BUILDING BLOCKS FOR SUPPLY CHAIN MANAGEMENT – A STUDY OF INVENTORY MODELLING

by

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ABSTRACT

Title: Building Blocks for Supply Chain Management – A Study of Inventory Modelling

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This thesis presents a study of stochastic models of continuous review of inventory systems of perishable and non-perishable products, as well as inventory systems operating in random environment. It contains five chapters. The first chapter is introductory in nature, containing the motivation for the study and the techniques required for the analysis of respective models described in the remaining chapters.

Chapter 2 provides a model of perishable product inventory system operating in a random environment. For the sake of simplicity, the stochastic environment is considered to alternate randomly over time between two states 0 and 1 according to an alternating renewal process. When the environment is in state k, the items in inventory have a perishable rate μ_k, the demand rate is λ_k and the replenishment cost is CR_k. The performance of various measures of the system evolution are obtained, assuming instantaneous replenishment at the epoch of the first demand after the stock-out and associating a Markov renewal process with the inventory level.

In Chapter 3, a continuous review single product perishable inventory model is considered. Items deteriorate in two phases and then perish. Independent demands occur at constant rates for items in phase I and in phase II. Demand that occurs for an item in phase I during its stock-out period is satisfied by an item in phase II with some

probability. However a demand for an item in phase II occurring during its stock-out period is lost. The reordering policy is an adjustable (S,s) policy with the lead-time following an arbitrary distribution. Identifying the stochastic process as a renewal process, the probability distribution of the inventory level at any arbitrary instant of time is obtained. The expressions for the mean stationary rates of demands lost, demands substituted, perished units and scrapped units are also derived. A numerical example is considered to highlight the results obtained.

Chapter 4 is a study of a two-commodity inventory system under continuous review. The maximum storage capacity for the *i*-th item is S_i (i=1, 2). The demand points for each commodity are assumed to form an independent Poisson process, with unit demand for one item and bulk demand for the other. The order level is fixed as s_i for the *i*-th commodity (i=1, 2) and the ordering policy is to place an order for Q_i (= $S_i - s_i$, i = 1,2) items for the *i*-the commodity when both the inventory levels are less than or equal to their respective reorder levels. The lead-time is assumed to be exponential. The joint probability distribution for both commodities is obtained in both transient and steady state cases. Various measures of systems performance and the total expected cost rate in the steady state are derived. The results are illustrated with numerical examples.

Chapter 5 provides an analysis of a continuous review of two-product system with two types of demands and with individual (S,s) ordering policy. The lead-time distribution of product 1 is arbitrary and that of product 2 exponential. Two types of demands occur at constant rates either for both products or for product 2 alone. Expressions for the stationary distribution of the inventory level are obtained by identifying the underlying stochastic processes as a semi-regenerative process. The mean stationary rates of the lost demands, the demands that are satisfied and the number of reorders are obtained and these measures are used to provide an expression for the cost rate.

The main objective of this thesis is to improve the state of art of continuous review inventory systems. The salient features of the thesis are summarized below:

(a) Consideration of
  (i) The impact of the stochastic environment on inventory systems;
  (ii) The interactions existing among the products in multi-product systems;
  (iii) Individual and joint-ordering policies;

(b) Discussion of inventory systems with perishable products;
(c) Effective use of the regeneration point technique to derive expressions for various system measures;
(d) Illustration of the various results by extensive numerical work;
(e) Relevant optimization problems

Key Words:

Inventory optimization
Inventory modelling
Inventory systems in a stochastic environment
Perishable product inventory systems
Continuous review inventory systems
Two-commodity inventory system
Individual and joint-ordering policies
(S, s) order policy
Regeneration point technique
Renewal process

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CHAPTER 1

INTRODUCTION

1.1 SUPPLY CHAIN MANAGEMENT

1.1.1 Background

A new era has dawned in Supply Chain Management with the advent of globalization. This has led to increased competition and in order to achieve and sustain competitive advantage, companies must be able to respond quickly to customer demand and deliver a high level of customer service. The need for companies to be flexible and to be able to customize their products is also becoming more important. This added pressure on supply chains, coupled with global deregulation, is encouraging many companies to move the sourcing of components and low-value added operations offshore, to lower cost countries (Ross, 2003) - this result in supply chains which increase in distance and complexity.

With global markets and suppliers, companies need to have a supply chain that is lean on inventory and responsive to customer demand. To ensure an efficient supply chain, all aspects of such a supply chain need to be monitored continually and inputs need to be managed in order to anticipate any uncertainty in supply, demand and cost and to ensure that appropriate contingencies are in place.

According to Lakahl et al (2001) companies must concentrate on their core competencies to help sustain competitive advantage. Non-strategic activities that can be performed more effectively by a third party need to be externalized. A company's core competencies depend heavily on its resources and how they are utilized and if a company is able to develop and allocate resources in a way, which creates more value for customers than their competitors can, it creates a sustainable competitive advantage. A superior supply chain strategy maximizes the value added by internal activities while developing solid partnerships leading to high value external activities.

Supply chain management is plagued with conflicting objectives and supply chain managers must make appropriate tradeoffs to ensure optimal functioning of the supply chain. Traditionally inventory was used to ensure compliance with customer demand and to guard against uncertain delivery lead times. Economies of scale is another reason for inventory accumulation - fixed costs are lowered by producing or ordering in large quantities, transportation discounts can be achieved and it guards against uncertainties. The problem with high inventories however is that capital is tied up and high inventory holding costs is incurred. The inability to meet customer demand, in turn, leads to lost profits and in the long run, possibly the loss of clients. Thus the trade off between customer satisfaction and inventory holding costs is one of the most important decisions that a supply chain manager has to make.

The problem of providing customer satisfaction under conditions of demand variability is usually addressed with safety stock. In the literature, safety stock are considered from the traditional inventory theory viewpoint and it fails to address key features of realistic supply chain problems such as multiple products sharing multiple production facilities with capacity constraints and demand originating from multiple customers. Safety stock levels are dependant on factors such as probabilistic distributions of demand, the demand-capacity ratio as well as the dependence of overall customer satisfaction levels on meeting demands for several different products produced at the same facility (Jung et al, 2004).

In order to manage the supply chain, a supply chain manager needs accurate, timely information. To produce corporate planning solutions, one, or a combination of enterprise planning methods are used, these include manual processes, proprietary planning solutions, Enterprise Resource Planning (ERP) and Advanced Planning and Scheduling (APS).

To support the increasingly complex analysis associated with extended supply chains, decision support tools have to lead key strategic, tactical and operational decisions at every stage of the supply chain. These tools have to provide insight into the tradeoffs that have to be made among alternative strategies regarding, for example, site location, transportation strategies, inventory strategies, resource allocation and supply chain operations (Padmos et al, 1999). In addition to this, these tools and the methods that they employ need to take the uncertainties that are characteristic of supply chains (e.g. demand uncertainty), into consideration.

The objective is to have a supply chain were all participants act as if they are part of one entity in an effort to maximize the timely arrival of good quality raw material, minimum lead times and minimum reasonable inventory – this will contribute to a "seamless supply chain" (Kerbache & Smith, 2004).

1.1.2 Literature Review of Supply Chain Optimization

A study was undertaken to consider various supply chain optimization approaches available in literature. Literature with regards to supply chain optimization is abundant and no attempt is made to do a complete review. In agreement with the observation that Kerbache & Smith (2004) made, it is observed that the literature has taken three directions:

1. *Purchasing and supply perspective*: The interest here is directed toward the upstream supply chain.
2. *Transportation and logistics perspective*: Interest focused on the downstream supply chain activities.
3. *Complete supply chain perspective*: Attempts are made to deal with the supply chain as a whole (De Kok & Graves, 2003) .

The interest for this paper was focused on literature that takes the third direction – that is, literature that considers the complete supply chain.

Such literature seemed to be subdivided into three categories:

a. Modelling the supply chain using mathematical programming (Operations Research Techniques) (Stadtler & Kilger, 2002)
b. Modelling the supply chain through simulation modelling
c. Modelling the supply chain using IT-driven techniques – these includes object oriented modelling and intelligent agent technology

A brief discussion of the approach within each of these three groups is provided.

a. Modelling the supply chain using mathematical programming (Operations research techniques)

Operations Research models are either deterministic or stochastic.

(i) Deterministic Programming Models

Deterministic models are used to address strategic and tactical decisions through the use of mixed integer linear programming (MILP) or mixed integer programming (MIP). The objective of these models is usually to maximize after-tax profit or minimize supply chain costs. Because of the complexity of some of the models, heuristics are often used to attain solutions.

Linear programming and mixed integer programming models are developed to address various decisions that have to be made in the supply chain. These solutions give answers to strategic, tactical and operational decisions.

In an effort to make strategic investment decisions easier, for example with regards to alternative products and development projects, Fandel & Stammen (2004) designed a general linear mixed integer model by considering the total product life cycle, including development and recycling. The goal of their approach was to optimize after-tax profit and to fix the product program and the extended supply chain network.

To investigate strategic networking issues, Lakhal et al (2001) developed a large mixed integer programming (MIP) problem that aims to find the networking strategy that maximizes the value added by internal activities of the company (they equate this to maximizing profits). Because of the complexity of the problem, the MIP is relaxed and a heuristic is used to obtain solutions for an illustrative example. The authors however admit that the static nature of the model poses an important limitation, as supply chains are inherently dynamic.

(ii) Stochastic programming models

Stochastic operations research models incorporate multi-objective mixed integer linear programming (MILP) and mixed integer non-linear programming (MINLP) in an attempt to resolve strategic and operational problems. These problems aims to maximize supply chain profit and customer satisfaction. For tactical decision making a non-deterministic (NP) hard problem is used, but because of the complexity a suitable heuristic is develop.

In a multi-objective stochastic MILP problem, Guillén et al (2005) consider strategic and operational decision-making. Decisions such as the capacity and location of plants and warehouses, the amount of products to be made at each plant and the flow of material between each two nodes of the supply chain are addressed in a hypothetical example. The objective of the problem is to maximize supply chain profit and customer satisfaction and also takes uncertainty into account by means of the concept of financial risk. The problem is solved using a standard-constraint method and branch and bound techniques.

In a novel approach, Seferlis & Giannelos (2004) uses a two-layer optimization-based control approach for use in operational decision-making. The control strategy applies multivariable model-predictive control principles to the entire supply chain. This is done whilst safety inventory levels are maintained through the use of dedicated feedback controllers for every product and storage node. These inventory controllers are embedded in the optimization framework as additional equality constraints. The optimization-based controller aims to satisfy multiple objectives: that is to maximize customer satisfaction and minimize operating costs. It is not clear from the source which operational research method is employed although extensive detailed equations, assumptions and constraints are described. Illustrative simulations are used to demonstrate that the model can accommodate supply chain networks of realistic size under a variety of stochastic and deterministic disturbances.

(iii) Queuing network models

Using a queuing network, Arda and Hennet (2004) represent a simple two-level supply chain. With this network, the producer uses a base-stock inventory control policy that keeps the inventory position level (current inventory plus pending replenishment orders) constant. The decision variables are the reference inventory position level and the percentages of orders sent to the different suppliers. In the model, the percentages of orders are implemented as Bernoulli branching parameters. The expected cost is obtained as a complex non-linear function of the decision variables. A centralized inventory control model is incorporated to combine supply and demand randomness in the queuing network model. Because of the complexity of the problem, a decomposed approach is proposed for solving the optimization problem in an approximate manner. When applied to a test case, the approximate solutions' quality is evaluated when it is compared with the numerically computed values. This can however only be done for simple cases and the main drawback of this model is its simplicity in that it can only show you the economic advantages for the producer of using several suppliers instead of just one.

Kerbache and Smith (2004) also use queuing network systems to model and analyze supply chains. They focus on using closed queuing network systems to evaluate performance measures such as throughput, cycle time and WIP. The methodology employ analytical queuing networks coupled with nonlinear optimization in order to maximize the throughput of the system offset by the cost of providing the service. A case study is used to demonstrate the use of the model and to show that it provides a useful tool with which to analyze congestion problems and to evaluate the performance of the network.

b. Modelling the supply chain using simulation

Supply chain modelling with simulation can be divided into descriptive and normative/optimization models. Simulation proves to be problematic as that experts are needed to construct realistic models. This is time consuming and even if a realistic model is constructed it is even more problematic and time consuming to gather the input data for the model (Bansal 2002).

Notwithstanding these problems, simulation is still used in supply chain optimization. A discrete-event simulation model, which have a linear programming model embedded in it, is used to minimize costs, maximize customer satisfaction and sustain acceptable inventory levels.

Kalasky (1996) presents an application of discrete-event simulation in modelling the supply chain for consumer products. The author employs a linear program (LP) to provide for cost models of the supply chain. The objective of the LP is to satisfy multiple objectives, namely minimize costs, maximize customer service levels and sustain acceptable inventory levels. The combined technologies of simulation and optimization provide a viable and useful tool for planning and operation of supply chains.

c. Modelling the supply chain using IT-driven techniques

IT-driven approaches suggested to optimize and model supply chains, are object oriented modelling and intelligent agent technology. Object oriented modelling employs generic building blocks in a simulation model. Operations research techniques (LP and MIP) are embedded in the object-oriented model to help with strategic, tactical and operational decision-making.

1.2 INVENTORY OPTIMIZATION

1.2.1 Inventory Management

Managing inventory within the supply chain is a key aspect of almost any business, that is the ability to provide the right goods or materials at the right price, place and time. Inventory is one of the most visible and tangible aspects of doing business and, as a result, all the problems of a business often end up in inventory. The role of inventory management is to coordinate the actions of sales, marketing, production and purchasing to ensure that the correct level of stocks are held to satisfy customers demand at the lowest possible cost. Inventory management aims to balance the supply and demand equation by regulating the supply of goods to affect their availability in such a way that they match demand conditions as closely as possible (Wheller, 2004). Inventory management involves methods or processes and is a fundamental requirement prior to considering inventory or supply chain optimization.

1.2.2 Inventory Optimization in Software Applications

"Few supply-chain problems have proved as difficult as inventory optimization" according to Murphy (2003:1). He compares managing inventory levels across the supply chain, so as to consistently meet customer requirements at the least possible cost, to squeezing a balloon: air that gets pressed out in one place pops up somewhere else. One reason is that functional solutions tend to optimize a single point in the chain without taking into account the impact of these changes on other areas. Moreover, determining just the right amount of each product to make, how much to place where, when to re-order and in what quantities, are very hard problems to solve. Supply and demand variability precludes the use of linear algorithms that is used to optimize other areas of the supply chain.

A report by Aberdeen Group found that more than 60% of companies use overly simplistic inventory management methods, such as ABC inventory policies or simple weeks-of-supply rules for products. These companies frequently have 15-30 % more inventory than they need and lower service levels. Less that 5 % of companies surveyed are factoring in total supply chain variability when determining inventory policies (Enslow 2004:1-17).

According to Murphy (2003:1) companies are trying to deal with the inventory problem from an execution perspective. They use visibility and alerting tools to get an early view of where the plan is wrong in order to ensure that corrective action can be taken. While helpful, this approach is not a substitute for optimizing inventory levels across the chain. Execution tools can go a long way toward solving supply disruptions, but often resolution is not responsive enough, resulting in more buffer inventories.

Academic research has resulted in significant breakthroughs in stochastic modelling (problems with a high degree of variability). Mathematical algorithms invented in the 1990's and tested over several years at individual companies, are now coming to the market in the form of new Inventory Optimization products. These solutions promise to change the way companies set policies on safety stock, not just for finished goods, but across entire supply chains, with huge potential for savings. These optimization engines are highly sophisticated with algorithms that consider consumption, supply, various lead-times and then determine the amount of inventory required at different location. “It’s a myth to think anyone will ever get to zero inventory, but inventory optimization engines are the next step in that direction,” according to Mary Haigis of Clarkston Consulting, Durham, N.C. (Murphy 2003:1).

The Aberdeen benchmark study found that companies using new optimization methods commonly drove 20-30 % reductions in on-hand inventory and 10-20 % improvements in time to market (Enslow 2004:1-17). The study also found that nearly half of respondents have shifted away from purchase orders or release notices for some of their suppliers. Instead, these companies are setting a minimum and maximum inventory target level for an item at a plant or other company location, and then ask the supplier to take responsibility for ensuring that inventory is maintained within that range – in essence, Vendor Managed Inventory. Inventory reduction of 30 % and more has been realized in these enterprises and stock-outs have been drastically reduced. New supplier collaboration technology is helping companies execute these min/max replenishment strategies in a way that enables suppliers to also reduce their own inventories. Companies need to be much more aggressive in using the new generation of multi-echelon inventory optimization technology and inventory collaboration technology.

The Inventory Optimization Tool marketplace is a niche market since all software vendors present in this market also deliver other software components such as Forecasting, Supply Chain Network Design, Enterprise Resource Planning (ERP), Retail software or Advance Planning Systems (APS). It is also interesting to note that all of the Inventory Optimization software vendors also deliver forecasting software, which is often closely linked to the Inventory Optimization functionalities, that is optimization of inventory levels based on future forecasting data (Cap Gemini Ernst & Young 2003:4).

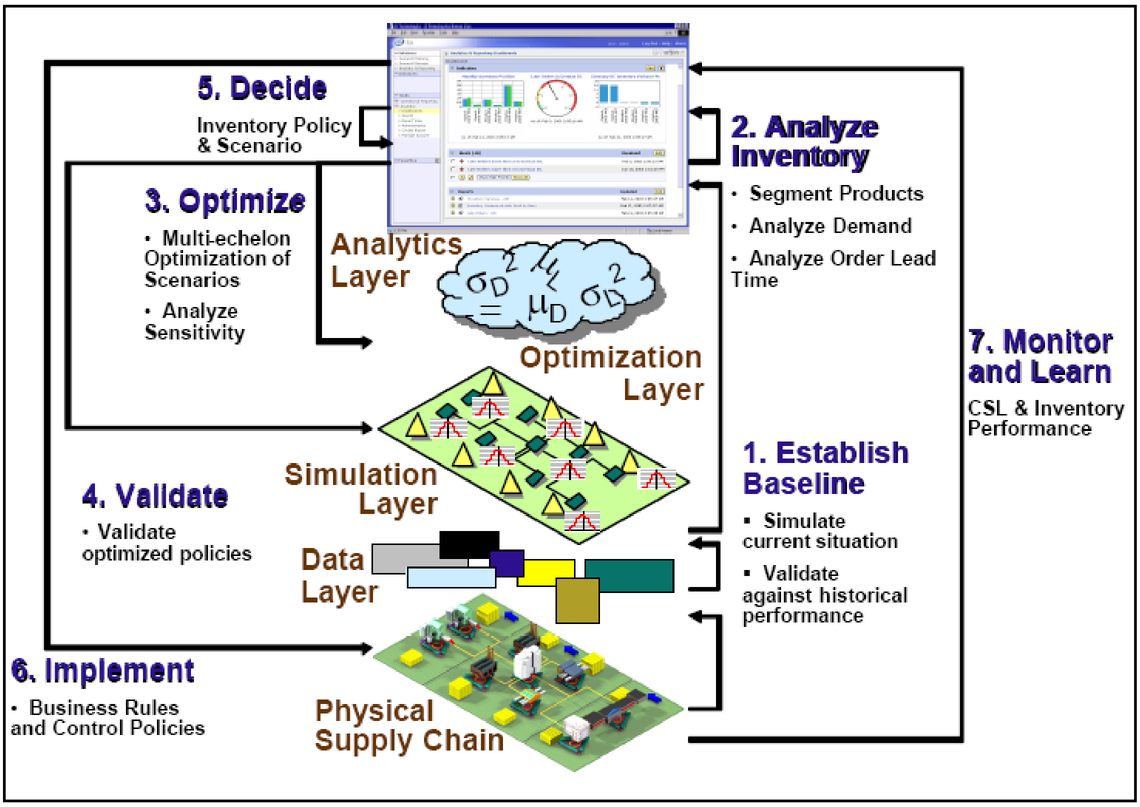
Inventory Optimization tools typically contain functionalities such as:

- Calculation of optimal safety stock levels based on customer service level parameters
- Calculation of ABC classifications
- Determination of the best ordering methodology
- Best before date management and optimal lot size calculations
- Analysis for stock / non-stock decisions
- Dynamic safety stock level management

Software vendors are increasingly realizing that the new direction in supply chain management will require them to have an inventory optimization module. In reaction to this optimization packages are emerging in two forms: Specialized Optimization Packages or Inventory Optimization modules as an addition to ERP systems.

1.2.3 *i2* Technology Seven Step Approach

As an example of an approach used in commercial information technology, ***i2*** Technology, a leading supply chain optimization solution provider, deploys inventory optimization through a seven-step process as depicted in Figure 1.2 below (***i2*** Technology, 2003).



Source: i2 Technology Inc. White Paper 2003:26)

Figure 1.2: The i2 Technology Seven Step Approach to Inventory Optimization

The ***i2*** Seven Step Approach is described below:

a. Establish Baseline – Simulate the current situation and validate against history

i2 starts the process by building a valid simulation model of the supply chain based on historical data.

b. Analyze Inventory – Segment products and analyze demand

Concurrently with the establishment of the baseline and through close cooperation with business leaders, an understanding of the companies' business priorities, the market environment and the customers are obtained. This leads to a profile of customers' buying behaviour and lead-time expectations. These insights are used to design an appropriate segmentation and stratification strategy for the companies' customers and products, driven by the companies' customer expectations and business priorities. Items may be analyzed for demand patterns, demand volumes, service criticality, product lifecycle, product structure similarities, lead times, and competitive posture.

c. Optimize scenarios and measure sensitivity

Candidate "to-be" scenarios are identified in cooperation with the companies' business leaders. Based on ***i2***'s optimization technology, which takes into account anticipated demand and the companies' supply chain constraints and business priorities, calculations are made in order to determine how much of what inventories must be carried and where and in what form, it should be.

d. Validate optimized policies with simulation

The optimized recommendations are then validated with a simulation run. The simulation allows ***i2*** to get more detailed expected performance metrics pertinent to the supply chain for each scenario.

e. Select best inventory policy scenario for business

i2's solution includes an analytics framework that provides metrics based on the SCOR model across the supply chain. The metrics can be tailored according to part, location, customer and time hierarchies and comparison of metrics at any level for part, location, customer and time, between scenarios or within each scenario can be made. This analysis helps the company to decide which "to-be" scenario is best for the business.

f. Implement business rules and inventory policies in the supply chain

i2's solution provides direct integration to supply chain planning business processes through standard API and ***i2***'s Supply Chain Operating System (SCOS) architecture.

g. Monitor service levels and inventory performance

i2 provides a structured framework for continuous learning and process improvement using Six Sigma concepts. Standard reports are provided to monitor actual performance against plans. These reports will also help the company understand if the assumptions that plans were based on, were valid or not. The analysis framework provides guided analysis paths that help to quickly identify root causes of execution problems.

i2 bases their inventory management technologies on the concept of response buffers. The response buffer is the inventory point from which material is consumed to fulfil a customer order. In the retail environment, for instance, the primary response buffer is at the customer facing location. The retail store's shelf is the response buffer. If the customer fails to find the item he wants on the shelf, he simply goes elsewhere and the store loses the order. On the other hand, in a manufacturing environment with component inventories, assemblies, and finished goods, the response buffer can be anywhere in the supply chain, predicated by the business model. In a make-to-stock setting for instance, the response buffer is downstream in the supply chain similar to the retail store. In contrast, in a build-to-order environment, the response buffer may be upstream in the form of raw material.

Response buffers play a fundamental role in inventory optimization strategies. ***i2*** has identified four fundamental strategies that define, according to them, world-class inventory management. These strategies are:

- *Optimized segmentation* **-** stratification of products based on common inventory characteristics and similar response buffer strategies
- *Optimized postponement* **-** deals with decisions around which echelon (node in the supply chain) to position the response buffers in the supply chain
- *Optimized inventory levels* **-** drives decision on how much inventory to carry in the response buffers
- *Continuous learning for process improvement* - enables ongoing, incremental improvement of the inventory management process

In addition to these strategies, ***i2*** describe three variables fundamental in performing inventory optimization:

- ***Demand distribution***

  Demand distributions reflect the expected volume and variability for demand Normal distributions are typically used for medium and high volume demand streams. Poisson distributions are typically good to represent low volume or intermittent demand streams. The system will automatically choose the appropriate distribution based on demand data.

- ***Order Lead Time distribution***

  The order lead-time (OLT) is the time between the last change-order date and customer request date (CRD).

- ***Supply Lead Time distribution***

  The supply path of an end-item has lead times for each of the upstream echelons of the supply chain. Sometimes the lead-time may be insignificant but typically this can range anywhere from a few days to a few weeks.

Added to the fundamental variables, ***i2*** believes that there are two key policy inputs to inventory optimization, namely:

- *Target Customer Service Level (CSL)*
- *Minimum Offered Lead Time*

  This is the minimum lead-time the planner would like to offer for a particular sub-scope of the supply chain. This means that the inventory optimization will plan for at least this much lead time regardless of the customer request date.

With these concepts in mind, ***i2*** then goes on to define an objective function and summarizes the inputs, outputs and decision outputs as follows (i2 Inventory Optimization User Manual 2005:6-10):

- *Objective Function*

  Minimize total expected inventory cost while meeting target CSL
- *Inputs*

  • CSL or delinquency target for end item buffers

  • Demand rate by time period

  • Demand variability

  • Cycle time & cycle time variability for each arc in the network
- *Key Outputs*

  • Inventory targets for all buffers

  • Inventory Turns

  • Revenue (R)

  • Inventory carrying costs (R)

  • Delinquency (R)

  The primary outputs of inventory optimization are the optimized inventory targets for every buffer in the supply chain. These targets are passed on to

Supply Chain or Replenishment Planning. Data is obtained from current ERP or legacy systems employed by the company.

1.3 INVENTORY MODELS

A storage point into and out of which commodities move or flow is termed an inventory system. The inflow is characterised by replenishment from production sources and demand processes induce the outflow. The net flow generates a cascade of problems pertaining to the control and maintenance of inventory systems. There are numerous factors pertaining to the functioning of an inventory system and considering only a small number of factors in the formulation of an inventory model can result in a very complex model. Accordingly, it is quite impossible to obtain a traitable mathematical model that will truly reflect the behaviour of an inventory system. However, several nearly realistic models have been proposed and studied extensively in the past giving importance to the inherent stochastic nature of these systems. Most of these models assume that the organisations maintaining the inventory have control in determining when and in what quantity the inventory have to be replenished, but have no control over the demand process. A systematic account of the early analyses of stochastic inventory systems can be found in Arrow et al (1951, 1958), Beckmann (1961) and Hadley and Whitin (1963). As the study of these systems progressed over time, several reviews have appeared to highlight the state-of-art (for example, see Aggarwal (1974), Nahmias (1978), and Raafat (1991). A review and critique of inventory problems that have been effectively solved is provided by Silver (1981), who also suggested some problems for future research. Girlich (1984) executed a survey of dynamic inventory problems and models that can be implemented.

1.3.1 Types of Inventory Models

The various models of stochastic analysis of inventory systems are broadly classified into two types namely, periodic review systems and continuous review systems. In periodic review systems the state of the system is examined only at specific time intervals at equally spaced points in time and decisions such as placing of orders and the quantity to be added to the inventory are made only at these review points. In continuous review systems, on the other hand, all events associated with the time evolution of the inventory are recorded and the stock level is reviewed continuously at the occurrence of each demand for the product in inventory. Continuous review systems have occupied a wider scope for application since failure of review of the inventory level even at a single time point may prove disastrous for organisations in the defence and medical industries. Inventory systems are also classified as either single product inventory systems or multi product inventory systems, based on the consideration of a single product or a variety of products in interaction.

