Chapter 3

Combined Advanced Failure Intensity Models

3.1 Introduction

In Chapter 2, advanced failure intensity models found in the literature were discussed and categorized in different model classes. The theoretical foundation, implementability and practical applicability of each model were evaluated. From this literature survey and evaluation, the following conclusions were drawn:

(i) The distinction between models applicable for non-repairable systems and repairable systems are not clear enough and are rarely emphasized in the literature. According to Ascher (1999) this contributes to the confusion between the two approaches.

(ii) Most models only consider relative risks. Relative risks are attractive because no assumption needs to be made about the underlying baseline, but it does not provide any information with regards to absolute probabilities. Absolute risks are required to utilize the techniques described in Section 1.4.2 and also to estimate residual life. For this study, models need to be fully parametric to be able to calculate absolute risks and hence residual life, as was stated in the problem statement in Section 1.6.

(iii) With the exception of the Extended Hazard Regression Model (EHRM) (see Section 2.3.3.2), models focus on one particular enhancement (such as addition, multiplication, frailty, mixed time-scales, etc.) rather than combining different enhancements. Models will be more practical if more than one enhancement is allowed in the same model.

(iv) Data sets are often modeled with only one type of model and the results of this model are accepted without comparing it to other models. Part of the reason for this practice is because it is such a laborious task to manipulate data, estimate coefficients and refine algorithms for any particular model.

Subsequent to these conclusions, a methodology was established to improve the shortcomings outlined. The methodology is illustrated in Figure 3.1 and elucidated below.

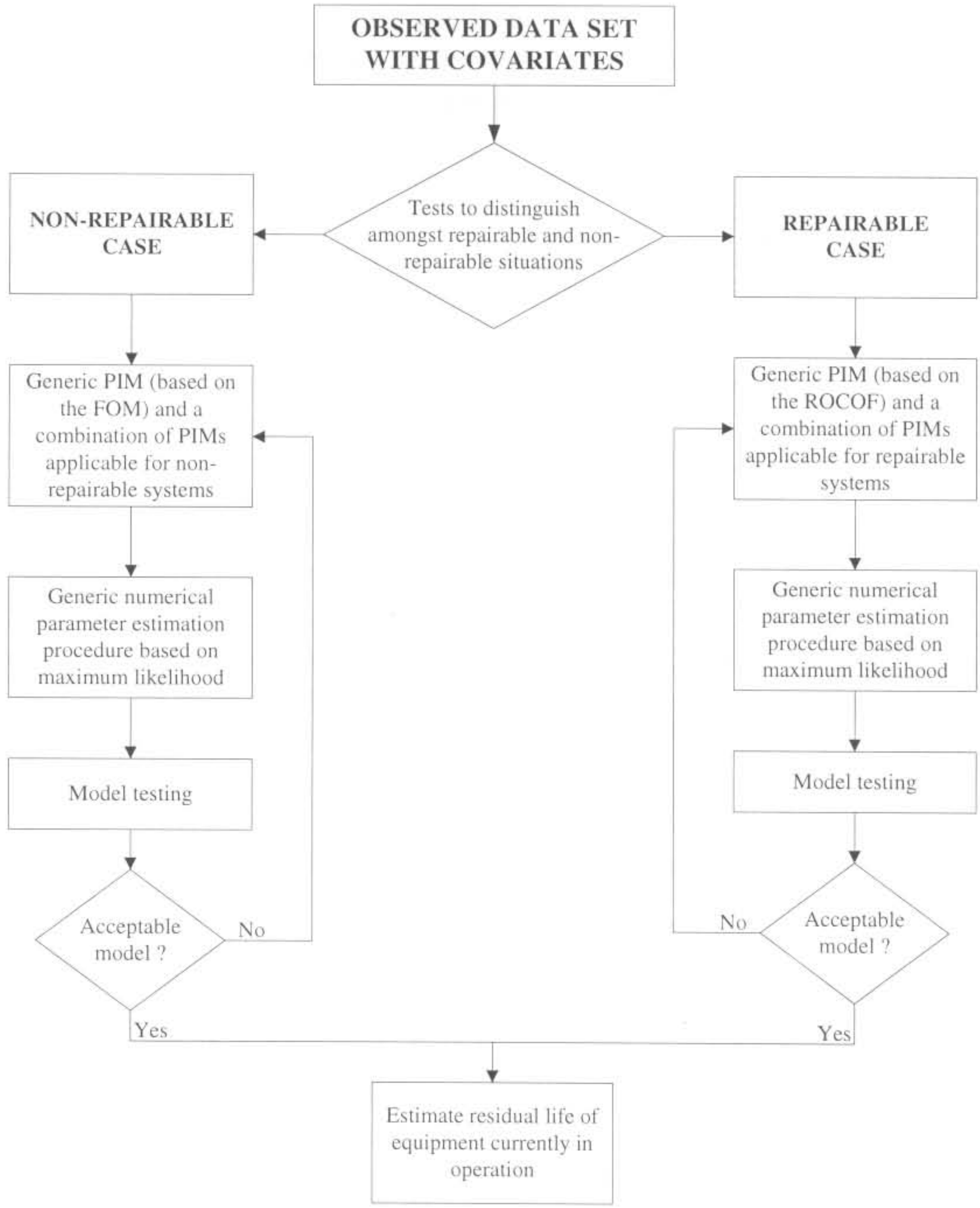

Figure 3.1: Modeling methodology

(i) Fundamentally different models will be constructed for non-repairable and repairable situations and it is therefore important to distinguish between these cases. Techniques such as those of De Laplace (1773), Bates (1955), Bartholomew (1956a), Bartholomew (1956b), Boswell (1966), Cox and Lewis (1966), Boswell and Brunk (1969), Lorden and Eisenberger (1973), Saw (1975), Bain, Engelhardt, and Wright (1985), Lawless and Thiagarajah (1996), Martz and Kvam (1996) and Vaurio (1999), as mentioned in Section 1.2.2, will be used on data sets to determine whether non-repairable or repairable systems theory is more appropriate. These techniques are applied in Chapter 5.

(ii) For both the non-repairable and repairable case, generic fully parametric PIMs will be developed that are able to simplify to the majority of models (or combination of models) described in Section 2.3. For non-repairable cases, the full intensity or conditional intensity is used, i.e. FOM, and for repairable cases the mean intensity or unconditional intensity is used, i.e. ROCOF. (See Table 2.1). Such generic models have the advantage that data can be modeled with the aid of more than one of the conventional enhancements.

(iii) Numerical parameter estimation techniques and algorithms will be developed for the generic PIMs based on maximum likelihood techniques. These algorithms will also be able to estimate the parameters of any simplification of the generic models. This simplifies the modeling processes because different models can be tested without having to develop an estimation algorithm for every special case of the generic PIMs.

(iv) Statistical techniques similar to those described by Kay (1984), Anderson (1982) and Moreau, O'Quigly, and Mesbah (1985) will be used as part of the testing of models' quality. Model quality will also be evaluated by "forecasting" observed events, following case studies by Vlok (1999) and Vlok, Coetzee, Banjevic, Jardine, and Makis (2001) that have shown that models with relatively poor statistical performance can provide very useful practical results. The motivation for this approach is discussed as part of the residual life estimation procedure in Chapter 4.

The methodology above addresses the shortcomings outlined earlier in this section. In the remainder of Chapter 3, the generic PIMs for both the non-repairable and repairable cases are developed with likelihood construction for parameter estimation. Several assumptions are made while developing the theory. These assumptions are motivated in Section 3.4 at the end of this chapter where the practical implementation of the combined advanced failure intensity models is discussed.

