CHAPTER 2

A PERISHABLE PRODUCT INVENTORY SYSTEM OPERATING IN A RANDOM ENVIRONMENT

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2.1 INTRODUCTION

Various stochastic models of inventory systems have been studied recently by Yadavalli & Joubert [2003], Yadavalli et al [2004]. Studies on perishable product inventory systems have gained much importance in literature (Kumaraswamy and Sankarasubramanian [1981], Kalpakam and Arivarignan [1988], Pal [1990], Liu [1990], Raafat [1991] and Kalpakam and Sapna [1994, 1996]). In the stochastic analysis of such inventory systems, it is generally assumed that the distributions of the random variables representing the number of demands over a period of time, the lifetime of the product and the lead-time remain the same and do not change throughout the domain of the analysis. However, there are external factors that affect these random variables. Seasonal changes can affect the demand rate, the perishing rate, the selling price and the cost of replenishment. The demand for umbrellas and rain shoes are higher in winter than in summer. The perishing rates of vegetables are higher in summer. The selling price and the cost of replenishment also fluctuate over time due to reasons such as inflation and non-availability of the product. The state of the environment in which the system is operating may randomly change due to several factors, including weather conditions and breakdown of storage facilities. Consideration of the impact of the random environment on such inventory systems is, therefore, absolutely essential. Only a few authors have considered inventory systems operating in random environments (Feldman [1978], Pal [1990], Song and Zipkin [1993] and Girlich [1998]). These authors considered non-perishable product inventory evolving in random environments. The survey of Raafat [1991] presents only literature on deteriorating inventory models in non-changing environments. Kalpakam and Sapna [1996] considered inventory models where the items have constant perishing rates only.

A perishable product inventory system operating in a random environment is studied in this chapter. For the sake of simplicity, the stochastic environment is considered to alternate randomly over time between two states, 0 and 1, according to an alternating renewal process. When the environment is in state k, the items in the inventory have a perishing rate μ_k, the demand rate is λ_k and the replenishment cost is CR_k. Assuming instantaneous replenishment at the epoch of the first demand after the stock-out and associating a Markov renewal process with the inventory system, the stationary distribution of the inventory level and the performance of various measures of the system evolution are obtained.

This chapter is structured as follows:

Section 2.2 provides the assumptions and notation of a model of an inventory system operating in a random environment and certain auxiliary functions are obtained in Section 2.3. An associated Markov renewal process is analyzed in Section 2.4. In Section 2.5, the stationary distribution of the inventory level is given and the stationary measures of performance of the system are obtained in Section 2.6. A cost analysis for the model of the inventory system is presented in Section 2.7. Section 2.8 considers a particular case of the general model and obtains the probability distribution of the total sales proceeds up to any time t. In Section 2.9, another particular case of the general model is considered and the total replenishment cost incurred up to t is studied, followed by a numerical illustration in Section 2.10.

2.2 ASSUMPTIONS AND NOTATION

2.2.1 Assumptions

A continuous review inventory system operating in a random environment is considered. The random environment is assumed to alternate between two states, 0 and 1. The durations of stay in the state 0 are given by the sequence of i.i.d. random variables $\{X_n\}$, having a common exponential distribution with parameter ν_0, and the durations of stay in the state 1 are given by the sequence of i.i.d. random variables $\{Y_n\}$, having a common exponential distribution with parameter ν_1. A renewal of one state occurs at the termination of the other. The two families { X_n } and { Y_n } are independent.

Other applicable assumptions are the following:

(i) The items under consideration are perishable. The rate of perishing depends on the state of the random environment. The lifetime distribution of an item in the inventory is exponential with parameter μ_k when the environment is in state k, (k = 0, 1).

(ii) Demands occur according to a double stochastic Poisson process. The demand occurs with rate λ_k when the environment is in state k, (k = 0, 1).

(iii) Replenishment is instantaneous for $S+1$ units and is made at the epoch of the occurrence of the first demand that occurs during the stock-out period. The cost of replenishment is CR_k when the environment is in state k (k = 0, 1).

2.2.2 Notation

$\xi(t)$	:	The state of the environment at time t
π	:	Event that an item perishes
©	:	Convolution symbol
H(i-j)	:	Heaviside function $=\begin{cases} 1 \text{ if } i \geq j \\ 0 \text{ if } i < j \end{cases}$

2.3 AUXILIARY FUNCTIONS

In this section, the underlying stochastic process is identified as a Markov renewal process. In order to study its transient behaviour, certain auxiliary functions are obtained in this section.

2.3.1 Function $P(j,t;i,k)$

An interval in which there is no replenishment and the environment remains in a fixed state, the inventory level process $L(t)$ behaves like a death process. To describe the behaviour of this process, the function

$$P(j,t;i,k) = P[L(t) = j \mid L(0) = i, \xi(0) = k]$$

is defined, where $0 \le i, j \le S$ and $k = 0,\ 1$. To derive an expression for $P(j,t;i,k)$ consider that if $L(t) \ne 0$, a change in the state of $L(t)$ occurs due to any one of the following mutually exclusive and exhaustive cases:

(i) A demand for the product occurs

(ii) An item perishes and is removed instantaneously from the inventory

Accordingly

Case 1: $i = 0,\ j = 0$

$$P(0,t;0,k) = e^{-\lambda_k t} \tag{2.1}$$

Case 2: $j > i$

$$P(j,t;i,k) = 0 \tag{2.2}$$

Case 3: $i = j \neq 0$

$$P(j,t;i,k) = e^{-(\lambda_k + j\mu_k)t} \tag{2.3}$$

Case 4: $0 \leq j < i$

$$P(j,t;i,k) = (\lambda_k + i\mu_k)e^{-(\lambda_k + i\mu_k)t} \copyright P(j,t;i-1,k) \tag{2.4}$$

