Appendix A

A DERIVATION OF THE PLANAR EQUATIONS OF MOTION

A.1 Newton’s second law

Kinetics is the study of motion and its relationship with the forces that produce the motion (see [65]). The simplest body arising in the study of motion is a particle, or point mass, defined by Nikravesh [65] as a mass concentrated at a point. According to Newton’s second law, a particle will accelerate when it is subjected to unbalanced forces. More specifically, Newton’s second law as applied to a particle is

\[ \vec{f} = m^{(p)} \vec{a} \]  

(A.1)

where \( \vec{f} \), \( m^{(p)} \) and \( \vec{a} \) respectively represents the total force acting on the particle, the mass of the particle and the acceleration of the particle (see Figure A.1).

![Figure A.1: A particle moving in a global coordinate system (after [65]).](image)

Since \( \vec{f} \) and \( \vec{a} \) are three-dimensional physical vectors, they may be represented as three-vectors in the global fixed coordinate system, i.e. \( \vec{f} = [f_x, f_y, f_z]^{T} \) and \( \vec{a} = [a_x, a_y, a_z]^{T} \). The position vector \( \vec{r} \) shown in Figure A.1 locates the particle in the global coordinate system and is represented in the global reference frame by \( \vec{r} = [x, y, z]^{T} \). The acceleration \( \vec{a} \) of the particle is the second time...
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derivative of the position vector, i.e. \( \mathbf{a} = \ddot{\mathbf{r}} = [\ddot{x}, \ddot{y}, \ddot{z}]^T \), and the global representation of expression (A.1) therefore is

\[
\mathbf{f} = m^{(p)} \ddot{\mathbf{r}} \tag{A.2}
\]

For a system of \( p \) particles, the application of expression (A.2) may be extended to describe the motion of each particle \( i, i = 1, 2, \ldots, p \) in the system, i.e.

\[
m_i^{(p)} \ddot{r}_i = f_i + \sum_{j=1}^{p} f_{ij}, \quad i = 1, 2, \ldots, p \tag{A.3}
\]

where \( f_i \) is the global representation of the total externally applied force \( \ddot{\mathbf{f}}_i \) acting on particle \( i \), and \( f_{ij} \) is the global representation of the internal force \( \ddot{\mathbf{f}}_{ij} \) extended by particle \( j \) on particle \( i \). Note that \( \ddot{f}_{ii} = 0 \).

Summing expression (A.3) over all \( p \) particles in the system results in

\[
\sum_{i=1}^{p} m_i^{(p)} \ddot{r}_i = \sum_{i=1}^{p} f_i + \sum_{i=1}^{p} \sum_{j=1}^{p} f_{ij} \tag{A.4}
\]

and since \( \ddot{f}_{ij} = -\ddot{f}_{ji} \) (Newton's third law), it follows that \( \sum_{i=1}^{p} \sum_{j=1}^{p} f_{ij} = 0 \), and consequently expression (A.4) reduces to

\[
\sum_{i=1}^{p} m_i^{(p)} \ddot{r}_i = \sum_{i=1}^{p} f_i \tag{A.5}
\]

or simply

\[
m\ddot{\mathbf{r}} = \mathbf{f} \tag{A.6}
\]

where \( m = \sum_{i=1}^{p} m_i^{(p)} \) is the total mass of the system of particles,

\[
\mathbf{r} = \frac{1}{m} \sum_{i=1}^{p} m_i^{(p)} \ddot{r}_i \quad \text{is the center of mass of the system of particles, and}
\]

\[
\mathbf{f} = \sum_{i=1}^{p} f_i \quad \text{is the total external force acting on the system of particles.}
\]

The translational equation of motion of a system of particles (A.6), also holds for a general continuous rigid body or continuum, the center of mass of which is given by

\[
\mathbf{r} = \frac{1}{m} \int \mathbf{r} \, dm \tag{A.7}
\]

where \( \mathbf{r} \) locates an infinitesimally small mass element \( dm \) as shown in Figure A.2.

Appendix A
Since vector \( \mathbf{r}^P \) is the sum of two vectors \( \mathbf{r} + \mathbf{s} \), it follows that
\[
\mathbf{r} = \frac{1}{\text{vol}} \int \mathbf{r}^P \, d\mathbf{m}
\]
\[
= \frac{1}{\text{vol}} \int (\mathbf{r} + \mathbf{s}) \, d\mathbf{m}
\]
\[
= \mathbf{r} + \frac{1}{\text{vol}} \int \mathbf{s} \, d\mathbf{m}
\]
which implies that
\[
\int \mathbf{s} \, d\mathbf{m} = 0 \tag{A.8}
\]
and taking first and second order time derivatives gives
\[
\int \dot{s} \, d\mathbf{m} = 0 \tag{A.9}
\]
and
\[
\int \ddot{s} \, d\mathbf{m} = 0 \tag{A.10}
\]

### A.2 Planar equations of motion

For the purposes of analyzing a planar machining center it is required to derive the planar equations of motion. Consider Figure A.3 showing an external force \( \mathbf{f}_i \) acting on the \( i \)-th particle of a system. For planar motion, the center of mass \( \mathbf{C} \) of the system remains in the \( \text{Gxy} \)-plane, and coincides with the origin \( 0 \) of the body-fixed \( \xi\eta\zeta \)-coordinate system. The centroidal body-fixed \( \xi\eta\zeta \)-coordinate system is chosen in such a way that the \( \zeta \)-axis is parallel to the \( z \)-axis. Note that the origin of the global \( \text{xyz} \)-reference frame is denoted by \( G \).
A.2.1 Planar translational equations of motion

The first necessary condition for the system in Figure A.3 to experience planar motion in the Gxy-plane, is that the external force $\vec{f}$ must be parallel to the xy-plane of motion, which is only possible if the z-component $f_z$ of the force $\vec{f}$ is zero (see [65]).

More specifically, expression (A.6) gives the two planar translational equations of motion, for the system shown in Figure A.3:

$$m\ddot{x} = f_x$$  \hspace{1cm} (A.11)

and

$$m\ddot{y} = f_y$$  \hspace{1cm} (A.12)

with of course $f_z = 0$.

Expressions (A.11) and (A.12) may also be combined in matrix form:

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} f_x \\ f_y \end{bmatrix}$$  \hspace{1cm} (A.13)

with $m$ representing the total mass of the system, $\ddot{x}$ and $\ddot{y}$ respectively representing the x- and y-accelerations of the center of mass C, and $f_x$ and $f_y$ respectively representing the x- and y-components of the total external force acting on the mass system.
**A.2.2 Planar rotational equations of motion**

The *second necessary condition* for the system in Figure A3 to experience *planar motion*, is that it is only to rotate about the $\zeta$-axis (see [65]). This rotation is due to the *resultant moment* about C of the external forces acting on the system. Since the $\zeta$-axis is parallel to the z-axis, the global representation of the resultant moment of the forces about C must, for planar motion, only have a z-component.

Consider the moment about G as a result of the external forces $\vec{f}_i$ and $\sum_{j=1}^{p} \vec{f}_j$ acting on particle $i$ (see Figure A3):

$$\vec{n}_i^G = \vec{r}_i^p \times (\vec{f}_i + \sum_{j=1}^{p} \vec{f}_j)$$  \hspace{1cm} (A.14)

With particle $i$ located in the global reference frame by

$$\vec{r}_i^p = \vec{r} + \vec{s}_i$$  \hspace{1cm} (A.15)

the global representation of expression (A.15) is

$$\vec{r}_i^p = \vec{r} + \vec{s}_i$$  \hspace{1cm} (A.16)

Hence, expression (A.14) may be represented in the global reference frame by

$$\vec{n}_i^G = \vec{r}_i^p (\vec{f}_i + \sum_{j=1}^{p} \vec{f}_j)$$  \hspace{1cm} (A.17)

with $\vec{r}_i^p$ representing the expansion of $\vec{r}_i^p$ into a skew symmetric matrix, i.e.:

$$\vec{r}_i^p = \begin{bmatrix} 0 & -z^p & y^p \\ z^p & 0 & -x^p \\ -y^p & x^p & 0 \end{bmatrix}$$  \hspace{1cm} (A.18)

Substituting expression (A.3) into (A.17) yields

$$\vec{n}_i^G = \vec{r}_i^p (\vec{f}_i + \sum_{j=1}^{p} \vec{f}_j) = \vec{r}_i^p (m_i^p \vec{r}_i^p)$$  \hspace{1cm} (A.19)

Summing expression (A.19) over all $i = 1,2,...,p$, gives the sum of the moments with respect to G as

$$\vec{n}^G = \sum_{i=1}^{p} \vec{r}_i^p \vec{f}_i + \sum_{i=1}^{p} \sum_{j=1}^{p} \vec{r}_i^p \vec{f}_j = \sum_{i=1}^{p} \vec{r}_i^p (m_i^p \vec{r}_i^p)$$  \hspace{1cm} (A.20)

However, since $\vec{r}_i^p \vec{f}_j = -\vec{r}_j^p \vec{f}_i$ (see expressions (A.4) and (A.5)), it follows that $\sum_{i=1}^{p} \sum_{j=1}^{p} \vec{r}_i^p \vec{f}_j = 0$.

