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Contribution to qualitative and constructive treatment of the heat equation with domain singularities

by

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Declaration

I the undersigned, hereby declare that the thesis submitted herewith for the degree Philosophiae Doctor to the University of Pretoria contains my own, independent work and has not been submitted for any degree at any other University.

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Name: Pius Wiysanyuy Molo, Chin

Date: September 2011.

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Dedication

This thesis is dedicated to the almighty God whose protective eyes have allow me to witness this glorious period of my existence. It is also in loving memory of my parents Mr/Mrs Chin Peter. My loving wife Chin-Molo nee Lange Maryann and my three girls, Chin-Molo Kinyuy, Asherinyuy and Bongnyang also form a bigger part of this dedication, as their prayers, love, patience and support led me to overcome this difficult task.

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Abstract

The diffusion phenomenon arises in several real-life situations in engineering, science and technology. Typical examples include heat flow, reaction diffusion, advection/convection-diffusion, chemotaxis, nonlocal mechanisms, models for animal dispersal and the spread of diseases.

Mathematically, diffusion problems are modeled by parabolic equations which are classically studied in the ideal framework of smooth domains. In this thesis, we focus on the model parabolic equation, which is defined by linear heat equation. This equation associated with an initial condition and the Dirichlet boundary condition is considered on a non-smooth domain namely a polygonal domain. In considering such a domain with edge singularities, one main difficulty arises: the variational solution is not smooth and this negatively impacts on the accuracy and performance of any classical numerical method. In this thesis, we clarify as optimally as possible the singular nature of the variational solution. More precisely, we show that the variational solution admits a decomposition into a regular part and a singular part, which captures the rough geometry of the domain. Furthermore, we show that the solution achieves global regularity in weighted Sobolev spaces in which the rough nature of the domain is once again suitably incorporated.

On the constructive side, the global regularity result is used to design and analyze an optimally convergent semi-discrete Finite Element Method (FEM) in which the mesh of the triangulation is adequately refined. Two types of fully discrete mesh refinement (FEM) are constructed. The first method is made of Fourier series discretization in time while the second method is the Non-standard Finite Difference (NSFD) discretization. It is shown that these fully discrete methods converge optimally in relevant norms, with the coupled NSFD and mesh refined FEM presenting the additional advantage of replicating the dynamics of the heat equation in the limit case of space independent equation.

The tool used throughout the thesis is the Laplace transform of vector-valued distributions, a topic on which we elaborated substantially in order to show that any (tempered)

vector-valued distribution can be approximated by a sequence of finite operators. Applied to the heat equation, the Laplace transform leads to a family of Helmholtz equations for a complex parameter $p \in \mathbb{C}$. This raises a second main challenge that we dealt with successfully by using another type of weighted Sobolev spaces. The said challenge is to obtain the solutions of the Helmholtz problems with a priori estimates with the same constant that is independent of the parameter p.

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Key Notation

$\mathbb{N}, \quad \mathbb{Z}, \quad \mathbb{R}, \quad \mathbb{C}$	Sets of natural numbers, (positive and negative) integers, real numbers and complex numbers.
$\mathbb{R}^2_+$	Half-plane $\{x = (x_1, x_2, ..., x_{n-1}) \in \mathbb{R}^2\};\{x_n > 0\}$.
$C^m(\Omega),\ m \geq 0$ integer	Space of m-times continuously differentiable real-valued functions on Ω.
$C^m_b(\Omega)$	Space of bounded continuous functions on Ω.
$\mathcal{D}(\Omega) \equiv C^\infty_0(\Omega)$	Space of infinitely differentiable real-valued functions in Ω with compact support contained in Ω.
$L^p(\Omega), 1 \leq p < \infty, \quad \lVert \cdot \rVert_{0,p},$	Lebesgue space of classes of measurable real-valued functions f on Ω such that $x \rightarrowtail \lvert f(x)\rvert^p$ is integrable on Ω, with its natural norm.
$L^p_{loc}(\Omega), \quad 1 \leq p < \infty$	Space of classes of measurable functions f on Ω such that $x \rightarrowtail \lvert f(x)\rvert^p$ is integrable on any compact set contained in Ω.
$\mathcal{S}(\mathbb{R}^n)$	Space of $C^\infty(\mathbb{R}^n)$ functions f which together with their derivatives are rapidly decreasing at infinity i.e. $\lvert x\rvert^k \lvert D^\alpha f(x)\rvert \to 0$ as $\lvert x\rvert \to \infty, \ \forall\, k \in \mathbb{N},\ \alpha \in \mathbb{N}^n$.
$\mathbb{O}_M(\mathbb{R}^n)$	Space of $C^\infty(\mathbb{R}^n)$ functions f which together with all their derivatives are slowly increasing at infinity $\forall \alpha \in \mathbb{N}^n,\ \exists K \in \mathbb{N}$ such that $\lvert x\rvert^{-k} \lvert D^\alpha f(x)\rvert \to 0$ as $\lvert x\rvert \to \infty$. The subscript M refers to the fact that $\mathbb{O}_M(\mathbb{R}^n)$ is a multiplicator of $\mathcal{S}'(\mathbb{R}^n)$ defined below.
$\mathcal{D}'(\Omega)$	Space of distributions on Ω.
$\mathcal{S}'(\mathbb{R}^n)$	Space of tempered distributions on $\mathbb{R}^n$.
$\mathcal{D}'_+(\mathbb{R})$ or $\mathcal{D}'_-(\mathbb{R})$	Space of distributions on $\mathbb{R}$ with support limited to the left or right.
$L_+(\mathbb{R})$	Space of distributions on $\mathbb{R}$ which have a Laplace transform.

$L^p\left[(-\infty,+\infty);X\right]$	Lebesgue space of functions on $\mathbb{R}$ with values in X, where X is here and after either a Hilbert with inner product $(\cdot,\cdot)_X$ or Banach space with norm $\|\cdot\|_X$, X' being the dual of X.
$\mathcal{D}'(X)\equiv\mathcal{L}(\mathcal{D}(\mathbb{R}),\,X)$ or $\mathcal{S}'(X)=\mathcal{L}(\mathcal{S}(\mathbb{R},\,X)$	Spaces of distributions or tempered distributions on $\mathbb{R}$ with values in X.
$H^m(\Omega),\ \|\cdot\|_{m,\Omega},\ \lvert\cdot\rvert_{m,\Omega}$	Sobolev space of non-negative integer of order m, with its natural Hilbert norm and semi-norm.
$W^{m,p}(\Omega);\ 1\leq p<\infty$ $\|\cdot\|_{m,p,\Omega},\lvert\cdot\rvert_{m,p,\Omega}$	The general Sobolev space of order m, with its natural Banach norm and semi-norm.
$H_0^m(\Omega)$	Closure of $\mathcal{D}(\Omega)$ in $H^m(\Omega)$.
$H^{-m}(\Omega)$	The dual space of $H_0^m(\Omega)$.
$\mathcal{L}(v)(p)\equiv\widehat{v}(p)$	Laplace transform of the function or distribution v at the point $p=\xi+i\eta$.
$\mathcal{F}(w)(\eta)$	Fourier transform of the function or distribution w at the point $\eta\in\mathbb{R}$.
$\mathcal{F}^{-1}(w)(t)$	Inverse Fourier transform of the function or distribution w at the point $t\in\mathbb{R}$.
$\langle\cdot,\cdot\rangle$	Duality pairing between $\mathcal{S}'(\mathbb{R}^n$ and $\mathcal{S}(\mathbb{R}^n)$ or $\mathcal{D}'(\Omega)$ and $\mathcal{D}(\Omega)$.
$v*_t w$	The convolution product of v and w over the argument t.
I_T	Set $I_T:=\left\{\xi\in\mathbb{R}:e^{-\xi t}T\in\mathcal{S}'(\mathbb{R})\right\}$ for $T\in\mathcal{D}'(\mathbb{R}_t)$ where $\mathbb{R}_t$ means that the distributions are taken with argument t.
$H^2(O)$ and $H^2\left[O,X\right]$	Hardy-Lebesgue scalar and vector-valued spaces.
$E\otimes Y$	Tensor product of the spaces E and Y.
$\mathcal{D}_K(\mathbb{R})$,	The subspace of $\mathcal{D}_K(\mathbb{R})$ consisting of functions with compact support in K.
$(P_{K,m})\ m\geq 1$	A sequence of semi-norms on $\mathcal{D}_K(\mathbb{R})$ defined by $P_{K,m}(\rho)=\sup_{x\in K}\left\lvert\frac{d^m\rho(x)}{dx^m}\right\rvert$
$V(m,\epsilon)$	A fundamental system of neighborhoods of the origin 0 for the topology of $\mathcal{D}_K(\mathbb{R})$.
$V(\{m_j\},\{\epsilon_j\})$	A fundamental system of neighborhoods of the origin 0 for the topology of $\mathcal{D}(\mathbb{R})$, where the sequences $\{m_j\}$ and $\{\epsilon_j\}$ vary arbitrarily.
$N(\{m_j\},\{\epsilon_j\})$	A family of semi-norms that generates the topology of $\mathcal{D}(\mathbb{R})$.