1.3.2 Single Product Inventory Systems

Several models for single product inventory systems have been proposed. Optimal ordering policies have been developed and studied extensively in the past by several researchers both for periodic and continuous review cases. For example, see Beckmann (1961), Dirickx and Koevoets (1977), Wijngaard and Winkel (1974), Kalpakam and Arivarignan (1985, 88), Horrowitz and Doganso (1986), Beckmann and Srinivasan (1987), Ramanarayanan and Jacob (1987), Ravichandran (1988), Weiss (1988), Srinivasan (1989), Krishnamoorthy and Laxmy (1990), Kalpakam & Sapna (1996), Hargreaves (2002) and Krishnamoorthy and Manoharan (1990).

1.3.3 Multi-product Inventory Systems

Many real life situations exist in which multi-product inventories are required. For example a pharmacist keeps a number of medicines of different brands, a ready-made clothing shop keeps dresses of different designs, colours, and sizes, a shoe store stocks shoes of various styles and sizes. Hence the study of multi- product inventory models has drawn special attention recently. Page and Paul (1976) and Chakravarthy (1981), Sung and Chang (1986), Oneiva and Larraneta (1987), Aksoy and Erengue (1988), Amiya and Martin (1988), Goyal (1988) and Correnu (1990) have analysed multi-product inventory systems.

a. Ordering Policies

In a multi-product inventory system the inventory control policies and the nature of demands may be different from that of a single product system. First we consider inventory control policies. The inventory of each product may be controllable independently or there may exist an interaction among the items and a joint control of the inventory may be required. For example demand for tyres for off-road vehicles will not affect the demand for truck tyres available at the same dealership. Inventory of such items can be controlled individually. The demand for new and retreads of trucks may be highly dependent and need to be controlled jointly. Hence we may have the following two types of re ordering policies for the control of inventory on products:

(i) Individual order policy

This policy determines that each item is ordered according to its own single item policy.

(ii) Joint order policy

This policy determines that all jointly controlled items is ordered whenever an order for specific product order is triggered, irrespective of the inventory level of the other items. That is wherever replenishment occurs; every product is replenished to a specified inventory level.

b. Demand Interaction

Considering the nature of demand, a demand may be for a single product or several products. For example, the inventory of a dealership for new cars, in addition to new vehicles, consists of replacement parts for maintenance and optional accessories such as special trimming. The buyer has the option to take one or more of these accessories. It is also possible that a demand for a particular product during its stock-out may be substituted with another similar product in the inventory. Examples of products having at least partial substitutability include:

- Consumer products such as different brands of toothpastes and different types of pastas or cereals.
- Building products such as different brand of paints and containers of different sizes of the same brand.
- Clothing products such as dresses in the same design and brand but in different colours.
- Electrical products such as fluorescent light bulbs of different makes and ceiling fans of different brands.

When this type of interaction occurs, large stock quantities of a particular product can be avoided, as it is substitutable by another similar product. The available total inventory storage space can be shared optimally as to reduce the lost demand due to unavailability. Kamat (1971) studied substitutability of demands by considering a two substitutable product inventory model with a prescribed order period and obtained a cost function. McGillivray and Silver (1978) investigated the effect of substitutable demands on stock control rules and a heuristic approach for establishing the value of control parameters (the order up to levels) for the case of two products. Parlar and Goyal (1984) considered a model of two substitutable products as an extension of the classical single period *news-boy* problem. They have shown that the optimal order quantities can be found for each product by maximizing the expected profit function, which is strictly concave for the wide range of parameter values. Parlar (1988) used *game-theoretic* concepts (two person continuous game) to analyse an inventory problem with two substitutable products having random demands.

1.3.4 Perishable Product Inventory

Apart from these considerations, the perishability of products also plays a vital role in inventory theory. Several inventory models of perishable products have been proposed and studied extensively. A review of work done on perishable inventory can be found in Nahmias (1982). Further and Weiss (1986), Nahmias and Schmidt (1986), Sarma (1987), Abdel, Malek and Ziegle (1988), Ravichandran (1988), Mandal and Phaujdar (1989) and Perry and Posner (1990) have analysed perishable inventory models. In his survey article, Raafat (1991) has consolidated the work done on continuously deteriorating inventory models. Kalpakam and Sapna (1994, 96) studied a perishable inventory model with (s, S) policy and arbitrary lead times.

a. Demand Interaction

A different type of interaction can occur in the case of perishable inventory. Products such as vegetables, fish, etc. have a short life span and deteriorate in quality due to ageing. The same applies to fashion clothing losing its value due to changing seasons or new trends. In these cases there may also be a demand for an item slightly deteriorated in quality if the cost is reduced compared to the new or fresh product. A multi product perishable inventory system with economic substitution, which deals with a product that perishes in a single period has been proposed and studied by Deuermeyer (1980). Parlar (1985) has also developed a Markov decision model to generate ordering policies for perishable (in two periods) and substitutable products.

1.3.5 Random Environment

In the stochastic analysis of inventory systems, it is generally assumed that the distributions of the random variables, representing the number of demands over a period of time, the life of the product (in case of a perishable product) and the lead-time, remain the same and do not change through the domain of the analysis. However, there are external factors that affect these random variables. Seasonal changes can affect the demand rate, the perishing rate, the selling price and the cost of replenishment. The demand for umbrellas and rain shoes are higher in the rainy season than at other times of the year. The selling price and the cost of replenishment also fluctuate over time due to inflation, non-availability of the products, cost of transport, etc. The state of the environment in which the system is operating may randomly change due to weather, breakdown of storage facilities, etc. Consequently, consideration of the impact of the random environment on such inventory systems is absolutely essential.

1.3.6 Deteriorating Inventory

Balkhi (1999) developed a unified inventory model for integrated production systems with a single product. The production, demand and deterioration rates for the finished product and the deterioration rates for raw materials are assumed to be known functions of time.

The objective of the author is to determine the optimal values of the length of the production stage and the length of the inventory cycle that minimizes the total variable cost of the inventory system. The problem is converted to an unconstrained minimization problem, and when a solution to the underlying inventory system exists, it is the unique global optimal solution. A rigorous mathematical formulation proves the global optimality of the solution. The article is concluded with a numerical example that illustrates the solution procedure.

Rau et al (2003) worked on an integrated inventory model for deteriorating items under a multi-echelon supply chain environment**.** Demand, production and deterioration rate is assumed to be deterministic and constant with production rate greater than demand rate. Only a single supplier, producer, buyer and product are considered**.** A model that gives the optimal joint total cost from an integrated perspective among the supplier, producer and buyer is obtained and Matlab is used to obtain the optimal solution. A numerical example illustrates the use of the model and it shows that an integrated approach results in the lowest joint total cost as compared with the independent decision strategies.

1.3.7 Techniques Used in the Study of Inventory Models

a. Renewal Theory

One of the important types of stochastic processes is the renewal process. Several researchers in the theory of renewal processes have made outstanding contributions, e.g. Feller (1965), Cox and Smith (1958), Smith (1958) and Neuts (1978). A systematic account of renewal theory and its applications to diversified fields can be found in Cox (1962), Parzen (1962), Sahin (1990) and Medhi (1994). A renewal process is a sequence of independent, non-negative and identically distributed random variables, which are not all zero with a probability of one.

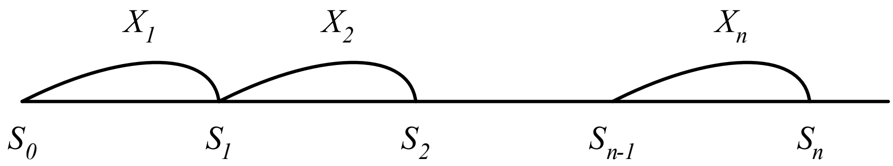
(i) Definition

Let $\{X_n ; n = 1, 2, \ldots\}$ be a collection of non-negative random variables which are independent and identically distributed. Then $\{X_n\}$ is called a renewal process.

We assume that each of the random variable X_i has a finite meaning. A renewal process is completely determined by $f(\cdot)$, the pdf of X_i. Let

$$S_0 = 0$$

$S_n = X_1 + X_2 + \ldots\ldots + X_n$, $n = 1, 2, \ldots$



$N(t) = \max\{n : S_n \leq t\}$, $t > 0$

Then $N(t)$ is called the number of renewal up to time (t). The expected value of $N(t)$, namely $E[N(t)]$ is called the renewal function and is denoted by $H(t)$. The derivative $H(t)$, whenever it exists, is called the renewal density and is denoted by $h(t)$.

(ii) Renewal Equation

The quantity of $h(t)\ dt$ has the probabilistic interpretation that it denotes probability that the renewal occurs in the interval $(t, t + dt)$. Since this renewal may be either the first or the subsequent renewal, the function $h(t)$ satisfies the equation.

$$h(t) = f(t) + \int_0^t h(u) f(t-u) du$$

This equation is called the renewal equation.

(iii) Key Renewal Theorem

Let $Q(t)$ be non-negative and non-increasing for $t > 0$ such that

$$\int_0^t Q(u)du < \infty$$

then

$$\lim_{t\to\infty}\int_0^t Q(u-x)dH(x) = \frac{1}{\mu}\int_0^{co} Q(u)du$$

where $\mu = E[X_i]$

b. Markov Renewal Processes

These stochastic processes are generalisations of renewal processes and have become indispensable in inventory applications. A systematic and in depth study can be found in Pyke (1961a,b), Cinlar (1975a,b) and Medhi (1994).

Let E be a finite set, N be the set of non-negative integers and $R_+ = [0,\infty]$. Suppose we have on a probability space (Ω, X, P), random variables,

$$X_n : \Omega \to E, T_n : \Omega \to R_+$$

defined for each $n \in N$, so that

$$0 = T_0 \le T_1 \le \ldots \le T_n$$

Definition 1: The stochastic process $(X,T) = \{X_n, T_n; n \in N\}$ is said to be a Markov renewal process with state space E provided that

$$P[X_{n+1} = j,\ T_{n+1} - T_n \le t \mid X_0, X_1, \ldots, X_n;\ T_0, T_1, \ldots, T_n]$$

$$= P[X_{n+1} = j,\ T_{n+1} - T_n \leq t \mid X_n]$$

for all $n \in N$, $j \in N$ and $t \in R_+$

Assuming that *(X, T)* is time homogeneous, that is, for any $i, j \in E$ and $t \in R_+$,

$$P[X_{n+1} = j,\ T_{n+1} - T_n \leq t \mid X_n = i] = Q\,(i, j, t) \text{ is independent of } n.$$

The family of probabilities

$$\boldsymbol{Q} = \{Q(i, j, t); i, j \in E, t \in R_+\}$$

is called a semi-Markov Kernel over *E*. We assume that $Q(i, j, t) = 0$ for all *i, j* in *E*.

For each pair *(i,j)* the function $t \to Q(i, j, t)$ has all properties of a distribution function except that

$$P(i, j) = \lim_{t \to \infty} Q(i, j, t)$$

is not necessarily 1. It can be seen that

$$P(i, j) \geq 0$$

$$\sum P(i, j) = 1, j \in E$$

That is, $P(i, j)$ are the transition probabilities for some Markov chain with state space *E*. It follows from Definition 1 and the above that

$$P\left[X_{n+1}=j\,|\,X_0,X_1,...,X_n;T_0,T_1,...,T_n\right]=P\left(X_n,j\right)$$

for all $n\in N, j\in E$. This implies that

$$X=\{X_n;n\in N\}$$

is a Markov chain with state space *E* and transition matrix *P*.

We write $P_i\left(A\right)$ for the conditional probability $P\left(A\,|\,X_0=i\right)$ and similarly $E_i\left(X\right)$ for the conditional expectation of X given $\{X_0=i\}$. We also assume that $P_i\left[T_0=T_1=T_2=...=0\right]=0$. We define

$$Q^n\left(i,j,t\right)=P_i\left[X_n=j,T_n=t\right];\qquad i,t\in E,t\in R_+$$

for all $n\in N$. Then

$$Q^0\left(i,j,t\right)=\delta_{ij}=\begin{cases}1\,if\;\,i=j\\0\;otherwise\end{cases}$$

for all $t\geq 0$; and $n\geq 0$, we have the recursive relation. δ_{ij} is the Kronecker's delta function.

$$Q^{n+1}\left(i,k,t\right)=\sum_{j\in E}\int_0^t Q\left(i,j,du\right)Q^n\left(j,k,t-u\right)$$

where the integration is over $\left(0,t\right]$

The expression $R(i, j,t)$, which gives the expected number of renewals of the position j in the interval $(0,t]$, is given by

$$R(i, j,t) = \sum_{n=0}^{\infty} Q^n(i, j,t)$$

This is finite for any $i, j \in N$ and $t < \infty$. The $R(i, j,t)$ are called Markov renewal functions and the collection $\boldsymbol{R} = \{R(i, j,t); i, j \in E, t \in R_+\}$ of these functions is called the Markov renewal Kernel corresponding to Q. We note that for fixed $i, j \in E$ the function $t \to R(i, j,t)$ is a renewal function.

We can easily see from the various expressions above that $R_\alpha = [I - Q_\alpha]^{-1}$ where I is the unit matrix, and

$$Q_\alpha(i, j) = \int_0^\infty e^{-\alpha t} Q(i, j,t)\,dt; \qquad \alpha > 0$$

$$R_\alpha(i, j) = \int_0^\infty e^{-\alpha t} R(i, j,t)\,dt; \qquad \alpha > 0$$

The class B of functions which we will be working with, is the set of all functions $f : E \times R_+ \to R$ such that for every $i \in E$, the function $t \to f(i,t)$ is Borel measurable and bounded over finite intervals, and for every fixed $j \in E$, the functions $(i, j) \to Q^n(i, j,t)$ and $(i, j) \to R(i, j,t)$ both belong to B.

For any function of $f \in B$, the function $Q © f$ defined by

$$Q © f(i,t) = \sum_{j \in E} \int_0^t Q(i,j,ds) f(t,t-s)$$

is well defined and $Q © P$ belongs to B again. Hence the operation can be repeated, and the n[th] iteration is given by

$$Q^n © f(i,t) = \sum_{j \in E} \int_0^t Q^n(i,t,ds) f(j,t-s)$$

We can replace Q by R and note that $R © f$ is again a well defined function; that is $f \in B$

$$R © f = \sum_{j \in E} \int_0^t R(i,j,ds) f(j,t-s)$$

A function $f \in B$ is said to satisfy a Markov renewal equation if for all $i \in E$ and $t \in R_+$

$$f(i,t) = g(i,t) + \sum_{j \in E} \int_0^t Q(i,j,ds) f(j,t-s)$$

for some function of $g \in B$.

Limiting ourselves to function $f, g \in B$, which are non negatives and denoting this set by B_+, the Markov renewal equation now becomes

$$f = g + Q © f \; ; \; f, g \in B_+$$

This Markov renewal equation has a solution $R © g$. Every solution *f* is of the form

$$f = R © g + h$$

where h satisfies

$$h = Q © h, h \in B_+$$

c. Semi-Markov Processes

Let (X,T) be a Markov renewal process with state space E and semi-Markov Kernel Q. Define $L = \underset{n}{Sup}\, T_n$. Then L is the lifetime of (X,T). If E is finite or if X is irreducible recurrent, then $L=+\infty$ almost surely. By weeding out those $\omega \in \Omega$ for which $\underset{n}{Sup}\, T_n(\omega) < \infty$, we assume that $\underset{n}{Sup}\, T_n(\omega) = +\infty$ for all ω. Then for any $\omega \in \Omega$ and $t \in R$, there is some integer $n \in N$. Such that $T_n(\omega) \leq t < T_{n+1}(\omega)$. We can therefore define a continuous time parameter $Y = (Y_t)_{t \in R_+}$ which stake space E by putting $Y_t = X_n$ on $\{T_n \leq t < T_{n+1}\}$. The process $Y = (Y_t)_{t \in R_+}$ so defined is called a Semi-Markov process with state space E and Semi Markov transition Kernel $Q = \{Q(i, j, t)\}$.

d. Semi-Regenerative Processes

Let a stochastic process $Z = (Z_t)_{t \in R_+}$ be a stochastic process with a topological state space F, and suppose that the function $t \rightarrow Z_t(\omega)$ is right continuous and has left-hand limits for almost all $\omega \in \Omega$. A random variable $T : \Omega \rightarrow [0,\infty)$ is called stopping time for Z provided that for any $t \in R_+$, the occurrence or non occurrence of the event $\{T \leq t\}$ can be determined once the history $H_t = \sigma(Z_u : u \leq t)$ of Z before t is known. If T is the stopping time for Z, then we denote by H the history of Z

before T. The process $Z= \{Z_t \; ; \; t \geq 0\}$ is called a regenerative if there exists a sequence S_0, S_1, S_2 .... of stopping times such that

(i) $S = \{S_n ; n \in N\}$ is a renewal process

(ii) For any $n,\; m \in N; t_1, t_2, ..., t_n \in R_+$ and any bounded function f defined on E^n

$$E\left[f\left(Z_{s_m+t_1}, Z_{s_m+t_2}, ..., Z_{s_m+t_m} \right) | Z_u ; u \leq S_m \right] = E\left[f\left(Z_{t_1}, Z_{t_2}, ..., Z_{t_n} \right) \right]$$

Definition 2: Let $Z = (Z_t)_{t \in R_+}$ be a stochastic process topological state space F, and suppose that the function $t \rightarrow Z(\omega)$ is right continuous and has left hand limits for almost all ω. The process Z is said to be semi-regenerative if there exists a Markov renewal process (X,T) with infinite lifetime satisfying the following:

i) for each $n \in N, T_n$ is a stopping time for Z

ii) for each $n \in N, X_n$ is determined by $\{Z_u : u \leq T_n\}$

iii) for each $n \in N, M \geq 1, 0 \leq t_1 < t_2 < ... < t_m$ and function f defined on F^m

$$E_i\left[f\left(Z_{T_n+t_1}, Z_{T_n+t_2}, ..., Z_{T_n+t_m} \right) | Z_u ; u \leq T_m \right] = E_j\left[f\left(Z_{t_1}, Z_{t_2}, ..., Z_{t_m} \right) \right] \text{ on } [X_n = j]$$

In this definition E_i and E_j refer to the expectations given the initial state for the Markov chain X.

Detailed treatment and MRP can be found in Pyke (1961a,b), Levy (1954), Cinlar (1975b) and Ross (1970). The survey of Cinlar (1975a) demonstrates the usefulness of the theory MRP and SMP in applications.

e. Stochastic Point Processes

Stochastic point processes form a class of processes more general than those considered in the previous sections. Since point processes have been more studied by many with varying backgrounds there have been several definitions of the point processes each appearing quite natural from the viewpoint of the particular problem under study. [See for example Bhabha (1950), Khinchine (1960), Harris (1963) and Bartlett (1966)]. A stochastic point process is the mathematical abstraction, which arises from considering such phenomena as randomly located population or a sequence of events in time. Typically there is envisaged a state space X and a set of points X_n, from X representing the locations of the different members of the population or the times at which the events occur. Because a realization (or sample path) of any of these phenomena is just a set of points in time or space, a family of such realizations has come to be called a point process. (Daley and Vere-Jones, 1971)

A comprehensive definition of point process is due to Moyal (1962) who deals with such processes in a general space, which is not necessarily Euclidian. Consider a set of objects, each of whose state is described by a point x of a fixed set X of points. Such a collection of objects, which we may call a population, may be stochastic if there exists a well-defined probability distribution P on σ some field β of subsets of the space Φ of all states. We shall assume that members of the population are indistinguishable from one another. The state of the population is defined as an unordered set $x^n = \{x_1, x_2, \ldots, x_n\}$ representing the situation where the population has n members with one each in the states $x_1, x_2, \ldots, x_n$. Thus the population state space Φ is the collection of all x^n with $n = 0,1,2\ldots$where x^0 denotes the empty population. A point process is defined to be the triplet (Φ, β, P). For a detailed treatment of stochastic point processes with special reference to their applications, refer to Srinivasan (1974). A point process is called a regular point process if the probability of occurrence of more than one event *(0, Δ)* is *0(Δ)*, where *Δ* is very small.

(i) Product Densities

One of the ways of characterising a general stochastic point process is enough product densities (Ramakrishnan 1950, 1958) and Srinivasan (1974). These densities are analogous to the renewal density in the case of non-renewal processes.

Let $N(t,x)$ denote the random variable representing the number of events in the interval $(t, t+x)$, $d_xN(t,x)$ the events in the interval $(t+x, t+x+dx)$ and $P_n(n,t,x)= P[N(t,x) = n]$. The product density of order n is defined as:

$$h_n\left(x_1, x_2, \ldots, x_n\right) = \lim_{\Delta_1, \Delta_2, \ldots, \Delta_n \to 0} P\left[\frac{N\left(x_i, \Delta_i\right) \geq 1, i = 1, 2, \ldots, n}{\Delta_1 \Delta_2 \ldots \Delta_n}\right]$$

where $x_1 \neq x_2 \neq \ldots \neq x_n$,

or equivocally for a regular process

$$h_n\left(x_1, x_2, \ldots, x_n\right) = \lim_{\Delta_1, \Delta_2, \ldots, \Delta_n \to 0} E\left[\frac{\prod_{i=1}^{n} N\left(x_i, \Delta_i\right)}{\Delta_1 \Delta_2 \ldots \Delta_n}\right]$$

where $x_1 \neq x_2 \neq \ldots \neq x_n$

These densities represent the probability of an event in each of the intervals $(x_1, x_1 + \Delta x_1)$, $(x_2, x_2+\Delta x_2)$, … , $(x_n, x_n+\Delta x_n)$. Even though the functions h_n $(x_1,$

$x_2, \ldots, x_n$) are called density, it is important to note that their integrates will not give probabilities, but will yield the factorial moments. The stationary moments can be obtained by relaxing the condition that all x_i are different.

1.3.8 Measures of System Performance

In this section some of the important measures of inventory systems are explained. Let *I(t)* be the inventory level at time *t* and *S* be the maximum capacity of the inventory. Then the next inventory level distribution $P(i,t|k)$ at any time t is given by

$$P(i,t|k) = P[I(t) = i|I(0) = k];\ i,\ k = 0,\ 1,\ \ldots,\ S$$

The limiting distribution *P(i)*, if it exists, is defined as:

$$P(i) = \lim_{t\to\infty} P(i,t|k)$$

For a two product system let the state of the system be represented by the ordered pair *(X(t), Y(t))*, where *X(t)* is the inventory level of product 1 and *Y(t)* is the inventory level of product 2 . Then the inventory level distribution $P(i,j,t|k,l)$ at time t is given by

$$P[i,j,t|k,l] = P\Big[\big(X(t),Y(t)\big) = (i,j)\Big|\big(X(0),Y(0)\big) = (k,l)\Big]$$

$$i\ ;\ k = 0,1,2,\ldots,S_1;\ j;\ \ l = 0,1,\ldots,S_2$$

where S_1 and S_2 are the maximum inventory levels of product 1 and product 2 respectively. The limiting of distribution *P(i,j)*, if it exists, is defined as:

$$P(i,j) = \lim_{t\to\infty} P(i,j,t|k,l)$$

The expected stock on hand of mean inventory level *E(L)*, at any time for a single product system in the steady state is given by:

$$E(L) = \sum_{i=0}^{S} iP(i)$$

In any inventory model, apart from the distribution of the inventory level, the mean number of re orders places, replenishments made, demand satisfied demands lost in an arbitrary interval of time are also some of the important measures.

In the context of a multi-product system allowing substitution, the number of demands for a particular product satisfied by a different product deserves considerations. The stationary roles of these events are used in the cost analysis of the system. To find these measures, we follow the procedure given below.

Let $N(\eta,t)$ denote the number of a specific type of event η (like re-orders, replenishment, demand for a product satisfied by the same product, demand for a product satisfied by another product, demands lost, etc.) in $(o,t]$. Then the expected number of n events in $(o,t]$ is given by:

$$E\left[N(\eta,t)\right] = \int_{0}^{t} h(u)\,du$$

Where *h(u)* is the first order product density corresponding to the event under consideration. In the long term, the stationary role of η events is given by:

$$E(\eta) = \lim_{t\to\infty} \frac{E\left[N(\eta,t)\right]}{t} = \lim_{t\to\infty} h(t)$$

1.3.9 Cost Analysis

a. Inventory Related Costs

We consider the following costs in the analysis of the inventory models.

(i) Holding Costs

This not only includes the expenses incurred by storage facilities but also the amount invested that could have earned a return on investment elsewhere. This cost at any time depends upon the level of stock on hand.

(ii) Re-ordering Costs

When the stock in hand comes down to a level where re-order is necessary, an order is placed. This involves additional expenses with regard to transactions, paperwork, inspection and material handling costs.

(iii) Cost for Demand Lost

When demand is not met and also not backordered, the profit that would have been made is lost together with some goodwill.

(iv) Procurement Cost

This is the price at which the items are bought either from a manufacturer or from the market. Most inventory control procedures recognise price fluctuations, and they are treated accordingly in this study.

b. Cost Optimization

There are a number of objectives that may be sought after by inventory managers. These usually involve the minimisation (maximisation) of costs (profits) function, which could be either discounted or undiscounted. The planning period of horizon may be finite or infinite, In stochastic models the mean value of costs are measured and the criterion consists in the minimisation of the total expected cost pen unit time or of the expected discount cost over a finite or infinite horizon. The cost function will, in general, consist of the additive contribution of the procurement cost, the holding cost and the storage cost.

Under the *(S, s)* policy, the objective function will, in general, be expressible as a function of two variables *S* and *s.* The resultant optimization problem consists in determining the optimal values of *S* and *s* to achieve the selected extension. For a multi-product system the maximum inventory levels of the various products and the re-order levels can be considered as variables for optimization.

In this regard, it should be pointed out that there are two distinct approaches in formulating and solving the stochastic inventory problems both in theory and in practise.

In the first approach the system is viewed as a multi-stage decision process and the technique of dynamic programming is employed in finding the optimal policy that minimises the total expected cost over the duration of the process.

The following second approach is often used when the duration process is infinite: an ordering policy of a given type is chosen and the stationary behaviour of the inventory levels is analysed without reference to the cost structure of the problem. Such entities as the expected frequencies of orders and the expected quantity on hand, etc. are computed. A cost structure is then imposed on the system and the stationary total expected cost rate for operating the inventory system is minimised. In this thesis, the stationary approach is adopted for optimal analysis.