Furthermore, expression (A.16) implies that

$$\vec{r}_i^p = \vec{r} + \vec{s}_i$$  \hspace{1cm} (A.21)

Expression (A.20) may therefore be written as
DERIVATION OF THE PLANAR EQUATIONS OF MOTION

\[ n^G = \bar{r}f + \sum_{i=1}^{p} \bar{s}_i f_i = \sum_{i=1}^{p} m_i^p \bar{r}^p + \bar{s}_i \quad (A.22) \]

where \( \sum_{i=1}^{p} \bar{s}_i f_i = n \) the sum of the moments of the external forces about \( C \).

For a continuous body, \( m_i^p \) represents an infinitesimally small mass element \( dm \), i.e. \( m_i^p = dm \), and therefore expression (A.22) becomes

\[ n^G = \bar{r}f + n = \int_{\text{vol}} \bar{r}^p \bar{s}_i dm \quad (A.23) \]

with the position of the center of mass of the continuum \( r \) given by expression (A.7).

Since the center of mass of the body under consideration is to remain in the Gxy-plane, the z-component of \( r \) is identically equal to zero, i.e. \( r = [x, y, 0]^T \). The expansion of \( r \) into a skew-symmetric matrix is therefore

\[ \bar{r} = \begin{bmatrix} 0 & 0 & y^p \\ 0 & 0 & -x^p \\ -y^p & x^p & 0 \end{bmatrix} \quad (A.24) \]

The force, \( f \) in expression (A.23) is the global representation of the resultant external force acting on the body, and \( n^G \) and \( n \) respectively represent the resultant moment of the external forces acting on the body about the origin of the global reference frame \( G \), and the center of mass of the body \( C \).

In satisfying the second necessary condition for planar motion, the expansion of expression (A.23) may only yield a single non-zero scalar equation corresponding to the z-component of the resultant moment acting on the body with respect to \( G \):

\[ n_{(z)}^G = x f_{(y)} - y f_{(x)} + n_{(z)} = \int_{\text{vol}} \bar{r}^p \bar{s}_i dm \quad (A.25) \]

From expression (A.16) it follows that for a continuous body

\[ r^p = r + s = \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} s_{(x)} \\ s_{(y)} \\ s_{(z)} \end{bmatrix} \quad (A.26) \]

with first and second time derivatives given by \( \dot{r}^p = \dot{r} + \dot{s} = [\dot{x}, \dot{y}, \dot{z}]^T + [\dot{s}_{(x)}, \dot{s}_{(y)}, \dot{s}_{(z)}]^T \) and \( \ddot{r}^p = \ddot{r} + \ddot{s} = [\ddot{x}, \ddot{y}, \ddot{z}]^T + [\ddot{s}_{(x)}, \ddot{s}_{(y)}, \ddot{s}_{(z)}]^T \) respectively.

Substituting expression (A.26) into the right hand side of expression (A.25) results in
\[ n_{(x)}^0 = x f_{(y)} - y f_{(x)} + n_{(z)} \]
\[ = \int_{\text{vol}} (\tilde{r} + \tilde{s})(\tilde{r} + \tilde{s}) \, dm \]  
(A.27)

From expressions (A.8) and (A.10) it follows that \[ \int_{\text{vol}} \tilde{s} \tilde{r} \, dm = 0 \] and \[ \int_{\text{vol}} \tilde{r} \tilde{s} \, dm = 0 \], and therefore expression (A.27) reduces to

\[ n_{(x)}^0 = x f_{(y)} - y f_{(x)} + n_{(z)} = x (m \dot{y}) - y (m \dot{x}) + \int_{\text{vol}} s_{(x)} s_{(y)} - s_{(y)} s_{(x)} \, dm \]  
(A.28)

From expressions (A.11) and (A.12) it also follows that \[ x f_{(y)} - y f_{(x)} = x (m \dot{y}) - y (m \dot{x}) \], and therefore (A.28) becomes

\[ n_{(x)} = \int_{\text{vol}} [s_{(x)} s_{(y)} - s_{(y)} s_{(x)}] \, dm \]  
(A.29)

For general three-dimensional motion, the above argument would give

\[ \mathbf{n} = \int_{\text{vol}} \tilde{s} \tilde{s} \, dm \]  
(A.30)

where \( \tilde{s} \) represents a skew-symmetric matrix of the form

\[ \tilde{s} = \begin{bmatrix} 0 & -s_{(z)} & s_{(y)} \\ s_{(z)} & 0 & -s_{(x)} \\ -s_{(y)} & s_{(x)} & 0 \end{bmatrix} \]  
(A.31)

Nikravesh [65] shows that

\[ \tilde{s} \tilde{s} = -\tilde{s} \tilde{s} \omega - \tilde{\omega} \tilde{s} \tilde{s} \omega \]  
(A.32)

with \( \omega = [\omega_{(x)}, \omega_{(y)}, \omega_{(z)}]^{T} \) the global representation of the angular velocity vector.

In agreement with the second necessary condition for planar motion, the body shown in Figure A.3 that rotates only about the \( \zeta \)-axis, has an angular velocity vector of which only the \( z \)-component may be non-zero, i.e.:

\[ \omega = [0, 0, \omega_{(z)}]^{T} \]  
(A.33)

with \( \omega_{(z)} = \dot{\phi} \) (see Figure A.3).

Consequently, the skew-symmetric matrix \( \tilde{\omega} \) in (A.32) is given by:

\[ \tilde{\omega} = \begin{bmatrix} 0 & -\dot{\phi} & 0 \\ \dot{\phi} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]  
(A.34)

The time derivative of expression (A.33) is

\[ \dot{\omega} = [0, 0, \dot{\omega}_{(z)}]^{T} \]  
(A.35)

with \( \dot{\omega}_{(z)} = \ddot{\phi} \).
Hence, by substituting expressions (A.33) - (A.35) into the right hand side of expression (A.32), and isolating the third component of the resultant vector, gives for planar motion:

\[ s_{(x)} \ddot{s}_{(x)} - s_{(y)} \ddot{s}_{(y)} = \ddot{\phi}(s_{(x)}^2 + s_{(y)}^2) \]  
(A.36)

Since for planar motion, the \( \zeta \)-axis is parallel to the \( z \)-axis (see Figure A.3), it follows that:

\[ s_{(x)} \ddot{s}_{(y)} - s_{(y)} \ddot{s}_{(x)} = \ddot{\phi}(s_{(x)}^2 + s_{(y)}^2) = \ddot{\phi}(s_{(z)}^2 + s_{(y)}^2) \]  
(A.37)

Substituting expression (A.37) into expression (A.29) finally gives

\[ n_{(z)} = \ddot{\phi} \int_{\text{vol}} (s_{(x)}^2 + s_{(y)}^2) \, dm \]  
(A.38)

The integral in expression (A.38):

\[ j_{(z)} = \int_{\text{vol}} (s_{(x)}^2 + s_{(y)}^2) \, dm \]  
(A.39)

is called the mass moment of inertia of the body about the \( \zeta \)-axis through \( C \).