Taking Laplace transforms, the equations (2.1) to (2.4) yield the following:

$$P^*(j,s;i,k) = \begin{cases} 0 & i < j \\ \dfrac{1}{s+\lambda_k} & i = j = 0 \\ \dfrac{1}{(s+\lambda_k + i\mu_k)} & i = j \neq 0 \\ \dfrac{u(i,j,k,s)}{s+\lambda_k} & 0 = j < i \\ \dfrac{u(i,j,k,s)}{s+\lambda_k + j\mu_k} & 1 \leq j < i \end{cases} \tag{2.5}$$

where

$$u(i,j,k,s) = \frac{\prod_{m=j+1}^{i} (\lambda_k + m\mu_k)}{\prod_{m=j+1}^{i} (s+\lambda_k + m\mu_k)} \qquad ;0 \leq j < i$$

Inverting Equation 2.5 obtains the following:

$$P(j,t;i,k)=\begin{cases} e^{-\lambda_k t} & ;i=j=0 \\ e^{-(\lambda_k+i\mu_k)t} & ;i=j\neq 0 \\ v(i,j,k)e^{-(\lambda_k+j\mu_k)t}(1-e^{-\mu_k t})^{i-j} & ;1\leq j<i \\ 1-i\mu_k v(i,0,k)\sum_{m=1}^{i}(-1)^{m-1}\binom{i-1}{m-1}\dfrac{e^{-(\lambda_k+m\mu_k)t}}{\lambda_k+m\mu_k} & ;0=j<i \\ 0 & ;otherwise \end{cases} \qquad (2.6)$$

where

$$v(i,j,k)=\frac{\prod_{m=j+1}^{i}(\lambda_k+m\mu_k)}{\mu_k{}^{i-j}.(i-j)!}$$

2.3.2 Function $f_{r,k}(t)$

Consider the point process of r-events occurring in an interval in which there is no change in the state of the environment. Let

$$f_{r,k}(t)=\lim_{\Delta\to 0}\frac{P[r-event\ in\ (t,t+\Delta),\ N(r,t)=0 \mid r-event\ at\ t=0,\ \xi(0)=k]}{\Delta};\quad k=0,1$$

The function $f_{r,k}(t)$ represents the pdf of the interval between any two successive occurrences of replenishment when the state of the environment remains at k throughout the interval under consideration. Note that

$$f_{r,k}(t)=P(0,t;S,k)\lambda_k \quad ;k=0,1$$

2.3.3 Function $h_{r,k}(t)$

Considering the point process of r-events occurring in an interval in which there is no change in the state of the environment, the function $h_{r,k}(t)$ are defined as follows:

$$h_{r,k}(t) = \lim_{\Delta \to 0} \frac{P[r - event\ in\ (t, t+\Delta) \mid r - event\ at\ t = 0, \xi(0) = k]}{\Delta}; \quad k = 0, 1$$

The function $h_{r,k}(t)$ represents the renewal density of r-events in an interval in which the state of the environment remains as k throughout the interval. Note that

$$h_{r,k}(t) = \sum_{n=1}^{\infty} f_{r,k}{}^{(n)}(t); \quad k = 0, 1 \qquad (2.7)$$

2.3.4 Function $W(j,t;i,k)$

Consider an interval in which there is no change in the state of environment. The function $W(j,t;i,k)$ is defined as follows:

$$W(j,t;i,k) = P[L(t) = j \mid L(0) = i, \xi(0) = k]$$

where $0 \le i, j \le S$ and $k = 0,1$.

This function gives the distribution of the inventory level at any time t if the environment is in state k, k = 0, 1, throughout the interval $(0,t]$. To obtain an expression for $W(j,t;i,k)$, the following mutually exclusive and exhaustive cases are considered:

(i) No replenishment occurs in $(0,t]$

(ii) Only one replenishment occurs in $(0,t]$

(iii) More than one replenishment occurs in $(0,t]$

Accordingly

$$W(j,t;i,k) = H(i-j)P(j,t;i,k) + \lambda_k P(0,t;i,k) © P(j,t;S,k)$$
$$+ \ \lambda_k P(0,t;i,k) © h_{r,k}(t) © P(j,t;S,k) \qquad (2.8)$$

where $0 \le i, j \le S$ and $k = 0, 1$.

2.4 INVENTORY LEVEL

Let $0 = T_0, T_1, T_2 \ldots$ be the successive epochs at which the environment changes its state and

$$L_n = L(T_n+); \qquad \xi_n = (T_n+); \qquad n = 0, 1, 2, \ldots$$

Setting $Z_n = (L_n, \xi_n)$, it follows that $(Z,T) = \{Z_n, T_n; n = 0,1,2\ldots\}$ is a Markov renewal process (Cinlar [1975a]) with the state space $E = E_2 \cup E_3$, where

$$E_2 = \{(i,0), i = 0,1,2,\ldots,S\}; \quad E_3 = \{(i,1), i = 0,1,2,\ldots,S\}$$

Defining

$$Q(j_2,k_2,t \mid j_1,k_1) = P[Z_{n+1} = (j_2,k_2), T_{n+1} - T_n \le t \mid Z_n = (j_1,k_1)] \qquad ;(j_1,k_1),(j_2,k_2) \in E_1$$

The function $Q(j_2, k_2, t | j_1, k_1)$ has the following interpretation. Given that the environment over its state to k_1, at time T_n and that the inventory level at T_n is j_1, the probability is $Q(j_2, k_2, t | j_1, k_1)$ that the subsequent change of the state of the environment takes place at time T_{n+1} not later than a duration t from T_n and that the state of Z at T_{n+1} is (j_2, k_2).

Since T_n's are epoch transitions of the process $\xi(t)$,

$$Q(j_2, k_2, t | j_1, k_1) = 0 \text{ for } k_1 = k_2 \tag{2.9}$$

For $k_1 \neq k_2$,

$$Q(j_2, 1, t \; | j_1, 0) = \int_0^t W(j_2, u; j_1, 0)\nu_0 e^{-\nu_0 u} du \tag{2.10}$$

$$Q(j_2, 0, t \; | j_1, 1) = \int_0^t W(j_2, u; j_1, 1)\nu_1 e^{-\nu_1 u} du \tag{2.11}$$

where $0 \le j_1, j_2 \le S$.