3.2 The non-repairable case

A single model that incorporates all the conventional model enhancements related to non-repairable systems (discussed in Chapter 2) is required. In this section such a model is developed. The following assumptions are made:

(i) Multiple system copies are nominally similar and are operating in similar conditions.
(ii) All items considered are behaving according to renewal processes.
(iii) The Weibull distribution is used to parameterize models, except for the case of the POM.
(iv) Covariates are assumed to be positive.

The validity and practical implications of these assumptions are discussed in Section 3.4.

3.2.1 Model development

Suppose $k = 1, 2, .., w$ nominally similar single-part system copies are studied and the event times on each system are recorded. This scenario is illustrated in Figure 3.2.

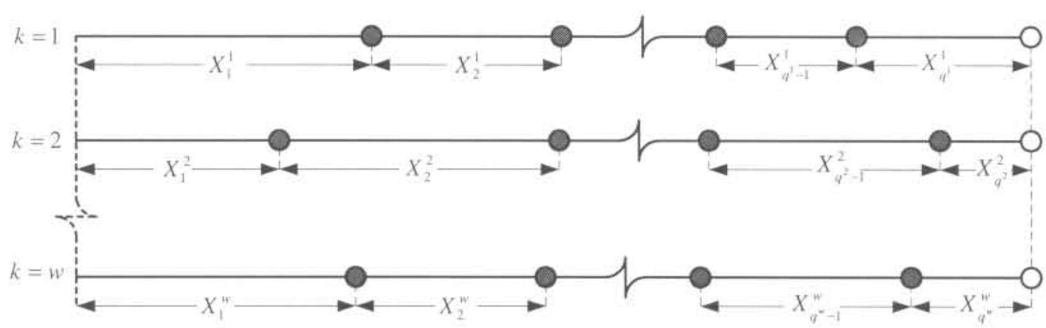

Figure 3.2: w nominally similar single-part system copies (renewed / replaced after each failure) with m time-dependent covariate measurements on each copy (Dots denote failures, circles denote suspensions)

It is assumed that tests revealed that all the particular systems can be modeled with non-repairable systems theory. Event data for each system is recorded in a $q^k \times 2$ matrix, where each row contains X_i^k, the time to event and C_i^k, the event type indicator, i.e. $C_i^k = 0$ denotes suspension and $C_i^k = 1$ denotes failure. This is similar to the approach of Wei, Lin, and Weissfeld (1989). The total observation period for any system k is $\sum X_i^k$ (for $i = 1, 2, ..., q^k$). On each system m time-dependent covariates are measured, i.e. $\boldsymbol{z}_i^k = [z_{i_1}^k(x)\ z_{i_2}^k(x)\ \dots\ z_{i_m}^k(x)]^*$, where x refers to the local time of system k during lifetime i. It is also assumed that the data is categorized in $s = 1, 2, ..., r$ different strata, where r is the highest stratum of item k. A general model that represents the FOM of any of the observed system copies is given by,

$$h(x, \boldsymbol{\theta}) = \zeta_s^k \left(g_s^k(x, \tau_s^k, \psi_s^k) \cdot \lambda(\boldsymbol{\gamma}_s^k \cdot \boldsymbol{z}_s^k) + \nu(\boldsymbol{\alpha}_s^k \cdot \boldsymbol{z}_s^k) \right) \tag{3.1}$$

*For notational convenience, the indication of the time dependence of covariates, "(x)", is suppressed in expressions to follow.

where $\boldsymbol{\theta}^{\dagger}$ consists of

- k: the system copy indicator
- s: the current stratum indicator
- ζ_s^k: a random variable that acts as a frailty in the model that could be system copy- and stratum-specific
- g_s^k: a fully parametric baseline function that could be system copy- and stratum-specific
- τ_s^k: a factor that acts additively on x in g_s^k to represent a time jump or time setback that could be system copy- and stratum-specific
- ψ_s^k: a factor that acts multiplicatively on x in g_s^k to result in an acceleration or deceleration of time that could be system copy- and stratum-specific
- λ_s^k: a multiplicative functional term that is determined by $\boldsymbol{z}_s^k$ and that acts on g_s^k
- ν_s^k: an additive functional term determined by $\boldsymbol{z}_s^k$
- $\boldsymbol{z}_s^k$: a vector of time-dependent covariates

Before any comment is made on (3.1), the model will be fully parameterized first. The Weibull distribution is used throughout this thesis for a parametric baseline function because of its versatility, except for the special case of the Proportional Odds Model (see Section B.1.2). (β and η denote the Weibull shape and scale parameters respectively). Both the multiplicative and additive terms are assumed to be exponential. Every element in $\boldsymbol{\theta}$ has potentially unique values for every k and every s, i.e. $\boldsymbol{\gamma}_s^k = [\gamma_{s_1}^k \ \gamma_{s_2}^k \cdots \gamma_{s_m}^k]$, $\zeta_s^k \in \{\zeta_1^k, \zeta_2^k, ..., \zeta_r^k\}$, $\psi_s^k \in \{\psi_1^k, \psi_2^k, ..., \psi_r^k\}$, $\tau_s^k \in \{\tau_1^k, \tau_2^k, ..., \tau_r^k\}$, $\beta_s^k \in \{\beta_1^k, \beta_2^k, ..., \beta_r^k\}$ and $\eta_s^k \in \{\eta_1^k, \eta_2^k, ..., \eta_r^k\}$. For any given value of s, k and x, the baseline function g_s^k is,

$$g_s^k(x, \boldsymbol{\theta}) = \frac{\beta_s^k}{\eta_s^k}\left(\frac{\psi_s^k\left(x - \tau_s^k\right)}{\eta_s^k}\right)^{\beta_s^k - 1} \tag{3.2}$$

Similarly, for the functional terms,

$$\lambda_s^k(x, \boldsymbol{\theta}) = \exp\left(\sum_{j=1}^{m} \gamma_{s_j}^k \cdot z_{i_j}^k\right) \tag{3.3}$$

$$\nu_s^k(x, \boldsymbol{\theta}) = \exp\left(\sum_{j=1}^{m} \alpha_{s_j}^k \cdot z_{i_j}^k\right) \tag{3.4}$$

† $\boldsymbol{\theta}$ also include any additional parameters used in the parametric baseline.

The FOM of any item, k, under consideration at any point in time, x, and for any stratum, s, is thus given by,

$$h(x,\boldsymbol{\theta}) = \zeta_s^k \left(\frac{\beta_s^k}{\eta_s^k} \left(\frac{\psi_s^k \left(x - \tau_s^k\right)}{\eta_s^k} \right)^{\beta_s^k - 1} \cdot e^{\sum_{j=1}^{m} \gamma_{s_j}^k \cdot z_{i_j}^k} + e^{\sum_{j=1}^{m} \alpha_{s_j}^k \cdot z_{i_j}^k} \right) \tag{3.5}$$

The model in (3.5) is probably unrealistic from a reliability modeling point of view because of the large number of parameters that need to be estimated. Huge data sets would be required to fit such a model. The objective with this model however, is not to use it in its complete form as presented above, but to simplify it to conventional enhanced models or to combinations of models with different enhancements by applying restrictions on some of the parameters. For example, to obtain a conventional Weibull-parameterized PHM from w system copies, the restrictions summarized in Table 3.2 is applied on (3.5).