Finally substituting expression (A.39) into expression (A.38) gives the planar rotational equation of motion:

\[ n_{(z)} = \ddot{\phi} j_{(z)} \]  
(A.40)

Note that the global representation of the angular velocity vector \( \omega = [0,0,\dot{\phi}]^T \) is equal to the local representation of the angular velocity vector \( \omega' = [0,0,\dot{\phi}]^T \).
Appendix B

B  Flowchart of the OCAS Trajectory-Planning Methodology
FLOWCHART OF THE OCAS TRAJECTORY-PLANNING METHODOLOGY

Figure B.1: OCAS flowchart.
FLOWCHART OF THE OCAS TRAJECTORY-PLANNING METHODOLOGY

Determine the dependence of the curve length on parameter t (see Sections 3.1.1.3 & 3.3)

1. Isolate nodal point \( N_{\text{MD}} \) (see Section 3.3.2)
   - Determine time instant \( T_{\text{MD}} \) and the 4 unknown coefficients of the cubic polynomial function in time \( s(t) \) specifying \( s(T_{\text{MD}}) = v^* \) (see Section 3.1.1.3)
   - Determine initial acceleration \( s_i(0) \) (expression (3.25))

2. Shift node \( N_i \) to coincide with node \( N_{\text{MD}} \)
   - Determine time instant \( T_i \) and the 4 unknown coefficients of the cubic polynomial function in time \( s_i(t) \) specifying \( s_i(T_i) = v^* \) (see Section 3.1.1.3)
   - Determine initial acceleration \( s_i(0) \) (expression (3.25))

3. Is central speed \( v^* \) attained at nodal point \( N_{\text{MD}} \) such that \( s(0) \leq \frac{\text{SALLOW}}{2} \)?
   - Yes: Set \( i = 1 \)
   - No: Determine time instant \( T_{\text{MD}} \) and the 4 unknown coefficients of the cubic polynomial function in time \( s(t) \) specifying \( s(T_{\text{MD}}) = v^* \) (see Section 3.1.1.3)

4. Shift node \( N_i \) to coincide with node \( N_{\text{MD}} \)
   - Determine time instant \( T_i \) and the 4 unknown coefficients of the cubic polynomial function in time \( s_i(t) \) specifying \( s_i(T_i) = v^* \) (see Section 3.1.1.3)
   - Determine initial acceleration \( s_i(0) \) (expression (3.25))

5. Is central speed \( v^* \) attained at nodal point \( N_i \) such that \( s(0) \leq \frac{\text{SALLOW}}{2} \)?
   - Yes: Shift node \( N_i \) to coincide with node \( N_{\text{MD}} \)
   - No: Shift node \( N_i \) to coincide with node \( N_{\text{MD}} \)

6. Determine time instant \( T_{\text{MD}} \) and the 4 unknown coefficients of the cubic polynomial function in time \( s(t) \) specifying \( s(T_{\text{MD}}) = v^* \) (see Section 3.1.1.3)

7. Set \( j = 1 \)
   - Shift node \( N_j \) to coincide with node \( N_{\text{MD}} \)

8. Determine time instant \( T_\text{M} \) (see Section 3.3.1)
   - Determine time instant \( T_m \) and the 4 unknown coefficients of the cubic polynomial function in time \( s(t) \) specifying \( s_m(T_m) = v^* \) (see Section 3.3.1)
   - Determine final acceleration \( |\dot{s}_m(T_m)| \) (expression (3.41))

9. Is deceleration from \( v^* \) at nodal point \( N_k \) to zero at nodal point \( N_n \) such that \( |\dot{s}_m(T_m)| \leq \frac{\text{SALLOW}}{2} \)?
   - Yes: Shift node \( N_n \) to coincide with node \( N_{\text{MD}+1} \)
   - No: Set \( j = j + 1 \)

10. Determine time instant \( T_\text{M} \) and the 4 unknown coefficients of the cubic polynomial function in time \( s(t) \) specifying \( s(T_{\text{MD}}) = v^* \) (see Section 3.1.1.3)

Figure B.2: Detail flowchart of determining the dependence of the curve length on t.

Appendix B
C THE LFOPC MATHEMATICAL OPTIMIZATION

ALGORITHM

C.1 Background

The dynamic trajectory method (also called the “leap-frog” method) for the unconstrained minimization of a scalar function $F(X)$ of $n$ real variables represented by the vector $X = [X_1, X_2, \ldots, X_n]^T$ was originally proposed by Snyman [64, 74]. The original algorithm has recently been modified to handle constraints by means of a penalty function formulation (Snyman et al [75, 76]). The method possesses the following characteristics:

- It uses only function gradient information $\nabla F(X)$.
- No explicit line searches are performed.
- It is extremely robust and handles steep valleys and discontinuities in functions and gradients with ease.
- The algorithm seeks low local minimum and can therefore be used as a basic component in a methodology for global optimization.
- The method is not as efficient as classical methods on smooth and near-quadratic functions.

C.2 Basic dynamic model

The algorithm is modeled on the motion of a particle of unit mass in an $n$-dimensional conservative force field with potential energy at $X$ given by $F(X)$. At $X$, the force on the particle is given by

$$a = \ddot{X} = -\nabla F(X) \quad (C.1)$$

from which it follows that for the time interval $[0, t]$:

$$\frac{1}{2} \|\dot{X}(t)\|^2 - \frac{1}{2} \|X(0)\|^2 = F(X(0)) - F(X(t))$$
$$T(t) - T(0) = F(0) - F(t) \quad (C.2)$$
or

\[ F(t) + T(t) = \text{constant} \ \{\text{conservation of energy}\} \]

Note that since \( \Delta F = -\Delta T \) as long as \( T \) increases \( F \) decreases. This forms the basis of the dynamic algorithm.

**C.3 LFOP: Basic algorithm for unconstrained problems**

Given \( F(X) \) and a starting point \( X(0) = X^0 \)

- Compute the dynamic trajectory by solving the initial value problem (IVP)
  \[
  \begin{align*}
  \dot{X}(t) &= -\nabla F(X(t)) \\
  X(0) &= 0, \quad X(0) = X^0 
  \end{align*}
  \tag{C.3}
  \]

- Monitor \( \dot{X}(t) = v(t) \). Clearly as long as \( T = \frac{1}{2}\|v(t)\|^2 \) increases \( F(X(t)) \) decreases and descent follows as required
- When \( \|v(t)\| \) decreases apply some interfering strategy to extract energy and thereby increasing the likelihood of descent.
- In practice a numerical integration “leap-frog” scheme is used to integrate the IVP (C.3). Compute for \( k = 0,1,2,... \) and time step \( \Delta t \)

  \[
  \begin{align*}
  X^{k+1} &= X^k + v^k \Delta t \\
  v^{k+1} &= v^k + a^{k+1} \Delta t 
  \end{align*}
  \tag{C.4}
  \]

  where \( a^k = -\nabla F(X^k), \quad v^0 = \frac{1}{2} a^0 \Delta t \)

- A typical interfering strategy is:
  If \( \|v^{k+1}\| \geq \|v^k\| \) continue
  else

    \[
    \begin{align*}
    v^k &= \frac{v^{k+1} + v^k}{4}, \\
    x^k &= \frac{x^{k+1} + x^k}{2}
    \end{align*}
    \tag{C.5}
    \]

    compute new \( v^{k+1} \) and continue.
- Further heuristics are used to determine an initial \( \Delta t \), to allow for magnification and reduction of \( \Delta t \), and to control the step size.
C.4 LFOPC: Modification for constrained problems

Constrained optimization problems are solved by the application, in three phases, of LFOP to a penalty function formulation of the problem [64, 76]. Given a function \( F(X) \), \( X \in \mathbb{R}^n \) with equality constraints \( H_i(X) = 0 \) \( (i = 1, 2, \ldots, p < n) \) and inequality constraints \( C_j(X) \leq 0 \) \( (j = 1, 2, \ldots, m) \) and penalty parameter \( \mu >> 0 \), the penalty function problem is to minimize

\[
P(X, \mu) = F(X) + \sum_{i=1}^{p} \mu H_i^2(X) + \sum_{j=1}^{m} \beta_j C_j^2(X)
\]

where \( \beta_j = \begin{cases} 0 & \text{if } G_j(X) \leq 0 \\ \mu & \text{if } G_j(X) > 0 \end{cases} \)

