T}{p_0}\right)^{H_0\left(\mu - \frac{3}{2H_0}\right)}\right] + B\left[H_0\left(-\mu - \frac{1}{2H_0}\right)\left(\frac{p_T}{p_0}\right)^{H_0\left(-\mu - \frac{3}{2H_0}\right)}\right] = 0, \tag{3.128}$$

which reduces to

$$B = A\frac{\left(\mu - \frac{1}{2H_0}\right)}{\left(\mu + \frac{1}{2H_0}\right)}\left(\frac{p_T}{p_0}\right)^{2H_0\mu}. \tag{3.129}$$

3.5.5.5 A second relationship between μ and c

From substituting (3.125) and (3.126) into (3.123) and applying (3.129) we obtain that:

$$\left(\frac{p_0 + p_T}{p_0}\right)^{H_0\left(\mu - \frac{1}{2H_0}\right)} + \left(\frac{\mu - \frac{1}{2H_0}}{\mu + \frac{1}{2H_0}}\right)\left(\frac{p_T}{p_0}\right)^{2H_0\mu}\left(\frac{p_0 + p_T}{p_0}\right)^{H_0\left(-\mu - \frac{1}{2H_0}\right)} =$$
$$\frac{g}{c^2}\left\{\left(\frac{1}{\mu + \frac{1}{2H_0}}\right)\left[\left(\frac{p_0 + p_T}{p_0}\right)^{H_0\left(\mu + \frac{1}{2H_0}\right)} - \left(\frac{p_T}{p_0}\right)^{H_0\left(\mu + \frac{1}{2H_0}\right)}\right]\right.$$
$$\left. + \left(\frac{\mu - \frac{1}{2H_0}}{\mu + \frac{1}{2H_0}}\right)\left(\frac{p_T}{p_0}\right)^{2H_0\mu}\left(\frac{1}{-\mu + \frac{1}{2H_0}}\right)\left[\left(\frac{p_0 + p_T}{p_0}\right)^{H_0\left(-\mu + \frac{1}{2H_0}\right)} - \left(\frac{p_T}{p_0}\right)^{H_0\left(-\mu + \frac{1}{2H_0}\right)}\right]\right\}, \tag{3.130}$$

which reduces to

$$\left(\mu + \frac{1}{2H_0}\right)\left(\frac{p_0 + p_T}{p_0}\right)^{H_0\left(\mu - \frac{1}{2H_0}\right)} + \left(\mu - \frac{1}{2H_0}\right)\left(\frac{p_T}{p_0}\right)^{2H_0\mu}\left(\frac{p_0 + p_T}{p_0}\right)^{H_0\left(-\mu - \frac{1}{2H_0}\right)} =$$
$$\frac{g}{c^2}\left[\left(\frac{p_0 + p_T}{p_0}\right)^{H_0\left(\mu + \frac{1}{2H_0}\right)} - \left(\frac{p_T}{p_0}\right)^{2H_0\mu}\left(\frac{p_0 + p_T}{p_0}\right)^{H_0\left(-\mu + \frac{1}{2H_0}\right)}\right]. \tag{3.131}$$

It may be noted that in the limit, when p_T approaches 0, (3.131) reduces to

$$\mu + \frac{1}{2H_0} = \frac{g}{c^2}. \tag{3.132}$$

This corresponds exactly to the result obtained by Miller and White (1984), for the MP equations, as formulated originally with $p_T = 0$. It is not clear how the simultaneous equations (3.118) and (3.131) may solved for the general case of $p_T > 0$, in order to obtain an expression for c in terms of the reference state parameters and physical constants. However, an approximated relationship may be obtained by making use of a Taylor series expansion for c^2 in (3.131). To this end, note that from (3.131) it follows that

$$c^2 = g\left\{\left[\left(p_0 + p_T\right)/p_0\right]^{H_0(\mu+1/2H_0)} - \left(p_T/p_0\right)^{2H_0\mu}\left[\left(p_0 + p_T\right)/p_0\right]^{H_0(-\mu+1/2H_0)}\right\}/$$

$$\left\{(\mu + 1/2H_0)\left[\left(p_0 + p_T\right)/p_0\right]^{H_0(\mu-1/2H_0)}\right.$$

$$\left. + (\mu - 1/2H_0)\left(p_T/p_0\right)^{2H_0\mu}\left[\left(p_0 + p_T\right)/p_0\right]^{H_0(-\mu-1/2H_0)}\right\}. \tag{3.133}$$

