Appendix A

Examples concerning ergodicity

A.1 On the definition of ergodicity

This section is devoted to the construction of a *-dynamical system \((\mathfrak{A}, \varphi, \tau)\) with the property that if \(\|\tau(A) - A\|_\varphi = 0\), then \(\|A - \alpha\|_\varphi = 0\) for some \(\alpha \in \mathbb{C}\), but for which the fixed points of the operator \(U\) defined in Proposition 2.3.3 in terms of some cyclic representation, form a vector subspace of \(\mathfrak{H}\) with dimension greater than one. This will prove the necessity of a sequence, rather than a single element, in Definition 2.3.2, in order for Proposition 2.3.3 to hold.

First some general considerations. Consider a dense vector subspace \(\mathfrak{G}\) of a Hilbert space \(\mathfrak{H}\), and let \(\mathcal{L}(\mathfrak{H})\) be the bounded linear operators \(\mathfrak{H} \to \mathfrak{H}\). Set

\[ \mathfrak{A} := \{ A|_\mathfrak{G} : A \in \mathcal{L}(\mathfrak{H}), A\mathfrak{G} \subset \mathfrak{G} \text{ and } A^*\mathfrak{G} \subset \mathfrak{G} \} \]

where \(A|_\mathfrak{G}\) denotes the restriction of \(A\) to \(\mathfrak{G}\), then \(\mathfrak{A}\) is clearly a vector subspace of \(\mathcal{L}(\mathfrak{G})\). For any \(A \in \mathfrak{A}\), denote by \(\widetilde{A}\) the (unique) bounded linear extension of \(A\) to \(\mathfrak{H}\). Now define an involution on \(\mathfrak{A}\) by

\[ A^* := \widetilde{A}^*|_\mathfrak{G} \]

for all \(A \in \mathfrak{A}\), then it is easily verified that \(\mathfrak{A}\) becomes a unital *-algebra. (For \(A, B \in \mathfrak{A}\) it is clear that \(AB\) is a bounded linear operator \(\mathfrak{G} \to \mathfrak{G}\) which therefore has the extension \(\widetilde{A}\widetilde{B} \in \mathcal{L}(\mathfrak{H})\) for which \(\widetilde{A}\widetilde{B}\mathfrak{G} \subset \mathfrak{G}\) and \((\widetilde{A}\widetilde{B})^* \mathfrak{G} = \widetilde{B}^*\widetilde{A}^* \mathfrak{G} \subset \mathfrak{G}\) by the definition of \(\mathfrak{A}\). Hence \(AB \in \mathfrak{A}\), which means that \(\mathfrak{A}\) is a subalgebra of \(\mathcal{L}(\mathfrak{G})\). Also, \((AB)^* = (\widetilde{A}\widetilde{B})^*|_\mathfrak{G} = (\widetilde{B}^*\widetilde{A}^*)|_\mathfrak{G} = \widetilde{B}^*\left(\widetilde{A}^*|_\mathfrak{G}\right) = \widetilde{B}^*\widetilde{A}^* = B^*A^*\). Similarly for the other defining properties of an involution.) Note that for \(A \in \mathfrak{A}\) and \(x, y \in \mathfrak{G}\) we have

\[ \langle x, Ay \rangle = \langle x, \widetilde{A}y \rangle = \langle \widetilde{A}^*x, y \rangle = \langle A^*x, y \rangle . \]
A. CONCERNING a norm one vector $n \in \mathcal{B}$ we define a state $\varphi$ on $\mathfrak{A}$ by

$$\varphi(A) = \langle \Omega, A\Omega \rangle.$$ 

Next we construct a cyclic representation of $(\mathfrak{A}, \varphi)$. Let

$$\pi: \mathfrak{A} \to L(\mathcal{G}): A \mapsto A$$

then clearly $\pi$ is linear with $\pi(1) = 1$ and $\pi(AB) = \pi(A)\pi(B)$. Note that for any $x, y \in \mathcal{G}$ we have $(x \otimes y)^* = y \otimes x$, hence $(x \otimes y)\mathcal{G} \subseteq \mathcal{G}$ and $(x \otimes y)^*\mathcal{G} \subseteq \mathcal{G}$, so $(x \otimes y)|_{\mathcal{G}} \in \mathfrak{A}$. Now, $\pi((x \otimes \Omega)|_{\mathcal{G}})\Omega = x (\Omega, \Omega) = x$, hence $\pi(\mathfrak{A})\Omega = \mathcal{G}$. Furthermore,

$$\langle \pi(A)\Omega, \pi(B)\Omega \rangle = \langle A\Omega, B\Omega \rangle = \langle \Omega, A^*B\Omega \rangle = \varphi(A^*B).$$

Thus $(\mathcal{G}, \pi, \Omega)$ is a cyclic representation of $(\mathfrak{A}, \varphi)$.