Phase 0: Given some \( X^0 \), then with the overall penalty parameter \( \mu = \mu_0 (= 10^2) \) apply LFOP to \( P(X, \mu_0) \) to give \( X'(\mu_0) \)

Phase 1: With \( X^0 = X'(\mu_0) \), \( \mu = \mu_1 (= 10^4) \) apply LFOP to \( P(X, \mu_1) \) to give \( X'(\mu_1) \) and identify active constraints \( i_a = 1, 2, \ldots, n_a ; \ G_i(X'(\mu_1)) > 0 \)

Phase 2: With \( X^0 = X'(\mu_1) \), use LFOP to minimize

\[
P_a(X, \mu_1) = \sum_{i=1}^{p} \mu_i H_i^2(X) + \sum_{i_a=1}^{n_a} \mu_i A_i^2(X)
\]


to give \( X^* \).
D PHYSICAL SPECIFICATION FOR SIMULATION AND OPERATIONAL CONSTRAINTS OF THE TEST-MODEL

D.1 Introduction

This Appendix describes the physical specifications required for simulation of the motion of the test-model. In addition it deals in detail with the incorporation of further physical constraints to prevent mechanical interference during the operation of the test-model.

D.2 Physical specifications for simulation of the test-model

A photograph of the test-model is shown in Figure D.1. Figure D.2 is a scaled two-dimensional view of the test-model where the eight bodies comprising the mechanical system are numbered in accordance with Figure 2.5.

Figure D.1: Photograph of the test-model
Figure D.2: Geometrical representation of the test-model as a mechanical system of eight bodies.

D.2.1 Operational geometry

Note that the same five design variables $X = [X_1, X_2, X_3, X_4, X_5]^T$, introduced in Section 4.2.1, are used here to describe the adjustable operational geometry of the physical test-model.
Design variable $X_1$ indicates the distance between revolute joints A and B on the moving platform. The center of mass of the moving platform is taken as midway between revolute joints A and B, hence 

$$\xi^a_1 = -\frac{X_1}{2} \quad \text{and} \quad \xi^b_1 = \frac{X_1}{2}.$$ 

Note that the absolute position of revolute joint D on the base twin tubular rails is fixed. Design variables $X_2$, $X_3$, and $X_4$ therefore determine the relative position of the global $Oxy$-coordinate system, relative to the positionally fixed revolute joint D. More specifically, $x = 0$ is at $-X_2 - X_4$ relative to revolute joint D, and $y = 0$ is at $-X_3$ relative to revolute joint D.

For the fixed workpiece case of Section 2.4.1 and 2.6.4.2.1, the prescribed tool path is specified in terms of the global $Oxy$-coordinate system. Hence, a change in design variables $X_2$, $X_3$, and $X_4$ will shift the position of the prescribed tool path relative to revolute joint D.

The use of the test-model is currently limited to the execution of prescribed paths specified in the fixed workpiece. With an extension of the test-model capabilities, it would also be possible to demonstrate the machining operation for the fixed tool scenario as explained in Sections 2.4.2 and 2.6.4.2.2.

For the fixed tool case, the fixed tool tip traces the prescribed tool path specified in terms of local $0_\xi_\eta$-coordinate system, and a change in design variables $X_2$, $X_3$, and $X_4$ will vary the position of the fixed tool tip relative to revolute joint D.

Finally design variable $X_3$ indicates the relative distance between revolute joints D and E, so that for simulation purposes, the global positions of revolute joints C, D and E are given by expression (4.3):

$$(x,y)_c = (X_4, X_3), \; (x,y)_d = ((X_4 + X_2),(X_3)) \quad \text{and} \quad (x,y)_e = ((X_4 + X_2 + X_3),(X_3)).$$

### D.2.2 Local coordinates

Given a prescribed tool path, as well as an operational geometry $X = [X_1, X_2, X_3, X_4, X_5]^T$, the physical motion of the test-model platform may be simulated to find the overall maximum magnitudes along the path of the individual actuator forces $f_k$, $k = 1,2,3$. This objective function (expression (4.4)) is formally defined in Section 4.2.2.

In evaluating the objective function (see Section 4.3.1), the inverse kinematic analysis of the test-model requires that the global coordinates of revolute joints C, D and E, as well as the local coordinates $\xi_2^a$, 

Appendix D
The coordinates of joints C, D and E are given by substituting the specific values of the relevant design variables into expression (4.3) (see also the last paragraph of Section D.2.1). The value of design variable \( X_i \) fixes the values of local coordinates \( \xi^{A}_{3} \) and \( \xi^{B}_{4} \) as explained in the second paragraph of Section D.2.1.

Corresponding to Section 2.3, the origin of each local \( \xi_{i}, \eta_{i} \)-coordinate system, \( i = 1, 2, \ldots, 7 \) is chosen to coincide with the center of mass of respective bodies \( 1, 2, \ldots, 7 \). Using the known dimensions and densities of the components that make up each individual body of the test-model, the positions of each body’s center of mass may be calculated. With reference to Figure D.2, the following local coordinates of the test-model are fixed:

\[
\begin{align*}
\xi^{A}_{2} &= 0.1904 \text{ m} \\
\xi^{A}_{3} &= 0.2921 \text{ m} \\
\xi^{B}_{4} &= 0.1751 \text{ m} \\
\xi^{C}_{5} &= 0.050 \text{ m} \\
\xi^{D}_{6} &= 0.050 \text{ m} \\
\xi^{E}_{7} &= 0.050 \text{ m}
\end{align*}
\]  

(D.1)

The evaluation of the objective function, furthermore involves solving the inverse dynamic equations of motion (expression (2.124)) for the unknown Lagrange multipliers and actuator forces \( \mathbf{f} \) (see Section 4.3.1).

### D.2.3 Gravitational and frictional external forces

The specific entries of the constant diagonal mass matrix \( \mathbf{M} \) in expression (2.124) consist of the specific masses and moments of inertia of the individual bodies of the test-model (see Chapter 5), and may also be determined using the known dimensions and densities of the parts constituting each body. With reference to the eight numbered bodies of Figure D.2, the entries of the mass matrix \( \mathbf{M} \) are

\[
\mathbf{M} = \text{diag} [ \mathbf{M}^T_1, \mathbf{M}^T_2, \ldots, \mathbf{M}^T_8 ]
\]  

(D.2)

with

\[
\begin{align*}
\mathbf{M}_1 &= [2.1246, 2.1246, 0.0829]^T \\
\mathbf{M}_2 &= [0.7671, 0.7671, 0.0355]^T \\
\mathbf{M}_3 &= [0.5696, 0.5696, 0.0263]^T \\
\mathbf{M}_4 &= [0.8341, 0.8341, 0.0377]^T \\
\mathbf{M}_5 &= \mathbf{M}_6 = \mathbf{M}_7 = [3.0725, 3.0725, 6.17 \times 10^{-3}]^T \\
\mathbf{M}_8 &= [0, 0, 0]^T
\end{align*}
\]

and specified in SI units.
The vector of known external forces $\mathbf{g}^{(k)}$ in expression (2.124) consists of the cutting force $\mathbf{g}^{(\text{cutting})}$ acting on the moving platform (body 1) and the weights of individual bodies comprising the planar Gough-Stewart platform $\mathbf{g}^{(\text{gravity})}_i$, $i = 1, 2, ..., 7$ (see expression (2.124)).