From (3.133) the function $\partial c^2/\partial p_T$ may be calculated, and c^2 may be approximated with a Taylor series expansion around $p_T = 0$ as

$$c^2(\mu, p_T) = c^2(\mu, 0) + \left(\frac{\partial c^2}{\partial p_T}(\mu, 0)\right)(p_T) + O\left(p_T^2\right) \tag{3.134}$$

Evaluating c^2 and $\partial c^2/\partial p_T$ at $p_T = 0$ and substituting in (3.134) leads to the approximated relationship

$$c^2 \approx \frac{g}{\mu + 1/2H_0} + \left(\frac{gH_0}{p_0}\right)\left(1 - \frac{\mu - 1/2H_0}{\mu + 1/2H_0}\right)p_T. \tag{3.135}$$

3.5.5.6 The influence of the height of the model top on the phase speed of the Lamb waves

Equations (3.118) and (3.135) may be used to obtain an approximated expression for c in terms of the reference state parameters and physical constants. To this end, note that c^2 may be eliminated from (3.135) by the use of (3.118), to give

$$\frac{N^2}{1/4H_0^2 + k^2 - \mu^2} = \frac{g}{\mu + 1/2H_0} + \left(\frac{gH_0}{p_0}\right)\left(1 - \frac{\mu - 1/2H_0}{\mu + 1/2H_0}\right)p_T. \tag{3.136}$$

Rearranging the terms in (3.136) gives

$$g\left(1 + p_T/p_0\right)\mu^2 + N^2\mu - \left(1/4H_0^2 + k^2\right)g\left(1 + p_T/p_0\right) + N^2/2H_0, \tag{3.137}$$

which has solutions

$$\mu = \frac{-N^2 \pm \sqrt{N^4 + 4g\,(1 + p_T/p_0)\,[(1/4H_0^2 + k^2)\,g\,(1 + p_T/p_0) - N^2/2H_0]}}{2g\,(1 + p_T/p_0)}. \tag{3.138}$$

Equation (3.138) gives the relationship between μ and the reference state parameters and physical constants. It may be noted that only the positive root represents a physical solution, since $\mu > 0$ by definition. From (3.118) it follows that

$$c = \pm N/\sqrt{(1/4H_0^2 + k^2 - \mu^2)}. \tag{3.139}$$

Equation (3.138) may be used to eliminate μ from (3.139) to give a relationship between c and the reference state parameters and physical constants. The dependency of c on wave number and the height of the model top is depicted in Fig. 3.1, by the use of (3.138) and (3.139), for the choice of $p_T = 0\,hPa$ (yellow line), $p_T = 135\,hPa$ (green line) and $p_T = 442\,hPa$ (black line). The true sound wave speed is depicted by the red line. Formulas (3.138) and (3.139) were applied for $T_0 = 300\,K$; all the other constants have been defined earlier.

It is clear from Fig. 3.1 that for the choice $p_T = 0$ the Lamb wave phase speed is retarded at all wave lengths. The retardation is the strongest at the shortest resolvable scales, whilst the Lamb wave phase speed approaches the true sound wave speed in the long wave length limit. For the case $p_T = 135\,hPa$ and $p_T = 442\,hPa$, the Lamb waves are also retarded at the shortest resolvable scales. However, at lower wave numbers the Lamb waves are accelerated, reaching almost 400 m/s in the long wave length limit for the case $p_T = 442\,hPa$.