If $C(t)$ represents the total cost in $[0, t]$, then the expected cost rate, $E(C)$, is given by:

$$E(C) = \lim_{t \to \infty} \frac{E[C(t)]}{t}$$

Notation:

λ_i :	Demand rate of product i, i = 1,2
μ_i :	Perishable rate of product i, i = 1,2
S_i :	Maximum inventory level of product i, i = 1,2
s_i :	Re-order level of product i, i = 1,2
S_i-s_i :	Quantity of product i re-ordered, i = 1,2
d_i :	Event that a demand for product i is satisfied with product i, i = 1,2
g :	Event that a demand for product 1 is satisfied by product 2
l_i :	Event that a demand for product i is lost, i = 1,2
$N(\eta,t)$:	Number of η events in interval (0,t]
δ_{ij} :	Kronecker's delta function
$H(\bullet)$:	Heaviside function
© :	Convolution Symbol
$\xi^*(s)$:	Laplace transform of $\xi(t)$
$f^{(n)}(t)$:	n-fold convolution of $f(t)$

$$\bar{F}(t) = 1 - F(t) = 1 - \int_0^t f(u)du$$

CHAPTER 2

A PERISHABLE PRODUCT INVENTORY SYSTEM OPERATING IN A RANDOM ENVIRONMENT

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2.1 INTRODUCTION

Various stochastic models of inventory systems have been studied recently by Yadavalli & Joubert [2003], Yadavalli et al [2004]. Studies on perishable product inventory systems have gained much importance in literature (Kumaraswamy and Sankarasubramanian [1981], Kalpakam and Arivarignan [1988], Pal [1990], Liu [1990], Raafat [1991] and Kalpakam and Sapna [1994, 1996]). In the stochastic analysis of such inventory systems, it is generally assumed that the distributions of the random variables representing the number of demands over a period of time, the lifetime of the product and the lead-time remain the same and do not change throughout the domain of the analysis. However, there are external factors that affect these random variables. Seasonal changes can affect the demand rate, the perishing rate, the selling price and the cost of replenishment. The demand for umbrellas and rain shoes are higher in winter than in summer. The perishing rates of vegetables are higher in summer. The selling price and the cost of replenishment also fluctuate over time due to reasons such as inflation and non-availability of the product. The state of the environment in which the system is operating may randomly change due to several factors, including weather conditions and breakdown of storage facilities. Consideration of the impact of the random environment on such inventory systems is, therefore, absolutely essential. Only a few authors have considered inventory systems operating in random environments (Feldman [1978], Pal [1990], Song and Zipkin [1993] and Girlich [1998]). These authors considered non-perishable product inventory evolving in random environments. The survey of Raafat [1991] presents only literature on deteriorating inventory models in non-changing environments. Kalpakam and Sapna [1996] considered inventory models where the items have constant perishing rates only.

A perishable product inventory system operating in a random environment is studied in this chapter. For the sake of simplicity, the stochastic environment is considered to alternate randomly over time between two states, 0 and 1, according to an alternating renewal process. When the environment is in state k, the items in the inventory have a perishing rate μ_k, the demand rate is λ_k and the replenishment cost is CR_k. Assuming instantaneous replenishment at the epoch of the first demand after the stock-out and associating a Markov renewal process with the inventory system, the stationary distribution of the inventory level and the performance of various measures of the system evolution are obtained.

This chapter is structured as follows:

Section 2.2 provides the assumptions and notation of a model of an inventory system operating in a random environment and certain auxiliary functions are obtained in Section 2.3. An associated Markov renewal process is analyzed in Section 2.4. In Section 2.5, the stationary distribution of the inventory level is given and the stationary measures of performance of the system are obtained in Section 2.6. A cost analysis for the model of the inventory system is presented in Section 2.7. Section 2.8 considers a particular case of the general model and obtains the probability distribution of the total sales proceeds up to any time t. In Section 2.9, another particular case of the general model is considered and the total replenishment cost incurred up to t is studied, followed by a numerical illustration in Section 2.10.

2.2 ASSUMPTIONS AND NOTATION

2.2.1 Assumptions

A continuous review inventory system operating in a random environment is considered. The random environment is assumed to alternate between two states, 0 and 1. The durations of stay in the state 0 are given by the sequence of i.i.d. random variables $\{X_n\}$, having a common exponential distribution with parameter ν_0, and the durations of stay in the state 1 are given by the sequence of i.i.d. random variables $\{Y_n\}$, having a common exponential distribution with parameter ν_1. A renewal of one state occurs at the termination of the other. The two families { X_n } and { Y_n } are independent.

Other applicable assumptions are the following:

(i) The items under consideration are perishable. The rate of perishing depends on the state of the random environment. The lifetime distribution of an item in the inventory is exponential with parameter μ_k when the environment is in state k, (k = 0, 1).

(ii) Demands occur according to a double stochastic Poisson process. The demand occurs with rate λ_k when the environment is in state k, (k = 0, 1).

(iii) Replenishment is instantaneous for $S+1$ units and is made at the epoch of the occurrence of the first demand that occurs during the stock-out period. The cost of replenishment is CR_k when the environment is in state k (k = 0, 1).

2.2.2 Notation

$\xi(t)$	:	The state of the environment at time t
π	:	Event that an item perishes
©	:	Convolution symbol
H(i-j)	:	Heaviside function $=\begin{cases} 1 \; if \; i \geq j \\ 0 \; if \; i < j \end{cases}$

2.3 AUXILIARY FUNCTIONS

In this section, the underlying stochastic process is identified as a Markov renewal process. In order to study its transient behaviour, certain auxiliary functions are obtained in this section.

2.3.1 Function $P(j,t;i,k)$

An interval in which there is no replenishment and the environment remains in a fixed state, the inventory level process $L(t)$ behaves like a death process. To describe the behaviour of this process, the function

$$P(j,t;i,k) = P[L(t) = j \mid L(0) = i, \xi(0) = k]$$

is defined, where $0 \le i, j \le S$ and $k = 0,\ 1.$ To derive an expression for $P(j,t;i,k)$ consider that if $L(t) \ne 0$, a change in the state of $L(t)$ occurs due to any one of the following mutually exclusive and exhaustive cases:

(i) A demand for the product occurs

(ii) An item perishes and is removed instantaneously from the inventory

Accordingly

Case 1: $i = 0,\ j = 0$

$$P(0,t;0,k) = e^{-\lambda_k t} \tag{2.1}$$

Case 2: $j > i$

$$P(j,t;i,k) = 0 \tag{2.2}$$

Case 3: $i = j \neq 0$

$$P(j,t;i,k) = e^{-(\lambda_k + j\mu_k)t} \tag{2.3}$$

Case 4: $0 \leq j < i$

$$P(j,t;i,k) = (\lambda_k + i\mu_k)e^{-(\lambda_k + i\mu_k)t} © P(j,t;i-1,k) \tag{2.4}$$

Taking Laplace transforms, the equations (2.1) to (2.4) yield the following:

$$P^*(j,s;i,k) = \begin{cases} 0 & i < j \\ \dfrac{1}{s+\lambda_k} & i = j = 0 \\ \dfrac{1}{(s+\lambda_k + i\mu_k)} & i = j \neq 0 \\ \dfrac{u(i,j,k,s)}{s+\lambda_k} & 0 = j < i \\ \dfrac{u(i,j,k,s)}{s+\lambda_k + j\mu_k} & 1 \leq j < i \end{cases} \tag{2.5}$$

where

$$u(i,j,k,s) = \frac{\prod_{m=j+1}^{i} (\lambda_k + m\mu_k)}{\prod_{m=j+1}^{i} (s+\lambda_k + m\mu_k)} \qquad ;0 \leq j < i$$

Inverting Equation 2.5 obtains the following:

$$P(j,t;i,k)=\begin{cases} e^{-\lambda_k t} & ;i=j=0 \\ e^{-(\lambda_k+i\mu_k)t} & ;i=j\neq 0 \\ v(i,j,k)e^{-(\lambda_k+j\mu_k)t}(1-e^{-\mu_k t})^{i-j} & ;1\leq j<i \\ 1-i\mu_k v(i,0,k)\sum_{m=1}^{i}(-1)^{m-1}\binom{i-1}{m-1}\dfrac{e^{-(\lambda_k+m\mu_k)t}}{\lambda_k+m\mu_k} & ;0=j<i \\ 0 & ;otherwise \end{cases} \quad (2.6)$$

where

$$v(i,j,k)=\frac{\prod_{m=j+1}^{i}(\lambda_k+m\mu_k)}{\mu_k{}^{i-j}.(i-j)!}$$

2.3.2 Function $f_{r,k}(t)$

Consider the point process of r-events occurring in an interval in which there is no change in the state of the environment. Let

$$f_{r,k}(t)=\lim_{\Delta\to 0}\frac{P[r-event\ in\ (t,t+\Delta),\ N(r,t)=0\,|\ r-event\ at\ t=0,\ \xi(0)=k]}{\Delta};\quad k=0,1$$

The function $f_{r,k}(t)$ represents the pdf of the interval between any two successive occurrences of replenishment when the state of the environment remains at k throughout the interval under consideration. Note that

$$f_{r,k}(t)=P(0,t;S,k)\lambda_k \quad ;k=0,1$$

2.3.3 Function $h_{r,k}(t)$

Considering the point process of r-events occurring in an interval in which there is no change in the state of the environment, the function $h_{r,k}(t)$ are defined as follows:

$$h_{r,k}(t) = \lim_{\Delta \to 0} \frac{P[r - event\ in\ (t, t+\Delta) \mid r - event\ at\ t = 0, \xi(0) = k]}{\Delta}; \quad k = 0, 1$$

The function $h_{r,k}(t)$ represents the renewal density of r-events in an interval in which the state of the environment remains as k throughout the interval. Note that

$$h_{r,k}(t) = \sum_{n=1}^{\infty} f_{r,k}{}^{(n)}(t); \quad k = 0, 1 \tag{2.7}$$

2.3.4 Function $W(j,t;i,k)$

Consider an interval in which there is no change in the state of environment. The function $W(j,t;i,k)$ is defined as follows:

$$W(j,t;i,k) = P[L(t) = j \mid L(0) = i, \xi(0) = k]$$

where $0 \le i, j \le S$ and $k = 0,1$.

This function gives the distribution of the inventory level at any time t if the environment is in state k, $k = 0, 1$, throughout the interval $(0,t]$. To obtain an expression for $W(j,t;i,k)$, the following mutually exclusive and exhaustive cases are considered:

(i) No replenishment occurs in $(0,t]$

(ii) Only one replenishment occurs in $(0,t]$

(iii) More than one replenishment occurs in $(0,t]$

Accordingly

$$W(j,t;i,k) = H(i-j)P(j,t;i,k) + \lambda_k P(0,t;i,k) \copyright P(j,t;S,k)$$
$$+\ \lambda_k P(0,t;i,k) \copyright h_{r,k}(t) \copyright P(j,t;S,k) \qquad (2.8)$$

where $0 \le i, j \le S$ and $k = 0, 1$.

2.4 INVENTORY LEVEL

Let $0 = T_0, T_1, T_2 ...$ be the successive epochs at which the environment changes its state and

$$L_n = L(T_n+); \qquad \xi_n = (T_n+); \qquad n = 0, 1, 2, ...$$

Setting $Z_n = (L_n, \xi_n)$, it follows that $(Z,T) = \{Z_n, T_n; n = 0,1,2...\}$ is a Markov renewal process (Cinlar [1975a]) with the state space $E = E_2 \cup E_3$, where

$$E_2 = \{(i,0), i = 0,1,2,..., S\}; \quad E_3 = \{(i,1), i = 0,1,2,..., S\}$$

Defining

$$Q(j_2,k_2,t | j_1,k_1) = P[Z_{n+1} = (j_2,k_2), T_{n+1} - T_n \le t | Z_n = (j_1,k_1)] \qquad ;(j_1,k_1),(j_2,k_2) \in E_1$$

The function $Q(j_2,k_2,t|j_1,k_1)$ has the following interpretation. Given that the environment over its state to k_1, at time T_n and that the inventory level at T_n is j_1, the probability is $Q(j_2,k_2,t|j_1,k_1)$ that the subsequent change of the state of the environment takes place at time T_{n+1} not later than a duration t from T_n and that the state of Z at T_{n+1} is (j_2,k_2).

Since T_n's are epoch transitions of the process $\xi(t)$,

$$Q(j_2,k_2,t|j_1,k_1)=0 \text{ for } k_1=k_2 \tag{2.9}$$

For $k_1 \neq k_2$,

$$Q(j_2,1,t\ |j_1,0)=\int_0^t W(j_2,u;j_1,0)\nu_0 e^{-\nu_0 u}du \tag{2.10}$$

$$Q(j_2,0,t\ |j_1,1)=\int_0^t W(j_2,u;j_1,1)\nu_1 e^{-\nu_1 u}du \tag{2.11}$$

where $0 \le j_1, j_2 \le S$.

The semi-Markov kernel $Q(t)$ of the Markov renewal process is given by the following $(2S+2)\times(2S+2)$ order matrix:

$$\mathbf{Q}(t)=\begin{array}{c|cc} & E_2 & E_3 \\ \hline E_2 & 0 & A(t) \\ E_3 & B(t) & 0 \end{array}$$

where $A(t)$ is a matrix of order $(S+1)\times(S+1)$ whose elements are given by (2.10) and the matrix $B(t)$ is of order $(S+1)\times(S+1)$ whose elements are given by (2.11).

For any two elements (j_1,k_1) and $(j_2,k_2)\in E_1$,

$$R(j_2,k_2,t|j_1,k_1)=\sum_{n=0}^{\infty}Q^{(n)}(j_2,k_2,t|j_1,k_1) \tag{2.12}$$

where

$$Q^{(n)}(j_2,k_2,t\ |j_1,k_1)=\sum_{(j,k)\in E_1}\int_0^t Q(j,k,du\ |j_1,k_1)Q^{(n-1)}(j_2,k_2,t-u\ |j,k)$$

$R(j_2,k_2,t|j_1,k_1)$ represents the expected number of renewals of the state (j_2,k_2) in the interval $(0,t)$ and is called Markov renewal function. The Markov renewal kernel $R(t)$ of the process (Z,T) is given by the $(2S+2)\times(2S+2)$ order matrix

$$R(t)=[R(j_2,k_2,t|j_1,k_1)].$$

If $R^*(s)$ is the matrix Laplace transform defined by (see Girlich, 2003)

$$R^*(s)=[R^*(j_2,k_2,s|j_1,k_1)],$$

then, from the theory of Markov renewal process,

$$R^*(s)=[I-Q^*(s)]^{-1}$$

$$=\begin{bmatrix}(I-A^*(s)B^*(s))^{-1} & A^*(s)(I-A^*(s)B^*(s))^{-1}\\ B^*(s)(I-A^*(s)B^*(s))^{-1} & (I-A^*(s)B^*(s))^{-1}\end{bmatrix} \tag{2.13}$$

where $Q^*(s), A^*(s)$ and $B^*(s)$ are the matrices of Laplace transforms corresponding to $Q(t), A(t)$ and $B(t)$ respectively. Inversion of the elements of $R^*(s)$ yields the elements of $R(t)$. Using these elements, the probability distribution of the inventory level is defined as follows:

$$P(j,t|i,k) = P[L(t) = j | L(0) = i, \xi(0) = k] \quad ; \quad 0 \le j \le S, \quad (i,k) \in E_1$$

$P(j,t|i,k)$ is the probability that the inventory level is j at time t given that initially, at time $t = 0$, the inventory level is i and the environment level is k. To obtain an expression for $P(j,t|i,k)$, the vector process $(L(t), \xi(t))$ is semi-regenerative (Cinlar [1975]) with state space E_1 and the Markov renewal process (Z,T) embedded in it. Its probability function is defined by

$$\beta(j_2,k_2,t|j_1,k_1) = P[L(t) = j_2, \xi(t) = k_2 | L(0) = j_1, \xi(0) = k_1]$$

where (j_1,k_1) and $(j_2,k_2) \in E_1$.

An auxiliary function is defined as follows:

$$\gamma(j_2,k_2,t|j_1,k_1) = P[L(t) = j_2, \xi(t) = k_2, T_1 > t | L(0) = j_1, \xi(0) = k_1] \quad ; \quad (j_1,k_1),(j_2,k_2) \in E_1$$

This function has the following probabilistic interpretation:

Given that the inventory level is j_1 and that the environment is in state k_1, at time $t = 0$, the probability is $\gamma(j_2,k_2,t|j_1,k_1)$ that the next change of state of the environment takes place after a time t and that the levels of the inventory and the environment at time t are j_2 and k_2 respectively.

Since T_1 corresponds to the epoch of change of the state of the environment from the state of the process, the following conditions apply:

(i) $\gamma(j_2,k_2,t \mid j_1,k_1) = 0 \quad \text{for } k_1 \neq k_2$

(ii) $\gamma(j_2,1,t \mid j_1,1) = W(j_2,t;\, j_1,1)$

(iii) $\gamma(j_2,0,t \mid j_1,0) = W(j_2,t;\, j_1,0)$ (2.14)

Conditioning on the random variable T_1,

$$\beta(j_2,k_2,t \mid j_1,k_1) = \gamma(j_2,k_2,t \mid j_1,k_1) + \sum_{(j_3,k_3)\in E_1} \int_0^t Q(j_3,k_3,du \mid j_1,k_1) \quad \beta(j_2,k_2,t-u \mid j_3,k_3) \tag{2.15}$$

The solution of (2.15) is given by

$$\beta(j_2,k_2,t \mid j_1,k_1) = \sum_{(j_3,k_3)\in E_1} \int_0^t R(j_3,k_3,du \mid j_1,k_1) \quad \gamma(j_2,k_2,t-u \mid j_3,k_3) \tag{2.16}$$

Using the function $\beta(j_2,k_2,t \mid j_1,k_1)$,

$$P(j,t \mid i,k_1) = \sum_{k=0}^{1} \beta(j,k,t \mid i,k_1) \tag{2.17}$$

2.5 LIMITING DISTRIBUTION OF THE INVENTORY LEVEL

Considering the Markov chain $\{L_n, \xi_n\}$ and defining

$$A = \lim_{t\to\infty} A(t) \qquad ; B = \lim_{t\to\infty} B(t),$$

the one-step transition probability matrix of the Markov chain $\{L_n, \xi_n\}$ is given by

$$\boldsymbol{Q} = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \tag{2.18}$$

The structure of $\boldsymbol{Q}$ implies that the chain is periodic with period 2. Since every element of A is greater than 0, the chain $\{L_n, \xi_n\}$ is irreducible (Feller [1965]).

Consequently, the stationary distribution of $\{L_n, \xi_n\}$ exists. Let $\tilde{\pi} = (\tilde{\pi}_1, \tilde{\pi}_2)$ be the stationary distribution where

$$\tilde{\pi}_1 = \left(\pi(0,0), \pi(1,0), \dots, \pi(S,0)\right)$$

and $$\tilde{\pi}_2 = \left(\pi(0,1), \pi(1,1), \dots, \pi(S,1)\right),$$

such that $\tilde{\pi}_1 AB = \tilde{\pi}_1$ and $\tilde{\pi}_2 = \tilde{\pi}_1 A$.

Since (L_n, ξ_n) has a stationary distribution, the semi-regenerative process $(L(t), \xi(t))$ also has a stationary distribution defined by

$$\phi(j_2, k_2) = \lim_{t \to \infty} \beta(j_2, k_2, t \mid j_1, k_1) \tag{2.19}$$

where (j_1, k_1) and $(j_2, k_2) \in E_1$.

To obtain $\phi(j_2, k_2)$ consider the mean sojourn time of the Markov renewal process (Z, T) in a state (j_1, k_1) of E_1 defined by

$$m(j_1, k_1) = E[T_{n+1} - T_n \mid Z_n = (j_1, k_1)] \tag{2.20}$$

From the definition of $Q(j_2, k_2, t | j_1, k_1)$, Cinlar [1975] indicates that

$$m(j_1, k_1) = \int_0^\infty [1 - \sum_{(j_2,k_2)\in E_1} Q(j_2, k_2, t | j_1, k_1)]dt \tag{2.21}$$

By applying a theorem on semi-regenerative process,

$$\phi(j_2, k_2) = \frac{\sum_{(j_1,k_1)\in E_1} \pi(j_1, k_1) \int_0^\infty \gamma(j_2, k_2, t | j_1, k_1)dt}{\tilde{\pi}.\tilde{m}} \tag{2.22}$$

where $\quad \tilde{m} = (m(0,0),\ m(1,0),\ldots,\ m(S,0),\ m(0,1),\ \ldots,\ m(S,1))$

and $\quad \tilde{\pi}.\tilde{m} = \sum_{(j_1,k_1)\in E_1} \pi(j_1, k_1) m(j_1, k_1) \qquad (2.23)$

The stationary distribution of $L(t)$ can be obtained, defined by

$$\theta(j) = \lim_{t\to\infty} P[L(t) = j | L(0) = i, \xi(0) = k] \tag{2.24}$$

where $0 \le j \le S$, $(i,k) \in E_1$.

Note that

$$\begin{aligned} \theta(j) &= \lim_{t\to\infty} \sum_{k_2=0}^{1} \beta(j, k_2, t | i, k_1) \\ &= \sum_{k_2=0}^{1} \phi(j, k_2) \end{aligned} \tag{2.25}$$

2.6 MEASURES OF SYSTEM PERFORMANCE

2.6.1 Mean Number of Replenishments

Let $h_r(t)$ be the first order product density of the point process of r-events. Since at the epoch of an r-event the environment may be either in state 0 or 1,

$$h_r(t) = \sum_{k=0}^{1} \beta(0,k,t \mid j_1,k_1)\lambda_k$$

where $(j_1,k_1) \in E_1$

The mean number of replenishments in $(0,t]$ is given by

$$E[N(r,t)] = \int_0^t h_r(u)du$$

Hence the mean-stationary rate of replenishments is

$$\begin{aligned} E(r) &= \lim_{t\to\infty} \frac{E[N(r,t)]}{t} \\ &= \lim_{t\to\infty} h_r(t) \\ &= \sum_{k=0}^{1} \phi(0,k)\lambda_k \end{aligned}$$

2.6.2 Mean Number of Demands

Since replenishment is instantaneous, any demand that occurs is satisfied. Define

$$h_d(t) = \lim_{\Delta\to 0} \frac{P[a\ d - event\ in\ (t,t+\Delta) \mid Z_0 = (j_1,k_1)]}{\Delta} \ ;\ (j_1,k_1) \in E_1 .$$

Then $h_d(t)$ is the first-order product density of the d-events and

$$h_d(t) = \sum_{j_2=0}^{S} \sum_{k_2=0}^{1} \beta(j_2, k_2, t | j_1, k_1)$$

The mean number of demands occurring in $(0, t]$ is given by

$$E[N(d,t)] = \int_0^t h_d(u) du \,.$$

Consequently, the mean stationary rate of demands is given by

$$E(d) = \sum_{j=0}^{S} \sum_{k=0}^{1} \phi(j,k) \lambda_k \,.$$

Let $h_d^k(t)$ be the product density of d-events occurring while the environment is in state k, $k = 0, 1$. Then,

$$h_d^k(t) = \lim_{\Delta \to 0} \frac{P[N(d, t+\Delta) - N(d,t) = 1, \xi(t) = k | Z_0 = (j,k)]}{\Delta} \; ; (j,k) \in E_1$$

and $\qquad h_d^k(t) = \sum_{j=0}^{S} \beta(j, k, t | j_1.k_1); \quad k = 0, 1.$

Consequently, $h_d(t) = h_d^0(t) + h_d^1(t)$

2.6.3 Mean Number of Perished Items

For the first-order product density $h_\pi(t)$ of the point process of π-events,

$$h_\pi(t) = \sum_{j=0}^{S}\sum_{k=0}^{1}\beta(j,k,t \mid j_1,k_1)\, j\mu_k$$

where $(j_1,k_1) \in E_1$.

The mean number of items that perish in the interval $(0,t]$ is then given by

$$E[N(\pi,t)] = \int_0^t h_\pi(u)du$$

and the mean-stationary rate of items that perish is

$$\begin{aligned}E(\pi) &= \lim_{t\to\infty}\frac{E[N(\pi,t)]}{t}\\ &= \lim_{t\to\infty} h_\pi(t)\\ &= \sum_{j=0}^{S}\sum_{k=0}^{1}\phi(j,k)\, j\mu_k\end{aligned}$$

2.7 COST ANALYSIS

The profit per unit time can be formulated as follows:

$$P_f = \sum_{k=0}^{1} E(d_k)c_{d_k} - [E(r)(S+1)c_b + \sum_{k=0}^{1} E(r_k)CR_k] - \sum_{j=0}^{s} \theta(j)c_j - E(\pi)c_\pi$$

where

c_{d_k} : Selling price of one item when the environment is in state k, k = 0, 1

c_b : Buying cost of one item

CR_k : Cost of replenishment when the environment is in state k, k = 0, 1

c_j : Holding cost when the inventory level is j

c_π : Salvage cost of one perished item

P_f : Profit per unit time in the long run

2.8 TOTAL SALE PROCEEDS

Assuming the following:

(i) The demand rate is a constant and is the same for all time $t > 0$.

(ii) The selling price of one item is c_k when the environment is in state k, k = 0, 1.

For the stochastic process $\beta(t)$ defined by

$$\beta(t) = \int_0^t \xi(u)du$$

Then $\beta(t)$ represents the total time in $(0,t)$ during which the environment is in state 1.

Consequently, the total time in $(0,t)$ during which the environment is in state 0 is $t-\beta(t)$. Tackacs (1957a,b) has investigated and obtained the distribution function of $\beta(t)$ as

$$\Omega(t,x)=\sum_{n=0}^{\infty} H^{(n)}(x)[G^{(n)}(t-x)-G^{(n+1)}(t-x)]$$

where $G(x)=P[X_n \le x]$,

$H(x)=P[Y_n \le x]$,

and $H^{(0)}(x)=\begin{cases}1 \; if \;\; x \ge 0\\ 0 \; if \;\; x<0\end{cases}$

$G^{(0)}(x)=1$

Since $N(d,t)$ represents the total number of demands which have occurred up to time t, the total sale proceeds to time t is given by

$$S(t)=c_0+c_0N(d,t-\beta(t))+c_1N(d,\beta(t)) \tag{2.26}$$

Assuming that $N(d,t)$ is a stationary renewal process, equation (8.1) can be expressed as

$$S(t)=c_0+c_0N(d,t)+(c_1-c_0)N(d,\beta(t)) \tag{2.27}$$

For simplicity, assume that $c_1=mc_0$, where m is a fixed positive integer. Setting

$$\tilde{S}(t)=\frac{[S(t)-c_0]}{c_0}$$

The equation (2.27) simplifies as

$$\tilde{S}(t) = N(d,t) + (m-1)N(d,\beta(t)) \tag{2.28}$$

In order to determine the probability distribution of $S(t)$, the joint probability distribution of $N(d,t)$ and $N(d,\beta(t))$ is required.

Define $\alpha(i, j, t) = P[N(d,t) = 1; N(d,\beta(t)) = j]$

Since $N(d,\beta(t))$ and $N(d,t-\beta(t))$ are stochastically independent,

$$\alpha(i, j, t) \; = \; P[N(d,\beta(t)) = j, N(d,t-\beta(t)) = i - j]$$

$$= e^{-\lambda t} \; \frac{\lambda^i}{i!} \binom{i}{j} \int_0^t u^j (t-u)^{i-j} d_u \; \Omega(t,u) \tag{2.29}$$

For any non-negative integer k, the event $(\tilde{S}(t) = k)$ occurs if and only if one of the following events occurs:

$$[N(d,\beta(t)), N(d,t) = k - (m-1)j]; \quad j = 0, 1, 2, \ldots, \; r$$

where r is the largest integer less than or equal to $\left[\dfrac{k}{m}\right]$.