With the masses of the individual bodies of the platform test-model known (see Chapter 5), their respective weights follow from expression (2.104), i.e.:

$\mathbf{g}^{(\text{gravity})}_1 = [0, -20.842 \text{ N}, 0]^T$

$\mathbf{g}^{(\text{gravity})}_2 = [0, -7.526 \text{ N}, 0]^T$

$\mathbf{g}^{(\text{gravity})}_3 = [0, -5.588 \text{ N}, 0]^T$

$\mathbf{g}^{(\text{gravity})}_4 = [0, -8.182 \text{ N}, 0]^T$

$\mathbf{g}^{(\text{gravity})}_5 = \mathbf{g}^{(\text{gravity})}_6 = \mathbf{g}^{(\text{gravity})}_7 = [0, -30.141 \text{ N}, 0]^T$

The platform test-model represents a special case of the fixed work-piece scenario, with a zero tool length $\eta^T_1 = 0$ (see Section 2.4.1). This is because the pen, used for demonstration purposes, is mounted on the moving platform (body 1) at local coordinates $(\xi, \eta)_1 = (0, 0)$.

The cutting force $\mathbf{g}^{(\text{cutting})}_1$ of the fixed workpiece case is described in Section 2.6.4.2.1. Note that the cutting force is modeled as a friction force, the magnitude of which is linearly dependent on the magnitude of the cutting velocity (see expression (2.107)). The moving platform of the test-model experiences frictional forces, not only as a result of the pen tracing the prescribed tool path on the Perspex side panel, but also because of the spring loaded lateral stiffeners (see Chapter 5) sliding against the Perspex side panels during the motion of the moving platform.

The magnitude of the resultant frictional force may be measured by means of a simple experiment using a spring balance. In particular, the moving platform is disconnected from the three actuator legs, and hung unto a string. The experiment is executed by connecting the moving platform to the spring balance, and pulling the moving platform in a horizontal direction while the pen and spring loaded lateral stiffeners slide against the Perspex side panels. While moving at a constant speed along a known distance, the “constant speed motion” time and spring balance reading are measured.

Since the string supporting the moving platform is very long (1.38 m) compared to the horizontal motion (80 mm) of the moving platform, the vertical displacement of the moving platform may be neglected, hence the reading on the spring balance approximately equals the resultant frictional force.
Figure D.3: The experimental setup for measuring the frictional force using a spring balance.

Figure D.3 shows a photograph of the experimental setup used to measure the resultant frictional force on the moving platform using the spring balance. Table D.1 lists the readings that were obtained, as well as the calculation of the average cutting force constant, $C_{\text{cut}}$. The reasonable assumption is made here that the friction forces acting on the moving platform may be merged and simulated as a single cutting force (see Section 2.6.4.2.1). In order to determine the average cutting force constant, $C_{\text{cut}}$ used in expression (2.107), the experimental constant speed translations listed in Table D.1 were also measured and timed to yield the magnitudes of the constant speeds.

<table>
<thead>
<tr>
<th>Spring balance reading No.</th>
<th>Resultant friction force [N]</th>
<th>Distance [mm]</th>
<th>Time [s]</th>
<th>Constant speed [m/s]</th>
<th>$C_{\text{cut}}$ [Ns/m]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7.85</td>
<td>80 mm</td>
<td>3.74</td>
<td>0.02139</td>
<td>366.894</td>
</tr>
<tr>
<td>2</td>
<td>9.81</td>
<td>80 mm</td>
<td>1.54</td>
<td>0.05195</td>
<td>188.843</td>
</tr>
<tr>
<td>3</td>
<td>11.28</td>
<td>80 mm</td>
<td>1.13</td>
<td>0.07080</td>
<td>159.351</td>
</tr>
<tr>
<td>4</td>
<td>8.34</td>
<td>80 mm</td>
<td>3.57</td>
<td>0.02241</td>
<td>372.106</td>
</tr>
<tr>
<td>5</td>
<td>9.81</td>
<td>80 mm</td>
<td>1.23</td>
<td>0.06504</td>
<td>150.829</td>
</tr>
<tr>
<td>6</td>
<td>10.3</td>
<td>80 mm</td>
<td>1.39</td>
<td>0.05755</td>
<td>178.971</td>
</tr>
<tr>
<td>7</td>
<td>11.77</td>
<td>80 mm</td>
<td>0.77</td>
<td>0.10390</td>
<td>113.306</td>
</tr>
<tr>
<td>8</td>
<td>8.34</td>
<td>80 mm</td>
<td>3.17</td>
<td>0.02524</td>
<td>330.413</td>
</tr>
</tbody>
</table>
**Table D.1: Experimental readings in determining the average “cutting force constant”**

<p>| | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>9.81</td>
<td>80 mm</td>
<td>1.42</td>
<td>0.056338</td>
<td>174.128</td>
</tr>
<tr>
<td>10</td>
<td>11.28</td>
<td>80 mm</td>
<td>1.39</td>
<td>0.05755</td>
<td>196.016</td>
</tr>
<tr>
<td>11</td>
<td>10.79</td>
<td>80 mm</td>
<td>1.17</td>
<td>0.06838</td>
<td>157.818</td>
</tr>
<tr>
<td>12</td>
<td>7.85</td>
<td>80 mm</td>
<td>3.43</td>
<td>0.02332</td>
<td>336.483</td>
</tr>
<tr>
<td>13</td>
<td>8.83</td>
<td>80 mm</td>
<td>4.16</td>
<td>0.01923</td>
<td>459.108</td>
</tr>
<tr>
<td>14</td>
<td>19.81</td>
<td>80 mm</td>
<td>1.58</td>
<td>0.05063</td>
<td>193.748</td>
</tr>
<tr>
<td>15</td>
<td>8.34</td>
<td>80 mm</td>
<td>4.89</td>
<td>0.01636</td>
<td>509.691</td>
</tr>
<tr>
<td>16</td>
<td>9.81</td>
<td>80 mm</td>
<td>1.71</td>
<td>0.04678</td>
<td>209.689</td>
</tr>
<tr>
<td>17</td>
<td>11.28</td>
<td>80 mm</td>
<td>1.01</td>
<td>0.07921</td>
<td>142.429</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Average</td>
<td>249.4</td>
</tr>
</tbody>
</table>

The average value for $C_{cut}$ as determined from the experimental measurements is 249.4. The specific value used in all for the simulations of the motion of the platform test-model, is 250.

**D.3 Specification of the physical operational constraints of the test-model**

In illustrating the optimization methodology, the configurational constraints (expressions (4.5) and (4.6)), relating to dimensional limitations of the individual components of the manipulator, were the only inequality constraints specified to ensure a feasible design for the hypothetical planar machining center. For the real test-model some of the physical limitations of the planar mechanism may also be incorporated in these configurational constraints, while others, specifically those relating to the prevention of *mechanical interference*, must be dealt with separately.

In this section, the latter constraints are first explained in general terms below, followed by a categorization of the test-model physical limitations, and an explanation of the necessary inequality constraints with which a feasible test-model design may be obtained.

**D.3.1 Inequality constraint specification for the prevention of mechanical interference**

In general the instantaneous perpendicular distance between a line in body $j$ and a point in body $i$ may easily be determined for the special case where the line is parallel to the $\xi$-axis of body $j$.

Consider for example Figure D.4 showing the schematic representation of bodies $i$ and $j$ experiencing planar motion.
It is assumed here that the following are known:

1. the instantaneous positions and orientations of respective bodies \( i, q_i = [r^T, \phi]^T = [x, y, \phi]^T \), and \( j, q_j = [r^T, \phi]^T = [x, y, \phi]^T \), as defined by expression (2.2).

2. the fixed local coordinates of point \( P \) in body \( i \), \( (\xi^p, \eta^p) \), and

3. the fixed local \( \eta \)-coordinate of line QR in body \( j \), \( \eta^j_{R^i} \)

Using the transformation given by expression (2.1), the instantaneous global x- and y coordinates of point \( P \) may be determined, i.e.,

\[
\begin{align*}
x^p &= x_i + \xi^p \cos \phi_i - \eta^p \sin \phi_i, \\
y^p &= y_i + \xi^p \sin \phi_i + \eta^p \cos \phi_i
\end{align*}
\]

The instantaneous \( \eta \)-coordinate of point \( P \) relative to the local coordinate system of body \( j \), \( \eta^j_{R^i} \), may also be determined using the inverse of the transformation given by expression (2.1), i.e.,

\[
s^p_j = A_j^{-1} s^p_j \quad (D.4)
\]

with \( A_j^{-1} = \begin{bmatrix} \cos \phi_j & \sin \phi_j \\ -\sin \phi_j & \cos \phi_j \end{bmatrix} \).