σ	The collection of all bounded subsets $\mathbb{A}$ of $\mathcal{D}(\mathbb{R})$.
$\mathcal{W}_I = \{q_\alpha\}_{\alpha \in I}$	Family of semi-norms that generate the topology of a locally convex topological vector space Y.
$q_{\alpha,A}$	Semi-norm defined on $\mathcal{L}\left(\mathcal{D}(\mathbb{R}), Y\right))$ with $\alpha \in I$.
$\mathcal{W}_{I,\sigma} = \{q_{\alpha,\mathbb{A}}\}_{q_\alpha \in \mathcal{W}_I,\ \mathbb{A} \in \sigma}$	A family of semi-norms $q_{\alpha,A}$ that generate the σ-topology $\mathcal{L}\left(\mathcal{D}(\mathbb{R}), Y\right))$ with the topology of uniform convergence on bounded subsets.
$\mathcal{L}_\sigma\left(\mathcal{D}(\mathbb{R}), Y\right))$	The space $\mathcal{L}(\mathcal{D}(\mathbb{R}))$ equipped with the σ-topology.
σ_f	The collection of finite union of bounded set σ.
$\mathbb{B} = \{V(A, M), \mathbb{A} \in \sigma_f\}$	A fundamental system of neighborhoods of the origin 0 for the σ-topology of $\mathcal{L}\left(\mathcal{D}(\mathbb{R}), Y\right)$.
$N_A(\{m_j\}, \{\epsilon_j\})$	A family of semi-norms that generate the σ-topology $\mathcal{L}\left(\mathcal{D}(\mathbb{R}), \mathcal{D}(\mathbb{R})\right)$.
V_j	The rectangle $[-\alpha, \alpha] \times [-\beta, \beta]$ in a new co-ordinate system $(x = x_{1,j}, x_{2,j})$.
V_j^+	The set $\{(x_1, x_2) \in \Omega : \ -\beta < x_2 < \varphi_j(x_1), \ \ -\alpha < x_1 < \alpha\}$.
V_j^-	The set $\{(x_1, x_2) \in \Omega : \ \beta > x_2 > \varphi_j(x_1), \ -\alpha < x_1 < \alpha\}$.
V_j^0	The set $\{(x_1, x_2) \in \Gamma : x_2 = \varphi_j(x_1), \ \ -\alpha < x_1 < \alpha\}$.
Q	The unit square described by $\{(y_1, y_2) : \lvert y_1\rvert < 1, \lvert y_2\rvert < 1\}$.
Q_+	Positive half of the unit square i.e the set consisting of $(y_1, y_2) \in Q$ such that $y_2 > 0$.
Q_-	Negative half of the unit square i.e the set consisting of $(y_1, y_2) \in Q$ such that $y_2 < 0$.
Q_0	Intersection of the unit square Q with the horizontal line $y_2 = 0$.
G	A sector described in polar co-ordinates (r, θ) centered at a vertex of Γ the origin of the plane such that $G = \{(r\cos\theta, r\sin\theta) : r > 0, \ 0 < \theta < \omega\}$.

$P_2^k(G)$	Kondratiev weighted Sobolev space of all distributions v in G such that $r^{\|\alpha\|-k} D^\alpha v \in L^2(G)\ \forall\ \|\alpha\| \leq k$ where k is a non-negative integer with its natural norm $\\| \cdot \\|_{P_2^k(G)}$.
$H^{2,\beta}(\Omega)$	Weighted Sobolev space of all distributions $w \in H^1(\Omega)$ such that $r^\beta D^\alpha w \in L^2(\Omega) \quad \forall\ \alpha$ such that $\|\alpha\| = 2$ with its natural norm $\\| \cdot \\|_{H^{2,\beta}(\Omega)}$.
$\widetilde{H}^m\left[(0,+\infty); L^2(\Omega)\right]$	Space of functions $v \in H^m\left[(0,+\infty); L^2(\Omega)\right]$ such that the extension $\widetilde{v}$ by zero outside $(0,+\infty)$ belong to $H^m\left[(-\infty,+\infty);\ L^2(\Omega)\right]$ with its natural norm $\\| \cdot \\|_{\widetilde{H}^m[(0,+\infty);\ L^2(\Omega)]}$.

Chapter 1

Introduction

Diffusion and heat flow processes occur extensively in science, engineering and in fact in real life situations. In the linear case, these processes are mathematically modeled by parabolic partial differential equations of the form

$$\frac{\partial u}{\partial t} - Lu = f \ \text{ in } \ \Omega \times (0, +\infty) \tag{1.0.1}$$

where

- $L = \sum_{|\alpha| \leq 2m} a_\alpha D^\alpha$ is a strongly elliptic operator of order $2m$ with constant coefficients,
- Ω is a domain of $\mathbb{R}^n$, with boundary $\partial\Omega \equiv \Gamma$.
- f is a given real-valued function.

The parabolic equation (1.0.1) coupled with suitable boundary and initial conditions has been extensively studied in the literature under some smoothness assumptions on the domain Ω. Following Lions and Magenes [38], the most popular assumption is to consider Ω to be a bounded open set with boundary Γ being a C^∞-manifold of dimension $n-1$, the set Ω being locally located at one side of Γ. In other words, $\bar{\Omega}$ is a compact manifold with boundary Γ of class C^∞.

Under this smoothness assumption, the famous qualitative result by Agmon, Douglis and Nirenberg [2] regarding elliptic problems can be stated as follows:
Let u in the Sobolev space $H_0^m(\Omega)$ be such that $Lu \in L^2(\Omega)$. Then u is optimally regular in the sense that

$$u \in H^{2m}(\Omega) \ \text{ and } \ \|u\|_{H^{2m}(\Omega)} \leq C\|Lu\|_{L^2(\Omega)}$$

for some constant $C > 0$ which does not depend on u.
In this smooth framework, similar results for parabolic and hyperbolic problems as well as further contributions to elliptic problems can be found in [38] and [39].

The qualitative analysis in the more difficult case when the domain Ω is non-smooth was considered relatively later. In this regard, the historical reference is Kondratiev [36] who investigated the singular behavior of solutions of elliptic equations in domains with conical and angular points. Since this seminal contribution of Kondratiev, there has been a surge of works on elliptic problems in non-smooth domains ranging from the case of operators of mathematical physics in simple two dimensional geometry (see [29], [31]) to more complicated cases that involve both conical and edge singularities (see [19], [30], [44], [49]). The specific two dimensional case of the parabolic problem (1.0.1) is investigated in the thesis [46]. Our work is mostly based on this thesis [46]. Given the level of generalization and complexity in [46], the purpose of our thesis is:

- To analyze and better understand the results obtained.
- To obtain results that are as explicit as possible;
- To visualize the impact of the rough geometry Ω in the result;
- To enrich and complete the theoretical study of [46] with reliable numerical methods in which the singularities of the continuous problem are relatively easily incorporated.

To achieve the above objectives, the setting of this thesis is made explicit as follows:

- The domain Ω is a polygon ($n = 2$)
- The operator L is taken to be

$$Lu = -\Delta u + \lambda u, \quad \lambda \geq 0$$

where

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$$

is the Laplace operator.

In other words, we are dealing with the following boundary value problem for the two dimensional heat/diffusion operator on a polygon:

$$\frac{\partial u}{\partial t} - \Delta u + \lambda u = f \;\text{ in }\; \Omega \times (0, +\infty)$$

$$u(x, 0) = 0,\; x \in \Omega$$

$$u(x, t) = 0, \;\; (x, t) \in \partial\Omega \times (0, +\infty).$$

There exist several methods for solving evolution problems (see for example [21]). In this work, we mainly employ the Laplace transform for the analytical part, whereas the finite element method (in space variable) coupled with the finite difference method or Fourier series method (in time variable) is used for the constructive part.

The use of the Laplace transform reduces the heat equation to a family of Helmholtz equations for a complex parameter $p \in \mathbb{C}$. The main challenge is to obtain for the solutions of the Helmholtz problems, a priori estimates with the same constant that is independent of the parameter p so that the inverse Laplace transform (i.e. the Paley-Wiener Theorem) is applicable to obtain solution of the heat equation in suitable function spaces. More precisely, our contribution can be outlined as follows:

1. We provide a comprehensive study of Laurent Schwartz extension of the Laplace transform to vector-valued distributions, which constitute a suitable framework for the heat equation;

2. We show that the family of solutions of the resulting Helmholtz equations satisfy the following properties:

   (a) the solutions belong to appropriate weighted Sobolev spaces and they depend continuously upon the family of the right hand side with the same constant that is independent of p;

   (b) the solutions admit decompositions into regular and singular parts, where the regular part (in usual Sobolev spaces) and the coefficients of the singular parts depend continuously upon the family of the right hand sides with independent constants;

3. We deduce from (2) above global regularity and singular decomposition results for the heat equation.

4. We design an optimally convergent mesh refinement finite element method for the Helmholtz equation, as a consequence of the regularity in (2) above.

5. We present two discrete methods for the heat equation. Firstly, we couple the Fourier series method (in the time variable) with the mesh refinement finite element method (FEM) (in the space variable). Secondly, we use the Non-standard finite difference (NSFD)method (in the time variable) in conjunction with the mesh refinement FEM (in the space variable).

   The idea of using the (NSFD) method for such problems is new. NSFD techniques introduced by Mickens [52] more than two decades ago have laid the foundation for designing methods that preserve the dynamics of the continuous differential models. In our context, the NSFD-FEM we obtained preserves some intrinsic properties of the solution of the heat equation.

The results of this thesis are published in the papers [14] and [13]. In view of our focus to better understand the complex issue of singularities, we deliberately spend a lot of time on some crucial details. This contributes to give a self-contained flavor to the thesis, which is essential given the amount of tools and deep concepts from various areas that are needed in this work. This also explains why despite the title of the thesis on the heat equation, much time and space are devoted to the Helmholtz problem, which is the backbone of the analysis of the heat equation.