Consequently,

$$P[\tilde{S}(t)=k]=\sum_{j=0}^{r}P[N(d,\beta(t))=j,N(d,t)=k-(m-1)j]$$

$$=\sum_{j=0}^{r}\alpha(k-(m-1)j,j,t) \qquad (2.30)$$

Further specializing to the case where

$$G(x)=\begin{cases}1-e^{-ax} & if\ \ x>0\\ 0 & otherwise\end{cases}$$

$$H(x-k)=\begin{cases}1 & if\ \ x>k\\ 0 & otherwise\end{cases}$$

The following results from the work of Tackacs (1957a,b):

$$\Omega(t,x)=\sum_{n=0}^{\infty}e^{-a(t-x)}\frac{[a(t-x)]^n}{n!}U(x-nk)$$

where $U(.)$ stands for the Heaviside function. Now, for this particular case, the pdf of $\beta(t)$ is given by

$$\omega(t,x)=\frac{e^{-a(t-x)}[a(t-x)]^{\left[\frac{x}{k}\right]}}{\left[\frac{x}{k}\right]!}\delta(x-\left[\frac{x}{k}\right]k)+\frac{ae^{-a(t-x)}[a(t-x)]^{\left[\frac{x}{k}\right]}}{\left[\frac{x}{k}\right]!};\quad 0\le x\le t \qquad (2.31)$$

Using (2.31) in (2.29) the expression for $\alpha(i,j,t)$ is derived:

$$\alpha(i,j,t)=e^{-\lambda t}\frac{\lambda^t}{i!}\binom{i}{j}\int_0^t x^j(t-x)^{i-j}\omega(t,x)dx \qquad (2.32)$$

The following cases are applicable:

Case (i) Let $k > t$, then from (2.31)

$$\omega(t,x) = e^{-a(t-x)}\delta(x) + ae^{-a(t-x)}$$

and hence, from (2.32) we get

$$\alpha(i,j,t) = e^{-\lambda t}\frac{\lambda^i}{i!}\binom{i}{j}\int_0^t x^j(t-x)^{i-j}ae^{-a(t-x)}dx \tag{2.33}$$

Case (ii) Let $k < t$, note that, for some positive integer n, $nk < t \le (n+1)k$ and so,

$$\alpha(i,j,t) = e^{-\lambda t}\frac{\lambda^i}{i!}\binom{i}{j}(I_1 + I_2) \tag{2.34}$$

where

$$I_1 = a\int_0^k x^j(t-x)^{i-j}e^{-a(t-x)}dx \tag{2.35}$$

$$I_2 = \sum_{r=1}^{n}[(rk)^j(t-rk)^{i-j}e^{-a(t-rk)} + \sum_{r=1}^{n-1}\int_{rk}^{(r+1)k} x^j(t-x)^{i-j}ae^{-a(t-x)}\frac{[a(t-x)]^r}{r!}dx$$

$$+\int_{nk}^{t} x^j(t-x)^{i-j}ae^{-a(t-x)}\frac{[a(t-x)]^n}{n!}dx \tag{2.36}$$

As $\alpha(i,j,t)$ is explicitly known in all the cases, the probability distribution of $\tilde{S}(t)$ is obtained from (2.30).

2.9 THE TOTAL COST OF REPLENISHMENT

Let the cost of replenishment be CR_k when the environment is in state k, $k = 0,1$ and $C(t)$ be the total cost of replenishments up to time t. Proceeding as in Section 2.8,

$$C(t) = CR_0 + CR_0 N(r,t) + (CR_1 - CR_0) N(r, \beta(t)) \tag{2.37}$$

Where $N(r,t)$ represents the number of replenishments made in the interval $(0,t]$.

Setting $\tilde{C}(t) = \dfrac{(C(t) - CR_0)}{CR_0}$ and taking $CR_1 = m\, CR_0$ in (2.37), where m is a positive integer constant,

$$\tilde{C}(t) = N(r,t) + (m-1) N(r, \beta(t)) \tag{2.38}$$

Consequently,

$$P[\tilde{C}(t) = k] = \sum_{j=0}^{n} P[N(r,\beta(t)) = j, N(r,t) = k - (m-1)j] \tag{2.39}$$

where n is the largest integer less than or equal to $\left[\dfrac{k}{m}\right]$.

Since the event $\{N(r,\beta(t)) = j, N(r,t) = k-(m-1)j\}$ is equivalent to the event $\{N(r,\beta(t)) = j, N(r,t-\beta(t)) = k - mj\}$, $N(r,\beta(t))$ and $N(r,t-\beta(t))$ are independent, and that

$$P[N(r,t) = j] = P[j(S+1) \le N(d,t) < (j+1)(S+1)]$$
$$= \sum_{i=j(S+1)}^{(j+1)(S+1)-1} P[N(d,t) = i]$$
$$= \sum_{i=j(S+1)}^{(j+1)(S+1)-1} \frac{e^{-\lambda t}(\lambda t)^i}{i!}$$
$$= \sigma(j,t) \quad ;(say)$$

Equation (2.39) yields explicitly that

$$P[\tilde{C}(t) = k] = \sum_{j=0}^{n} \int_0^t \sigma(j,u)\sigma(k-mj,t-u)d_u\Omega(t,u)$$

2.10 NUMERICAL ILLUSTRATION

In this section, numerical examples illustrate the functioning of the inventory system operating in a random environment.

2.10.1 Analysis of Measures of System Performance

First, considering the various measures obtained in Section 6 and 7, their behaviour under the following cases are obtained:

Case (i): λ_0 varies from 10.0 to 200; S = 3, λ_1 = 50.0, μ_0 = 10.0, μ_1 = 20.0, ν_0 = 1.5, ν_1 = 2.5.

Case (ii): λ_1 varies from 50.0 to 250; S = 3, λ_0 = 10.0, μ_0 = 10.0, μ_1 = 20.0, ν_0 = 1.5, ν_1 = 2.5.

Case (iii): μ_0 varies from 10.0 to 20.0; λ_0 = 10.0, λ_1 = 50.0, μ_1 = 20.0, ν_0 = 1.5, ν_1 = 2.5.

Case (iv): μ_0 varies from 10.0 to 20.0; λ_0 = 10.0, λ_1 = 50.0, μ_0 = 10.0, ν_0 = 1.5, ν_1 = 2.5.

The results for each of these cases are given in Tables 2.2 to 2.5. In all the above four cases, the following values is assumed for the costs in order to determine the mean-rate of the total profit (PF):

C_{d0} = 100.0, C_{d1} = 150.0, C_{R0} = 10.0, C_{r1} = 20.0, C_j = 5.0, C_b = 50.0, C_π = 3.0

A consolidated overview of the results are provided in Table 2.1 below:

	Mean Rate of			
	Replenishment (RR)	**Demands (RD)**	**Perished Items (RP)**	**Total Profit (PF)**
λ_0 increases	Increases	Increases		Increases
λ_1 increases	Increases	Increases		Increases
μ_0 increases			Increases	Decreases
μ_1 increases			Increases	Decreases

Table 2.1: Overview of the Analysis of System Performance Measures

2.10.2 Analysis of Probability Distributions

The probability distribution of the total sale proceeds obtained in Section 2.8 is considered and evaluated numerically by assuming the following values for the parameters:

m = 2, k = 10, t = 10

Fixing the demand rate $\lambda = 0.3$, the value of a is increased to obtain the values of the probability $P[S(10) = 10]$ corresponding to the cases $k = 8$ and $k = 20$ (Table 2.6).

Fixing $a = 0.00006$, the demand rate of λ is increased to obtain the values of $P[S(10) = 10]$ corresponding to $k = 8$ and $k = 20$ (Table 2.7).

The time dependent behaviour of $P[S(t) = k]$, in the interval $0 < t < 10$ is also illustrated. For this purpose, $k = 5$, $a = 0.00001$ and $k = 6$ to obtain $P[S(t) = 5]$, $0 < t < 10$ for three cases $\lambda = 0.1$, $\lambda = 0.2$ and $\lambda = 0.3$ (Table 2.8). It is noted that the probability increases as time increases in (0, 10) and that the probability increases as the demand rate λ increases.

Finally, the probability distribution of the total cost of replenishment obtained in Section 2.9 are considered and evaluated numerically by assuming the following values for the parameters:

$$m = 2,\ k = 10,\ t = 10$$

Fixing the demand rate $\lambda = 3.0$, the value of a is increased. Note that the probability $P[C(10) = 10]$ increases for both cases $k = 8$ and $k = 10$ as detailed in Table 2.9.

Fixing $a = 0.00006$ and increasing λ, note that the probability decreases for both cases $k = 8$ and $k = 10$ as per Table 2.10.

The time-dependent behaviour of $P[C(t) = k]$, $0 < t < 10$ is illustrated by assuming $k = 20$, $a = 0.00006$ and considering three cases: $\lambda = 3.0, 3.2, 3.4$ as detailed in Table 2.11.

S =3, λ_1 = 50.0, μ_0 = 10.0, μ_1 = 20.0, ν_0 = 1.5, ν_1 = 2.5

λ_0	**RR**	**RD**	**RP**	**PF**
10.0	6.25000	25.00000	20.62500	2011.25000
20.0	7.81250	31.24999	20.62500	2308.12500
30.0	9.37500	37.50000	20.62500	2605.00000
40.0	10.93750	43.74999	20.62500	2901.87500
50.0	12.50000	49.99999	20.62500	3198.75000
60.0	14.06250	56.24998	20.62499	3495.62500
70.0	15.62499	62.49998	20.62499	3792.50000
80.0	17.18751	68.75002	20.62500	4089.37500
90.0	18.75000	74.99999	20.62500	4386.25100
100.0	20.31250	81.24999	20.62500	4683.12500
110.0	21.87499	87.49996	20.62499	4979.99900
120.0	23.43749	93.74997	20.62499	5276.87400
130.0	25.00000	99.99999	20.62500	5573.75000
140.0	26.56249	106.25000	20.62499	5870.62400
150.0	28.12498	112.49990	20.62499	6167.49700
160.0	29.68750	118.75000	20.62500	6464.37500
170.0	31.24999	124.99990	20.62499	6761.24800
180.0	32.81248	131.24990	20.62499	7058.12300
190.0	34.37499	137.50000	20.62500	7355.00100
200.0	35.93749	143.75000	20.62499	7651.87400

Table 2.2: **Measures of Performance versus Demand Rate varying in environment in State 0**

S = 3, λ_0 = 10.0, μ_0 = 10.0, μ_1 = 20.0, ν_0 = 1.5, ν_1 = 2.5

λ_1	**RR**	**RD**	**RP**	**PF**
50.0	6.25000	25.00000	20.62500	2011.25000
60.0	7.18750	28.75000	20.62500	2367.50000
70.0	8.12500	32.50000	20.62500	2723.75000
80.0	9.06250	36.25002	20.62501	3080.00200
90.0	9.99999	39.99997	20.62498	3436.24800
100.0	10.93749	43.74998	20.62499	3792.49900
110.0	11.87500	47.49999	20.62500	4148.75000
120.0	12.81250	51.25000	20.62500	4505.00000
130.0	13.74999	54.99997	20.62499	4861.24900
140.0	14.68750	58.75000	20.62500	5217.50000
150.0	15.62499	62.49998	20.62499	5573.75000
160.0	16.56249	66.24998	20.62499	5930.00000
170.0	17.50001	70.00002	20.62501	6286.25300
180.0	18.43749	73.74995	20.62498	6642.49700
190.0	19.37498	77.49995	20.62499	6998.74800
200.0	20.31248	81.24990	20.62498	7354.99200
210.0	21.25000	84.99998	20.62500	7711.25000
220.0	22.18750	88.75002	20.62500	8067.50200
230.0	23.12497	92.49989	20.62498	8423.74300
240.0	24.06249	96.24995	20.62499	8779.99700
250.0	24.99999	99.99998	20.62500	9136.24900

Table 2.3: Measures of Performance versus Demand Rate varying in environment in State 1

$S = 3$, $\lambda_0 = 10.0$, $\lambda_1 = 50.0$, $\mu_1 = 20.0$, $\nu_0 = 1.5$, $\nu_2 = 2.5$

λ_0	**RR**	**RD**	**RP**	**PF**
10.0	6.25000	25.00000	20.62500	2011.25000
10.5	6.25000	25.00000	21.09375	2009.84400
11.0	6.25000	25.00000	21.56250	2008.43800
11.5	6.25000	25.00000	22.03125	2007.03200
12.0	6.25000	25.00000	22.50000	2005.62500
12.5	6.25000	25.00000	22.96875	2004.21900
13.0	6.25000	25.00000	23.43750	2002.81300
13.5	6.25000	25.00000	23.90624	2001.40600
14.0	6.25000	25.00000	24.37500	2000.00000
14.5	6.25000	25.00000	24.84375	1998.59400
15.0	6.25000	25.00000	25.31250	1997.18800
15.5	6.25000	25.00000	25.78125	1995.78100
16.0	6.25000	25.00000	26.25000	1994.37500
16.5	6.25000	25.00000	26.71875	1992.96900
17.0	6.25000	25.00000	27.18750	1991.56300
17.5	6.25000	25.00000	27.65625	1990.15600
18.0	6.25000	25.00000	28.12500	1988.75000
18.5	6.25000	25.00000	28.59375	1987.34400
19.0	6.25000	25.00000	29.06250	1985.34400
19.5	6.25000	25.00000	29.53125	1984.53100
20.0	6.25000	25.00000	30.00000	1983.12500

Table 2.4: **Measures of Performance versus Demand Rate varying in environment in State 0**

S = 3, λ_0 = 10.0, λ_1 = 50.0, μ_0 = 10.0, ν_0 = 1.5, ν_1 = 2.5

λ_0	**RR**	**RD**	**RP**	**PF**
10.0	6.25000	25.00001	15.00001	2028.12600
10.5	6.25000	25.00000	15.28125	2027.28100
11.0	6.25000	24.99999	15.56250	2026.43700
11.5	6.25000	24.99999	15.84375	2025.59400
12.0	6.25000	25.00000	16.12500	2024.75000
12.5	6.25000	25.00000	16.40625	2023.90600
13.0	6.25000	25.00001	16.68751	2023.06300
13.5	6.25000	24.99999	16.96875	2022.21900
14.0	6.25000	25.00000	17.25000	2021.37500
14.5	6.25000	25.00000	17.53125	2020.53200
15.0	6.25000	25.00001	17.81250	2019.68800
15.5	6.25000	25.00001	18.09375	2018.84400
16.0	6.25000	25.00000	18.37500	2018.00000
16.5	6.25000	25.00000	18.65625	2017.15600
17.0	6.25000	25.00000	18.93750	2016.31300
17.5	6.25000	25.00000	19.21875	2015.46900
18.0	6.25000	25.00000	19.50001	2014.62600
18.5	6.25000	25.00000	19.78125	2013.78100
19.0	6.25000	25.00000	20.06250	2012.93800
19.5	6.25000	25.00000	20.34375	2012.09400
20.0	6.25000	25.00000	20.62500	2011.25000

Table 2.5: **Measures of Performance versus Perishing Rate varying in environment in State 1**

$\lambda = 0.3$

	P[S(10) = 10]	
a	k = 8	k = 20
0.00006	0.0460570	0.0000166
0.00011	0.0460597	0.0000304
0.00016	0.0460625	0.0000442
0.00021	0.0460652	0.0000579
0.00026	0.0460679	0.0000717
0.00031	0.0460707	0.0000855
0.00036	0.0460734	0.0000993
0.00041	0.0460761	0.0001131
0.00046	0.0460789	0.0001268
0.00051	0.0460816	0.0001406

Table 2.6: P[S(10) = 10] versus Environment Rate

a = 0.00006

	P[S(10) = 10]	
a	k = 8	k = 10
0.25000	0.0290197	0.0000099
0.26000	0.0322745	0.0000112
0.27000	0.0356271	0.0000124
0.28000	0.0390563	0.0000138
0.29000	0.0425402	0.0000151
0.30000	0.0460570	0.0000166
0.31000	0.0495848	0.0000180
0.32000	0.0531022	0.0000195
0.33000	0.0565883	0.0000210
0.34000	0.0600231	0.0000225
0.35000	0.0633876	0.0000241
0.36000	0.0666638	0.0000256

Table 2.7: P[S(10) = 10] versus Demand Rate

k = 6 and a = 0.00001

	P[S(t) = 5]		
t	λ = 0.1	λ = 0.2	λ = 0.3
0.50	0.000000000	0.000000000	0.000000001
1.00	0.000000000	0.000000003	0.000000009
1.50	0.000000002	0.000000013	0.000000040
2.00	0.000000006	0.000000039	0.000000112
2.50	0.000000013	0.000000088	0.000000243
3.00	0.000000027	0.000000168	0.000000447
3.50	0.000000047	0.000000287	0.000000736
4.00	0.000000078	0.000000453	0.000001116
4.50	0.000000119	0.000000670	0.000001590
5.00	0.000000175	0.000000945	0.000002155
5.50	0.000000246	0.000001279	0.000002805
6.00	0.000000336	0.000001678	0.000003548
6.50	0.004705349	0.019678810	0.034720840
7.00	0.008988675	0.035907300	0.060612490
8.00	0.012914050	0.049407630	0.079737830
8.50	0.016536890	0.060749660	0.094119200
9.00	0.019905290	0.070382460	0.104909600
9.50	0.026039840	0.085853610	0.119100000
10.00	0.028873060	0.092181770	0.123630600

Table 2.8: **P[S(t) = 5] versus Time t**

λ = 3.0

	P[C(10) = 10]	
a	k = 8	k = 20
0.00006	0.0787214	0.0000745
0.00011	0.0787648	0.0001366
0.00016	0.0788081	0.0001986
0.00021	0.0788514	0.0002606
0.00026	0.0788947	0.0003226
0.00031	0.0789380	0.0003845
0.00036	0.0789812	0.0004465
0.00041	0.0790244	0.0005083
0.00046	0.0790676	0.0005702
0.00051	0.0791108	0.0006320

Table 2.9: **P[C(10) = 10] versus Environment Rate**

a = 0.00006

λ	P[C(10) = 10] k = 8	P[C(10) = 10] k = 20
3.00000	0.0787214	0.0000745
3.10000	0.0664267	0.0000735
3.20000	0.0548080	0.0000720
3.30000	0.0442755	0.0000702
3.40000	0.0350609	0.0000681
3.50000	0.0272459	0.0000658
3.60000	0.0207988	0.0000632
3.70000	0.0156118	0.0000604
3.80000	0.0115328	0.0000573
3.90000	0.0083918	0.0000540
4.00000	0.0060197	0.0000504
4.10000	0.0042604	0.0000467
4.20000	0.0029774	0.0000428
4.30000	0.0020564	0.0000389
4.40000	0.0014049	0.0000350
4.50000	0.0009502	0.0000311

Table 2.10: **P[C(10) = 10] versus Demand Rate**

k = 20 and a = 0.00006

t	P[C(10) = 10] λ = 3.0	λ = 3.2	λ = 3.4
0.50000	0.0000000	0.0000000	0.0000000
1.50000	0.0000000	0.0000000	0.0000000
2.50000	0.0000000	0.0000000	0.0000000
3.50000	0.0000001	0.0000002	0.0000004
4.50000	0.0000016	0.0000026	0.0000041
5.50000	0.0000083	0.0000121	0.0000165
6.50000	0.0000235	0.0000299	0.0000358
7.50000	0.0000432	0.0000492	0.0000533
8.50000	0.0000604	0.0000632	0.0000638
9.50000	0.0000714	0.0000704	0.0000679

Table 2.11: **P[C(10) = 10] versus Time t**

2.11 CONCLUSION

A model of a perishable product inventory system operating in a random environment is studied in this chapter. For the sake of simplicity, the stochastic environment is considered to alternate randomly over time between two states 0 and 1 according to an alternating renewal process. When the environment is in state k, the items in the inventory have a perishing rate μ_k, the demand rate is λ_k and the replenishment cost is CR_k. Assuming instantaneous replenishment at the epoch of the first demand after the stock-out and associating a Markov renewal process with the inventory system, the stationary distribution of the inventory level and the performance of various measures of the system evolution are obtained. Numerical examples illustrated the results obtained.

CHAPTER 3

A SINGLE PRODUCT PERISHING INVENTORY MODEL WITH DEMAND INTERACTION

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3.1 INTRODUCTION

In inventory models of perishing products the lifetime of the products in the inventory model is described in alternative ways. One assumption is that the product has a fixed lifetime and if no demand occurs for the product within its lifetime, it is considered as perished and removed from the inventory. Nahmias (1982) has given an exhaustive survey of the fixed-life perishable inventory literature. Another description of the lifetime is that the product deteriorates continuously in quality over time and eventually perishes. Raafat (1991) has presented a review of the literature on deteriorating (decaying) inventory models. Apart from the lifetime consideration, the deteriorating items have one important kind of interaction on the demand process in the sense that, in addition to the usual demand, there may also be a separate demand for items slightly deteriorated in quality if the cost is comparatively lesser than a new one. For example, vegetables, food, meat and fish loose their lustre as time elapse. A day old vegetable is slightly inferior in quality compared to a new one. Such items may be accepted by some customers in the event of non-availability of new ones. There may also be a significant number of demands for slightly deteriorated items due to the fact that they are less expensive. Some of continuous review inventory models have been studied recently by Beyer and Girlich (1994), Yadavalli et al (2001), Yadavalli & Joubert (2003) and Yadavalli et al (2004).

In this chapter, an attempt is made to incorporate the above kind of interaction in the study of deteriorating product inventory systems. Specifically, a continuous review of perishing inventory models is considered with the assumption that if there is no demand for product in inventory, it passes through two phases and then perishes. An item in Phase I is fresh and in Phase II slightly deteriorated. On leaving Phase II, it is considered as being perished and removed from inventory or scrapped. Independent demand takes place at constant rates for items in both phases. Demand for an item during Phase I stock-out may be satisfied by an item in Phase II based on a probability measure. Demand for product in phase II during stock-out is lost. Using the regeneration point technique, various measures of the inventory model are obtained.

The organization of this chapter is as follows: Section 3.2 lists various assumptions and notations in the description of the inventory model and also provides the auxiliary functions which are needed to describe the behaviour of the process between two successive regeneration points of the underlying stochastic process describing the inventory model. Various performance measures of the inventory model are obtained in Section 3.3. A cost analysis is provided in Section 3.4 and some numerical results are presented in Section 3.5.

3.2 ASSUMPTIONS AND AUXILIARY FUNCTION

The following assumptions are considered in the continuous inventory model with:

1. The item under consideration is perishable.

2. The lifetime distribution of an item is a generalized Erlang distribution with two phases. For convenience the items in Phase I are designated as Product 1 and that in Phase II as Product 2.

3. The demand for product i occurs at a constant rate λ_i, $i = 1, 2$.

4. Maximum storage capacity or total capacity of the inventory level is S and re-order takes place if the total inventory level is s.

5. At the epoch of replenishment, all items of Product 2 are scrapped (deleted) and the inventory level is raised to S.

6. The lead-time is arbitrary with pdf $f(.)$, and survivor function $\overline{F}(t) = 1 - F(t)$, where *F(t)* is the cdf. The arbitrary distribution is selected as an approximation of complex problems.

7. A demand for Product 1 occurring during the stock-out period can be substituted by an item of Product 2 with probability p if available, $0 \le p \le 1$.

8. A demand for Product 2 occurring during the stock-out period is lost, that is no backlogging is possible.

The following notation are used in this chapter:

a_j : Event that a re-order takes place when the inventory level of Product 2 is j, $0 \le j \le s$.

a : Any a_j-event, $0 \le j \le s$.

r_{ij} : Event that a stock replenishment occurs. $S - i$ units of Product 1 are added to the inventory and j units of Product 2 scrapped from the inventory.

r : Any r_{ij}-event, $0 \le i,\ j,\ \ i + j \le s$.

l_j : Event that a demand for product j is lost, j = 1, 2

g : Event that a demand for Product 1 is substituted by Product 2.

d_i : Event that a demand for product i is satisfied with product i, $i = 1, 2$.

k_1 : Event of Product 1 transitting as Product 2.

k_2 : Event of Product 2 perishing and being removed from the inventory.

$L_i(t)$: Inventory level of product i at time t; $i = 1, 2$.

$Z(t)$: $\left(L_1(t), L_2(t)\right)$.

λ_i : The demand rate of product i, $i = 1, 2$.

μ_i : The perishing rate of product i, $i = 1, 2$.

N(η,t): Number of η events in (0,t].

$E[N(a_j, \infty)]$: The mean stationary rate of re-order.

$E[N(k_1, \infty)]$: The mean stationary rate of transit of Product 1 as Product 2.

$E[N(k_2, \infty)]$: The mean stationary rate of perishing and removed from the inventory.

CR : Re-ordering cost.

CL_i : Cost of lost demand for product i, $i = 1, 2$.

CP : Salvage cost per scrapped (deleted) unit.

CB : Purchase price of one unit.

$C(S, s)$: Total expected cost per unit time.

© : Convolution symbol.

In order to study the stochastic process $(L_1(t), L_2(t))$, note that the r-events constitute a renewal process (see Figure 3.1 below). Consequently, it is sufficient to describe the behaviour of the inventory process between two successive renewals.

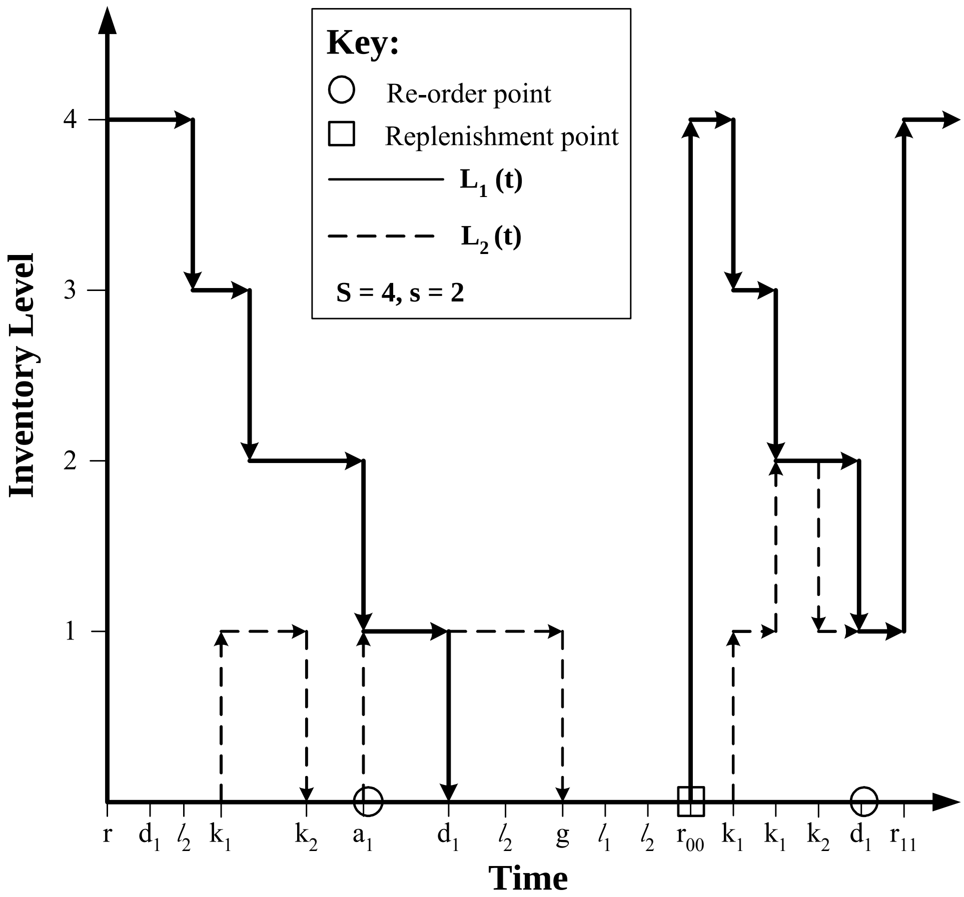

Figure 3.1: Realization of Events

The necessary auxiliary functions are introduced:

3.2.1 FUNCTION $P\left(k,l,t \middle| i,j\right)$

We define

$$P(k,l,t|i,j) = P\left[Z(t)=(k,l), N(\eta,t)=0 \middle| Z(0)=(i,j)\right] \quad , \quad \eta = a, r \, .$$

$P(k,l,t|\ i,j)$ represents the probability distribution of the inventory level in an interval in which neither reorder nor replenishment can occur. To derive an expression for this function, we note that a change in the inventory level may occur due to any one of the following possibilities:

1. A demand for Product i occurs and is satisfied by product i, $(i=1,2)$
2. A unit of Product 1 perishes and transits as Product 2.
3. A unit of Product 2 perishes.
4. A demand for a unit of Product 1 occurs during the stock-out period and is substituted by Product 2 with probability p if it is available.