Substituting \( s^p_j = r^p_j - r_j \) (see expression (2.1)) into expression (D.4), yields
and therefore

$$\eta^p_j = (y^p - y_j) \cos \phi_j - (x^p - x_j) \sin \phi_j \quad (D.5)$$

Hence, the instantaneous perpendicular distance between line QR and point P is given by

$$\delta_{PQR} = \eta^p_j - \eta^QR_j \quad (D.6)$$

Note that for the relative positioning of bodies i and j shown in Figure D.4, $\delta_{PQR}$ will become smaller as the two bodies move closer to each other. Hence, if the two bodies shown in Figure D.4 move relative to each other over a time interval $[0, T] = [0, T_{\text{time}}]$, the instantaneous perpendicular distance $\delta_{PQR}$ (expression (D.6)) may be monitored at discrete time instants $t_j$ to find the overall minimum perpendicular distance, i.e. $\delta_{PQR}^{\text{min}} = \min_j [\delta_{PQR}(t_j)]$ for $t_j = j \Delta t$; $j = 0, 1, 2, ..., M_{\text{time}}$; where $M_{\text{time}} = \frac{T}{\Delta t}$ and $\Delta t$ is a suitably small chosen monitoring time interval.

The alternative relative positioning of bodies i and j is shown in Figure D.5. Here the instantaneous perpendicular distance $\delta_{PQR}$ (expression (D.6)) becomes larger as the two bodies move closer to each other. Hence, if the two bodies shown in Figure D.5 move relative to each other over a time interval $[0, T] = [0, T_{\text{time}}]$, the instantaneous perpendicular distance $\delta_{PQR}$ (expression (D.6)) may be monitored at discrete time instants $t_j$ to find the overall maximum perpendicular distance, i.e. $\delta_{PQR}^{\text{max}} = \max_j [\delta_{PQR}(t_j)]$ for $t_j$ as defined above.
PHYSICAL SPECIFICATIONS FOR SIMULATION AND OPERATIONAL CONSTRAINTS OF TEST-MODEL

Figure D.5: Schematic representation of bodies $i$ and $j$ experiencing planar motion (alternative relative positioning of the two bodies).

With the judicious selection of point $P$ on body $i$, and points $Q$ and $R$ on body $j$, mechanical interference between bodies $i$ and $j$ may be prohibited for both situations as depicted in Figure D.4 and Figure D.5 respectively: Firstly, an allowable perpendicular distance $|\delta_{PQR}^{allow}|$ is chosen. Secondly, one of the following two inequality constraints is specified:

$$|\delta_{PQR}^{allow}| \leq \delta_{PQR}^{max} \quad (D.7)$$

or

$$\delta_{PQR}^{max} \leq -|\delta_{PQR}^{allow}| \quad (D.8)$$

Inequality constraint (D.7) is used for the relative positioning of bodies $i$ and $j$ as shown in Figure D.4, while inequality constraint (D.8) is used for the relative positioning of bodies $i$ and $j$ as shown in Figure D.5.

**D.3.2 Linearly adjustable revolute joints**

The configurational constraints limiting the allowable relative distances between the linearly adjustable revolute joints of the fixed base ($X_1$ and $X_5$) and the moving platform ($X_4$) (see expression (4.5)) are directly applicable on the physical test-model. With reverence to Figure D.6 showing the adjustable capability of the test-model, the specific bounds (given in meters) on design variables $X_1$, $X_2$ and $X_5$ are

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\[
0.1 \leq X_i \leq 0.45 \\
0.113 \leq X_2 \leq 0.465 \\
0.113 \leq X_3 \leq 0.27
\]  
\[(D.9)\]

\[
X_1 = 0.45 \\
X_2 = 0.465 \\
X_3 = 0.27
\]

**Figure D.6: Adjustable capability of the test-model.**

The lower and upper bounds specified in expression (D.9) correspond exactly to the lower and upper bounds that were specified for the optimization test run of the hypothetical planar machining center (see expression (4.14)). Hence, in agreement with the inequality constraints given by expression (4.7), the first six inequality constraints used to ensure a feasible test-model design, are

\[
\begin{align*}
C_1(X) &= X_1 - 0.45 \leq 0 \\
C_2(X) &= 0.1 - X_1 \leq 0 \\
C_3(X) &= X_2 - 0.465 \leq 0 \\
C_4(X) &= 0.113 - X_2 \leq 0 \\
C_5(X) &= X_3 - 0.27 \leq 0 \\
C_6(X) &= 0.113 - X_3 \leq 0
\end{align*}
\]  
\[(D.10)\]

**D.3.3 Extreme motion constraints**

The extreme motion of the hypothetical moving platform is bounded by the *allowable* minimum and maximum actuator leg lengths (see expressions (4.6) and (4.8)). On the other hand, the motion of the physical moving platform is to be confined within the four frame boundaries represented by the four sides of rectangle FGHI as annotated in Figure D.7.
D.3.3.1 Upper frame boundary

The upper frame boundary (line HI) cannot be exceeded by the lateral stiffeners on the moving platform with specification that the allowable maximum actuator leg length $\ell_k$ for all three actuator legs $k = 1, 2, 3$ is 0.525 m. Figure D.8 serves to illustrate this fact.
Test-model inequality constraints 7-9 are given by

\[
\begin{align*}
C_7(X) &= \ell_1^{\max}(X) - 0.525 \leq 0 \\
C_8(X) &= \ell_2^{\max}(X) - 0.525 \leq 0 \\
C_9(X) &= \ell_3^{\max}(X) - 0.525 \leq 0
\end{align*}
\] (D.11)

where the overall maximum leg lengths for any prescribed path are given by \( \ell_k^{\max}(X) \), \( k = 1, 2, 3 \) as explained in Section 4.3.2. These constraints (expression (4.8)) correspond exactly to the constraints \( C_{k+6}(X) \equiv \ell_k^{\max}(X) - 0.525 \leq 0 \), \( k = 1, 2, 3 \) specified for the optimization of the hypothetical platform see expression (4.8).

**D.3.3.2 Lower frame boundary**

The lower frame boundary of the test-model is represented by line FG in Figure D.7. Here it is important to prevent mechanical interference between the moving platform and the bottom frame cross members indicated by shaded regions a and b in Figure D.7. Line FG coincides with the top plane of the two bottom frame cross members.

With the moving platform in a horizontal orientation, the bottom ends of the adjustable brackets of revolute joints A and B are the lowest points on the moving platform (see Figure D.7). Due to the fact that the relative positions of the revolute joints may be adjusted, it is highly unlikely that the adjustable
brackets of revolute joints A and B will collide with the bottom frame cross members (shaded regions a
and b), even with the moving platform in a horizontal orientation \( \phi = 0 \). Furthermore, for a large
enough CCW rotation of the moving platform, point J indicated in Figure D.7 is the lowest point on the
moving platform. Similarly, for a large enough CW rotation of the moving platform, point K (see Figure
D.7) is the lowest point on the moving platform.

The lower frame boundary is therefore treated here by monitoring the perpendicular distances between
line FG on the frame, and respective points J and K on the moving platform following the methodology
explained in Section D.3.1. In particular, assumptions 1 - 3 listed in Section D.3.1 are valid here, since
the instantaneous position \((x_i, y_i)\) and orientation \(\phi_i\) of the moving platform (body 1 in Figure D.2) are
known as the prescribed path is traced (see expressions (2.27) - (2.29)). Furthermore, the global position
\((x_g, y_g)\) and orientation \(\phi_g\) of the frame (body 8 in Figure D.2) are fixed: \([x_g, y_g, \phi_g]^T = [0,0,0]^T\). The
fixed local coordinates (in meters) of points J and K are \((\xi_j, \eta_j) = (-0.285,-0.004)\) and
\((\xi_k, \eta_k) = (0.285,-0.004)\), and the fixed local \(\eta\)-coordinate of line FG is \(\eta_{FG} = X_3 + 0.065\). Note that
\(X_3\) is the design variable representing the \(y\)-coordinate of the three base revolute joints C, D and E (see
Figure D.2). Hence, the respective instantaneous perpendicular distances \(\delta_{j,FG}(X, t_{ij})\) and \(\delta_{k,FG}(X, t_{ij})\)
may be determined at any time instant \(t_{ij}, i = 0,1,2,...,N-1, j = 0,1,2,...,n_{time}\) (see Section 4.2.2) in
accordance with expressions (D.3) - (D.6). The default value for \(n_{time} = 10\) (see Appendix B).