As a matter of principle comments as to how our thesis fits in the literature are generally made throughout the text next to where the results are stated and proved. See for example Remark 4.3.3 regarding the literature on singularities.

We outline now chapter by chapter the content of the thesis. Chapter 2 is devoted to some basic tools mostly related to function spaces (e.g Sobolev spaces, etc) we need. A key aspect of this chapter is the analysis of the Laplace transform of vector-valued distributions, which requires from us to elaborate substantially on Laurent Schwartz's canonical topology of the space of test functions $\mathcal{D}$ in order to prove the density of the space of finite rank distributions into the space of vector-valued distributions [61].

Chapter 3 and 4 deal with the quantitative and qualitative analysis of the Dirichlet problem for the Helmholtz operator involving a parameter $p \in \mathbb{C}$. The quantitative analysis amounts to the well-posedness (with constant independent of p) of the problem in appropriate Sobolev spaces. The qualitative analysis takes care of two aspects. Firstly, in Chapter 3, we deal with the case when the domain is smooth and the Agmon, Douglis and Nirenberg [2] regularity results are presented. Secondly, in Chapter 4 when the domain is a polygon, the decomposition of solutions into regular and singular parts is investigated and this is exploited to establish the global regularity of the solutions into a weighted Sobolev space in such a way that the solutions depend continuously on the data with a constant independent of the parameter $p \in \mathbb{C}$.

The uniform (with respect to p) estimates obtained in the previous chapters combined with the Paley-Wiener theorem permit in Chapter 5 to establish for the heat equation, the existence of a unique variational solution, the tangential regularity (in the time variable) of the solution, the singular decomposition of the solution and its global regularity in vector-valued weighted Sobolev spaces.

Chapter 6 is reserved for numerical approximations of the heat equation. First, we design a semi-discrete (in time) mesh refinement finite element method which is optimally convergent. Next the time variable is discretized by the Fourier series method and the space variable by the mesh refinement FEM. This leads to a full discrete method which is optimally convergent in both the time and the space variables. Finally, we use an alternative approach of discretizing the time variable by the NSFD scheme while the mesh refinement FEM is used for the space variable. In addition to the optimal convergence, this NSFD-FEM procedure preserves some qualitative property of the continuous model of the solution such as the decay property in the limit case of space independent equation. These theoretical results are supported by numerical experiments.

Concluding remarks are gathered in chapter 7. They underline how this work fits in the literature and how it can be extended.

Chapter 2

Basic Tools

The study of boundary value problems such as the heat equation conventionally takes as its starting point the idea of function spaces in which the solution of the problem will be handled. For this reason, we will start this thesis with some introductory aspects of function spaces. The underlying domain on which the functions are defined is presented in section 2.1. The most prominent function spaces of interest in our study will be the spaces of continuous functions, Lebesgue space (section 2.2), Distributions (section 2.3) and Sobolev spaces (section 2.4). Some relevant results on Laplace transform (the second tool used in our study) are described in section 2.5.

2.1 The domain Ω

In what follows, we shall work with functions defined on a domain $\Omega \subset \mathbb{R}^2$, i.e., an open and connected set, with boundary denoted by $\partial\Omega$ or Γ. The domain Ω or its boundary Γ is supposed to satisfy some regularity conditions. Following Grisvard [29], our standard reference for function spaces, the regularity conditions can be grouped into the two categories.

The first category is to view $\Gamma \equiv \partial\Omega$ as being locally the graph of a function φ. The regularity of Γ is then described through the differentiability properties of φ. The precise definition reads as follows:

Definition 2.1.1. *We say that the boundary Γ is continuous (respectively, Lipschitz, m times continuously differentiable, etc.) if for every $x \in \Gamma$, there exist a neighborhood V of x in $\mathbb{R}^2$ and a new system of co-ordinates (y_1, y_2) such that,*

1. *V is a rectangle in the new co-ordinate system:*

$$V := \{y = (y_1, y_2) : -a_1 < y_1 < a_1, \;\; -a_2 < y_2 < a_2\},$$

2. *there exists a function $\varphi : (-a_1, a_1) \to \mathbb{R}$ which is continuous (respectively Lipschitz, m times continuously differentiable etc) and satisfies the following conditions:*

$$\begin{aligned} |\varphi(y_1)| &< \frac{a_2}{2} \;\; \textit{for every} \;\; y_1 \in V' := (-a_1, a_1), \\ \Omega \cap V &= \{y = (y_1, y_2) \in V : \quad y_2 < \varphi(y_1)\}, \\ \Gamma \cap V &= \{y = (y_1, y_2) \in V : \quad y_2 = \varphi(y_1)\}. \end{aligned}$$

More generally, Γ is called of class $\mathcal{H}$ when the above function φ is of class $\mathcal{H}$.

Definition 2.1.1 is illustrated in Figure 2.1

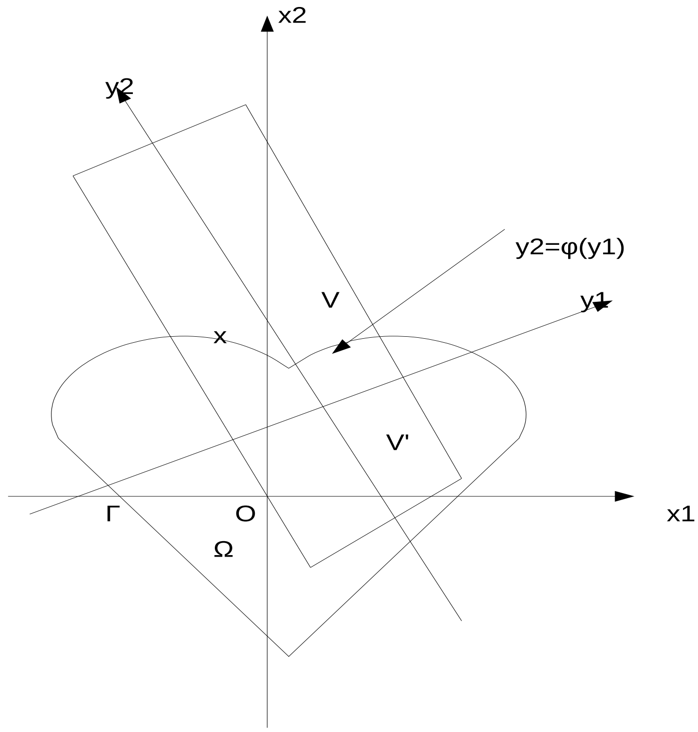

Figure 2.1: Lipschitz boundary Γ

Definition 2.1.1 implies that Ω is locally on one side of its boundary Γ. Indeed, it follows that $\Omega \cap V$ is below the graph of φ and $\Gamma \cap V$ is the graph. Consequently domains with cusps do not satisfy Definition 2.1.1.

The second category is to consider the closure $\overline{\Omega}$, of the domain Ω as a 2-dimensional manifold with the boundary imbedded in $\mathbb{R}^2$. The regularity assumptions are then added on the manifold.

Definition 2.1.2. *We say that $\overline{\Omega}$ is a 2-dimensional continuous (respectively, Lipschitz, m times continuously differentiable etc.) sub-manifold with boundary in $\mathbb{R}^2$, if for every $x \in \Gamma$ there exists a neighborhood V of x in $\mathbb{R}^2$ and a mapping T from V into $\mathbb{R}^2$ such that*

1. *T is injective,*
2. *T together with T^{-1} (defined on $T(V)$) are continuous (respectively, Lipschitz, m times continuously differentiable),*
3. *$\Omega \cap V = \{y \in \Omega : T_2(y) < 0\}$ where $T_2(y)$ denotes the 2th component of $T(y)$.*

Definition 2.1.2 is illustrated in Figure 2.2.

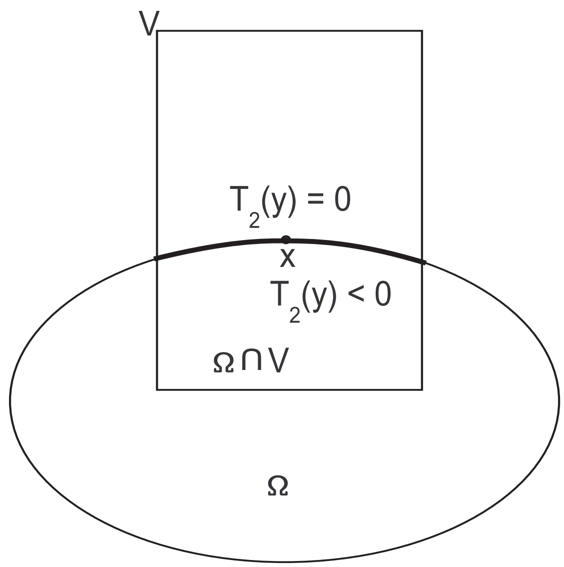

Figure 2.2: Local charts of the boundary Γ

As a result of condition (3) of Definition 2.1.2, the boundary Γ of Ω is defined locally by the equation $T_2(y) = 0$.