Table 3.2: Parameter restrictions for equation (3.5) to obtain a conventional Weibull-parameterized PHM from w system copies

Parameter	Restriction		
k:	k	$=$	$1, 2, ..., w$
s:	s	$=$	1, for all values of i^k
ζ_s^k:	ζ_s^k	$=$	1, for all values of s and k
ψ_s^k:	ψ_s^k	$=$	1, for all values of s and k
τ_s^k:	τ_s^k	$=$	0, for all values of s and k
α_s^k:	$\alpha_{s_j}^k$	$=$	$-\infty$, for $j = 1, 2, ..., m$ and all values of s and k
γ_s^k:	$\gamma_{s_j}^k$	$=$	γ, for $j = 1, 2, ..., m$ and all values of s and k
β_s^k:	β_s^k	$=$	β, for all values of s and k
η_s^k:	η_s^k	$=$	η, for all values of s and k

The restrictions in Table 3.2 applied to (3.5) gives,

$$h(x,\boldsymbol{\theta}) = \frac{\beta}{\eta} \left(\frac{x}{\eta} \right)^{\beta - 1} \cdot e^{\sum_{j=1}^{m} \gamma_j \cdot z_j} \tag{3.6}$$

To further illustrate the usefulness of (3.5), a special case of the Weibull-parameterized PWP Model 2 (similar to (2.37)) is constructed. Suppose the following requirements for the model are set:

(i) No frailty.

(ii) No accelerative or decelerative component.

(iii) No PAR or PAS component, i.e. no time jump or setback.

(iv) No additive component.

(v) Strata are defined as $s = i^k$, i.e. $s = 1$ for $x \leq X_1^k$, $s = 2$ for $X_1^k < x \leq X_2^k$, etc.

(vi) Regression coefficients in the multiplicative functional term are stratum-specific but not system copy specific.

(vii) The Weibull shape parameter is neither stratum nor system copy specific.

(viii) The Weibull scale parameter is system copy specific but not stratum-specific.

To obtain the desired model, certain restrictions are applied on (3.5). These restrictions are summarized in Table 3.3.

Table 3.3: Parameter restrictions for equation (3.5) to obtain a special case of the Weibull-parameterized PWP Model 2 from w system copies

Parameter	Restriction		
k:	k	$=$	$1, 2, ..., w$
s:	s^k	$=$	i^k, for all values of i^k
ζ_s^k:	ζ_s^k	$=$	1, for all values of s and k
ψ_s^k:	ψ_s^k	$=$	1, for all values of s and k
τ_s^k:	τ_s^k	$=$	0, for all values of s and k
α_s^k:	$\alpha_{s_j}^k$	$=$	$-\infty$, for $j = 1, 2, ..., m$ and all values of s and k
γ_s^k:	γ_s^k	$=$	γ_s, for $j = 1, 2, ..., m$ and all values of s and k
β_s^k:	β_s^k	$=$	β, for all values of s and k
η_s^k:	η_s^k	$=$	η^k, for all values of s and k

Applying the restrictions in Table 3.3 on (3.5), results in:

$$h(x, \boldsymbol{\theta}) = \frac{\beta}{\eta^k} \left(\frac{x}{\eta^k} \right)^{\beta - 1} \cdot e^{\sum_{j=1}^{m} \gamma_{s_j} \cdot z_{i_j}^k} \tag{3.7}$$

Equation (3.5) can be generalized even further by allowing for system copies that consist of multiple parts in series, where the total system success is dependent on the success of each individual part. On failure of any part, the total system is renewed or replaced. A situation of competing risks arise in such a case.

Reconsider the configuration in Figure 3.2, but suppose that each system copy now consists of $l = 1, 2, ..., n$ parts in series (see Figure 3.3). The success of the entire system is therefor dependent on the success of each individual part. It is assumed that tests confirmed the validity of non-repairable systems theory on all l parts of each of the k systems. On failure

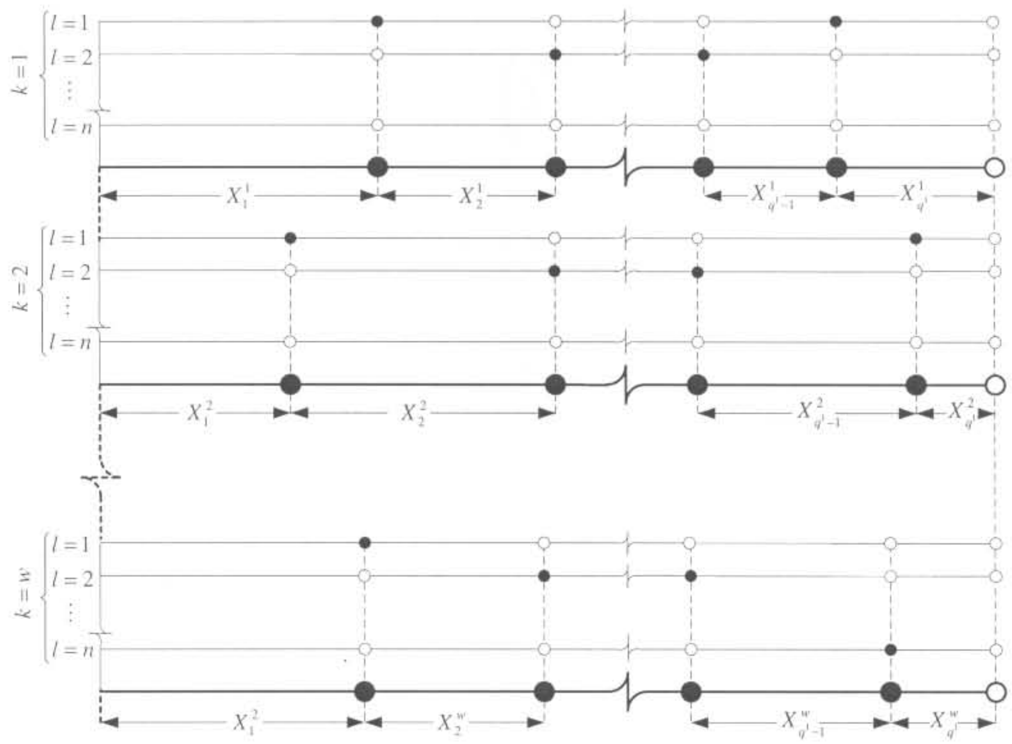

Figure 3.3: w nominally similar system copies containing n parts each with m_l time-dependent covariate measurements on each copy (Dots denote failures, circles denote suspensions)

of any of the n parts, all the parts are renewed or replaced before the system is put back into service. The event history of every part in every system is recorded in a $q^{k_l} \times 2$ matrix, consisting of event times, $X_i^{k_l}$ and event type indicators, $C_i^{k_l}$. Every system has a similar event history matrix that can be deduced from the part histories, i.e. $X_i^k = \min\{X_i^{k_l}\}$ for $l = 1, 2, ..., n$ and C_i^k corresponds to the event type of $\min\{X_i^{k_l}\}$. On each part in each system, m_l time-dependent covariates are measured, i.e. $\boldsymbol{z}_i^{k_l} = [z_{i_1}^{k_l}\ z_{i_2}^{k_l}\ ...\ z_{i_{ml}}^{k_l}]^{\ddagger}$, where x denotes the local time during lifetime i of system k. Event data is categorized in $s = 1, 2, ..., r^l$ strata, where r^l is the highest stratum of any part l. A general model for such a situation (analogous to 3.1) is given by,

$$h(x, \boldsymbol{\theta}) = \sum_{l=1}^{n} \varsigma_s^{k_l} \left(g_s^{k_l}(x, \tau_s^{k_l}, \psi_s^{k_l}) \cdot \lambda(\boldsymbol{\gamma}_s^{k_l} \cdot \boldsymbol{z}_s^{k_l}) + \nu(\boldsymbol{\alpha}_s^{k_l} \cdot \boldsymbol{z}_s^{k_l}) \right) \tag{3.8}$$

where the baseline function g for any item l, associated with system k in stratum s becomes,

$$g_s^{k_l}(x, \boldsymbol{\theta}) = \frac{\beta_s^{k_l}}{\eta_s^{k_l}} \left(\frac{\psi_s^{k_l}\,(x - \tau_s^{k_l})}{\eta_s^{k_l}} \right)^{\beta_s^{k_l}-1} \tag{3.9}$$

‡As before, "(x)" is suppressed.