The semi-Markov kernel $Q(t)$ of the Markov renewal process is given by the following $(2S+2)\times(2S+2)$ order matrix:

$$\mathbf{Q}(t) = \begin{array}{c c} & \begin{array}{cc} E_2 & E_3 \end{array} \\ \begin{array}{c} E_2 \\ E_3 \end{array} & \left[\begin{array}{c|c} 0 & A(t) \\ \hline B(t) & 0 \end{array}\right] \end{array}$$

where $A(t)$ is a matrix of order $(S+1)\times(S+1)$ whose elements are given by (2.10) and the matrix $B(t)$ is of order $(S+1)\times(S+1)$ whose elements are given by (2.11).

For any two elements (j_1,k_1) and $(j_2,k_2)\in E_1$,

$$R(j_2,k_2,t|j_1,k_1)=\sum_{n=0}^{\infty}Q^{(n)}(j_2,k_2,t|j_1,k_1) \tag{2.12}$$

where

$$Q^{(n)}(j_2,k_2,t\ |j_1,k_1)=\sum_{(j,k)\in E_1}\int_0^t Q(j,k,du\ |j_1,k_1)Q^{(n-1)}(j_2,k_2,t-u\ |j,k)$$

$R(j_2,k_2,t|j_1,k_1)$ represents the expected number of renewals of the state (j_2,k_2) in the interval $(0,t)$ and is called Markov renewal function. The Markov renewal kernel $R(t)$ of the process (Z,T) is given by the $(2S+2)\times(2S+2)$ order matrix

$$R(t)=[R(j_2,k_2,t|j_1,k_1)].$$

If $R^*(s)$ is the matrix Laplace transform defined by (see Girlich, 2003)

$$R^*(s)=[R^*(j_2,k_2,s|j_1,k_1)],$$

then, from the theory of Markov renewal process,

$$R^*(s)=[I-Q^*(s)]^{-1}$$

$$=\begin{bmatrix}(I-A^*(s)B^*(s))^{-1} & A^*(s)(I-A^*(s)B^*(s))^{-1}\\ B^*(s)(I-A^*(s)B^*(s))^{-1} & (I-A^*(s)B^*(s))^{-1}\end{bmatrix} \tag{2.13}$$

where $Q^*(s), A^*(s)$ and $B^*(s)$ are the matrices of Laplace transforms corresponding to $Q(t), A(t)$ and $B(t)$ respectively. Inversion of the elements of $R^*(s)$ yields the elements of $R(t)$. Using these elements, the probability distribution of the inventory level is defined as follows:

$$P(j,t|i,k) = P[L(t) = j|L(0) = i, \xi(0) = k] \quad ; \quad 0 \le j \le S, \quad (i,k) \in E_1$$

$P(j,t|i,k)$ is the probability that the inventory level is j at time t given that initially, at time $t = 0$, the inventory level is i and the environment level is k. To obtain an expression for $P(j,t|i,k)$, the vector process $(L(t), \xi(t))$ is semi-regenerative (Cinlar [1975]) with state space E_1 and the Markov renewal process (Z,T) embedded in it. Its probability function is defined by

$$\beta(j_2,k_2,t|j_1,k_1) = P[L(t) = j_2, \xi(t) = k_2|L(0) = j_1, \xi(0) = k_1]$$

where (j_1,k_1) and $(j_2,k_2) \in E_1$.

An auxiliary function is defined as follows:

$$\gamma(j_2,k_2,t|j_1,k_1) = P[L(t) = j_2, \xi(t) = k_2, T_1 > t|L(0) = j_1, \xi(0) = k_1] \quad ; \quad (j_1,k_1),(j_2,k_2) \in E_1$$

This function has the following probabilistic interpretation:

Given that the inventory level is j_1 and that the environment is in state k_1, at time $t = 0$, the probability is $\gamma(j_2,k_2,t|j_1,k_1)$ that the next change of state of the environment takes place after a time t and that the levels of the inventory and the environment at time t are j_2 and k_2 respectively.

Since T_1 corresponds to the epoch of change of the state of the environment from the state of the process, the following conditions apply:

(i) $\gamma(j_2,k_2,t|j_1,k_1)=0$ for $k_1 \neq k_2$

(ii) $\gamma(j_2,1,t|j_1,1)=W(j_2,t;j_1,1)$

(iii) $\gamma(j_2,0,t|j_1,0)=W(j_2,t;j_1,0)$ (2.14)

Conditioning on the random variable T_1,

$$\beta(j_2,k_2,t|j_1,k_1)=\gamma(j_2,k_2,t|j_1,k_1)+\sum_{(j_3,k_3)\in E_1}\int_0^t Q(j_3,k_3,du|j_1,k_1)\quad \beta(j_2,k_2,t-u|j_3,k_3)$$

(2.15)

The solution of (2.15) is given by

$$\beta(j_2,k_2,t\ |j_1,k_1)=\sum_{(j_3,k_3)\in E_1}\int_0^t R(j_3,k_3,du|j_1,k_1)\quad \gamma(j_2,k_2,t-u|j_3,k_3) \tag{2.16}$$

Using the function $\beta(j_2,k_2,t|j_1,k_1)$,

$$P(j,t\ |i,k_1)=\sum_{k=0}^{1}\beta(j,k,t|i,k_1) \tag{2.17}$$

2.5 LIMITING DISTRIBUTION OF THE INVENTORY LEVEL

Considering the Markov chain $\{L_n,\xi_n\}$ and defining

$$A=\lim_{t\to\infty}A(t) \qquad ;B=\lim_{t\to\infty}B(t),$$

the one-step transition probability matrix of the Markov chain $\{L_n,\xi_n\}$ is given by

$$\boldsymbol{Q} = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \tag{2.18}$$

The structure of $\boldsymbol{Q}$ implies that the chain is periodic with period 2. Since every element of A is greater than 0, the chain $\{L_n, \xi_n\}$ is irreducible (Feller [1965]).