Since the relative positioning of the moving platform with respect to the lower frame boundary conforms
to the relative positioning of the bodies \(i\) and \(j\) as depicted in Figure D.4, the lower frame boundary
inequality constraints formulated here correspond to the inequality (D.7):

\[
\begin{align*}
C_{10}(X) &= 0.006 - \delta_{j,FG}^{\min}(X) \leq 0 \\
C_{11}(X) &= 0.006 - \delta_{k,FG}^{\min}(X) \leq 0
\end{align*}
\]

(D.12)

where \(\delta_{j,FG}^{\min}(X) = \min_{i,j}[\delta_{j,FG}(X, t_i + t_j)]\), and \(\delta_{k,FG}^{\min}(X) = \min_{i,j}[\delta_{k,FG}(X, t_i + t_j)]\), and with \(t_i\) and \(t_j\) as
defined above.

The same value of 0.006 m is used for both allowable perpendicular distances \(|\delta_{j,FG}^{\text{allow}}|\) and \(|\delta_{k,FG}^{\text{allow}}|\) in
expression (D.12). This value was chosen, so that the shortest attainable actuator leg length, without
violating inequality constraints \(C_{10}\) and \(C_{11}\) (expression (D.12)), is 0.075 m as shown in Figure D.9.
This length is also the allowable minimum actuator leg length specified for the hypothetical platform in
expression (4.15).
Since the shortest attainable actuator leg length corresponds exactly to the allowable minimum actuator leg length, inequality constraints \( C_{k,9}(X) = \ell_k - \ell_k^m(X) \leq 0, \ k = 1,2,3 \) (expression (4.8)) specified for the hypothetical platform are redundant in the optimization of the platform test-model.

### D.3.3.3 Left hand frame boundary

The left- and right hand frame boundaries limit the horizontal movement of the physical moving platform. With the use of the spring loaded lateral stiffeners (see Chapter 5), point A is restricted to the right hand side of line FI, and point B is restricted to the left hand side of line GH (see Figure D.7).

Consider for the moment the left hand frame boundary. Point A is on the moving platform (body 1 in Figure D.2) at local coordinates \((\xi^A, \eta^A) = \left( -\frac{X_1}{2}, 0 \right)\), and the global position and orientation of body 1 \([x_1, y_1, \phi_1]^T\) are known at each time instant as the prescribed path is traced. The global position of point A may therefore be determined in accordance with expression (D.3).

Line FI on the frame (body 8 in Figure D.2) is dealt with in a special manner. According to the definitions given in Section 2.3, the fixed body 8 is considered as the ground of the planar Gough-Stewart platform mechanism. Figure D.2 shows that the origin of body 8 is chosen to coincide with the origin of the global 0xy-reference frame, \((x_8, y_8) = (0,0)\), and that the local \(0_s \xi_s \eta_s\) -coordinate system and the global 0xy-reference frame are identically orientated, i.e. \(\phi_s = 0\). This implies that the fixed
vertical line FI is parallel to the η-axis of body 8. However, the proposed collision prevention methodology explained in Section D.3.1 is based on the assumption that the line in body j is parallel to the ξ-axis of body j (see Figure D.4). The special treatment of line FI consists of the specification that body 8 is angled at 90°, i.e. \( \phi_8 = \frac{\pi}{2} \text{ rad} \), and is allowable, since each inequality constraint is treated separately and independent of the kinematic and kinetic analysis (Chapter 2).

The global x-coordinate of line FI is \( X^I = X_4 + X_2 - 0.6 \) (see Figure D.7), with \( X_4 \) and \( X_2 \) two of the five design variables describing the adjustable geometry of the planar Gough-Stewart platform machining center (see Figure 4.1). With the specification that \( [x_8, y_8, \phi_8]^T = [0, 0, \frac{\pi}{2}]^T \), the local η-coordinate of line FI may be determined using the transformation given by expression (D.4): \( \eta_8^I = -(X_4 + X_2 - 0.6) \).

The instantaneous perpendicular distance between point A on the moving platform and line FI on the frame \( \delta_{A-FI} \) may therefore be determined in accordance with expression (D.6). The relative positioning of the moving platform with respect to the left hand frame boundary agrees with the relative positioning of bodies i and j as depicted in Figure D.5. As a result of this, the instantaneous perpendicular distance \( \delta_{A-FI} \) becomes larger as the moving platform moves closer to the left hand frame boundary. The left hand frame boundary inequality constraint is therefore given by

\[
C_{L1}(X) = \delta_{A-FI}^\text{max}(X) + 0.015 \leq 0
\]

with \( \delta_{A-FI}^\text{max}(X) = \max_{i,j}[\delta_{A-FI}(X, t_i + t_j)] \) and \( t_i + t_j \) as defined in Section D.3.3.2. Since the radius of the spring loaded lateral stiffener is 15 mm, a value of 0.015 m is assigned to the allowable perpendicular distance \( |\delta_{A-FI}^\text{actual}| \) in the above expression. Note also that expression (D.13) corresponds to inequality (D.8) derived for the general situation depicted in Figure D.5.

**D.3.3.4 Right hand frame boundary**

The right hand frame boundary restricts point B on the moving platform to the left hand side of line GH in Figure D.7. Point B on the moving platform (body 1 in Figure D.2) is at local coordinates \( (\xi^n, \eta^n)_B = \left( \frac{X_1}{2}, 0 \right) \) (see Section D.2.1). Line GH on the frame (body 8 in Figure 2.5) is treated here in a similar manner to line FI of the left hand frame boundary (see Section D.3.3.3) with the specification...
that \( [x_g, y_g, \phi_g]^T = \left[ 0, 0, \frac{\pi}{2} \right]^T \). The global \( x \)-coordinate of line GH is \( x_{GH} = x_4 + x_5 + 0.4 \), from which the local \( \eta \)-coordinate, \( \eta^\text{GH}_x = -(x_4 + x_5 + 0.4) \), may be determined (see expression (D.4)).

Note that since the perpendicular distance between line GH and point B becomes smaller as the platform moves closer to the right hand frame boundary, the right hand frame boundary inequality constraint is given by

\[
C_{15}(X) = 0.015 - \delta_{\text{min}}^{\text{GH}}(X) \leq 0
\]

with \( \delta_{\text{min}}^{\text{GH}}(X) = \min_{i,j} \{ \delta_{\text{B-GH}}(X, t_i + t_j) \} \), and \( \delta_{\text{B-GH}}(X, t_i + t_j) \) solved for in accordance with expression (D.6) at each time instant \( t_i + t_j \). Once again, an allowable perpendicular distance \( |\delta_{\text{min}}^{\text{allow}}| = 0.015 \text{ m} \) is specified to compensate for the 15 mm radius of the spring loaded lateral stiffener (see Chapter 5).

**D.3.4 Revolute joint mechanical interference constraints**

There are no explicit constraints specified for the relative rotations about the revolute joints of the hypothetical platform. In practice however, the allowable rotations about the revolute joints of the physical test-model are limited as a result of mechanical interferences.

The design and assembly of the test-model is explained in detail in Chapter 5. Figure D.10 shows an annotated two-dimensional view of the planar Gough-Stewart platform test-model, where the different components involved in the revolute joint mechanical interferences are annotated.
Consider for the moment the fixed base platform assembly. In essence, and with reference to Figure D.10 each of the three revolute joints C, D and E is connected to a pair of carrier blocks, which are linearly adjustable along the base tubular twin rails. These base tubular twin rails are connected to the frame by means of mounting brackets. The base revolute joints C, D and E carry the actuator leg drive units, each consisting of a motor and gearbox assembly.