The functional terms become,

$$\lambda_s^{k_l}(x,\boldsymbol{\theta}) = \exp\left(\sum_{j=1}^{m_l} \gamma_{s_j}^{k_l} \cdot z_{i_j}^{k_l}\right) \tag{3.10}$$

$$\nu_s^{k_l}(x,\boldsymbol{\theta}) = \exp\left(\sum_{j=1}^{m_l} \alpha_{s_j}^{k_l} \cdot z_{i_j}^{k_l}\right) \tag{3.11}$$

The FOM for any part l in system copy k at any time in stratum s is given by,

$$h(x,\boldsymbol{\theta}) = \zeta_s^{k_l}\left(\frac{\beta_s^{k_l}}{\eta_s^{k_l}}\left(\frac{\psi_s^{k_l}\left(x-\tau_s^{k_l}\right)}{\eta_s^{k_l}}\right)^{\beta_s^{k_l}-1} \cdot e^{\sum_{j=1}^{m_l}\gamma_{s_j}^{k_l}\cdot z_{i_j}^{k_l}} + e^{\sum_{j=1}^{m_l}\alpha_{s_j}^{k_l}\cdot z_{i_j}^{k_l}}\right) \tag{3.12}$$

while the FOM of the entire system is represented by,

$$h(x,\boldsymbol{\theta}) = \sum_{l=1}^{n} \zeta_s^{k_l}\left(\frac{\beta_s^{k_l}}{\eta_s^{k_l}}\left(\frac{\psi_s^{k_l}\left(x-\tau_s^{k_l}\right)}{\eta_s^{k_l}}\right)^{\beta_s^{k_l}-1} \cdot e^{\sum_{j=1}^{m_l}\gamma_{s_j}^{k_l}\cdot z_{i_j}^{k_l}} + e^{\sum_{j=1}^{m_l}\alpha_{s_j}^{k_l}\cdot z_{i_j}^{k_l}}\right) \tag{3.13}$$

Equation (3.13) is more general than (3.5) and can be reduced to (3.5) by letting $n = 1$ for all values of k. Thus, the model constructed in (3.7), can also be achieved by (3.13). An advantage of the model in (3.13) is that different failure modes in systems are accommodated even if the system's condition is monitored by only one set of covariates, as is often the case in practice. For such a situation, $\boldsymbol{z}_s^{k_l} = \boldsymbol{z}_s^k$, $\boldsymbol{\gamma}_s^{k_l} = \boldsymbol{\gamma}_s^k$ and $\boldsymbol{\alpha}_s^{k_l} = \boldsymbol{\alpha}_s^k$, for all l.

In Appendix B it is shown that (3.8), and hence (3.13), can be reduced to the majority of models discussed in Section 2.3.

3.2.2 Likelihood construction

The general approach of Anderson, Borgan, Gill, and Keiding (1993) combined with the method used by Prentice, Williams, and Peterson (1981) is used to construct the likelihood for equation (3.13). Suspensions are accommodated in the likelihood, but should preferably only be used for calendar suspensions and for cases where the system was withdrawn from service preventively, according to Ascher (1999). Even if a system was withdrawn from service before failure, it is usually done for good reason, i.e. it is believed that the system is near to the end of its lifetime. In such a case it is more meaningful to include the observation as a failure as apposed to a suspension in the intensity model.

To simplify the calculation of the likelihood, data should be structured in the following way. Suppose d_s events are observed in stratum s. An auxiliary $d_s \times 4$ matrix is introduced, consisting of the chronologically ordered event times of stratum s in column 1, the corresponding event type indicators in column 2, the system on which the event occurred in column 3 and the part on the system which caused the event in column 4. In this matrix, the events are denoted $\underset{\sim}{X}_b^s$, the event indicators $\underset{\sim}{C}_b^s$, the system identifiers in $\underset{\sim}{k}_b^s$ and the part identifiers in $\underset{\sim}{l}_b^s$, for $b = 1, 2, ..., d_s$. The general form of the likelihood for (3.13) is given by,

$$L(x, \boldsymbol{\theta}) = \sum_{l=1}^{n} \left[\prod_{s=1}^{r^l} \prod_{b=1}^{d_s} h(\underset{\sim}{X}_b^s, \boldsymbol{\theta})^{\underset{\sim}{C}_b^s} \cdot \prod_{s=1}^{r^l} \prod_{b=1}^{d_s} e^{-\int_0^{\underset{\sim}{X}_b^s} h(x,\boldsymbol{\theta}) \mathrm{d}x} \right] \tag{3.14}$$

For numerical convenience, the natural logarithm of the likelihood, i.e. $\ln L(x, \boldsymbol{\theta})$, is maximized because,

$$\underset{x,\boldsymbol{\theta}}{\operatorname{argmax}} \, L(x, \boldsymbol{\theta}) = \underset{x,\boldsymbol{\theta}}{\operatorname{argmax}} \ln L(x, \boldsymbol{\theta}) \tag{3.15}$$

which leads to,

$$\ln L(x, \boldsymbol{\theta}) = \prod_{l=1}^{n} \left[\underbrace{\sum_{s=1}^{r^l} \sum_{b=1}^{d_s} \ln h(\underset{\sim}{X}_b^s, \boldsymbol{\theta})^{\underset{\sim}{C}_b^s}}_{\text{Term 1}} - \underbrace{\sum_{s=1}^{r^l} \sum_{b=1}^{d_s} \int_0^{\underset{\sim}{X}_b^s} h(x, \boldsymbol{\theta}) \mathrm{d}x}_{\text{Term 2}} \right] \tag{3.16}$$

Term 1 in equation (3.16) is,

$$\sum_{s=1}^{r^l} \sum_{b=1}^{d_s} \ln \left[\zeta_s^{k_l} \left(\frac{\beta_s^{k_l}}{\eta_s^{k_l}} \left(\frac{\psi_s^{k_l} \left(\underset{\sim}{X}_b^s - \tau_s^{k_l} \right)}{\eta_s^{k_l}} \right)^{\beta_s^{k_l}-1} \cdot e^{\sum_{j=1}^{m_l} \gamma_{s_j}^{k_l} \cdot z_{s_j}^{k_l}} + e^{\sum_{j=1}^{m_l} \alpha_{s_j}^{k_l} \cdot z_{s_j}^{k_l}} \right) \right]^{\underset{\sim}{C}_b^s} \tag{3.17}$$

where $k = \underset{\sim}{k}_b^s$ and $l = \underset{\sim}{l}_b^s$. Term 2 is,

$$\sum_{s=1}^{r^l} \sum_{b=1}^{d_s} \int_0^{\underset{\sim}{X}_b^s} \zeta_s^{k_l} \left(\frac{\beta_s^{k_l}}{\eta_s^{k_l}} \left(\frac{\psi_s^{k_l} \left(x - \tau_s^{k_l} \right)}{\eta_s^{k_l}} \right)^{\beta_s^{k_l}-1} \cdot e^{\sum_{j=1}^{m_l} \gamma_{s_j}^{k_l} \cdot z_{s_j}^{k_l}} + e^{\sum_{j=1}^{m_l} \alpha_{s_j}^{k_l} \cdot z_{s_j}^{k_l}} \right) \mathrm{d}x \tag{3.18}$$

also with $k = \underset{\sim}{k}_b^s$ and $l = \underset{\sim}{l}_b^s$.