Consequently, the stationary distribution of $\{L_n, \xi_n\}$ exists. Let $\tilde{\pi} = (\tilde{\pi}_1, \tilde{\pi}_2)$ be the stationary distribution where

$$\tilde{\pi}_1 = \left(\pi(0,0),\, \pi(1,0),\, \ldots,\, \pi(S,0)\right)$$

and $$\tilde{\pi}_2 = \left(\pi(0,1),\, \pi(1,1),\, \ldots,\, \pi(S,1)\right),$$

such that $\tilde{\pi}_1 AB = \tilde{\pi}_1$ and $\tilde{\pi}_2 = \tilde{\pi}_1 A$.

Since (L_n, ξ_n) has a stationary distribution, the semi-regenerative process $(L(t), \xi(t))$ also has a stationary distribution defined by

$$\phi(j_2, k_2) = \lim_{t \to \infty} \beta(j_2, k_2, t \mid j_1, k_1) \tag{2.19}$$

where (j_1, k_1) and $(j_2, k_2) \in E_1$.

To obtain $\phi(j_2, k_2)$ consider the mean sojourn time of the Markov renewal process (Z, T) in a state (j_1, k_1) of E_1 defined by

$$m(j_1, k_1) = E[T_{n+1} - T_n \mid Z_n = (j_1, k_1)] \tag{2.20}$$

From the definition of $Q(j_2, k_2, t | j_1, k_1)$, Cinlar [1975] indicates that

$$m(j_1, k_1) = \int_0^\infty [1 - \sum_{(j_2,k_2)\in E_1} Q(j_2, k_2, t | j_1, k_1)]dt \tag{2.21}$$

By applying a theorem on semi-regenerative process,

$$\phi(j_2, k_2) = \frac{\sum_{(j_1,k_1)\in E_1} \pi(j_1, k_1) \int_0^\infty \gamma(j_2, k_2, t | j_1, k_1)dt}{\tilde{\pi}.\tilde{m}} \tag{2.22}$$

where $\tilde{m} = (m(0,0),\ m(1,0), \ldots,\ m(S,0),\ m(0,1), \ldots,\ m(S,1))$

and $\tilde{\pi}.\tilde{m} = \sum_{(j_1,k_1)\in E_1} \pi(j_1, k_1) m(j_1, k_1)$ (2.23)

The stationary distribution of $L(t)$ can be obtained, defined by

$$\theta(j) = \lim_{t\to\infty} P[L(t) = j | L(0) = i, \xi(0) = k] \tag{2.24}$$

where $0 \le j \le S$, $(i,k) \in E_1$.

Note that

$$\begin{aligned}\theta(j) &= \lim_{t\to\infty} \sum_{k_2=0}^{1} \beta(j, k_2, t | i, k_1) \\ &= \sum_{k_2=0}^{1} \phi(j, k_2)\end{aligned} \tag{2.25}$$

2.6 MEASURES OF SYSTEM PERFORMANCE

2.6.1 Mean Number of Replenishments

Let $h_r(t)$ be the first order product density of the point process of r-events. Since at the epoch of an r-event the environment may be either in state 0 or 1,

$$h_r(t) = \sum_{k=0}^{1} \beta(0,k,t | j_1,k_1)\lambda_k$$

where $(j_1,k_1) \in E_1$

The mean number of replenishments in $(0,t]$ is given by

$$E[N(r,t)] = \int_0^t h_r(u)du$$

Hence the mean-stationary rate of replenishments is

$$\begin{aligned} E(r) &= \lim_{t\to\infty} \frac{E[N(r,t)]}{t} \\ &= \lim_{t\to\infty} h_r(t) \\ &= \sum_{k=0}^{1} \phi(0,k)\lambda_k \end{aligned}$$

2.6.2 Mean Number of Demands

Since replenishment is instantaneous, any demand that occurs is satisfied. Define

$$h_d(t) = \lim_{\Delta\to 0} \frac{P[a\ d-event\ in\ (t,t+\Delta) | Z_0 = (j_1,k_1)]}{\Delta} \ ;\ (j_1,k_1) \in E_1 .$$

Then $h_d(t)$ is the first-order product density of the d-events and

$$h_d(t) = \sum_{j_2=0}^{S} \sum_{k_2=0}^{1} \beta(j_2, k_2, t | j_1, k_1)$$

The mean number of demands occurring in $(0,t]$ is given by

$$E[N(d,t)] = \int_0^t h_d(u)du \, .$$

Consequently, the mean stationary rate of demands is given by

$$E(d) = \sum_{j=0}^{S} \sum_{k=0}^{1} \phi(j,k)\lambda_k \, .$$

Let $h_d^k(t)$ be the product density of d-events occurring while the environment is in state k, $k = 0, 1$. Then,

$$h_d^k(t) = \lim_{\Delta \to 0} \frac{P[N(d,t+\Delta) - N(d,t) = 1, \xi(t) = k | Z_0 = (j,k)]}{\Delta} \, ; (j,k) \in E_1$$

and $\quad h_d^k(t) = \sum_{j=0}^{S} \beta(j,k,t | j_1.k_1); \quad k = 0, 1.$

Consequently, $h_d(t) = h_d^0(t) + h_d^1(t)$

2.6.3 Mean Number of Perished Items

For the first-order product density $h_\pi(t)$ of the point process of π-events,

$$h_\pi(t) = \sum_{j=0}^{S}\sum_{k=0}^{1}\beta(j,k,t|j_1,k_1)\,j\mu_k$$

where $(j_1,k_1)\in E_1$.