Suppose we have a unitary operator $U: \mathfrak{H} \to \mathfrak{H}$ such that $U\mathcal{G} = \mathcal{G}$ and $U\Omega = \Omega$. Then $U^*\mathcal{G} = U^{-1}\mathcal{G} = \mathcal{G}$, so $V := U|_{\mathcal{G}} \in \mathfrak{A}$, and $V^* = U^*|_{\mathcal{G}}$. It follows that $VAV^* \in \mathfrak{A}$ for all $A \in \mathfrak{A}$, hence we can define a linear function $\tau: \mathfrak{A} \to \mathfrak{A}$ by

$$\tau(A) = VAV^*.$$ 

Clearly $V^*V = 1 = VV^*$, so $\tau(1) = 1$ and $\varphi(\tau(A)^*\tau(A)) = \varphi(VA^*AV^*) = \langle U^*\Omega, A^*AU^*\Omega \rangle = \varphi(A^*A)$, since $U^*\Omega = U^{-1}\Omega = \Omega$. Therefore $(\mathfrak{A}, \varphi, \tau)$ is a dynamical system. Note that $U|_{\mathcal{G}}$ satisfies equation (3.1) of Section 2.3, namely $U\pi(A)\Omega = UAU\Omega = \tau(A)\Omega = \varphi(\tau(A))\Omega$, hence $U$ is the operator which appears in Proposition 2.3.3.

Assume $\{x \in \mathcal{G}: Ux = x\} = \mathbb{C} \Omega$. If $\|\tau(A) - A\|_\varphi = 0$, it then follows for $x = \iota(A)$, with $\iota$ given by equation (2.1) of Section 2.2, that $\|Ux - x\| = \|\iota(\tau(A) - A)\| = \|\tau(A) - A\|_\varphi = 0$, so $x = \alpha\Omega$ for some $\alpha \in \mathbb{C}$. Therefore $\|A - \alpha\|_\varphi = \|\iota(A - \alpha)\| = \|x - \alpha\Omega\| = 0$.

In other words, assuming that the fixed points of $U$ in $\mathcal{G}$ form the one-dimensional subspace $\mathbb{C}\Omega$, it follows that $\|\tau(A) - A\|_\varphi = 0$ implies that $\|A - \alpha\|_\varphi = 0$ for some $\alpha \in \mathbb{C}$.

It remains to construct an example of a $U$ with all the properties mentioned above, whose fixed point space in $\mathfrak{H}$ has dimension greater than one. The following example was constructed by L. Zsidó:

Let $\mathfrak{H}$ be a separable Hilbert space with an orthonormal basis of the form

$$\{\Omega, y\} \cup \{u_k : k \in \mathbb{Z}\}$$

(that is to say, this is a total orthonormal set in $\mathfrak{H}$) and define the linear operator $U: \mathfrak{H} \to \mathfrak{H}$ via bounded linear extension by

$$U\Omega = \Omega,$$

$$Uy = y,$$

$$Uu_k = u_{k+1}, \quad k \in \mathbb{Z}.$$
ON THE DEFINITION OF ERGODICITY

Clearly $U$ is isometric, while $U\mathcal{H}$ is dense in $\mathcal{H}$, hence $U$ is surjective, since $\mathcal{H}$ is complete. Since $U$ is a surjective isometry, it is unitary. Let $\mathcal{G}$ be the linear span of 

$$\{\Omega\} \cup \{y + u_k : k \in \mathbb{Z}\}.$$ 

Then $U\mathcal{G} = \mathcal{G}$. Furthermore, $\mathcal{G}$ is dense in $\mathcal{H}$. Indeed,

$$\|y - \frac{1}{n} \sum_{k=1}^{n} (y + u_k)\| = \frac{1}{n} \sum_{k=1}^{n} u_k = \frac{1}{\sqrt{n}} \to 0$$

implies that $y \in \mathcal{G}$, the closure of $\mathcal{G}$, hence also

$$u_k = (y + u_k) - y \in \mathcal{G}$$

for $k \in \mathbb{Z}$.