The maximum value of equation (3.16) is found where,

$$\frac{\partial \ln L(x, \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = 0 \tag{3.19}$$

for all values of $\boldsymbol{\theta}$. Numerical optimization techniques with which (3.19) can be obtained are described in Appendix C.

3.3 The repairable case

A single model that incorporates all the conventional model enhancements related to repairable systems (discussed in Chapter 2) is required. In this section such a model is developed. The following assumptions are made:

(i) Multiple system copies are nominally similar and are operating in similar conditions.

(ii) All items considered are by the definition of Section 1.2.1, repairable systems.

(iii) NHPPs of log-linear and power-law forms are used to parameterize models.

(iv) Covariates are assumed to be positive.

The validity and practical implications of these assumptions are discussed in Section 3.4.

3.3.1 Model development

Suppose $k = 1, 2, .., w$ nominally similar single-part system copies are studied (see Figure 3.4) on which q^k observations were recorded. Assume that tests revealed that repairable systems theory is suitable to model the reliability of each of these systems. Event data of each system is saved in a $q^k \times 2$ matrix, where each row contains T_i^k, the arrival time a particular event and C_i^k, the event type indicator, i.e. $C_i^k = 0$ in case of suspension and $C_i^k = 1$ otherwise. On each system m time-dependent covariates are measured, i.e. $\boldsymbol{z}^k = [z_1^k(t)\ z_2^k(t)\ ...\ z_m^k(t)]$[§] for $i = 1, 2, ..., q^k$, where t refers to the global time of system k. It is also assumed that the data of each system is categorized in $s = 1, 2, ..., r$ different strata, where r is the highest stratum of system k.

A general model that represents the mean intensity, i.e. ROCOF, of any of the observed system copies is given by,

$$v(t, \boldsymbol{\theta}) = \zeta_s^k \left(g_s^k(t, \tau_s^k, \psi_s^k) \cdot \lambda(\boldsymbol{\gamma}_s^k \cdot \boldsymbol{z}_i^k) + \nu(\boldsymbol{\alpha}_s^k \cdot \boldsymbol{z}_i^k) \right) \tag{3.20}$$

where $\boldsymbol{\theta}$[¶] consists of
- k: the system copy indicator
- s: the current stratum indicator
- ζ_s^k: a random variable that acts as a frailty in the model that could be system copy- and stratum-specific
- g_s^k: a fully parametric baseline function that could be system copy- and stratum-specific
- τ_s^k: a factor that acts additively on t in g_s^k to represent a time

[§]For notational convenience, the indication of the time dependence of covariates, "(t)", is suppressed in expressions to follow.

[¶]$\boldsymbol{\theta}$ also include any additional parameters used in the parametric baseline.

jump or time setback that could be system copy- and stratum-specific

ψ_s^k: a factor that acts multiplicatively on t in g_s^k to result in an acceleration or deceleration of time that could be system copy- and stratum-specific

λ_s^k: a multiplicative functional term that is determined by $\boldsymbol{z}_s^k$ and that acts on g_s^k

ν_s^k: an additive functional term determined by $\boldsymbol{z}_s^k$

$\boldsymbol{z}_s^k$: a vector of time-dependent covariates

The same symbols that were used to present the general combined model for the FOM of a number of system copies in (3.5), are used in (3.20) to introduce the general combined model for the ROCOF of a number of system copies. It is assumed that the different variables will be interpreted in context, i.e. as FOMs in the non-repairable case and as ROCOFs in the repairable case.

NHPP models have gained general acceptance for non-repairable situations as described in Section 1.2.2. In this thesis two types of NHPP models are used namely the log-linear process where $\rho_1 = \exp(\Gamma + \Upsilon t)$ and power-law process where $\rho_2 = \kappa\beta t^{\beta-1}$. Theory will only be developed for the log-linear process but the same principles apply for the power-law process. Both the multiplicative and additive terms are assumed to be exponential. Every element in $\boldsymbol{\theta}$ has potentially unique values for every k and every s, i.e. $\boldsymbol{\gamma}_s^k = [\gamma_{s_1}^k\ \gamma_{s_2}^k \cdots \gamma_{s_m}^k]$, $\zeta_s^k \in \{\zeta_1^k, \zeta_2^k, ..., \zeta_{r_k}\}$, $\psi_s^k \in \{\psi_1^k, \psi_2^k, ..., \psi_{r_k}\}$, $\tau_s^k \in \{\tau_1^k, \tau_2^k, ..., \tau_{r_k}\}$, $\Gamma_s^k \in \{\Gamma_1^k, \Gamma_2^k, ..., \Gamma_{r_k}\}$ and $\Upsilon_s^k \in \{\Upsilon_1^k, \Upsilon_2^k, ..., \Upsilon_{r_k}\}$. For any given value of s, k and t, the baseline function g_s^k is,

$$g_s^k(t,\boldsymbol{\theta}) = \exp(\Gamma_s^k + \psi_s^k \Upsilon_s^k (t - \tau_s^k)) \tag{3.21}$$

Similarly, for the functional terms,

$$\lambda_s^k(t,\boldsymbol{\theta}) = \exp\left(\sum_{j=1}^{m} \gamma_{s_j}^k \cdot z_{i_j}^k\right) \tag{3.22}$$

$$\nu_s^k(t,\boldsymbol{\theta}) = \exp\left(\sum_{j=1}^{m} \alpha_{s_j}^k \cdot z_{i_j}^k\right) \tag{3.23}$$

The peril rate of any item, k, under consideration at any point in time, t, and for any stratum, s, is thus given by,

$$\rho_1(t,\boldsymbol{\theta}) = \zeta_s^k \left(e^{\Gamma_s^k + \psi_s^k \Upsilon_s^k (t-\tau_s^k) + \sum_{j=1}^{m} \gamma_{s_j}^k \cdot z_{i_j}^k} + e^{\sum_{j=1}^{m} \alpha_{s_j}^k \cdot z_{i_j}^k} \right) \tag{3.24}$$