The comparison of Definition 2.1.1 and Definition 2.1.2 is an issue of interest. To this end, assuming that Definition 2.1.1 holds, let us define T by

$$T(y) = \{y_1, y_2 - \varphi(y_1)\}. \tag{2.1.1}$$

The function in (2.1.1) has its inverse given by

$$T^{-1}(z) = \{z_1, z_2 + \varphi(z_1)\}.$$

It is clear that T in (2.1.1) fulfils all the conditions in Definition 2.1.2 with the same amount of differentiability for T and T^{-1} as the function φ. In other words, Definition 2.1.1 implies Definition 2.1.2. However, the converse is partly true, namely when T is at least of class C^1. Indeed, assuming that Definition 2.1.2 holds,

$$T_2(y_1, y_2) = 0 \ \text{ for } \ (y_1, y_2) \in \Gamma \cap V. \tag{2.1.2}$$

Let $(y_1^\star, y_2^\star) \in \Gamma \cap V$ be such that $\frac{\partial T_2}{\partial y_2}(y_1^\star, y_2^\star) \neq 0$. Then by the implicit function theorem, there exists open neighborhoods $U \subset \mathbb{R}^2$ of $(y_1^\star, y_2^\star)$ and $V' \subset \mathbb{R}$ of $y_1^\star$ as well as C^1 function $\varphi : V' \to \mathbb{R}$ such that

$$(y_1, y_2) \in U \ \text{ solves } \ (2.1.2) \ \text{ if and if } y_2 = \varphi(y_1), \ y_1 \in V'.$$

The above constraint on the use of the implicit function theorem, motivates why we prefer Definition 2.1.1. In this regard, a typical example on which our thesis is based is given in the next result taken from [55].

Proposition 2.1.3. *A domain Ω with polygonal boundary Γ is Lipschitz in the sense of Definition 2.1.1.*

Proof. We take Ω to be the unit square represented by

$$\Omega = (-1, 1) \times (-1, 1),$$

as illustrated in Figure 2.3.

Let $z \in \Gamma$ not be a vertex. We let the new co-ordinate system y_1, y_2 centered at z be such that the y_1-line coincides with the side of the square that contains z, while the y_2-line is perpendicular to the y_1-line (see Figure 2.4).We then take $V = [-\alpha + z_1, \alpha + z_1] \times [-\beta + z_2, \beta + z_2]$ in the new system and $\varphi(y_1) = 0$. It is clear that y_2 is of class C^∞. Next we consider the case when z is a vertex. In view of the symmetry of the square Ω, it is enough to restrict ourselves to the point $z = (1, 1)$. By Definition 2.1.1, we consider new co-ordinate system as follows, in in view of Figure 2.4. We pass to the co-ordinate (y_1, y_2) from (x_1, x_2) after performing a rotation through an angle of $\pi/4$ and a translation of $(3/4, 3/4)$. These transformations yield the follow equation

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 3/4 \\ 3/4 \end{bmatrix}.$$

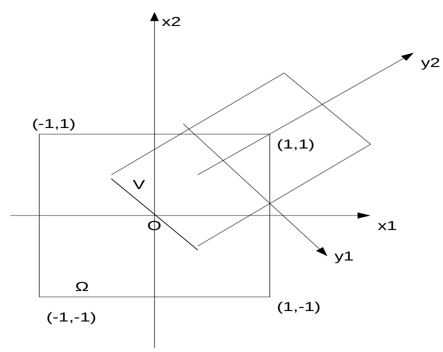

Figure 2.3: Polygon as Lipschitz domain (a)

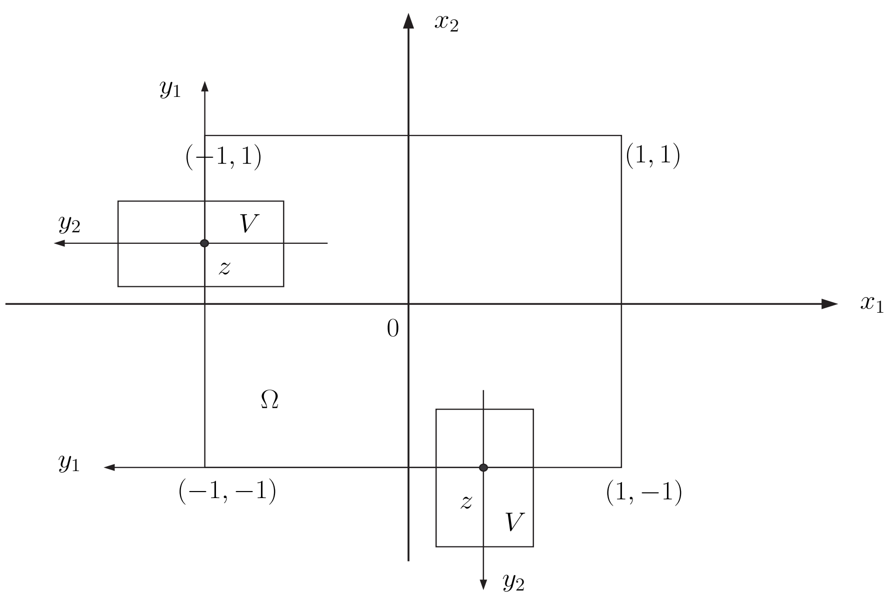

Figure 2.4: Polygon as Lipschitz domain (b)

In the new co-ordinate system, we take

$$V=\left(-\sqrt{2}/4,\sqrt{2}/4\right)\times\left(3\sqrt{2}/4,3\sqrt{2}/4\right);$$

for a neighborhood of the point $(1,1)$. For the function φ, we consider

$$\varphi(y_1)=\sqrt{2}/4-|y_1|.$$

With reference to Definition 2.1.1, we can check that

$$\begin{array}{rcl} |\varphi(y_1)| & < & \dfrac{3\sqrt{2}}{8} \text{ for every } y_1\in V'=\left(-\sqrt{2}/4,\sqrt{2}/4\right), \\ \Omega\cap V & = & \{y=(y_1,y_2)\in V: \quad y_2<\varphi(y_1)\}, \\ \Gamma\cap V & = & \{y=(y_1,y_2)\in V: \quad y_2=\varphi(y_1)\}. \end{array}$$

For $y_1\neq 0$ we have

$$\frac{\varphi(y_1)-\varphi(0)}{y_1}=\begin{cases} -1 & \text{if } y_1\geq 0 \\ \\ 1 & \text{if } y_1\leq 0, \end{cases}$$

which implies that

$$|\varphi(y_1)-\varphi(0)|\leq|y_1|,\forall\, y_1\in V'.$$

Now for arbitrary y_1 and y_1' in V', we have

$$|\varphi(y_1)-\varphi(y_1')|=|y_1-y_1'|,$$

if the signs of y_1 and y_1' are the same. In the case where the signs are different we have

$$\begin{array}{rcl} |\varphi(y_1)-\varphi(y_1')| & \leq & |\varphi(y_1)-\varphi(0)|+|\varphi(0)-\varphi(y_1')| \\ & \leq & |y_1|+|y_1'| \\ & = & |y_1-y_1'| \end{array}$$

We then have that

$$|\varphi(y_1) - \varphi(y_1')| \leq |y_1 - y_1'|, \;\; \forall \, y_1, y_1' \in V',$$

which conclude the proof. $\square$

With Definition 2.1.1, we associate once and for all the notation below which will be used in future. For all $z \in \Gamma$ there exists a neighborhood V_z defined in a new co-ordinate system $x_z = (x_{1,z}, x_{2,z}) \equiv (x_1, x_2) \equiv x$ by

$$V_z = \{x = (x_1, x_2) : -a_{1,z} < x_1 < a_{1,z}, \; -a_{2,z} < x_2 < a_{2,z}\}.$$

Since the boundary Γ is compact, there exist $z_1, \; z_2, \; ,.....z_k \in \Gamma$ such that $\Gamma \subset \cup_{j=1}^{k} V_j$, where $V_j \equiv V_{z,j}, \;\; a_{1,zj} \equiv a_{1,j}$ and $a_{2,zj} \equiv a_{2,j}$. In view of this notations, we can find an open set V_0 with $\bar{V_0} \subset \Omega$ such that the family of open sets $V_j, j = 0, 1, 2, ...k$ is a covering of $\bar{\Omega}$.

Without loss of generality and following Necas [54], we can assume that for all $1 \leq j \leq k$

$$V_j = \{x = (x_1, x_2) :, \; -\alpha < x_1 < \alpha, \;\; -\beta < x_2 < \beta\}, \text{ for some } \alpha, \beta > 0$$

where we recall that (x_1, x_2) in the right hand side should be viewed as in the new co-ordinate system $(x_{1,j}, x_{2,j})$.Furthermore, we have the following regions of $\mathbb{R}^2$ demarcated by:

$$V_j^0 = \Gamma \cap V_j = \{(x_1, x_2) : x_2 = \varphi_j(x_1), \;\; -\alpha < x_1 < \alpha\},$$

$$V_j^+ = V_j \cap \Omega = \{(x_1, x_2) \in V_j : \varphi_j(x_1) - \beta < \varphi_j(x_1), \;\; -\alpha < x_1 < \alpha\},$$

$$V_j^- = V_j \cap (\mathbb{R}^2/\Omega) = \{(x_1, x_2) \in V_j : \varphi_j(x_1) + \beta > \varphi_j(x_1), \;\; -\alpha < x_1 < \alpha\}.$$

For a fixed $j, \;\; 1 \leq j \leq k$, we consider the T_j with its inverse T_j^{-1}

$$T_j : V_j \rightarrow Q \;\text{ and }\; T_j^{-1} : Q \rightarrow V_j,$$

defined by

$$T_j(x) \equiv T_j(x_1, x_2) = \left(\frac{x_1}{\alpha},\ \frac{\varphi_j(x_1) - x_2}{\beta} \right), \tag{2.1.3}$$

and

$$T_j^{-1}(y) \equiv T_j^{-1}(y_1, y_2) = (\alpha y_1,\ \varphi_j(\alpha y_1) - \beta y_2)\,. \tag{2.1.4}$$

where

$$Q = \{(y_1, y_2) : |y_1| < 1,\ \ |y_2| < 1\}\,,$$

is the unit square. The smoothness of T_j and T_j^{-1} is determined by that of the map φ_j in Definition 2.1.1. Furthermore, under the transformation T_j

$$V_j^+ \text{ becomes } Q_+ = \{(y_1, y_2) :\ |y_1| < 1,\ 0 < y_2 < 1\}\,,$$

$$V_j^- \text{ becomes } Q_- = \{(y_1, y_2) :\ |y_1| < 1,\ -1 < y_2 < 0\}\,,$$

$$V_j^0 \quad \text{becomes} \quad Q_0 = \{(y_1, 0) : \quad |y_1| < 1\}\,,$$

as seen in Figure 2.1.2.

