

# Valuation models for credit portfolios and collateralised debt obligations

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## Declaration

I, Paul Jacobus Erasmus declare that the dissertation, which I hereby submit for the degree Magister Scientiae at the University of Pretoria, is my own work and has not previously been submitted by me for a degree at this or any other tertiary institution.

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## Abstract

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In this dissertation we study models for the valuation of portfolios of credit risky securities and collateralised debt obligations. We start with models for single security of the reduced form type and investigate means of extending these to the portfolio level concentrating on default dependence between obligors.

The Gaussian copula model has become a market standard and we study how the model deals with dependence between portfolio constituents. We implement the model and confirm analytical formulae for certain risk measures.

Simplifying assumptions made eases implementation of this model but causes inconsistencies with observed market prices. Evidence of this is the observed correlation smile, highlighted by the recent global credit crises. This has caused researchers to look to extensions of the model to better fit current market pricing. We study a number of these extensions and compare the credit losses for various tranches to those under the standard model.

A number of these extensions are able to replicate observed prices by accounting for some observed feature overlooked by the standard model. Of these the most promising appear to be those having default and recovery rates negatively correlated. Various empirical studies have found this to hold true. Another promising advancement is in the area of stochastic correlation.

The main problems with such extensions is that no single one has been adopted as standard while all require more sophisticated numerical implementation than the convenient recursive algorithm available for the standard model. Even if such problems are overcome questions still remain. No current usable model is able to provide simultaneously both a term structure of credit spreads for the portfolio and individual constituents. This prevents the valuation of the next generation of credit products. An answer may well be beyond capabilities of the now familiar copula framework which has served the market for the last decade.

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## List of acronyms and notation

ABS – Asset Backed Security

$B(t, T)$  – Price of a default risk free bond maturing at  $T$  evaluated at time  $t$

$c$  – Default barrier for the security

CDO – Collateralised Debt Obligation

CDS – Credit Default Swap

$D(t, T)$  – Price of a defaultable bond maturing at  $T$  evaluated at time  $t$

$\delta$  – Recovery rate specified as a number between 0 and 1

$F(t)$  – The distribution function of  $\tau$  or  $P[\tau \leq t]$

$\lambda_t$  – The hazard rate or intensity of default process which determines distribution of  $\tau$

$M$  – Stochastic variable representing the common market (systematic) risk factors

LGD – Loss Given Default, usually expressed as a percentage of security value

MBS – Mortgage Backed Security

$N$  – Number of obligors (names) making up a CDO (index)

OTC – Over the Counter

PD – Probability of Default

$(q|m)$  – Unconditional probability of default given  $M$  or  $P[Z < c|M = m]$

$r_t$  – The risk free short rate of interest process

$S(t)$  – The survival function of  $\tau$  or  $P[\tau > t]$

TRS – Total Return Swap

$\tau$  – Positive real valued number representing the time at which default occurs

$tq_x$  – The conditional probability of defaulting in the  $t$  years starting  $x$  years from now

$tp_x$  – The conditional probability of not defaulting in the  $t$  years starting  $x$  years from now

$T$  – Maturity date of contract or security

$X_i$  – The idiosyncratic risk factor for obligor  $i$

## List of definitions

### Asset Backed Security

An asset backed security is a security whose value and income payments are derived from and secured by a specified pool of underlying assets. The pool of assets is typically a group of small and illiquid assets that are unable to be sold individually. Pooling the assets into financial instruments allows them to be sold to general investors in a process called securitisation. This allows the risk of investing in the underlying assets to be diversified.

### Collateralised Debt Obligation

Collateralised debt obligations are a type of structured asset backed security whose value and payments are derived from a portfolio of fixed-income underlying assets. Collateralised debt obligations are split into different risk classes, or tranches, whereby senior tranches are considered the safest securities. Interest and principal payments are made in order of seniority, so that junior tranches offer higher coupon payments (and interest rates) or lower prices to compensate for additional default risk.

### Credit Default Swap

A credit default swap is a bilateral contract between a protection buyer and a protection seller in which the protection buyer makes a series of premium payments to the protection seller and, in exchange, receives a payoff if a credit instrument (typically a bond or loan) goes into default. The credit instrument is called the reference entity and need not be owned by any of the parties to the contract.

### Mortgage Backed Security

A mortgage backed security is an asset backed security whose value and income payments are derived from and secured by a pool of mortgage loans.

### Probability of Default

The probability of default (also call expected default frequency or EDF) is the likelihood that a loan will not be repaid and will fall into default. Default occurs when a debtor has not met his or her legal obligations according to the debt contract, e.g. has not made a scheduled payment, or has violated a covenant (condition) of the debt contract.

## Chapter 1 Introduction

### 1.1 Background

The markets for credit risky securities and their derivatives have grown rapidly in recent years. Collateralised Debt Obligations (CDOs) are of central importance in the credit derivatives market. In 2006 the collateralised debt obligation (CDO) market had a record year with \$552 billion worth of instruments issued worldwide<sup>1</sup>. Of this total \$312 billion was issued in the United States, this represented an increase of more than 100% over the previous year. The total size of the CDO market as at the end of 2006 was estimated to be \$2 trillion. The size of the CDO market has enormous implications for the broader financial system.

This growth arose from the need to manage credit risk, one of the major components of financial risk. This need was met by developing innovative new securities that re-packaged and transferred credit risk between market participants.

Supporting this growth is the use of advanced mathematical models and quantitative methodologies to value such securities. Along with their increased use and importance these models have seen a rapid increase in the research effort devoted to them.

The result is that today extensive theoretical work has been done to develop such models. Many models appear to reflect reality but lack the required data to implement them. Additionally there are very few studies devoted to empirical results in the field of credit risk. The implication is that much of the models themselves or their results remain untested in real world scenarios.

This weakness became apparent when unexpectedly high numbers of borrowers in the United States started to default on their mortgages. Many of these mortgages were used as underlying assets for CDO type securities. This led to unforeseen losses for holders of these securities and subsequent lack of confidence and liquidity in the CDO and, eventually, credit markets as a whole.

The resulting ongoing financial turmoil experienced since mid 2007 has become known as the credit crunch. Some commentators blame much of this crisis on the complexity of CDO products and the reckless or ignorant use of their valuation models<sup>2</sup>.

Due to the complexity of CDO instruments the structuring and underwriting thereof attracted high fees. This led to the CDO market becoming a very profitable one for investment banks. Today none

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<sup>1</sup> Figures from Securities Industry and Financial Markets Association  
[http://archives1.sifma.org/assets/files/SIFMA\\_CDOIssuanceData2007q1.pdf](http://archives1.sifma.org/assets/files/SIFMA_CDOIssuanceData2007q1.pdf)

<sup>2</sup> Credit Crisis Interview: Susan Wachter on Securitisations and Deregulation Published : June 20, 2008 in Knowledge@Wharton

of the so-called big five Wall Street investment banks remain in their original form. One was forced to file for bankruptcy and the others became or were rescued by commercial banks.

Many insurance companies provided credit guarantees for some CDO securities, enabling issuers to achieve superior credit ratings. Due to its exposure to such instruments AIG, the world's largest insurance company, was taken over by the United States government. This was also the fate of Freddie Mac and Fannie Mae, two large mortgage lending companies.

At the end of 2008 the United States, European Union and OECD region as a whole were in the midst of an economic recession as a result of the crisis. The worldwide financial turmoil has also spread to South Africa resulting in the first domestic recession in 17 years. This economic slowdown is expected to continue through much of 2009<sup>1</sup> while losses from U.S. loans and securitized assets are expected to reach \$1.4 trillion according to a recent report by the IMF.<sup>2</sup>

The cause of the crisis is of much concern to global leaders as evident in the following statement by leaders from the G20 countries. "During a period of strong global growth, growing capital flows, and prolonged stability earlier this decade, market participants sought higher yields without an adequate appreciation of the risks and failed to exercise proper due diligence. At the same time, weak underwriting standards, unsound risk management practices, increasingly complex and opaque financial products, and consequent excessive leverage combined to create vulnerabilities in the system. Policy-makers, regulators and supervisors, in some advanced countries, did not adequately appreciate and address the risks building up in financial markets, keep pace with financial innovation, or take into account the systemic ramifications of domestic regulatory actions."<sup>3</sup>

In light of the above developments it would appear that users of CDO models did not fully understand their working and effects of simplifying assumptions used. There is a need for a review of structured credit products and their valuation models. In this dissertation we will examine such models focusing on the latest offerings applied to products such as CDO securities.

## 1.2 The research problem

Increased liquidity in the credit markets fuelled by the introduction of credit default swaps on indices like the Dow Jones iTraxx and standardisation of credit derivatives led to new ways of quoting prices for these securities. This is especially true in the case on index tranches which gives an investor exposure to credit losses that fall within a certain percentage of the total potential loss.

In essence an index tranche is a position in two call options, one long and the other short, with different strikes on the index loss. The probability that losses fall within a certain range is heavily influenced by the loss distribution and hence the dependence or correlation between defaults of the

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<sup>1</sup> "Economic Projections for the US, Japan & Euro area" statement released by OECD on 13 November 2008

<sup>2</sup> "Global Financial Stability Report – October 2008" semi-annual report released by IMF

<sup>3</sup> "Declaration of the Summit on Financial Markets and the World Economy" statement by G20 leaders on 15 November 2008

constituents of the index. As such correlation instead of credit spread has become a standard means of quoting prices on these securities much like implied volatility is used in equity markets. In the same way volatility can be traded in the options markets correlation can be traded in the credit derivatives market.

In order for market participants to quote prices based on some correlation parameter a standard market model to calculate the price is required. For equities option prices are quoted based on the volatility parameter in the Black-Scholes model, this is known as the implied volatility of the option. Similarly CDO tranches are quoted based on the implied correlation using the standard market model. The similarities between implied volatility and implied correlation are more thoroughly discussed in Ağca, Agrawal and Islam (2008). We give a summary of the main points below.

**Table 1 – Comparison of Black Scholes to Gaussian copula model**

Black Scholes	Gaussian Copula
Dynamic model of underlying	Default times are static
Obvious economic interpretation	Tenuous economic interpretation via Merton model
Delivered volatility uniquely determines price	Delivered correlation is a complex function of Greeks and realised defaults
Natural extensions linked to market implied volatility skew	Less obvious on how to extend to overcome shortcomings

The standard market model is a Gaussian copula model introduced to the credit field by Li (2000). The model uses a single parameter to summarize all correlations between the default times of the securities in the index. The implied correlation of a tranche is this uniform asset correlation that makes the computed tranche spread equal to the observed market spread.

Arguably even more so than the case of Black-Scholes option pricing the strong simplifying assumptions used causes severe limitations in the use of the model. The most important of these assumptions are discussed below.

Firstly, the model assumes constant and equal deterministic pair-wise asset correlation between all reference entities in the portfolio. This simple correlation structure is not sufficient to reflect the heterogeneity in the underlying assets since the complex relationship between different default times cannot be summarized to a single number.



In addition to this the correlation parameter required is that of the asset values while only the equity values are observable. In order to parameterize the model the correlation of equity returns are often used as proxy for the correlation of assets returns.

Secondly, constant and equal deterministic recovery rates on all securities in the portfolio are assumed. Particularly that recovery rates are independent of the number of defaults. A number of studies have found evidence of negative correlation between default and recovery rates, Chava, Stefanescu and Turnbull (2008) give an overview of these results.

Thirdly, it is assumed a single common risk factor drives all dependence between default rates and that this risk factor is normally distributed.

Fourthly, the model assumes constant and equal deterministic credit spreads for all securities in the portfolio.

Given these simplifying assumptions it comes as no surprise that the model fails to fit observed market prices. In essence this means that market participants do not agree with the assumptions underlying the model. The existence of a correlation smile is evidence to support this notion.

Similar to the volatility smile in options markets a correlation smile is observed in CDO markets. This means that different tranches on the same underlying portfolio trade at different implied correlations. When implied correlation is plotted against tranche attachment points a “U” shaped pattern is often observed.

Recently an increasing number of researchers have turned their attention to the correlation smile and by implication the assumptions underlying the standard market model. This need became particularly acute during the height of the credit crises when it became impossible for market participants to calibrate correlation parameters to market data for more senior tranches. Laurent, Amraoui, Cousot and Hitier (2009) contribute this to the under specification of the recovery distribution and its dependence on defaults.

The purpose of this dissertation is to investigate the standard market model and the role its assumptions play in the valuation of CDO securities. We explore various extensions of the standard model used to make the assumptions more reasonable with the aim to model the correlation smile effect. We then consider what these imply for the values of different CDO tranches.

### 1.3 Research design and methodology

The first modern model for credit risk is given by Merton (1974). This approach to credit modelling is known as the structural or option-theoretic approach. The capital structure of a firm, equity and debt, are modelled as options on the underlying asset value of the firm. Using information about the equity and debt of the firm and standard option pricing theory the probability that the firm defaults within a certain horizon is evaluated.

Over the years many variants of the original model by Merton have emerged. Most of these aimed to introduce more realistic assumptions such that it replicates the dynamics of actual securities more closely. Black and Cox (1976) extended the original model in several directions; most importantly

allowing for default to occur before maturity. Geske (1977) and Geske and Johnson (1984) treat the valuation of corporate liabilities as compound options. Longstaff and Schwartz (1995) extended the approach to account for interest rate risk as well as default risk by assuming the short rate follows a Vasicek model. Saá-Requejo and Santa Clara (1999) allow for a more general approach to the dynamics of the short rate process. To allow for non-zero short term credit spreads Zhou (1996) introduced a jump diffusion approach which allows unexpected default by the firm caused by a sudden drop in value.

Two major drawbacks of the structural approach are that credit spreads for short maturities are close to zero and the lack of observations of the asset values of firms. The second shortcoming is dealt with on a theoretical level by Duffie and Lando (2001) but many practical problems remain. This led to most recent research effort being devoted to other approaches of modelling credit risk.

The second main approach to credit modelling is known as the intensity-based or reduced form approach. This approach was formalized by Jarrow and Turnbull (1995) and Madan and Unal (1998). Specification of the probability of default, given that it has not yet occurred, is central to these models. This is usually done by using a hazard rate or intensity of default process.

Defaults are modelled as arrivals of Cox processes. Cox processes are generalisations of Poisson processes where the intensities are stochastic. As such these models have much in common with modelling time to event data, an area of statistics known as reliability theory or survival analysis. In the field of credit risk the event under interest is the default on some security. In his paper Lando (1998) provides an overview of the construction of a Cox process along with applications to the field of credit risk.

Examples of reduced form models can be found in Artzner and Dalbaen (1995), Duffie and Singleton (1997), Lando (1998), Duffie and Singleton (1999a) and Jeanblanc and Rutkowski (2002).

Madan and Unal (1998) combined the basic ideas from both types of model such that the hazard rate is dependent on the value of the firm's assets. Reduced form models of this type are known as hybrid models. Chen (2003) shows how the structural model of Geske (1977) can be extended in such a way that it can be made consistent with the reduced form models proposed by Jarrow and Turnbull (1995) and Duffie and Singleton (1999a).

Bohn (2000) and Bielecki and Rutkowski (2002) provide an overview of both of the main types of credit models.

The models listed above are for a single obligor and security. A CDO model has to value credit losses from a portfolio of securities over multiple time horizons. The reason for this is that a CDO provides multiple payments to investors with payments based on the amount of principle remaining in a particular tranche at that time after credit losses have been deducted.

Jorion and Zhang (2007) provide empirical evidence that defaults happen in clusters as the default of one obligor leads to default of another. Altman et al (2005) find evidence that recovery rates are lower when default rates are high.

A CDO model has to account for these dependencies between losses from different obligors in the portfolio. In later Chapters we show that default dependence has a major influence on the expected losses and hence value of a particular tranche. In addition changes in dependence do not affect all tranches the same way; for some the expected loss will increase while for others it will decrease.

Although there is much empirical evidence for loss dependence the concept of dependence and how it is modelled have many practical pitfalls as shown in Embrechts et al (1999).

Commercial models for credit portfolios have been available for some time. Early examples, both of the structural variety, are CreditMetrics from JP Morgan and the KMV model by the KMV Corporation. Later models based on the reduced form framework are CreditRisk+ by Credit Suisse Financial Services and CreditPortfolioView by consulting firm McKinsy.

These models can be considered as first generation portfolio models and do not provide a consistent measure of credit exposure and portfolio loss distribution. These models are adequate for portfolios of simple bonds and loans but not for credit derivatives or structured credit products like CDO's. All of these models assume deterministic interest rates. Comparisons between these models can be found in Gordy (2000) and Crouhy, Galai and Mark (2000). Frey and McNeil (2003) investigate how these models deal with dependence between defaults.

Reduced form models have proved more tractable than the structural ones and thus better suited to deal with the added complexity required of a CDO model. The main reason for this is that reduced form models have a technical setting as found in models of non-defaultable bonds and the term structure of interest rates. In addition their most important properties are easily extracted using familiar Cox processes. An example of this can be found in Duffie and Singleton (1997).

The first and most widely used type of CDO model is called a factor model. These models assume that, conditional on some latent state variables, defaults are independent. This simplifies the computation of aggregate loss distributions since the dimension of the problem is reduced. In some cases semi-explicit expressions can be found for the pricing of CDO tranches as seen in Laurent and Gregory (2003). The factor approach is thus suited to portfolios with a large number of obligors. Anderson and Sidenius (2005) give a general framework and survey of factor models.

Much of the effort devoted to these models focus on the joint distribution of default times. There are currently three main ways in which such a joint distribution can be specified.

The first of these methods is by specifying dependent intensities of default in the reduced form framework. Typical examples are Jarrow and Yu (2000), Giesecke (2001) and Duffie and Garleanu (2001). The main drawback of this approach is that it is difficult to achieve reasonable levels of dependence. To do this jumps in the intensity process needs to be introduced which means that pricing of CDOs require Monte Carlo simulation. Another method is to allow multiple defaults at the same time as seen in Duffie and Singleton (1999b).

Hull et al (2005) provides a second alternative based on the structural approach. In this approach the asset prices of obligors are dependent. This approach is computationally more cumbersome and not found to fit market data better than the third approach below.

The third, and currently most popular, way to specify such a joint distribution is through the use of a copula function. A copula function provides a convenient way to link marginal distributions to a joint distribution and the result is essentially a multivariate extension of the Cox process approach introduced by Lando (1998). Copula functions are well known in the actuarial literature (see Frees and Valdez (1998)) while Schönbucher and Schubert (2001) consider their use in the area of credit risk.

The main feature of this approach is that it reduces the technical issues faced when modelling credit risk to those face when modelling the default free term structure of interest rates. The approach allows the intensity of default process to follow a Cox process dependent on economic or state variables.

The copula approach is very flexible and a wide range of models can be obtained through different specifications of the copula function.

The Gaussian copula model introduced by Li (2000) has become the industry standard and variations of this model is used to price most CDO securities. This model can be thought of as an extension to the original CreditMetrics model by Gupton et al (1997).

Giesecke (2003) uses a Marshall-Olkin copula function which allows for simultaneous defaults and non smooth joint distribution functions. This approach is thus related to the one followed by Duffie and Singleton (1999b).

The Clayton copula used by Schönbucher and Schubert (2001) is related to the dependent intensity of default approaches.

The market standard Gaussian copula model cannot simultaneously replicate market prices for different tranches on the same underlying portfolio. The main reason for this is that the model assumes constant and equal pair wise correlation and recovery rates for all obligors. This correlation parameter is used to quote CDO prices much like volatility is used to quote option prices. This implied correlation has a number of drawbacks as seen in Hager and Schöbel (2006a).

The most important amongst these is the existence of a correlation smile. This means different correlations are quoted on the same underlying portfolio depending on tranche and term. From this we can conclude that the market rejects the assumptions underlying the model for if they were true no correlation smile would exist. Much recent work has been done on developing extensions of the standard model to address issues with the correlation smile and term-structure of defaults. In chapter 6 we explore some of these efforts.

Factor models as a whole have a major drawback in that they lack proper dynamics for credit spreads and thus a term-structure of defaults as argued in Sidenius et al (2006). This means the loss distribution does not evolve dynamically and models lack the ability to price instruments such as options on CDO tranches and forward starting CDO tranches.

## 1.4 Remaining chapters

The second chapter of this dissertation contains an introduction to credit risk. We define the components of credit risk and explore some common securities subject to credit risk and the management of credit risk using credit derivatives. We discuss how a CDO is structured and provide a real world example of how these securities function and the risks involved. In conclusion, we consider the main approaches used to model credit risk.

The statistical framework and background needed to value individual credit securities is covered in Chapter 3. We concentrate on the reduced form modelling framework as this forms the basis for the CDO models covered in later chapters.

Chapter 4 introduces the concept of dependence and measures of dependence as it applies to credit risk. We consider how copulas provide a simple method to specify a dependence structure and compare the dependence structure for a number of popular copulas.

In Chapter 5 we study in detail the standard market model and the Gaussian copula used to specify its dependence structure. We discuss some of the advantages of using the time to default as opposed to the default rate. The sensitivities of different tranches to the model parameters are explored. In particular we calculate some Greeks for the equity and senior tranches theoretically and confirm the results numerically.

Chapter 6 considers the various shortcomings of the standard market model and explores various ways in which the model can be extended to deal with these. We consider changes to the correlation parameter by using a structured correlation matrix, changing the correlation structure by using an alternative copula or changing the distribution of the common risk factor. In addition we consider the effects of correlated and stochastic recovery rates as well as stochastic correlation.

We conclude with Chapter 7 in which we present results and summarize our findings. The requirements of the next generation of models used to price CDO securities are discussed.

## Chapter 2 Overview of Credit Risk

### 2.1 Introduction

In this Chapter we aim to introduce the basic modelling building blocks required for valuing a single security subject to credit risk. We start by giving a short overview of the credit securities available and the types of credit risk they carry. This is followed by a short discussion of the main type of models available to value them.

### 2.2 Definition of credit risk

Credit risk is the risk associated with any type of credit related event such as changes in credit quality, variations in credit spreads and default. Spread risk and default risk are thus the main components of credit risk.

A default event occurs when one of the counterparties to a financial contract does not fulfil a contractual commitment to meet obligations stated in the contract.

Spread risk is the risk that the market will change its view on the probability of either the default event occurring or the loss associated with default.

To analyse complex agreements it is necessary to distinguish between counterparty credit risk and reference credit risk.

#### 2.2.1 Counterparty credit risk

Counterparty credit risk is the credit risk incurred by the parties entering into a contract due to the counterparty of the contract. This risk can be either bilateral or unilateral.

The bilateral case can best be illustrated when considering an over-the-counter (OTC) derivative. An important feature of these contracts is that they are not guaranteed by an exchange or clearinghouse. Counterparties are thus exposed to the default risk of the other party. Often this risk is managed by requiring the posting of collateral by the parties or the regularly marking to market of the contract.

The unilateral case applies when the default risk of one of the parties is negligible. Securities offered by such parties are usually called default free, examples of which are bonds offered by certain governments. Pricing these securities and their derivatives is the aim of interest rate models for the term structure of risk free interest rates. Some of these models are an

important component of a number of models for credit risky securities, especially those of the reduced form variety.

### 2.2.2 Reference credit risk

Reference credit risk is when the credit risk of some reference entity plays a central role in the settlement of the contract when both parties to the contract are assumed to be default free. The reference risk is that part of the contract's credit risk that is due to some external third party.

Reference risk is the main focus of credit derivatives. These contracts allow market participants to isolate and trade the credit risk of the reference entity. Reference credit risk can thus be transferred between the parties to the contract. Usually one of the parties effectively buys insurance against a possible credit event of the reference risk. This party is called the protection buyer while the other party is known as the protection seller.

Most credit derivatives are over the counter instruments and as such usually carry an element of counterparty credit risk in addition to any reference credit risk.

## 2.3 Types of securities carrying credit risk

Bielecki and Rutkowski (2002) identify the following three main types of securities carrying credit risk. In addition to credit risk most of these securities will also carry some element of market risk. For instance a general change in interest rates will affect the market value of a bond even if the credit risk remains unchanged.

In addition market risk can influence the credit risk of a security by affecting both the default and loss given default. Higher interest rates may increase the risk of default on a security paying a floating rate of interest. Market risk may also affect the value of collateral and hence the loss given default, as an example we can mention property prices and a portfolio of mortgages. If market and credit risk are not considered together in a consistent manner the overall risk of a particular portfolio may be underestimated.

### 2.3.1 Defaultable claims

Defaultable claims include corporate debt and default prone sovereign bonds. The issuer commits itself to make regular specified payments to bondholders in return for an initial payment. When the issuer fails to meet a legal obligation stated in the agreement between the issuer and bondholders a default occurs. A default may occur when the issuer is unable or unwilling to make the full payment due or when some covenant in the agreement is violated.

The financial loss incurred by bondholders in case of default will depend on the amount they can subsequently recover from the issuer. In practice the debt structure of most firms is complex with specific recovery rules for debt issued. Recovery rules govern the priority and timing of payments if default occurs before maturity based on the seniority of the debt.

The price of defaultable bonds thus depend both on the default and recovery rates, an aspect which will be discussed when we focus on specific models of credit risk.

The credit spread on a defaultable bond is the excess return earned on such a bond compared to a similar default free bond to compensate for the credit risk taken by the buyer. This excess can be expressed as differences in yield to maturity or instantaneous forward rates. The term structure of these differences is referred to as the term structure of credit spreads.

This term structure is the ultimate output from most credit risk models, which in conjunction with a model of the default-free term structure allows for the pricing of credit risky securities.

### 2.3.2 Vulnerable claims

Vulnerable claims are contingent claims traded over-the-counter between default prone parties. The credit risk of one or both of the parties is an important component of the market risk related to the contract.

An example of a claim with unilateral default risk would be a European call option on a security. The payoff at maturity depends on whether the option writer has defaulted or not prior to maturity. The default risk of the option holder is not relevant since the option writer cannot suffer a loss on the contract should default occur.

Defaultable swaps are examples of contracts with bilateral default risk. Under a typical swap contract net payments are exchanged between parties at specific times. If the party making the net payment defaults it is usually assumed that no payment is made. When the party due to receive the payment defaults there are a number of different settlement rules that can be applied. For example the payment can be received or withheld.

Hybrid credit derivatives are derivatives with both counterparty and reference credit risk. An example being an OTC option purchased on a defaultable bond. The valuation of these instruments must consider both types of credit risk.

### 2.3.3 Credit derivatives

Credit derivatives are privately negotiated instruments that derive their value from an underlying security subject to credit risk. In contrast to vulnerable claims where credit risk is a secondary consideration in contract valuation it is the primary component of credit derivatives.

Credit derivatives can be structured in a large number of ways in order to meet the needs of an investor. All credit derivatives allow for the transfer of credit risk from one party to another making them a convenient tool to control credit risk exposure separate from market risk.

Credit derivatives can be grouped into three types of contracts, most of which are available as forward contracts or options.

The first type of contract is concerned with the default event itself and not the credit quality of the underlying. Examples are default swaps and default options.



The second type of contract focuses on the credit quality or credit spread of the underlying of which credit spread swaps and credit spread options are examples.

The third class of derivative is the primary focus of this paper and allow for the total transfer of risk of the underlying asset between the counterparties. Total return swaps and synthetic securitisations are examples of such derivatives.

## 2.4 Credit risk mitigation and management

A popular way to manage credit risk is by the use of credit derivatives.

### 2.4.1 Credit default swaps

A credit default swap (CDS) is a contract between a protection buyer and a protection seller. The protection buyer makes a series of regular (usually quarterly) payments to the protection seller. In return he receives the par value of a specified bond issued by a reference entity (who is not a party to the contract) in case the reference entity defaults and physically delivers the defaulted bond to the protection seller. The contract may also be settled in cash rather than the physical delivery taking place.

A credit default swap is thus a form of insurance against the default of the reference entity. The parties to the contract need not have any other exposure to the reference entity and can use the CDS to speculate on the credit worthiness of the reference entity.

### 2.4.2 Total return swaps

A total return swap (TRS) transfers both the credit and market risk of the reference asset and is therefore not purely a credit derivative. A total return swap effectively transfers the ownership of the asset from the protection buyer to the protection seller without a physical sale taking place.

The protection buyer will transfer all cash flows from the underlying asset to the protection seller as well as any positive capital gains on the value of the underlying. In return the protection seller will pay the protection buyer a floating rate like LIBOR plus a spread as well as any negative changes in capital value of the asset. Should the asset default it is valued at zero and the protection seller must pay the original value of the asset at the inception of the contract to the protection buyer.

Total return swaps are popular in the structuring of CDO's since they can be used to leverage the returns for investors. The CDO will sell protection on the underlying assets which gives them exposure to the interest payable without requiring the outright purchase of the asset.

### 2.4.3 Credit spread options

A credit spread option is another method to transfer credit risk from one party to another. In return for an initial premium the option buyer will receive a payoff if the spread between two

reference entities widen or narrows. The reference entities are usually some underlying bond and a benchmark such as LIBOR.

The credit spread option can either be a call or put which allows speculation or hedging on either a widening or narrowing of credit spreads.

#### 2.4.4 Credit linked notes

A credit linked note is a form of credit derivative structured as a security but with a build-in credit default swap. The issuer of the security is not obliged to repay the debt if a specified credit event occurs; in return the investors in the credit linked notes receive a higher return.

For example a bank may offer a loan to a company and finance the lending by issuing a credit linked note with the particular company as reference entity. If the company does go bankrupt the credit risk is borne by the note holders as they then receive the loan due by the company and become one of its creditors while the issuing bank need not repay the note holders.

## 2.5 Securitisations and collateralised debt obligation

Securitisation is a fairly recent innovation and this chapter thus aims to provide the background required for a basic understanding of how these securities operate.

Total risk transfer can be achieved simply by one party selling and the other buying a particular security. Securitisation differs in that the securities are sold to a third party who uses them as collateral to issue new securities bought by the buyer.

Securitisation has typically involved the complete removal of the asset from the balance sheet of the seller. The buyer takes on all risks associated with the cash flows from the assets used as collateral for the securities purchased without recourse to the original owner.

The largest and oldest class of securitised assets is mortgage backed securities (MBS) as supported by US government agencies. This is not surprising since the mortgage market in the United States has become one of the largest asset classes of which about 60% of outstanding balances are securitised. By outstanding balance it exceeds the market for U.S. Treasury notes and bonds. The above figures are from Fabozzi (2006), Chapter 1, which provides a detailed review MBS securities and methods for their valuation.