The mean number of items that perish in the interval $(0,t]$ is then given by

$$E[N(\pi,t)] = \int_0^t h_\pi(u)du$$

and the mean-stationary rate of items that perish is

$$\begin{aligned}
E(\pi) &= \lim_{t\to\infty}\frac{E[N(\pi,t)]}{t} \\
&= \lim_{t\to\infty} h_\pi(t) \\
&= \sum_{j=0}^{S}\sum_{k=0}^{1}\phi(j,k)\,j\mu_k
\end{aligned}$$

2.7 COST ANALYSIS

The profit per unit time can be formulated as follows:

$$P_f = \sum_{k=0}^{1} E(d_k)c_{d_k} - [E(r)(S+1)c_b + \sum_{k=0}^{1} E(r_k)CR_k] - \sum_{j=0}^{s} \theta(j)c_j - E(\pi)c_\pi$$

where

c_{d_k} : Selling price of one item when the environment is in state k, k = 0, 1

c_b : Buying cost of one item

CR_k : Cost of replenishment when the environment is in state k, k = 0, 1

c_j : Holding cost when the inventory level is j

c_π : Salvage cost of one perished item

P_f : Profit per unit time in the long run

2.8 TOTAL SALE PROCEEDS

Assuming the following:

(i) The demand rate is a constant and is the same for all time $t > 0$.

(ii) The selling price of one item is c_k when the environment is in state k, k = 0, 1.

For the stochastic process $\beta(t)$ defined by

$$\beta(t) = \int_0^t \xi(u)du$$

Then $\beta(t)$ represents the total time in $(0,t)$ during which the environment is in state 1.

Consequently, the total time in $(0,t)$ during which the environment is in state 0 is $t-\beta(t)$. Tackacs (1957a,b) has investigated and obtained the distribution function of $\beta(t)$ as

$$\Omega(t,x)=\sum_{n=0}^{\infty}H^{(n)}(x)[G^{(n)}(t-x)-G^{(n+1)}(t-x)]$$

where $G(x)=P[X_n\le x]$,

$H(x)=P[Y_n\le x]$,

and $H^{(0)}(x)=\begin{cases}1\ if\ \ x\ge 0\\ 0\ if\ \ x<0\end{cases}$

$G^{(0)}(x)=1$

Since $N(d,t)$ represents the total number of demands which have occurred up to time t, the total sale proceeds to time t is given by

$$S(t)=c_0+c_0N(d,t-\beta(t))+c_1N(d,\beta(t)) \tag{2.26}$$

Assuming that $N(d,t)$ is a stationary renewal process, equation (8.1) can be expressed as

$$S(t)=c_0+c_0N(d,t)+(c_1-c_0)N(d,\beta(t)) \tag{2.27}$$

For simplicity, assume that $c_1=mc_0$, where m is a fixed positive integer. Setting

$$\tilde{S}(t)=\frac{[S(t)-c_0]}{c_0}$$

The equation (2.27) simplifies as

$$\tilde{S}(t) = N(d,t) + (m-1)N(d,\beta(t)) \tag{2.28}$$

In order to determine the probability distribution of $S(t)$, the joint probability distribution of $N(d,t)$ and $N(d,\beta(t))$ is required.

Define $\alpha(i, j, t) = P[N(d,t) = 1; N(d,\beta(t)) = j]$

Since $N(d,\beta(t))$ and $N(d,t-\beta(t))$ are stochastically independent,

$$\alpha(i, j, t) \;=\; P[N(d,\beta(t)) = j, N(d,t-\beta(t)) = i - j]$$

$$= e^{-\lambda t} \;\frac{\lambda^i}{i!}\binom{i}{j}\int_0^t u^j (t-u)^{i-j} d_u \;\Omega(t,u) \tag{2.29}$$

For any non-negative integer k, the event $(\tilde{S}(t) = k)$ occurs if and only if one of the following events occurs:

$$[N(d,\beta(t)), N(d,t) = k - (m-1)j]; \quad j = 0, 1, 2, \ldots, r$$

where r is the largest integer less than or equal to $\left[\dfrac{k}{m}\right]$.

Consequently,

$$P[\tilde{S}(t)=k]=\sum_{j=0}^{r}P[N(d,\beta(t))=j,N(d,t)=k-(m-1)j]$$

$$=\sum_{j=0}^{r}\alpha(k-(m-1)j,j,t) \tag{2.30}$$

Further specializing to the case where

$$G(x)=\begin{cases}1-e^{-ax} & if\ \ x>0\\ 0 & otherwise\end{cases}$$

$$H(x-k)=\begin{cases}1 & if\ \ x>k\\ 0 & otherwise\end{cases}$$

The following results from the work of Tackacs (1957a,b):

$$\Omega(t,x)=\sum_{n=0}^{\infty}e^{-a(t-x)}\frac{[a(t-x)]^{n}}{n!}U(x-nk)$$

where $U(.)$ stands for the Heaviside function. Now, for this particular case, the pdf of $\beta(t)$ is given by

$$\omega(t,x)=\frac{e^{-a(t-x)}[a(t-x)]^{\left[\frac{x}{k}\right]}}{\left[\frac{x}{k}\right]!}\delta\left(x-\left[\frac{x}{k}\right]k\right)+\frac{ae^{-a(t-x)}[a(t-x)]^{\left[\frac{x}{k}\right]}}{\left[\frac{x}{k}\right]!};\quad 0\le x\le t \tag{2.31}$$

Using (2.31) in (2.29) the expression for $\alpha(i,j,t)$ is derived:

$$\alpha(i,j,t)=e^{-\lambda t}\frac{\lambda^{t}}{i!}\binom{i}{j}\int_{0}^{t}x^{j}(t-x)^{i-j}\omega(t,x)dx \tag{2.32}$$

The following cases are applicable:

Case (i) Let $k > t$, then from (2.31)

$$\omega(t,x) = e^{-a(t-x)}\delta(x) + ae^{-a(t-x)}$$

and hence, from (2.32) we get

$$\alpha(i,j,t) = e^{-\lambda t}\frac{\lambda^i}{i!}\binom{i}{j}\int_0^t x^j (t-x)^{i-j} ae^{-a(t-x)} dx \qquad (2.33)$$

Case (ii) Let $k < t$, note that, for some positive integer n, $nk < t \le (n+1)k$ and so,

$$\alpha(i,j,t) = e^{-\lambda t}\frac{\lambda^i}{i!}\binom{i}{j}(I_1 + I_2) \qquad (2.34)$$

where

$$I_1 = a\int_0^k x^j (t-x)^{i-j} e^{-a(t-x)} dx \qquad (2.35)$$

$$I_2 = \sum_{r=1}^{n}[(rk)^j (t-rk)^{i-j} e^{-a(t-rk)} + \sum_{r=1}^{n-1}\int_{rk}^{(r+1)k} x^j (t-x)^{i-j} ae^{-a(t-x)} \frac{[a(t-x)]^r}{r!} dx$$

$$+ \int_{nk}^{t} x^j (t-x)^{i-j} ae^{-a(t-x)} \frac{[a(t-x)]^n}{n!} dx \qquad (2.36)$$

As $\alpha(i,j,t)$ is explicitly known in all the cases, the probability distribution of $\tilde{S}(t)$ is obtained from (2.30).