With these notation in mind, there exist non-negative functions $\theta_j \in \mathcal{D}(V_j)$, $\theta_j \leq 1$, $0 \leq j \leq k$
satisfying

$$\forall x \in \bar{\Omega},\ \sum_{j=0}^{k} \theta_j(x) = 1 \ \text{ and } \ \forall x \in \Gamma,\ \sum_{j=1}^{k} \theta_j(x) = 1. \tag{2.1.5}$$

The family $(\theta_j)_{j=0}^k$ and $(\theta_j)_{j=1}^k$ are called C^∞-partition of unity on $\bar{\Omega}$ and Γ subordinated to the open coverings $(V_j)_{j=0}^k$ and $(V_j)_{j=1}^k$ respectively.

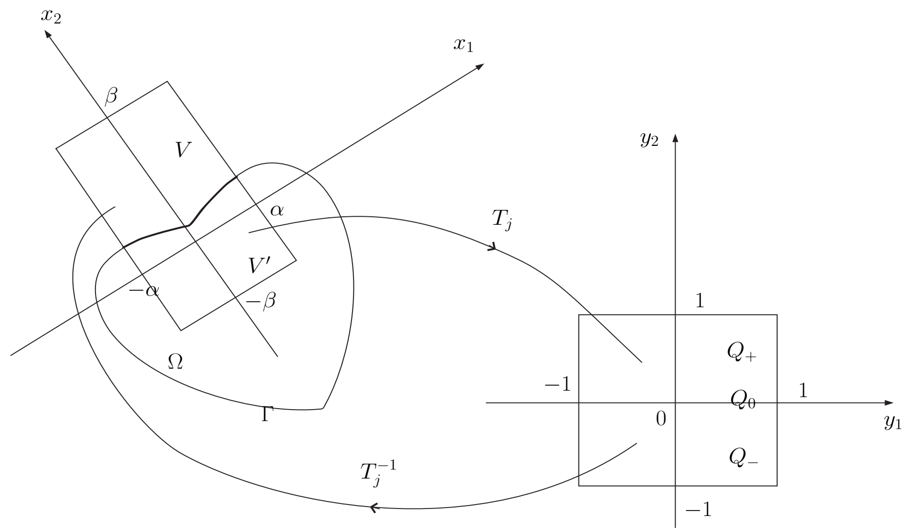

Figure 2.5: Boundary Γ of piecewise C^m class

2.2 Usual Function Spaces

With the domain Ω, we associate the following classical function spaces that we will use:

Definition 2.2.1. ([*40*])
Given an integer $m \geq 0$, we define

- $C^m(\Omega) = \{v : \Omega \to \mathbb{R};\ D^\alpha v$ *is continuous on* $\Omega\ \forall\ |\alpha| \leq m\}$. *This is the space of m times continuously differentiable functions on Ω.*

- $C_b^m(\Omega) := \{v \in C^m(\Omega),\ D^\alpha v$ *is bounded* $\forall\ |\alpha| \leq m\}$, $C_b^m(\Omega)$ *is a Banach space under the norm*

$$\|v\|_{m,\infty,\Omega} := \max_{|\alpha|\leq m} \sup_{x\in\Omega} |D^\alpha v(x)|. \tag{2.2.1}$$

- $C_0^m(\Omega) := \{v \in C^m(\Omega) : v$ *has a compact support contained in* $\Omega\}$.

- $\mathcal{D}(\Omega) \equiv C_0^\infty(\Omega) = \bigcap_{m\geq 0} C_0^m(\Omega)$. *This is the space of test functions, which consists of infinitely differentiable functions* $v : \Omega \to \mathbb{R}$ *with compact support in* Ω.

- $C^m(\bar{\Omega}) := \{v \in C^m(\Omega); \forall\ |\alpha| \leq m, x \to D^\alpha v(x)$ *is bounded and uniformly continuous on* $\Omega\}$.

- $C^{m,\theta}(\bar{\Omega}) := \left\{v \in C^m(\bar{\Omega});\ \exists\, C \geq 0 : |D^\alpha v(x) - D^\alpha v(y)| \leq C|x-y|^\theta\ \forall\ x, y \in \Omega\ \forall\ |\alpha| = m\right\}$ *is the Hölder space of order* m *and exponent* $\theta \in (0, 1]$.

Definition 2.2.2. ([*40*])
Let $1 \leq p \leq +\infty$ *be a real number. The Lebesgue space* $L^p(\Omega)$ *consists of classes of measurable functions* v *on* Ω *such that*

$$\|v\|_{0,p,\Omega} = \begin{cases} V^1 < +\infty, & if\, p < \infty \\ \\ V^{11} < +\infty, & if\, p = \infty, \end{cases} \tag{2.2.2}$$

where

$$V^1 = \left(\int_\Omega |v(x)|^p dx\right)^{1/p}$$

and

$$V^{11} = ess \sup_{x\in\Omega} |v(x)| := \inf\left\{k \geq 0 : |v(x)| \leq k\ a.e\ on\, \Omega\right\}.$$

Notice that $L^1(\Omega)$ is the space of classes of measurable functions on Ω which are Lebesgue-integrable. Notice also that $L^p(\Omega)$ is a Banach space under the natural norm in (2.2.2) while $L^2(\Omega)$ is a Hilbert space for the inner product

$$(u, v)_{0,\Omega} := \int_\Omega u(x)v(x)dx. \tag{2.2.3}$$

Definition 2.2.3. *The space of locally integrable functions is denoted by* $L^1_{loc}(\Omega)$ *and defined by*

$$\begin{aligned} L^1_{loc}(\Omega) \quad &:= \quad \left\{v : \phi v \in L^1(\Omega),\ \forall\ \phi \in \mathcal{D}(\Omega)\right\} \\ &= \quad \left\{v : v\chi_K \in L^1(\Omega),\ \forall\ K \subset \Omega,\ K\ compact\ in\, \mathbb{R}^2\right\}, \end{aligned}$$

where χ_K *is the characteristic function of the set* K.

Remark 2.2.4. *Spaces of functions $C_0^\infty(\Omega)$ and $L^1_{loc}(\Omega)$ are the smallest and the largest spaces of functions of interest in applications as depicted in Figure 2.6.*

$$
\begin{array}{ccccc}
C_0^\infty(\Omega) \subset C_0^m(\Omega) \subset L^p(\Omega) \subset & L^1_{loc}(\Omega) \\
 & \cup \\
C_b^m(\Omega) \subset \qquad\qquad & C^m(\Omega) \\
 & \cup \\
 & C^m(\bar{\Omega}) & .
\end{array}
$$

Figure 2.6: Smallest and Largest spaces

2.3 Distributions

Functions in the smallest space $\mathcal{D}(\Omega)$ have many nice properties that functions in the largest space $L^1_{loc}(\Omega)$ fail to have. By duality on $\mathcal{D}(\Omega)$ we will construct a much larger space which contains $L^1_{loc}(\Omega)$ and possess the said nice properties in a weaker sense.

Definition 2.3.1. ([*40*])**(Pseudo-topology of $\mathcal{D}(\Omega)$)**
A sequence $(\varphi_n)_{n\geq 1}$ in $\mathcal{D}(\Omega)$ converges to $\varphi \in \mathcal{D}(\Omega)$ if

1. *There exists a compact set K of $\mathbb{R}^2$ such that $K \subset \Omega, \quad supp(\varphi_n) \subset K, \ \forall n \geq 1, \ \ supp(\varphi) \subset K$;*

2. *For every multi-index $\alpha, (D^\alpha \varphi_n)$ converges to $(D^\alpha \varphi)$ uniformly on K.*

We will elaborate a bit more on the topology of $\mathcal{D}(\Omega)$ in subsection 2.5.3 below.