Many other types of assets are also securitised to and used as collateral for asset-backed securities (ABS). Examples of such assets are credit card receivables, automotive loans, commercial mortgages, personal loans and leases.

The simplest MBS or ABS is a so called pass-through security which merely receives the amounts payable and transfers these to investors. Much of the innovation in the MBS market is linked to the pricing and management of prepayment risk.

In contrast a collateralised debt obligation (CDO) differs from a MBS or ABS because securities with widely different risk characteristics are created from a single portfolio of securities. Cash

flows from the collateral are used to support multi-class securities or “tranches” where investors are grouped into a number of classes which receive payment in a predetermined order.

If an investor buys a risky tranche, he will have to pay off the first percentage of losses for the entire portfolio. Someone who buys a safe or conservative tranche will not be called on to pay for losses in the CDO until many more defaults have occurred. Each of the tranches has different yields depending on how risky the tranche is; which means the higher the risk, the higher the yield. The riskiest tranche is called the equity tranche followed by the mezzanine tranches and then the senior tranche being the most secure. Further details on CDO securities can be found in Hull (2006) Chapter 21.

Structuring CDO instruments in this way enabled issuers to obtain favourable credit ratings for the senior tranche, typically triple-A. The equity tranche usually remains unrated and is often retained by the issuer.

To value a securitisation a model of the joint loss distribution for a portfolio of credit risky securities at certain future times are needed. The model must thus be of a dynamic nature because the pay-off depends on the exact timing of the loss.

In general a joint distribution will depend on both the margins and some dependence parameter. For example we can specify a multivariate normal distribution by specifying the margins and a correlation matrix. The joint loss distribution is thus influenced by both the marginal default probability for specific obligors and the dependence of default events for different obligors.

The quantification of this dependence is thus an essential component for the valuation of such securities. The introduction of copula functions to the area of credit risk provided a convenient way to quantify this dependence.

Once the default dependence was quantified in valuation models, bonds could easily be pooled and priced, and so could a credit default swap (CDS) with the bonds as reference entity.

Investors could create and sell off slices of a portfolio of credit default swaps just as they were doing with bonds or mortgages. The difference is that a company can sell as many credit default swaps (or CDS's) as it wants, and therefore create as many CDO's as it wants, unlike for mortgages and bonds, which are limited. CDO's structured this way are called synthetic CDO's since the CDO does not own any assets but gains exposure to them through CDS's. The fact that a company can sell as many credit default swaps as it wants, whether individually or in tranches, results in leverage, meaning that a small change in the underlying can have a large effect on the credit default swaps.

The events of May 2005 described in the *Wall Street Journal* (Whitehouse (2005)) give an example of the effects leverage can have on returns for CDO investors. We repeat the gist of the article in the remainder of this chapter on collateralised securities.

*“It should be noted that it is the practice to quote the price of an equity tranche different from that of the other tranches. A market price quoted, of say 25%, means that the investor will receive an immediate upfront payment of 25% of tranche principal as well as 500 basis points per year on the remaining principal. For the other tranches there is no upfront payment received and the market quote is the spread the investor receives on the outstanding principal.”*

Someone who sells a synthetic CDO's equity tranche, agreeing to protect the pool against its first \$10 million in default losses, might receive an immediate payment of \$5 million up front, plus \$500,000 a year, for taking on this risk.

They would get this \$5 million without investing anything, just for a pledge to pay in case of a default, much like what an insurance company does.

Some investors, to prove they can pay if there is a default, might have to put up some collateral, but even then it would be only 15% or so of the amount they can potentially lose, or \$1.5 million in this example.

This setup makes such an investment very tempting for many hedge-fund managers, especially for funds just starting out as the large upfront payment will attract their attention.

To hedge such a position a trader will often take an opposite position in a more senior tranche. This hedge is only effective if credit spreads of the constituents of the portfolio move together and can still lead to losses should this assumption be violated. In effect this position is still subject to basis risk.

Consider the following trade that tripped up some hedge funds during May 2005. It involved selling insurance on the equity tranche of a synthetic CDO containing General Motors and Ford as reference entities and then hedging the position. Investors calculated that they could hedge the default risk by buying twice the exposure on a more conservative tranche.

For selling protection on the equity tranche they agreed to pay as much as \$10 million to cover the pool's first default losses and collected a \$3.5 million upfront payment and an additional \$500,000 yearly.

Hedging the risk would cost the investor a mere \$415,000 annually, the price to buy protection on a \$20 million conservative tranche.

On May 5, while the outlook for most bond issuers stayed about the same, General Motors and Ford both got downgraded to below investment grade by Standard & Poor's. That caused a jump in the price of protection on General Motors and Ford bonds. Within two weeks, the premium payment on the equity tranche of the CDO increased to about \$6.5 million upfront.

An investor who had sold protection on the riskiest slice for \$3.5 million had a loss of nearly \$3 million. That's because if the investor wanted to get out of the investment, he would have to buy a like amount of insurance from somebody else for \$6.5 million, \$3 million more than he received initially.

Since the outlook for most other bonds in the portfolio stayed the same the conservative tranche hedge did not increase in value sufficiently to compensate. The basis has thus moved against these investors causing them losses while the assumption made when the hedge was constructed was that credit spreads will move together.

Arbitrage is the primary reason behind the construction of most CDO's<sup>1</sup>. Investors seek to gain from the spread between the relatively high yielding assets and lower yielding rated liabilities.

Secondarily CDO's can also be used to remove assets from the issuer balance sheet. This reduces capital requirements and enables the issuer to originate new assets without the need to raise additional capital.

## 2.6 Models for credit risk

The earliest models for credit risky securities, though simple, contain numerous unrealistic assumptions that limit their usefulness in real world application and adherence to empirical observations. For example, the original model by Merton (1974) assumed that the debt was a single zero coupon bond that can only default at maturity. Earlier research attempts aimed to improve the models by relaxing some of these assumptions.

The result is that models for credit risk, like the securities they model, have become increasingly complex. Some lost mathematical tractability while others have unrealistic data requirements for calibration.

It must be noted that the requirements, and hence complexity of a model, will vary depending on the instrument being valued.

To value a single defaultable security the only requirements may be the probability of a credit event and the likely loss if it should occur. Models for credit spread options require as output the change in credit quality or spread risk, irrespective of a credit event occurring. Securitisations require a model to value potential losses from a portfolio of multiple securities over various time horizons.

Because of the complex nature of the latest credit risky securities models for their valuation will always be fairly intricate. The main reason for this is the large number of factors that can influence credit risky securities as well as the relationships between such factors.

The first factor is the risk free interest that can be earned on securities carrying no credit risk. This rate can be either deterministic or chosen from one of the many stochastic models available.

The second factor is the probability that the obligor would default and is commonly known as the PD (Probability of Default). This is the focus of most models of credit risk. A probability of default is always associated with some time horizon, usually one year. This time horizon will vary depending on the type of security and the purpose of the valuation. For an illiquid bond a buyer may be interested in the probability of default over a number of years. In contrast a

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<sup>1</sup> Figures from Securities Industry and Financial Markets Association  
[http://archives1.sifma.org/assets/files/SIFMA\\_CDOIssuanceData2007q1.pdf](http://archives1.sifma.org/assets/files/SIFMA_CDOIssuanceData2007q1.pdf)

trader may only be interested in default occurring over some contractual period for which he is exposed to the default risk of the security.

The third factor is the likely loss that would be suffered if the obligor should default. This is commonly known as the LGD (Loss Given Default). Often models specify this as the recovery rate. The recovery rate has a simple relation to the LGD since 100% minus the LGD is the recovery rate, i.e. the portion not recovered is the loss.

Lastly the relationships between these three factors need to be considered not only for a single obligor but possibly for a number of obligors. The key insight of recent models is that risk needs to be measured in the context of a portfolio as well as on a single security basis.

As noted earlier few empirical results are available on these relationships predicted by various models. Empirical studies are important for two main reasons according to Bohn (2000).

The first reason is that it will help to characterize desirable features in new models that value credit risky securities. Secondly such studies will help to sort through the numerous existing models put forward in order to determine which can be supported by empirical evidence.

Most empirical studies have focused on the relationship between PD and LGD and concluded that these are negatively correlated. The results of a recent study can be found in Altman et al (2005).

Another empirical observation is that defaults often happen in clusters or that default events appear to depend positively on one another. There are numerous suggestions why the default of one firm might signal the increased risk of default in another. This positive correlation is also referred to as default contagion.

Empirical evidence of this being the case is put forward by Collin-Dufresne et al (2003) and Jorion and Zhang (2007). Jorion and Zhang (2007) also find evidence of negative correlation for credit events in certain circumstances. This they attribute to the competition effect. This occurs when the demise of one firm signals improved operating conditions for its competitors. This might be due to increased pricing power and number of customers.

A credit event might thus trigger both contagion and competition effects with the observed result being the net effect of the two. The type of credit event is significant in determining the type of correlation. When liquidation takes place competition effects dominate while contagion effects dominate when a firm is restructured.

### **2.6.1 The structural approach**

The structural approach uses obligor specific information and treats debt as a contingent claim on the value of the firm. These models are thus concerned by projections of the future asset value and capital structure of the firm.

Structural models use the evolution of firms' structural variables, such as asset and debt values, to determine the time of default. Merton (1974) specified the first modern model of default and is considered the first structural model. In Merton's model, a firm defaults if, at the time of servicing the debt, its asset value is below the value of its outstanding debt.

A second approach, within the structural framework, was introduced by Black and Cox (1976). In this approach defaults occur as soon as firm's asset value falls below a certain threshold. In contrast to the Merton approach, default can occur at any time.

Most structural models consider only one type of credit event. In most cases this is default by the firm. The time to default is usually specified as the first instant a stochastic process, such as the firm's value, reaches a lower limit or default barrier. This barrier can be specified exogenously or endogenously in terms of total firm value. Recovery is often specified as a function of the firm's value.

In some models a distinction is made between total firm value and the value of the firm's assets. Total firm value allows for tax deductions and bankruptcy costs. If these items are ignored firm value will equal asset value.

The structural approach is often referred to as the option-theoretic approach since it is directly inspired by the Black-Scholes-Merton model for the valuation of financial options.

Here debt holders are assumed to have sold a put option on the assets of the firm with strike price equal to the face value of the debt. In the event of default the option is exercised and the debt is settled leaving debt holders with the remainder of the assets. The creditors will then lose the difference between the value of the debt and the liquidation value of the assets. The value of the firm's debt is thus equal to a similar risk free bond less the value of this put option.

Similarly equity holders have a call option on the assets of the firm with strike price equal to the face value of the debt. At maturity the equity holders have the option to repay the debt and in return receive the assets of the firm; if default occurs the option is not exercised.

As with equity options put-call parity holds and in this case we can write

$$Call + Cash = Put + Underlying$$

or

$$Equity + Face\ value\ of\ debt - Put = Firm\ assets$$

or

$$Equity + Economic\ value\ of\ debt = Firm\ assets.$$

A natural extension of the structural approach is then to assume that shareholders can decide, since they hold a call option, whether to declare bankruptcy or not. This leads to the problem for shareholders of dynamically optimizing the default decision and the capital structure of the firm. Such extensions are considered by Leland (1994) and Leland and Toft (1996).

### 2.6.2 The reduced form approach

The second approach is of the reduced form or intensity based type. This ignores issues of firm valuation and works with market information. Default risk is modelled from what is implied in market prices and credit spreads. These models are not concerned with the capital structure of the firm or the value of its assets.

The key concept of this approach is the survival probability or time to default of a firm and specifically the hazard rate representing the intensity of default. Models of this hazard rate can vary from simple hazard functions to complex hazard processes. The main focus of such models is to characterize the random time to default in terms of such hazard functions, hazard processes and martingale hazard processes.

Reduced form models can in turn be broadly grouped into three types.

The first type is known as the default-based approach. This links the price of a defaultable bond to that of a similar default-free bond by using some exchange rate or conversion factor.

The second type is an extension of the first type and is known as a rating-transition based approach. Instead of a single default state multiple rating categories are possible with the probability of future default dependent on the current bond rating. Mathematically this approach is similar to the first.

The last type of reduced form model is spread type models. Here the excess yield on a defaultable bond is split into two components representing default and recovery. The default and recovery processes are used together with a conventional risk free interest rate process to price defaultable bonds.

We shall concentrate on the reduced form approach, and attempt to describe the modeling of the default intensity or hazard rate in the single obligor case in the next chapter. Extension to the framework to the modeling of default dependencies across many obligors is covered in later chapters.

### 2.6.3 The hybrid approach

The hybrid approach can be seen as a variant of the reduced form approach that incorporates state variables. The time to default is modeled using a stochastic hazard rate but the conditional probability of default is directly related to some observable or unobservable market variable. If such a variable is observable it is called a state variable, examples of which are the value of a firm's equity or certain economic indicators. If the variable is unobservable it is called a latent variable. The hybrid approach thus combines ideas of the structural and reduced form approaches.



## Chapter 3 Probabilistic framework

### 3.1 Introduction

In this chapter we aim to provide a summary of the statistical background and some general notation commonly found in the field of reduced form credit models. These basic building blocks of the reduced form framework are treated in detail in a number of texts. This chapter closely follows Bielecki and Rutkowski (2002), Chapter 8 and McNeil Frey and Embrechts (2005), Chapter 9.

We shall always assume that economic uncertainty is modelled with the specification of a filtered probability space  $\Pi = (\Omega, F, (F_t), P)$ , where  $\Omega$  is the set of possible states of the economic world, and  $P$  is a probability measure. The filtration  $(F_t)$  represents the flow of information over time. Let  $F = \sigma(\cup_{t \geq 0} F_t)$  be a  $\sigma$ -algebra, a family of events for which probabilities can be assigned in a consistent way.

We further assume that we can fix a unique physical or real probability measure  $P^\sim$  and we consider the filtered probability space  $\Pi^\sim = (\Omega, F, (F_t), P^\sim)$ . The choice of the probability space will vary in some respects, according to the particular problem under consideration. We shall regularly make use of a probability measure  $P$  that will be assumed to be equivalent to  $P^\sim$ .

Firstly the model for the default-free term structure of interest rates is given by a nonnegative, bounded and  $(F_t)$  adapted default-free short-rate process  $r_t$ . The money market account, or risk free assets process is given by  $B_t = \exp\left(\int_0^t r_s ds\right)$ .

We shall use the class of equivalent probability measures  $P$ , where non-dividend paying risky security processes discounted by the money market account are  $((F_t), P)$ -martingales. Such an equivalent measure is called a risk neutral measure. Under this probability measure a risk neutral investor is indifferent between investing in the money market account or the risky security.

By using the risk free asset as numeraire and assuming the market is arbitrage free we can ensure the existence of such an equivalent martingale measure  $P$ . This will hold if we limit the class of admissible trading strategies (Björk (2004), Chapter 10).

It is worthwhile to notice that we only assume absence of arbitrage but not that the market is complete. This will lead to a number of possible measures  $P$  for which the discounted security prices are  $((F_t), P)$ -martingales.

We assume the existence on  $\Pi$  of a  $R^J$  valued Markov process  $M_t = (M_{1t}, \dots, M_{jt})$  representing economy wide variables. These variables can either be observable (state) or unobservable (latent).

Further we specify  $N$  counting processes  $N_{it}, i = 1, \dots, N$  initialized at zero that represent the default process of each of the  $N$  different obligors in the economy. A default occurs for the  $i^{th}$  obligor when  $N_{it}$  jumps from 0 to 1.

The filtrations  $(\mathcal{G}_{M,t}) = \sigma(M_s, 0 \leq s \leq t)$  and  $(\mathcal{G}_{i,t}) = \sigma(N_{is}, 0 \leq s \leq t)$  are generated by  $M_t$  and  $N_{it}$  respectively. The information on the development of market variables is contained in  $(\mathcal{G}_{M,t})$  while  $(\mathcal{G}_{i,t})$  only contains information on the default status of firm  $i$ .

The filtration  $(F_t)$  contains all the information generated by the economic variables and the default processes of the firms  $(F_t) = (\mathcal{G}_{M,t}) \vee (\mathcal{G}_{1,t}) \vee \dots \vee (\mathcal{G}_{N,t})$  while we define  $(F_{it}) = (\mathcal{G}_{M,t}) \vee (\mathcal{G}_{i,t})$ .

### 3.2 Poisson and Cox processes

Poisson processes provide a convenient way to model the arrival of uncertain events such as default. Default of the  $i^{th}$  obligor is taken to be the first jump in the Poisson process  $N_{it}$  from 0 to 1. The parameters specifying this default intensity are inferred from market data and taken under the probability measure  $P$ .

A Poisson process is a continuous time counting process with the following properties:

1. Initial value is zero  $N_{i0} = 0$
2. Independent increments  $N_{it} - N_{is}$  independent of  $N_{iu} - N_{iv}$  if  $(s, t) \cap (u, v) = \emptyset$
3. Stationary increments  $N_{it} - N_{is}$  depends only on the interval length i.e.  $t - s$
4. No simultaneous events  $\lim_{\Delta t \rightarrow 0} P(N_{i,t+\Delta t} - N_{i,t} > 1 | N_{i,t+\Delta t} - N_{i,t} \geq 1) = 0$

As a consequence of these properties are that the probability distribution of  $N_{i,t}$  is a Poisson distribution. This distribution is characterized by the rate parameter  $\lambda$  which is also known as the intensity. Under this distribution we have that

$$P(N_{it} - N_{is} = k) = \frac{e^{-\lambda(t-s)} (\lambda(t-s))^k}{k!} \quad k = 0, 1, \dots$$

Just as a Poisson random variable is characterized by its scalar parameter  $\lambda$  a Poisson process is characterized by its rate parameter  $\lambda$ . This parameter is the expected number of events (or defaults) per unit time.

So far we have considered only the case of a homogeneous Poisson process where the rate parameter (or default intensity) is a constant, we can easily allow the default intensity to be time dependent  $\lambda_t = \lambda(t)$ . Such a process is called a non-homogeneous Poisson process. Under a non-homogeneous Poisson process we have that

$$P(N_{it} - N_{is} = k) = \frac{e^{-\int_s^t \lambda(u) du} (\int_s^t \lambda(u) du)^k}{k!} \quad k = 0, 1, \dots$$

When the default intensity is stochastic the process is called a Cox process. Examples of such would be when we assume that  $\lambda_t$  depends on the state variables such that  $\lambda_t = f(t, X_t)$  the result being a hybrid model. Another specification can be that  $\lambda_t$  follows a diffusion process  $\lambda_t = \mu(t, \lambda_t)dt + \sigma(t, \lambda_t)dW_t$ .

Central to all reduced form models is the characterization of default as the first jump of some Cox process parameterized by some stochastic default intensity. This is often referred to as a doubly-stochastic model since both the time of default and the process governing its arrival are stochastic. The time of default will then be defined as  $\tau = \inf(t \in \mathbb{R}^+ | N_{it} > 0)$ .

The random variable  $\tau$  is analogous to the future lifetime of some specimen under study in the statistical fields of survival analysis or reliability theory. These fields deal with death in biological organisms and failure in mechanical systems. In those contexts the random variable  $\tau$  is often denoted by  $T$ . To avoid confusion we will use  $\tau$  for the random variable that is the time until default while  $T$  will be the fixed expiry date of the security under question.

The following results allow us to continue our exposition of the reduced form framework and are only quoted here; their derivations can be found in Chapter 5.

$$P[N_{it} = 0] = P[\tau \geq t] = E[\exp(-\int_0^t \lambda_s ds)]$$

If  $F(t) = P[\tau < t]$  and  $f(t)$  is the density of  $F(t)$  then

$$\frac{f(t)}{1 - F(t)} = \lambda_t$$

The distribution function of the time until default is thus  $F(t)$ . The probability that default does not occur within  $t$  years or  $P[\tau \geq t]$  is given by  $1 - F(t)$ , this will be denoted by  $S(t)$ . The density function  $f(t)$  of  $\tau$  is thus  $S(t) \times \lambda_t$ .

Define the conditional default and survival probabilities for a firm after  $x$  years as

Equation 1

$${}^tq_x = P[t + x > \tau | \tau > x] = 1 - E[\exp(-\int_x^{x+t} \lambda_{x+s} ds)]$$

Equation 2

$${}^tp_x = P[t + x \leq \tau | \tau > x] = E[\exp(-\int_x^{x+t} \lambda_{x+s} ds)]$$

In using the above notation we often omit the leading  $t$  if we measure probabilities over one year and write  $q_x$  or  $p_x$  instead of  $1q_x$  or  $1p_x$ .

The probability that a firm will default in the next  $t$  years following  $x$  conditional that it not default in the first  $x$  years is given by  ${}^tq_x$ . The complimentary conditional probability is that the firm does not default in the  $t$  years following  $x$  is given by  ${}^tp_x$ .

It is worthwhile to note that the distribution function of  $\tau$  is completely specified by the default intensity process  $\lambda_t$ . All information regarding the time to default can be extracted from  $\lambda_t$ . Secondly we observe that the default intensity represents the instantaneous default rate at time  $t$  conditional on the fact that default has not yet occurred at that time.

Next we turn our attention to some basic results for the pricing of securities subject to default.

### 3.3 Valuation of defaultable bonds

The first step in pricing defaultable securities is to start with the pricing of securities with no default risk. Let  $P(t, T)$  be the price of a default-free zero coupon bond at time  $t$  with face value 1 maturing at time  $T$  with  $T > t$ . The value of  $P(t, T)$  is given by

$$P(t, T) = B(t)E \left[ \frac{P(T, T)}{B(T)} \middle| F_t \right] = E \left[ \exp \left( - \int_t^T r_s ds \right) \middle| F_t \right].$$

Consider now the value of a defaultable bond  $D(t, T)$  at time  $t$  with face value 1 maturing at time  $T$ . In the case of default at time  $\tau < T$  the recovery is  $\delta_\tau$  units with  $\delta_\tau = 0$  for  $\tau \geq T$  then

$$D(t, T) = B(t)E \left[ \frac{D(T, T)}{B(T)} \middle| F_t \right] = B(t)E \left[ \frac{\mathbb{1}_{\tau > T}}{B(T)} \middle| F_t \right] + B(t)E \left[ \frac{\delta_\tau}{B(\tau)} \middle| F_t \right].$$

Clearly the defaultable bond is the sum of its no default and recovery values respectively. Each of these values are weighted by the risk neutral probability of occurrence and discounted at the risk free interest rate.

Following the notation introduced above we can write this as

$$D(t, T) = E \left[ \exp \left( - \int_t^T \lambda_s + r_s ds \right) \middle| F_t \right] + E \left[ \int_t^T \delta_s \lambda_s \exp \left( - \int_t^s \lambda_u + r_u du \right) ds \middle| F_t \right].$$

It is of interest to note that factoring in the default risk of the bond is equivalent to valuing the bond at an increased rate of interest. This is due to the similarities in the formulas for the survival probabilities  $S(t)$  and the price of a default free bond  $P(t, T)$ . The implication is that much of the theory for short rate models is readily transferrable to the reduced form modelling framework.

The above formula is a general result, reduced form models differ in the assumptions made about the short rate, default intensity and recovery process.

The short rate dynamics can be specified from any of the well known models in the literature. For example any of the Vasicek, Cox-Ingersoll-Ross, Ho-Lee, Hull-White or similar models can be used.

We have already specified the default intensity as a Cox-process either dependent on some state variables or following a diffusion process. The possible choices for the recovery process  $\delta_\tau$  deserve some further attention.

### 3.4 Recovery rates

Recovery rates usually follow one of these three main specifications encountered in the literature.

#### 3.4.1 Recovery of treasury

Recovery of treasury assumes that recovery is some fraction of the default-free zero coupon bond. This specification of  $\delta_\tau = \delta_\tau P(\tau, T)$  was first suggested by Jarrow and Turnbull (1995). At maturity the value of the defaultable bond is given by

$$D(T, T) = \mathbb{I}_{\tau > T} + \delta_\tau \mathbb{I}_{\tau \leq T}.$$

When the fraction recovered is deterministic i.e.  $\delta_\tau = \delta$  the price of the bond simplifies to

$$D(T, T) = \delta + (1 - \delta) \mathbb{I}_{\tau > T}.$$

This last formula can easily be evaluated if we know the price of the claim  $\mathbb{I}_{\tau > T}$ .

#### 3.4.2 Recovery of face value

Under this assumption the bond holder immediately receives a payment of  $\delta_\tau$  at the time of default. At maturity the value of the defaultable bond is thus given by

$$D(T, T) = \mathbb{I}_{\tau > T} + \frac{\delta_\tau \mathbb{I}_{\tau \leq T}}{P(\tau, T)}.$$

From the above formula it is clear that the value of the recovery payment depend on the exact time of default. Even if we assume a deterministic recovery rate the pricing formula will not simplify to a simpler form like the case for recovery of treasury.

#### 3.4.3 Recovery of market value

The recovery of market value assumption has become popular after the paper by Duffie and Singleton (1999a). The main reason for this is that it leads to simpler pricing formulae for defaultable bonds. The assumption is that the recovery value is a fraction of the pre-default value of the bond such that

$$\delta_\tau = \delta_{\tau^-} \times D(\tau^-, T).$$

The value of the bond at time  $\tau^-$ , just before default occurs is denoted by  $D(\tau^-, T)$ .

Obviously this is a recursive relationship as the pre-default value of the bond depends on the recovery rate which in turn depends on the pre-default value of the bond.

Duffie and Singleton (1999a) show that, with certain assumptions, the bond can be priced as a normal default-free bond with the short rate process replaced by a default adjusted short rate. They also consider a more general model incorporating liquidity risk in which case the adjusted short rate equals

Equation 3

$$R_t = r_t + \lambda_t(1 - \delta_t) + l_t.$$

This formula includes the interest rate (market) risk, default (credit) risk and liquidity risk. In the above formula  $\lambda_t(1 - \delta_t)$  is the risk neutral loss rate process (probability of default multiplied by loss given default) and  $l_t$  the additional spread due to liquidity risk. Under the recovery of market value assumption the price of the defaultable bond can then simply be expressed as

Equation 4

$$D(t, T) = E \left[ \exp \left( - \int_t^T R_s ds \right) \middle| F_t \right].$$

The main advantage of this specification is that the loss rate process does not depend on the value of the defaultable bond. Well know term short rate processes can be used to model  $R_t$ .

In addition dependence between  $r_t$  and  $\lambda_t(1 - \delta_t)$  can be introduced via the state variables. In their paper Duffie and Singleton consider such a specification with 3 state variables following a Cox-Ingersoll-Ross type process with the short rate and credit spread affine functions of these such that

$$\begin{aligned} r_t &= \alpha_0 + \alpha_1 M_t^1 + \alpha_2 M_t^2 + \alpha_3 M_t^3 \\ s_t &= \gamma_0 + \gamma_1 M_t^1 + \gamma_2 M_t^2 + \gamma_3 M_t^3 \end{aligned}$$

with  $s_t = R_t - r_t$ .

Duffee (1999) provides one of the first empirical investigations into reduced form models using this specification and concludes that the model is able to replicate observed credit spreads while producing positive credit spreads regardless of the credit quality of the security. This suggests the model successfully captures some liquidity risk component. In addition the model naturally produces an important feature of observed credit spreads with the term structure of such spreads steeper upward sloping for riskier securities.

The drawbacks are that the model implies that the volatility of default risk follows that of a square root diffusion process. Duffee finds that data indicate an additional form of persistent variation in default volatility.

The previous section is only concerned with a single security, further in the dissertation we discuss common assumptions for recovery rates in the portfolio case and how these affect CDO valuation.

### 3.5 Summary

In this chapter we introduced the statistical framework commonly underlying reduced form models. Instead of fixing a time horizon and specifying a default probability we introduce a random variable  $\tau$  that denotes the time until default. Once the distribution of  $\tau$  is known the probability of default over any time horizon can be calculated. Implicitly we assume that all firms will eventually default.

The distribution of  $\tau$  can be completely specified by the hazard rate process which is assumed to follow some Poisson or Cox process. The value of this process may be linked to certain state variables such as economic indicators.

The price of a defaultable bond can be calculated by using the probability of default occurring and the portion of the value recovered should default occur. Different reduced form models assume different recovery rules and timing of recovery payments. Currently the most popular assumption is that a portion of the market value just before default is recovered since it is computationally convenient.

Given the valuation framework of a single bond the next chapter we will explore the concept of dependence and means of extending the model to multiple securities.

## Chapter 4 Dependence concepts in credit modelling

### 4.1 Introduction

In the previous chapter it was noted that risk needs to be measured in the context of a portfolio. This is certainly the case for securities such as securitisations and CDO tranches. With this comes the problem of specifying and quantifying the degree of dependence between the risks in the portfolio.

Dependence measures the degree to which the probability of one event happening changes with the probability of another event happening. In this chapter we study common measures of dependence for joint distributions. Our focus will be on copula functions which are used to specify dependence structures in models of credit portfolios. Examples are the Gaussian copula used in the model studied in the following chapter.

### 4.2 Measures of dependence

Measures of dependence allow us to quantify the degree of dependence between variables by summarizing the dependence to a single number.

Let  $\delta(\cdot, \cdot): (X, Y) \rightarrow \mathbb{R}$  be a measure of dependence between the random variables  $X$  and  $Y$ .

Embrechts et al (1999) give five desired properties of a dependence measure.

- A1:  $\delta(X, Y) = \delta(Y, X)$  (Symmetry)
- A2:  $-1 \leq \delta(X, Y) \leq 1$  (Normalization)
- A3:  $\delta(X, Y) = 1 \Leftrightarrow X, Y$  (Comonotomic)  
 $\delta(X, Y) = -1 \Leftrightarrow X, Y$  (Countermonotomic)
- A4: For  $T: \mathbb{R} \rightarrow \mathbb{R}$ , a strictly monotone function on the range of  $X$   
 $\delta(T(X), Y) = \delta(X, Y)$   $T$  increasing (Invariant under monotone transforms)  
 $\delta(T(X), Y) = -\delta(X, Y)$   $T$  decreasing
- A5:  $\delta(X, Y) = 0 \Rightarrow X$  and  $Y$  independent (Independence is the zero measure)



### 4.2.1 Linear correlation

The linear correlation between two variables is defined as:

$$\rho_{X,Y} = \frac{E[XY] - E[X]E[Y]}{\sqrt{\sigma^2(X)\sigma^2(Y)}}.$$

This is also known as the Pearson product-moment correlation coefficient and is a measure of the linear dependence between two variables. This measure of dependence is closely related to linear regression. The coefficient can be thought of as the relative reduction in the variance of  $Y$  by linear regression on  $X$ .