Accordingly, we have for

$$0 \le k+l \le i+j \le s \text{ or } s+1 \le k+l \le i+j \le S,$$

Case 1: $k > i$.

$$P(k,l,t|\ i,j) = 0. \tag{3.1}$$

Case 2: $i > 0,\ j > 0,\ 0 < k < i,\ k+l < i+j$.

$$P(k,l,t|\ i,j) = \lambda_1\, e^{-(\lambda_1+\lambda_2+i\mu_1+j\mu_2)t} \copyright\ P(k,l,t|\ i-1,j) + i\mu_1 e^{-(\lambda_1+\lambda_2+i\mu_1+j\mu_2)t}$$
$$\copyright\ P(k,l,t|\ i-1,j+1) + (\lambda_2 + j\mu_2)\, e^{-(\lambda_1+\lambda_2+i\mu_1+j\mu_2)t} \copyright\ P(k,l,t|\ i,j-1). \tag{3.2}$$

Case 3: $i > 0,\ j = 0,\ 0 \le k < i,\ l \ge 0,\ k+l < i$.

$$P[k,l,t|\ i,0] = \lambda_1\ e^{-(\lambda_1+\lambda_2+i\mu_1+j\mu_2)t} \copyright\ P[k,l,t|i-1,0] + i\mu_1 e^{-(\lambda_1+\lambda_2+i\mu_1+j\mu_2)t}$$
$$\copyright\ P(k,l,t|i+1). \tag{3.3}$$

Case 4: $i > 0,\ j > 0,\ k = i,\ l = j$.

$$P[i,j,t|\ i,j] = e^{-(\lambda_1+\lambda_2+i\mu_1+j\mu_2)t}. \tag{3.4}$$

Case 5: $i > 0,\ j > 0,\ k = i,\ 0 \le l < j$.

$$P[i,l,t|\ i,j] = (\lambda_2 + j\mu_2)e^{-(\lambda_1+\lambda_2+i\mu_1+j\mu_2)t} \copyright\ P[i,l,t|i,j-1]. \tag{3.5}$$

Case 6: $i > 0,\ j = 0,\ k = i,\ l = 0$.

$$P[i,0,t|\ i,0] = e^{-(\lambda_1+i\mu_1)t}. \tag{3.6}$$

Case 7: $i = 0,\ j > 0,\ k = 0,\ l = j$.

$$P[0,j,t|\ 0,j] = e^{-(\lambda_1 p+\lambda_2+j\mu_2)t}. \tag{3.7}$$

Case 8: $i = 0,\ l \ge 0,\ k = 0,\ l < j$

$$P[0,l,t|\ 0,j] = (\lambda_1 p + \lambda_2 + j\mu_2)e^{-(\lambda_1 p+\lambda_2+j\mu_2)t} \copyright\ P[0,l,t|\ 0,j-1]. \tag{3.8}$$

Case 9: $i = j = k = l = 0$.

$$P[0,0,t|0,0] = 1. \tag{3.9}$$

3.2.2 FUNCTION $\phi_j(t)$

We define

$$\phi_j(t) = \lim_{\Delta \to 0} \frac{P[a_j - event\ in\ (t, t+\Delta),\ N(r,t) = 0 \mid r - event\ at\ t = 0]}{\Delta}.$$

The function $\phi_j(t)dt$ represents the probability that an a_j-event occurs in $(t, t+\Delta)$ and there is no replenishment in $(0,t]$, given that an r-event has occurred at $t = 0$. Hence, we have

$$\phi_j(t) = P[k+1, j, t \mid S, 0]\lambda_1 \overline{F}(t) + P[k, j+1, t \mid S, 0][\lambda_2 + (j+1)\mu_2 + \delta_{k0}\lambda_1 p]\overline{F}(t) \quad (3.10)$$

where $k + j = s, 0 \le k, j \le s$, and δ_{k0} is a Kronecker's delta function.

3.2.3 FUNCTION $W(i, j, t)$

We define

$$W(i, j, t) = P[Z(t) = (i, j), N(r, t) = 0 \mid Z(0) = (S, 0)].$$

Then the function $W(i, j, t)$ represents the probability that the inventory level is (i, j) at the time t, where t is the time elapsed since the last renewal. To obtain $W(i, j, t)$, we consider

Case 1: $0 \le i + j \le s$

In this case, exactly one re-order is made in $(0,t)$ and it does not materialize up to time t. Precisely, we have

(i) The system is in state $(S,0)$ at $t=0$.

(ii) The system enters the state (k,l) in $(u,u+du)$ where $k+l=s$ and $0<u<t$.

(iii) A re-order is placed in $(u,u+du)$.

(iv) The re-order does not materialize up to time t.

(v) The system enters the state (i,j) at time t.

Using probabilistic arguments,

$$W(i,j,t)=\sum_{l=0}^{s}\phi_l(t)\copyright\left\{\overline{F}(t)P(i,j,t|k,l)\right\}, \text{ where } 0\le k,l\le s \text{ and } k+l=s. \quad (3.11)$$

Case 2: $s+1\le i+j\le S$

In this case no re-order takes place in $(0,t)$. Hence,

$$W(i,j,t)=P[i,j,t|S,0] \quad (3.12)$$

The steady-state probabilities of the system are given by

$$W(i,j)=\lim_{t\to\infty}W(i,j,t) \quad (3.13)$$

$$=\lim_{\Delta\to 0}W^*(i,j,s)$$

Where $W^*(i,j,s)$ is the Laplace transform of $W(i,j,t)$ (See Girlich, 2003)

$$W^*(i,j,s)=\int_0^\infty e^{-st}W(i,j,t)\,dt$$

3.3 MEASURES OF SYSTEM PERFORMANCE

To obtain explicit expressions for various performance measures of the presented model, we proceed to define the first-order product density

$$h_\eta(t) = \lim_{\Delta \to 0} \frac{P[\eta - event\ in\ (t, t+\Delta) | Z(0) = (S,0)]}{\Delta}.$$

where $\eta = r, r_{ij}, a, a_j, d_1, d_2, l_1, l_2, g, k_1, k_2$.

3.3.1 MEAN NUMBER OF RE-ORDERS

Since a re-order is defined as an a_j-event, the expressions for $h_{ij}(t)$ are derived to obtain the mean number of re-orders. Note that a re-order takes place when the total inventory level enters s. Hence,

$$h_{a_j}(t) = \sum_{i+j=s}^{\infty} [W(i+1, j, t)\lambda_1 + W(i, j+1, t)\{\delta_{i0}\lambda_1 p + \lambda_2 + (j+1)\mu_2\}]. \tag{3.14}$$

The mean number of re-orders in $(0, t]$ is given by

$$E[N(a_j, t)] = \int_0^t h_{a_j}(u)du. \tag{3.15}$$

Consequently, the mean stationary rate of re-orders is given by

$$E[N(a_j, \infty)] = \lim_{t \to \infty} \frac{1}{t} E[N(a_j, t)] = \lim_{t \to \infty} h_{a_j}(t)$$

$$= \sum_{i+j=s}^{\infty}[W(i+1,j)\lambda_1 + W(i,j+1)\{\delta_{i0}\lambda_1 p + \lambda_2 + (j+1)\mu_2\}]. \quad (3.16)$$

3.3.2 MEAN NUMBER OF DEMANDS FOR A PARTICULAR PRODUCT WHICH IS SATISFIED BY THE SAME PRODUCT

A demand for Product 1 being satisfied by Product 1 is represented by the d_1-event. Hence an expression for $h_{d_1}(t)$ is derived. Observe that a d_1-event occurs whenever a demand for Product 1 occurs when the inventory levels is (i, j) where, $1 \le i \le S$, $0 \le j \le S$ and $0 < i + j \le S$. Hence,

$$h_{d_1}(t) = \sum\sum_{\substack{0 \le i+j \le S \\ i \ge 1,\, j \ge 0}} W(i,j)\lambda_1 \ , \quad (3.17)$$

so that

$$E[N(d_1,t)] = \int_0^t h_{d_1}(u)du \, .$$

Therefore,

$$E[N(d_1,\infty)] = \sum\sum_{\substack{0 \le i+j \le S \\ i \ge 1,\, j \ge 0}} W(i,j)\lambda_1 \, . \quad (3.18)$$

In the same way,

$$h_{d_2}(t) = \sum\sum_{\substack{0 \le i+j \le S \\ i \ge 1,\, j \ge 0}} W(i,j,t)\lambda_2 \, , \quad (3.19)$$

so that

$$E[N(d_2,t)] = \int_0^t h_{d_2}(u)du$$

$$E[N(d_2,\infty)] = \sum\sum_{\substack{0\le i+j\le S \\ i\ge 1,\, j\ge 0}} W(i,j)\lambda_2 \,. \tag{3.20}$$

3.3.3 MEAN NUMBER OF LOST DEMAND

A demand for Product 1 is lost when the total inventory level is zero or when the inventory level of Product 1 is zero and that of Product 2 is positive, but when the demand is not substituted with Product 2. Therefore,

$$\begin{aligned} h_{l_1}(t) &= W(0,0,t)\lambda_1 + \sum_{j=1}^{S} W(0,j,t)\lambda_1(1-p) \\ &= \sum_{j=0}^{S} W(0,j,t)\{1-p+p\delta_{j0}\}\lambda_1. \end{aligned} \tag{3.21}$$

The mean number of lost demands for Product 1 is given by

$$E[N(l_1,t)] = \int_0^t h_{l_1}(u)du \,,$$

so that the mean stationary rate of lost demand for Product 1 is given by

$$E[N(l_1,\infty)] = \sum_{j=0}^{S} W(0,j)[1-p+p\delta_{j0}]\lambda_1 \,. \tag{3.22}$$

In the same way, for the events l_2,

$$h_{l_2}(t) = \sum_{i=0}^{S} W(i,0,t)\lambda_2 \ , \qquad (3.23$$

$$E[N(l_2,t)] = \int_0^t h_{l_2}(u)du$$

and

$$E[N(l_2,\infty)] = \sum_{i=0}^{S} W(i,0)\lambda_2 \ . \qquad (3.24)$$

3.3.4 MEAN NUMBER OF DEMANDS OF PRODUCT 1 BEING SUBSTITUTED BY PRODUCT 2

A demand for Product 1 being substituted by Product 2 is denoted by the g-event. Note that a g-event occurs in $(t, t+\Delta)$ if the inventory level of the system at time t equals $(0, j), 1 \le j \le S$ and if a demand for Product 1 occurs in $(t, t+\Delta)$ being substituted by Product 2. Hence,

$$h_g(t) = \sum_{j=1}^{S} W(0, j, t)\lambda_1 p \qquad (3.25)$$

and

$$E[N(g,t)] = \int_0^t h_g(u)du \ .$$

Therefore,

$$E[N(g,\infty)] = \sum_{j=1}^{S} W(0, j)\lambda_1 p \ . \qquad (3.26)$$

3.3.5 MEAN NUMBER OF UNITS DETERIORATED FROM PRODUCT 1 AND TRANSITTED AS PRODUCT 2

Since a k_1-event pertains to the event of a unit of Product 1 deteriorates and transits as Product 2 and a k_1-event occurs in $(t, t+\Delta)$ if the system is in state (i, j) at time t, $1 \le i \le S$, $0 \le j \le S$ and $1 \le i + j \le S$ and a unit in Product 1 transits as Product 2 in $(t, t+\Delta)$, we have

$$h_{k_1}(t) = \sum_{\substack{0 \le i+j \le S \\ i \ge 1,\, j \ge 0}} \sum W(i, j, t) i \mu_1 \tag{3.27}$$

The mean number of units of Product 1 that have transitted as Product 2 in $(0, t]$ is given by

$$E[N(k_1, t)] = \int_0^t h_{k_1}(u) du$$

and the mean stationary rate of units of Product 1 transiting as Product 2 is given by

$$E[N(k_1, \infty)] = \sum_{\substack{0 \le i+j \le S \\ i \ge 1,\, j \ge 0}} \sum W(i, j) i \mu_1 . \tag{3.28}$$

3.3.6 MEAN NUMBER OF PRODUCT 2 PERISHED AND REMOVED FROM THE INVENTORY

The first order product density of k_2 is given by

$$h_{k_2}(t) = \sum_{\substack{0 \le i+j \le S \\ i \ge 1,\, j \ge 0}} \sum W(i, j, t) j \mu_2 \ . \tag{3.29}$$

Hence the mean number of units of Product 2 that have perished and removed from the inventory in $(0,t]$ is given by

$$E[N(k_2,t)] = \int_0^t h_{k_2}(u)du \, .$$

Consequently, the mean stationary rate of perishing of Product 2 is given by

$$E[N(k_2,\infty)] = \sum_{\substack{0\le i+j\le S \\ i\ge 1,\, j\ge 0}}\sum W(i,j)\, j\mu_2 \, . \qquad (3.30)$$

3.3.7 MEAN NUMBER OF REPLENISHMENTS

Consider the renewal process of r-events and derive its first-order product density $h_r(t)$. Firstly, an expression for the pdf $g(t)$ of the interval between two successive occurrences of the r-events is derived. By definition,

$$g(t) = \lim_{\Delta\to 0} \frac{P[r-event\ in\ (t,t+\Delta),\, N(r,t)=0 \,|\, Z(0)=(S,0)]}{\Delta} \, .$$

In order to derive $g(t)$, its survival function $\overline{G}(t)$ is determined. Since $\overline{G}(t)$ denotes the probability that a replenishment has not occurred up to t, we have two mutually exclusive cases for $\overline{G}(t)$:

(i) A re-order does not occur up to time t.

(ii) A re-order is placed in $(u,u+\Delta)$, $0<u<t$, but it has not been realized up to time t.

Hence,

$$\overline{G}(t) = \sum_{\substack{s+1\le k+l\le S \\ k\ge 0, l\ge 0}}\sum P(k,l,t|\ S,0) + \sum_{l=0}^{s} \phi_l(t) © \left\{ \overline{F}(t) + \sum_{k_1=0}^{s-l}\sum_{l_1=0}^{l} P(k_1,l_1,t|\ s-l,l \right\}. \quad (3.31)$$

However,

$$h_r(t) = \sum_{n=1}^{\infty} g^{(n)}(t),$$

and

$$E[N(r,t)] = \int_0^t h_r(u)du .$$

Hence, by renewal theory, the mean stationary rate of replenishment is given by

$$E[N(r,\infty)] = \lim_{t\to\infty}\frac{1}{t}\int_0^t h_r(u)du = \frac{1}{\int_0^\infty \overline{G}(t)dt} . \quad (3.32)$$

3.3.8 MEAN NUMBER OF REPLENISHMENTS

First, the product density is defined

$$h_{r_{ij}}(t) = \lim_{\Delta\to 0}\frac{P[r_{ij} - event\ in\ (t,t+\Delta)|\ Z(0) = (S,0)]}{\Delta}$$

Next a relation between $h_{r_{ij}}(t)$ and $h_r(t)$ is obtained.

Therefore, the following function is define

$$f_{ij}(t) = \lim_{\Delta \to 0} \frac{P[r_{ij} - event\ in\ (t, t+\Delta), N(r,t) = 0 |\ Z(0) = (S,0)]}{\Delta}$$

Observe that

$$f_{ij}(t) = \sum_{\substack{0 \le k+l \le S \\ k \ge 0, l \ge 0}} \sum [P(k+1, l, t |\ S, 0)\lambda_1 + P(k, l+1, t |\ S, 0)\{\lambda_2 + (l+1)\mu_2 + \delta_{k0}\lambda_1 p\}] ©$$

$$f(t) P(i, j, t |\ k, l). \tag{3.33}$$

Consequently,

$$h_{r_{ij}}(t) = f_{ij}(t) + h_r(t) © \ f_{ij}(t) \tag{3.34}$$

and

$$E[N(r_{ij}, t)] = \int_0^t h_{r_{ij}}(u) du .$$

Hence,

$$E[N(r_{ij}, \infty)] = \lim_{t \to \infty} \frac{1}{t} \int_0^t h_{r_{ij}}(u) du$$

$$= E[N(r, \infty)] \lim_{\theta \to 0} f_{ij} * (\theta) . \tag{3.35}$$

Since at the occurrence of each r_{ij}-event, $S - i$ units of Product 1 are added to the inventory, the mean number of Product 1 items added to the inventory per unit time is given by

$$\sum_{\substack{0 \le i+j \le S \\ i \ge 0,\, j \ge 0}} \sum E[N(r_{ij},\infty)](S-i) = E[N(r,\infty)] \sum_{\substack{0 \le i+j \le S \\ i \ge 0,\, j \ge 0}} \sum \lim_{\theta \to 0} f_{ij}{}^*(\theta). \tag{3.36}$$

3.3.9 MEAN NUMBER OF UNITS SCRAPPED FROM THE INVENTORY

Since, at the occurrence of an r_{ij}-event, j units of Product 2 are scrapped from the inventory per unit time, we have

$$\sum_{\substack{0 \le i+j \le S \\ i \ge 0,\, j \ge 0}} \sum E[N(r_{ij},\infty)]j = E[N(r,\infty)] \sum_{\substack{0 \le i+j \le S \\ i \ge 0,\, j \ge 0}} \sum \lim_{\theta \to 0} f_{ij}{}^*(\theta). \tag{3.37}$$

3.4 COST ANALYSIS

Since $E[N(l_1,\infty)]$ and $E[N(l_2,\infty)]$ are respectively the mean stationary rates of the two types of lost demands. The cost due to lost demand is given by

$$E[N(l_1,\infty)]CL_1 + E[N(l_2,\infty)]CL_2 \tag{3.38}$$

The cost corresponding to items of Product 2 perished and removed from the inventory is $E[N(k_2,\infty)]CP$. The number of items of Product 2 that are scrapped from the inventory per unit time is

$$\sum_{\substack{0\le i+j\le S\\ i\ge 0,\, j\ge 0}}\sum E[N(r_{ij},\infty)]\,j\,. \tag{3.39}$$

The cost due to this is

$$\sum_{\substack{0\le i+j\le S\\ i\ge 0,\, j\ge 0}}\sum E[N(r_{ij},\infty)]\,jCP\,. \tag{3.40}$$

Hence the total expected cost per unit time is:

$$C(S,s) = E[N(a,\infty)]CR + E[N(l_1,\infty)]CL_1 + E[N(l_2,\infty)]CL_2$$
$$+[E[N(k_2,\infty)] + \sum_{\substack{0\le i+j\le S\\ i\ge 0,\, j\ge 0}}\sum E[N(r_{ij},\infty)]\,j]CP + \sum_{\substack{0\le i+j\le S\\ i\ge 0,\, j\ge 0}}\sum E[N(r_{ij},\infty)](s-i)CB\,. \tag{3.41}$$

3.5 NUMERICAL EXAMPLE

For illustration purposes, consider the following numerical example. Let

$f(t) = \theta e^{-\theta t}, t > 0, \theta > 0$

$\lambda_1 = 4.0,$

$\lambda_2 = 6.0,$

$\mu_1 = 2.5,$

$\mu_2 = 2.5,$

$\theta = 2.0,$

CR = 10.0,

CL_1 = 6.0,

CL_2 = 5.0,

CP = 4.0, and

CB = 10.0

By varying the probability p from 0.1 to 0.9 and varying S from 2 to 4, with corresponding possible values for s, the values of the mean stationary rates of the following variables are obtained:

(i) Demand satisfied (ED_1, ED_2)
(ii) Demands substituted (EG)
(iii) Lost demands (EL_1, EL_2)
(iv) Items perished (EK_2)
(v) Re-orders (ES)
(vi) Replenishments (RRATE)
(vii) Units replenished (EUR)
(viii) Units scrapped or deleted (EUS)
(ix) Total expected cost (COST)

The numerical results of the relationship between p and the above variables are summarised in Table 3.1 below:

	S=2, s=1	S=3, s=1	S=3, s=2	S=4, s=1	S=4, s=2	S=4, s=3
ED1	increases	increases	increases	increases	increases	increases
ED2	decreases	decreases	decreases	decreases	decreases	decreases
EG	increases	increases	increases	increases	increases	increases
EL1	decreases	decreases	decreases	decreases	decreases	decreases
EL2	increases	increases	increases	increases	increases	increases
EK2	decreases	decreases	decreases	decreases	decreases	decreases
EA	increases	increases	increases	increases	increases	increases
RRATE	increases	increases	increases	increases	increases	increases
EUR	increases	increases	increases	increases	increases	increases
EUS	decreases	decreases	decreases	decreases	decreases	decreases
COST	decreases	decreases	decreases	decreases	decreases	decreases

Table 3.1: Relationship between p and selected variables for varying S and s

Per illustration, the relationships of Total Expected Cost (COST) and Lost Demand (EL2) versus increasing values of p are shown graphically in Figure 3.2.

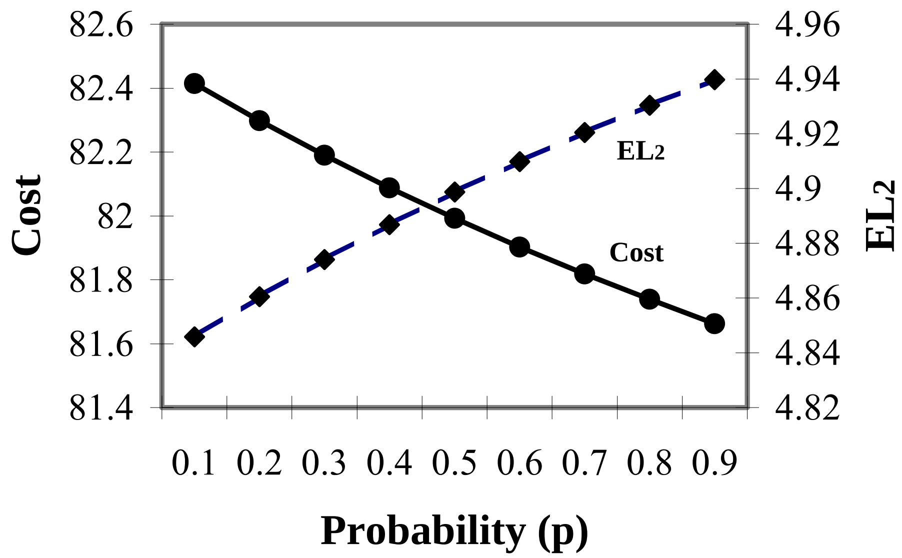

Figure 3.2: Relationship of COST and EL2 versus p for S =3, s =1

The detail results of the numerical example are given in Table 3.2 to Table 3.7 for varying values of S and s.

p	ED1	ED2	EG	EL1	EL2	EK2
0.1	1.447408	0.802104	0.030899	2.521693	5.197896	0.365150
0.2	1.448016	0.786138	0.059651	2.492333	5.213861	0.357462
0.3	1.448584	0.771246	0.086471	2.464945	5.228754	0.350288
0.4	1.449116	0.757323	0.111549	2.439334	5.242677	0.343578
0.5	1.449616	0.744276	0.135049	2.415335	5.255723	0.337289
0.6	1.450087	0.732027	0.157114	2.392799	5.267973	0.331382
0.7	1.450530	0.720503	0.177874	2.371595	5.279497	0.325824
0.8	1.450949	0.709642	0.197440	2.351611	5.290359	0.320585
0.9	1.451345	0.699389	0.215912	2.332742	5.300611	0.315637

p	EA	RRATE	EUR	EUS	COST
0.1	1.523762	1.523762	2.775306	0.129744	86.0899
0.2	1.524402	1.524402	2.776470	0.125240	85.9627
0.3	1.525000	1.525000	2.777560	0.120971	85.8441
0.4	1.525560	1.525560	2.778581	0.117015	85.7332
0.5	1.526087	1.526087	2.779540	0.113309	85.6293
0.6	1.526582	1.526582	2.780442	0.109832	85.5318
0.7	1.527049	1.527049	2.781292	0.106561	85.4400
0.8	1.527490	1.527490	2.782095	0.103480	85.3536
0.9	1.527907	1.527907	2.782854	0.100571	85.2720

Table 3.2: Numerical Results for S = 2, s =1

p	ED1	ED2	EG	EL1	EL2	EK2
0.1	2.015121	1.154118	0.029509	1.955370	4.845882	0.591228
0.2	2.016546	1.139571	0.057012	1.926442	4.860429	0.583235
0.3	2.017882	1.125985	0.082706	1.899412	4.874015	0.575761
0.4	2.019136	1.113267	0.106766	1.874098	4.886733	0.568758
0.5	2.020316	1.101338	0.129342	1.850342	4.898662	0.562181
0.6	2.021429	1.090125	0.150568	1.828003	4.909874	0.555993
0.7	2.022480	1.079566	0.170562	1.806958	4.920434	0.550161
0.8	2.023474	1.069605	0.189429	1.787098	4.930395	0.544654
0.9	2.024415	1.060193	0.207260	1.768325	4.939806	0.539447

p	EA	RRATE	EUR	EUS	COST
0.1	1.139148	1.139148	3.227835	0.104645	82.4150
0.2	1.139954	1.139954	3.230117	0.101013	82.2985
0.3	1.140709	1.140709	3.232256	0.097624	82.1897
0.4	1.141418	1.141418	3.234266	0.094455	82.0879
0.5	1.142085	1.142085	3.236156	0.091486	81.9924
0.6	1.142714	1.142714	3.237938	0.088698	81.9027
0.7	1.143308	1.143308	3.239621	0.086075	81.8182
0.8	1.143870	1.143870	3.241213	0.083603	81.7384
0.9	1.144402	1.144402	3.242721	0.081269	81.6631

Table 3.3: Numerical Results for S = 3, s = 1

p	ED1	ED2	EG	EL1	EL2	EK2
0.1	2.100194	1.209167	0.028260	1.871547	4.790833	0.619952
0.2	2.100425	1.195623	0.054702	1.844873	4.804378	0.612563
0.3	2.100644	1.182931	0.079498	1.819857	4.817068	0.605627
0.4	2.100852	1.171014	0.102799	1.796349	4.828986	0.599103
0.5	2.101049	1.159802	0.124737	1.774214	4.840198	0.592954
0.6	2.101236	1.149234	0.145429	1753335	4.850765	0.587149
0.7	2.101414	1.139256	0.164979	1.733607	4.860744	0.581660
0.8	2.101583	1.129818	0.183481	1.714936	4.870183	0.576461
0.9	2.101745	1.120878	0.201016	1.697240	4.879122	0.571530

p	EA	RRATE	EUR	EUS	COST
0.1	2.058707	1.436794	3.566840	0.231180	94.8435
0.2	2.058934	1.436952	3.567234	0.225902	94.7067
0.3	2.059149	1.437102	3.567606	0.220951	94.5783
0.4	2.059352	1.437244	3.567958	0.216298	94.4577
0.5	2.059545	1.437379	3.568292	0.211917	94.3441
0.6	2.059728	1.437507	3.568610	0.207783	94.2370
0.7	2.059903	1.437628	3.568912	0.203878	94.1357
0.8	2.060069	1.437744	3.569200	0.200181	94.0398
0.9	2.060227	1.437855	3.569474	0.196677	93.9489