Definition 2.3.2. ([*40*]) **(Pseudo-topology of $L^1_{loc}(\Omega)$)**
A sequence (v_n) converges to φ in $L^1_{loc}(\Omega)$ if

$$\forall \text{ compact } K \subset \Omega, \lim_{n\to\infty} \int_K |v_n - \varphi| dx = 0.$$

With all these structures we can then define distributions as follows:

Definition 2.3.3. ([*40*])

1. *By definition,* $\mathcal{D}'(\Omega)$ *the dual of* $\mathcal{D}(\Omega)$*, is the space of distributions on* Ω*. This means* $T \in \mathcal{D}'(\Omega)$ *if and only if the convergence to* 0 *in* $\mathcal{D}(\Omega)$ *of any sequence* (φ_n) *implies the linear convergence to* 0 *of the scalar sequence* $(\langle T, \varphi_n \rangle)$*. (The symbol* $\langle \cdot, \cdot \rangle_{\mathcal{D}' \times \mathcal{D}}$ *or* $\langle \cdot, \cdot \rangle$ *when there is no risk of confusion denotes the duality pairing between* $\mathcal{D}'(\Omega)$ *and* $\mathcal{D}(\Omega)$*).*

2. *A sequence* (T_n) *of distributions on* Ω *converges to* $0 \in \mathcal{D}'(\Omega)$ *if* $\langle T_n, \varphi \rangle \to 0,\ \forall \varphi \in \mathcal{D}'(\Omega)$*.*

Remark 2.3.4. *The definition of convergent sequence* T_n *of distributions given in Definition 2.3.3 and used often in the literature is incomplete but sufficient in applications. The complete definition of this concept will be clarified when we equip* $\mathcal{D}'(\Omega)$ *with the topology of uniform convergence on bounded subsets of* $\mathcal{D}(\Omega)$ *(see Proposition 2.5.17).*

Another type of space of test functions which will be useful to us in the context of Fourier transform of distributions, is given in the next definition.

Definition 2.3.5. ([*61*])
Schwartz's space $\mathcal{S}(\mathbb{R})$ *of test functions consists of* C^∞ *functions which together with all their derivatives are rapidly decreasing at infinity. In other words* $\varphi \in \mathcal{S}(\mathbb{R})$ *if* $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ *is infinitely differentiable and for all integers* $m, n \geq 0$*, there exists a constant* $C_{m,n} \geq 0$ *such that*

$$\sup\{|x|^m |\frac{d^n\varphi}{dx^n}(x)| : x \in \mathbb{R}\} < C_{m,n}. \tag{2.3.1}$$

This is equivalent to $\varphi \in C^\infty(\mathbb{R})$ *and* $\lim_{|x|\to\infty} |x|^m \frac{d^n\varphi(x)}{dx^n} = 0\ \ \forall\, m \in \mathbb{N},\ \forall\, n \in \mathbb{N}$*.*

The space $\mathcal{S}(\mathbb{R})$ has the structure of a locally convex topological space when equipped with Schwartz canonical topology. In terms of this topology, we have the following definitions:

Definition 2.3.6. ([*40*])**(Pseudo-topology of** $\mathcal{S}(\mathbb{R})$**)**
A sequence (φ_j) *converges to* φ *in* $\mathcal{S}(\mathbb{R})$ *whenever*

$$\lim_{j\longrightarrow 0} \sup_{x\in\mathbb{R}} |x^m \left(\frac{d^n\varphi_j}{dx^n} - \frac{d^n\varphi}{dx^n}\right)(x)| = 0,\ \forall n \in \mathbb{N},\ \forall m \in \mathbb{N}.$$

Definition 2.3.7. ([*40*])
By definition, the dual $\mathcal{S}'(\mathbb{R})$ of $\mathcal{S}(\mathbb{R})$ is the space of tempered distributions in $\mathbb{R}$. This means $T \in \mathcal{S}'(\mathbb{R})$ if and only if the convergence to 0 in $\mathcal{S}(\mathbb{R})$ of any sequence (φ_n) implies the convergence to 0 of numerical sequence $(\langle T, \varphi_n\rangle)$. (Again the symbol $\langle \cdot, \cdot\rangle$ denotes the duality pairing between $\mathcal{S}'(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$.)

Definition 2.3.8. ([*40*])
Given a distribution $T \in \mathcal{D}'(\Omega)$, its derivative with respect to x_i, $1 \leq i \leq 2$ is the distribution denoted by $\frac{\partial T}{\partial x_i}$ and defined by

$$\forall \varphi \in \mathcal{D}(\Omega),\ < \frac{\partial T}{\partial x_i}, \varphi >= - < T, \frac{\partial \varphi}{\partial x_i} > .$$

In general, for a multi-index $\alpha \in \mathbb{N}^2$, the derivative of T of order α is the distribution $D^\alpha T$ defined by

$$\forall \varphi \in \mathcal{D}(\Omega),\ < D^\alpha T, \varphi >= (-1)^{|\alpha|} < T, D^\alpha \varphi > .$$

2.4 Sobolev Spaces

For a fixed parameter which is either the time variable t or a complex number p, the solutions of the heat and Helmholtz equations that we will consider in this thesis belong to the class of Sobolev spaces that we outline now. Our standard reference for Sobolev spaces is [29], though we add from time to time those references that we used most.

Definition 2.4.1. ([*29*])
Let $m \geq 0$ be an integer. The Sobolev space $H^m(\Omega)$ is defined by

$$H^m(\Omega) := \left\{ v \in \mathcal{D}'(\Omega) : D^\alpha v \in L^2(\Omega),\ \ \forall\, |\alpha| \leq m \right\}. \tag{2.4.1}$$

In other words, $H^m(\Omega)$ is the collection of all functions in $L^2(\Omega)$ such that all distributional derivatives up to order m are also in $L^2(\Omega)$.

We make $H^m(\Omega)$ a Hilbert space under the norm

$$\|v\|_{m,\Omega} := \left(\sum_{|\alpha| \leq m} \int_\Omega |D^\alpha v|^2 dx \right)^{1/2} \tag{2.4.2}$$

and the inner product

$$(w, v)_{m,\Omega} := \sum_{|\alpha| \leq m} \int_\Omega D^\alpha w D^\alpha v dx.$$

We denote by $|\cdot|_{m,\Omega}$ the semi-norm

$$|v|_{m,\Omega} := \left(\sum_{|\alpha|=m} \int_\Omega |D^\alpha v|^2 dx \right)^{1/2}. \tag{2.4.3}$$

Clearly the Sobolev space of order 0 i.e $H^0(\Omega) = L^2(\Omega)$. Unless $\Omega = \mathbb{R}^2$, or $m = 0$ the space $\mathcal{D}(\Omega)$ is not dense in $H^m(\Omega)$. For this reason we introduce the following subspace.

Definition 2.4.2. ([*29*])
We define the Sobolev subspace $H_0^m(\Omega)$ as the closure of $\mathcal{D}(\Omega)$ in the space $H^m(\Omega)$.

Theorem 2.4.3. ([*29*])*(Poincaré-Friedrichs Inequality)*
Assume that Ω is bounded in one of the directions, say x_n. Then there exists a constant $C > 0$ depending upon Ω such that

$$\forall v \in H_0^1(\Omega),\ \|v\|_{0,\Omega} \leq C \|\frac{\partial v}{\partial x_n}\|_{0,\Omega}. \tag{2.4.4}$$

Consequently for $m \geq 1$ the semi-norm $|\cdot|_{m,\Omega}$ is a norm on $H_0^m(\Omega)$ equivalent to $\|\cdot\|_{m,\Omega}$. Occasionally, we will use the non-Hilbertian Sobolev space defined as follows:

Definition 2.4.4. *1. For $1 \leq p < \infty$ the Sobolev space of integer order $m \geq 0$ is denoted $W^{m,p}(\Omega)$ and defined by*

$$W^{m,p}(\Omega) := \{v \in \mathcal{D}'(\Omega) : D^\alpha v \in L^p(\Omega),\ \forall\ |\alpha| \leq m\}. \tag{2.4.5}$$

It is clear that $W^{m,p}(\Omega)$ coincides with the Hilbertian Sobolev space $H^m(\Omega)$. However for $p \neq 2$ the space $W^{m,p}(\Omega)$ is a Banach space (not a Hilbert space) with the norm and semi-norm defined respectively by

$$\|v\|_{m,p,\Omega} = \begin{cases} \left(\sum_{|\alpha|\leq m} \int_\Omega |D^\alpha v(x)|^p dx\right)^{1/p} & \textit{if} \quad p < \infty \\ \max_{|\alpha|\leq m} ess\sup_{x\in\Omega} |D^\alpha v(x)| & \textit{otherwise} \end{cases}$$

and

$$|v|_{m,p,\Omega} = \begin{cases} \left(\sum_{|\alpha|=m} \int_\Omega |D^\alpha v(x)|^p dx\right)^{1/p} & \text{if} \qquad p < \infty \\ \\ \max_{|\alpha|=m} ess\sup_{x\in\Omega} |D^\alpha v(x)| & \text{otherwise.} \end{cases}$$

Theorem 2.4.5. ([29])*(Sobolev Continuous Embedding Theorem and Rellich Kondrachov Compact Embedding Theorem).*
Assume that a bounded open set Ω has boundary $\partial\Omega \equiv \Gamma$ which is Lipschitz if $m = 1$ or is of class C^m if $m > 1$. Consider the number $p^\star$ defined by

$$\frac{1}{p^\star} = \frac{1}{p} - \frac{m}{2}, \quad 1 < p < \infty, \quad m \geq 1.$$

1. *If $\frac{1}{p^\star} \geq 0$, i.e. $m \leq \frac{2}{p}$, then we have, for any $q \in [1, p^\star]$, the continuous embedding*

   $$W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$$

   which is compact in the particular case when $q \neq p^\star$;

2. *If $\frac{1}{p^\star} < 0$, i.e. $m > \frac{2}{p}$, we have the continuous and compact embedding*

   $$W^{m,p}(\Omega) \hookrightarrow C^s(\bar{\Omega}).$$

   where s is the non-negative integer satisfying $s \leq m - \frac{2}{p} < s + 1$.