This measure can easily be extended to  $n$  variables by considering the pair wise correlations in an  $n \times n$  matrix.

In modern financial theory the notion of correlation is central to both the Capital Asset Pricing Model (CAPM) and Arbitrage Pricing Theory (APT). Both these frameworks rely heavily on the ease by which covariance and correlation, once determined, can be manipulated for linear transformations of variables.

Linear correlation is only one of a number of measures of stochastic dependence but is often used to refer to any notion of dependence leading to confusion. The popularity of correlation is due to a number of reasons according to Embrechts et al (1999).

1. Correlation requires only the second moments of the distribution. For many bivariate distributions these are easy to calculate.
2. Correlation and covariance are easily manipulated under linear transformations. This fact is exploited for use in portfolio theory.
3. Correlation is a natural measure of dependence for elliptical distributions of which the multivariate normal distribution is part.

Similarly the disadvantages of correlation can be mentioned.

1. Correlation can only be computed when the variances are finite. For many heavy tailed distributions this is not the case.
2. When variables are independent the correlation between them is zero. The converse does not apply in general and only holds true for multivariate normal distributions.
3. Correlation is not invariant under strictly increasing transformation of the variables.

Linear correlation thus only satisfies properties A1 and A2. It can be shown that no dependence measure satisfies both A4 and A5 (see Embrechts et al (1999)). If however we amend these properties to only allow for positive measures of dependence all of A1 to A5 can be satisfied (Schweizer and Wolff (1981)).

Correlation is widely used as a tool for portfolio optimization in a mean-variance framework and in risk measures such as Value at Risk. This can only be justified if risks are elliptically distributed (where contours of equal density are ellipsoids; the multivariate normal, t and logistic distributions being examples) and the portfolio is a linear combination of such risks.

The reasons for this are that the correlation matrix uniquely determines the dependence structure for elliptical distributions and linear combinations of elliptical variables are themselves elliptical. In addition to this Value at Risk is a coherent risk measure for such distributions according to Artzner et al (2002).

Despite its popularity the distributional assumptions of linear correlation are often violated. This is especially true for heavy tailed and skew distributions often found for credit losses. Over the years a typical credit portfolio will produce frequent small profits and occasional large losses. Derivative securities where portfolio risk is a non-linear function of the underlying risks also violate these assumptions.

#### 4.2.2 Rank correlation

Rank correlation is a non-parametric measure of dependence between variables. Like many non-parametric statistics it is based on the ranks of observations.

Rank correlation does not rely on assumptions about how variables are distributed. Instead it measures how well an arbitrary monotonic function (a function that preserves rank order) describes the dependence between variables. Unlike the case of linear correlation this function need not be linear. Rank correlation is thus a measure of concordance (agreement) between two variables. We consider the properties of concordance measures in the following section.

Consider the variables  $X$  and  $Y$  with marginal distribution functions with distribution functions  $F_X$  and  $F_Y$  respectively and joint distribution  $F$ .

##### *Spearman's rho*

Spearman's rank correlation is given by

$$\rho_S = \rho(F_X, F_Y)$$

where  $\rho$  is the usual Pearson linear correlation.

##### *Kendall's tau*

Kendall's measure of rank correlation is given by

$$\rho_\tau = P[(X_1 - X_2)(Y_1 - Y_2) > 0] - P[(X_1 - X_2)(Y_1 - Y_2) < 0]$$

where  $(X_1, Y_1)$  and  $(X_2, Y_2)$  are a pair of random observations from  $F$ .

Both these measures can be seen as measuring the degree of monotonic dependence between  $X$  and  $Y$ . Rank correlation can be extended to the  $n$  variable case by again considering an  $n \times n$  matrix of pair wise rank correlation measures.

Rank correlation measures are more robust than linear correlation and these measures satisfy properties A1 to A4 for dependence measures. The proofs for these are trivial and can be found in Embrechts et al (1999).

The main advantage of rank correlation over linear correlation is the invariance of the measure under monotone transformations. For this we sacrifice the useful variance-covariance manipulations afforded by linear correlation.

Rank order correlation is closely linked to the copula function for the joint distribution. Both the above measures can be written as a function of the copula for  $F$ . Next we turn our attention to copula functions and alternative measures of dependence.

## 4.3 Copula functions

### 4.3.1 Introduction and properties

Copula is a Latin word that means “to fasten or fit.” It describes a link between two things. In our case, the copula acts as a bridge between marginal distributions and a joint distribution.

In general, for a set of marginal distributions, some dependence parameter is needed to specify the joint distribution. For example a multivariate normal distribution is specified by the marginal distributions along with a correlation matrix. A given joint distribution, however, completely determines its marginal distributions.

A copula approach allows independent study of the dependence structure and the marginal distributions. The copula function then “couples” these to form a joint distribution.

By using a copula we do not specify the dependence between the variables of interest directly. Instead the variables are mapped to ones with more manageable properties and the dependence structure between those variables is defined.

#### *Definition*

*A copula is defined as a distribution function of a vector of random variables where all marginal distributions are standard uniform distributions. Equivalently a copula is a function  $C[0,1]^n \rightarrow [0,1]$  such that*

1.  $C(u_1, \dots, u_n)$  is increasing in each  $u_i$ .
2.  $C(1,1, \dots, u_i, \dots, 1,1) = u_i$  for all  $1 \leq i \leq n$  and  $u_i \in [0,1]$ .
3. For all  $(a_1, \dots, a_n), (b_1, \dots, b_n)$  with  $a_i \leq b_i$  and  $1 \leq i \leq n$  we have that  $\sum_{i_1=1}^2 \dots \sum_{i_n=1}^2 (-1)^{i_1+\dots+i_n} C(u_{1i_1}, \dots, u_{ni_n}) \geq 0$  where  $u_{i_1} = a_i$  and  $u_{i_2} = b_i$ ,  $1 \leq i \leq n$ .

The first property is required for  $C$  to be a distribution function.

The second is required for all marginal distributions to be standard uniform.

The third property ensures that  $P[a_1 \leq X_1 \leq b_1, \dots, a_n \leq X_n \leq b_n] \geq 0$ .

Combined these three conditions ensure that  $C$  is a joint distribution function on the unit hyper cube  $[0,1]^n$  with standard uniform marginal distributions.

#### Theorem 1

For the univariate distributions  $F_1(X_1), \dots, F_n(X_n)$  and the copula function  $C$ ,  $C(F_1(x_1), \dots, F_n(x_n)) = F(x_1, \dots, x_n)$  results in a multivariate distribution with uniform marginal distributions.

#### Proof

$$\begin{aligned} & C(F_1(x_1), \dots, F_n(x_n)) \\ &= P[U_1 \leq F_1(x_1), \dots, U_n \leq F_n(x_n)] \\ &= P[F_1^{-1}(U_1) \leq x_1, \dots, F_n^{-1}(U_n) \leq x_n] \\ &= P[X_1 \leq x_1, \dots, X_n \leq x_n] \\ &= F[x_1, \dots, x_n]. \end{aligned}$$

#### Sklar's Theorem

Sklar's Theorem (1959) establishes the converse of the above:

For every joint distribution  $F[x_1, \dots, x_n]$  there exists a copula function  $C$  such that  $F[x_1, \dots, x_n] = C(F_1(x_1), \dots, F_n(x_n))$ . Further if all the marginal distribution functions are continuous then  $C$  is unique.

This theorem is important because it allows us to study the dependence structure separately from the marginal distributions by only studying the copula function. In other words all the information about the dependence structure is contained in the copula function.

As noted by Frees and Valdez (1998) identifying the copula function is, however, not always convenient. This is usually less of a problem in financial applications since the use of a particular marginal distribution is not required. Rather a convenient way is sought to describe certain facts such as the clustering of defaults.

We now look at some results concerning copula functions limiting ourselves to bivariate distributions.

Again we consider the variables  $X$  and  $Y$  with marginal distribution functions  $F_X$  and  $F_Y$  respectively and joint distribution  $F$  with copula function  $C(u, v) = F(F_1^{-1}(u), F_1^{-1}(v))$  as defined above. We assume that  $X$  and  $Y$  are continuous variables.

#### The product copula - independence

Firstly we consider the case where  $X$  and  $Y$  are independent such that

$$F = F_X F_Y = C(F_X, F_Y) = uv = C(u, v).$$

This is called the product copula and is denoted by  $C^\perp$ .

### Frechet-Hoeffding bounds for the joint distribution function – perfect dependence

If  $C_1(u, v)$  and  $C_2(u, v)$  are copulas we say that  $C_1$  is smaller than  $C_2$  if  $C_1(u, v) \leq C_2(u, v)$  and  $\bar{C}_1(u, v) \leq \bar{C}_2(u, v)$  where  $\bar{C} = P[U \geq u, V \geq v] = 1 - u - v + C(u, v)$ . We write this as  $C_1 \preceq C_2$ .

Given the product copula and the above ordering it is natural to ask if there exists some lower and upper bounds for copula functions around it according to the dependence between  $X$  and  $Y$ . The answer is yes and these bounds are given by the Frechet-Hoeffding bounds inequality (Frechet (1957)) and hold for any copula  $C(u, v)$ .

1.  $C^-(u, v) = \max(1 - u - v, 0)$  (Lower Frechet bound)
2.  $C^+(u, v) = \min(u, v)$  (Upper Frechet bound)
3.  $C^- \preceq C \preceq C^+$

Further we can state the following two results of the Frechet-Hoeffding theorem as quoted in Verschuere (2006).

- $Y$  is almost surely an increasing function of  $X \Leftrightarrow F = C^+$ .
- $Y$  is almost surely a decreasing function of  $X \Leftrightarrow F = C^-$ .

Both the lower and upper bounds are themselves copula functions in the bivariate case. These bounds also hold for the multivariate case in general but the lower bound is then no longer a copula according to Embrechts Lindskog and McNeill (2001).

This partial ordering of the set of copulas is called a concordance ordering. The ordering is only partial since not every pair of copulas can be ordered in this way. Many parametric families of copula however are totally ordered in the sense that if  $\theta_1 < \theta_2$  then  $C_{\theta_1} \preceq C_{\theta_2}$  where  $\theta$  is a parameter for the copula.

### Concordance

Let  $(x_1, y_1)$  and  $(x_2, y_2)$  be two pairs of observations from the vector  $(X, Y)$  of continuous random variables  $X$  and  $Y$ . Then  $(x_1, y_1)$  and  $(x_2, y_2)$  is concordant if  $(x_1 - x_2)(y_1 - y_2) > 0$  and discordant if  $(x_1 - x_2)(y_1 - y_2) < 0$ .

A numeric measure  $\kappa$  of dependence between two continuous variables  $X$  and  $Y$  is called a measure of concordance if it satisfies the following according to Bouyé (2000):

- B1 :  $\kappa$  is defined for every pair of  $X$  and  $Y$
- B2 :  $-1 \leq \kappa(X, -X) \leq \kappa(X, Y) \leq \kappa(X, X) \leq 1$
- B3 :  $\kappa(X, Y) = \kappa(Y, X)$
- B4 : If  $X$  and  $Y$  are independent then  $\kappa(X, Y) = \kappa(C^\perp) = 0$
- B5 :  $\kappa(X, -Y) = -\kappa(X, Y)$
- B6 : If  $C_1 \preceq C_2$  then  $\kappa(C_1) \leq \kappa(C_2)$

The most important measures of concordance are Spearman's rho and Kendall's tau discussed earlier. Kendall's tau measure can be interpreted as the probability of concordance less the probability of discordance (Verschuere (2006)).

For proof that these are measures of concordance as defined above see Embrechts et al (1999). They are related to the copula of the joint distribution by the following functions

$$\rho = 12 \iint [C(u, v) - uv] dudv$$

$$\tau = 4 \iint C(u, v) dC(u, v) - 1.$$

Another measure of concordance is the Gini index which can be written using the copula function as

$$\gamma = 2 \iint (|u + v - 1| - |u - v|) dC(u, v).$$

The Gini index has a graphical interpretation in that it is twice the area between the Lorenz curve and the line of equality and ranges from 0 to 1.

Rank correlation measures how well a monotonic function describes dependence between variables. It would then make intuitive sense if these measures are invariant under monotone increasing transforms. Since these measures are functions of the copula we expect the copula to also be invariant under monotone increasing transforms. This is indeed the case.

### *Monotone transforms of marginal distributions*

Let  $\alpha$  and  $\beta$  be strictly monotone increasing or decreasing functions then the following relations hold:

1. If  $\alpha$  increasing and  $\beta$  increasing then  $C_{\alpha(X)\beta(Y)} = C_{XY}$ .
2. If  $\alpha$  increasing and  $\beta$  decreasing then  $C_{\alpha(X)\beta(Y)} = uC_{XY}(u, 1 - v)$ .

The proofs to the above can be found in Verschuere (2006).

Schweizer and Wolff (1981) first showed that the copula accounts for all the dependence between two random variables. They established this by proving the invariance of the copula function under monotone increasing transforms and by showing that Spearman's rho and Kendall's tau can be written in terms of only the copula function.

The invariance of the copula under monotone increasing transforms used together with Sklar's Theorem is the key to understanding its usefulness. All marginal distributions functions are monotone increasing functions of the marginal variable and uniformly distributed. All joint distribution functions can be written in copula form, a function of uniform marginal distributions. Using these two facts together we can preserve a given dependence structure while changing the marginal distributions as we please.

### Tail dependence

Tail dependence is a particularly useful measure of dependence when we are concerned about extreme values of distributions. In short the measure of tail dependence is the probability (in the bivariate case) that one variable has an extreme value given that the other variable takes on an extreme value.

Like the measures of rank correlation tail dependence is invariant under monotone transforms of the variables. Both upper and lower measures of tail dependence can be specified depending on if we are interested joint events from the upper or lower part of the marginal distributions. These measures are respectively given by

$$\lim_{u \rightarrow 1^-} P[Y > F_Y^{-1}(u) | X > F_X^{-1}(u)] = \lambda_U$$

and

$$\lim_{u \rightarrow 0^+} P[Y \leq F_Y^{-1}(u) | X \leq F_X^{-1}(u)] = \lambda_L.$$

Provided that the above limit exist and that  $\lambda_U \in (0,1]$  we can say the copula displays upper tail dependence or if  $\lambda_L \in (0,1]$  lower tail dependence.

These measures can be calculated directly if the copula function is known by using the following formulas

$$\frac{\lim_{u \rightarrow 1^-} (C(u, u) - 2u + 1)}{1 - u} = \lambda_U$$

$$\frac{\lim_{u \rightarrow 0^+} C(u, u)}{u} = \lambda_L.$$

Some copulas, like the Gaussian copula, do not exhibit tail dependence while some always have tail dependence, an example being the Student copula. Tail dependence can exist even when marginal variables have zero or negative correlation (Embrechts et al (1999)).

### 4.3.2 Families of copula functions

#### Elliptical copulas

Elliptical copulas are simply the copulas of the elliptical distributions. Simulation from such distributions and their copulas are easy.

#### Gaussian copula

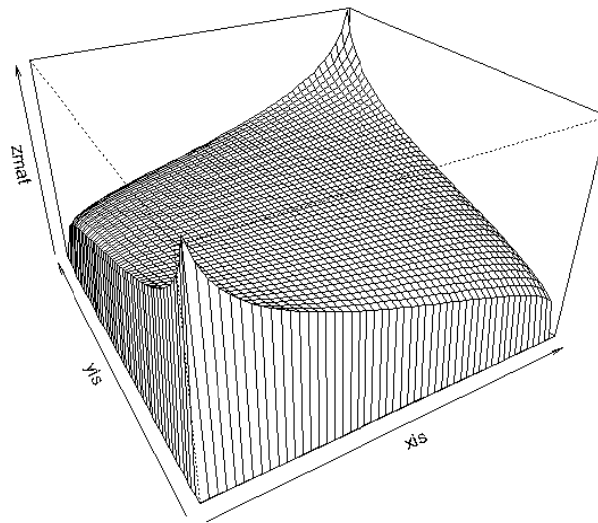
Let  $\Phi$  be the distribution function of a standard univariate normal distribution and  $\Phi_R$  the distribution functions of a standard multivariate normal distribution with covariance matrix  $R$ .

The Gaussian copula is then defined as  $C_{\Phi_R} = \Phi_R(\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_n))$ .

To sample from such a distribution we use the Cholesky<sup>1</sup> decomposition of the correlation matrix which is derived from the covariance matrix and an independent sample of random Gaussian variables. Such routines are available in most mathematical or statistical software packages.

The figures below are plots of the probability density (pdf) and cumulative distribution function (cdf) of a bivariate Gaussian copula with correlation parameter equal to .

Figure 1 – Gaussian copula probability density function



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<sup>1</sup> The Cholesky decomposition is the unique lower-triangular matrix such that



Figure 2 - Gaussian copula cumulative distribution function

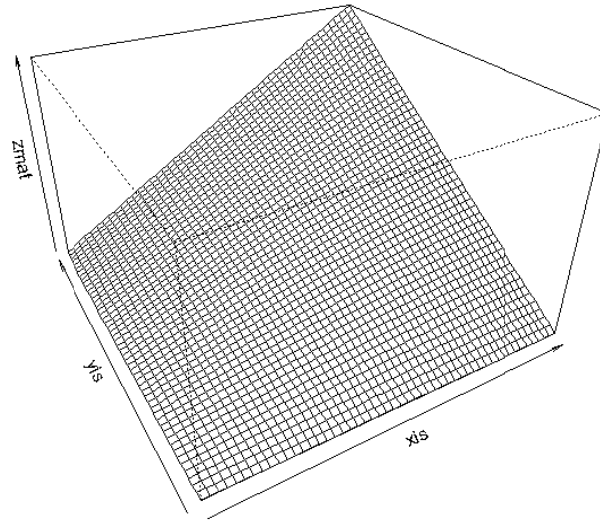
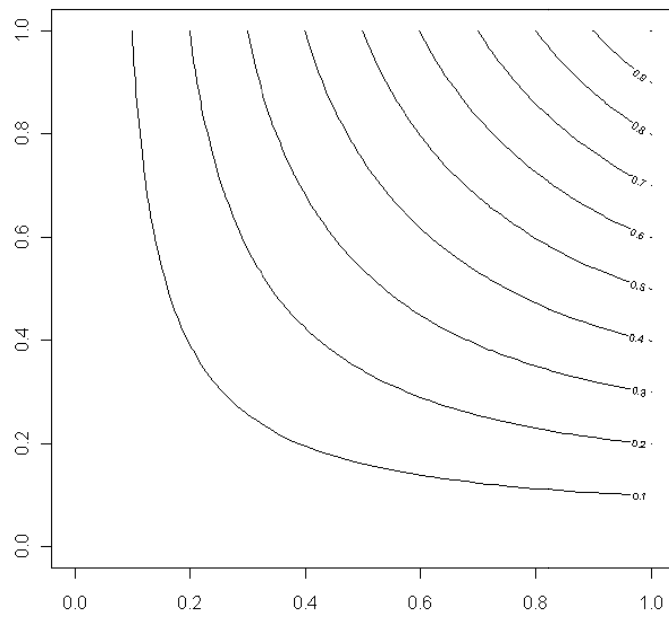


Figure 3 - Gaussian copula cumulative distribution function contour plot



## Student copula

Let  $F$  be the distribution function of a standard Student distribution with  $\nu$  degrees of freedom and  $F_{\Sigma}$  be the distribution function of the multivariate Student distribution with covariance matrix  $\Sigma$  and  $\nu$  degrees of freedom.

The Student copula is then defined as

Sampling from such a distribution is relatively easy since the Student variable is the ratio between a standard Gaussian variable and the square root of an independent Chi-square variable divided by its degrees of freedom. The process is explained in Verschuere (2006).

The figures below are plots of the probability density (pdf) and cumulative distribution function (cdf) of a bivariate Student copula with correlation parameter equal to  $\rho$  and  $\nu$  degrees of freedom.

Figure 4 - Student copula probability density function

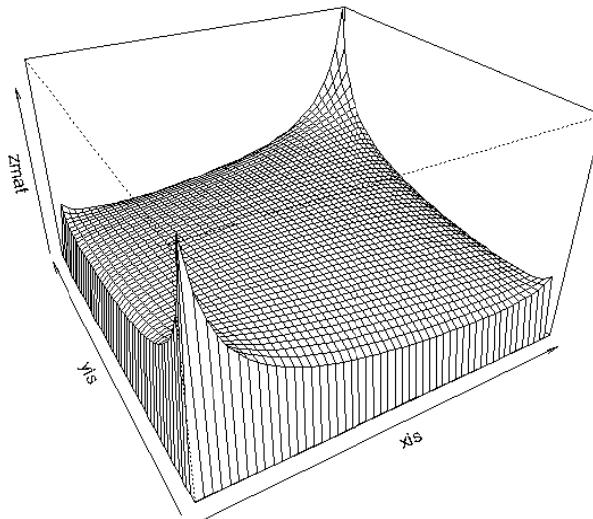


Figure 5 - Student copula cumulative distribution function

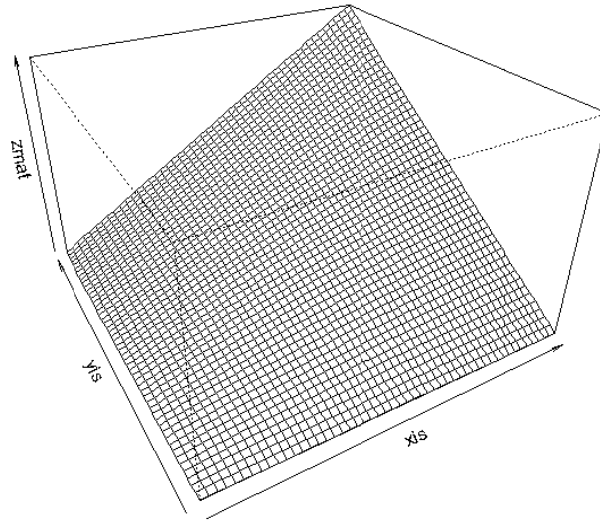
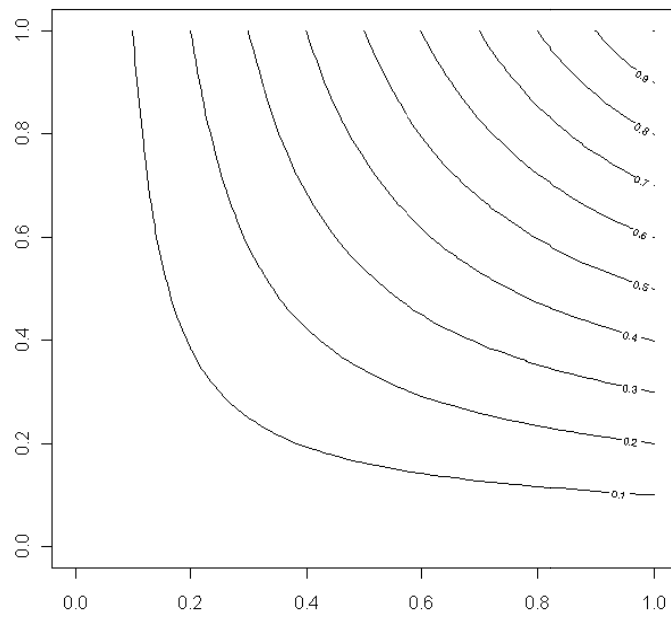


Figure 6 - Student copula cumulative distribution function contour plot



## Comparison of elliptical copulas

Perhaps surprisingly for most commonly used distributions in the elliptical class Kendall's rank correlation takes the same elegant form, being only a function of the correlation parameter according to Demarta and McNeil (2004). Despite having the same value for the correlation measure for a given correlation parameter the distributions differ in the way probabilities are assigned to extreme events.

Compared to the Gaussian copula the Student copula assigns higher probabilities of joint extreme events for a given correlation. The reason for this is that the Student copula displays a measure of both upper and lower tail dependence while the Gaussian copula does not. The level of tail dependence depends on the degrees of freedom, the lower the value the higher the level of tail dependence. In the limiting case with infinite degrees of freedom the Student distribution equates to the Gaussian distribution and no tail dependence is present.

**Table 2 – Comparison of Gaussian copula parameters and dependence measures**

Copula	$\rho_\tau$	$\lambda_U$	$\lambda_L$
<b>Gaussian</b>	$\frac{2}{\pi} \arcsin(\rho)$	0	0
<b>Student t</b>	$\frac{2}{\pi} \arcsin(\rho)$	$2t_{v+1}\left(-\sqrt{v+1} \frac{\sqrt{1-\rho}}{\sqrt{1+\rho}}\right)$	$2t_{v+1}\left(-\sqrt{v+1} \frac{\sqrt{1-\rho}}{\sqrt{1+\rho}}\right)$

In the figures below we plot observations from a Gaussian copula and Student copula to illustrate the effect of tail dependence. In both cases the correlation parameter is 0.6 while the Student copula has 2 degrees of freedom.

Figure 7 – 5000 Observations form a Gaussian copula with

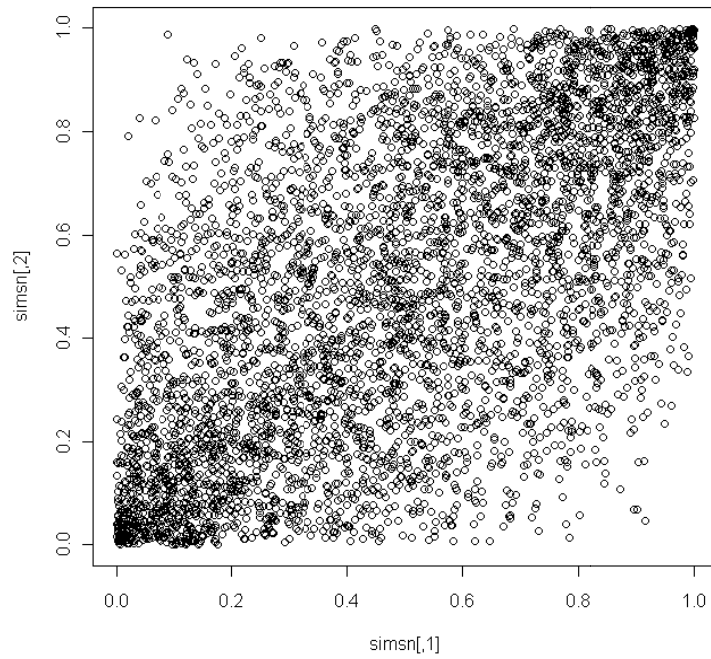
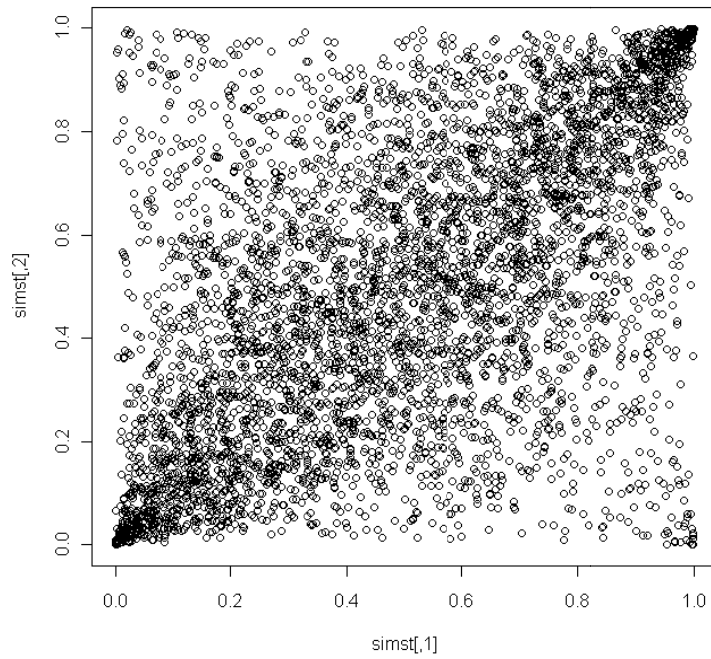


Figure 8 – 5000 Observations form a Student copula with and



## Archimedean copulas

Archimedean copulas allow us to reduce the study of a multivariate copula to that of a univariate function. Members of the Archimedean copula family are characterized by the function used to specify them. This function is called the generator function and must satisfy certain properties in order for it to be a valid parameter for the copula. There are as many Archimedean copulas as there are functions that satisfy the following properties.

Let  $\phi$  be the generator function for an Archimedean copula such that

1.  $\phi$  is convex and decreasing
2.  $\phi$  has domain  $(0,1]$  and range  $[0, \infty)$  such that  $\phi(1) = 0$ .

The Archimedean copula is then defined as  $C_\phi = \phi^{-1}(\phi(u_1) + \dots + \phi(u_n))$ .

The product copula is an Archimedean copula with  $\phi(t) = -\ln(t)$  but more specifically a Gumbel copula with parameter  $\theta = 1$ .

The form of the generator function defines certain types of Archimedean copulas. Below we look at three popular Archimedean copulas and conclude with a table comparing their characteristics.

### Gumbel copula

The generator function for the Gumbel copula is given by  $\phi(t) = (-\ln(t))^\theta$  where  $\theta \geq 1$ .

The figures below are plots of the probability density function (pdf) and cumulative distribution function (cdf) of a bivariate Gumbel copula with parameter equal to 2.