Table 3.4: Numerical Results for S = 3, s = 2

p	ED1	ED2	EG	EL1	EL2	EK2
0.1	2.461397	1.550978	0.022888	1.515716	4.449023	0.824795
0.2	2.462760	1.540001	0.044223	1.493017	4.460000	0.818603
0.3	2.464039	1.529750	0.064159	1.471803	4.470250	0.812810
0.4	2.465240	1.520156	0.082829	1.451930	4.479844	0.807379
0.5	2.466372	1.511157	0.100352	1.433276	4.488843	0.802276
0.6	2.467440	1.502699	0.116830	1.415730	4.497301	0.797472
0.7	2.468449	1.494735	0.132354	1.399197	4.505266	0.792943
0.8	2.469404	1.487222	0.147005	1.383592	4.512778	0.788664
0.9	2.470309	1.480124	0.160855	1.368836	4.519876	0.784615

p	EA	RRATE	EUR	EUS	COST
0.1	0.884151	0.884151	3.387431	0.080117	77.6749
0.2	0.884640	0.884640	3.389307	0.077324	77.5813
0.3	0.885100	0.885100	3.391067	0.074719	77.4939
0.4	0.885531	0.885531	3.392720	0.072284	77.4120
0.5	0.885938	0.885938	3.394278	0.070030	77.3352
0.6	0.886321	0.886321	3.395747	0.067862	77.2629
0.7	0.886684	0.886684	3.397136	0.065848	77.1949
0.8	0.887027	0.887027	3.398450	0.063950	77.1037
0.9	0.887352	0.887352	3.399695	0.062158	77.0700

Table 3.5: Numerical Results for S = 4, s = 1

p	ED1	ED2	EG	EL1	EL2	EK2
0.1	2.571096	1.635669	0.021343	1.407561	4.364332	0.872010
0.2	2.571408	1.625571	0.041319	1.387274	4.374429	0.866307
0.3	2.571703	1.616109	0.060056	1.368241	4.383891	0.860949
0.4	2.571984	1.607225	0.077667	1.350350	4.392776	0.855907
0.5	2.572251	1.598865	0.094251	1.333499	4.401135	0.851153
0.6	2.572504	1.590985	0.109898	1.317598	4.409014	0.846663
0.7	2.572746	1.583545	0.124684	1.302570	4.416455	0.842414
0.8	2.572976	1.576508	0.138680	1.288344	4.423493	0.838389
0.9	2.573195	1.569841	0.151948	1.274857	4.430159	0.834568

p	EA	RRATE	EUR	EUS	COST
0.1	1.557317	1.082949	3.764555	0.170350	87.6552
0.2	1.557506	1.083081	3.765011	0.166502	87.5522
0.3	1.557685	1.083205	3.765444	0.162895	87.4556
0.4	1.557855	1.083323	3.765855	0.159505	87.3647
0.5	1.558017	1.083436	3.766245	0.156314	87.2792
0.6	1.558170	1.083542	3.766616	0.155504	87.1984
0.7	1.558316	1.083644	3.766970	0.150461	87.1221
0.8	1.558456	1.083741	3.767308	0.147770	87.0498
0.9	1.558589	1.083834	3.767629	0.145219	86.9813

Table 3.6: Numerical Results for S = 4, s = 2

p	ED1	ED2	EG	EL1	EL2	EK2
0.1	2.639271	1.675820	0.020484	1.340245	4.324180	0.902930
0.2	2.639348	1.666208	0.039680	1.320971	4.333792	0.897530
0.3	2.639421	1.657192	0.057710	1.302868	4.342807	0.892449
0.4	2.639492	1.648719	0.074677	1.285831	4.351282	0.887662
0.5	2.639560	1.640739	0.090675	1.269766	4.359261	0.883141
0.6	2.639623	1.633211	0.105786	1.254591	4.366789	0.878865
0.7	2.639685	1.626096	0.120081	1.240234	4.373904	0.874815
0.8	2.639744	1.619362	0.133628	1.226628	4.380638	0.870973
0.9	2.639800	1.612977	0.146483	1.213717	4.387023	0.867323

p	EA	RRATE	EUR	EUS	COST
0.1	2.405319	1.401584	4.151391	0.363537	100.2953
0.2	2.405389	1.401625	4.151512	0.359440	100.1917
0.3	2.405456	1.401664	4.151628	0.355589	100.0942
0.4	2.405520	1.401702	4.151738	0.351962	100.0025
0.5	2.405581	1.401737	4.151844	0.348539	99.9159
0.6	2.405640	1.401771	4.151945	0.345305	99.8340
0.7	2.405696	1.401804	4.152041	0.342243	99.7565
0.8	2.405749	1.401835	4.152133	0.339339	99.6830
0.9	2.405801	1.401865	4.152223	0.336583	99.6133

Table 3.7: Numerical Results for S = 4, s = 3

3.6 CONCLUSION

This chapter described a single perishing product inventory model where items deteriorate in two phases and then perish. Independent demand takes place at constant rates for items in both phases. Demand for an item in Phase I not satisfied may be satisfied by an item in Phase II based on a probability measure. Demand for items in Phase II during stock-out is lost. The re-ordering policy is an adjustable (S, s) policy with the lead-time following an arbitrary distribution. Identifying the underlying stochastic process as a renewal process, the probability distribution of the inventory level at any arbitrary point in time is obtained. The expressions for the mean stationary rates of lost demand, substituted demand, perished units and scrapped units are also derived. A numerical example is considered to highlight the obtained results.

CHAPTER 4

TWO-COMMODITY CONTINUOUS REVIEW INVENTORY SYSTEM WITH BULK DEMAND FOR ONE COMMODITY

A modified version of this chapter has been accepted in Asia-Pacific Journal of Operational Research

4.1 INTRODUCTION

With the advent of advanced computing systems, many industries and firms deal with multi-commodity systems. In dealing with such systems, models were initially proposed with independently established reorder points. In situations where several products compete for limited storage space, or share the same transport facility, or items are produced on (procured from) the same equipment (supplier), the above strategy overlooks the potential saving associated with joint replenishment, reduction in ordering and setup costs and allowing the user to take advantage of quantity discounts.

In continuous review inventory systems, Ballintify (1964) and Silver (1974) have considered a coordinated reordering policy, which is represented by the triplet (**S, c, s**), where the three parameters S_i, c_i and s_i are specified for each item i with $s_i \leq c_i \leq S_i$. In this policy, if the level of i-th commodity at any time is below s_i, an order is placed for $S_i - s_i$ items and at the same time, for any other item $j(\neq i)$ with available inventory at or below its can-order level c_j , an order is placed so as to bring its level back to its maximum capacity S_j . Subsequently many articles have appeared with models involving the above policy. Another article of interest is due to Federgruen, Groenevelt and Tijms (1984), which deals with the general case of compound Poisson demands and non-zero lead times. A review of inventory models under joint replenishment is provided by Goyal and Satir (1989).

Kalpakam and Arivarignan (1993) have introduced (s,S) policy with a single reorder level s defined in terms of the total number of items in the stock. This policy avoids separate ordering for each commodity and hence a single processing of orders for both commodities has some advantages in situation where in procurement is made from the same supplies, items are produced on the same machine, or items have to be supplied by the same transport facility.

A natural extension of (s,S) policy to two-commodity inventory system is to have two reorder levels and to place order for each commodity independent of other. But this policy will increase the total cost, as separate processing of two orders is required. Anbazhagan and Arivarignan (2000) have considered a two commodity inventory system with independent reorder levels where a joint order for both the commodities is placed only when the levels of both commodities are less than or equal to their respective reorder levels. The demand points form an independent Poisson process and the lead-time is distributed as negative exponential. They also assumed unit demands for both commodities.

In this chapter, the above work is extended by assuming unit demand for one commodity and bulk demand for the other commodity. The number of items demanded for the latter commodity is assumed to be a random variable Y with probability function $p_k = Pr\{Y = k\}$, $k = 1, 2, 3, \ldots$. A reorder is made for both commodities when the inventory levels of these commodities are at or below the respective inventory levels.

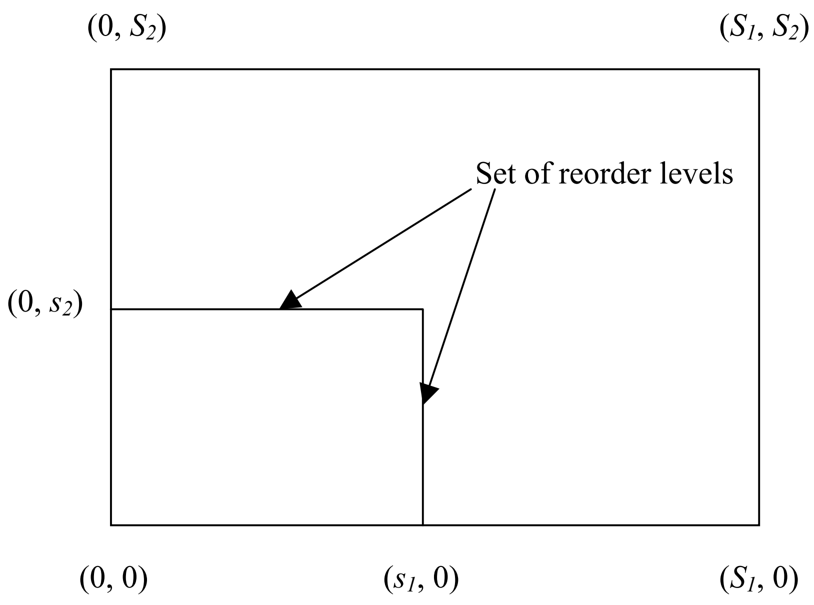

Figure 4.1: Space of Inventory Levels (s, S)

The joint probability distribution of the two inventory levels is obtained in both transient and steady state cases. Various measures of systems performance and the total expected cost rate in the steady state are also derived.

4.2 MODEL DESCRIPTION

Consider a two commodity inventory system with the maximum capacity S_i units for i-th commodity (i = 1, 2). We assume that demand for first commodity is for single item and demand for second commodity is for bulk items. The sequences of respective demand points for commodities 1, 2 and for both commodities are assumed to form independent Poisson processes with parameters λ_1, λ_2 and λ_{12} respectively. The number of items demanded for the second commodity at any demand point is a random variable Y with probability function $p_k = Pr\{Y = k\}$, $k = 1, 2, 3, \ldots$. The reorder level for the i-th commodity is fixed at $s_i(1 \leq s_i \leq S_i)$ and ordering quantity for i-th commodity is $Q_i(= S_i - s_i > s_i + 1)$ items when both inventory levels are less than or equal to their respective reorder levels. The requirement $S_i - s_i > s_i + 1$, ensure that after a replenishment the inventory level will be always above the respective reorder levels. Otherwise it may not be possible to place reorder which leads to perpetual shortage. That is if $L_i(t)$ represents inventory level of i-th commodity at time t, then a reorder is made when $L_1(t) \leq s_1$ and $L_2(t) \leq s_2$.

The lead-time is assumed to be distributed as negative exponential with parameter μ (> 0). The demands that occur during stock out periods are lost. The stochastic process $\{(L_1(t), L_2(t)), t \geq 0\}$ has the state space,

$$E = E_1 \times E_2$$

where $E_1 = \{0, 1, 2, \ldots, S_1\}$ and $E_2 = \{0, 1, \ldots, S_2\}$.

Notation:

0: zero matrix

I'_N : $(1, 1, \ldots, 1)_{1\times N}$

I : an identity matrix

$$<\mathrm{x}> = \begin{cases} x \text{ if } x > 0 \\ 0 \text{ if } x \le 0 \end{cases}$$

From the assumptions made on demand and on replenishment processes, it follows that $\{(L_1(t), L_2(t)), t \ge 0\}$ is a Markov process. To determine the infinitesimal generator $A = ((a(i, q; j, r))), (i, q), (j, r) \in E$, we use the following arguments:

The demand for the first commodity takes the state of the process from (i, q) to $(i-1, q)$ and the intensity of transition $a(i, q; i-1, q)$ is given by λ_1, $i = 1, 2, \ldots, S_1$. A bulk demand of k items for second commodity takes the state from (i, q) to $(i, <q-k>)$, $i = 0, 1, \ldots, S_1$, $q = 0, 1, \ldots, S_2$, $k = 1, 2, \ldots,$ and the respective intensity of transitions are given by $\lambda_2 p_k$ and $\lambda_2 \sum_{u=k}^{\infty} p_u$. A joint demand for single item of first commodity and for k items of second commodity takes the system from the state (i, q) to $(i-1, <q-k>)$, $i = 0, 1, 2, \ldots, S_1$, $q = 0, 1, 2, \ldots, S_2$, $k = 1, 2, \ldots,$ and the respective intensity of transition are given by $\lambda_{12}\, p_k$ and $\lambda_{12} \sum_{u=k}^{\infty} p_k$. From the state (i, q) $(\le(s_1, s_2))$ a replenishment takes the joint inventory level to $(i + Q_1, q + Q_2)$ and the intensity of transition for this is given by μ. For other transition from (i, q) to (j, r), when $(i, q) \ne (j, r)$, is zero. To obtain the intensity of passage, $a(i, q; i, q)$ of state (i, q), we note that the entries in any row of this matrix add to zero. Hence the diagonal entry is equal to the negative of the sum of the other entries in that row. More explicitly

$$a(i,q;i,q) = -\sum_{j}\sum_{r} a(i,j;j,r)$$

$$(j,r) \ne (i,q)$$

Hence we have,

$$a(i,q;j,r)=\begin{cases}\lambda_1 & r=q; & j=i-1\\ & q=0,1,2,\ldots,S_2; & i=1,2,3,\ldots,S_1,\\ \lambda_2 p_k & r=q-k; & j=i\\ & k=1,2,\ldots,q-1; & i=0,1,2,\ldots,S_1,\\ & q=2,3,\ldots,S_2, & \\ \lambda_2 p'_q & r=0; & j=i,\\ & q=1,2,3,\ldots,S_2; & i=0,1,2,\ldots,S_1,\\ \lambda_{12} p_k & r=q-k; & j=i-1\\ & k=1,2,\ldots,q-1; & i=1,2,\ldots,S_1,\\ & q=2,3,\ldots,S_2, & \\ \lambda_{12} p'_q & r=0; & j=i-1,\\ & q=1,2,3,\ldots,S_2; & i=0,1,2,\ldots,S_1,\\ \mu & r=q+Q_2; & j=i+Q_1,\\ & q=0,1,2,\ldots,s_2; & i=0,1,2,\ldots,s_1,\\ -(\lambda_1+\lambda_2+\lambda_{12}) & r=q; & j=i,\\ & q=1,2,3,\ldots,S_2; & i=s_1+1,s_1+2,\ldots,S_1,\\ -\lambda_1 & r=q; & j=i,\\ & q=0; & i=s_1+1,s_1+2,\ldots,S_1,\\ -(\lambda_1+\lambda_2+\lambda_{12}) & r=q; & j=i,\\ & q=s_2+1,s_2+2,\ldots,S_2, & i=0,1,2,\ldots,s_1,\\ -(\lambda_1+\lambda_2+\lambda_{12}+\mu) & r=q; & j=i,\\ & q=1,2,3,\ldots,s_2; & i=0,1,2,\ldots,s_1,\\ -(\lambda_1+\mu) & r=q; & j=i,\\ & q=0; & i=1,2,\ldots,s_1,\\ -\lambda_2 & r=q; & j=i,\\ & q=s_2+1,s_2+2,\ldots,S_2, & i=0\\ -(\lambda_2+\mu) & r=q; & j=i,\\ & q=1,2,3,\ldots,s_2; & i=0,\\ -\mu & r=q; & j=i,\\ & q=0; & i=0,\\ 0 & Otherwise, & \end{cases}$$

where $p'_q = \sum_{k=q}^{\infty} p_k$.

Denoting $\boldsymbol{m}$ = ((m, S_2), (m, S_2−1), . . . , (m, 1), (m, 0)) for m = 0, 1, 2, . . . , S_1, the infinitesimal generator A can be conveniently expressed as a block partitioned matrix:

$$A = \begin{array}{c} S_1 \\ S_1-1 \\ \vdots \\ s_1+1 \\ s_1 \\ s_1-1 \\ \vdots \\ 1 \\ 0 \end{array} \left(\begin{array}{cccccccccc} A_1 & B & & & & & & & & \\ & A_1 & B & & & & & & & \\ & & \cdots & & & & & & & \\ & & & \cdots & A_1 & B & & & & \\ C & & & & & A_2 & B & & & \\ & C & & & & & A_2 & & & \\ & & \cdots & & & & \cdots & & & \\ & & & & & & & \cdots & A_2 & B \\ & & & C & & & & & & D \end{array} \right)$$

where,

$$B = \begin{array}{c} \\ S_2 \\ S_2-1 \\ \vdots \\ \vdots \\ 1 \\ 0 \end{array} \begin{array}{c} \begin{array}{cccccc} S_2 & S_2-1 & S_2-2 & \cdots & 1 & 0 \end{array} \\ \left(\begin{array}{cccccc} \lambda_1 & \lambda_{12}p_1 & \lambda_{12}p_2 & \cdots & \lambda_{12}p_{S_2-1} & \lambda_{12}p'_{S_2} \\ 0 & \lambda_1 & \lambda_{12}p_1 & \cdots & \lambda_{12}p_{S_2-2} & \lambda_{12}p'_{S_2-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \lambda_1 & \lambda_{12}p'_1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_1 \end{array} \right) \end{array}$$

$$C = \begin{array}{c} S_2 \\ S_2-1 \\ \vdots \\ s_2 \\ s_2-1 \\ \vdots \\ 0 \end{array} \left(\begin{array}{cccccc} 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mu & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \mu & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \mu & \cdots & 0 \end{array} \right)$$

$$A_1 = \begin{pmatrix} -(\lambda_1+\lambda_2+\lambda_{12}) & \lambda_2 p_1 & \lambda_2 p_2 & \cdots & \lambda_2 p_{S_2-1} & \lambda_2 p'_{S_2} \\ 0 & -(\lambda_1+\lambda_2+\lambda_{12}) & \lambda_2 p_1 & \cdots & \lambda_2 p_{S_2-2} & \lambda_2 p'_{S_2-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -(\lambda_1+\lambda_2+\lambda_{12}) & \lambda_2 p'_1 \\ 0 & 0 & 0 & \cdots & 0 & -\lambda_1 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} -(\lambda_1+\lambda_2+\lambda_{12}) & \cdots & \lambda_2 p_{Q_2-1} & \lambda_2 p_{Q_2} & \cdots & \cdots & \lambda_2 p'_{S_2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & -(\lambda_1+\lambda_2+\lambda_{12}) & \lambda_2 p_1 & \cdots & \cdots & \lambda_2 p'_{s_2+1} \\ 0 & \cdots & 0 & d & \lambda_2 & \cdots & \lambda_2 p'_{s_2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & \cdots & d & \lambda_2 p'_1 \\ 0 & \cdots & 0 & 0 & \cdots & \cdots & -(\lambda_1+\mu) \end{pmatrix}$$

with $d = -(\lambda_1 + \lambda_2 + \lambda_{12} + \mu)$,

and

$$D = \begin{pmatrix} -\lambda_2 & \cdots & \lambda_2 p_{Q_2-1} & \lambda_2 p_{Q_2} & \cdots & \cdots & \lambda_2 p'_{S_2} \\ 0 & \cdots & \lambda_2 p_{Q_2-2} & \lambda_2 p_{Q_2-1} & \cdots & \cdots & \lambda_2 p'_{S_2-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & -\lambda_2 & \lambda_2 p_1 & \cdots & \cdots & \lambda_2 p'_{s_2+1} \\ 0 & \cdots & 0 & d_1 & \cdots & \cdots & \lambda_2 p'_{s_2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & \cdots & d_1 & \lambda_2 p'_1 \\ 0 & \cdots & 0 & 0 & \cdots & \cdots & -\mu \end{pmatrix}$$

with $d_1 = -(\lambda_2 + \mu)$.

4.3 TRANSIENT ANALYSIS

Define

$$\phi(i, q; j, r, t) = Pr\,[L(t) = j, X(t) = r \mid L(0) = i, X(0) = q],\ (i, q), (j, r) \in E.$$

Let $\phi_{ij}(t)$ denote a matrix whose $(q, r)^{th}$ element is $\phi(i, q; j, r, t)$ and $\Phi(t)$ denote a block partitioned matrix with the sub matrix $\phi_{ij}(t)$ at (i, j)th position. The Kolmogorov's differential equation can be written as

$$\Phi'(t) = \Phi(t)A,$$

the solution of which is given by

$$\Phi(t) = e^{At}$$

where e^{At} represents

$$I + \frac{At}{1!} + \frac{A^2t^2}{2!} + \ldots$$

Alternatively, if we use the notation $A^*(\alpha)$ to denote the Laplace transform of the function (or matrix) A (t) then we have

$$\Phi^*_\alpha = (\alpha I - A)^{-1}$$

The matrix $(\alpha I - A)$ has the following block partitioned form

$$(\alpha I - A) = \begin{array}{c} S_1 \\ S_1 - 1 \\ \cdots \\ s_1 \\ s_1 - 1 \\ \cdots \\ 1 \\ 0 \end{array} \begin{pmatrix} D_{S_1} & -B & 0 & \cdots & 0 & 0 \\ 0 & D_{S_1-1} & -B & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -C & 0 & 0 & \cdots & 0 & 0 \\ 0 & -C & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & D_1 & -B \\ 0 & 0 & 0 & \cdots & 0 & D_0 \end{pmatrix}$$

Where

$$D_i = \begin{cases} \alpha\, I - D, & i = 0, \\ \alpha\, I - A_2, & i = 1,2,\ldots,s_1, \\ \alpha\, I - A_1, & i = s_1 + 1,\ldots,S_1 \end{cases}$$

(Note that the rows and columns have been numbered in decreasing order if magnitude.)

It may be observed that $(\alpha I - A)$ is an almost lower triangular matrix in block partitioned form. That is, if we denote the $(i, j)^{\text{th}}$ sub matrix of $P(=\alpha I - A)$ by P_{ij}, then we have

$$P_{ij} = 0 \qquad i = 1, 2, \ldots, S_1;\ j > i - 1.$$

To compute $P^{-1} = (\alpha I - A)^{-1}$ we proceed as described below:

Consider a lower triangular matrix

$$U = \begin{array}{c} S_1 \\ S_1 - 1 \\ \cdots \\ 1 \\ 0 \end{array} \begin{pmatrix} U_{S_1 S_1} & 0 & \cdots & 0 & 0 \\ U_{(S_1-1)S_1} & U_{(S_1-1)(S_1-1)} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \\ U_{1S_1} & U_{1(S_1-1)} & \cdots & U_{11} & 0 \\ U_{0S_1} & U_{0(S_1-1)} & \cdots & U_{01} & U_{00} \end{pmatrix}$$

with $U_{ii} = 1$, $i = 0, 1, 2, \ldots, S_1$ and an almost lower triangular matrix

$$R = \begin{array}{c} S_1 \\ S_1 - 1 \\ \cdots \\ 1 \\ 0 \end{array} \begin{pmatrix} \mathbf{0} & -B & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -B & \cdots & \mathbf{0} & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & -B \\ R_{0S_1} & R_{0(S_1-1)} & R_{0(S_1-2)} & \cdots & R_{01} & R_{00} \end{pmatrix}$$

such that $PU = R$. We find the sub matrices U_{ij} and R_{0j} by computing the product PU and equating it to R. The $(i, j)^{\text{th}}$ sub matrix of PU, denoted by $[PU]_{ij}$ is given by

By equating the sub matrices of PU to the corresponding elements of R, we get

$$U_{ij} = \begin{cases} (B^{-1}D_{i+1})(B^{-1}D_{i+2})\ldots(B^{-1}D_j), & i = 0, 1, 2, \ldots, j+1 \\ & j = 1, 2, \ldots, Q_1 \\ & \text{or} \\ & i = j - Q_1, j - Q_1 + 1, \ldots, S_1 \\ & j + Q_1 + 1, Q_1 + 2, \ldots, S_1 \\ B^{-1}CU_{(i+Q_1+1)j} + B^{-1}D_{i+1}U_{(i+1)j}, & i = 0, 1, \ldots, j - Q_1 - 1 \\ & j = Q_1 + 1, Q_1 + 2, \ldots, S_1 \end{cases}$$

and

$$R_{0j} = \begin{cases} D_0, & j = 0 \\ D_0 U_{0,j}, & j = 1, 2, \ldots, Q_1 - 1 \\ CU_{Q_1 j} + D_0 U_{0,j}, & j = Q_1, Q_1 + 1, \ldots, S_1. \end{cases}$$

The equation PU = R implies

$$(PU)^{-1} = R^{-1}$$
$$U^{-1}P^{-1} = R^{-1}$$
$$P^{-1} = UR^{-1}.$$

It can be verified that the inverse of R is given by,

$$R^{-1} = \begin{pmatrix} R_{0S_1}^{-1}R_{0(S_1-1)}B^{-1} & R_{0S_1}^{-1}R_{0(S_1-2)}B^{-1} & \cdots & R_{0S_1}^{-1}R_{00}B^{-1} & R_{0S_1}^{-1} \\ -B^{-1} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -B^{-1} & \cdots & \mathbf{0} & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \cdots & -B^{-1} & \mathbf{0} \end{pmatrix}$$

Since the expression for R^{-1} involves the term $R_{0S_1}^{-1}$, it is demonstrated that the latter exists.