2.9 THE TOTAL COST OF REPLENISHMENT

Let the cost of replenishment be CR_k when the environment is in state k, $k = 0,1$ and $C(t)$ be the total cost of replenishments up to time t. Proceeding as in Section 2.8,

$$C(t) = CR_0 + CR_0 N(r,t) + (CR_1 - CR_0) N(r, \beta(t)) \tag{2.37}$$

Where $N(r,t)$ represents the number of replenishments made in the interval $(0,t]$.

Setting $\tilde{C}(t) = \dfrac{(C(t) - CR_0)}{CR_0}$ and taking $CR_1 = m\, CR_0$ in (2.37), where m is a positive integer constant,

$$\tilde{C}(t) = N(r,t) + (m-1) N(r, \beta(t)) \tag{2.38}$$

Consequently,

$$P[\tilde{C}(t) = k] = \sum_{j=0}^{n} P[N(r,\beta(t)) = j, N(r,t) = k - (m-1) j] \tag{2.39}$$

where n is the largest integer less than or equal to $\left[\dfrac{k}{m}\right]$.

Since the event $\{N(r,\beta(t)) = j, N(r,t) = k-(m-1)j\}$ is equivalent to the event $\{N(r,\beta(t)) = j, N(r,t-\beta(t)) = k - mj\}$, $N(r,\beta(t))$ and $N(r,t-\beta(t))$ are independent, and that

$$
\begin{aligned}
P[N(r,t) = j] &= P[j(S+1) \leq N(d,t) < (j+1)(S+1)] \\
&= \sum_{i=j(S+1)}^{(j+1)(S+1)-1} P[N(d,t) = i] \\
&= \sum_{i=j(S+1)}^{(j+1)(S+1)-1} \frac{e^{-\lambda t}(\lambda t)^i}{i!} \\
&= \sigma(j,t) \quad ;(say)
\end{aligned}
$$

Equation (2.39) yields explicitly that

$$
P[\tilde{C}(t) = k] = \sum_{j=0}^{n} \int_0^t \sigma(j,u)\sigma(k-mj, t-u) d_u \Omega(t,u)
$$

2.10 NUMERICAL ILLUSTRATION

In this section, numerical examples illustrate the functioning of the inventory system operating in a random environment.

2.10.1 Analysis of Measures of System Performance

First, considering the various measures obtained in Section 6 and 7, their behaviour under the following cases are obtained:

Case (i): λ_0 varies from 10.0 to 200; S = 3, λ_1 = 50.0, μ_0 = 10.0, μ_1 = 20.0, ν_0 = 1.5, ν_1 = 2.5.

Case (ii): λ_1 varies from 50.0 to 250; S = 3, λ_0 = 10.0, μ_0 = 10.0, μ_1 = 20.0, ν_0 = 1.5, ν_1 = 2.5.

Case (iii): μ_0 varies from 10.0 to 20.0; λ_0 = 10.0, λ_1 = 50.0, μ_1 = 20.0, ν_0 = 1.5, ν_1 = 2.5.

Case (iv): μ_0 varies from 10.0 to 20.0; λ_0 = 10.0, λ_1 = 50.0, μ_0 = 10.0, ν_0 = 1.5, ν_1 = 2.5.

The results for each of these cases are given in Tables 2.2 to 2.5. In all the above four cases, the following values is assumed for the costs in order to determine the mean-rate of the total profit (PF):

C_{d0} = 100.0, C_{d1} = 150.0, C_{R0} = 10.0, C_{r1} = 20.0, C_j = 5.0, C_b = 50.0, C_π = 3.0

A consolidated overview of the results are provided in Table 2.1 below:

	Mean Rate of			
	Replenishment (RR)	**Demands (RD)**	**Perished Items (RP)**	**Total Profit (PF)**
λ_0 increases	Increases	Increases		Increases
λ_1 increases	Increases	Increases		Increases
μ_0 increases			Increases	Decreases
μ_1 increases			Increases	Decreases

Table 2.1: Overview of the Analysis of System Performance Measures

2.10.2 Analysis of Probability Distributions

The probability distribution of the total sale proceeds obtained in Section 2.8 is considered and evaluated numerically by assuming the following values for the parameters:

m = 2, k = 10, t = 10

Fixing the demand rate $\lambda = 0.3$, the value of a is increased to obtain the values of the probability $P[S(10) = 10]$ corresponding to the cases $k = 8$ and $k = 20$ (Table 2.6).

Fixing $a = 0.00006$, the demand rate of λ is increased to obtain the values of $P[S(10) = 10]$ corresponding to $k = 8$ and $k = 20$ (Table 2.7).

The time dependent behaviour of $P[S(t) = k]$, in the interval $0 < t < 10$ is also illustrated. For this purpose, $k = 5$, $a = 0.00001$ and $k = 6$ to obtain $P[S(t) = 5]$, $0 < t < 10$ for three cases $\lambda = 0.1$, $\lambda = 0.2$ and $\lambda = 0.3$ (Table 2.8). It is noted that the probability increases as time increases in (0, 10) and that the probability increases as the demand rate λ increases.

Finally, the probability distribution of the total cost of replenishment obtained in Section 2.9 are considered and evaluated numerically by assuming the following values for the parameters:

$$m = 2,\ k = 10,\ t = 10$$

Fixing the demand rate $\lambda = 3.0$, the value of a is increased. Note that the probability $P[C(10) = 10]$ increases for both cases $k = 8$ and $k = 10$ as detailed in Table 2.9.