Figure 9 - Gumbel copula probability density function

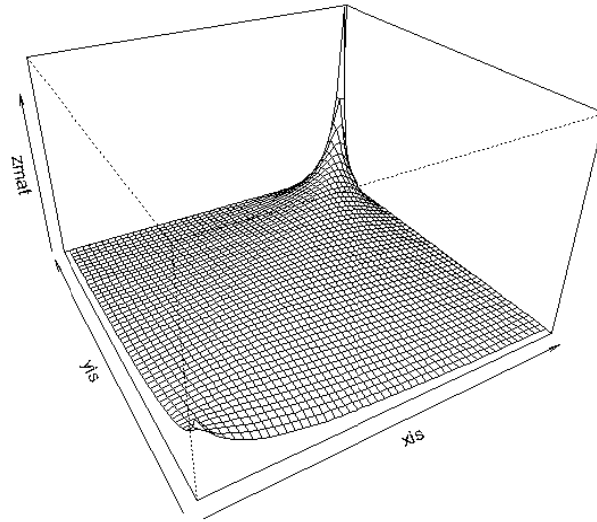


Figure 10 - Gumbel copula cumulative distribution function

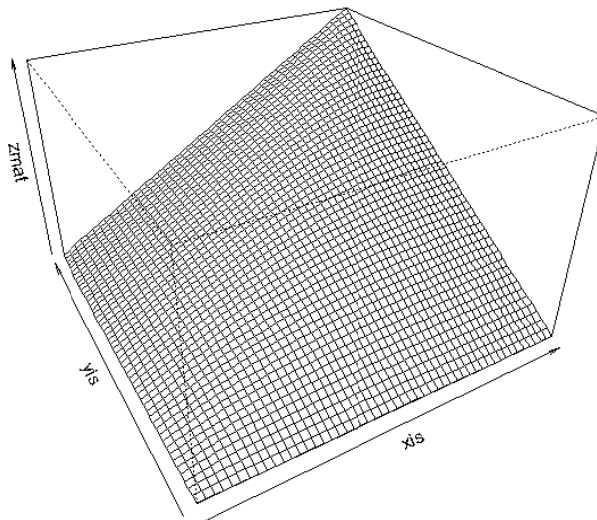
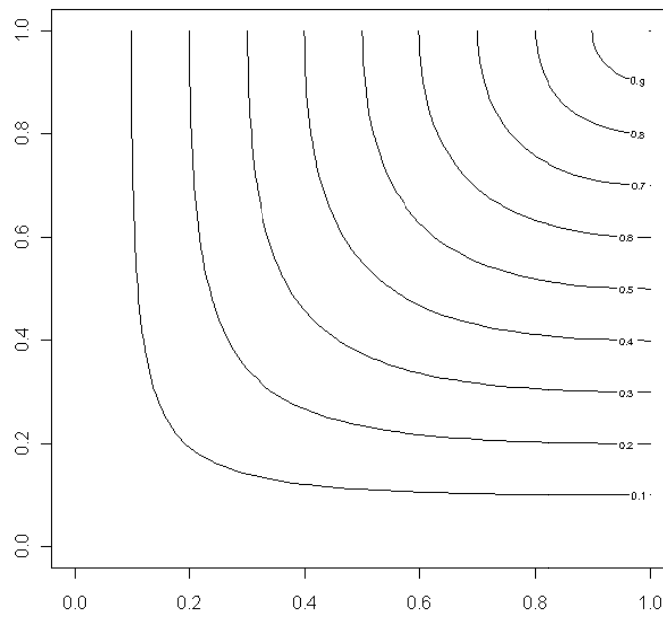


Figure 11 - Gumbel copula cumulative distribution function contour plot



### Clayton copula

The generator function for the Clayton copula is given by

where  $\theta > -1$ .

The figures below are plots of the probability density (pdf) and cumulative distribution function (cdf) of a bivariate Clayton copula with parameter equal to



Figure 12 - Clayton copula probability density function

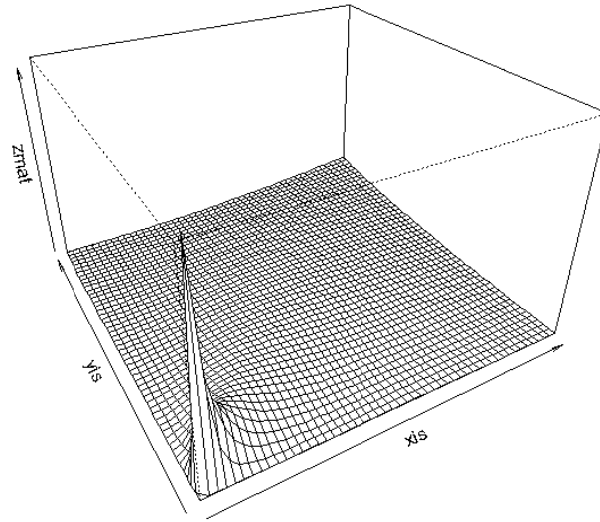


Figure 13 - Clayton copula cumulative distribution function

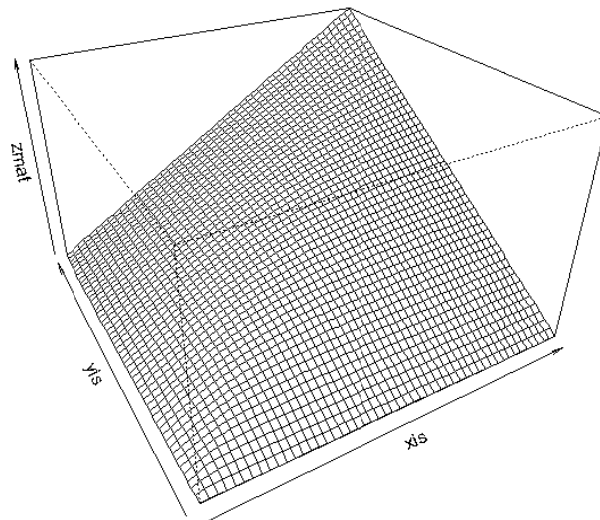
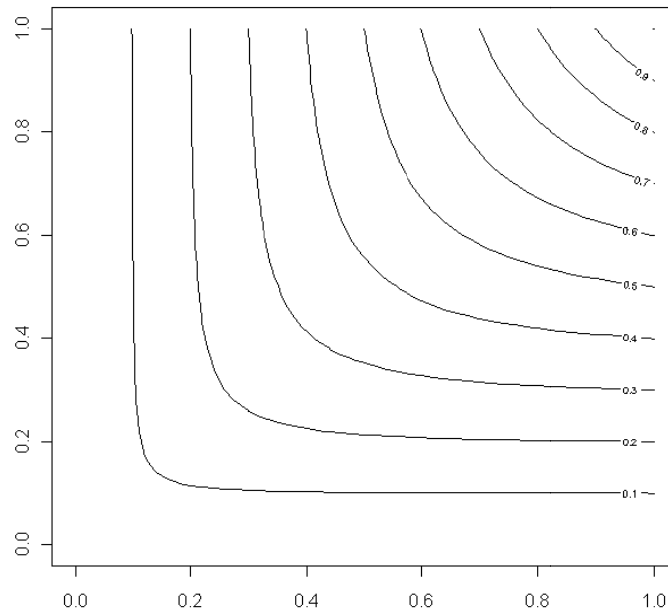


Figure 14 - Clayton copula cumulative distribution function contour plot



### Frank copula

The generator function for the Frank copula is given by

$$g(t) = \frac{e^{-t} - 1}{t} \quad \text{where } t > 0.$$

The figures below are plots of the probability density (pdf) and cumulative distribution function (cdf) of a bivariate Frank copula with parameter equal to 2.

Figure 15 - Frank copula probability density function

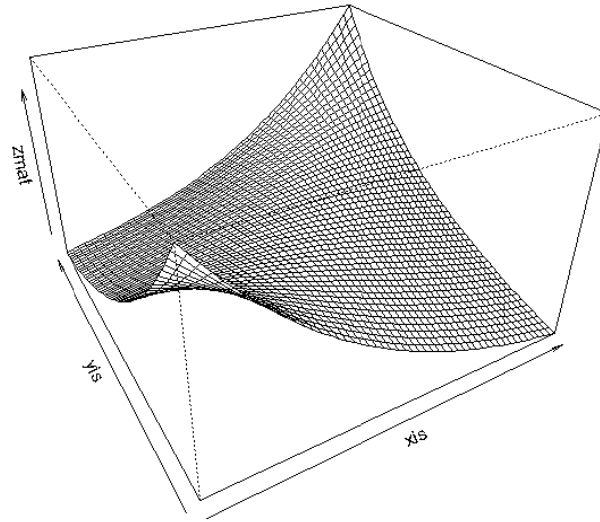


Figure 16 - Frank copula cumulative distribution function

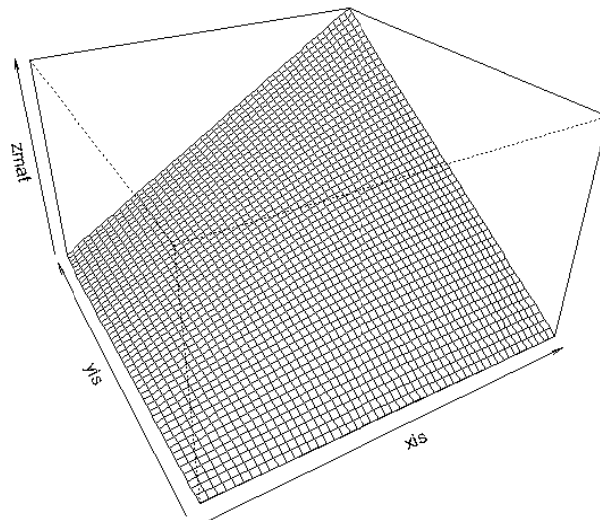
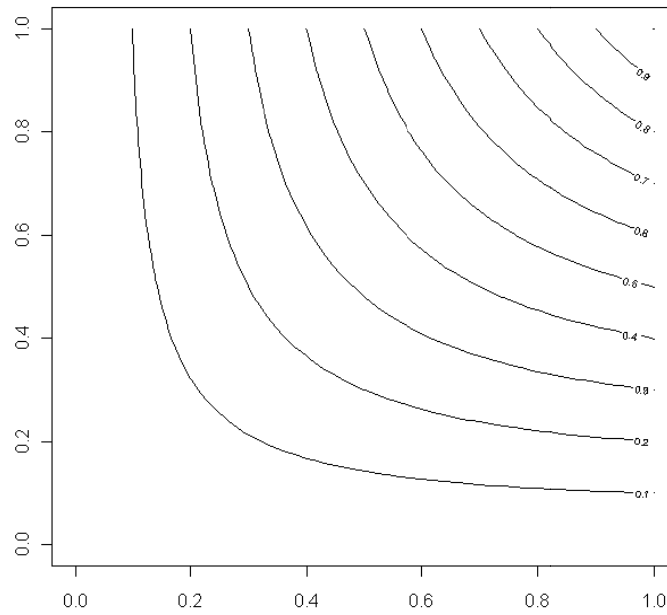


Figure 17 - Frank copula cumulative distribution function contour plot



### Comparison of Archimedean copulas

Table 3 – Comparison of Archimedean copula

Copula	Generator	Parameter	Minimum Value	Maximum Value
<b>Gumbel</b>				
<b>Clayton</b>	_____			
<b>Frank</b>	_____			

A natural question to ask is how the choice of parameter affects the degree of dependence for the different Archimedean copulas and if any of them display a measure of tail dependence.

Table 5.5 from McNeil Frey and Embrechts (2005) gives the relationship between the parameter, Kendall's tau and the upper and lower tail dependence for these copulas.

Table 4 – Comparison of Archimedean copula parameters and dependence measures

Copula	$\rho_\tau$	$\lambda_U$	$\lambda_L$
<b>Gumbel</b>	$1 - \frac{1}{\theta}$	$2 - 2^{1/\theta}$	0
<b>Clayton</b>	$\frac{\theta}{\theta + 2}$	0	$\begin{cases} 2^{-1/\theta}, & \theta > 0 \\ 0, & \theta \leq 0 \end{cases}$
<b>Frank</b>	$1 - \frac{4}{\theta}(1 - D(\theta))$ <sup>1</sup>	0	0

In the above  $D(\theta)$  is the Debye function  $D(\theta) = \theta^{-1} \int_0^\theta t / (\exp(t) - 1) dt$ .

The figures below illustrate the effect of tail dependence for the different Archimedean copulas. In each instance the parameter value was chosen such that roughly  $\rho_\tau = 0.6$  for all the copulas.

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<sup>1</sup>  $D_n(\theta)$  is a family of functions  $D_n(\theta) = n\theta^{-n} \int_0^\theta t^n / (\exp(t) - 1) dt$

Figure 18 – 5000 Observations from a Gumbel copula with

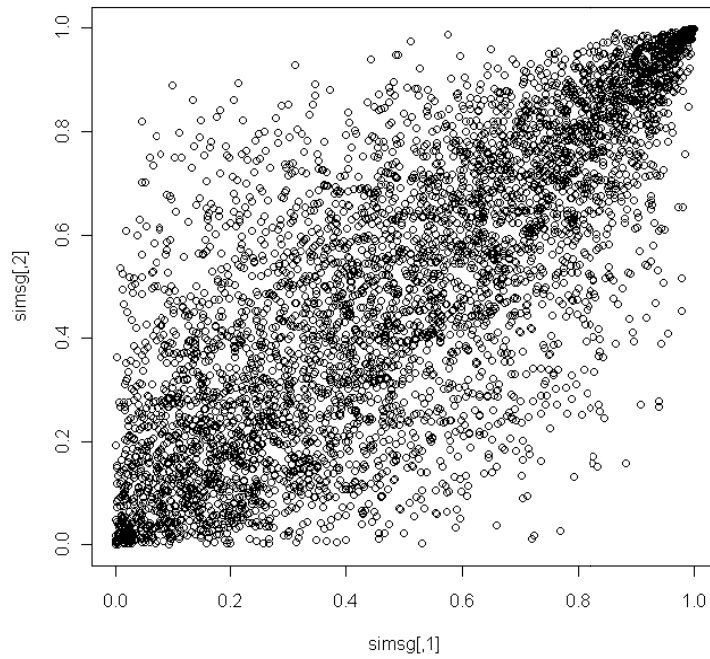


Figure 19 – 5000 Observations from a Clayton copula with

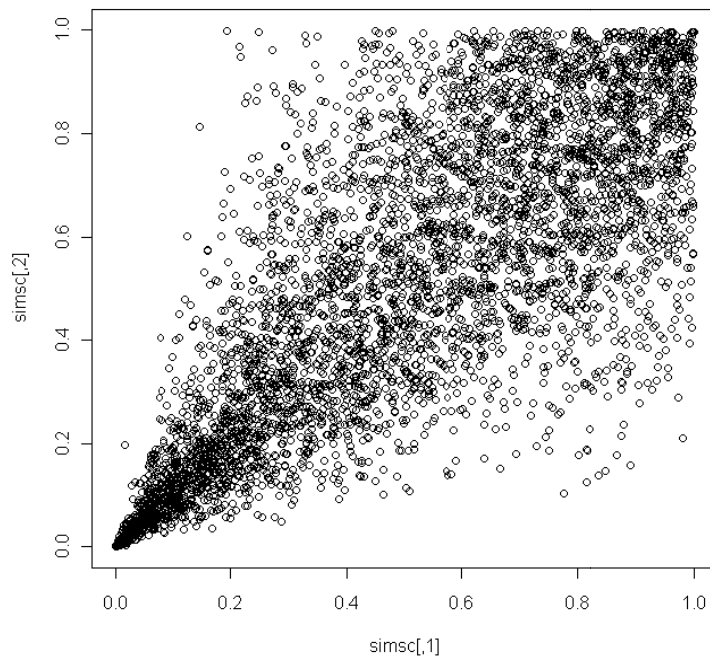
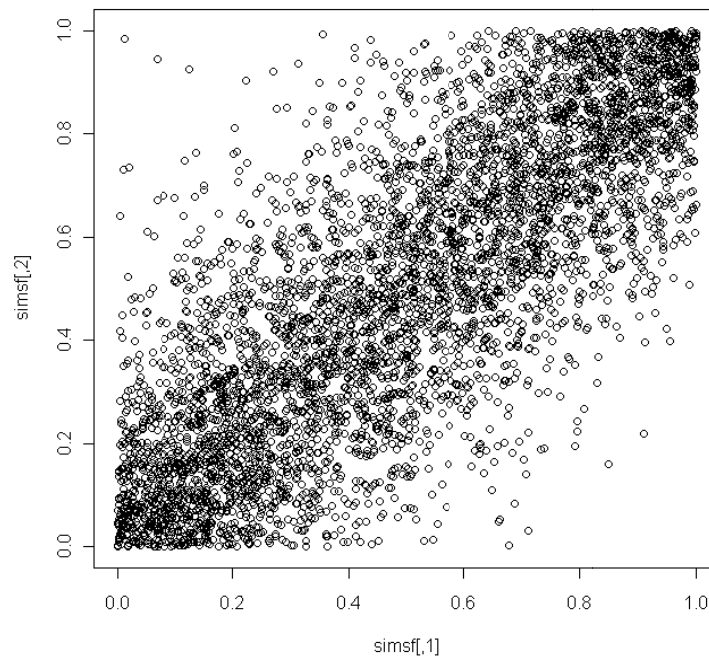


Figure 20 – 5000 Observations from a Frank copula with



Sampling from Archimedean copulas is a more involved process than that used for elliptical copulas. The method relies on a general algorithm for constructing copulas developed by Marshall and Olkin (1988). Examples of constructing copulas using this method can be found in Frees and Valdez (1998) with details on sampling from such copulas found in Verschuere (2006).

### *Marshall-Olkin copulas*

Marshall-Olkin copulas are especially attractive when modelling the joint distribution of object lifetimes when these lifetimes are dependent. As such they were first used by actuaries studying the joint lifetime distribution of, for example, married couples, members of the same family etc. Examples include the work of Clayton (1978) and Hougaard (1984).

Marshall Olkin copulas aim to produce multivariate distribution where the marginal variables are exponentially distributed. The parameter for a particular marginal distribution depends on both idiosyncratic and common risk factors. These common risk factors are the source of dependence in the model.

In particular for two entities and three hazard rates would be specified such that  $\lambda_1$ ,  $\lambda_2$  and  $\lambda_{12}$  denote the instantaneous probabilities of individual and joint failure respectively.

When working with positive random variables such as the future lifetime of an object the survival function is often used in lieu of the distribution function. The survival function is simply the probability that the future lifetime exceeds a given value. Continuing with the notation from the previous Chapter we denote this function as

In a similar fashion we denote the joint survival function by  $S(x, y) = P[X > x, Y > y]$ . For the bivariate case this is related to the joint distribution function of the lifetime by

$$F(x, y) = 1 - S(x) - S(y) + S(x, y).$$

The survival copula is then given by  $\bar{C}(u, v) = P[U > u, V > v]$  with  $S(x, y) = \bar{C}(S(x), S(y))$ .

The Marshall Olkin copula is defined as  $\bar{C}(u, v) = \min(vu^{1-\theta_1}, uv^{1-\theta_2})$  for the bivariate case with  $\theta \in [0, 1]^2$  where  $\theta_1 = \frac{\lambda_{AB}}{\lambda_A + \lambda_{AB}}$  and  $\theta_2 = \frac{\lambda_{AB}}{\lambda_B + \lambda_{AB}}$ .

The parameter vector  $\theta$  controls the dependence structure of the joint distribution. When  $\theta_1 = \theta_2 = 0$  then  $\bar{C}(u, v) = C^\perp$  while if  $\theta_1 = \theta_2 = 1$  then  $\bar{C}(u, v) = C^+$ .

The Marshall Olkin copula only allows for positive dependence between variables.

A model of correlated defaults using the Marshall Olkin copula as dependence structure is presented by Giesecke (2003). More traditional examples from the actuarial field can be found in Frees and Valdez (1998).

To simulate observations from the copula in the bivariate case one would start by simulating three independent observations  $r, s$  and  $z$  from a standard uniform distribution. We then set

$$t_1 = \min \left[ -\frac{\ln(r)}{\lambda_A}, -\frac{\ln(z)}{\lambda_{AB}} \right]$$

$$t_2 = \min \left[ -\frac{\ln(s)}{\lambda_B}, -\frac{\ln(z)}{\lambda_{AB}} \right]$$

and

$$u = \exp(-(\lambda_A + \lambda_{AB})t_1)$$

$$v = \exp(-(\lambda_B + \lambda_{AB})t_2).$$

The pair  $(u, v)$  will then be an observation from a Marshall Olkin copula.

A major drawback for this copula is the large number of variables that need to be simulated when sampling from this copula. For a portfolio of size  $n$  there are  $2^n - 1$  different ways in which defaults can happen jointly. Since this number grows exponentially with portfolio size the copula becomes unsuitable for larger portfolios.

In general this holds true for many alternative dependency models in risk management and currently hampers their implementation for large portfolios according to Schönbucher (2002). This is particularly true for application to credit risk where the relative scarcity of default data compared to say equity returns compounds the problem.

### 4.3.3 Choosing and calibrating a copula function

In order to choose a copula we should consider aspects such as the dependence structure each copula involves as well as the number of parameters we need to estimate.



The Gaussian copula does not allow either upper or lower tail dependence. Its use has been criticized for not assigning enough probability to extreme events when defaults are expected to cluster.

The Student copula is a natural answer to these criticisms since it displays tail dependence for appropriate degrees of freedom parameters.

The main problem with using either the Gaussian or Student copula is that the number of variables that need to be estimated grows rapidly with the dimension of the joint distribution. If we consider the joint distribution of  $n$  obligors we need to estimate the  $\frac{n(n-1)}{2}$  parameters for the covariance matrix. In addition the student copula requires the degrees of freedom to be specified.

Archimedean copulas are particularly attractive in when we consider the problem of estimating parameters. There are a large number of one parameter Archimedean copulas that can specify a wide range of dependence structures.

In addition Rogge and Schönbucher (2003) argue that Archimedean copulas should be preferred over normal or student copulas since these imply an unrealistic term structure of default dependencies with model results strongly date dependent. We will return to the term structure of default dependencies when considering extensions to the Gaussian copula framework. Schönbucher (2002) finds a closed form distribution for a large homogeneous portfolio using an Archimedean copula dependence structure.

The price paid for this simplicity is the exchangeability of Archimedean copulas. The dependence structure between any groups of marginal variables will be independent of the particular variables in the group. The problem of fitting the most appropriate Archimedean copula in empirical applications is discussed in De Matteis (2001) while Whelan (2004) presents sampling algorithms.

Methods for choosing a copula function and estimating its parameters are presented in Durrleman, Nikeghbali and Roncalli (2000) but there still remains uncertainty about the best copula to use. According to the authors there is no systematic rigorous method for the choice of the copula. There is no measurement that ensures the selected family of copula will converge to the real dependence structure underlying the data. This can provide biased results since according to the dependence structure selected the obtained results might be different. This uncertainty surrounding the choice of copula is an example of model risk.

#### 4.4 Incorporating default dependence in credit models

Dependence measures the degree to which the probability of one event happening moves in sync with the probability of another event happening. In terms of default dependence a dependence measure of zero means that the default of one obligor has no bearing on the default of another obligor – the obligors are completely independent of each other. Perfect positive dependence means that if one obligor defaults, the other will automatically follow suit. Perfect negative dependence means that if one obligor defaults the other one will certainly not, and vice versa.

#### 4.4.1 Structural models

In the single obligor case a default will occur when the value of the firm's assets fall below a certain threshold. In a structural model default correlation is introduced by assuming the asset values of different companies follow correlated stochastic processes.

For valuation purposes however, the structural approach suffers from a major drawback as it does not replicate observed market prices for single name instruments in the portfolio. In particular short term credit spreads are often too low as the model excludes the probability of a sudden extreme decrease in asset values.

According to Mashal, Naldi and Zeevi (2003) multi-name credit instruments are usually hedged with single name instruments (mainly credit default swaps). This inconsistency would usually rule out the use of structural models for valuation of multi-name instruments.

Hull et al (2005) concluded that the structural approach is a computationally viable approach but that the basic model was unable to replicate observed CDO tranche prices.

The structural approach can potentially offer two main advantages over reduced form models. Firstly the model is dynamic and credit quality can evolve over time and secondly the model holds some economic rationale leading to correlation parameters that can be empirically estimated.

#### 4.4.2 Reduced form models

Reduced form models are by design able to replicate observed market prices for single name instruments. In the literature there are three distinct methods that can be used to model default correlation for reduced form models.

##### *Conditionally independent defaults (CID)*

The first approach introduces correlation in the firms' default intensities making them dependent on a set of common state variables and on a firm specific factor. These models have received the name of conditionally independent defaults (CID) models because, conditioned to the realization of the state variables, the firm's default intensities are independent as are the default times that they generate.

The main drawback of these models is that they do not generate sufficiently high default correlations according to Hull and White (2001), Schönbucher and Schubert (2001) or Frey and Backhaus (2003). Yu (2002) indicates that this is not necessarily a problem of the model itself, but rather an indication of the lack of sophistication in the choice of state variables.

However it can be shown (see Schlögl and O'Kane (2003)) that even if the intensities are perfectly dependent the default correlation the model can produce is only of the same order of magnitude as the default probabilities. A commercial model using this assumption is CreditRisk+ and one might conclude that such a model is likely to underestimate the tails of the loss distribution, particularly for poorly diversified portfolios.

Two direct extensions of the CID approach try to increase the amount of default correlation achievable with such models.

The first approach is to introduce joint jumps in the default intensities (Duffie and Singleton (1999b)) or secondly the probability of joint default events (Kijima and Muromachi (2000)).

The criticism that the joint defaults approach has received stem from the fact that it is unrealistic that several firms default at exactly the same time and that after a common credit event the intensity of other related obligors that do not default does not change.

### *Contagion effects*

The idea of default contagion is that, when a firm defaults, the default intensities of related firms jump upwards. In these models default dependencies arises from direct links between firms. The default of one firm increases the default probabilities of related firms, which might even trigger the default of some of them.

Contagion models follow two main approaches. The first is the so called infectious defaults approach by Davis and Lo (1999). When a default occurs, the default intensity of all remaining firms is increased. This increased intensity remains for an exponentially distributed period of time, after which it returns to the initial specified intensity. During this period of increased intensity the default probabilities of all firms increase reflecting the risk of default contagion.

The second approach is that of Jarrow and Yu (2001) aimed at counterparty risk. In this model the intensity of default at a particular time depends on whether the rest of the firms have defaulted or not. This dependence structure introduces circularity to the model which hampers the derivation of the joint default distribution.

The authors amend the model to restrict a firm as either being a primary or secondary firm. Defaults intensities of primary firms are independent of the default state of other firms but if a primary firm defaults the default intensities of secondary firms are increased.

### *Copulas*

The main problems to be resolved for CID and contagion models are the difficulties in their implementation and calibration to market prices. To date no empirical calibration to market data or implementation of these models have been documented.

In CID and contagion models the specification of the individual intensities includes all the default dependence structure between firms. In contrast, the copula approach separates individual default probabilities from the credit risk dependence structure. In the next chapter we study a popular model based on copula dependence used in CDO markets.

## Chapter 5 The Gaussian copula approach and the Li model

### 5.1 Introduction

The initial challenge for credit models was how to deal with the correlation of credit events within the portfolio. Extreme credit events are by definition rare but their correlation accounts for the tail of the portfolio loss distribution. This makes the specification of the correlation structure of the underlying portfolio important for CDO securities.

This is especially true for the higher rated tranches which are only expected to suffer losses if portfolio losses reach the tail of the loss distribution. If the credit events within the portfolio are highly correlated losses on higher rated tranches will not differ much from the speculative tranches.

If the credit events in the portfolio have low correlation the expected losses and hence prices for the different tranches will be much different.

Much of the recent growth in the CDO market is due to the availability of tools to manage credit risk for a portfolio of credit risky securities. A major cause of this growth is the introduction in 2001 of the Gaussian copula type of models by Li (2000) to the credit industry. This allowed for the rapid specification of the correlation structure and pricing of CDO securities. A highly readable account of the background of Mr. Li and his model recently appeared in the *Wall Street Journal*<sup>1</sup>; below we will focus on the technical details.

Li, trained as an actuary, based his model on the statistics used for survival analysis which has been well studied. Some of his colleagues in actuarial science were working on the problem of how the death of one person could influence the death of another person, especially a loved one like a spouse. Li realized it could also work for the problem of default correlation; in this case we will treat a default from a portfolio of securities like the death of a member of some population.

Surprisingly default correlation has not been well defined and understood in finance even though the first CDO instruments were issued in 1987<sup>2</sup>. Previous attempts were based on dichotomized observations of default or non-default over critical time periods such as a year.

As a simple example we start with two entities A and B and, using the notation introduced in Chapter 3, let  $q_A = P[\tau_A < 1]$ ,  $q_B = P[\tau_B < 1]$  and  $q_{AB} = P[\tau_A < 1, \tau_B < 1]$  the probability that the respective entity defaults within the next year.

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<sup>1</sup> How a Formula Ignited Market Burned Some Big Investors. *Wall Street Journal* 15 September, 2005; page A1.

<sup>2</sup> Issued by now defunct Drexel Burnham Lambert Inc on behalf of Imperial Savings Association

The correlation between the two events is then simply given by

Equation 5

$$\rho_{A,B} = \frac{q_{AB} - q_A q_B}{\sqrt{q_A(1-q_A)q_B(1-q_B)}}.$$

This is defined as the discrete default correlation since it applies to a discrete time period, in this case one year. The default correlation will depend on the time period chosen which in turn might be based on empirical studies available. Most rating agencies quote one year default probabilities for the various rating classes.

The dependence of discrete correlation on the choice of time interval has several disadvantages.

Firstly, the probability of the event depends on the time interval chosen. The probability of a firm defaulting in the next year is not the same as defaulting in the next 100 years. In the case of human survival the former is fairly unlikely while the latter is an almost sure event. The correlation thus also depends on the time interval chosen resulting in different correlation between the same entities depending on an arbitrary observation period.

Secondly, concentrating on a single period wastes information. Default rates may depend on age since issue, similarly human survival depends on current age. Defaults also depend on the current state of the economic cycle. This is ignored when focusing on a single arbitrary period.

Thirdly, the valuation of credit derivative instruments and CDO securities may depend on the default correlation over an extended period. Typically, default rates are quoted for one year periods.

Finally, the calculation of default rates as simple proportions is only correct when there is no censoring in the sample. Censoring is a typical feature of survival data and occurs when an entity leaves the study before its conclusion for a reason other than default. For example, a company may be acquired by another before the end of the observation period. We are thus unable to say the company did not default during the period but only that it survived up to a point.

Such censored observations may also distort the results if it is deemed informative. For example if companies more likely to default are also more likely to leave the study for reasons other than default we will under estimate default rates.

The main contributions Li makes in his paper is to introduce some techniques from life contingencies and survival analysis to address the problems with discrete correlation. By using a copula to specify the dependence structure the study of the marginal distributions can be separated from the study of the dependence structure between them.

The remainder of the chapter is structured in five parts.

Firstly, we will introduce the concept of time to default as specified by the survival function and hazard rate. This specification of the marginal distribution of individual default time is an example of the reduced form approach to credit modelling.