From $PU = R$, we get

$$\det(PU) = \det(R)$$
$$\det(P)\det(U) = \det(R_{0S_1})\det(-B)\det(-B)\cdots\det(-B).$$

Since U is a lower triangular matrix and B is a upper triangular matrix, their determinant values are not equal to zero. Hence $\det(R_{0S_1})$ is not equal to zero. This proves the existence of the inverse of $R_{0S_1}^{-1}$. From $P^{-1} = UR^{-1}$, we can compute the (i, j)th sub matrix (denoted by Pij) of $P^{-1} = (\alpha I - A)^{-1}$ and it is given by,

$$P^{ij} = \begin{cases} U_{iS_1}R_{0S_1}^{-1}R_{0(j-1)}B^{-1} & i = j, j+1, \ldots, S_1; \quad j = 1, 2, \ldots, S_1 \\ U_{iS_1}R_{0S_1}^{-1} & i = 0, 1, 2, \ldots, S_1; \quad j = 0 \\ U_{iS_1}R_{0S_1}^{-1}R_{0(j-1)} - U_{i(j-1)}B^{-1} & i = 0, 1, \ldots, S_1 - 1; \quad j = i+1, \ldots, S_1. \end{cases}$$

4.4 Steady State Analysis

It can be seen from the structure of A that the homogeneous Markov process $\{(L_1(t), L_2(t)), t \geq 0\}$ on the state space E is irreducible. Hence the limiting distribution

$$\mathbf{\Phi} = (\phi^{S_1}, \phi^{S_1 - 1}, \ldots, \phi^1, \phi^0) \tag{4.1}$$

with $\phi^m = (\phi^{(m,S_2)}, \phi^{(m,S_2-1)}, \ldots, \phi^{(m,0)})$, where $\phi^{(i,j)}$ denotes the steady state probability for the state (i, j) of the inventory level process, exists and is given by

$$\Phi A = 0 \quad and \quad \sum_{(i,j)\in E} \sum \phi^{(i,j)} = 1\,. \tag{4.2}$$

The first equation of the above yields the following set of equations:

$$\begin{aligned}
\phi^1 B + \phi^0 D &= 0 & & \\
\phi^i B + \phi^{i-1} A_2 &= 0, & & i = 2, \ldots, s_1 + 1 \\
\phi^i B + \phi^{i-1} A_1 &= 0, & & i = s_1 + 2, \ldots, Q_1 \\
\phi^i B + \phi^{i-1} A_1 + \phi^{i-1-Q_1} C &= 0, & & i = Q_1 + 1, \ldots, S_1 \\
\phi^{S1} A_1 + \phi^{s_1} C &= 0. & &
\end{aligned}$$

Simplification yields the following:

$$\begin{aligned}
\phi^i &= (-1)^i \phi^0 DB^{-1}(A_2B^{-1})^{i-1} & & i = 1, 2, \ldots, s_1 + 1 \\
&= (-1)^i \phi^0 DB^{-1}(A_2B^{-1})^{s_1}(A_1B^{-1})^{i-s_1-1} & & i = s_1 + 2, \ldots, Q_1 \\
&= (-1)^i \phi^0 DB^{-1}(A_2B^{-1})^{s_1}(A_1B^{-1})^{i-s_1-1} - \phi_0 CB^{-1} & & i = Q_1 + 1 \\
&= (-1)^i \phi^0 DB^{-1}(A_2B^{-1})^{s_1}(A_1B^{-1})^{i-s_1-1} & & \\
&\quad +(-1)^{i-Q_1} \phi_0 CB^{-1}(A_1B^{-1})^{i-Q_1-1} + & & \\
&\quad (-1)^{i-Q_1} \phi^0 DB^{-1} \sum_{k=0}^{i-Q_1-2} (A_2B^{-1})^k CB^{-1}(A_1B^{-1})^{i-Q_1-2-k} & & i = Q_1 + 2, Q_1 + 2, \ldots, S_1.
\end{aligned}$$

where ϕ^0 can be obtained by solving,

$$\phi^{S1}A_1 + \phi^{s_1}C = 0 \quad \textit{and} \quad \sum_{i=0}^{S_1}\phi^i 1_{(S_2+1)\times 1} = 1,$$

that is

$$\phi^{\mathbf{0}}\left[(-1)^{S_1}DB^{-1}(A_2B^{-1})^{s_1}(A_1B^{-1})^{Q_1-1} + (-1)^{s_1}CB^{-1}(A_1B^{-1})^{s_1-1} + \right.$$

and

$$(-1)^{s_1}DB^{-1}\sum_{k=0}^{s_1-2}(A_2B^{-1})^k CB^{-1}(A_1B^{-1})^{i-Q_1-2-k} +$$

$$\left.(-1)^{s_1}DB^{-1}(A_2B^{-1})^{s_1-1}\right] = \mathbf{0}.$$

$$\phi^0\left[\sum_{i=1}^{s_1+1}(-1)^i DB^{-1}\left(A_2B^{-1}\right)^{i-1} + \sum_{i=s_1+2}^{Q_1}(-1)^i DB^{-1}\left(A_2B^{-1}\right)^{s_1}\left(A_1B^{-1}\right)^{i-s_1-1} + (-1)^{Q_1+1}DB^{-1}\left(A_2B^{-1}\right)^{s_1}\left(A_1B^{-1}\right)^{Q_1-s_1}\right.$$

$$\left. -CB^{-1} + \sum_{i=Q_1+2}^{S_1}\left\{ (-1)^i DB^{-1}\left(A_2B^{-1}\right)^{s_1}\left(A_1B^{-1}\right)^{i-s_1-1} + (-1)^{i-Q_1}DB^{-1}\sum_{k=0}^{i-Q_1-2}\left(A_2B^{-1}\right)^k CB^{-1}\left(A_1B^{-1}\right)^{i-Q_1-2-k} \right\}\right]1_{(S_2+1)\times 1} = 1$$

The marginal probability distribution $\{\phi_{i1}, i = 0, 1, 2, \ldots, S_1\}$ of the first commodity is given by

$$\phi_{i1} = \sum_{q=0}^{S_2}\phi^{(i,q)}, \qquad i = 0,1,2,\ldots,S_1,$$

and the marginal probability distribution $\{\phi_{2q}, q = 0, 1, 2, \ldots, S_2\}$ of the second commodity is given by

$$\phi_{2j} = \sum_{i=0}^{S_1}\phi^{(i,q)}, \qquad i = 0,1,2,\ldots,S_2,$$

The expected inventory level in the steady state, for the i-th commodity is given by

$$E[L_i] = \sum_{k=0}^{S_i}k\phi_{ik}, \qquad i = 1,2. \tag{4.3}$$

4.5 REORDERS AND SHORTAGES

In this section the reorder and shortages are studied. This requires the study of time points at which a transition occurs in the inventory level process.

Let $0 = T_0 < T_1 < T_2 < \cdots$ be the instances of transitions of the process. Let $(L_n^{(1)}, L_n^{(2)}) = (L_1(T_n+), L_2(T_n+))$, $n = 0, 1, 2, \ldots$. From the well known theory of Markov processes, $\{(L_n^{(1)}, L_n^{(2)}), n = 0, 1, 2, \ldots\}$ is a Markov chain and with the transition probability matrix (*tpm*)

$$P = ((\, p(i, j;\, k, l)\,))_{(i,j) \in E, (k,l) \in E},$$

where,

$$p(i, j;\, k, l) = \begin{cases} 0 & (i, j) = (k, l) \\ -a(i, j; k, l)/\theta_{ij}, & (i, j) \neq (k, l) \end{cases}$$

Here $\theta_{ij} = a(i, j;\, i, j)$ which is a negative value. Moreover for all n, we also have,

$$Pr\,[(L_1(T_{n+1}+), L_2(T_{n+1}+)) = (k, l), T_{n+1} - T_n > t \mid (L_1(T_n+), L_2(T_n+)) = (i, j)]$$

$$= p(i, j;\, k, l)e^{\theta_{ij} t}.$$

4.5.1 Reorders

A reorder for both commodities is made when the joint inventory level at any time t, drops to either (s_1, s_2) or (s_1, j), $j < s_2$ or (i, s_2), $i < s_1$.

We associate with a reorder a counting process $N(t)$. Define

$$\beta_1(i, j, t) = \lim_{\Delta \to 0} P_{ij}\,[N(t + \Delta) - N(t) = 1]\,\frac{1}{\Delta}$$

where $\bar{P}_{ij}[\cdots]$ represents $Pr[\cdots \mid (L_0^{(1)}, L_0^{(2)}) = (i, j)]$. The fact that the reorder at time t is either due to the first transition or due to a subsequent one, gives the following equations:

$$\begin{aligned}
\beta_1(i,j,t) &= \lim_{\Delta\to 0}\frac{1}{\Delta}\{P_{ij}\left[N(t+\Delta)-N(t)=1, t<T_1<t+\Delta\right] + \\
&\quad \sum_{(k,l)\in E}\int_0^t P_{ij}\left[(L_1(T_1+), L_2(T_1+)) = (k,l), u<T_1<u+\Delta\right] \\
&\quad Pr\left[N(t-u+\Delta)-N(t-u)=1 \mid (L_1(T_1+), L_2(T_1+)) = (k,l)\right]\} \\
&= \tilde{\beta}_1(i,j,t) + \sum_{(k,l)\in E}\int_0^t p(i,j;k,l)\theta_{ij}e^{\theta_{ij}u}\beta_1(k,l,t-u)du
\end{aligned}$$

where $\tilde{\beta}_1(i,j,t)$ is given by

$$\tilde{\beta}_1(i,j,t) = \begin{cases} \lambda_1 e^{\theta_{ij}t} & i = s_1+1, \quad j = 0,1,\ldots,s_2 \\ \lambda_2 e^{\theta_{ij}t}\sum\limits_{u=j-s_2}^{\infty} p_u & i = 0,1,\ldots,s_1 \quad j = s_2+1, s_2+2,\ldots,S_2 \\ \lambda_{12} e^{\theta_{ij}t}\sum\limits_{u=j-s_2}^{\infty} p_u & i = 1,2,\ldots,s_1+1 \quad j = s_2+1, s_2+2,\ldots,S_2 \\ 0 & \text{otherwise.} \end{cases}$$

In the above expression we have used the fact that, when $j = 0, 1, 2, \ldots, s_2$ and $i = s_1 + 1$, then the next demand for commodity 1 will trigger a reorder. When $i = 0, 1, 2, \ldots, s_1$ and $j = s_2 + k$, ($k = 1, 2, \ldots, Q_2$) then k or more than k demands for commodity 2 alone will trigger a reorder. A demand for both commodities will trigger a reorder if the number of demanded items for the second commodity is k ($k = 1, 2, \ldots,$) when $i = 1, 2, \ldots, s_1 + 1$ and $j = s_2 + k$.

As the Markov process $\{(L_1(t), L_2(t)), t \geq 0\}$ is irreducible and recurrent (due to finite state space),

$$\beta_1 = \lim_{t\to\infty}\beta_1(i,j,t)$$

exists and will be equal to the steady state mean reorder rate. Moreover, we have from Cinlar(1975),

$$\beta_1 = \sum_{(i,j)\in E} \pi^{(i,j)} \int_0^\infty \widetilde{\beta}_1(i,j,t)dt \,/ \sum_{(i,j)\in E} \pi^{(i,j)} m_{ij}, \tag{4.4}$$

where m_{ij} is the mean sojourn time in the inventory level (i, j) and is given by $1/\theta_{ij}$, and $\pi^{(i,j)}$ is the stationary distribution of the Markov chain

$$\{(L_n^{(1)}, L_n^{(2)}), n = 0, 1, 2, \ldots\}.$$

Since for a Markov process,

$$\phi^{(i.j)} = \pi^{(i,j)} m_{ij} \,/ \sum_{(k,l)\in E} \pi^{(k,l)} m_{kl}, \tag{4.5}$$

we have from (4.5)

$$\begin{aligned}
\beta_1 &= \sum_{(i,j)\in E} \left(\frac{\phi^{(i,j)}}{m_{ij}}\right) \int_0^\infty \tilde{\beta}_1(i,j,t)dt \\
&= \sum_{(i,j)\in E} \phi^{(i,j)} \theta_{ij} \int_0^\infty \tilde{\beta}_1(i,j,t)dt. \\
&= \lambda_2 \sum_{k=0}^{s_1} \sum_{j=1}^{Q_2} \phi^{(k,s_2+j)} \sum_{u=j}^{\infty} p_u + \lambda_1 \sum_{k=0}^{s_2} \phi^{(s_1+1,k)} + \lambda_{12} \sum_{k=1}^{s_1+1} \sum_{j=1}^{Q_2} \phi^{(k,s_2+j)} \sum_{u=j}^{\infty} p_u
\end{aligned}$$

4.5.2 Shortages

A shortage for a commodity occurs when a demand occurs during a stockout period. We associate with a shortage a counting process $M(t)$. Define

$$\beta_2(i,j,t) = \lim_{\Delta\to 0}\frac{1}{\Delta}Pr[M(t+\Delta) - M(t) = 1 \mid (I_0^{(1)}, I_0^{(2)}) = (i,j)] \tag{4.6}$$

which satisfies the equation,

$$\beta_2(i,j,t) = \tilde{\beta}_2(i,j,t) + \sum_{(k,l)\in E}\int_0^t p(i,j;k,l)\theta_{ij}e^{\theta_{ij}u}\beta_2(k,l,t-u)du.$$

We have used the fact that the shortage at time t is due to the first demand or a subsequent one. Hence

$$\tilde{\beta}_2(i,j,t) = \lim_{\Delta\to 0}\frac{1}{\Delta}Pr[M(t+\Delta) - M(t) = 1, t < T_1 < t+\Delta \mid (L_0^{(1)}, L_0^{(2)}) = (i,j)]$$

and is given by,

$$\tilde{\beta}(i,j,t) = \begin{cases} (\lambda_1+\lambda_{12})e^{\theta_{ij}t}, & i=0, \quad j=1,2,\ldots,S_2 \\ (\lambda_2+\lambda_{12})e^{\theta_{ij}t}\sum\limits_{u=j+1}^{\infty}p_u, & i=1,2,\ldots,S_1, \quad j=1,2,\ldots,S_2 \\ 0, & \text{Otherwise.} \end{cases} \tag{4.7}$$

Derivations similar to the one used to derive β_1 (refer subsection Reorder) yields,

$$\begin{aligned} \beta_2 &= \lim_{t\to\infty}\beta_2(i,j,t) \\ &= (\lambda_2+\lambda_{12})\sum_{i=1}^{S_1}\sum_{j=0}^{S_2}\phi^{(i,j)}\sum_{l=j+1}^{\infty}p_l + (\lambda_1+\lambda_{12})\sum_{j=0}^{S_2}\phi^{(0,j)}. \end{aligned}$$

4.5.3 Expected Cost

The long run expected cost rate $C(S_1, S_2, s_1, s_2)$, is given by,

$$C(S_1, S_2, s_1, s_2) = h_1E[L_1] + h_2E[L_2] + K\beta_1 + b\beta_2$$

where h_1 and h_2 are holding cost for first and second commodity respectively, K is the

$$\begin{aligned} C(S_1, S_2, s_1, s_2) &= h_1 \sum_{i=0}^{S_1} i\phi^{(i,k)} + h_2 \sum_{i=0}^{S_1} i\phi^{(i,k)} + K \left\{ \lambda_2 \sum_{k=0}^{s_1} \sum_{j=1}^{Q_2} \phi^{(k,s_2+j)} \sum_{u=j}^{\infty} p_u \right. \\ &\left. + \lambda_1 \sum_{k=0}^{s_2} \phi^{(s_1+1,k)} + \lambda_{12} \sum_{k=1}^{s_1+1} \sum_{j=1}^{Q_2} \phi^{(k,s_2+j)} \sum_{u=j}^{\infty} p_u \right\} \\ &+ b \left\{ (\lambda_2 + \lambda_{12}) \sum_{i=1}^{S_1} \sum_{j=0}^{S_2} \phi^{(i,j)} \sum_{l=j+1}^{\infty} p_l + (\lambda_1 + \lambda_{12}) \sum_{j=0}^{S_2} \phi^{(0,j)} \right\} \end{aligned}$$

fixed cost per order and b is the shortage cost. Then we have

4.6 NUMERICAL ILLUSTRATIONS

The limiting probability distribution of inventory level is computed for specific values of parameters. For the first example we have assumed,

$S_1 = 6$, $S_2 = 7$, $s_1 = 2$, $s_2 = 2$, $\lambda_1 = 1.5$, $\lambda_2 = 2$, $\lambda_{12} = 1$, $\mu = 1.5$, $p_1 = 0.25$, $p_2 = 0.20$, $p_3 = 0.15$, $p_4 = 0.05$, $p_5 = 0.01$, $p_6 = 0.005$, $p_7 = 0.001$, $h_1 = 0.2$, $h_2 = 0.3$, $b = 0.7$, $K = 50$.

		Commodity II			
		0	1	2	3
Commodity I	0	0.075012	0.001401	0.001621	0.00252
	1	0.065147	0.001181	0.001587	0.00204
	2	0.117119	0.002746	0.003406	0.00405
	3	0.205454	0.005726	0.007193	0.00704
	4	0.152802	0.005999	0.008525	0.00894
	5	0.086886	0.004373	0.006852	0.00754
	6	0.031357	0.001949	0.003524	0.00413

		Commodity II			
		4	5	6	7
Commodity I	0	0.00164	0.001108	0.000059	0.000025
	1	0.00155	0.001436	0.000064	0.000034
	2	0.00340	0.004236	0.000166	0.000103
	3	0.00656	0.012514	0.000411	0.000311
	4	0.00894	0.037037	0.000974	0.000935
	5	0.00792	0.035197	0.001012	0.001185
	6	0.00460	0.039340	0.001134	0.001967

Table 4.1: Limiting probability distribution of the inventory level – Example 1

This example gives the following results:

Expected reorder rate = 0.39466.

Expected shortage rate = 2.41792.

Expected inventory level for the commodity I = 3.25862.

Expected inventory level for the commodity II = 1.04528.

Total Expected Cost rate = 22.39090.

As a second example, the following values have been considered and the calculated joint probability distribution of the inventory level is given in Table 4.2:

$S_1 = 5$, $S_2 = 6$, $s_1 = 1$, $s_2 = 2$, $\lambda 1 = 1.2$, $\lambda_2 = 1.5$, $\lambda_{12} = 0.8$, $\mu = 1$, $p_1 = 0.3$, $p_2 = 0.20$, $p_3 = 0.15$, $p_4 = 0.05$, $p_5 = 0.01$, $p_6 = 0.005$, $h_1 = 0.3$, $h_2 = 0.3$,$b = 0.9$, $K = 75$.

		Commodity II			
		0	1	2	3
Commodity I	0	0.168866	0.004688	0.007036	0.005056
	1	0.116626	0.003385	0.004312	0.003621
	2	0.188656	0.006302	0.007149	0.006626
	3	0.145235	0.007318	0.009887	0.010186
	4	0.082216	0.005493	0.008781	0.009814
	5	0.021723	0.001687	0.002995	0.003492

		Commodity II		
		4	5	6
Commodity I	0	0.003363	0.002014	0.000053
	1	0.002929	0.002484	0.000067
	2	0.006133	0.007181	0.000196
	3	0.010919	0.020758	0.000573
	4	0.011697	0.059996	0.001671
	5	0.004383	0.033446	0.000967

Table 4.2: Joint probability distribution of the inventory level – Example 2

This example gives the following results:

Expected reorder rate = 0.288526.

Expected shortage rate = 1.718531.

Expected inventory level for the commodity I = 2.254718.

Expected inventory level for the commodity II = 1.033886.

Total Expected Cost rate = 24.172744.

4.7 CONCLUISION

This article analyses a two-commodity inventory system under continuous review. The maximum storage capacity for the *i*-th item is S_i (i = 1, 2). The demand points for each commodity are assumed to form an independent Poisson process. We also assume that unit demand for one item and bulk demand for the other. The reorder level is fixed as s_i for the *i*-th commodity (i = 1, 2) and the ordering policy is to place order for Q_i (= $S_i - s_i$, i = 1, 2) items for the *i*-th commodity when both the inventory levels are less than or equal to their respective reorder levels. The lead-time is assumed to be exponential. The joint probability distribution for both commodities is obtained in both transient and steady state cases. Various measures of systems performance and the total expected cost rate in the steady state are derived. The results are illustrated with numerical examples.

CHAPTER 5

A SUBSTITUTABLE TWO-PRODUCT INVENTORY SYSTEM WITH JOINT-ORDERING POLICY AND COMMON DEMAND

A modified version of this chapter has been published in Applied Mathematics Computation, 2005.

5.1 INTRODUCTION

In the study of multi-product inventory systems, the concept of common demand for some products arises (Yadavalli and Hargreaves, 2003). For example, when a desired customer arrives at a shop that sells two brands of soft drinks, he/she may be satisfied by a soft drink of a particular brand with probability p_1 or by the other with probability p_2, $0 < p_i < 1$, $p_1 + p_2 = 1$. If any one of the products is out of stock, due to the desire, the customer will accept with probability 1 the other product that may be available in the shop. Also when the supplier is the same for several products under consideration, the dealer would prefer to have a simultaneous replenishment of all the products due to several reasons like cost considerations. Joint ordering policies for periodic inventory systems have been studied by several researchers (Bahadur and Acharya (1986) and Goyal and Satir (1989). Parlar and Weng (1997) and Anbazhagan (2002) developed optimal coordination policies for the supply and manufacturing departments. They considered a problem where the responsibility of the manufacturing department was to meet the random demand of a product with a short life cycle. The responsibility the supply department was to provide a sufficient amount of raw materials, so that the required production level could be achieved. Girlich (1996) and Yadavalli & Joubert (2003) studied a problem of joint coordination between manufacturing and supplying department encountered in a short life cycle multi-product environment. On the other hand, the study of continuous review multi-product inventory systems with common demand has not been considered so far in the literature. In this paper, an attempt is made to fill the gap by providing a study of a substitutable two-product inventory system with joint-ordering policy and common demand. The layout of the paper is as follows: In Section 2, the model assumptions and notation are provided. Certain auxiliary functions that characterise the occurrence of various events pertaining to the model are derived in Section 3. Section 4 gives some of the measures of the performance of the system. A cost analysis is provided in Section 5. Section 6 deals with the numerical results, which highlights the behaviour of the system.

5.2 MODEL ASSUMPTIONS AND NOTATION

5.2.1 ASSUMPTIONS

The following assumptions applies to the continuous review two-product inventory model:

(i) The maximum inventory level of product i is S_i, $i = 1, 2$.

(ii) Demands occur according to a Poisson process with parameter λ.

(iii) When both products are available, a demand is satisfied either with Product 1 with probability p_1 or with Product 2 with probability p_2, $0 < p_i < 1,\ p_1 + p_2 = 1$. When one of the products is out of stock, the demand is satisfied with the other product with probability 1. When both the products are out of stock, all demand is lost. That is, no backorders are allowed.

(iv) A re-order for both the products is placed at the epoch when the inventory level of product i reaches s_i and that of the other product j is greater than s_j, $j = 1, 2$ and $i \neq j$.

(v) The lead-time follows an arbitrary distribution with pdf $f(.)$.

(vi) When the re-order materializes, the inventory level of each product is brought to its maximum level.

5.2.2 NOTATION

The following notation is used in this chapter:

$L_i(t)$: The inventory level of product i at time t, $t = 1, 2$.

$L(t)$: The ordered pair $(L_1(t), L_2(t))$ representing the inventory level of the system at time t.

r_{ij} : Event that a replenishment of stock occurs at the epoch when the inventory level is (i, j).

γ_{ij} : Event that a re-order is placed when the product i reaches s_i and the level of the other product j, where $(i = 1, s_2 < j \le S_2)$ or $(i = 2, s_1 < j \le S_1)$.

$$B_i(n,t) = \frac{e^{-\lambda p_i t}(\lambda p_i t)^n}{n!};\ i = 1, 2$$

$$B_3(n,t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

E_0 : Initial condition representing the occurrence of an r-event.

5.3 AUXILIARY FUNCTIONS

The inventory level of each product is brought to its maximum at every epoch of replenishment. Hence the r-events constitute a renewal process. To derive the expression for the various measures of performance of the system, we proceed to study the renewal process of r-events. For this, certain auxiliary functions are defined to characterise the performance of the system in one cycle, which is the time interval between any two successive r-events.

5.3.1 FUNCTION ${}_r\phi_{ij}(t)$

Defining ${}_r\phi_{ij}(t)$ as:

$$ {}_r\phi_{ij}(t) = \lim_{\Delta \to 0} \frac{P[r_{ij} \ in \ (t, t+\Delta)/E_0]}{\Delta} $$

where $(i = 1, s_2 < j \leq S_2)$ or $(i = 2, s_1 < j \leq S_1)$. Then the function ${}_r\phi_{ij}(t)$ has the interpretation that it represents the probability that the inventory level of product i enters the state s_i in $(t, t+dt)$, the inventory level of the other product at time t is j and a re-order is placed in $(t, t+dt)$ given that an r-event has occurred at time $t = 0$. Since a re-order is made at the epoch when the inventory level of product i reaches s_i and the inventory level of product j is greater than s_j $(j \neq i)$, where

$$ {}_r\phi_{1j}(t) = B_1(S_1 - s_1 - 1, t)B_2(S_2 - j, t)\lambda p_1, \quad s_2 < j \leq S_2 \tag{5.1} $$

$$ {}_r\phi_{2j}(t) = B_1(S_1 - j, t)B_2(S_2 - s_2 - 1, t)\lambda p_2, \quad s_1 < j \leq S_1 \tag{5.2} $$

5.3.2 FUNCTION ${}_r h_l(t)$

Defining ${}_r h_l(t)$ as:

$$ {}_r h_l(t) = \lim_{\Delta \to 0} \frac{P[an \ l - event \ in \ (t, t+\Delta)/E_0\}}{\Delta} $$

Then ${}_r h_l(t)dt$ represents the probability that a demand occurs in $(t, t+dt)$ and is lost given that an r-event has occurred at time $t = 0$. To derive an expression for ${}_r h_l(t)$, we characterize the occurrence of the l-event in the following diagram (Figure 1). Accordingly, we have

$$
\begin{aligned}
{}_r h_l(t) = &\sum_{i=s_2+1}^{S_2} {}_r\phi_{1i}(t) \copyright \bar{F}(t)[\{\sum_{i=1}^{i} B_1(s_1-1,t)B_2(i-k,t)\lambda p_1 \copyright B_3(k-1,t)\lambda\} \\
&+\{\sum_{r=1}^{s_1} B_1(s_1-r,t)B_2(i-1,t)\lambda p_2 \copyright B_3(r-1,t)\lambda\}]\lambda \\
&+\sum_{i=s_1+1}^{S_1} {}_r\phi_{2i}(t) \copyright \bar{F}(t)[\{\sum_{k=1}^{S_2} B_1(i-1,t)B_2(s_2-k,t) \copyright B_3(k-1,t)\lambda\} \\
&+\{\sum_{r=1}^{i} B_1(i-r,t)B_2(s_2-1,t)\lambda p_2 \copyright B_3(r-1,t)\lambda\}]\lambda
\end{aligned} \tag{5.3}
$$

5.3.3 FUNCTION ${}_r\psi_{ij}(t)$

Defining ${}_r\psi_{ij}(t)$ as:

$$
{}_r\psi_{ij}(t) = \lim_{\Delta\to 0} \frac{P[an\ r_{ij}\ event\ in\ (t, t+\Delta)/E_0]}{\Delta}
$$

Then ${}_r\psi_{ij}(t)$ represents the pdf of the interval between two successive replenishments and that the replenishment which occurs in $(t, t+\Delta)$ is of r_{ij} type. Note that at the time of occurrence of the r_{ij}-event, $S_1 - i$ units of product 1 and $S_2 - j$ units of product 2 are added to the stock. Accordingly, the following cases exist:

Case 1: $i > s_1$ and $j > s_2$

$$ {}_r\psi_{ij}(t) = 0 \tag{5.4} $$

Case 2: $i > s_1$ and $0 < j \le s_2$

In this case, a γ_{2k}-event should occur in $(u, u+\Delta)$, $0 < u < t$, $i \le k \le S_1$.

Consequently,

$$ {}_r\psi_{ij}(t) = \sum_{k=i}^{S_1} {}_r\phi_{2k}(t) © [B_1(k-i, t) B_2(s_2 - j, t)] f(t) \tag{5.5} $$

Case 3: $i > s_1$ and $j = 0$

Since the inventory level of product 2 is 0 at time t, a γ_{2k} event occurs in $(u, u+\Delta)$, $0 < u < t$, and the system enters the state $(k-k', 0)$ in $(v, v+\Delta)$, $u < v < t$ and is in state $(i, 0)$ at time t. Hence

$$ {}_r\psi_{ij}(t) = \sum_{k=i}^{S_1} \sum_{k'=0}^{k-i} {}_r\phi_{2k}(t) © [B_1(k', t) B_2(s_2 - 1, t) \lambda p_2 © B_3(k - k' - i, t)] f(t) \tag{5.6} $$

Case 4: $0 < i \le s_1$ and $j \ge B_2$

As in Case 2,

$$ {}_r\psi_{ij}(t) = \sum_{k=j}^{S_2} {}_r\phi_{1k}(t) © [B_1(s_1 - i, t) B_2(k - j, t)] f(t) \tag{5.7} $$

Case 5: $i = 0$ and $j > s_2$

This case is similar to Case 3 and

$$ {}_r\psi_{ij}(t) = \sum_{k=j}^{S_2} \sum_{k'=0}^{k-i} {}_r\phi_{1k}(t) \copyright [B_1(s_1 - i, t) B_2(k', t) \lambda p_1 \copyright B_3(k - k' - j, t)] f(t) \tag{5.8} $$

Case 6: $0 < i \le s_1$ and $0 < j \le s_2$

In this case, either a γ_{1k} event or a γ_{2k} event should occur in $(u, u+\Delta)$, $0 < u < t$. Hence

$$ {}_r\psi_{ij}(t) = \sum_{k=s_2+1}^{S_2} {}_r\phi_{1k}(t) \copyright [B_1(s_1 - i, t) B_2(k - j, t)] f(t) $$
$$ + \sum_{k=s_1+1}^{S_1} {}_r\phi_{2k}(t) \copyright [B_1(k - i, t) B_2(s_2 - j, t)] f(t) \tag{5.9} $$

Case 7: $0 < i \le s_1$ and $j = 0$

At time $t = 0$, the system is in state (S_1, S_2) and enters the state (s_1, k) $k > s_1$ or the state (k, s_2), $k > s_2$ in $(u, u+\Delta)$ when a re-order is placed.