In boundary layer modelling, air pollution studies and numerical tests designed to test the performance of numerical schemes, it is common practice to choose the model top well below 0 hPa. With these and meso-scale applications in mind, more insight into the Lamb wave behaviour may be obtained from considering Fig. 3.2. Here the Lamb wave phase speed is displayed as a function of horizontal wave length, for $0.2\,km \leq L_x \leq 200.2\,km$. Note that $Lx = 0.2\,km$ represents the shortest resolvable wave for the choice $2\Delta x = 200\,m$. The colour code used is as stated for Fig. 3.1. Close to the limit of the shortest resolvable waves, the retardation of the Lamb waves are considerable for all the choices of p_T, being about 50 % for wavelengths of about 20 km. Thus, for micro- and meso-scale applications over relatively small domains, the equations (3.82)-(3.85) offer significant computational advantages over the fully-elastic equations. It may be noted that for the choice of $p_T = 442\,hPa$, Lamb waves travelling faster than the true sound wave speed are only present for wave lengths longer than about $140\,km$.

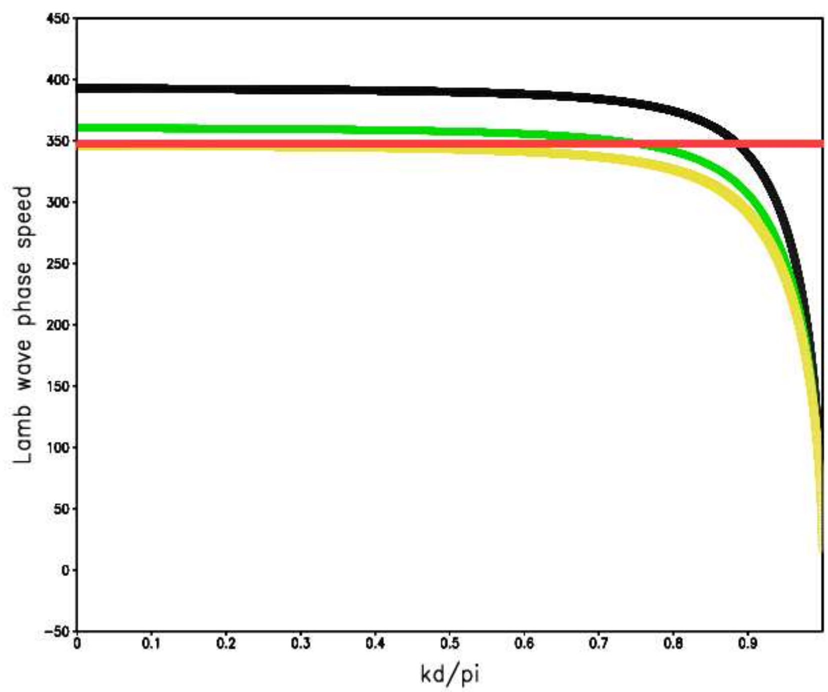

Figure 3.1: Lamb wave phase speed as a function of the normalised wave number, for various choices of the height of the model top: $p_T = 0\,hPa$ (yellow line); $p_T = 135\,hPa$ (green line); $p_T = 442\,hPa$ (black line). The true sound wave speed is depicted by the red line.