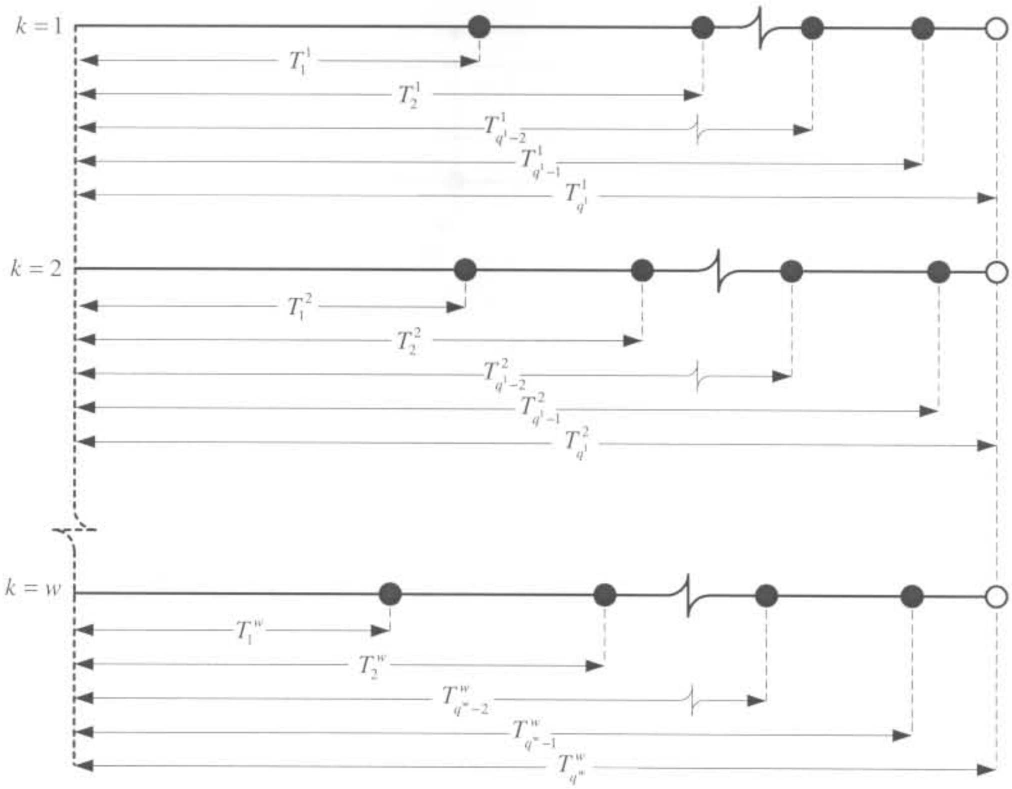

Figure 3.4: w nominally similar single-part system copies (repaired after each failure) with m time-dependent covariate measurements on each copy (Dots denote failures, circles denote suspensions)

As in the case of (3.5), the model in (3.24) is probably unrealistic from a reliability modeling point of view because of the data requirements. The objective with this model is also not to use it in its complete form as presented above, but to simplify it to conventional enhanced models or to combinations of models with different enhancements by applying restrictions on some of the parameters.

Equation (3.24) can be generalized further by allowing for system copies that consist of multiple parts in series, where the total system success is dependent on the success of each individual part. On failure of any part, only the particular part is repaired and the system is put back into service. A situation of competing risks arise in such a case.

Suppose a system is considered with parts $l = 1, 2, ..., n$ in series (see Figure 3.5) where the success of the system is dependent on the success of each individual part. Event data from each part in each system is recorded in $q^{k_l} \times 2$ matrices, consisting of event times, $T_i^{k_l}$ and event type indicators, $C_i^{k_l}$. Every system has a similar event history matrix that can be deduced from the part histories, i.e. $T_i^k = \min\{T_i^{k_l}\}$ for $l = 1, 2, ..., n$ and $C_i^{k_l}$ corresponds to

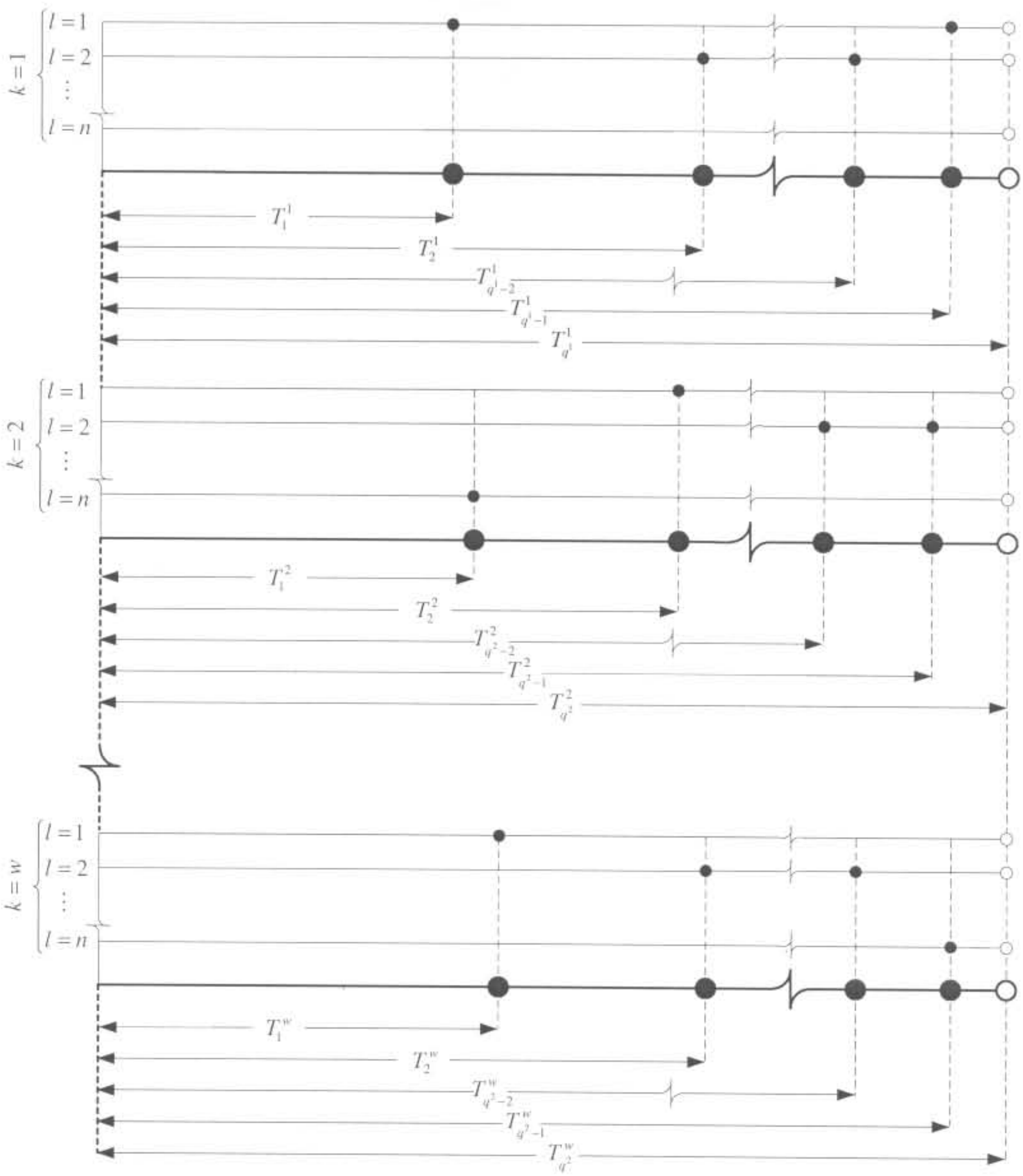

Figure 3.5: w nominally similar repairable system copies containing n parts each with m_l time-dependent covariate measurements on each copy (Dots denote failures, circles denote suspensions)

the event type of $\min\{T_i^{k_l}\}$. On each part m_l time-dependent covariates are measured, i.e. $\mathbf{z}^{k_l} = [z_1^{k_l}\ z_2^{k_l}\ ...\ z_{m_l}^{k_l}]$[‖], where t refers to the global time of system k. Event data is categorized in $s = 1, 2, ..., r^l$ different strata, where r^l is the highest stratum of any part l. The general

[‖] As before, "(t)" is suppressed.

model for such a situation (analogous to 3.20) is given by,

$$v(t,\boldsymbol{\theta}) = \sum_{l=1}^{n} \zeta_s^{k_l} \left(g_s^{k_l}(t, \tau_s^{k_l}, \psi_s^{k_l}) \cdot \lambda(\boldsymbol{\gamma}_s^{k_l} \cdot \boldsymbol{z}_i^{k_l}) + \nu(\boldsymbol{\alpha}_s^{k_l} \cdot \boldsymbol{z}_i^{k_l}) \right) \tag{3.25}$$

where the baseline function g for any item l, associated with system k in stratum s becomes,

$$g_s^{k_l}(t,\boldsymbol{\theta}) = \exp(\Gamma_s^{k_l} + \psi_s^{k_l} \Upsilon_s^{k_l}(t - \tau_s^{k_l})) \tag{3.26}$$