Secondly, we look at various ways to calibrate the marginal time to default for a single entity through construction of a credit curve using market data.

Thirdly, we demonstrate how copula functions can be used to specify a dependence structure between these marginal times to default to obtain a joint lifetime distribution for a portfolio of entities.

Fourthly, we explore a simplified version of a model using a Gaussian copula, the so called one factor model, which forms the reference model currently used for CDO pricing.

Finally, we provide some numerical examples illustrating the sensitivity of CDO tranche prices to the assumptions made in the one factor model.

## 5.2 Time to default

Instead of a default rate a continuous variable called the survival time representing the time until an entity or security defaults is introduced. To specify the survival time a time origin, time scale and precise definition of default is needed. For these the current time, one year scale and default definition as per some rating agency is used. Let  $\tau_A$  and  $\tau_B$  be the time until entities  $A$  and  $B$  respectively default.

The correlation between entities is then specified as the correlation between their future survival times

Equation 6

$$\rho_{AB} = \frac{\text{cov}(\tau_A, \tau_B)}{\sqrt{\text{var}(\tau_A)\text{var}(\tau_B)}}.$$

The discrete default and survival time correlation defined above are equivalent to the Pearson correlation coefficient, a measure of linear dependence between two variables described in Chapter 4. The survival time correlation, however, is a more general concept than the discrete correlation. Knowing the survival time correlation allows us to calculate the discrete correlation for any given time period. The discrete correlation however does not tell us anything about the survival time correlation in general.

### 5.2.1 Survival functions

Let  $\tau$  be the continuous random variable measuring the time from now until default occurs for some entity with  $F(t)$  the distribution function of  $\tau$  with

$$F(t) = P[\tau \leq t] \quad t > 0.$$

Now define the survival function  $S(t)$ , the probability that the entity survives another  $t$  years

$$S(t) = 1 - F(t) \quad t > 0.$$

We assume  $F(0) = 0$  which implies  $S(0) = 1$  and define the probability density of  $\tau$

$$f(t) = F'(t) = -S'(t) = \lim_{h \rightarrow 0} \frac{P[t < \tau < t+h]}{h}.$$

For a security that has survived  $x$  years we can define its remaining lifetime after  $x$  years as

$$\tau - x | \tau > x.$$

Recall from Chapter 3 that we defined the probabilities of this remaining lifetime being less than or exceeding  $t$  respectively as

$${}^tq_x = P[t + x > \tau | \tau > x]$$

$${}^tp_x = P[t + x \leq \tau | \tau > x].$$

Notice that the above probabilities are both conditional on survival for  $x$  years from the present while for the special case of  $x = 0$  we obtain the unconditional probabilities

$${}^tq_0 = F(t)$$

$${}^tp_0 = S(t).$$

A credit curve in the discrete setting is then simply the series of future one year marginal default probabilities  $q_0, q_1, \dots, q_n$ . This is similar to using a series of one year forward interest rates to construct a yield curve.

### 5.2.2 Specifying the time to default by the default intensity

In the previous section we specified the time to default by using the survival function  $S(t)$ . This can also be done by using the default intensity which is the instantaneous conditional probability of default at a given age expressed as an annual rate. As seen in Chapter 2 the default intensity and its specification form the basis for most of the reduced form models of credit risk. This intensity serves as input parameter for some Cox process, the first jump of which signals a default.

Define the default intensity at time  $x$  as  $\lambda_x$  with

$$\lambda_x = \lim_{h \rightarrow 0} \frac{P[x < \tau \leq x + h] | \tau > x}{h}$$

$$\lambda_x = \lim_{h \rightarrow 0} \frac{F(x + h) - F(x)}{h} \frac{1}{1 - F(x)}$$

$$\lambda_x = f(x) \frac{1}{S(x)}$$

$$\lambda_x = -\frac{S'(x)}{S(x)}.$$

The survival function can be expressed in terms of the hazard function by integration of the last equation above.

$$\lambda_x = -\frac{d}{dx}(\ln(S(x)))$$

$$\int_0^t \lambda_x dx = -\ln(S(x))$$

$$e^{-\int_0^t \lambda_x dx} = S(t).$$

Similarly we can derive the conditional survival probabilities from the hazard function

$$e^{-\int_0^t \lambda_{x+s} ds} = {}_t p_x.$$

The density function for the future lifetime is then given by taking the derivative of

$$F(t) = 1 - S(t) = 1 - e^{-\int_0^t \lambda_x dx}$$

to find

$$f(t) = F'(t) = S(t) \times \lambda_t.$$

Notice the similarities between the hazard rate and the short rate of interest. If the hazard rate were a short rate interest rate process then  $S(t)$  would be the price per nominal value of risk-free zero coupon bonds maturing in  $t$  years. In this case  $S(t)$  is the probability that a defaultable bond will not default within  $t$  years.

Modelling a default process is thus equivalent to modelling a hazard process since this completely specifies the survival function  $S(t)$  and distribution of  $\tau$ .

### *Assumptions for fractions of a year*

Default probabilities are often quoted as one year rates but it is often required to compute default probabilities for a fraction of a year. There are three main methods used in survival analysis to deal with this problem. All of them involve assumptions on how the hazard rate develops during the year. Let  $0 < u < 1$  represent a fraction of a year.

#### *Linearity of $uq_x$*

The first method assumes that  $uq_x$  is a linear function of  $u$  such that  $uq_x = u(q_x)$ . The hazard function is then given by

$$\lambda(x+u) = -\frac{d}{du} \ln(up_x)$$

$$\lambda(x+u) = -\frac{d}{du} \ln(1 - u(q_x))$$

$$\lambda(x+u) = \frac{q_x}{1 - up_x}$$

This would imply that the hazard function increases during the year.



## Constant hazard rate

The second popular assumption is that the hazard function remains constant during the year with  $\lambda(x + u) = -\ln(p_x)$  such that  $up_x = p_x^u$ .

Notice that under the assumption of a constant hazard rate  $\lambda_x = \lambda$  the density function of  $\tau$  is given by

$$f(t) = \lambda e^{-\lambda t}.$$

As expected this is the density function of an exponentially distributed random variable as it represents the time to arrival of the first event for a Poisson variable.

## Linearity of $(1 - u)q_{x+u}$

The third assumption is that  $(1 - u)q_{x+u} = (1 - u)q_x$ . The hazard rate can then be derived in a similar fashion as in the first case to give  $\lambda(x + u) = \frac{q_x}{1 - (1 - u)q_x}$ . This implies that the hazard rate decreases during the year.

## *Advantages of using a default intensity process*

According to Li (2000) there are a number of advantages to using default intensity to model a default process. The first is that it provides immediate information on the default risk of each entity at risk of default at a given time. Secondly, it allows for ready comparison between default risks for various entities. Thirdly, the hazard rate is a very flexible platform from which to extend the model to more complicated scenarios. The model can easily be extended to allow for more than one type of default or rating transitions. Finally, the similarity between the hazard rate and the short rate allows for a ready transfer of a number of techniques used to model the short rate.

## 5.3 Marginal distributions and construction of a credit curve

Li calls the hazard function used to specify the survival function a credit curve due to its similarities to a yield curve. The question is then how we would go about obtaining such a credit curve for a particular firm. The following are three main ways in which this can be done

### 5.3.1 Information from rating agencies

Rating agencies regularly publish one year default probabilities for various rating classes. In addition multi-year cumulative default rates are also given. From these cumulative rates we can obtain the marginal conditional future default rates for times beyond the first year.

The  $n$  year cumulative default rate given would correspond to  $nq_x$  in the notation above. By solving for  $q_{x+n}$  in  $(n + 1)q_x = nq_x + np_x q_{x+n}$  we can obtain the future conditional one year default rates, the condition being that default has not yet occurred at that time.

The hazard rate for a particular year can then be derived by using any of the assumptions in the preceding section. In particular if we assume a piecewise constant hazard rate function for the year ending  $n$  years from now then  $\lambda(x + n) = -\ln(p_{x+n}) = -\ln(1 - q_{x+n})$ .

### 5.3.2 Default probabilities from the structural approach

Using the structural approach to credit modelling one year default probabilities based on observable market variables like stock prices can be obtained. Delianedis and Geske (1998) extend the original work by Merton to produce a term structure of default probabilities. Using the same method as above a credit curve can be derived from these probabilities.

### 5.3.3 Market implied probabilities based on swap spreads and bond prices.

This approach is mostly used by derivative traders and assumes that there exists a series of bonds with equal seniority issued by the same company with increasing term to maturity. Based on the market prices of these bonds the yield to maturity is calculated and compared to that of similar treasury bonds to obtain a yield spread curve.

A credit curve is then constructed using this spread together with an exogenous assumption of recovery rates. In his paper Li prefers this method for the following reasons.

Firstly a trading desk will calculate profit and losses daily on a mark-to-market basis. This can only be done if the model depends on dynamic market data which reflects the agreed perception on the evolution of the market. This perception and current market prices may be very different from historical experience.

Secondly credit ratings are based on some classification of firm characteristics in the hope that homogeneous risks will remain in each rating class. This ignores some firm specific information that may be contained in market prices.

Thirdly rating agencies are much slower to respond to anticipated changes in credit quality than the market.

Fourthly credit ratings only indicate default frequency and not loss severity while the value of many credit derivatives depend on both.

Finally credit rating agencies usually provide the one year default probability for each rating class along with a transition matrix. Neither of these is stable over time and many credit derivatives have maturities greater than one year.

After constructing a credit curve for each obligor the next step is to extend this framework to a portfolio consisting of multiple securities by specifying the joint distribution of the time until default of all securities in the portfolio.

## 5.4 Specification of dependence structure using a Gaussian copula

Li uses a Gaussian copula to specify the joint lifetime distribution of portfolio constituents. This copula is also the one used in the commercial CreditMetrics model by JP Morgan but not referred to as such explicitly. The CreditMetrics model is of the structural variety and as such relies on the modelling of asset values to calculate default probabilities. CreditMetrics uses the correlation of the asset values of two firms to calculate the correlation in the default probability.

Obtaining a reliable correlation matrix for every pair of obligors is clearly an impossible task. To circumvent this problem CreditMetrics uses correlation indices. The correlation is then inferred based on industry, geographic region etc. More detail on asset correlation and the CreditMetrics model can be found in Gupton et al (1997).

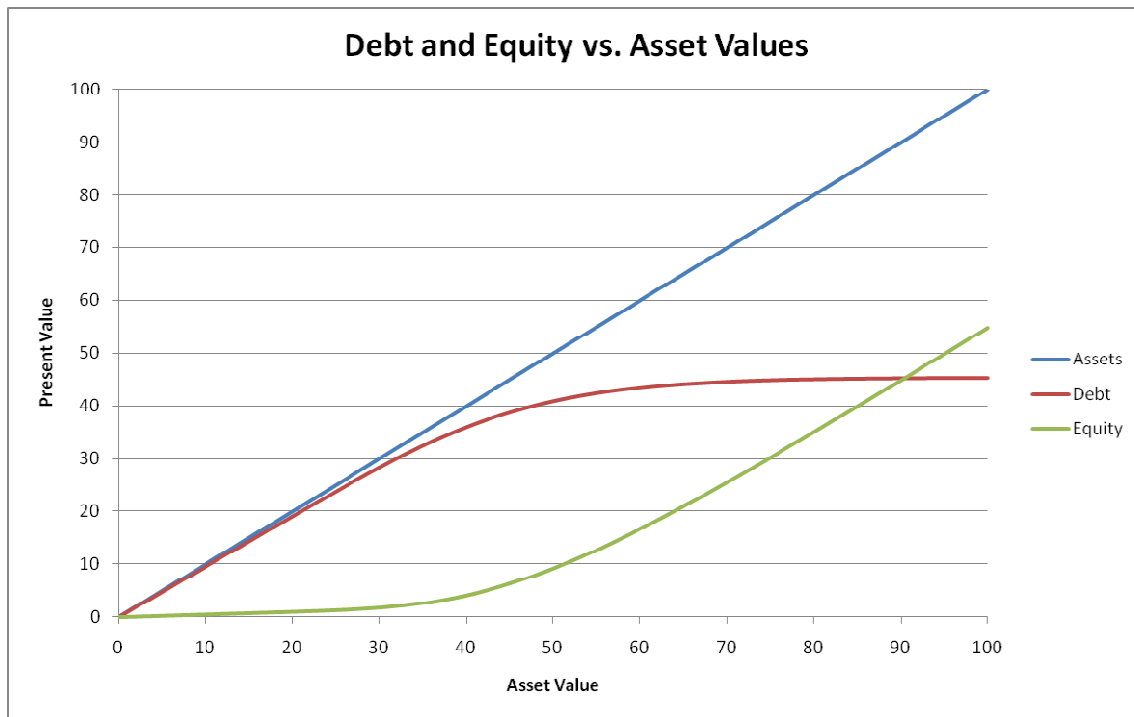
In practice the asset values of firms are unobservable with the correlation in equity prices often used as a proxy for the asset correlation.

The use of equity prices to refer the behaviour of asset values have been widely criticized due to the different leverage of assets and equity. For a high yield security a change in asset values will impact the market price of the debt as well as that of the equity. For a security with low probability of default the problem is less severe since a change in asset value will be mostly reflected by a change in equity prices.

The above can perhaps best be illustrated when we plot the debt and equity values vs. the assets value of the firm under the original Merton model. The graph below is for debt with a face value of 50 maturing in one year with asset variance of 30%. When the asset value far exceeds the face value of the debt the probability of default is low. In this case any change in asset value is mirrored by a change in equity prices. Conversely, when the asset value is low any change in asset value will mostly impact the value of the debt and not the equity.

Mashal, Naldi and Zeevi (2003) compare the dependence structure between asset and equity returns and investigate if the former behaves similarly to the latter. They also consider the magnitude of the possible error in using equity returns instead of actual asset returns.

Figure 21 – Debt and Equity vs. Asset Values



In the CreditMetrics model the joint default probability of two entities, A and B, are calculated as follows.

1. Obtain values  $\alpha$  and  $\beta$  such that  $\alpha$  and  $\beta$  are standard normal variables with marginal one year default probabilities  $p_A$  and  $p_B$  respectively.
2. Given the asset correlation  $\rho$  the joint default probability is then given by  $P(\alpha < -\alpha_A, \beta < -\alpha_B)$  where  $f$  and  $F$  are density and distribution functions respectively of the bivariate normal distribution with correlation parameter  $\rho$ .

Under the survival time framework we can calculate the same probability using a Gaussian copula with correlation parameter  $\rho$  as follows

This will hold provided that  $\alpha$  and  $\beta$  are standard normal variables since  $\alpha$  and  $\beta$  are standard normal variables and  $\rho$  is the correlation parameter.

Note that the correlation between survival times will be much smaller than the correlation of asset values. This is perhaps best illustrated with a numerical example; the following is taken from Hull (2006) p499, example 20.4.

Suppose that the probability that entity  $A$  defaults within one year is 0.01 and the probability that entity  $B$  defaults within one year is also 0.01 and  $\gamma = 0.2$ . We find that the joint default probability is  $\Phi_2(\Phi^{-1}(0.01), \Phi^{-1}(0.01), 0.2) = 0.000337$ .

Given this joint one year default probability we can calculate the discrete default correlation between  $A$  and  $B$  over a one year horizon as.

$$\begin{aligned}\rho_{A,B} &= \frac{q_{AB} - q_A q_B}{\sqrt{q_A(1 - q_A)q_B(1 - q_B)}} \\ &= \frac{0.000337 - 0.01 \times 0.01}{\sqrt{0.01(0.99)0.01(0.99)}} \\ &= 0.0239.\end{aligned}$$

The discrete default correlation is thus much smaller than the asset correlation. Intuitively the discrete default correlation should increase with the time horizon since there will be an increased tendency for both companies to have defaulted over a longer periods of time.

Suppose in the above example we assume a constant hazard rate for each entity of  $\lambda = -\ln(0.99)$  such that  $tq_x = 1 - e^{-\int_0^t -\ln(0.99)ds} = 1 - e^{t \times \ln(0.99)}$ . This will imply that  $q_A = q_B = 0.01$  as before and that  $2q_A = 2q_B = 0.0199$ . The joint default probability over the two year horizon is then given by  $\Phi_2(\Phi^{-1}(0.0199), \Phi^{-1}(0.0199), 0.2) = 0.0010908$ .

If we now compute the discrete default correlation over a two year time horizon we find that

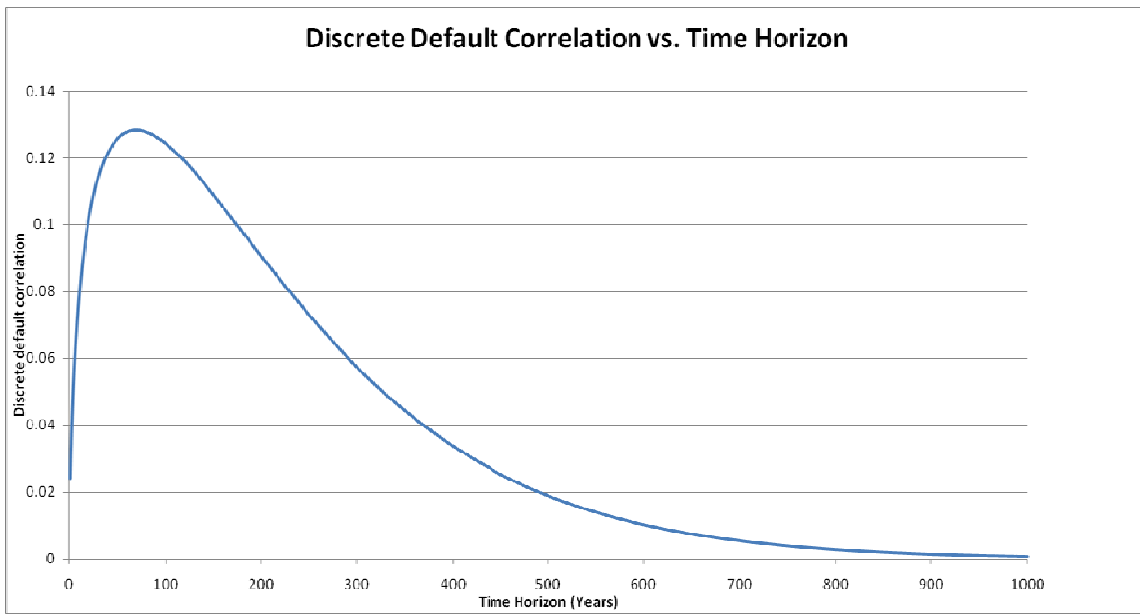
$$\begin{aligned}\rho_{A,B} &= \frac{q_{AB} - q_A q_B}{\sqrt{q_A(1 - q_A)q_B(1 - q_B)}} \\ &= \frac{0.0010908 - 0.0199 \times 0.0199}{\sqrt{0.0199(0.9801)0.0199(0.9801)}} \\ &= 0.0356.\end{aligned}$$

This is an increase over the one year default correlation and serves to illustrate the point that discrete default correlation is dependent of the time horizon over which it is measured.

The above joint default probabilities can be calculated by using the `mvtnorm` package in R that evaluates distribution functions for multivariate Gaussian and Student-t distributions. More details appear in the appendix.

In the figure below the discrete default correlation is plotted as a function of time. Note that the correlation increases rapidly at first and then tapers off as the time horizon is extended. Intuitively we can say that the probability that a firm has defaulted over a very long period is almost certain if we recall the reduced form framework that assumes all firms will eventually default. Over such a long time horizon the event is thus not influenced by the default times of other obligors. Similarly the default risk over a very short time horizon is almost zero regardless of what happens to other firms.

Figure 22 – Discrete default correlation as a function of time horizon



We can also extend this analysis to compute the correlation between  $\tau_1$  and  $\tau_2$  as the correlation between the times until default  $\tau_1$  and  $\tau_2$  using the formula introduced earlier.

\_\_\_\_\_

Now under the assumption of a constant hazard rate for each company  $\tau_i$  will be exponentially distributed with parameter  $\lambda_i$ . It then follows that

\_\_\_\_\_

The reason for this large variance compared to the mean is that the exponential distribution is the distribution of maximum entropy for positive continuous random variables. This means the exponential distribution contains the minimum inbuilt prior information about the variable.

To complete the calculation we need to compute  $\rho(\tau_1, \tau_2)$  which can be done numerically by simulating pairs of observations  $(\tau_1, \tau_2)$  from a bivariate normal distribution with correlation parameter 0.2. We then compute the observed values of  $\tau_1$  and  $\tau_2$  as  $\tau_1^*$  and  $\tau_2^*$ .

For 100,000 simulations done using the R software package (code to be found in appendix) the observed value for  $\rho(\tau_1, \tau_2)$  was found to be 11587.71. Completing the calculation we find that the value of  $\rho(\tau_1, \tau_2)$  is 0.2.

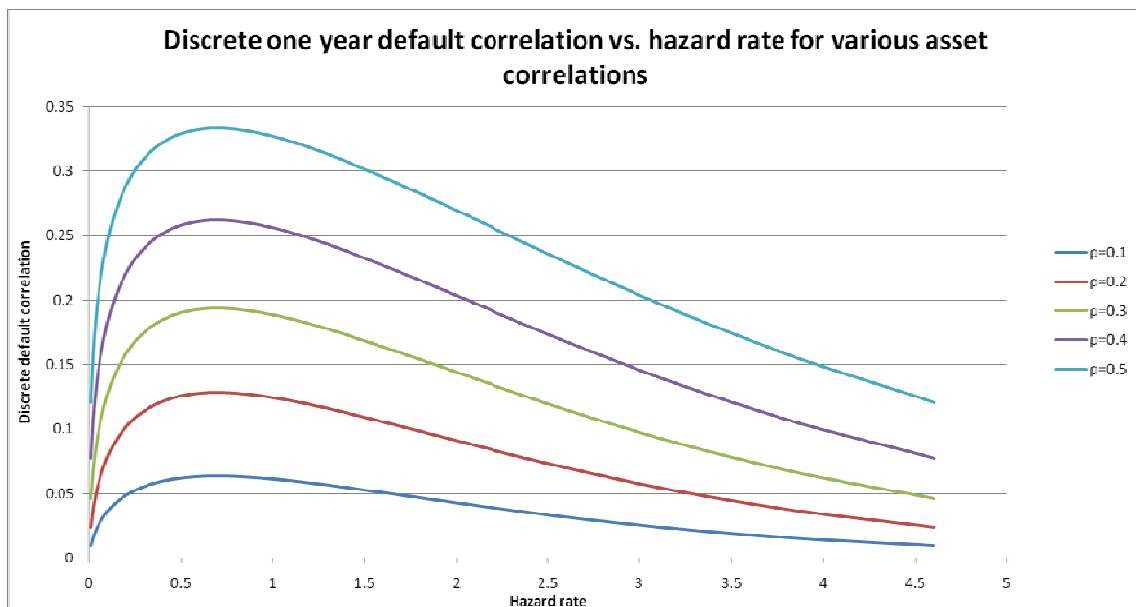
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This is higher than the discrete default correlation over any of the time horizons plotted in the previous graph but lower than the asset correlation.

Perhaps more importantly we can observe that it is the probability of default that determines the discrete default correlation for a given asset correlation. We can thus keep the time horizon fixed and vary the probability of default to investigate the effect on the discrete correlation. Investors will usually have more certainty about the horizon over which they invest but the actual default probability implied from market data can change suddenly.

The figure below shows the discrete default correlation as a function of the hazard rate we assume that . Discrete correlation increases rapidly with hazard rate and reaches a maximum when .

**Figure 23 – Discrete default correlation as a function of hazard rate**



When the dimensionality of the problem is increased to more than two companies the strengths of the Gaussian copula comes to the fore as sampling from a multivariate normal distribution is fairly easy using most statistical software packages.

The required inputs to calculate the joint distribution default times for any number of companies will be the correlation matrix of asset values and the marginal distribution function for each company.

## 5.5 The one factor model

In practice a one factor model is often used instead of specifying a correlation matrix. The assumption is that there is a common market factor  $M$  affecting all defaults as well as idiosyncratic factor  $X_i$  specific to each company  $i$  affecting only that company. Recall that CreditMetrics specifies the probability of default as  $tq_i = P[Z < Z_i]$  where  $Z$  is a standard normal variable. Under the one factor model we assume that

Equation 7

$$Z_i = a_i M + \sqrt{1 - a_i^2} X_i \text{ with } -1 \leq a_i \leq 1 \text{ for all } i.$$

and that the factors  $M$  and  $X_i$  are independent standard normal distributions for all  $i$ .

The covariance (and correlation) between  $Z_i$  and  $Z_j$  will then be given by  $a_i a_j$ .

Under the Gaussian copula model a default will occur when

$$Z_i < \Phi^{-1}(tq_i) = c_i$$

or

$$a_i M + \sqrt{1 - a_i^2} X_i < \Phi^{-1}(tq_i)$$

or

$$X_i < \frac{\Phi^{-1}(tq_i) - a_i M}{\sqrt{1 - a_i^2}}$$

Conditional on the value of  $M$  the probability that the  $i^{\text{th}}$  obligor defaults within  $t$  years is thus given by

$$P[Z < Z_i | M = m] = \Phi\left(\frac{\Phi^{-1}(tq_i) - a_i m}{\sqrt{1 - a_i^2}}\right) = (tq_i | m).$$

It is often convenient to think of  $Z_i$  and  $c_i$  as proxies for the asset and liquidation value of the firm respectively under the structural framework although they have no direct economic interpretation.

A particular case of the one factor model is where we assume constant and equal pairwise correlation such that  $a_i = a = \sqrt{\rho}$  and equal default probabilities  $tq_i = tq$  and  $c_i = c$ . For a one year horizon the above equation then becomes



Equation 8

$$(q|m) = \Phi\left(\frac{c - \sqrt{\rho} M}{\sqrt{1 - \rho}}\right).$$

The above equation gives the percentage of entities that will default by time  $t$  as a function of  $M$ . This result is due to Vasicek (1987) and is used for calculating credit VaR on a portfolio of loans under the Basel II accord if the internal ratings based approach is used.

Under this framework the correlation parameter  $\rho$  is specified depending on the type of exposure and type of product. The time horizon  $t$  is one year and the confidence level 99.9%. For each exposure the probability of default  $q$  and loss given default (LGD) will be estimated based on internal data and the capital required will be calculated as

$$\Phi\left(\frac{\Phi^{-1}(q) - \sqrt{\rho} \Phi^{-1}(0.999)}{\sqrt{1 - \rho}}\right) \times LGD.$$

For example on a retail mortgage portfolio a particular exposure of R1 million might be estimated to have a 2% probability of defaulting in the next year with a 20% LGD. The asset correlation measure for the portfolio is specified as 0.15. The required capital to be held against the exposure will then be calculated as

$$\Phi\left(\frac{\Phi^{-1}(0.02) - \sqrt{0.15} \Phi^{-1}(0.999)}{\sqrt{1 - 0.15}}\right) \times 0.2 \times R1,000,000$$

$$= 0.17632 \times 0.2 \times R1,000,000$$

$$= R35,265.79$$

which is the credit VaR at a 99.9% confidence level.

In the next section we explore some key results for the Gaussian copula model when applied to CDO valuation.

## 5.6 CDO valuation and tranche sensitivities

In this section we look at how the assumptions made in the Gaussian copula model affect the valuation of a CDO. Particularly we examine tranche sensitivities to changes in the correlation structure as well as credit spread on the portfolio of assets.

For a CDO the individual assets comprising the CDO are often referred to as credits or names while the portfolio of assets is called the index. Two important indices used by credit derivative traders are the 5- and 10-year CDX NA IG indices and the 5- and 10-year iTraxx Europe. Each consisting of 125 investment grade companies the former from North American and the latter from Europe. These indices are used by traders to easily obtain exposure to a portfolio of credit default swaps.

The portfolios underlying these indices are used to define standardized index tranches similar to the tranches of a CDO. In the case of the CDX NA IG 5 year index, successive tranches are

responsible for 0% to 3%, 3% to 7%, 7% to 10%, 10% to 15%, and 15% to 30% of the losses. In the case of the iTraxx Europe 5 year index, successive tranches are responsible for 0% to 3%, 3% to 6%, 6% to 9%, 9% to 12%, and 12% to 22% of the losses.

Following Meng and Sengupta (2008) the following simplifying assumptions are made:

1. We assume a homogeneous portfolio consisting of  $N$  obligors (in a homogeneous portfolio each credit has the same default probability) with default probability  $q$
2. We assume constant and equal pairwise asset correlation of  $\rho$  between obligors
3. We assume a recovery rate of zero in case of default by any obligor
4. We assume a loss of one unit should any particular obligor default
5. We assume the CDO comprises only two tranches (equity and senior)

The above assumptions mean the correlation structure is described by the one factor model from the previous section. Much like vanilla equity option prices are quoted using the volatility of the underlying CDO tranche prices are quoted based on the correlation factor.

A CDO involves a series of payments of losses and spreads but we limit ourselves to the losses over a single one of these periods. The loss payment for the full life of the CDO is a discounted-weighted sum of the single-period losses, and therefore may be deduced readily from the single-period loss results.

For example the  $i^{\text{th}}$  name defaults in year one when  $Z_i < \Phi^{-1}(q) = c_1$  and during the first two years when  $Z_i < \Phi^{-1}(2q) = c_2$ . The probability of defaulting in the second year is then given by

$$P[c_1 < Z_i < c_2].$$

In what follows we turn our attention to losses over the first period of the transaction. Let  $v$  count the number of defaults that occur over this period i.e.  $v$  is the number of names where  $Z_i < c$  where  $c = \Phi^{-1}(q)$ . The probability that exactly  $j$  of the  $N$  names defaulting over this period is given by

$$P[v = j] = \int_{-\infty}^{\infty} \binom{N}{j} (q|m)^j (1 - (q|m))^{N-j} \phi(m) dm$$

where, from Equation 8,

$$(q|m) = P[Z < c | M = m] = \Phi\left(\frac{c - \sqrt{\rho}m}{\sqrt{1-\rho}}\right).$$

This follows since; conditional on the value of  $M$ , the number of defaults in a homogeneous portfolio follows a binomial distribution. The unconditional distribution can be found by integrating the conditional distribution over the density of  $M$ . We define  $P[v = j]$  over a time horizon of  $t$  as  $\Pi_j^t$ ; since we only work over the first time horizon we will write this as  $\Pi_j$  to avoid confusion.