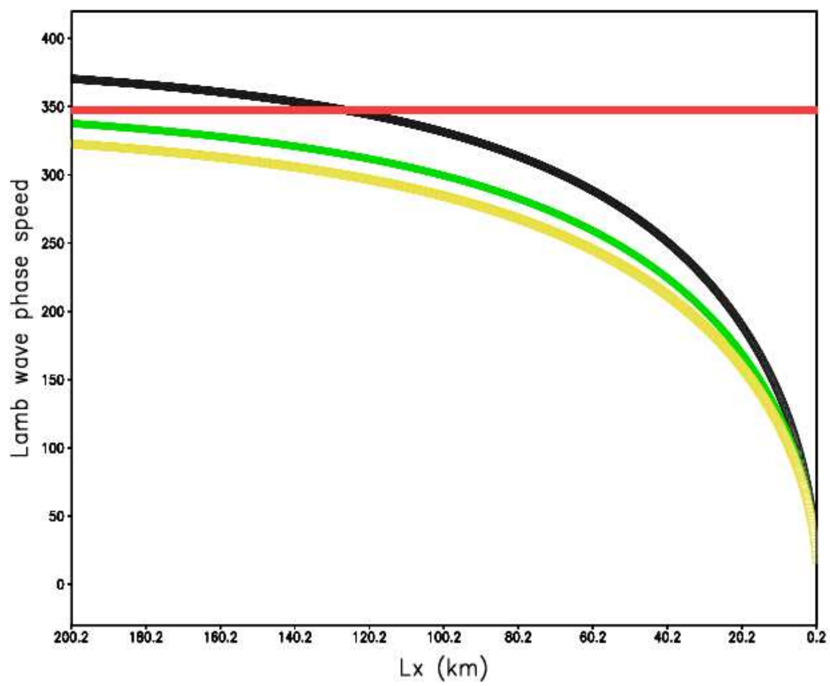

Figure 3.2: Lamb wave phase speed as a function of horizontal wave length, for various choices of the height of the model top: $p_T = 0\,hPa$ (yellow line); $p_T = 135\,hPa$ (green line); $p_T = 442\,hPa$ (black line). The true sound wave speed is depicted by the red line.

3.5.6 Solutions with sinusoidal variation in height

3.5.6.1 Form of the gravity wave solutions

Solutions of (3.116) with sinusoidal height variation (if they exist) have

$$m^2 = -\frac{1}{4H_0^2} + (\frac{N^2}{c^2} - k^2) > 0 \tag{3.140}$$

so that

$$\hat{\phi}' = [A \exp^{-imZ} + B \exp^{imZ}] \exp^{\frac{Z}{2H_0}} \tag{3.141}$$

and one may define $m > 0$. Equation (3.140) corrresponds to the frequency equation obained for gravity waves in the fully-elastic equations (see, for example, Eckart, 1960; Miller, 1974). The fully-elastic equations additionally describe horizontally and vertically propagating acoustic modes (e.g. Eckart, 1960; Miller, 1974), which are not present in (3.82) to (3.85), except for the Lamb waves described in section 3.5.5.

For later application (in Chapter 4) , it is useful to write (3.140) alternatively as

$$-\omega_T^2 \left(k^2 + m^2 + \frac{1}{4H_0^2} \right) c^2 + k^2 c^2 N^2 = 0, \tag{3.142}$$

with $\omega_T = kc$ the true frequency of the gravity waves.

3.5.6.2 Finding an expression for $\hat{\phi}'$ at $\sigma = 1$

From noting that by definition $Z = -H_0 \ln [(\sigma p_0 + p_T)/p_0]$, it follows that

$$\hat{\phi}' = \left(A \exp^{imH_0 ln[(\sigma p_0 + p_T)/p_0]} + B \exp^{-imH_0 ln[(\sigma p_0 + p_T)/p_0]} \right) \left(\frac{p_0}{\sigma p_0 + p_T} \right)^{1/2}. \tag{3.143}$$

From (3.143) it follows that

$$\hat{\phi}'(\sigma = 1) = \left(A \exp^{imH_0 ln[(p_0 + p_T)/p_0]} + B \exp^{-imH_0 ln[(p_0 + p_T)/p_0]} \right) \left(\frac{p_0}{p_0 + p_T} \right)^{1/2}. \tag{3.144}$$