Then the system enters the state $(k, 0)$ in $(v, v+\Delta)$, $u < v < t$ and the inventory level is in state $(i, 0)$ at time t and the re-order materializes in $(t, t+\Delta)$. Hence

$$ {}_r\psi_{ij}(t) = \sum_{k=s_2+1}^{S_2} \sum_{k'=0}^{s_1-i} {}_r\phi_{1k}(t) \copyright [B_1(s_1 - k', t) B_2(k-1, t) \lambda p_2 \tag{5.10} $$

$$\copyright B_3(s_1 - k' - i, t)]f(t)$$

Case 8: $i = 0$ and $0 < j \le s_2$

This case is similar to Case 7. Hence

$$\begin{aligned} {}_r\psi_{ij}(t) = &\sum_{k=s_2+1}^{S_2}\sum_{k'=0}^{k-j} {}_r\phi_{1k}(t) \copyright [B_1(s_1 - 1, t)B_2(k', t)\lambda p_1 \\ &\copyright B_3(k - k' - j, t)]f(t) + \sum_{k=s_1+1}^{S_1}\sum_{k'=0}^{s_2-j} {}_r\phi_{2k}(t) \\ &\copyright [B_1(k-1, t)B_2(k', t)\lambda p_1 \copyright B_3(s_2 - k' - j, t)]f(t) \end{aligned} \tag{5.11}$$

Case 9: $i = 0$ and $j = 0$

At time $t = 0$, the inventory level is (S_1, S_2) and it enters the state (s_1, k), $k > s_2$ or the state (k, s_2), $k > s_1$ in $(u, u + \Delta)$, where a re-order is also placed. That re-order does not materialize in $(0, t)$ and the system enters the state $(r, 0)$ or the state $(0, r)$ in $(v, v + \Delta)$, $0 < u < v < w < t$. The system then enters the state $(0,0)$ in $((w, w + \Delta)$, $0 < u < v < w < t$, and is in state $(0,0)$ at time t and the re-order materializes in $(t, t + \Delta)$.

Consequently,

$${}_r\psi_{ij}(t) = \sum_{k=s_2+1}^{S_2} {}_r\phi_{1k}(t) \copyright [\{\sum_{k'=0}^{k-i} B_1(s_1 - 1, t)B_2(k', t)\lambda p_1 \copyright B_3(k, k' - 1, t)\lambda \copyright 1]$$

$$+\{\sum_{k'=0}^{s_1-1} B_1(k',t)B_2(k-1,t)\lambda p_2 © B_3(s_1-k'-1,t)\lambda ©1\}]f(t)$$

$$+\sum_{k=s_1+1}^{S_1} {}_r\phi_{2k}(t) ©[\{\sum_{k'=0}^{s_2-1} B_1(k-1,t)B_2(k',t)\lambda p_1$$

$$© B_3(k-k'-1,t)\lambda ©1\}+\{\sum_{k'=0}^{k-1} B_1(k',t)B_2(s_2-1,t)\lambda p_2$$

$$© B_3(k-k'-1,t)\lambda ©1\}]f(t) \tag{5.12}$$

5.3.4 FUNCTION ${}_r p_{ij}(t)$

Defining ${}_r p_{ij}(t)$ as:

$${}_r p_{ij}(t) = P[Y(t) = j, \mathrm{N}(r,t) = 0 / E_0] \;\; i = 0, 1, ..., S_1; \;\; j = 0, 1, ..., S_2$$

The following cases exists:

Case 1: $i > s_1$ and $j > s_2$

$${}_r p_{ij}(t) = B_1(S_1 - i, t)B_2(S_2 - j, t) \tag{5.13}$$

Case 2: $0 < i < s_1$ and $j > s_2$

In this case a γ_{1k}-event, $j \le k < S_2$, should occur in $(u, u+\Delta)$, $0 < u < t$. Hence

$${}_r p_{ij}(t) = \sum_{k=i}^{S_1} {}_r\phi_{2k}(t) ©[B_1(s_1 - i, t)B_2(k - j, t)]\overline{F}(t) \tag{5.14}$$

Case 3: $i > s_1$ and $0 < j \leq s_2$

This case is similar to case 2. Thus

$$_r p_{ij}(t) = \sum_{k=i}^{S_1} {}_r\phi_{2k}(t) \copyright [B_1(k-i,t)B_2(s_2-j,t)]\overline{F}(t) \tag{5.15}$$

Case 4: $i > s_1$ and $j = 0$

Since the inventory level of product 2 is zero at t, a γ_{2k}-event occurs at $(u, u+\Delta)$, $0 < u < t$, and the inventory level enters the state $(k-k',0)$ in $(v, v+\Delta)$, $0 < u < v < t$ and is in state $(i,0)$ at time t. Hence

$$\begin{aligned} {}_r p_{ij}(t) = \sum_{k=i}^{S_1} \sum_{k'=0}^{k-j} {}_r\phi_{2k}(t) \copyright [B_1(k',t)B_2(s_1-1,t)\lambda p_2 \\ \copyright B_3(k-k'-i,t)]\overline{F}(t) \end{aligned} \tag{5.16}$$

Case 5: $i = 0$ and $j > s_2$

This case is similar to case 4. So, we obtain

$$\begin{aligned} {}_r p_{ij}(t) = \sum_{k=i}^{S_1} \sum_{k'=0}^{k-j} {}_r\phi_{1k}(t) \copyright [B_1(s_1-1,t)B_2(k'-1,t)\lambda p_1 \\ \copyright B_3(k-k'-j,t)]\overline{F}(t) \end{aligned} \tag{5.17}$$

Case 6: $0 < i < s_1$ and $j = 0$

At time $t = 0$, the system is in state (S_1, S_2) and either enters the state $(s_1, k),\ k > s_2$ or enters the state (k, s_2), $k > s_1$, a re-order is placed in $(u, u+\Delta)$, $0 < u < t$. And the inventory level enters the state $(m,0),\ m \geq i$ in $(v, v+\Delta)$, $0 < u < v < t$, and the inventory is in state $(i,0)$ at time t. Hence

$$
\begin{aligned}
{}_r p_{ij}(t) = {} & \sum_{k=s_2+1}^{S_2} {}_r\phi_{1k}(t) © [\sum_{k'=0}^{s_1-i} B_1(k', t) B_2(k-1, t) \lambda p_2 © B_3(s_1 - k' - i, t)] \overline{F}(t) \\
& + \sum_{k=s_1+1}^{S_1} {}_r\phi_{2k} © [\sum_{k'=0}^{k-i} B_1(k', t) B_2(s_2 - 1, t) \lambda p_2 © B_3(k - k' - i, t)] \overline{F}(t)
\end{aligned}
\tag{5.18}
$$

Case 7: $i = 0$ and $0 < j \leq s_2$

This is similar to Case 6. Hence

$$
\begin{aligned}
{}_r p_{ij}(t) = {} & \sum_{k=s_2+1}^{S_2} {}_r\phi_{1k}(t) © [\sum_{k'=0}^{k-j} B_1(s_1 - 1, t) B_2(k', t) \lambda p_1 © B_3(k - k' - j, t)] \overline{F}(t) \\
& + \sum_{k=s_1+1}^{S_1} {}_r\phi_{2k} © [\sum_{k'=0}^{s_2-j} B_1(k-1, t) B_2(k', t) \lambda p_1 © B_3(s_2 - k' - j, t)] \overline{F}(t)
\end{aligned}
\tag{5.19}
$$

Case 8: $0 < i \leq s_1$ and $0 < j \leq s_2$

In this case either a γ_{1k}-event or a γ_{2k}-event should occur in $(u, u+\Delta)$, $0 < u < t$. The following equation is obtained

$$
{}_r p_{ij}(t) = \sum_{k=s_2+1}^{S_2} {}_r\phi_{1k}(t) © [B_1(s_1 - i, t) B_2(k - j, t) \overline{F}(t) + \sum_{k=s_1+1}^{S_1} {}_\Omega\Phi_{2k}(t)
$$

$$©[B_1(k-i,t)B_2(s_2-j,t)]\overline{F}(t) \tag{5.20}$$

Case 9: $i=0$ and $j=0$

At time $t=0$, the system is in state (S_1,S_2) and enters the state (s_1,k), $k>s_2$ or enters the state (k,s_2), $k>s_1$ and corresponding re-order is placed in $(u,u+\Delta)$. And the re-order does not materialize in $(0,t)$ and the system enters the state $(0,m)$ or $(m,0)$, $m>0$ in $(v,v+\Delta)$, $0<u<v<t$ and then enters the state $(0,0)$ in $(w,w+\Delta)$, $0<u<v<w<t$ and is in state $(0,0)$ at time t. Accordingly,

$$\begin{aligned}{}_r p_{ij}(t) = &\sum_{k=s_2+1}^{S_2} {}_r\phi_{1k}(t) ©[\{\sum_{k'=0}^{k-1} B_1(s_1-1,t)B_2(k',t)\lambda p_1 © B_3(k-k'-1,t)\lambda ©1\} \\ &+\{\sum_{k'=0}^{s_1-1} B_1(k',t)B_2(k-1,t)\lambda p_2 © B_3(s_1-k'-1,t)\lambda ©1\}] \\ &+\sum_{k=s_1+1}^{S_1} {}_r\phi_{2k}(t) ©[\{\sum_{k'=0}^{s_2-1} B_1(k-1,t)B_2(k',t)\lambda p_1 © B_3(k-k'-1,t)\lambda ©1\} \\ &+\{\sum_{k'=0}^{s_1-1} B_1(k',t)B_2(s_2-1,t)\lambda p_2 © B_3(k-k'-1,t)\lambda ©1\}]\overline{F}(t)\end{aligned} \tag{5.21}$$

Based on the above auxiliary functions (5.1) to (5.21), some measures of system performance are presented in the next section.

5.4 MEASURES OF SYSTEM PERFORMANCE

5.4.1 MEAN NUMBER OF REPLENISHMENTS

The r-events correspond to the epoch of replenishments, and as such they constitute a renewal process. The first-order product density $h_r(t)$ corresponding to the r-events is given by

$$h_r(t) = \sum_{n=1}^{\infty} g^{(n)}(t)$$

where $g(t)$ is the pdf of the interval between two successive occurrences of r-events.

To obtain an expression for $g(t)$, an expression for the survivor function $\overline{G}(t)$ corresponding to $g(t)$ is defined. Since $\overline{G}(t)$ is the probability that replenishment has not occurred up to time t, the following probabilities exist:

(i) A re-order is not placed up to time t

(ii) A re-order is placed in $(u, u+\Delta)$, $0 < u < t$, but it has not materialized until t

$$\overline{G}(t) = \sum_{i=0}^{S_1-s_1-1} \sum_{j=0}^{S_2-s_2-1} B_1(i,t) B_2(j,t) + \sum_{j=s_2+1}^{S_2} {}_r\phi_{ij}(t) © \overline{F}(t) + \sum_{j=s_1+1}^{S_1} {}_r\phi_{2j}(t) © \overline{F}(t) \quad (5.22)$$

Consequently, the mean number of replenishments is given by

$$E[\mathrm{N}(r,t)] = \int_0^t h_r(u)du$$

and the expected stationary rate of replenishments is given by

$$E(r) = \lim_{t\to\infty} h_r(t)$$

5.4.2 MEAN NUMBER OF RE-ORDERS PLACED

Defining $h_{\gamma_{ij}}(t)$ as:

$$h_{\gamma_{ij}}(t) = \lim_{\Delta\to 0} \frac{P[a\ \gamma_{ij} - event\ in\ (t, t+\Delta)/E_0]}{\Delta}$$

Since an epoch of re-order corresponds to the occurrence of a γ-event the first-order product density $h_\gamma(t)$ corresponding to re-orders is given by

$$h_\gamma(t) = \sum_{j=s_2+1}^{S_2} h_{\gamma_{ij}}(t) + \sum_{j=s_1+1}^{S_1} h_{\gamma_{2j}}(t) \tag{5.23}$$

To derive an expression for $h_{\gamma_{ij}}(t)$, consider the following mutually exclusive and exhaustive possibilities

(i) No r-event has occurred up to time t

(ii) At least one r-event has occurred in $(0, t)$

$$h_{\gamma_{ij}}(t) = {}_r\phi_{ij}(t) + h_r(t) © {}_r\phi_{ij}(t) \tag{5.24}$$

Hence the mean number of re-orders placed in $(0, t)$ is given by

$$E[\mathrm{N}(\gamma,t)] = \sum_{j=s_2+1}^{S_2} \int_0^t h_{\gamma ij}(u)du + \sum_{j=s_1+1}^{S_1} \int_0^t h_{\gamma 2j}(u)du$$

The mean stationary rate of re-ordering is given by

$$E(\gamma) = \lim_{t\to\infty} h_\gamma(t)$$
$$= E(r)[\sum_{j=s_2+1}^{S_2} {}_r\phi_{1j}^*(0) + \sum_{j=s_1+1}^{S_1} {}_r\phi_{2j}^*(0)] \tag{5.25}$$

where ${}_r\phi_{ij}^*(.)$ is the Laplace transform of ${}_r\phi_{ij}(.)$, (see Girlich, 2003).

5.4.3 MEAN NUMBER OF LOST DEMANDS

Let $h_l(t)$ be the first-order product density corresponding to the epochs of occurrences of lost demands. Then the following expression can be derived:

$$h_l(t) =_r \phi_1(t) + h_r(t) © {}_r h_l(t)\,. \tag{5.26}$$

Hence the mean number of lost demands in $[0,t]$ is given by

$$E[\mathrm{N}(l,t)] = \int_0^t h_1(u)du$$

and the mean stationary rate of lost demands is given by

$$E(l) = \lim_{t\to\infty} h_1(t)$$
$$= E(r)_r h_l^*(0) \tag{5.27}$$

5.4.4 MEAN NUMBER OF UNITS REPLENISHED

At the occurrence of each r_{ij}-event $S_1 - i$ units of product 1 and $S_2 - j$ units of product 2 are replenished. Also note that $E(r_{ij})$ is the mean stationary state of r_{ij}-events and it is given by

$$E(r_{ij}) = \lim_{t \to \infty} h_{r_{ij}}(t)$$

where $h_{r_{ij}}(t)$ is the first order product density corresponding to r_{ij}-events. Then

$$h_{r_{ij}}(t) = {}_r\psi_{ij}(t) + h_r(t) \copyright {}_r\psi_{ij}(t) \tag{5.28}$$

Consequently,

$$\begin{aligned} E(r_{ij}) &= \lim_{s \to 0} s[1 + h_r^*(s)]_r\, \psi_{ij}^*(s) \\ &= E(r)_r\, \psi_{ij}^*(0) \end{aligned} \tag{5.29}$$

Thus, the mean number of Product 1 that may be added to the inventory in unit time in the long run is given by

$$\sum_{i=0}^{s1} \sum_{j=0}^{S2} E(r_{ij})(S_1 - i) + \sum_{i=s_1+1}^{S1} \sum_{j=0}^{s2} E(r_{ij})(S_1 - i)$$

and, in the same manner, the mean number of Product 2 that may be added to the inventory in unit time in the long run is given by

$$\sum_{i=0}^{s_1}\sum_{j=0}^{S_2} E(r_{ij})(S_2 - j) + \sum_{i=s_1+1}^{S_1}\sum_{j=0}^{s_2} E(r_{ij})(S_2 - j) \tag{5.30}$$

5.4.5 DISTRIBUTION OF THE INVENTORY LEVEL

The probability distribution of the inventory level is defined by

$$p_{ij}(t) = P[Y(Z) = (i, j) / E_0]$$

where $0 \le i \le S_1$ and $0 \le j \le S_2$.

Using renewal theoretic arguments,

$$p_{ij}(t) = {}_r p_{ij}(t) + h_r(t) © {}_r\, p_{ij}(t) \tag{5.31}$$

Consequently, the stationary distribution of the inventory level is given by

$$\Pi_{ij} = \lim_{t\to\infty} p_{ij}(t) = E(r)\,{}_r p_{ij}^{*}(0)$$

(5.32)

5.5 COST ANALYSIS

We have two types of re-orders, namely

(i) the re-order is placed when the inventory level of Product 1 reaches s_1 or

(ii) the re-order is placed when the inventory level of Product 2 reaches s_2

It can be assumed that the two types of re-orders placed are with two different suppliers and hence that the corresponding costs are different. Let CR_i be the cost corresponding to a re-order due to the inventory level of product i reaching s_i, $i = 1, 2$. Let CL be the cost corresponding to a lost demand. Since $E(r)\,{}_r h_1^*(0)$ is the mean rate of the lost demand, the cost due to lost demands is given by $E(r)\,{}_r h_1^*(0)\,CL$. In the same way, the cost corresponding to re-orders placed is given by

$$E(r)[\sum_{j=s_2+1}^{S_2} {}_r\phi_{1j}^*(0)CR_1 + \sum_{j=s_1+1}^{S_1} {}_r\phi_{2j}^*(0)CR_2]$$

Hence the total cost is given by

$$Total\ Cost = E(r)[{}_r h_1^*(0)\,CL + \sum_{j=s_2+1}^{S_2} {}_r\phi_{1j}^*(0)CR_1 + \sum_{j=s_1+1}^{S_1} {}_r\phi_{2j}^*(0)CR_2] \qquad (5.33)$$

The total cost can be considered as a function of s_1 and its optimal value can be obtained.

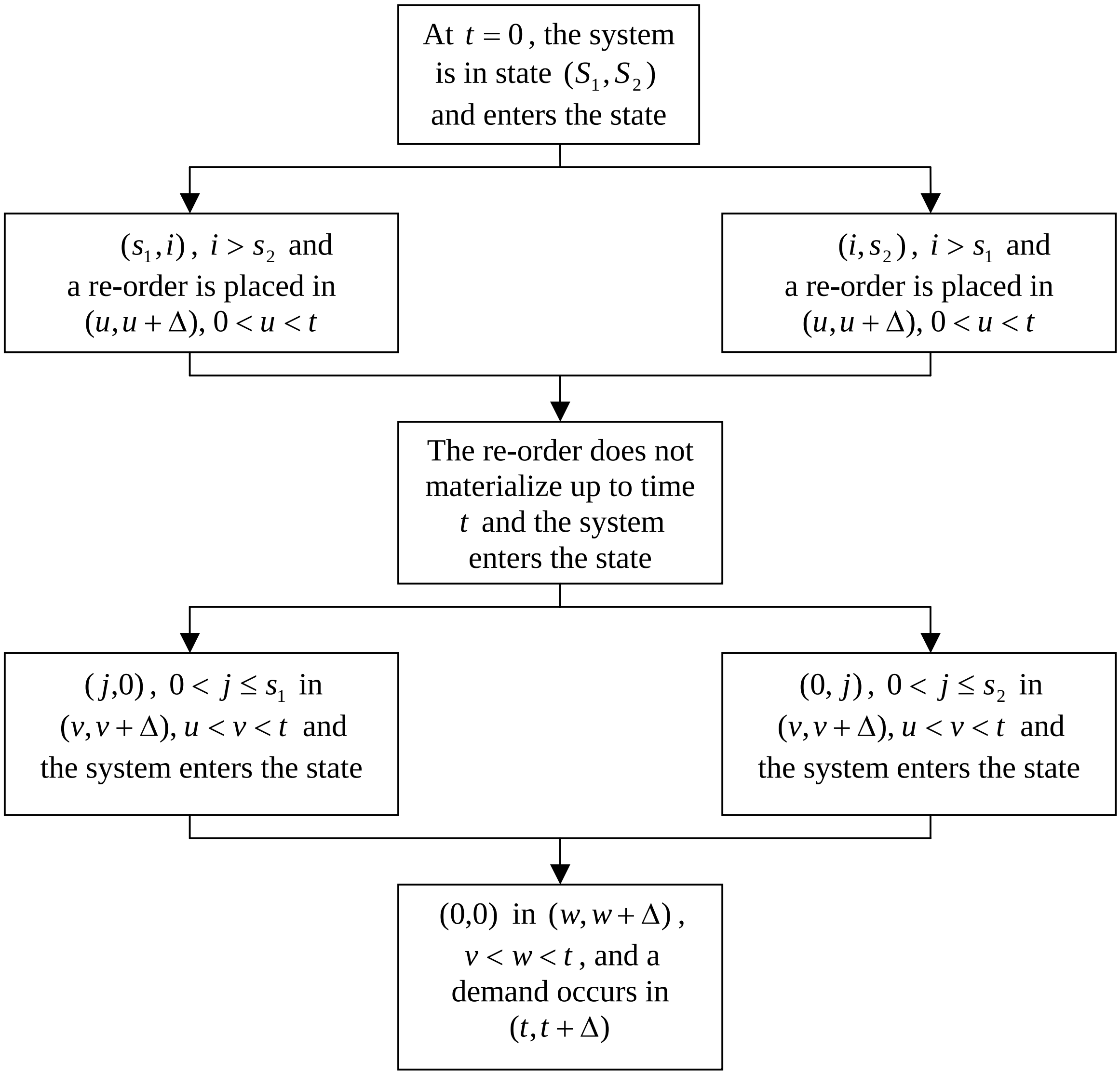

Figure 5.1: System State for Cost Function

5.6 NUMERICAL ILLUSTRATION

For the purpose of illustration, we assume that f(t) = a exp{-at} and the values of various parameters as follows:

λ = 1.2,
a = 0.5,
S_1 = 8,
S_2 = 5,
s_1 = 1,
CL =10,
CR_1 = 200,
CR_2 = 300

First, the re-order level for Product 1 is fixed as $s_1 = 2$ and the value of p_1 increased from 0.1 to 0.9 to obtain the behaviour of the mean rates of

(i) r-events,
(ii) γ_{ij}-events
(iii) Lost demands
(iv) Unit 1 replenished
(v) Unit 2 replenished
(vi) Total cost

From Table 5.1, it can be observed that, as p_1, the probability of demand for Product 1, increases,

(i) The mean rate of replenishments decreases and then increases
(ii) The mean rate of γ_{1j}-events increase and that of γ_{2j} decreases
(iii) The mean rate of lost demands increases

(iv) The mean rate of unit 1 replenished increases

(v) The mean rate of unit 2 replenished decreases

(vi) The mean rate of total cost decreases and then increases. The total cost is a minimum when $p_1 = 0.7$

Next, as p_1 is fixed and the re-order level for Product 1 increased, the results presented in Table 2 is obtained. The result is that, as s_1 increases with $p_1 = 0.7$,

(i) The mean rate of replenishments increases

(ii) The mean rate of γ_{1j} increases and that of γ_{2j} decreases

(iii) The mean rate of lost demands decreases

(iv) The mean rate of unit 1 replenished increases

(v) The mean rate of unit 2 replenished increases

(vi) The mean rate of total cost increases.

Re-Order Level for Product 1 Fixed at 2

p_1	p_2	LL1	RR	ERO1	ERO2	RLD	U1RR	U2RR	TCOST
0.1	0.9	2	2.852	0.000	2.852	0.003	0.908	7.534	855.613
0.2	0.8	2	1.496	0.005	1.492	0.004	1.113	4.107	448.500
0.3	0.7	2	1.022	0.026	0.996	0.008	1.362	2.909	304.097
0.4	0.6	2	0.790	0.078	0.711	0.014	1.689	2.300	229.198
0.5	0.5	2	0.675	0.171	0.503	0.026	2.117	1.934	185.494
0.6	0.4	2	0.643	0.310	0.333	0.056	2.683	1.679	162.366
0.7	0.3	2	0.699	0.510	0.189	0.141	3.498	1.464	160.152
0.8	0.2	2	0.908	0.831	0.078	0.421	4.944	1.251	193.664
0.9	0.1	2	1.633	1.620	0.014	1.668	9.013	1.030	344.680

p_1 : Probability of Demand for Product 1
p_2 : Probability of Demand for Product 2
LL1 : Re-Order Level for Product 1
RR : Rate of Replenishment
ERO1 : Rate of Type 1 Re-Order
ERO2 : Rate of Type 2 Re-Order
RLD : Rate of Lost Demand
U1RR : Rate of Units of Product 1 Replenishment
U2RR : Rate Of Units Of Product 2 Replenishment
TCOST : Rate of Total Cost

Table 5.1: Variation of Measures of System Performance Against the Probability of Demand for Product 1

Probability of Demand for Product 1 Fixed at Various Levels

p_1	p_2	LL1	RR	ERO1	ERO2	RLD	U1RR	U2RR	TCOST
0.3	0.7	1	1.014	0.011	1.003	0.104	1.330	2.889	303.227
0.3	0.7	2	1.022	0.026	0.996	0.008	1.362	2.909	304.097
0.3	0.7	3	1.041	0.060	0.981	0.005	1.408	2.946	306.296
0.5	0.5	1	0.634	0.109	0.525	0.116	1.888	1.826	180.451
0.5	0.5	2	0.675	0.171	0.503	0.026	2.117	1.934	185.494
0.5	0.5	3	0.741	0.269	0.472	0.009	2.284	2.041	195.454
0.7	0.3	1	0.600	0.390	0.210	0.942	2.690	1.224	150.363
0.7	0.3	2	0.699	0.510	0.189	0.141	3.498	1.464	160.152
0.7	0.3	3	0.848	0.684	0.165	0.019	3.977	1.633	186.306

p_1 : Probability of Demand for Product 1
p_2 : Probability of Demand for Product 2
LL1 : Re-Order Level for Product 1
RR : Rate of Replenishment
ERO1 : Rate of Type 1 Re-Order
ERO2 : Rate of Type 2 Re-Order
RLD : Rate of Lost Demand
U1RR : Rate of Units of Product 1 Replenishment
U2RR : Rate Of Units Of Product 2 Replenishment
TCOST : Rate of Total Cost

Table 5.2: Variation of Measures of System Performance Against Re-Order Level for Product 1

5.7 CONCLUSION

A substitutable two-product inventory system with joint-ordering policy is considered in this chapter. Common demands occur according to a Poisson process. A demand is satisfied either with an item of Product 1 with probability p_1 or with an item of Product 2 with probability p_2 ($p_1 + p_2 = 1$). When one of the products is out of stock, the demand is satisfied with the other available product with probability 1. Analyzing the imbedded renewal process describing the system, expressions for the stationary distribution of the inventory level and the stationary rates of the replenishments, the re-orders placed, the lost demands, and the units replenished are obtained. A cost analysis is also provided and a numerical example illustrates the results obtained.

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