In their paper Hull and White (2004) present a method for calculating  $\pi_j^t$  recursively without using Monte Carlo simulation. The paper also extends the one factor model to a number of factors.

The model used to produce the numerical results for the remainder of the chapter can be found in the appendix. We follow the five simplifying assumptions made above to illustrate the results in the remainder of the chapter even though the model allows us to relax some of these. The resulting parameter values chosen are the following

1.  $N = 100$  (Number of names in portfolio)
2.  $\lambda = 0.02$  (Intensity of default or hazard rate)
3.  $\rho = 0.5$  (Asset correlation)
4.  $r = 0$  (Risk free short rate)
5.  $k = 10$  (Equity tranche detachment point)
6.  $\delta = 1$  (Loss given default)
7.  $T = 1$  (Time to maturity)

Each time we vary the specific variable under investigation to demonstrate the effect on tranche prices.

### 5.6.1 Tranche sensitivities to asset correlation

Consider the equity tranche with detachment point  $k$  and denote the losses on the tranche by  $l_k^e = \min\{v, k\}$ . Similarly the senior tranche will have attachment point  $k$  and losses given by  $l_k^s = v - \min\{v, k\}$ .

Intuitively the equity tranche should be long correlation and the senior tranche short correlation. This means that the value of the equity tranche should rise when correlation increases while falling with a decrease in correlation. The effect on the senior tranche should be the opposite since total portfolio loss is insensitive to the correlation parameter.

Another way to come to this conclusion is to note that as correlation increases the portfolio of assets start to behave like a single asset since either all or none of the names will default. The value of the equity and senior tranches must then converge as correlation increases to  $\rho = 1$  since the losses in each tranche will be the same percentage of the tranche principal.

What might not be intuitively clear is if the above holds for all equity tranches regardless of attachment point. The proof found in Meng and Sengupta (2008) and repeated below establishes that it holds for all attachment points.

We note that the overall portfolio expected loss is independent of the default correlation structure of the assets in the portfolio. However the expected loss of single tranches in the portfolio will depend on the correlation structure. As the correlation structure changes it distributes losses differently between tranches. Expected losses on some tranches will increase while for others it will decrease with the quantum dependent on the particular tranche

sensitivity to changes in the correlation structure. Equity and senior tranches are usually more sensitive to changes in correlation structure than mezzanine tranches.

For mezzanine tranches a change in correlation can either decrease or increase the expected losses in that tranche. The outcome will depend on the attachment and detachment points as well as the correlation parameter. This also implies that there may not be a unique correlation parameter that will reproduce observed spreads for these tranches. In general mezzanine tranches are less sensitive to changes in correlation than equity or senior tranches.

### Theorem 2

For the equity tranche with detachment point  $k$  and senior tranche with attachment point  $k$  and tranche losses given by  $l_k^e$  and  $l_k^s$  respectively we have that for all  $1 \leq k < N$   $\frac{dE[l_k^e]}{d\rho} < 0$  and  $\frac{dE[l_k^s]}{d\rho} > 0$  for all  $1 \leq k < N$  with  $\rho$  the correlation parameter in the one factor model.

### Proof

$$l_k^e + l_k^s = v$$

$$E[l_k^e] + E[l_k^s] = \mathbb{E}[v]$$

$$E[v] = E[\sum_{i=1}^N \mathbb{I}_{Z_i < c}]$$

$$= N\Phi(c)$$

This is independent of  $\rho$  meaning we only need to prove  $\frac{dE[l_k^e]}{d\rho} < 0$

$$E[l_k^e] = 1\Pi_1 + 2\Pi_2 + \dots + (k-1)\Pi_{(k-1)} + k(1 - \Pi_1 - \Pi_2 - \dots - \Pi_{(k-1)})$$

$$= k - \sum_{j=0}^k (k-j)\Pi_j$$

By substituting the expression for  $\Pi_j$  we find that

$$\frac{dE[l_k^e]}{d\rho} = \int_{-\infty}^{\infty} I_k(q|m) \frac{\partial(q|m)}{\partial\rho} \phi(m) dm$$

with

$$I_k(q|m) = -\sum_{j=0}^k (k-j) \binom{N}{j} [j(q|m)^{j-1} (1-(q|m))^{N-j} - (N-j)(1-(q|m))^{N-j-1} (q|m)^j].$$

In Lemma 1 in Appendix B it is proven that  $I_k(\cdot)$  is a monotone decreasing function, a fact which will be used below.

Now

$$\begin{aligned}\frac{\partial(q|m)}{\partial\rho} &= \frac{\partial\Phi\left(\frac{c-\sqrt{\rho}m}{\sqrt{1-\rho}}\right)}{\partial\rho} \\ &= -\frac{m-c\sqrt{\rho}}{2\sqrt{\rho}(1-\rho)^{3/2}}\phi\left(\frac{c-\sqrt{\rho}m}{\sqrt{1-\rho}}\right)\end{aligned}$$

thus

$$\frac{dE[l_k^e]}{d\rho} = -\int_{-\infty}^{\infty} I_k(q|m) \frac{m-c\sqrt{\rho}}{2\sqrt{\rho}(1-\rho)^{3/2}} \phi\left(\frac{c-\sqrt{\rho}m}{\sqrt{1-\rho}}\right) \phi(m) dm.$$

Substituting  $y = m - c\sqrt{\rho}$  we get

$$\frac{dE[l_k^e]}{d\rho} = -\int_{-\infty}^{\infty} I_k(q|y) \frac{y}{2\sqrt{\rho}(1-\rho)^{3/2}} \phi\left(\frac{c(1-\rho)-\sqrt{\rho}y}{\sqrt{1-\rho}}\right) \phi(y+c\sqrt{\rho}) dy$$

where  $(q|m) = \Phi\left(\frac{c-\sqrt{\rho}m}{\sqrt{1-\rho}}\right) = \Phi\left(\frac{c(1-\rho)-\sqrt{\rho}y}{\sqrt{1-\rho}}\right) = (q|y)$  leading to

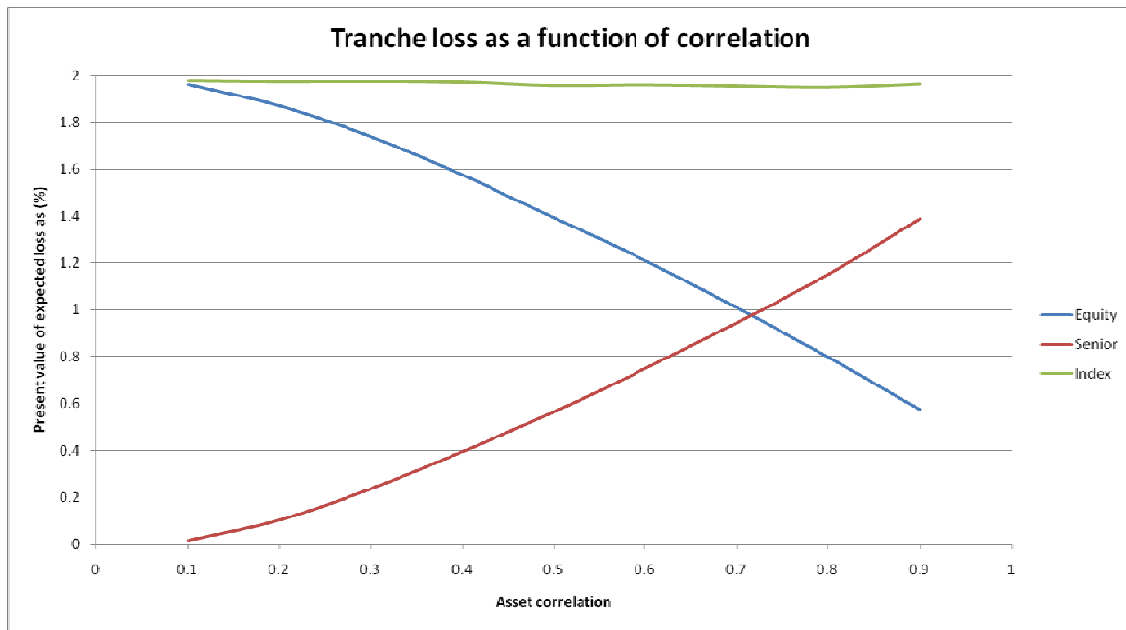
$$\begin{aligned}\frac{dE[l_k^e]}{d\rho} &= -\int_{-\infty}^{\infty} I_k(q|y) \frac{y}{2\sqrt{\rho}(1-\rho)^{3/2}} \frac{1}{2\Pi} e^{-\frac{y^2}{2(1-\rho)} - \frac{c^2}{2}} dy \\ &= -\int_0^{\infty} [I_k(q|y) - I_k(q|-y)] \frac{y}{2\sqrt{\rho}(1-\rho)^{3/2}} \frac{1}{2\Pi} e^{-\frac{y^2}{2(1-\rho)} - \frac{c^2}{2}} dy \\ &< 0.\end{aligned}$$

Since according to Lemma 1  $I_k(\cdot)$  is a monotone decreasing function; leading to the result we set out to prove.

It is of interest to note the similarities between the correlation parameter for CDO valuation and the volatility parameter for equity option valuation. For a CDO the loss variance or volatility will increase with an increase in correlation. This is because the probability of either very few or many defaults is increased. In both cases prices are quoted based on this parameter.

In the figures below we plot the tranche expected loss for the reference portfolio described in page 80 for various correlation parameters. The results are as expected given the preceding theorem.

Figure 24 – Tranche loss as a function of correlation



Using the result from the previous theorem we can calculate ——— for our base portfolio to be ——— . The table below shows the expected tranche losses from 10000 simulations and allows us to confirm the results numerically. The estimated sensitivity to changes in correlation from the base case is ——— .

Table 5 – Changes in expected tranche loss (%) by correlation parameter

Correlation	Equity	Senior	Index
0.45	1.4857	0.4815	1.9672
0.46	1.4682	0.4972	1.9654
0.47	1.4505	0.5133	1.9638
0.48	1.4314	0.5321	1.9635
0.49	1.4109	0.5482	1.9591
0.50	1.3932	0.5640	1.9572
0.51	1.3743	0.5809	1.9552
0.52	1.3581	0.5971	1.9552
0.53	1.3406	0.6145	1.9551
0.54	1.3225	0.6322	1.9547
0.55	1.3018	0.6510	1.9528

### 5.6.2 Tranche sensitivities to changes in default probabilities

Often a position in one of the tranches is hedged by an opposite position on the entire index. The notional investment in the index that hedges the investor against movement in credit spreads is called the delta for that tranche.

In the homogeneous portfolio the default probabilities only depend on the default threshold  $c$ . As such we will be interested in changes in tranche and index loss as the default threshold and hence default probabilities change.

As before  $l_k^e$  and  $l_k^s$  are the losses in the equity and senior tranche respectively with  $l_N$  being the loss on the entire index such that  $l_N = \sum_{i=1}^N \mathbb{I}_{Z_i < c}$ . In what follows we will investigate the delta for the equity tranche of our homogenous portfolio.

By delta we thus mean the factor  $\Delta_{spread}$  such that the hedged position is stationary to first order to changes in  $c$  meaning

$$\frac{dE[l_k^e]}{dc} = \Delta_{spread} \times \frac{dE[l_N]}{dc}.$$

### Theorem 3

The delta of the equity tranche with detachment point  $k$  is  $\Delta_{k,spread} = \Delta_{spread}(\{0,1, \dots, k\})$  where  $\Delta_{k,spread}$  is a probability measure on subsets of  $\{0,1, \dots, k\}$  given by  $\Delta_{spread}(S) = \sum_{k \in S} p_{\Delta_s}(k)$  where  $p_{\Delta_s}(k) = \int_{-\infty}^{\infty} \binom{N-1}{k-1} (q|y)^{k-1} (1-q|y)^{N-k} \frac{1}{\sqrt{2\Pi(1-\rho)}} e^{-\frac{y^2}{2(1-\rho)}} dy$  with  $p_{\Delta_s}(0) = 0$

### Proof

Recall that

$$E[l_N] = E\left[\sum_{i=1}^N \mathbb{I}_{Z_i < c}\right] = N\Phi(c)$$

meaning

$$\frac{dE[l_N]}{dc} = N\phi(c) = \frac{N}{\sqrt{2\Pi}} e^{-\frac{c^2}{2}}$$

while

$$E[l_k^e] = k - \sum_{j=0}^k (k-j) \Pi_j$$

with

$$\Pi_j = \int_{-\infty}^{\infty} \binom{N}{j} (q|m)^j (1-(q|m))^{N-j} \phi(m) dm$$

and

$$(q|m) = \Phi\left(\frac{c-\sqrt{\rho}m}{\sqrt{1-\rho}}\right).$$

Similar to before we then have that

$$\frac{dE[l_k^e]}{dc} = \int_{-\infty}^{\infty} I_k(q|y) \frac{1}{2\Pi\sqrt{(1-\rho)}} e^{-\frac{y^2}{2(1-\rho)} - \frac{c^2}{2}} dy$$

with

$$I_k(q|m) = -\sum_{j=0}^k (k-j) \binom{N}{j} [j(q|m)^{j-1} (1-(q|m))^{N-j} - (N-j)(1-(q|m))^{N-j-1} (q|m)^j].$$

Now

$$\frac{\partial(q|m)}{\partial c} \phi(m) = \frac{1}{2\Pi\sqrt{(1-\rho)}} e^{-\frac{(m-c\sqrt{\rho})^2}{2(1-\rho)} - \frac{c^2}{2}}$$

And after substituting  $y = m - c\sqrt{\rho}$  and  $\Phi\left(\frac{c(1-\rho)-\sqrt{\rho}y}{\sqrt{1-\rho}}\right) = (q|y)$  as before we get

$$\frac{dE[l_k^e]}{dc} = \int_{-\infty}^{\infty} I_k(q|y) \frac{1}{2\Pi\sqrt{(1-\rho)}} e^{-\frac{y^2}{2(1-\rho)} - \frac{c^2}{2}} dy$$

thus

$$\Delta_{spread} = \frac{1}{N} \int_{-\infty}^{\infty} I_k(q|y) \frac{1}{\sqrt{2\Pi(1-\rho)}} e^{-\frac{y^2}{2(1-\rho)}} dy.$$

From Lemma 1 we have that

$$I_k(p) = N \sum_{j=1}^k \binom{N-1}{j-1} p^{j-1} (1-p)^{N-j}.$$

meaning

$$\Delta_{spread} = \sum_{j=1}^k p_{\Delta_s}(j) \text{ where } p_{\Delta_s}(j) = \int_{-\infty}^{\infty} \binom{N-1}{j-1} (q|y)^{j-1} (1-q|y)^{N-j} \frac{1}{\sqrt{2\Pi(1-\rho)}} e^{-\frac{y^2}{2(1-\rho)}} dy.$$

The tranche delta for the Gaussian model is thus a probability measure on the loss levels. For the senior tranche the delta will be 1 less the delta of the equity tranche in our portfolio of two tranches.

In addition we can observe that changes in delta with respect to changes in spreads are negative for the equity tranche and positive for the senior tranche.

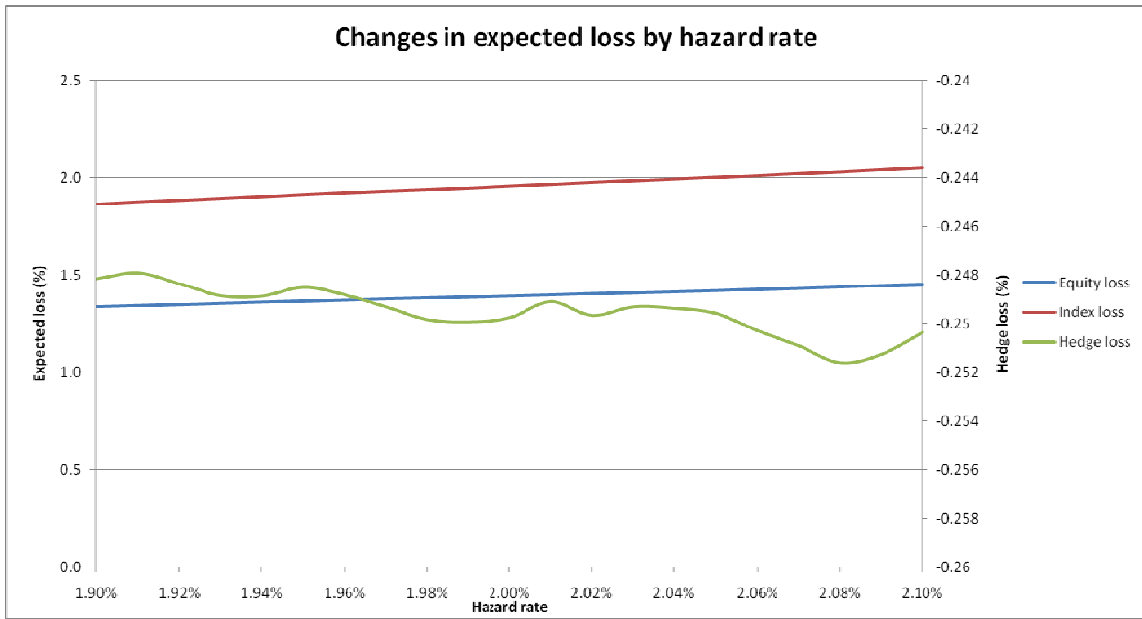
This follows by looking at  $\Delta_{spread} = \frac{1}{N} \int_{-\infty}^{\infty} I_k(q|y) \frac{1}{\sqrt{2\Pi(1-\rho)}} e^{-\frac{y^2}{2(1-\rho)}} dy$  and recalling that  $I_k(\cdot)$  is monotonically decreasing while  $(q|y)$  is increasing in  $c$ .

Continuing with our numerical illustrations we calculated the delta for the equity tranche,  $\Delta_{10,0.02}$ , to equal 0.5842 using the previous theorem. The details of the calculations can be found in the appendix. We now calculate the present value of the expected loss for a hedged position consisting of a short unit notional in the equity tranche and an opposite



position of 0.5842 units in the index for changes in our base hazard rate based on 10,000 simulations.

Figure 25 – Changes in expected loss by hazard rate for delta hedged position



The numbers represented in the above figure are repeated in the table below. From this we can see the hedge to be effective as changes in the expected loss of the position are less than 0.1 basis points for small changes in the hazard rate around the value of 2%.

Table 6 – Changes in expected loss (bps) by hazard rate for delta hedged position

Hazard	c	Equity loss	Senior loss	Index loss	Hedge loss
1.90%	-2.0787	1.3379	0.5274	1.8653	-0.2481
1.91%	-2.0766	1.3435	0.5318	1.8753	-0.2479
1.92%	-2.0745	1.3495	0.5353	1.8848	-0.2483
1.93%	-2.0724	1.3553	0.5386	1.8939	-0.2488
1.94%	-2.0703	1.3611	0.5427	1.9038	-0.2488
1.95%	-2.0682	1.3667	0.5473	1.9140	-0.2485
1.96%	-2.0661	1.3725	0.5509	1.9234	-0.2488
1.97%	-2.0640	1.3779	0.5539	1.9318	-0.2493
1.98%	-2.0620	1.3827	0.5564	1.9391	-0.2498
1.99%	-2.0599	1.3880	0.5600	1.9480	-0.2499
2.00%	-2.0579	1.3932	0.5640	1.9572	-0.2498
2.01%	-2.0558	1.3982	0.5687	1.9669	-0.2491
2.02%	-2.0538	1.4042	0.5720	1.9762	-0.2497
2.03%	-2.0518	1.4091	0.5761	1.9852	-0.2493
2.04%	-2.0498	1.4143	0.5797	1.9940	-0.2494
2.05%	-2.0478	1.4196	0.5831	2.0027	-0.2496
2.06%	-2.0458	1.4261	0.5865	2.0126	-0.2503
2.07%	-2.0438	1.4322	0.5898	2.0220	-0.2509
2.08%	-2.0418	1.4384	0.5930	2.0314	-0.2516
2.09%	-2.0398	1.4443	0.5978	2.0421	-0.2513
2.10%	-2.0379	1.4497	0.6032	2.0529	-0.2503

From the above table we can estimate the hedge ratio as  $\frac{(1.3667-1.4196)}{(1.9140-2.0027)} = 0.5964$  which is similar to the theoretically calculated value of 0.5842.

### 5.6.3 Tranche convexity with respect to credit spreads

Consider a portfolio consisting of a position in the equity tranche hedged with an opposite exposure to the index. The expected loss of this portfolio is given by the negative of

$$V_k(\Delta) = \Delta_{spread} \times E[l_N] - E[l_k^e].$$

The convexity of the position is defined as the second order increment of the value of the position with respect to changes in credit spreads. We denote the convexity by  $\Gamma_k$  where

$$\Gamma_k = \frac{\partial^2 V_k(\Delta)}{\partial c^2}.$$

**Theorem 4**

Let  $\Gamma_k = \frac{\partial^2 V_k(\Delta)}{\partial c^2}$  then  $\Gamma_k > 0 \forall k \in \{1, 2, \dots, N-1\}$ .

**Proof**

From the previous theorem we have that

$$\frac{dE[l_k^e]}{dc} = \int_{-\infty}^{\infty} I_k(q|y) \frac{1}{2\Pi\sqrt{(1-\rho)}} e^{-\frac{y^2}{2(1-\rho)} - \frac{c^2}{2}} dy$$

where

$$I_k(p) = N - (N-k)k \binom{N}{k} \int_0^p t^{k-1} (1-t)^{N-k-1} dt$$

and

$$\frac{dE[l_N]}{dc} = \frac{N}{\sqrt{2\Pi}} e^{-\frac{c^2}{2}}.$$

Taking the derivatives of the above we find that

$$\frac{\partial^2 E[l_k^e]}{\partial c^2} = \frac{1}{2\Pi\sqrt{(1-\rho)}} \int_{-\infty}^{\infty} \frac{\partial I_k(q|y)}{\partial c} e^{-\frac{y^2}{2(1-\rho)} - \frac{c^2}{2}} - c I_k(q|y) e^{-\frac{y^2}{2(1-\rho)} - \frac{c^2}{2}} dy$$

where

$$\frac{\partial I_k(q|y)}{\partial c} = -(N-k)k \binom{N}{k} (q|y)^{k-1} (1-q|y)^{N-k-1} \frac{\sqrt{1-\rho}}{\sqrt{2\Pi}} e^{-\frac{(c(1-\rho) - \sqrt{\rho}y)^2}{2(1-\rho)}}$$

with

$$\frac{\partial^2 E[l_N]}{\partial c^2} = \frac{-cN}{\sqrt{2\Pi}} e^{-\frac{c^2}{2}}.$$

We then have

$$\Gamma_k = \frac{\partial^2 V_k(\Delta)}{\partial c^2} = \Delta_{spread} \times \frac{\partial^2 E[l_N]}{\partial c^2} - \frac{\partial^2 E[l_k^e]}{\partial c^2}$$

where, from the previous theorem,

$$\Delta_{spread} = \frac{1}{N} \int_{-\infty}^{\infty} I_k(q|y) \frac{1}{\sqrt{2\Pi(1-\rho)}} e^{-\frac{y^2}{2(1-\rho)}} dy$$

meaning

$$\begin{aligned}\Gamma_k &= \frac{-cN}{\sqrt{2\Pi}} e^{-\frac{c^2}{2}} \frac{1}{N} \int_{-\infty}^{\infty} I_k(q|y) \frac{1}{\sqrt{2\Pi(1-\rho)}} e^{-\frac{y^2}{2(1-\rho)}} dy \\ &\quad - \frac{1}{2\Pi\sqrt{(1-\rho)}} \int_{-\infty}^{\infty} \frac{\partial I_k(q|y)}{\partial c} e^{-\frac{y^2}{2(1-\rho)} - \frac{c^2}{2}} - c I_k(q|y) e^{-\frac{y^2}{2(1-\rho)} - \frac{c^2}{2}} dy \\ &= -\frac{1}{2\Pi\sqrt{(1-\rho)}} \int_{-\infty}^{\infty} \frac{\partial I_k(q|y)}{\partial c} e^{-\frac{y^2}{2(1-\rho)} - \frac{c^2}{2}} dy.\end{aligned}$$

Now because  $k \in \{1, 2, \dots, N-1\}$

$$\frac{\partial I_k(q|y)}{\partial c} < 0$$

which leads us to conclude

$$\Gamma_k > 0.$$

By proceeding in a similar fashion as before we calculated the convexity of the equity tranche to be 1.8187 by using the previous theorem, as before the details are to be found in Appendix A. To estimate the convexity from simulated observations we again use Table 6 – Changes in expected loss (bps) by hazard rate for delta hedged position and estimate

$$\Gamma_k \approx \left( \frac{-0.2498+0.2481}{2.0579-2.0787} - \frac{-0.2503+0.2498}{2.0379-2.0579} \right) / (-2.0478 + 2.0682) = 2.3327.$$

## 5.7 The correlation smile

The introduction of credit default swaps on the Dow Jones CDX and iTraxx indices along with standardized tranches on these indices increased the liquidity and transparency of the CDO market. This increasingly led to the quotation of tranche prices not in terms of yield or price but rather through implied correlation. Implied correlation is the value of the correlation parameter which, for that tranche, will make its spread equal to the observed market spread using the standard market model. This is similar to the way in which implied volatility in the Black-Scholes model is used to price equity options.

The standard market model is the one factor Gaussian copula model described in the previous section. Here we use a single, constant, parameter  $\rho$  to summarise all dependence between the default times of the various obligors. We can list the main assumptions we made in the preceding section as follows:

1. The dependence structure of asset returns of the names in the index is given by a Gaussian copula.
2. The correlations among asset returns are driven by a single common factor.

3. All pair-wise correlations among asset returns are identical. Resulting in only one dependence parameter in the model.
4. All the credits in the index have identical spreads.
5. Recovery rates on underlying credits are homogeneous across credits and are independent of default rates.

The obvious attraction of these assumptions is the resulting simplicity of the model and the ease with which it can be implemented. Because the correlation structure is represented by a low number of systematic factors the portfolio loss distribution and tranche prices can be computed efficiently using recursive methods or Fourier transforms. Examples of such methods can be found in Anderson, Sidenius and Basu (2003).

This simplicity might explain the popularity of the model in the market and the reason for it becoming the reference for pricing portfolio credit derivatives.

As is usually the case there is a price to be paid for such tractability. When computing implied correlations for various tranches on the same index these should, in theory, be equal. In practice this is not observed and a correlation smile, analogous to the volatility smile for the Black-Scholes model, is found when plotting implied correlation against tranche detachment points.

According to Ağca, Agrawal and Islam (2008) the main reasons for the existence of the correlation smile can be attributed to the following.

1. The use of a Gaussian copula instead of a more fat-tailed copula
2. The use of homogeneous correlation instead of heterogeneous correlation amongst pairs of names.
3. Homogeneous constant credit spreads instead of heterogeneous credit spreads for the various names constituting the index.
4. Uncorrelated constant default and recovery rates rather than correlated ones.

The following are the main problems caused by the existence of a correlation smile according to Hager and Schöbel (2006a)

Firstly the implied correlation on traded tranches cannot be used to trade in off-market tranches with different attachment points.

Secondly implied correlation suffers existence and uniqueness problems since for mezzanine tranches expected losses are not monotone in correlation while some tranche spreads may not be attainable by a choice of correlation.

Thirdly the existence of the smile shows that market participants do not believe the model is correctly distributing the risk of the index to the equity, mezzanine and senior tranches. The Gaussian copula is known to overestimate the risk of only a few names defaulting and underestimating the risk of a very high or low number defaulting.

Finally the existence of the correlation smile might lead to wrong relative value assessments since the real dependence structure of the index is neglected. The strong simplifying assumptions above can cause implied correlation to be different from the true correlations leading smile shaped pattern even if tranches are fairly priced.

The main focus of recent CDO research is aimed at the relaxation of the assumptions of the standard model in order to replicate the correlation smile. This is accomplished by introducing additional parameters and hence degrees of freedom to the model in order to facilitate the fitting of the model to an observed correlation smile. In the next chapter we study various such adjustments.

## Chapter 6 Extensions to the market model

### 6.1 Introduction

The results of Burtschell, Gregory and Laurent (2005) suggest that credit spread dynamics are of lesser importance when valuing CDO tranches. Of the main assumptions made by the standard model the most important when considering CDO pricing are the correlation parameter, correlation structure and link between default and recovery rates.

In this chapter we study various methods for introducing additional degrees of freedom to the standard model by relaxing some of these assumptions. The aim is to obtain a model that is flexible enough to replicate observed tranche prices and the correlation smile yet tractable enough for real world application.

### 6.2 Correlation parameter

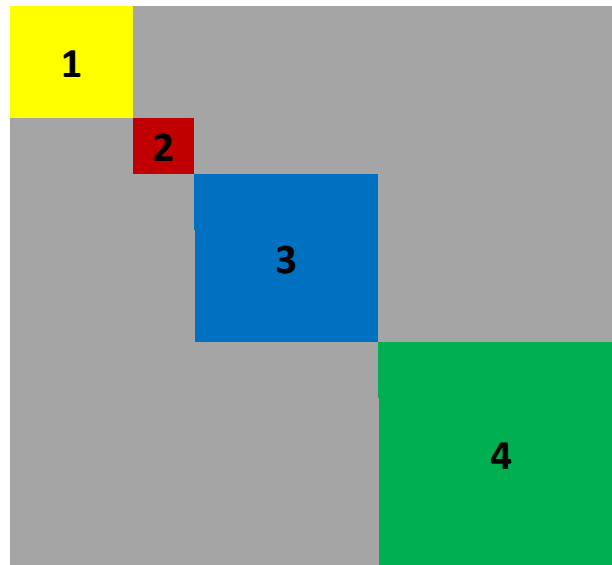
The standard model already has a number of free parameters in the correlation matrix that go unused because of the assumption of equal pairwise correlations. The simplest extension to the model can be to utilize the degrees of freedom readily available and specify a more complex correlation structure via the correlation matrix. This would imply that we drop the assumption of equal pairwise correlation and permit any correlation matrix. This method is capable of reproducing correlation smiles and is illustrated in Gregory and Laurent (2004).

The problem then becomes one of finding a correlation matrix that will fit the observed tranche spreads. Hager and Schöbel (2006b) investigate various numerical methods for doing this. Typically there will not be a unique matrix that fits the given tranche spreads.