The coefficients $\beta_s^{k_l}$, $\eta_s^{k_l}$, $\psi_s^{k_l}$ and $\tau_s^{k_l}$ can not be represented as matrices because different systems could be in different strata. The functional terms become,

$$\lambda_s^{k_l}(t,\boldsymbol{\theta}) = \exp\left(\sum_{j=1}^{m_l} \gamma_{s_j}^{k_l} \cdot z_j^{k_l} \right) \tag{3.27}$$

$$\nu_s^{k_l}(t,\boldsymbol{\theta}) = \exp\left(\sum_{j=1}^{m_l} \alpha_{s_j}^{k_l} \cdot z_j^{k_l} \right) \tag{3.28}$$

The peril rate for any part l in a system copy k at any time t in stratum s is given by,

$$\rho_1(t,\boldsymbol{\theta}) = \zeta_s^{k_l} \left(e^{\Gamma_s^{k_l} + \psi_s^{k_l} \Upsilon_s^{k_l}(t-\tau_s^{k_l}) + \sum_{j=1}^{m_l} \gamma_{s_j}^{k_l} \cdot z_j^{k_l}} + e^{\sum_{j=1}^{m_l} \alpha_{s_j}^{k_l} \cdot z_j^{k_l}} \right) \tag{3.29}$$

while the peril rate of an entire system is represented by,

$$\rho_1(t,\boldsymbol{\theta}) = \sum_{l=1}^{n} \zeta_s^{k_l} \left(e^{\Gamma_s^{k_l} + \psi_s^{k_l} \Upsilon_s^{k_l}(t-\tau_s^{k_l}) + \sum_{j=1}^{m_l} \gamma_{s_j}^{k_l} \cdot z_j^{k_l}} + e^{\sum_{j=1}^{m_l} \alpha_{s_j}^{k_l} \cdot z_j^{k_l}} \right) \tag{3.30}$$

Equation (3.30) is more general than (3.24) and can be reduced to (3.24) by letting $n = 1$ for all values of k. The biggest advantage of the model in (3.30) is that different failure modes in systems are accommodated even if the system's condition is monitored by only one set of covariates, as is often the case in practice. For such a situation, $\boldsymbol{z}_s^{k_l} = \boldsymbol{z}_s^{k}$, $\boldsymbol{\gamma}_s^{k_l} = \boldsymbol{\gamma}_s^{k}$ and $\boldsymbol{\alpha}_s^{k_l} = \boldsymbol{\alpha}_s^{k}$, for all l.

In Appendix B it is shown that (3.30) can be reduced to the majority of models discussed in Section 2.3.

3.3.2 Likelihood construction

The general approach of Anderson, Borgan, Gill, and Keiding (1993) combined with the method used by Prentice, Williams, and Peterson (1981) is used to construct the likelihood for equation (3.30). Suspensions are accommodated in the likelihood, but should preferably only be used for calendar suspensions and for cases where the system was withdrawn from service preventively, according to Ascher (1999). Even if a system was withdrawn from service before failure, it is usually done for good reason, i.e. it is believed that the system is near to the end of its lifetime. In such a case it is more meaningful to include the observation as a failure as apposed to a suspension in the intensity model.

To simplify the calculation of the likelihood, data should be structured in the following way. Suppose d_s events are observed in stratum s. An auxiliary $d_s \times 4$ matrix is introduced, consisting of the chronologically ordered event times of stratum s in column 1, the corresponding event type indicators in column 2, the system on which the event occurred in column 3 and the part on the system that caused the event in column 4. In this matrix, the events are denoted $\underset{\sim}{T}_b^s$, the event indicators $\underset{\sim}{C}_b^s$, the system identifiers as $\underset{\sim}{k}_b^s$ and the part identifiers as $\underset{\sim}{l}_b^s$ for $b = 1, 2, ..., d_s$.

The general form of the likelihood for (3.30) is given by,

$$L(t,\boldsymbol{\theta}) = \sum_{l=1}^{n} \left[\prod_{s=1}^{r^l} \left[\prod_{b=1}^{d_s} \rho_1(\underset{\sim}{T}_b^s, \boldsymbol{\theta})^{\underset{\sim}{C}_b^s} \cdot \left(e^{-\int_0^{\underset{\sim}{T}_b^s} \rho_1(t,\boldsymbol{\theta})\mathrm{d}t} \right)^{\underset{\sim}{C}_b^s} \right] \cdot e^{-\int_0^{\underset{\sim}{T}_{d_s}^s} \rho_1(t,\boldsymbol{\theta})\mathrm{d}t} \right] \tag{3.31}$$

For numerical convenience, the natural logarithm of the likelihood, i.e. $\ln L(t,\boldsymbol{\theta})$, is maximized because,

$$\underset{t,\boldsymbol{\theta}}{\operatorname{argmax}}\, L(t,\boldsymbol{\theta}) = \underset{t,\boldsymbol{\theta}}{\operatorname{argmax}} \ln L(t,\boldsymbol{\theta}) \tag{3.32}$$

which leads to,

$$\ln L(t,\boldsymbol{\theta}) = \prod_{l=1}^{n} \left[\sum_{s=1}^{r^l} \left[\left[\underbrace{\sum_{b=1}^{d_s} \underset{\sim}{C}_b^s \ln \rho_1(\underset{\sim}{T}_b^s, \boldsymbol{\theta})}_{\text{Term 1}} - \underbrace{\sum_{b=1}^{d_s} \underset{\sim}{C}_b^s \int_0^{\underset{\sim}{T}_b^s} \rho_1(t,\boldsymbol{\theta})\mathrm{d}t}_{\text{Term 2}} \right] - \underbrace{\int_0^{\underset{\sim}{T}_{d_s}^s} \rho_1(t,\boldsymbol{\theta})\mathrm{d}t}_{\text{Term 3}} \right] \right] \tag{3.33}$$

In equation (3.33) Term 1 is,

$$\sum_{b=1}^{d_s} \underset{\sim}{C}_b^s \ln \zeta_s^{k_l} \left(e^{\Gamma_s^{k_l} + \psi_s^{k_l} \Upsilon_s^{k_l} (\underset{\sim}{T}_b^s - \tau_s^{k_l}) + \sum_{j=1}^{m_l} \gamma_{s_j}^{k_l} \cdot z_{b_j}^{k_l}} + e^{\sum_{j=1}^{m_l} \alpha_{s_j}^{k_l} \cdot z_{b_j}^{k_l}} \right) \tag{3.34}$$

where $k = \underset{\sim}{k}_b^s$ and $l = \underset{\sim}{l}_b^s$. Term 2 is,

$$\sum_{b=1}^{d_s} \underset{\sim}{C}_b^s \int_0^{\underset{\sim}{T}_b^s} \zeta_s^{k_l} \left(e^{\Gamma_s^{k_l} + \psi_s^{k_l} \Upsilon_s^{k_l} (t - \tau_s^{k_l}) + \sum_{j=1}^{m_l} \gamma_{s_j}^{k_l} \cdot z_{b_j}^{k_l}} + e^{\sum_{j=1}^{m_l} \alpha_{s_j}^{k_l} \cdot z_{b_j}^{k_l}} \right) \mathrm{d}t \tag{3.35}$$

with $k = \underset{\sim}{k}_b^s$ and $l = \underset{\sim}{l}_b^s$ and Term 3 is,

$$\int_0^{\underset{\sim}{T}_b^s} \zeta_s^{k_l} \left(e^{\Gamma_s^{k_l} + \psi_s^{k_l} \Upsilon_s^{k_l} (t - \tau_s^{k_l}) + \sum_{j=1}^{m_l} \gamma_{s_j}^{k_l} \cdot z_{b_j}^{k_l}} + e^{\sum_{j=1}^{m_l} \alpha_{s_j}^{k_l} \cdot z_{b_j}^{k_l}} \right) \mathrm{d}t \tag{3.36}$$

with $k = \underset{\sim}{k}_b^s$ and $l = \underset{\sim}{l}_b^s$.