Varying the actuator leg lengths, not only causes the moving platform to change its position and orientation, but also causes the relative orientations of the actuator legs to vary. The relative orientations of the actuator legs and drive units correspond exactly, hence the potential danger exists of mechanical interference between the drive units and the different components of the fixed base frame.

**D.3.4.1 Revolute joint C mechanical interference constraints**

The relative position of revolute joint C on the base tubular twin rails is determined by design variable \( X_2 \) (see Section D.2.1). Depending on the magnitude of \( X_2 \), an excessively large CW rotation of
actuator leg 1 will cause the drive unit carried by revolute joint C to collide with either the left hand mounting brackets, or with the base tubular twin rails.

The proposed inequality constraint methodology for the prevention of mechanical interference (Section D.3.1) may however be applied here to formulate two separate inequality constraints with which both potential collisions may be avoided:

\[
\begin{align*}
C_{14}(X) &= \delta_{M1-PQ}^\text{max}(X) + 0.005 \leq 0 \quad (D.15) \\
C_{15}(X) &= 0.005 - \delta_{L-M1M2}^\text{min}(X) \leq 0 \quad (D.16)
\end{align*}
\]

Figure D.11: Inequality constraint \( C_{14} \) (expression (D.15)) active.

Figure D.11 shows a scaled two-dimensional view of the test-model with \( X_z = 0.360 \text{ m} \), where drive unit C is about to collide with the base twin tubular rails, but not with the mounting brackets. Note that the perpendicular distance between line PQ and point M1 is \( \delta_{M1-PQ} = 0.005 \text{ m} \), rendering inequality constraint \( C_{14} \) (expression (D.15)) active. In spite of this, the perpendicular distance between line M1M2 and point L is \( \delta_{L-M1M2} = 0.028 \text{ m} \), so that inequality constraint \( C_{15} \) (expression (D.16)) is not active.
The evaluation of inequality constraints $C_{14}$ is summarized in Table D.2 below:

<table>
<thead>
<tr>
<th>General case analogy</th>
<th>Inequality constraint $C_{14}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta_{P-QR}^{\text{max}} \leq -\frac{\delta_{P-QR}^{\text{low}}}{\delta_{P-QR}^{\text{low}}} \text{ (expression (D.8))}$</td>
<td>$C_{14}(X) = \delta_{m1-PQ}^{\text{max}}(X) + 0.005 \leq 0 \text{ (expression (D.15))}$</td>
</tr>
</tbody>
</table>

- **Assumption 1**: (see Section D.3.1)
  - $[x, y, \phi]_I^T$ must be known
  - $[x, y, \phi]_I^T$ known from the inverse kinematic analysis (see Section 2.5)

- **Assumption 2**: (see Section D.3.1)
  - $(\xi_5^P, \eta^o)_{i}$ must be known
  - $(\xi_5^{M1}, \eta^{M1})_{i} = (-0.06439, 0.03)$ (see Figure D.13)

- **Assumption 3**: (see Section D.3.1)
  - $\eta_{18}^{o8}$ must be known
  - $\eta_{18}^{o8} = X_3 - 0.015$

**Table D.2: Evaluation of constraint $C_{14}$ (expression (D.15)).**

![Figure D.12: Inequality constraint $C_{15}$ (expression (D.16)) active.](image-url)
On the other hand, Figure D.12 shows a scaled two-dimensional view of the test-model with $X_2 = 0.465$ m, where drive unit C is about to collide with the left hand mounting bracket, and not with the base twin tubular rails. As expected, the perpendicular distance between line PQ and point M1, is greater than 0.005 m ($\delta_{M1-PQ} = 0.031$ m) so that inequality $C_{14}$ (expression (D.15)) is not active, while $\delta_{L-M1M2} = 0.005$ m, so that inequality $C_{15}$ (expression (D.16)) becomes active.

The evaluation of inequality constraint $C_{15}$ are summarized in Table D.3 below:

<table>
<thead>
<tr>
<th>General case analogy</th>
<th>Inequality constraint $C_{15}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta_{allow} \leq \delta_{P-QR}^{\text{min}}$ (expression (D.7))</td>
<td>$C_{15}(X) \equiv 0.005 - \delta_{L-M1M2}^{\text{min}}(X) \leq 0$ (expression (D.16))</td>
</tr>
</tbody>
</table>

Assumption 1: (see Section D.3.1)

$[x,y,\phi]^T$ must be known

$[x,y,\phi]^T = [0,0,0]^T$ fixed frame position and orientation (see Figure D.2)

Assumption 2: (see Section D.3.1)

$[x,y,\phi]^T$ must be known

$[x,y,\phi]^T$ known from the inverse kinematic analysis (see Section 2.5)

Assumption 3: (see Section D.3.1)

$(\xi^*,\eta^*)_i$ must be known

$(\xi^*,\eta^*)_i = ((X_4 + X_2 - 0.555),(X_3 - 0.025))$

$\eta_{IM2}^Q = 0.03$ (see Figure D.13)

Table D.3: Evaluation of constraint $C_{15}$ (expression (D.16)).

Figure D.13: Local coordinates on drive unit C.
D.3.4.2 Revolute joint D mechanical interference constraints

The relative position of revolute joint D on the base twin tubular rails is fixed (see Section D.2.1). However, the adjustable relative position of revolute joint E, given by the magnitude of design variable $X_s$, influences whether drive unit D will collide with the base twin tubular rails, or with the carrier blocks of revolute joint E. These potential collisions will of course only occur for excessively large CCW rotations of actuator leg 2. A third possible collision may occur between actuator leg 2 and drive unit E, depending on the value of $X_s$, as well as the relative orientations of actuator legs 2 and 3.

In order to prevent all three potential collisions from happening, three inequality constraints are formulated (refer to the annotations in Figure D.10 and Figure D.14 respectively):

$$C_{16}(X) = \delta_{R1-PQ}^{\text{max}}(X) + 0.005 \leq 0 \quad (D.17)$$

$$C_{17}(X) = \delta_{S-R1R2}^{\text{max}}(X) + 0.005 \leq 0 \quad (D.18)$$

$$C_{18}(X) = \delta_{T5-R3R4}^{\text{max}}(X) + 0.005 \leq 0 \quad (D.19)$$

Figure D.14: Local coordinates on drive unit D.

D.3.4.3 Revolute joint E mechanical interference constraints

As previously stated in Sections D.2.1 and D.3.4.2, the magnitude of design variable $X_s$ determines the relative position of revolute joint E on the base twin tubular rails. For an extreme CCW orientation of actuator leg 3, and depending on the magnitude of design variable $X_s$, drive unit E will collide either with the base twin tubular rails, or with the right hand mounting bracket. There also exists the possibility of mechanical interference between actuator leg 3 and drive unit D, depending on the magnitude of design variable $X_s$, and the relative orientations of actuator legs 2 and 3.
Following the same recipe as before, three inequality constraint equations are formulated to prevent the above mentioned potential collisions from happening (refer to the annotations in Figure D.10 and Figure D.15 respectively):

\[
C_{19}(X) = \delta_{\text{V1-PQ}}^{\text{max}}(X) + 0.005 \leq 0 \tag{D.20}
\]
\[
C_{20}(X) = \delta_{\text{V2-V1V2}}^{\text{max}}(X) + 0.005 \leq 0 \tag{D.21}
\]
\[
C_{21}(X) = 0.005 - \delta_{\text{R5T3T4}}^{\text{min}}(X) \leq 0 \tag{D.22}
\]

Figure D.15: Local coordinates on drive unit E.

**D.3.4.4 Revolute joint A mechanical interference**

Depending on the relative orientation of the moving platform (body 1 in Figure D.2) and actuator leg 1 (body 2 in Figure D.2), point J1 on the moving platform left hand bracket may collide with actuator leg 1. In order to avoid such a collision from happening, the following inequality constraint is formulated (see Figure D.16):

\[
C_{22}(X) = 0.005 - \delta_{\text{J1-M3M4}}^{\text{min}}(X) \leq 0 \tag{D.23}
\]
Figure D.16: Inequality constraint $C_{22}$ (expression (D.23)) active.