In theory one can construct such correlation matrix by grouping the names comprising the CDO into a number of homogeneous sectors. Each sector will then be assigned an intra sector correlation and a lower inter sector correlation common to all sectors. The result is that names within the same sector will have a higher correlation with each other than names in another sector. The resulting correlation matrix has a block like structure represented in the figure below.

Each of the coloured blocks represents a single industry sector with all correlations for that sector being equal; this intra sector correlation may vary from sector to sector. The grey area represents the entries with the lower inter sector correlation. This reminds one of the correlation indices first used by CreditMetrics.

Figure 26 – Typical correlation matrix based on industry sectors



To illustrate a change in the correlation matrix can result in a correlation smile we extend our model used in the previous chapter by

1. addition of a mezzanine tranche with attachment 3% and detachment 10%
2. extending the term to maturity to five years
3. replacing the exchangeable correlation matrix with a matrix structured like Figure 26 – Typical correlation matrix based on industry sectors
4. assuming a loss given default of 40%
5. assuming a constant short rate of interest of 5%
6. assuming constant credit spreads of 1%

The result of these changes is that the model is now parameterised in similar way that found in Hager and Schöbel (2006a).

First we illustrate that the correlation matrix does not uniquely specify tranche yields. We calculate the new tranche prices using the previous flat correlation parameter of 0.5 and repeat with a structured correlation matrix. This matrix consists of 5 equal clusters of 20 names, each with a specific intra sector correlation. The intra sector correlations are 0.9754, 0.8994, 0.6069, 0.4700 and 0.4281 respectively with a inter sector correlation of 0.3911. As previously mentioned the code used can be found in Appendix A.



Table 7 – Simulation results illustrating correlation parameter does not uniquely specify tranche prices

	Correlation	Equity	Mezzanine	Senior	Index
<b>PV Loss %</b>	Flat	0.49	0.54	0.70	1.73
	Structured	0.49	0.53	0.71	1.73
<b>Yield%</b>	Flat	6.72	2.45	0.20	0.47
	Structured	6.75	2.42	0.21	0.47

The next step will be to show how the model can generate a correlation smile. To do this we changed the correlation matrix and altered the sector correlations to 0.9, 0.8, 0.5, 0.8 and 0.9 respectively with the inter sector correlation of 0.2. Next we calculate the implied correlation for each tranche as the correlation parameter in the standard model that will give the same expected tranche loss. The results in the following table show a clear correlation smile.

Table 8 – Implied correlations for simulations using a correlation parameter containing sector correlations

	Equity	Mezzanine	Senior	Index
<b>PV Loss %</b>	0.52	0.62	0.59	1.73
<b>Yield %</b>	10.47	2.97	0.14	0.47
<b>Implied Correlation</b>	0.46	0.35	0.425	

In practice when the correlation matrix is set to represent industry or geographic correlations the resulting correlation smile is usually much weaker than that implied by market prices.

Gregory and Laurent (2004) show how Equation 7 can be modified to accommodate a structured correlation matrix. In essence the correlation within a specific sector will follow the familiar one factor structure with the market risk factors replaced by a sector factor. A similar equation will then relate the different sector factors to the market factor. We can then write

Equation 9

$$Z_i = a_i b_i M + a_i \sqrt{1 - b_i^2} X_i + \sqrt{1 - a_i^2} Y_i$$

where  $Z, M, X$  and  $Y$  are independent normal variables.

To calibrate the model we set

$$a_i = \sqrt{\rho_k} \text{ and } b_i = \sqrt{\frac{\rho}{\rho_k}}$$

with  $\rho_k$  the intra sector correlation for name  $i$  and  $\rho$  the inter sector correlation.

This approach can be extended by making the correlation of a particular name dependent on the credit spread of that name. Equation 7 then becomes

$$Z_i = f(s_i)M + \sqrt{1 - (f(s_i))^2} X_i$$

where  $f(s_i)$  is a function mapping the credit spread to a correlation parameter.

To replicate the correlation smile seen in practice  $f(s_i)$  needs to be decreasing since the investor in the senior tranche is more affected by the names with lower credit spreads. According to Anderson and Sidenius (2004) this approach has a number of drawbacks when fitted to market data.

1. There is no empirical evidence that low default risk firms are more correlated with the market than higher risk ones.
2. Some entries in the correlation matrix can be unrealistically close to 0 or 1.
3. For homogeneous credit spreads there will be no correlation smile
4. Correlation will change daily as credit spreads are updated.

## 6.3 Correlation structure

### 6.3.1 The Student t copula

Recall that for Equation 7 it is often convenient to think of  $Z_i$  and  $c_i$  as the asset and liquidation value of the firm respectively. The implicit assumption in the one factor model is thus that the assets values of firms are given by a multivariate normal distribution, in particular one with a single correlation value specified for the entire correlation matrix. In the previous section we extended the approach to allow for any correlation structure while keeping the assumption of normally distributed asset values.

However there is much evidence to suggest that like equity returns asset returns are not normally distributed. In particular the normal distribution underestimates the risk of extreme co-movements in equity values. Although asset returns are not observable the dependence structure of equity returns are often used as a proxy with the potential drawbacks mentioned in section 5.4.

To avoid these problems Mashal, Naldi and Zeevi (2003) uses an approach based on standard option pricing theory to infer the value and volatility of assets values from equity prices by deleveraging the equity. This is similar to the method employed by KMV's CreditEdge product. They conclude that the dependence structure of asset returns is similar to that of equity returns and that the assumption of a Gaussian dependence structure can be rejected with a high degree of confidence.

Based on a formal test they find that the Student copula fits assets return data better than other popular copulas. In their paper Breymann, Dias and Embrechts (2003) also find this copula to fit financial data better than others over longer time horizons.

To extend the Gaussian copula to the Student-t copula is fairly simple. We start by recalling Equation 7 for the Gaussian model

$$Z_i = a_i M + \sqrt{1 - a_i^2} X_i.$$

We now modify this equation such that  $Z_i$  follows a Student t distribution with  $\nu$  degrees of freedom letting

Equation 10

$$Z_i = \sqrt{W}(a_i M + \sqrt{1 - a_i^2} X_i)$$

where  $\frac{\nu}{W}$  follows a  $\chi^2$  distribution with parameter  $\nu$  with  $W$  independent from  $M$  and  $X_i$ .

The covariance between  $Z_i, Z_j, i \neq j$  is given by  $a_i a_j \frac{\nu}{\nu-2}$  and the time until default of a particular entity by  $\tau_i = F_i^{-1}(t_\nu(Z_i))$ .

We again make the simplifying assumption that  $a_i = \sqrt{\rho}$  for all  $i$ . Conditional on the values of  $M$  and  $W$  defaults are independent and Equation 8 becomes

Equation 11

$$q|m = P[Z < c|M = m, W = w] = \Phi\left(\frac{W^{-\frac{1}{2}} t_\nu^{-1}(q) - \sqrt{\rho} m}{\sqrt{1-\rho}}\right).$$

The model has now been extended to two factors,  $M$  and  $W$ , while previously we only had  $M$  as the common factor influencing time until default.

We know that the Student t copula will have the same Kendall rank correlation measure as the Gaussian model but need to consider how the tail dependence influence tranche values and if the model is able to produce a correlation smile.

We use same parameters for our model as given at the start of this chapter but replace the Gaussian copula with a Student t copula with 6 and 12 degrees of freedom respectively.

Table 9 – Simulation results for Student t copula

	DF	Equity	Mezzanine	Senior	Index
PV Loss %	6	0.40	0.49	0.85	1.74
	12	0.44	0.51	0.75	1.71
	$\infty$	0.49	0.54	0.70	1.73

From the above table we can see that using a Student copula has much the same effect as increasing the correlation between all entities. Clearly the degrees of freedom used have a large impact on the tails of the distribution; this impact increases as one moves further into the tails of the distribution (Frey and McNeil (2001)). The table below shows the resulting implied correlation. The copula produces a weak correlation smile effect.

Table 10 – Implied correlation for the Student t copula

	DF	Equity	Mezzanine	Senior
<b>Implied</b>	6	0.61	0.58	0.61
<b>Correlation</b>	12	0.56	0.54	0.54

To match market data the model must increase the proportion of losses allocated to the equity tranche and the most senior tranches and decrease the losses allocated to the intermediate tranches.

To achieve this Hull and White (2004) alter Equation 7 such that both the market factor and idiosyncratic factor follow t-distributions. This equation then becomes

Equation 12

$$Z_i = a_i \sqrt{\frac{v_M - 2}{v_M}} M + \sqrt{1 - a_i^2} \sqrt{\frac{v_X - 2}{v_X}} X_i,$$

where  $M$  and  $X_i$  follow independent Student t distributions with  $v_M$  and  $v_X$  degrees of freedom. Since the Student t distribution is not stable under convolutions  $Z_i$  no longer has a Student t distribution and the copula of  $Z_i$  is not a Student copula. The distribution function  $H$  of  $Z_i$  will thus need to be computed numerically. The conditional probability of a name defaulting is given by a modified version of Equation 8

Equation 13

$$(q|m) = t_{v_X} \left( \frac{\sqrt{\frac{v_X}{v_X - 2}} H^{-1}(F(1)) - \sqrt{\rho} \sqrt{\frac{v_M - 2}{v_M}} M}{\sqrt{1 - \rho}} \right),$$

as found in Burtschell, Gregory and Laurent (2005). This approach is an example of a more general class of one factor mean-variance mixtures more examples of which are found in Frey and McNeil (2003).

Hull and White (2004) have found this double t copula model to fit market data quite well. If the tails of the systematic factor  $M$  are bigger than that of the idiosyncratic factor  $X$  extreme values  $Z$  are more likely to be due to extreme values of  $M$  than extreme values of  $X$ . Since  $M$  affects the values of all  $Z$  this has a similar effect of increasing the correlation parameter.

Burtschell, Gregory and Laurent (2005) quote some measures of tail dependence for the double t copula.

If  $v_M < v_X$  the systematic factor dominates in the tails of the distribution and  $\lambda_U = \lambda_L = 1$ , conversely if  $v_X < v_M$  there is no tail dependence and  $\lambda_U = \lambda_L = 0$ .

If  $v_X = v_M = v$  then the measure of tail dependence is given by

$$\lambda_U = \lambda_L = \frac{1}{1 + \left(\frac{\sqrt{1-\rho}}{\sqrt{\rho}}\right)^v}$$

### 6.3.2 The Clayton copula

Rogge and Schonbucher (2003) argue that the Gaussian and, even more so, the Student t copula imply unrealistic term structures of default probabilities. By term structure of default probabilities we mean the distribution of the default times conditional on information that may be available at a later date. An example of this information will be the default times of obligors who have defaulted by that time while for non-defaulters we only know that the default time exceeds that particular date.

By using the Clayton copula we implicitly assume that the systematic risk factor  $M$  has a Gamma distribution with shape parameter  $1/\theta$  where  $\theta > 0$  and scale parameter 1. The Laplace transform of  $M$  is given by

$$\psi(s) = E[e^{-sM}] = (1 + s)^{-1/\theta},$$

with the inverse of  $\psi(s)$  and the generating function of the copula given by

$$\psi^{-1}(u) = u^{-\theta} - 1.$$

The idiosyncratic factors  $X_i$  are assumed independently uniformly distributed and independent of  $M$ .

Equation 7 then takes the form

Equation 14

$$Z_i = \psi\left(\frac{-\ln(X_i)}{M}\right),$$

with  $Z_i$  uniformly distributed and the time until default given by

Equation 15

$$\tau = \inf\{t: F(t) \geq Z_i\} = F^{-1}(Z_i),$$

and Equation 8 now becoming

Equation 16

$$(q|m) = \exp(m(1 - F(1)^{-\theta})).$$

This equation immediately reminds of Equation 2 and we observe that the factor  $M$  enters the hazard rate multiplicatively. For low values of  $M$  the probability of early default is increased.

Since conditional on the value of  $M = m$  defaults are independent we can find the joint distribution of default times by following Laurent and Gregory (2003) and writing

$$\begin{aligned}
 & F(t_1, \dots, t_N) \\
 &= P[\tau_1 < t_1, \dots, \tau_N < t_N] \\
 &= \int_0^\infty \prod_{i=1}^N \exp\left(m(1 - F(t_i)^{-\theta})\right) f_M(m) dm \\
 &= \psi\left(-\sum_{i=1}^N (1 - F(t_i)^{-\theta})\right) \\
 &= \psi\left(\sum_{i=1}^N \psi^{-1}(F(t_i))\right) \\
 &= \psi(\psi^{-1}(u_1) + \dots + \psi^{-1}(u_N)).
 \end{aligned}$$

This is the form of an Archimedean copula with generating function  $\psi^{-1}$  and more specifically a Clayton copula with parameter  $\theta$ .

Schönbucher and Schubert (2001) explain in detail how the choice of copula influences the term structure of default probabilities and in particular the hazard rate process. Up until now we have used the hazard rate  $\lambda$  as introduced in Chapter 2 to characterize the marginal distributions of default times. This hazard rate is referred to as the pseudo hazard rate by Schönbucher and Schubert.

The reason for this is that it will only coincide with the actual hazard rate if defaults are independent, when we restrict our information to that of a single obligor or at time  $t = 0$ . In general, at future times, this pseudo hazard rate will not equal the actual hazard rate.

To see why this is the case consider the fact that the values of  $Z_i$  and hence  $\tau_i$  are dependent on each other by way of the common factor  $M$ . As time progresses the distribution function of default times needs to be evaluated as a conditional distribution function. Without loss of generality if the first  $k$  obligors have defaulted by time  $t$  the conditional distribution function will be

$$P_t[\boldsymbol{\tau} \leq T] = P[\boldsymbol{\tau} \leq T | \tau_i = t_i \{1 \leq i \leq k\} \wedge \tau_j > t \{k < j \leq N\}].$$

The conditional distribution will still be a distribution function on the unit hyper cube but the marginal distributions will not be uniform.

From Equation 2 we have that for a single obligor

$$P[\tau_i \leq T | \tau_i > t] = 1 - \exp\left(-\int_t^T \tilde{\lambda}_i(s) ds\right)$$

where  $\tilde{\lambda}_i$  is the actual hazard rate. In general we will only have  $\tilde{\lambda}_i = \lambda_i$  when  $t = 0$ .

It is evident that the information about the default or survival of other obligors must influence the dynamics of  $\tilde{\lambda}_i$  since they influence the conditional default probability. Schönbucher and Schubert show that the dynamics of  $\tilde{\lambda}_i$  will depend the dynamics of  $\lambda_i$  as well as the particular copula function governing the dependence structure.

For the Clayton copula the dynamics of  $\tilde{\lambda}_i$  simplifies to

$$\tilde{\lambda}_i(t) = \left( \frac{\bar{C}(F(t))}{S(t)} \right)^\theta \lambda_i(t)$$

prior to the default of any obligor. When a default occurs the hazard rate of all remaining obligors change by the same factor and we have for  $j \neq i$

$$\tilde{\lambda}_i(\tau_j) = (1 + \theta)\tilde{\lambda}_i(\tau_j^-).$$

This is same idea put forward by Davis and Lo (1999) where the default of one obligor puts the remainder of the portfolio at increased risk of default. Here this is expressed in terms of a copula function which simplifies the calculation of the distribution of times until default.

To compare our results to those produced previously we choose  $\theta = 1$  such that Kendall's rank correlation measure is the same as for the Gaussian and Student copulas with  $\rho = 0.5$ .

**Table 11 – Simulation results for Clayton copula**

	Copula	Equity	Mezzanine	Senior	Index
<b>PV Loss</b>	Clayton	0.21	0.33	1.20	1.73
<b>%</b>	Gaussian	0.49	0.54	0.70	1.73

From the simulation results it is clear that the Clayton copula gives much different tranche premiums to the Gaussian and Student copulas for the same level of rank correlation. These results are consistent to those found in Burtschell, Gregory and Laurent (2005). In particular the Clayton copula shows increased probability of multiple defaults for the same rank correlation when compared to the Gaussian copula. This does not come as a surprise when we recall that lower tail dependence is a feature of this copula.

To better examine the impact of this copula we reduced the rank correlation such that the equity tranche has the same expected loss as under the standard model. We find that the expected loss for the mezzanine and senior tranches are very similar to those computed under the Gaussian copula. The figures below were obtained with  $\theta = 0.32$  which implies a rank correlation of 0.138 compared to the 0.3 of the Gaussian copula.

Table 12 – Clayton copula compared to Gaussian copula with same equity tranche loss

	Copula	Equity	Mezzanine	Senior	Index
<b>PV Loss</b>	Clayton	0.49	0.55	0.68	1.73
<b>%</b>	Gaussian	0.49	0.54	0.70	1.73

## 6.4 Stochastic recovery rates

The importance of recovery rate modelling in the assessment of credit risk is well known. A number of studies have highlighted the negative correlation between default and recovery rates (Altman et al (2005), Acharya, Bharath and Srinivasan (2007) and Frye (2000))

Prior to the credit crisis of 2007 CDO models could fit market data without turning to stochastic recovery rates. CDO instruments were quoted on the implied correlation and a constant recovery rate, typically 40%, was assumed (Hull and White (2004)). The result was that the recovery distribution and its correlation with default rates were severely underspecified.

As the credit crisis progressed market participant could no longer calibrate their models to the spreads observed for senior tranches. Laurent Amraoui Cousot and Hitier (2009) contribute this breakdown to the correlation between default and recovery rates.

Losses are suffered on senior tranches when the economy is in a bad state and consequently recovery rates will be low. From the viewpoint of an investor in a senior tranche the recovery rate in which he or she should be interested in is the recovery rate conditional on the tranche suffering losses. For an investor in a senior tranche this will not equal the expected recovery rate used for pricing a normal credit default swap on the same underlying.

As research progressed and alternatives to the Gaussian copula were investigated and a number of authors considered stochastic recovery rates (Anderson and Sidenius (2004), Hull and White (2004) and Gregory and Laurent (2004)). By using state dependent stochastic recovery rates we can link recovery rates to the same common factor driving defaults and thus introduce correlation between default and recovery rates.

If recovery rates are stochastic but not correlated to default rates the impact of stochastic recovery rates are small. However, when the two rates are correlated the impact is substantial, particularly for the right tail of the portfolio loss distribution.

To introduce stochastic recovery rates we proceed by letting recovery rates depend on a market factor  $M_T$ . We set



Equation 17

$$R_i = a_r M_r + \sqrt{1 - a_r^2} Y_i$$

where  $M_r$  is the market factor and  $a_r^2$  is the correlation of recovery rates with the market factor, we assume  $M_r$  and all  $Y_i$  are standard normal variables and independent of each other. In addition we recall the similar specification of the default factor from Equation 7

$$Z_i = a_i M + \sqrt{1 - a_i^2} X_i.$$

We thus have a two factor Gaussian structure where we assume the market and idiosyncratic factors are independent but allowing for correlation between  $M$ ,  $M_r$  and  $Y_i$ ,  $X_i$  respectively. This is the same specification as found in Gregory and Laurent (2004).

The realized recovery is obtained in the same way as the default times by a percentile to percentile mapping, setting the recovery rate equal to

$$\delta_i = H^{-1}(\Phi(R_i))$$

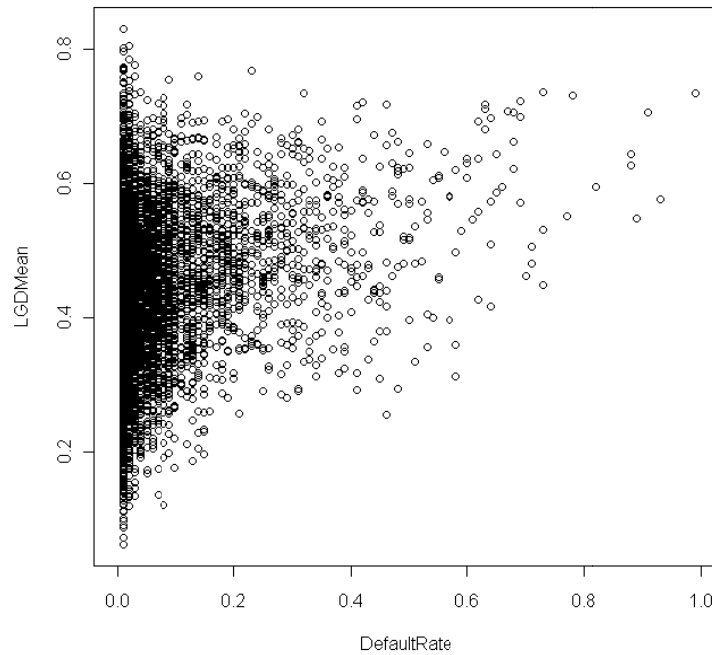
with  $H$  the distribution function of the recovery rate.

Like Gregory and Laurent (2004) we chose a Beta distribution for the recovery rate. This automatically ensures that  $\delta_i \in [0,1]$ . Evidently this is not the only choice available, for example Anderson and Sidenius (2004) use a cumulative Gaussian distribution.

To keep our results consistent we let  $\delta_i$  follow a Beta distribution with expected value 60% and standard deviation of 15%. The expected LGD will then remain at 40% as previously specified. For an initial implementation we choose  $a_r^2 = 0.5$  with the correlation of  $M$  and  $M_r$  set at 0.5. The idiosyncratic factors for default and loss given default remain uncorrelated.

The results below show 10,000 simulations using the above specification. Correlation introduced between default and recovery rates is clear with scenarios where default rates are higher also leading the higher average losses.

Figure 27 – Index default and average loss given default with correlated stochastic recovery rates



According to Hull and White (2004) the expected loss for the portfolio will increase when defaults and recoveries are correlated while there will be little change to the loss distribution if recoveries are stochastic but independent of defaults. This is because if recoveries are correlated we will place a bigger weight on high default scenarios meaning the effect or correlated recoveries will be greatest for the more senior tranches. The table below shows tranche losses for the correlated model compared to the standard model.

Table 13 – Tranche losses for correlated and stochastic recovery rates compared to constant recovery rates.

	Copula	Equity	Mezzanine	Senior	Index
<b>PV Loss %</b>	Gaussian – Constant LGD	0.49	0.54	0.70	1.73
	Gaussian – Correlated LGD	0.52	0.62	0.94	2.08

To obtain the same expected loss the default probabilities thus need to be decreased (Hull and White (2004)). To set the correct level for the hazard rates we calculate the spread on a single name CDS such that it is the same as for the uncorrelated case. The tranche prices are then computed using this reduced hazard rate.

From our previous results the annual CDS premium on a single name is  $\quad$  basis points under the assumptions given in Section 6.2. With the recovery specification as above the hazard rates needs to be reduced to 0.82% in order for the CDS premium to equal the

previously computed value. We now compute the tranche premiums under stochastic recovery rates and reproduce Table 13 with the reduced hazard rate.

Table 14 – Tranche losses for correlated and stochastic recovery rates compared to constant recovery rates.

	Copula	Equity	Mezzanine	Senior	Index
<b>PV Loss %</b>	Gaussian – Constant LGD	0.49	0.54	0.70	1.73
	Gaussian – Correlated LGD	0.47	0.53	0.73	1.73

As expected the more senior tranches are shown to be more risky than when recoveries are uncorrelated with defaults. The increased risk of correlated losses for the senior tranches has partly been offset by the lower probability of default for individual names. We thus expect there to be fewer instances where senior tranches suffer losses but for losses to be larger if they do occur.

The introduction of state dependent stochastic recovery rates increases the number of parameters available to calibrate the model to market prices while incorporating empirically observed facts in the model. In addition it serves as a useful tool to fatten the right tail of the loss distribution. This framework can also easily be extended to other factor copulas like the Clayton copula or the Student copula.

## 6.5 Stochastic correlation and random factor loadings

Stochastic correlation is another method used to fit models to market data. Most of these models are based on mixtures of Gaussian copulas and are simple extensions of the standard model. The copula of this mixed distribution however will not remain Gaussian. Burtschell Gregory and Laurent (2007) suggest a general class of stochastic correlation model with the aim to explain the market rather than merely fit market data.

The simplest type of stochastic correlation model is simply one where the factor loadings are stochastic. Burtschell, Gregory and Laurent (2005) provide the following specification for the factor loadings and modify Equation 7 as follows

Equation 18

$$Z_i = B_i(a_iM + \sqrt{1 - a_i^2}X_i) + (1 - B_i)(b_iM + \sqrt{1 - b_i^2}X_i)$$

by letting  $B_i$  be independent Bernoulli variables with parameter  $p$  and  $0 \leq a_i \leq b_i \leq 1$  some correlation parameters. The model thus describes two possible states of the world for factor exposure. In one state the exposure is  $a_i$  with probability  $p$  and in the second the exposure is  $b_i$  with probability  $1 - p$ .

A more realistic specification might be the one found in Anderson and Sidenius (2004) where the factor loadings depend on the value of the factor. Specifically we would like the loadings to decrease as the factor increases meaning the factors play a stronger role in the default time when their values are low. One might interpret this as all exposures becoming more dependent on economic conditions when the economy performs poorly. Our baseline equation would then become

Equation 19

$$Z_i = \mathbb{I}_{M \leq \theta} (a_i M + \sqrt{1 - a_i^2} X_i) + \mathbb{I}_{M > \theta} (b_i M + \sqrt{1 - b_i^2} X_i).$$

This specification is simple enough to allow closed form default probabilities and remains tractable for linear factor loadings.

From the above we are not merely trying to mimic empirical observations but also to generate a correlation skew as observed in the market. The logic here follows similarly to that of correlated losses investigated in the previous section. Senior tranches will only experience losses if several names default together which will happen when factor values are low. This will coincide with high factor loadings making it appear to the investor that correlation is high.

Equity investors expect to suffer some losses regardless of the factor value; to them correlation will appear to be the weighted average of the possible values.

For an initial implementation we chose the factor loadings give correlations of 0.45 and 0.7 respectively. When the factor value is below the 20<sup>th</sup> percentile the higher correlation value is used while the lower value applies for higher factor values. The average correlation is still expected to be 0.5 as previously specified. We would thus expect the value of the equity tranche to remain very similar while the senior tranche will have higher expected loss due to the higher observed correlation.

The table below shows the results for the above specification of stochastic correlation.

Table 15 – Tranche losses for stochastic correlation compared to standard static correlation

	Model	Equity	Mezzanine	Senior	Index
PV Loss %	Gaussian Copula	0.49	0.54	0.70	1.73
	Stochastic Correlation	0.50	0.51	1.01	2.02

From the results it is clear that stochastic correlation has a dramatic impact on the senior tranches in the portfolio, particularly when correlation increases when factor values are low.

## 6.6 Summary

The results in this chapter implies that any model not cognisant of the empirical observed facts of correlated default and recovery rates as well as the existence of more complex dependence relationships between entities will in essence be mis-specified. In particular such a model will tend to underestimate the probability of both larger losses and smaller losses occurring. This implies that not enough weight is given to the tails of the distribution according to Hager and Schöbel (2006a).

## Chapter 7 Results and conclusions

The study of CDO models cannot be done in isolation of the securities they intend to model and the markets in which these are traded. One can argue that not since the market-wide adoption of the Black Scholes option pricing model has a particular security been as dependent on the valuation method behind it as CDO's were. We can thus summarize our results in two parts. The first is the boom in the CDO market and the model that ignited it while the second is the market collapse partly caused by the overlooking of, or ignorance to, some of the assumptions made.

The initial challenge in construction of CDO models around the turn of the century lay with the specification of some joint distribution of default events while maintaining the calibration to market prices for the constituents of the portfolio. The available commercial models of that time were aimed at calculating credit VaR for portfolios of simple loans and not suitable for CDO valuation as they did not consider multiple time horizons and the exact timing of defaults.

The most tractable models of default risk are of the reduced form type as they share similar characteristics to familiar models for the short rate of interest. We studied a general reduced form model for valuing single credit risky securities and in particular the common assumptions used for specifying recovery values.

Attempts at extending these to the portfolio setting led to models that either failed to achieve significant correlation or became too intractable for practical implementation. This problem was seemingly overcome with the introduction of copulas to the field of credit risk by Li (2000). The main benefits of the copula approach are that the correlation structure can be specified independent of the marginal distributions while the model remains tractable enough for implementation.

In this dissertation we studied a number of alternative measures of dependence as well as various popular copulas used in finance, particularly of the elliptical and Archimedean class. After simulating and graphing observations from these copulas we showed that they behave differently even when specified with parameters leading them to have equal dependence measures. Such differences can be attributed to features such as tail dependence.

Market participants quickly adopted the Gaussian copula framework which on the face of it provided an elegant solution with a single parameter dictating correlation independent of the marginal distributions. The correlation parameter reminds one of the volatility parameter in the Black Scholes world as it similarly determines the dispersion of the value of the underlying and hence option price. In this context a CDO tranche is analogous to a combined long and short position on total portfolio loss with different strikes. The parameters also share the feature that they are not directly observable in the market.

We discussed the standard Gaussian model in some detail and note some of the main contributions to the field made by Li (2000) in its introduction. Important amongst these is the move towards specifying a distribution of joint lifetimes rather than joint default probabilities over a fixed horizon. Such a specification provides much more information on the correlation between default events.

The recursive probability bucketing technique for pricing CDO tranches proposed by Hull and White (2004) provided a convenient semi-analytical approach for implementing the model. This approach also allowed for the calculation of the Greeks for a particular tranche of the CDO and hence the means to hedge the exposure. We explored the analytical formulae presented by Meng and Sengupta (2008) for calculating tranche Greeks for equity and senior tranches and confirmed the results through simulation.