Fixing $a = 0.00006$ and increasing λ, note that the probability decreases for both cases $k = 8$ and $k = 10$ as per Table 2.10.

The time-dependent behaviour of $P[C(t) = k]$, $0 < t < 10$ is illustrated by assuming $k = 20$, $a = 0.00006$ and considering three cases: $\lambda = 3.0, 3.2, 3.4$ as detailed in Table 2.11.

S =3, λ_1 = 50.0, μ_0 = 10.0, μ_1 = 20.0, ν_0 = 1.5, ν_1 = 2.5

λ_0	**RR**	**RD**	**RP**	**PF**
10.0	6.25000	25.00000	20.62500	2011.25000
20.0	7.81250	31.24999	20.62500	2308.12500
30.0	9.37500	37.50000	20.62500	2605.00000
40.0	10.93750	43.74999	20.62500	2901.87500
50.0	12.50000	49.99999	20.62500	3198.75000
60.0	14.06250	56.24998	20.62499	3495.62500
70.0	15.62499	62.49998	20.62499	3792.50000
80.0	17.18751	68.75002	20.62500	4089.37500
90.0	18.75000	74.99999	20.62500	4386.25100
100.0	20.31250	81.24999	20.62500	4683.12500
110.0	21.87499	87.49996	20.62499	4979.99900
120.0	23.43749	93.74997	20.62499	5276.87400
130.0	25.00000	99.99999	20.62500	5573.75000
140.0	26.56249	106.25000	20.62499	5870.62400
150.0	28.12498	112.49990	20.62499	6167.49700
160.0	29.68750	118.75000	20.62500	6464.37500
170.0	31.24999	124.99990	20.62499	6761.24800
180.0	32.81248	131.24990	20.62499	7058.12300
190.0	34.37499	137.50000	20.62500	7355.00100
200.0	35.93749	143.75000	20.62499	7651.87400

Table 2.2: **Measures of Performance versus Demand Rate varying in environment in State 0**

S = 3, λ_0 = 10.0, μ_0 = 10.0, μ_1 = 20.0, ν_0 = 1.5, ν_1 = 2.5

λ_1	**RR**	**RD**	**RP**	**PF**
50.0	6.25000	25.00000	20.62500	2011.25000
60.0	7.18750	28.75000	20.62500	2367.50000
70.0	8.12500	32.50000	20.62500	2723.75000
80.0	9.06250	36.25002	20.62501	3080.00200
90.0	9.99999	39.99997	20.62498	3436.24800
100.0	10.93749	43.74998	20.62499	3792.49900
110.0	11.87500	47.49999	20.62500	4148.75000
120.0	12.81250	51.25000	20.62500	4505.00000
130.0	13.74999	54.99997	20.62499	4861.24900
140.0	14.68750	58.75000	20.62500	5217.50000
150.0	15.62499	62.49998	20.62499	5573.75000
160.0	16.56249	66.24998	20.62499	5930.00000
170.0	17.50001	70.00002	20.62501	6286.25300
180.0	18.43749	73.74995	20.62498	6642.49700
190.0	19.37498	77.49995	20.62499	6998.74800
200.0	20.31248	81.24990	20.62498	7354.99200
210.0	21.25000	84.99998	20.62500	7711.25000
220.0	22.18750	88.75002	20.62500	8067.50200
230.0	23.12497	92.49989	20.62498	8423.74300
240.0	24.06249	96.24995	20.62499	8779.99700
250.0	24.99999	99.99998	20.62500	9136.24900

Table 2.3: Measures of Performance versus Demand Rate varying in environment in State 1

$S = 3$, $\lambda_0 = 10.0$, $\lambda_1 = 50.0$, $\mu_1 = 20.0$, $\nu_0 = 1.5$, $\nu_2 = 2.5$

λ_0	**RR**	**RD**	**RP**	**PF**
10.0	6.25000	25.00000	20.62500	2011.25000
10.5	6.25000	25.00000	21.09375	2009.84400
11.0	6.25000	25.00000	21.56250	2008.43800
11.5	6.25000	25.00000	22.03125	2007.03200
12.0	6.25000	25.00000	22.50000	2005.62500
12.5	6.25000	25.00000	22.96875	2004.21900
13.0	6.25000	25.00000	23.43750	2002.81300
13.5	6.25000	25.00000	23.90624	2001.40600
14.0	6.25000	25.00000	24.37500	2000.00000
14.5	6.25000	25.00000	24.84375	1998.59400
15.0	6.25000	25.00000	25.31250	1997.18800
15.5	6.25000	25.00000	25.78125	1995.78100
16.0	6.25000	25.00000	26.25000	1994.37500
16.5	6.25000	25.00000	26.71875	1992.96900
17.0	6.25000	25.00000	27.18750	1991.56300
17.5	6.25000	25.00000	27.65625	1990.15600
18.0	6.25000	25.00000	28.12500	1988.75000
18.5	6.25000	25.00000	28.59375	1987.34400
19.0	6.25000	25.00000	29.06250	1985.34400
19.5	6.25000	25.00000	29.53125	1984.53100
20.0	6.25000	25.00000	30.00000	1983.12500

Table 2.4: **Measures of Performance versus Demand Rate varying in environment in State 0**

S = 3, λ_0 = 10.0, λ_1 = 50.0, μ_0 = 10.0, ν_0 = 1.5, ν_1 = 2.5

λ_0	**RR**	**RD**	**RP**	**PF**
10.0	6.25000	25.00001	15.00001	2028.12600
10.5	6.25000	25.00000	15.28125	2027.28100
11.0	6.25000	24.99999	15.56250	2026.43700
11.5	6.25000	24.99999	15.84375	2025.59400
12.0	6.25000	25.00000	16.12500	2024.75000
12.5	6.25000	25.00000	16.40625	2023.90600
13.0	6.25000	25.00001	16.68751	2023.06300
13.5	6.25000	24.99999	16.96875	2022.21900
14.0	6.25000	25.00000	17.25000	2021.37500
14.5	6.25000	25.00000	17.53125	2020.53200
15.0	6.25000	25.00001	17.81250	2019.68800
15.5	6.25000	25.00001	18.09375	2018.84400
16.0	6.25000	25.00000	18.37500	2018.00000
16.5	6.25000	25.00000	18.65625	2017.15600
17.0	6.25000	25.00000	18.93750	2016.31300
17.5	6.25000	25.00000	19.21875	2015.46900
18.0	6.25000	25.00000	19.50001	2014.62600
18.5	6.25000	25.00000	19.78125	2013.78100
19.0	6.25000	25.00000	20.06250	2012.93800
19.5	6.25000	25.00000	20.34375	2012.09400
20.0	6.25000	25.00000	20.62500	2011.25000