Next we show that

$$\{x \in \mathcal{G} : Ux = x \} = \mathbb{C}\Omega. \quad (1.1)$$

If $\alpha\Omega + \sum_{k=-n}^{n} \beta_k(y + u_k) \in \mathcal{G}$ is left fixed by $U$, then

$$\alpha\Omega + \sum_{k=-n}^{n} \beta_k y + \sum_{k=-n}^{n} \beta_k u_{k+1} = \alpha\Omega + \sum_{k=-n}^{n} \beta_k y + \sum_{k=-n}^{n} \beta_k u_k$$

and it follows that $\beta_{-n} = 0$, and that $\beta_{k+1} = \beta_k$ for $k = -n, \ldots, n - 1$. Thus

$$\alpha\Omega + \sum_{k=-n}^{n} \beta_k(y + u_k) = \alpha\Omega$$

proving (1.1).

On the other hand,

$$\{x \in \mathcal{H} : Ux = x \}$$

clearly contains the two-dimensional vector space spanned by $\Omega$ and $y$. 
A.2 An example of an ergodic system

Here we give the proof that Example 2.5.7 is indeed ergodic. It is clear that \( \tau \) is linear and that \( \tau(1) = 1 \). Let

\[
A = \begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix}
\]

and

\[
B = \begin{pmatrix}
b_{11} & b_{12} \\
b_{21} & b_{22}
\end{pmatrix}
\]

be complex matrices. Then

\[
\tau(A)^* = \begin{pmatrix}
a_{22} & c_{2}a_{21} \\
c_{1}a_{12} & a_{11}
\end{pmatrix}
\]


and

\[
\tau(A)^*\tau(A) = \left( \frac{|a_{22}|^2 + |c_{2}a_{21}|^2}{c_{1}a_{12}a_{22} + a_{11}c_{2}a_{21}} \quad \frac{a_{22}c_{1}a_{12} + c_{2}a_{21}a_{11}}{a_{11}^2 + |a_{12}|^2 + |a_{11}|^2} \right)
\]

while

\[
A^* = \begin{pmatrix}
a_{11} & a_{21} \\
\overline{a_{12}} & \overline{a_{22}}
\end{pmatrix}
\]


and

\[
A^*A = \left( \frac{|a_{11}|^2 + |a_{21}|^2}{a_{12}a_{11} + a_{22}a_{21}} \quad \frac{\overline{a_{11}}a_{12} + \overline{a_{21}}a_{22}}{|a_{12}|^2 + |a_{22}|^2} \right)
\]

so

\[
\varphi(\tau(A)^*\tau(A)) = \frac{1}{2} \left( |a_{22}|^2 + |c_{2}a_{21}|^2 + |c_{1}a_{12}|^2 + |a_{11}|^2 \right)
\]

\[
\leq \frac{1}{2} \left( |a_{22}|^2 + |a_{21}|^2 + |a_{12}|^2 + |a_{11}|^2 \right)
\]

\[
= \varphi(A^*A)
\]

for all \( A \), meaning that \( \mathfrak{A}, \varphi, \tau \) is a *-dynamical system, if and only if \( |c_{1}| \leq 1 \) and \( |c_{2}| \leq 1 \), which is what we will assume.

Next we prove that it is ergodic. For even \( k \geq 0 \) we have

\[
\tau^k(B) = \begin{pmatrix}
b_{11} & c_{1}^{k}b_{12} \\
c_{2}^{k}b_{21} & b_{22}
\end{pmatrix}
\]
and therefore
\[ A^k(B) = \begin{pmatrix} a_{11}b_{11} + a_{12}c_k b_{21} & a_{11}c_k b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}c_k b_{21} & a_{21}c_k b_{12} + a_{22}b_{22} \end{pmatrix} \]

which means
\[ \varphi(A^k(B)) = \frac{1}{2} (a_{11}b_{11} + a_{12}c_k b_{21} + a_{21}c_k b_{12} + a_{22}b_{22}) \]