3.5.6.3 Applying the linearized continuity equation

From substituting (3.143) in (3.117) and integrating it may be shown that

$$\hat{\pi} = \frac{1}{c^2 (1 + 4m^2 H_0^2)} \times$$

$$\left\{A\,(2-4mH_0 i)\left[\left(\frac{p_0+p_T}{p_0}\right)^{1/2}\exp^{imH_0\ln[(p_0+p_T)/p_0]}-\left(\frac{p_T}{p_0}\right)^{1/2}\exp^{imH_0\ln(p_T/p_0)}\right]+\right.$$

$$\left.B\,(2+4mH_0 i)\left[\left(\frac{p_0+p_T}{p_0}\right)^{1/2}\exp^{-imH_0\ln[(p_0+p_T)/p_0]}-\left(\frac{p_T}{p_0}\right)^{1/2}\exp^{-imH_0\ln(p_T/p_0)}\right]\right\}. \tag{3.145}$$

The details of the derivation may be found in Appendix E. Similar to the approach used to analyse the sound waves in the linearized equations, equations (3.144) and (3.145) may be substituted into equation (3.123), in order to find the required second relationship between c and m. In order to eliminate constants A and B from the resulting equation, it is necessary to find the relationship between the constants. This relationship is derived in the next subsection.

3.5.6.4 Utilizing the upper boundary condition on $d\hat{\phi}'/d\sigma$

From differentiating (3.143) with respect to σ it follows that

$$\frac{d\hat{\phi}'}{d\sigma}=\left(\frac{p_0}{\sigma p_0+p_T}\right)\left[A\left(-\frac{1}{2}+imH_0\right)\exp^{imH_0\ln[(\sigma p_0+p_T)/p_0]}\right.$$
$$\left.+B\left(-\frac{1}{2}-imH_0\right)\exp^{-imH_0\ln[(\sigma p_0+p_T)/p_0]}\right] \tag{3.146}$$

Recalling that for the case $p_T>0$, $d\hat{\phi}'/d\sigma=0$ at $\sigma=0$ (see section 3.5.5.4), an equation relating A and B may be obtained by the use of (3.146):

$$A\left(-\frac{1}{2}+imH_0\right)\exp^{imH_0\ln(p_T/p_0)}+B\left(-\frac{1}{2}-imH_0\right)\exp^{-imH_0\ln(p_T/p_0)}=0 \tag{3.147}$$

which may be written as

$$B=A\frac{(-2+4imH_0)}{(2+4imH_0)}\exp^{i2mH_0\ln(p_T/p_0)}. \tag{3.148}$$

3.5.6.5 A second relationship between m and c

From substituting (3.144) and (3.145) into (3.123) and applying (3.148) it is obtained that:

$$\left[\exp^{imH_0\ln[(p_0+p_T)/p_0]}+\left(\frac{-2+4imH_0}{2+4imH_0}\right)\exp^{imH_0\{2\ln(p_T/p_0)-\ln[(p_0+p_T)/p_0]\}}\right]\left(\frac{p_0}{p_0+p_T}\right)^2=$$

$$\frac{gH_0}{c^2\left(1+4m^2H_0^2\right)}\times$$

$$\left\{(2-4mH_0 i)\left[\left(\frac{p_0+p_T}{p_0}\right)^{1/2}\exp^{imH_0\ln[(p_0+p_T)/p_0]}-\left(\frac{p_T}{p_0}\right)^{1/2}\exp^{imH_0\ln(p_T/p_0)}\right]+\right.$$

$$\left(\frac{-2+4imH_0}{2+4imH_0}\right)(2+4imH_0)\times$$

$$\left.\left[\left(\frac{p_0+p_T}{p_0}\right)^{1/2}\exp^{imH_0\{2\ln(p_T/p_0)-\ln[(p_0+p_T)/p_0]\}}-\left(\frac{p_T}{p_0}\right)^{1/2}\exp^{imH_0\ln(p_T/p_0)}\right]\right\} \tag{3.149}$$

which reduces to

$$(2+4imH_0)\exp^{imH_0\ln[(p_0+p_T)/p_0]}+(-2+4imH_0)\exp^{imH_0\{2\ln(p_T/p_0)-\ln[(p_0+p_T)/p_0]\}}$$

$$=\frac{4gH_0}{c^2}\left(\exp^{imH_0\ln[(p_0+p_T)/p_0]}-\exp^{imH_0\{2\ln(p_T/p_0)-\ln[(p_0+p_T)/p_0]\}}\right). \tag{3.150}$$