The maximum value of equation (3.33) is found where,

$$\frac{\partial \ln L(x, \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = 0 \tag{3.37}$$

for all values of $\boldsymbol{\theta}$. Numerical optimization techniques with which (3.37) can be obtained are described in Appendix C.

3.4 Practical implementation of the combined models

In this section some comments are made with regards to the practical implementation of the combined models. As part of this, the validity of the assumptions for the models are also considered.

3.4.1 Comments on the assumption that covariates are always positive

Covariates were restricted to be positive during the model development in order to simplify the specification of restrictions. For example, in the non-repairable case, to restrict λ to 0 it is simply required to fix all elements of $\boldsymbol{\gamma}$ to $-\infty$, i.e. $\lambda(x, \boldsymbol{\theta}) = \exp(\sum -\infty \cdot z_j^{k_l}) = 0$, for all valid values of j. The assumption of positive covariates has no other influence on the combined models. Positive and negative covariates and also decreasing covariates do play a role in the estimation of residual life. This is discussed in Chapter 4.

3.4.2 Different modeling scenarios

Four different modeling scenarios are identified and numbered in Figure 3.6.

The different scenarios correspond to the following equations: (1) Equation (3.5); (2) Equation (3.13); (3) Equation (3.24); and (4) Equation (3.30). While developing these equations, it was assumed for Scenarios (1) and (2) that all parts of all system copies form part of renewal processes and that all parts of all system copies form part of NHPPs for Scenarios (3) and (4). In practice this will probably seldom be true but by separating renewal processes from NHPPs in "mixed" data sets, this can be overcome. In Table 3.5 below, scenarios

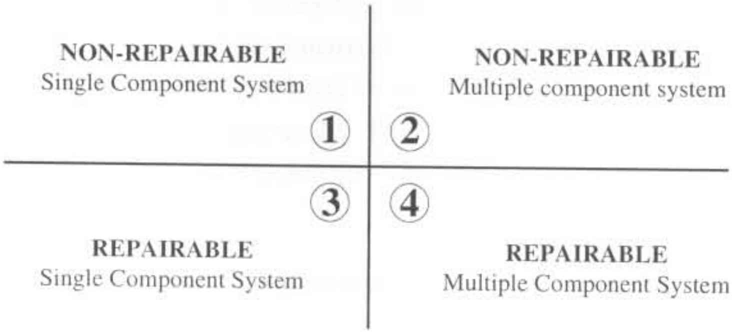

Figure 3.6: Modeling scenarios

other than the four in Figure 3.6 are sketched, i.e. mixed scenarios, with proposed modeling methodologies.

Table 3.5: Methodologies to model mixed scenarios

Scenario	Proposed modeling approach
Model w single part system copies of which y forms part of renewal processes and $w-y$ forms part of NHPPs.	Model the y renewal systems with Scenario (1) of Figure 3.6 and the $w-y$ remaining system with Scenario (3) of Figure 3.6. To estimate the next event time of a renewal system, use the model calculated for the y systems and to estimate the next event time for a repairable system, use the model for the $w-y$ systems.
Model w system copies consisting of n parts each. Of part l, y^l copies follow a renewal process while $n-y^l$ behave according to NHPP processes.	Model the $\sum y^l$ renewal copies with Scenario (2) of Figure 3.6 and the $\sum(n-y^l)$ remaining copies with Scenario (3) of Figure 3.6. To estimate the next event time of a renewal system, use the model calculated for the $\sum y^l$ copies and to estimate the next event time for a repairable system, use the model for the $\sum(n-y^l)$ systems.

Table 3.5 emphasizes the importance of separating renewal processes and repairable systems as was discussed in Section 1.2.2.

3.5 Conclusion

The models developed in this chapter primarily arose from a need to include more than one conventional enhancement in the same model. Generic models were developed to address this need with a clear distinction between the non-repairable and repairable cases.

For non-repairable cases, a generic model was constructed to estimate the FOM while for repairable cases, a generic model was constructed to estimate the peril rate. The Weibull distribution and log-linear NHPP were used to parameterize the generic model for the non-repairable and repairable cases, respectively. This was done to be able to calculate absolute risks and eventually estimate residual life (in Chapter 4). A summary of these models is presented in Table 3.6.

Table 3.6: Summary of generic models

Non-repairable Case**

w single part system copies (all forming part of a renewal process):

$$h(x,\boldsymbol{\theta}) = \zeta_s^k \left(\frac{\beta_s^k}{\eta_s^k} \left(\frac{\psi_s^k \left(x - \tau_s^k\right)}{\eta_s^k} \right)^{\beta_s^k - 1} \cdot e^{\sum\limits_{j=1}^{m} \gamma_{s_j}^k \cdot z_{i_j}^k} + e^{\sum\limits_{j=1}^{m} \alpha_{s_j}^k \cdot z_{i_j}^k} \right)$$

w system copies consisting of n parts in series each, where every part forms part of a renewal process:

$$h(x,\boldsymbol{\theta}) = \sum_{l=1}^{n} \zeta_s^{k_l} \left(\frac{\beta_s^{k_l}}{\eta_s^{k_l}} \left(\frac{\psi_s^{k_l} \left(x - \tau_s^{k_l}\right)}{\eta_s^{k_l}} \right)^{\beta_s^{k_l} - 1} \cdot e^{\sum\limits_{j=1}^{m_l} \gamma_{s_j}^{k_l} \cdot z_{i_j}^{k_l}} + e^{\sum\limits_{j=1}^{m_l} \alpha_{s_j}^{k_l} \cdot z_{i_j}^{k_l}} \right)$$

Repairable Case††

w single part system copies (all forming part of a NHPP):

$$\rho_1(t,\boldsymbol{\theta}) = \zeta_s^k \left(e^{\Gamma_s^k + \psi_s^k \Upsilon_s^k (t - \tau_s^k) + \sum\limits_{j=1}^{m} \gamma_{s_j}^k \cdot z_{i_j}^k} + e^{\sum\limits_{j=1}^{m} \alpha_{s_j}^k \cdot z_{i_j}^k} \right)$$

w system copies consisting of n parts in series each, where every part forms part of a NHPP:

$$\rho_1(t,\boldsymbol{\theta}) = \sum_{l=1}^{n} \zeta_s^{k_l} \left(e^{\Gamma_s^{k_l} + \psi_s^{k_l} \Upsilon_s^{k_l} (t - \tau_s^{k_l}) + \sum\limits_{j=1}^{m_l} \gamma_{s_j}^{k_l} \cdot z_j^{k_l}} + e^{\sum\limits_{j=1}^{m_l} \alpha_{s_j}^{k_l} \cdot z_j^{k_l}} \right)$$

The models in Table 3.6 are generic and it was proved in Appendix B that, in most cases, they can be reduced to the models considered in the literature survey of Section 2.3. Data constraints encountered in practice make these generic models unrealistic in their complete form but provide a basis from which simpler models (with more than one enhancement) can be derived. This concludes point (i) of Section 1.6 - the problem statement.

**Variables for the models corresponding to the non-repairable case are declared and described in Section 3.2

††Variables for the models corresponding to the repairable case are declared and described in Section 3.3