   Furthermore, if $m - \frac{2}{p}$ is not an integer, we have the continuous embedding

   $$W^{m,p}(\Omega) \hookrightarrow C^{s,\theta}(\bar{\Omega})$$

   where $\theta = m - \frac{2}{p} - s$ and $C^{s,\theta}(\bar{\Omega})$ the Hölder space equipped with the norm

$$\|v\|_{C^{s,\theta}(\bar{\Omega})} := \max_{|\alpha|\leq s} \sup_{x\in\Omega} |D^\alpha v(x)| + \max_{|\alpha|=s} \sup_{x\in\Omega} \frac{|D^\alpha v(x) - D^\alpha v(y)|}{|x-y|^\theta}$$

Remark 2.4.6. *Theorem 2.4.5 is valid in the one-dimensional case (i.e Ω is an interval) provided that 2 is replaced with 1 in the identity that defines $p^\star$.*

2.5 Laplace transform

The evolution equations that we study will be transformed into complex-parameter family of elliptic equations through the Laplace transform, which we outline in this section.

2.5.1 Laplace transform of functions

Given a test function $v \in \mathcal{D}(0, +\infty)$, its Laplace transform is denoted and defined by

$$(\mathcal{L}v)(p) \equiv \widehat{v}(p) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} e^{-pt} v(t) dt, \quad p = \xi + i\eta \in \mathbb{C}. \tag{2.5.1}$$

The connection of the Laplace transform with the Fourier transform is straight forward on extending the function $v \in \mathcal{D}(0, +\infty)$ to $\widetilde{v} \in \mathcal{D}(-\infty, +\infty)$ given by

$$\widetilde{v}(t) = \begin{cases} v(t) & \text{for} \quad t \geq 0 \\ \\ 0 & \text{for} \quad t < 0. \end{cases} \tag{2.5.2}$$

Indeed from (2.5.1), we have

$$\begin{aligned} \widehat{v}(p) &= \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} e^{-i\eta t} e^{-\xi t} v(t) dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i\eta t} e^{-\xi t} \widetilde{v}(t) dt. \end{aligned}$$

Thus

$$\widehat{v}(p) = \mathcal{F}\left(e^{-\xi t} \widetilde{v}(t)\right)(\eta), \tag{2.5.3}$$

where

$$\mathcal{F}(w)(\eta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i\eta t} w(t) dt, \tag{2.5.4}$$

is the Fourier transform of $w \in \mathcal{D}(-\infty, +\infty)$ and

$$w(t) = \mathcal{F}^{-1}(\mathcal{F}(w))(t) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{+\infty} e^{i\eta t} \mathcal{F}(w)(\eta) d\eta \tag{2.5.5}$$

is the inverse Fourier transform of $\mathcal{F}(w)$.

Given a function $v \in \mathcal{D}(0, +\infty)$, it is easy to show by integration by parts that the Laplace transform of the derivative $\frac{d^k v}{dt^k}$ is given by the relation

$$\mathcal{L}\left(\frac{d^k v}{dt^k}\right)(p) = p^k \mathcal{L}(v)(p) \quad \text{for} \quad k \in \mathbb{N}. \tag{2.5.6}$$

If another function $w \in \mathcal{D}(0, +\infty)$ is considered, we have for $\xi \in \mathbb{R}$ the Parseval identity

$$\int_0^{+\infty} v(t)w(t)e^{-2\xi t}dt = \int_{-\infty}^{+\infty} \widehat{v}(\xi + i\eta) \cdot \overline{\widehat{w}}(\xi + i\eta)d\eta, \tag{2.5.7}$$

which implies that the Laplace transform satisfies the relation

$$\left(\int_0^{+\infty} |v(t)e^{-\xi t}|^2 dt\right)^{\frac{1}{2}} = \left(\int_{-\infty}^{+\infty} |\widehat{v}(\xi + i\eta)|^2 d\eta\right)^{\frac{1}{2}}. \tag{2.5.8}$$

Furthermore, we have

$$\int_0^{+\infty} \widehat{v}(\xi + i\eta)w(\eta)d\eta = \int_0^{+\infty} e^{-\xi t}v(t)\overline{\mathcal{F}(w)}(t)dt. \tag{2.5.9}$$

It is clear that the Laplace transform of a function $v \in L^1(0, \infty)$ is well-defined by the integral (2.5.1) whenever the condition

$$\xi \geq 0, \tag{2.5.10}$$

is satisfied.

Theorem 2.5.1. *Let $g(t) \in L^2(-\infty, +\infty)$ have its support in the unbounded interval $I_\alpha = (-\infty, \alpha)$ or $I_\alpha = (\alpha, +\infty)$ where $\alpha \in \mathbb{R}$. Then the Laplace transform*

$$\widehat{g}(p) = \frac{1}{\sqrt{2\pi}} \int_{I_\alpha} e^{-pt} g(t)dt,$$

is defined for $Re(p) = \xi > 0$ if $I_\alpha = (\alpha, +\infty)$ or for $Re(p) = \xi < 0$ if $I_\alpha = (-\infty, \alpha)$.

Furthermore, $\widehat{g}(p)$ *is a holomorphic function in the complex region*

$$\mathbb{C}_\alpha = \begin{cases} p; \quad \xi > 0 & if \quad I_\alpha = (\alpha, +\infty) \\ \\ p; \quad \xi < 0 & if \quad I_\alpha = (-\infty, \alpha) \end{cases}$$

such that, for $p \in \mathbb{C}_\alpha$ *with a fixed* ξ*, the function* $\eta \to \widehat{g}(\xi + i\eta)$ *is of class* $L^2(-\infty, +\infty)$ *and satisfies the relation*

$$\int_{-\infty}^{+\infty} |\widehat{g}(\xi + i\eta)|^2 d\eta \leq e^{-2\xi\alpha} \int_{I_\alpha} |g(t)|^2 dt.$$

Proof. We prove the theorem for the case when $I_\alpha = (\alpha, +\infty)$, the situation $I_\alpha = (-\infty, \alpha)$ being analogue. We show that for $\xi > 0$, the function $t \to e^{-\xi t} g(t)$ is of class $L^1(\alpha, +\infty)$. Indeed, we have

$$\begin{aligned} |\widehat{g}(\xi + i\eta)| &= \frac{1}{\sqrt{2\pi}} \left| \int_\alpha^{+\infty} e^{-i\eta t} e^{-\xi t} g(t) dt \right| \\ &\leq \frac{1}{\sqrt{2\pi}} \int_\alpha^{+\infty} e^{-\xi t} |g(t)| dt \\ &\leq \frac{1}{\sqrt{2\pi}} \left(\int_\alpha^{+\infty} e^{-2\xi t} dt \right)^{\frac{1}{2}} \left(\int_\alpha^{+\infty} |g(t)|^2 dt \right)^{\frac{1}{2}}; \end{aligned}$$

where the previous inequality is due to Cauchy Schwarz inequality. This shows that $\widehat{g}(\xi + i\eta)$ is defined for $\xi > 0$ and also holomorphic by differentiation under the sum symbol.

On the other hand Plancherel-Parseval theorem yields for $\xi > 0$

$$\begin{aligned} \int_{-\infty}^{+\infty} |\widehat{g}(\xi + i\eta)|^2 d\eta &= \int_\alpha^{+\infty} |e^{-\xi t} g(t)|^2 dt \\ &= \int_\alpha^{+\infty} e^{-2\xi t} |g(t)|^2 dt \\ &\leq e^{-2\xi\alpha} \int_\alpha^{+\infty} |g(t)|^2 dt. \end{aligned}$$

□

2.5.2 Laplace transform of distributions

We want to define the Laplace transform of more general objects; namely, distributions in such a way that properties (2.5.6) and (2.5.8) remain valid. However, since the space $\mathcal{D}(\mathbb{R})$ is not invariant under the Fourier transform, we use Schwartz [61] space of test functions $\mathcal{S}(\mathbb{R})$ introduced in Definition 2.3.5. The estimate (2.3.1) guarantees that the Fourier transform of $\phi \in \mathcal{S}(\mathbb{R})$ is well-defined by the relation (2.5.4). More importantly, we have the following result.

Theorem 2.5.2. ([*21*])
The Fourier transform $\mathcal{F}$ is an isometric isomorphism, (with inverse $\mathcal{F}^{-1}$ given in (2.5.5)) from $\mathcal{S}(\mathbb{R})$ onto $\mathcal{S}(\mathbb{R})$ when $\mathcal{S}(\mathbb{R})$ is equipped with the $L^2(\mathbb{R})$-norm.

Motivated by the relations (2.5.3) and (2.5.9), we give the following definition:

Definition 2.5.3. ([*21*])
For a tempered distribution $T \in \mathcal{S}'(\mathbb{R})$, its Fourier transform is the tempered distribution denoted by $\mathcal{F}(T)$ and given by

$$\langle \mathcal{F}(T), \ \ \varphi\rangle_{\mathcal{S}'(\mathbb{R}_\eta)\times\mathcal{S}(\mathbb{R}_\eta)} = \langle T, \ \ \mathcal{F}(\varphi)\rangle_{\mathcal{S}'(\mathbb{R}_t)\times\mathcal{S}(\mathbb{R}_t)} \quad \forall \ \ \varphi \in \mathcal{S}(\mathbb{R}_\eta). \tag{2.5.11}$$

Here and after, the notation $\mathbb{R}_t$ means that distributions and test functions are considered with the argument t.