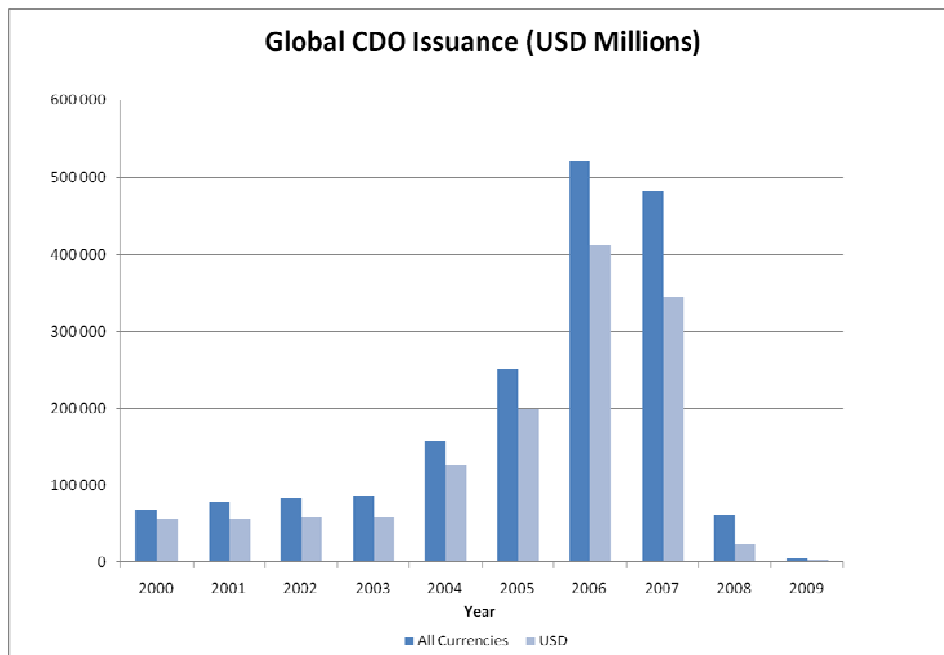
Armed with a widely adopted model to price CDO securities and a convenient recipe for implementation and hedging the liquidity in the CDO market was set to increase. More startling though is the dramatic increase in the volume of securities issued with the market growing six fold in the three years from 2003 to 2006 (see Figure 28).

There are many factors that led to this, increasing both demand and supply of CDO securities. In short we can mention low interest rate policies by the Federal Reserve, rising asset prices, pressure to offer more affordable products to borrowers, erosion of underwriting standards in the U.S. mortgage market and the originate and distribute model which absolved the originators from the ultimate credit experience on their loans.

As noted earlier arbitrage is the main purpose behind the creation of most CDO's with the issuer hoping to profit from the difference in spread between the assets and liabilities. In addition the structuring and management fees payable to the manager of the CDO are linked to the amount of assets under management.

All of the above incentives made for a large amount of assets originated that can readily be used as collateral for CDO securities. This together with the lucrative nature of the business of creating such securities fuelled the growth in the market.

Figure 28 – Global CDO issuance volumes (Source – Securities Industry and Financial Markets Association)



The credit crises which started during 2007 put an abrupt halt to this growth and at the time of writing the CDO market is yet to recover from the doldrums experienced since then.

The existence of a correlation skew in the market similar to that of the volatility smile observed in option markets would have alerted participants to the fact that the standard model is mis-specified. The correlation smile can be attributed to various assumptions made by the model as noted by Ağca, Agrawal and Islam (2008). Particularly those of constant homogeneous correlations and credit spreads, uncorrelated default and recovery rates and the choice of Gaussian copula instead of a fatter tailed distribution. These assumptions all appear quite dubious when placed next to the empirical evidence presented in various papers, a more recent example of which is Acharya, Bharath and Srinivasan (2007).

Even though the above is known it is not uncommon for market participants to rely on implied correlations as a true measure of underlying correlations or to pursue relative value strategies between tranches based on the correlation smile.

Despite its shortcomings the model alone cannot be blamed for all losses in the CDO market, much of this has to be placed on the various risk management functions at various institutions and oversight by industry regulators. In its Shareholder Report dated 18 April 2008 UBS, a prominent Swiss bank, outlines some of the overarching causes of the losses on its CDO positions during 2007 (amounting to \$14 billion at that time).

Most of these losses (63%) were related to AMPS (Amplified Mortgage Portfolio Super seniors) trades. The basic idea was that UBS went long on the super senior tranche of a residential mortgage CDO and purchased protection on a percentage (typically 2%-4%) of the nominal value of the position. The CDO desk considered such a position fully hedged based on statistical analyses of historical price movements that indicated that such protection was sufficient to protect UBS from any losses on the position. When the AMPS protection became exhausted, UBS was exposed to write-downs on losses to the extent that they exceeded the protection purchased.

In essence the model employed underestimated the risk to the super senior tranches and thus the protection required to hedge against losses. The above shareholder report had the following to say about the role of this in the subsequent losses made.

AMPS model: The AMPS model was certified by Quantitative Risk Control, but with the benefit of hindsight appears not to have been subject to sufficiently robust stress testing. Further, the CDO desk did not carry out sufficient fundamental analysis as market conditions deteriorated, or conduct 'look-through' analysis to re-assess potential issues in the AMPS structure or the underlying CDO structure. The cost of hedging through a negative basis trade was approximately 11 bps, whereas the cost of hedging through an AMPS trade was approximately 5 – 6 bps. The reasons for the differential pricing of hedging strategies that from a risk metrics perspective were deemed equivalent appears not to have been closely scrutinised at desk or other levels.

The above comments may well be extended to the standard market model in general with its industry wide adoption seen as a form of certification. During the time it became prevalent credit conditions were benign and the model remained untested under high stress scenarios. Market pricing and empirical data was known to be inconsistent with model output and assumptions yet escaped closer scrutiny by most participants.



Up to the 2007 crisis, research within CDO models mainly concentrated on the dependence between defaults. However, due to the substantial increase in the market price of systemic credit risk protection, recent attention has shifted to addressing flaws in the model in an effort to replicate current market pricing.

We examined a number of these which includes a more general correlation matrix as proposed by Gregory and Laurent (2004) with numerical implementation discussed in Hager and Schöbel (2006a). This extension is fairly flexible and can generally fit market prices although some observed prices may not be attainable for any choice of correlation matrix. If a solution exists the resulting correlation matrix obtained may well lack any economic intuition. In addition we demonstrated the drawback the model shares with the standard model in that tranche prices do not uniquely depend on the correlation matrix used. This extension does not appear to enjoy any widespread popularity.

Secondly we considered substituting the Gaussian copula with another copula. In particular the t-copula or mean-variance mixtures using the t-distribution, for example the one proposed by Hull and White (2004), seem to fit market data better. We demonstrate the effect of tail dependence in the copula by showing that a larger share of losses gets proportioned to senior tranches for the Clayton and t-copula. This is consistent with market observations of increased implied correlation for senior tranches. We noted that any choice of copula will bear an element of model risk since no rigorous method exists for choosing the optimal copula.

Currently the most promising areas of extension are towards stochastic recovery and stochastic correlation models. These models remain fairly tractable while allowing the Gaussian copula model to better calibrate to tranche prices and produce a correlation skew effect.

Stochastic recovery for the Gaussian copula was introduced in Gregory and Laurent (2004) and further advanced in Laurent, Amraoui, Cousot and Hitier (2009). A constant recovery assumption typically leaves the most senior tranches risk free and a model unable to calibrate to current market prices as spreads are quoted on such tranches. Our results show that stochastic recovery can dramatically increase the expected loss on senior tranches when correlated with default rates as recoveries are expected to be particularly low when these tranches suffer losses. A drawback of this method is that due to differences across names regarding the conditional losses given default, the standard recursion approach for implementing the model becomes problematic.

Stochastic correlation extensions in the form of random factor loadings for Gaussian copula were first proposed by Anderson and Sidenius (2004) and further extended in Burtschell Gregory and Laurent (2007).

All of the above models lack consistent dynamics for market variables such as credit spreads. This prevents the valuation of next generation products like tranche options and forward contracts on tranches. Li (2009) proposes a model incorporating both stochastic recovery and dynamic correlation to address these problems but stops short from specifying full credit spread dynamics. Such issues may well need to be addressed outside a copula framework.

We can conclude by noting that the state of CDO models is not dissimilar to that of the CDO market itself. Current extensions appear to be to be transient in nature, employed as a stop

gap to replicate market prices by providing a deeper understanding of the correlation smile. They still rely on the Gaussian copula and do not yet attain the goal of a more mature modelling framework but are a step in that direction.

## Appendix A – Mathematical results

### Lemma 1

Let

$$I_k(p) = - \sum_{j=0}^k (k-j) \binom{N}{j} [jp^{j-1}(1-p)^{N-j} - (N-j)(1-p)^{N-j-1}p^j]$$

where  $N$  and  $k$  are positive integers with  $k < N$  and  $p \in [0,1]$  then

$$I_k(q|m) = N - (N-k)k \binom{N}{j} \int_0^p t^{k-1}(1-t)^{N-k-1} dt$$

and in particular  $I_k(\cdot)$  is monotonically decreasing.

Proof

$$\begin{aligned} I_k(p) &= - \sum_{j=0}^k (k-j) \binom{N}{j} [jp^{j-1}(1-p)^{N-j} - (N-j)(1-p)^{N-j-1}p^j] \\ &= - \sum_{j=0}^{k-1} [\binom{N}{j+1}(k-j-1)(j+1) - \binom{N}{j}(k-j)(N-j)] p^j (1-p)^{N-j-1} \\ &= \sum_{j=0}^{k-1} \binom{N}{j} (N-j) p^j (1-p)^{N-j-1} \\ &= N \sum_{j=1}^k \binom{N-1}{j-1} p^{j-1} (1-p)^{N-j} \\ &= N(1-p)^{N-1} + \sum_{j=1}^{k-1} \binom{N}{j} (N-j) p^j (1-p)^{N-j-1} \end{aligned}$$

Taking the derivative we find

$$I'_k(p) = \sum_{j=1}^{k-1} \binom{N}{j} (N-j) j p^{j-1} (1-p)^{N-j-1}$$

$$- \sum_{j=0}^{k-1} \binom{N}{j} (N-j) p^j (N-j-1) (1-p)^{N-j-2}$$

$$I'_k(p) = \sum_{j=0}^{k-2} \{ \binom{N}{j+1} (N-j-1)(j+1) - \binom{N}{j} (N-j)(N-j-1) \} p^j (1-p)^{N-j-2}$$

$$- \binom{N}{k-1} (N-k+1)(N-k) p^{k-1} (1-p)^{N-k-1}$$

Now

$$\{ \binom{N}{j+1} (N-j-1)(j+1) - \binom{N}{j} (N-j)(N-j-1) \} = 0$$

which means that, after rewriting the last term, we have

$$I'_k(p) = -(N-k)k \binom{N}{k} p^{k-1} (1-p)^{N-k-1} < 0$$

leading us to conclude  $I_k(p)$  is a decreasing function in  $p$ .

Setting  $I_k(0) = N$  we get

$$I_k(p) = N - (N-k)k \binom{N}{k} \int_0^p t^{k-1} (1-t)^{N-k-1} dt.$$

## Appendix B – Software source code

All code used is for the R statistical package (<http://www.r-project.org/>) which can be downloaded freely. The functions used require the “copula” package (see Yan (2004)) to be installed; which in turn requires a number of other packages to function. All packages used are available for free download from the same source as the R software. Below each submitted statement commentary is given in italics explaining what the statement does. The software is case sensitive a fact which should be kept in mind to avoid errors.

### Graphing copula functions

#### Gaussian copula

```
cop.norm <- ellipCopula(family = "normal", dim = 2, dispstr = "un", param=0.2)
```

*Defines a elliptical copula from the normal family with dimension 2 and unstructured dispersion matrix with a parameter of 0.2*

```
persp(cop.norm,dcopula,xlim=c(0,1),ylim=c(0,1))
```

*Draws a perspective plot of the pdf function of the above defined copula*

```
persp(cop.norm,pcopula,xlim=c(0,1),ylim=c(0,1))
```

*Draws a perspective plot of the cdf function of the above defined copula*

```
contour(cop.norm,pcopula,xlim=c(0,1),ylim=c(0,1))
```

*Draws a contour plot of the cdf function of the above defined copula*

#### Student copula

```
cop.t<- ellipCopula(family = "t", dim = 2, dispstr = "un",param=0.2,df=8)
```

*Defines a elliptical copula from the Student family with dimension 2 and unstructured dispersion matrix with a parameter of 0.2 and 8 degrees of freedom*

```
persp(cop.t,dcopula,xlim=c(0,1),ylim=c(0,1))
```

*Draws a perspective plot of the pdf function of the above defined copula*

```
persp(cop.t,pcopula,xlim=c(0,1),ylim=c(0,1))
```

*Draws a perspective plot of the cdf function of the above defined copula*

```
contour(cop.t,pcopula,xlim=c(0,1),ylim=c(0,1))
```

*Draws a contour plot of the cdf function of the above defined copula*

## Gumbel copula

```
cop.gumbel <- archmCopula(family = "gumbel", dim = 2, param = 2)
```

*Defines a Gumbel copula from the Archimedean family with dimension 2 and parameter 2*

```
persp(cop.gumbel,dcopula,xlim=c(0,1),ylim=c(0,1))
```

*Draws a perspective plot of the pdf function of the above defined copula*

```
persp(cop.gumbel,pcopula,xlim=c(0,1),ylim=c(0,1))
```

*Draws a perspective plot of the cdf function of the above defined copula*

```
contour(cop.gumbel,pcopula,xlim=c(0,1),ylim=c(0,1))
```

*Draws a contour plot of the cdf function of the above defined copula*

## Clayton copula

```
cop.clayton <- archmCopula(family = "clayton", dim = 2, param = 2)
```

*Defines a Clayton copula from the Archimedean family with dimension 2 and parameter 2*

```
persp(cop.clayton,dcopula,xlim=c(0,1),ylim=c(0,1))
```

*Draws a perspective plot of the pdf function of the above defined copula*

```
persp(cop.clayton,pcopula,xlim=c(0,1),ylim=c(0,1))
```

*Draws a perspective plot of the cdf function of the above defined copula*

```
contour(cop.clayton,pcopula,xlim=c(0,1),ylim=c(0,1))
```

*Draws a contour plot of the cdf function of the above defined copula*

## Frank copula

```
cop.frank <- archmCopula(family = "frank", dim = 2, param = 2)
```

*Defines a Frank copula from the Archimedean family with dimension 2 and parameter 2*

```
persp(cop.frank,dcopula,xlim=c(0,1),ylim=c(0,1))
```

*Draws a perspective plot of the pdf function of the above defined copula*

```
persp(cop.frank,pcopula,xlim=c(0,1),ylim=c(0,1))
```

*Draws a perspective plot of the cdf function of the above defined copula*

```
contour(cop.frank,pcopula,xlim=c(0,1),ylim=c(0,1))
```

*Draws a contour plot of the cdf function of the above defined copula*

## Simulating observations from the copula

### Gaussian copula

```
cop.norm <- ellipCopula(family = "normal", dim = 2, dispstr = "un", param=0.6)
```

*Defines a elliptical copula from the normal family with dimension 2 and unstructured dispersion matrix with a parameter of 0.6*

```
sims<- rcopula(cop.norm,5000)
```

*Simulates 5000 observations from the copula*

```
plot(sims)
```

*Plots the simulated observations*

### Student copula

```
cop.t<- ellipCopula(family = "t", dim = 2, dispstr = "un", param=0.6,df=4)
```

*Defines a elliptical copula from the Student family with dimension 2 and unstructured dispersion matrix with a parameter of 0.6 and 4 degrees of freedom*

```
sims<- rcopula(cop.t,5000)
```

*Simulates 5000 observations from the copula*

```
plot(sims)
```

*Plots the simulated observations*

### Gumbel copula

```
cop.gumbel <- archmCopula(family='gumbel',dim=2,param=2.5)
```

*Defines an Archimedean copula from the Gumbel family with dimension 2 and parameter of 2.5*

```
sims<- rcopula(cop.t,5000)
```

*Simulates 5000 observations from the copula*

```
plot(sims)
```

*Plots the simulated observations*

### Clayton copula

```
cop.clayton <- archmCopula(family = "clayton", dim = 2, param = 3)
```

*Defines an Archimedean copula from the Clayton family with dimension 2 and parameter of 2.5*

```
sims<- rcopula(cop.t,5000)
```

*Simulates 5000 observations from the copula*

```
plot(sims)
```

*Plots the simulated observations*

## Frank copula

```
cop.frank<- archmCopula(family='frank',dim=2,param=8)
```

*Defines an Archimedean copula from the Frank family with dimension 2 and parameter of 8*

```
sims<- rcopula(cop.t,5000)
```

*Simulates 5000 observations from the copula*

```
plot(sims)
```

*Plots the simulated observations*

## Multivariate normal distributions

```
pmvnorm(lower= -Inf, upper=c(qnorm(0.0199), qnorm(0.0199)), mean=c(0,0),
```

```
corr=matrix(c(1,0.2,0.2,1),ncol=2), algorithm=Miw(steps = 128))
```

*Calculates the probability  $\Phi_2(\Phi^{-1}(0.0199), \Phi^{-1}(0.0199), 0.2)$*

## Simulating from a joint distribution with given copula and margins

```
library(copula)
```

*Loads the copula package*

```
set.seed(1)
```

*Sets the random number generator seed to 1*

```
cop.norm <- ellipCopula(family = "normal", dim = 2, dispstr = "un",param=0.2)
```

*Defines a elliptical copula from the normal family with dimension 2 and unstructured dispersion matrix with a parameter of 0.2*

```
mvd.norm <- mvdc(copula=cop.norm,margins=c("exp","exp"),paramMargins=list(list(rate=-log(0.99)),list(rate=-log(0.99))))
```

*Defines a multivariate distribution with above defined copula and with both marginal distributions exponential with rate parameter lambda equal to  $-\log(0.99)$*

```
sims<-rmvdc(mvd.norm,100000)
```

*Simulates 100000 observations from this distribution*

```
product<-sims[,1]*sims[,2]
```

*Calculates the product of the observed lifetimes for each simulation*

```
mean(product)
```



*Calculates the mean of the product of the observed lifetimes*

## Simulating CDO tranche losses and premiums

```
library(copula)
```

*Loads the copula package*

```
Sims<-10000
```

*Specify number of simulations to run*

```
N<-100
```

*Specify number of names in the index*

```
RFree<-0.05
```

*Specify the constant short rate of interest*

```
Hazard<-0.01
```

*Specify the credit spread, need not be the same for each name*

```
LGD<-0.4
```

*Specify the LGD*

```
Maturity<-5
```

*Specify the maturity*

```
PayFreq<-5
```

*Specify the time between margin payments*

```
Tranches<-c(0,0.03,0.06,0.09,0.12,0.22,1)
```

*Specify the tranche attachment / detachment points*

```
FaceValue<-1
```

```
FaceValues=rep(FaceValue,N)
```

*Specify the face value of individual names in this case all are 1 but values need not be equal*

```
set.seed(1)
```

```
Cop.Norm <- ellipCopula(family = "normal", dim = N, dispstr = "un", param = Corr)
```

```
Copula<-rcopula(Cop.Norm, Sims)
```

*Specify the random seed, copula to use and its parameters*

```
LT=length(Tranches)
```

```
Dates<-c(0,1:(Maturity/PayFreq)*PayFreq)
```

```
Payments<-mat.or.vec(Sims, LT)
```

```
Margin<-mat.or.vec(Sims, LT)
```

```
Attachment<-Tranches*sum(FaceValues)
```

```
Tau=-log(1-Copula)/Hazard
```

*Initialize some matrices used for output and calculate the default times for the simulations*

for (i in 1: Sims){

*Starts calculation of swap payments and margins for a given scenario*

DefaultRate[i,1]=sum(Tau[i,]<Maturity)/N

Defaults=Tau[i,][Tau[i,]<=Maturity]

IndexP=1

if (length(Defaults!= 0)){

FaceVal<-FaceValues[(1:length(Tau[i,]))[Tau[i,]<=Maturity]]

LGDVal=LGD[i,][Tau[i,]<=Maturity]

LGDMean[i,1]<-mean(LGDVal)

Defaults<-rbind(Defaults,FaceVal,LGDVal)

CumLoss<-cumsum(FaceVal)

CumLoss<-c(0,CumLoss)

for (j in 1:length(FaceVal)){

Residual=FaceVal[j]

while (Residual>0){

CumLoss[j],Attachment[IndexP+1]-Attachment[IndexP]

Residual=Residual-Additional

Payments[i,IndexP]=Payments[i,IndexP]+Additional\*exp(-

RFree\*Defaults[1,j])\*Defaults[3,j]

if (Residual > 0) {IndexP=IndexP+1}

}

}

for (j in 1:(LT-1)){

Payments[i,LT]=Payments[i,LT]+Payments[i,j]

}

}

FaceRemain<-rep(0,LT)

for (k in 1:(LT-1)){

FaceRemain[k]=Attachment[k+1]-Attachment[k]

}

FaceRemain[LT]=Attachment[LT]

*Calculates the present value of the credit losses for each tranche and the index*

IndexM=1

for (j in 2:length(Dates)){

Defaults=Tau[i,][Tau[i,]>Dates[j-1] & Tau[i,]<=Dates[j]]

if (length(Defaults) != 0) {

FaceVal<-FaceValues[(1:length(Tau[i,]))[Tau[i,]>Dates[j-1] &

Tau[i,]<=Dates[j]]]

Defaults<-rbind(Defaults,FaceVal)

Defaults=t(Defaults)

Defaults[rank(Defaults[,1]),]<-Defaults[c(1:nrow(Defaults)),]



```

Defaults=t(Defaults)
CumFaceVal<-cumsum(FaceVal)
CumFaceVal<-c(0,CumFaceVal)
CumFaceVal=CumFaceVal+(Attachment[LT]-FaceRemain[LT])
  for(k in 1:length(FaceVal)){
    Residual=FaceVal[k]
    while (Residual>0){
      Additional=min(Residual,Attachment[IndexM+1]-
CumFaceVal[k],Attachment[IndexM+1]-Attachment[IndexM])
      Residual=Residual-Additional
      FaceRemain[IndexM]=FaceRemain[IndexM]-Additional
      Margin[i,IndexM]=Additional*(Defaults[1,k]-Dates[j-
1])/PayFreq*exp(-RFree*Defaults[1,k])
      if (Residual > 0) {IndexM=IndexM+1}
    }
  }
  }
FaceRemain[LT]=sum(FaceRemain[1:(LT-1)])
Margin[i,LT]=sum(Margin[i,1:(LT-1)])

  for (k in 1:LT){
    Margin[i,k]=Margin[i,k]+FaceRemain[k]*exp(-RFree*Dates[j])
  }
}

```

*Calculates the present value of the margin payments for each tranche and the index as well as the face value of debt remaining in each tranche*

}

*Ends the calculation of swap payments and margins for a given scenario*

EPayments<-rep(0,LT)

EMargin<-rep(0,LT)

EYield<-rep(0,LT)

for (i in 1:LT){

EPayments[i]=mean(Payments[,i])

EMargin[i]=mean(Margin[,i])

EYield[i]=EPayments[i]/EMargin[i]/PayFreq\*100

}

*Calculates the expected present value of credit losses and margin payments across all scenarios. Note that this excludes any upfront margin payments to the equity tranche.*

EPayments

EMargin

EYield

*Displays results*

## Calculating CDO tranche price sensitivities

```
Maturity=1
Hazard=0.02
N=100
k=10
rho=0.5
c=qnorm(1-exp(-Hazard*Maturity))
```

*Sets the parameters to use, k specifies the detachment point of the equity tranche.*

```
Qy<-function(y){pnorm((c*(1-rho)-sqrt(rho)*y)/(sqrt(1-rho)))}
```

*Calculates  $\Phi\left(\frac{c(1-\rho)-\sqrt{\rho}y}{\sqrt{1-\rho}}\right) = (q|y)$*

```
IkQy<-function(y,N,k){x<-0
for(m in 1:k){x<-x+(choose(N-1,m-1)*Qy(y)^(m-1)*(1-Qy(y))^(N-m))}
x=x*N
return(x)}
```

*Calculates the value of  $I_k(p)$  for the probability y*

```
IkQdy<-function(y){-(N-k)*k*choose(N,k)*Qy(y)^(k-1)*(1-Qy(y))^(N-k-1)*sqrt(1-
rho)/sqrt(2*pi)*exp(-(c*(1-rho)-sqrt(rho)*y)^2/(2*(1-rho)))}
```

*Calculates  $\frac{\partial I_k Q_y}{\partial c}$*

```
CorrInt<-function(y){-(IkQy(y,N,k)-IkQy(-y,N,k))*y/(2*sqrt(rho)*(1-rho)^(3/2)*2*pi)*exp(-
y^2/(2*(1-rho))-c^2/2)}
integrate(CorrInt,0,Inf)
```

*Calculates  $\frac{dE[I_k^e]}{d\rho}$  (vega of tranche)*

```
DeltaInt<-function(y){1/N*IkQy(y,N,k)/sqrt(2*pi*(1-rho))*exp(-(y^2)/(2*(1-rho)))}
integrate(DeltaInt,-Inf,Inf)
```

*Calculates  $\Delta_{k,spread}$  (delta of tranche)*

```
Gammalnt<-function(y){-1/(2*pi*sqrt(1-rho))*IkQdy(y)*exp(-y^2/(2*(1-rho))-c^2/2)}
integrate(Gammalnt,-Inf,Inf)
```

*Calculates  $\Gamma_k$  (gamma of tranche)*

## Changing correlation parameter

```
M1<-matrix(rep(0.9754,400),ncol=20)
for(i in 1:20){
M1[i,i]=1
}
```



```
M2<-matrix(rep(0.8994,400),ncol=20)
for (i in 1:20){
M2[i,i]=1
}
M3<-matrix(rep(0.6069,400),ncol=20)
for (i in 1:20){
M3[i,i]=1
}
M4<-matrix(rep(0.4700,400),ncol=20)
for (i in 1:20){
M4[i,i]=1
}
M5<-matrix(rep(0.4281,400),ncol=20)
for (i in 1:20){
M5[i,i]=1
}
M0<-matrix(rep(0.3911,400),ncol=20)
```

```
R1<-cbind(M1,M0,M0,M0,M0)
R2<-cbind(M0,M2,M0,M0,M0)
R3<-cbind(M0,M0,M3,M0,M0)
R4<-cbind(M0,M0,M0,M4,M0)
R5<-cbind(M0,M0,M0,M0,M5)
```

```
CorrMat<-rbind(R1,R2,R3,R4,R5)
```

```
Corr<-mat.or.vec(0,1)
```

```
for (i in 1:99){
for (j in (i+1):100){
Corr=c(Corr,CorrMat[i,j])
}
}
```

*This specifies a structured correlation matrix containing 5 industry sectors.*

```
set.seed(1)
```

```
Cop.Norm <- ellipCopula(family = "normal", dim = N, dispstr = "un",param =Corr)
```

```
Copula<-rcopula(Cop.Norm,10000)
```

*This reruns the model with the changed correlation structure.*

## Changing correlation structure

### Student copula

```
set.seed(1)
```

```
Cop.T <- ellipCopula(family = "student", dim = N, dispstr = "un",param =Corr,df=6)
```

```
Copula<-rcopula(Cop.Norm,10000)
```

## Clayton copula

```
set.seed(1)
```

```
Cop.Clay <- archmCopula(family="clayton",dim=N,param=1)
```

```
Copula<-rcopula(Cop.Norm,10000)
```

*This reruns the model with the changed correlation structure.*

## Correlated recoveries

```
CorrPD=0.6
```

*This specifies the individual name correlation with the market factor for PD.*

```
CorrLGD=0.4
```

*This specifies the individual name correlation with the market factor for LGD.*

```
CorrMarket=0.4
```

*This specifies the correlation between the market factors for PD and LGD.*

```
CorrSpecific=0
```

*This specifies the correlation between the idiosyncratic factors for PD and LGD.*

```
MarketFactorPD=sqrt(CorrPD)
```

```
MarketFactorLGD=sqrt(CorrLGD)
```

```
SpecificFactorPD=sqrt(1-CorrPD)
```

```
SpecificFactorLGD=sqrt(1-CorrLGD)
```

```
SpecificPD <-matrix(rnorm(Sims*N,0,1),nrow=Sims,ncol=N)
```

```
SpecificLGD<-matrix(rnorm(Sims*N,0,1),nrow=Sims,ncol=N)
```

```
SpecificLGD<-CorrSpecific*SpecificPD+sqrt(1-CorrSpecific^2)*SpecificLGD
```

```
Copula<-matrix(0,nrow=Sims,ncol=N)
```

```
LGD<-matrix(0,nrow=Sims,ncol=N)
```

```
MarketPD<-matrix(rnorm(Sims,0,1),nrow=Sims,ncol=1)
```

```
MarketPD<-rep(MarketPD,N)
```

```
MarketPD<-matrix(MarketPD,Sims,N)
```

```
MarketLGD<-matrix(rnorm(Sims,0,1),nrow=Sims,ncol=1)
```

```
MarketLGD<-rep(MarketLGD,N)
```

```
MarketLGD<-matrix(MarketLGD,Sims,N)
```

```
MarketLGD<-CorrMarket*MarketPD+sqrt(1-CorrMarket^2)*MarketLGD
```

```
MarketPD<-MarketPD*MarketFactorPD
```

```
SpecificPD<-SpecificPD*SpecificFactorPD
```

```
Copula=pnorm(MarketPD+SpecificPD,0,1)
```

*This returns the default time copula allowing for above specified correlations.*

```
MarketLGD<-MarketLGD*MarketFactorLGD
SpecificLGD<-SpecificLGD*SpecificFactorLGD
LGD<-1-qbeta(pnorm(MarketLGD+SpecificLGD,0,1),58/10,58/15)
```

*This returns the LGD assuming the above correlations and a beta distribution.*

## Random factor loadings

```
CorrPD1=0.45
```

```
CorrPD2=0.7
```

*This sets the possible correlation values.*

```
MarketFactorPD1=sqrt(CorrPD1)
MarketFactorPD2=sqrt(CorrPD2)
SpecificFactorPD1=sqrt(1-CorrPD1)
SpecificFactorPD2=sqrt(1-CorrPD2)
```

```
SpecificPD <-matrix(rnorm(Sims*N,0,1),nrow=Sims,ncol=N)
Copula<-matrix(nrow=Sims,ncol=N)
```

```
MarketPD<-matrix(rnorm(Sims,0,1),nrow=Sims,ncol=1)
for (i in 1:Sims){
  if (MarketPD[i,1]<= -0.8416212) {
    MarketPD[i,1]=MarketFactorPD2*MarketPD[i,1]
    for (j in 1:N) {
      Copula[i,j]=pnorm(MarketPD[i,1]+SpecificFactorPD2*SpecificPD[i,j],0,1)
    }
  }
  else {
    MarketPD[i,1]=MarketFactorPD1*MarketPD[i,1]
    for (j in 1:N) {
      Copula[i,j]=pnorm(MarketPD[i,1]+SpecificFactorPD1*SpecificPD[i,j],0,1)
    }
  }
}
```

*This returns the copula allowing for the random factor loadings.*

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