It is not clear how (3.150) and (3.140) may be combined to find an expression which explicitly describes the phase speed c of the gravity waves as a function of the reference state parameters and physical constants. However, equation (3.150) implies the approximated relationship

$$c^2\left|2+4imH_0\right|\leq 4gH_0, \tag{3.151}$$

which may be written as

$$c^2\leq\frac{gH_0}{1+4m^2H_0^2}. \tag{3.152}$$

In the limit, where m approaches zero (the long vertical wave length limit), it holds that

$$c\leq\sqrt{gH_0}. \tag{3.153}$$

Equation (3.153) indicates an upper limit of the phase speed of the gravity waves in the quasi-elastic equations. Note that (3.153) is reminiscent of the phase speed $c = \sqrt{gH}$ of gravity waves in the shallow-water equations, with H the mean depth of the fluid (see Chapter 4). Equations (3.140) and (3.150) may possibly be solved numerically in order to describe the relationship between the gravity wave phase speed c and the reference state parameters more exactly (similar to the result obtained for the Lamb wave phase speed in section 3.5.5). Most importantly though, is that the form of the gravity wave solutions (3.140) correspond to that of the fully elastic equations (e.g. Eckart, 1960; Miller, 1974). This illustrates that buoyancy modes are left undistorted by the approximations introduced to obtain the MP equations and the equations of White (1989).

3.6 Discussion

The main objective of the present chapter was to derive the σ coordinate equivalent of the extended pressure coordinate equations of White (1989). This was achieved by a direct transformation of the pressure coordinate equations to σ coordinates (section 3.4). The numerical solution of the σ coordinate equations, as in the case of the pressure coordinate equations of White (1989) and the MP model, makes use of an elliptic equation in the geopotential. The σ coordinate version of this equation may be derived by means of a coordinate transformation of the associated pressure coordinate equation (section 3.4) or directly from the σ coordinate equations (Appendix A). Equations (3.64) to (3.68) and elliptic equation (3.81) represent the first formulation of White's pressure coordinate equations in σ coordinates. The equations derived make possible the construction of a numerical model based on the σ coordinate equivalent of White's pressure coordinate equations. This numerical model is realized in Chapter 4.

When the derived σ coordinate equations are linearized around an isothermal reference state of no motion, the resulting equations correspond exactly to the linearized MP equations in σ coordinates. The equations of White therefore offers the same advantages as the MP equations with respect to the properties of fast-travelling waves that form part of the solution set of the equations. Equation (3.140) obtained for the phase speed of gravity waves in the σ coordinate equations, is the same as that obtained for the fully-elastic equations. It illustrates that the buoyancy modes in the equations remain undistorted by the simplifications introduced to the unapproximated equations. The linear analysis of the sound waves present in the system, indicates that vertically travelling sound waves are filtered from the equations. The phase speed of Lamb waves that are present in the σ coordinate equations are significantly retarded at short wavelengths (Fig. 3.1), although this advantage is reduced when the model top is chosen to be below $0\,hPa$ (Fig. 3.1). The filtering of the sound waves propagating in the vertical, in combination with the retardation of the Lamb waves, yields the equation set to be "quasi-elastic".

The advantage offered by the extended σ coordinate equations, compared to the MP equations in σ coordinates, is that they are formulated independent of the use of a reference profile. Even the elliptic equation (3.81) is non-linear and may be numerically solved without applying linearization of any kind (see Chapter 4). The extended σ coordinate equations are therefore likely to remain applicable to situations where the basic state is not horizontally homogeneous. The corresponding MP model is likely to be a less accurate description of the flow in such a situation, because of its formulation in terms of a reference profile that depends on pressure only.