Table 2.5: **Measures of Performance versus Perishing Rate varying in environment in State 1**

$\lambda = 0.3$

	P[S(10) = 10]	
a	k = 8	k = 20
0.00006	0.0460570	0.0000166
0.00011	0.0460597	0.0000304
0.00016	0.0460625	0.0000442
0.00021	0.0460652	0.0000579
0.00026	0.0460679	0.0000717
0.00031	0.0460707	0.0000855
0.00036	0.0460734	0.0000993
0.00041	0.0460761	0.0001131
0.00046	0.0460789	0.0001268
0.00051	0.0460816	0.0001406

Table 2.6: P[S(10) = 10] versus Environment Rate

a = 0.00006

	P[S(10) = 10]	
a	k = 8	k = 10
0.25000	0.0290197	0.0000099
0.26000	0.0322745	0.0000112
0.27000	0.0356271	0.0000124
0.28000	0.0390563	0.0000138
0.29000	0.0425402	0.0000151
0.30000	0.0460570	0.0000166
0.31000	0.0495848	0.0000180
0.32000	0.0531022	0.0000195
0.33000	0.0565883	0.0000210
0.34000	0.0600231	0.0000225
0.35000	0.0633876	0.0000241
0.36000	0.0666638	0.0000256

Table 2.7: P[S(10) = 10] versus Demand Rate

k = 6 and a = 0.00001

t	P[S(t) = 5]		
	λ = 0.1	λ = 0.2	λ = 0.3
0.50	0.000000000	0.000000000	0.000000001
1.00	0.000000000	0.000000003	0.000000009
1.50	0.000000002	0.000000013	0.000000040
2.00	0.000000006	0.000000039	0.000000112
2.50	0.000000013	0.000000088	0.000000243
3.00	0.000000027	0.000000168	0.000000447
3.50	0.000000047	0.000000287	0.000000736
4.00	0.000000078	0.000000453	0.000001116
4.50	0.000000119	0.000000670	0.000001590
5.00	0.000000175	0.000000945	0.000002155
5.50	0.000000246	0.000001279	0.000002805
6.00	0.000000336	0.000001678	0.000003548
6.50	0.004705349	0.019678810	0.034720840
7.00	0.008988675	0.035907300	0.060612490
8.00	0.012914050	0.049407630	0.079737830
8.50	0.016536890	0.060749660	0.094119200
9.00	0.019905290	0.070382460	0.104909600
9.50	0.026039840	0.085853610	0.119100000
10.00	0.028873060	0.092181770	0.123630600

Table 2.8: **P[S(t) = 5] versus Time t**

λ = 3.0

a	P[C(10) = 10]	
	k = 8	k = 20
0.00006	0.0787214	0.0000745
0.00011	0.0787648	0.0001366
0.00016	0.0788081	0.0001986
0.00021	0.0788514	0.0002606
0.00026	0.0788947	0.0003226
0.00031	0.0789380	0.0003845
0.00036	0.0789812	0.0004465
0.00041	0.0790244	0.0005083
0.00046	0.0790676	0.0005702
0.00051	0.0791108	0.0006320

Table 2.9: **P[C(10) = 10] versus Environment Rate**

a = 0.00006

λ	P[C(10) = 10]	
	k = 8	k = 20
3.00000	0.0787214	0.0000745
3.10000	0.0664267	0.0000735
3.20000	0.0548080	0.0000720
3.30000	0.0442755	0.0000702
3.40000	0.0350609	0.0000681
3.50000	0.0272459	0.0000658
3.60000	0.0207988	0.0000632
3.70000	0.0156118	0.0000604
3.80000	0.0115328	0.0000573
3.90000	0.0083918	0.0000540
4.00000	0.0060197	0.0000504
4.10000	0.0042604	0.0000467
4.20000	0.0029774	0.0000428
4.30000	0.0020564	0.0000389
4.40000	0.0014049	0.0000350
4.50000	0.0009502	0.0000311

Table 2.10: **P[C(10) = 10] versus Demand Rate**

k = 20 and a = 0.00006

	P[C(10) = 10]		
t	λ = 3.0	λ = 3.2	λ = 3.4
0.50000	0.0000000	0.0000000	0.0000000
1.50000	0.0000000	0.0000000	0.0000000
2.50000	0.0000000	0.0000000	0.0000000
3.50000	0.0000001	0.0000002	0.0000004
4.50000	0.0000016	0.0000026	0.0000041
5.50000	0.0000083	0.0000121	0.0000165
6.50000	0.0000235	0.0000299	0.0000358
7.50000	0.0000432	0.0000492	0.0000533
8.50000	0.0000604	0.0000632	0.0000638
9.50000	0.0000714	0.0000704	0.0000679

Table 2.11: **P[C(10) = 10] versus Time t**

2.11 CONCLUSION

A model of a perishable product inventory system operating in a random environment is studied in this chapter. For the sake of simplicity, the stochastic environment is considered to alternate randomly over time between two states 0 and 1 according to an alternating renewal process. When the environment is in state k, the items in the inventory have a perishing rate μ_k, the demand rate is λ_k and the replenishment cost is CR_k. Assuming instantaneous replenishment at the epoch of the first demand after the stock-out and associating a Markov renewal process with the inventory system, the stationary distribution of the inventory level and the performance of various measures of the system evolution are obtained. Numerical examples illustrated the results obtained.