For odd \( k > 0 \) we then get
\[ \varphi(A^k(B)) = \frac{1}{2} (a_{11}b_{22} + a_{12}c_k b_{31} + a_{21}c_k b_{12} + a_{22}b_{11}) \]

by switching \( b_{11} \) and \( b_{22} \). For \( c \in \mathbb{C} \) it is clear that \( U : \mathbb{C} \to \mathbb{C} : x \mapsto cx \) is a linear operator with \( \|U\| \leq 1 \) if and only if \( |c| \leq 1 \), and for \( c \neq 1 \) the only fixed point of \( U \) is 0, in which case

\[ \frac{1}{n} \sum_{k=0}^{n-1} c^k x = \frac{1}{n} \sum_{k=0}^{n-1} U^k x \to 0 \]

for all \( x \in \mathbb{C} \) as \( n \to \infty \), by the Mean Ergodic Theorem 2.4.1. Hence, for \( c_1 \neq 1 \) and \( c_2 
eq 1 \) it follows that

\[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(A^k(B)) = \varphi(A) \varphi(B) \]

which means that \((\mathcal{A}, \varphi, \tau)\) is ergodic, by Proposition 2.5.6(ii).

On the other hand, if \( c_1 = 1 \) and \( c_2 
eq 1 \), then we have by a similar calculation that

\[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(A^k(B)) = \varphi(A) \varphi(B) + \frac{a_{21} b_{12}}{2} \]

Likewise for the other cases where either \( c_1 \) or \( c_2 \) or both are equal to 1. So \((\mathcal{A}, \varphi, \tau)\) is ergodic if and only if \( c_1 \neq 1 \) and \( c_2 \neq 1 \).
A.2.1 Remark. It is easily seen that \( \tau \) is not a homomorphism, namely

\[
\tau(AB) = \left( \begin{array}{cc}
   a_{21} b_{12} + a_{22} b_{22} & c_1 (a_{11} b_{12} + a_{12} b_{22}) \\
   c_2 (a_{21} b_{11} + a_{22} b_{21}) & a_{11} b_{11} + a_{12} b_{21}
\end{array} \right)
\]

while

\[
\tau(A) \tau(B) = \left( \begin{array}{cc}
   a_{22} b_{22} + c_1 c_2 a_{12} b_{21} & c_1 (a_{23} b_{12} + a_{12} b_{11}) \\
   c_2 (a_{21} b_{22} + a_{11} b_{21}) & c_1 c_2 a_{21} b_{12} + a_{11} b_{11}
\end{array} \right).
\]

In fact, unless \( c_1 c_2 = 1 \), it follows that we don’t even have \( \tau(A^2) = \tau(A)^2 \) for all \( A \). Nor, for that matter, do we have \( \tau(A^*) = \tau(A)^* \) for all \( A \), unless \( c_2 = \overline{c_1} \). This is opposed to the situation for a measure theoretic dynamical system as defined in Section 2.1, where \( \tau \) in equation (1.1) of that section is always a \( * \)-homomorphism. It therefore makes sense not to assume that \( \tau \) is a \( * \)-homomorphism in Definition 2.3.1, since we now have an example where it isn’t.

A.2.2 Remark. We note that \( \varphi(\tau(A)) = \varphi(A) \), i.e. \( \varphi \) is \( \tau \)-invariant, but this fact in itself does not imply that \( \varphi(\tau(A)^* \tau(A)) \leq \varphi(A^* A) \), since \( \tau \) is not a \( * \)-homomorphism, by Remark A.2.1.

Furthermore, \( \varphi(AB) = \varphi(BA) \) for all \( A, B \in \mathfrak{A} \), so \( \varphi \) is commutative (so to speak) even though \( \mathfrak{A} \) is not. Also, while \( \tau(AB) \neq \tau(BA) \) for some \( A, B \in \mathfrak{A} \), we still have \( \varphi(\tau(AB)) = \varphi(AB) = \varphi(BA) = \varphi(\tau(BA)) \), so \( \tau \) is noncommutative (so to speak), but with respect to \( \varphi \) it is again commutative. We conclude that while \( \mathfrak{A} \) is noncommutative, \( (\mathfrak{A}, \varphi, \tau) \) is still in many respects commutative simply because \( \varphi(AB) = \varphi(BA) \) for all \( A \) and \( B \).

A.2.3 Question. Is there an example of an ergodic \( * \)-dynamical system \( (\mathfrak{A}, \varphi, \tau) \) in which \( \varphi(AB) \neq \varphi(BA) \) for some \( A, B \in \mathfrak{A} \)?
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The remark at the end of each reference indicates where in this thesis (apart from the Introduction) the reference appears.