Remark 2.5.4. *Note that Definition 2.5.3 does not make sense for an arbitrary distribution $T \in \mathcal{D}'(\mathbb{R})$ in view of the fact that $\mathcal{F}(\varphi) \notin \mathcal{D}(\mathbb{R})$ for $\varphi \in \mathcal{D}(\mathbb{R})$. Therefore we had to use the largest space of test functions $\mathcal{S}(\mathbb{R})$ into which $\mathcal{D}(\mathbb{R})$ is densely and continuously embedded in order for Definition 2.5.3 to work for the small space $\mathcal{S}'(\mathbb{R})$ of tempered distributions which is densely and continuously embedded in $\mathcal{D}'(\mathbb{R})$.*

One of the important properties of the Fourier transform of distributions we shall need in this study is the Fourier transform of the derivative with respect to the time t. For $T \in \mathcal{S}'(\mathbb{R})$ and any non-negative integer n, we have

$$\mathcal{F}\frac{d^nT}{dt^n} = (i\eta)^n\mathcal{F}(T) \quad \in \mathcal{S}'(\mathbb{R}_\eta). \tag{2.5.12}$$

Indeed if $\varphi \in \mathcal{S}(\mathbb{R}_\eta)$, we have

$$\begin{aligned}
\left\langle \mathcal{F}(\frac{d^nT}{dt^n})(\eta),\ \varphi \right\rangle_{\mathcal{S}'(\mathbb{R}_\eta)\times\mathcal{S}(\mathbb{R}_\eta)} &= \left\langle \frac{d^nT}{dt^n},\ \mathcal{F}(\varphi) \right\rangle_{\mathcal{S}'(\mathbb{R}_t)\times\mathcal{S}(\mathbb{R}_t)}, \quad \text{by (2.5.11)}\\
&= (-1)^n \left\langle T,\ \frac{d^n}{dt^n}(\mathcal{F}(\varphi)) \right\rangle_{\mathcal{S}'(\mathbb{R}_t)\times\mathcal{S}(\mathbb{R}_t)}, \quad \text{by Definition 2.3.8, which}\\
&\qquad \text{is the same for tempered distributions}\\
&= (-1)^n \langle T,\ (i\eta)^n\mathcal{F}(\varphi)\rangle_{\mathcal{S}'(\mathbb{R}_t)\times\mathcal{S}(\mathbb{R}_t)}, \quad \text{by the properties of Fourier}\\
&\qquad \text{transform of usual functions}\\
&= \langle (i\eta)^n\mathcal{F}(T),\ \varphi\rangle_{\mathcal{S}'(\mathbb{R}_\eta)\times\mathcal{S}(\mathbb{R}_\eta)} \quad \text{by (2.5.11).}
\end{aligned}$$

With the above in mind, we are led to study the subspace $\mathcal{D}'_+(\mathbb{R})$ of $\mathcal{D}'(\mathbb{R})$ consisting of distributions T with support limited to the left. i.e. $supp(T) \subset [\alpha, +\infty)$ $\alpha \in \mathbb{R}$. Notice that distributions $\mathcal{D}'_+(\mathbb{R})$ are tested against functions φ in the space $\mathcal{D}_-(\mathbb{R})$ where $\varphi \in \mathcal{D}(\mathbb{R})$ is such that $supp(\varphi) \subset (-\infty, \beta]$, $\beta \in \mathbb{R}$. (The spaces $\mathcal{D}'_-(\mathbb{R})$ and $\mathcal{D}_+(\mathbb{R})$ are defined analogously). For $T \in \mathcal{D}'_+(\mathbb{R})$ we want to connect its Laplace transform to the Fourier transform of distributions via the analogue (2.5.3) and (2.5.11). To investigate this connection, we consider an important set introduced in [22].

Definition 2.5.5. ([*22*])
With a distribution $T \in \mathcal{D}'(\mathbb{R}_t)$, we associate the set I_T of real numbers given by

$$I_T = \left\{\xi \in \mathbb{R} : \quad e^{-\xi t}T \in \mathcal{S}'(\mathbb{R})\right\}. \tag{2.5.13}$$

The properties of the set I_T are summarized in the following result:

Proposition 2.5.6. ([*22*])

1. *For $T \in \mathcal{D}'(\mathbb{R})$, I_T is a convex set which may be empty;*

2. *If $T \in \mathcal{D}'_+(\mathbb{R})$ and if $I_T \neq \emptyset$, then $I_T = \mathbb{R}$ or $[\xi_0, +\infty)$ with $\xi_0 \in \mathbb{R}$.*

The next proposition specifies some useful properties of tempered distributions associated with $T \in \mathcal{D}'(\mathbb{R})$ and I_T.

Proposition 2.5.7. ([22]).
Let $T \in \mathcal{D}'(\mathbb{R})$. Denote by $int(I_T)$ the interior of I_T and suppose that it is non-empty. Then:

1. *For all $\xi \in int(I_T)$ the Fourier transform $\mathcal{F}\left(e^{-\xi t}T\right)(\eta)$ of the distribution $e^{-\xi t}T$ is a function of $\mathbb{O}_M$ where $\mathbb{O}_M$ is the space of C^∞ functions which together with all their derivatives are slowly increasing at infinity. That is, $v \in \mathbb{O}_M \Leftrightarrow v \in C^\infty(\mathbb{R}), \forall j \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that*

$$\lim_{|x|\to\infty} |x|^{-N}|v^{(j)}(x)| = 0,$$

2. *The function*

$$p = \xi + i\eta \to \mathcal{F}\left(e^{-\xi t}T\right)(\eta)$$

is holomorphic in the band $int(I_T) \times \mathbb{R}$.

In view of Proposition, 2.5.6 and 2.5.7, we can define the Laplace transform of a distribution as follows:

Definition 2.5.8. ([22]).
Let $T \in \mathcal{D}'(\mathbb{R})$ be such that $int(I_T) \neq \emptyset$. The holomorphic function denoted by $\mathcal{L}(T) : p \longrightarrow \mathcal{L}(T)(p)$ and defined for $p \in int(I_T) \times \mathbb{R}$ by

$$\widehat{T}(p) \equiv \mathcal{L}(T)(p) := \mathcal{F}\left(e^{-\xi t}T\right)(\eta) \tag{2.5.14}$$

is called the Laplace transform of the distribution $T \in \mathcal{D}'(\mathbb{R})$.

As mentioned earlier, the properties (2.5.6) and (2.5.9) are valid in this general setting of Definition 2.5.8 as shown below. For $T \in \mathcal{D}'(\mathbb{R})$ with $I_T \neq \emptyset$, and $\varphi \in \mathcal{S}(\mathbb{R})$, we have

$$\begin{aligned}
\langle \mathcal{L}(T)(p), \varphi \rangle_{\mathcal{S}'(\mathbb{R}_\eta)\times\mathcal{S}(\mathbb{R}_\eta)} &= \left\langle \mathcal{F}(e^{-\xi t}T)(\eta), \varphi \right\rangle_{\mathcal{S}'(\mathbb{R}_\eta)\times\mathcal{S}(\mathbb{R}_\eta)} \text{ by (2.5.14)} \\
&= \left\langle e^{-\xi t}T, \mathcal{F}(\varphi) \right\rangle_{\mathcal{S}'(\mathbb{R}_t)\times\mathcal{S}(\mathbb{R}_t)} \text{ by (2.5.11)} \\
&= \langle T, \mathcal{L}(\varphi)(p) \rangle_{\mathcal{S}'(\mathbb{R}_t)\times\mathcal{S}(\mathbb{R}_t)} .
\end{aligned}$$

This is the analogue of (2.5.9). On the other hand, by (2.5.11)

$$\begin{aligned}\left\langle \mathcal{L}\left(\frac{d^kT}{dt^k}\right)(p),\ \ \varphi\right\rangle &= \left\langle \mathcal{F}\left(e^{-\xi t}\frac{d^kT}{dt^k}\right)(\eta),\ \varphi\right\rangle_{\mathcal{S}'(\mathbb{R}_\eta)\times\mathcal{S}(\mathbb{R}_\eta)} \\ &= \left\langle e^{-\xi t}\frac{d^kT}{dt^k},\ \ \mathcal{F}(\varphi)\right\rangle_{\mathcal{S}'(\mathbb{R}_t)\times\mathcal{S}(\mathbb{R}_t)}. \end{aligned} \tag{2.5.15}$$

Since

$$\frac{d}{dt}\left(e^{-\xi t}T\right) = -\xi e^{-\xi t}T + e^{-\xi t}\frac{dT}{dt},$$

then (2.5.15), for $k = 1$, yields

$$\begin{aligned}\left\langle \mathcal{L}\left(\frac{dT}{dt}\right)(p),\ \varphi\right\rangle &= \left\langle \xi e^{-\xi t}T + \frac{d}{dt}\left(e^{-\xi t}T\right),\ \mathcal{F}\left(\varphi\right)\right\rangle_{\mathcal{S}'(\mathbb{R}_t)\times\mathcal{S}(\mathbb{R}_t)} \\ &= \left\langle \xi\mathcal{F}\left(e^{-\xi t}T\right)(\eta) + \mathcal{F}\left(\frac{d}{dt}\left(e^{-\xi t}T\right)\right)(\eta),\ \ \varphi\right\rangle \\ &= \left\langle \xi\mathcal{F}\left(e^{-\xi t}T\right)(\eta) + i\eta\mathcal{F}\left(e^{-\xi t}T\right)(\eta),\ \ \ \varphi\right\rangle \ \text{ by (2.5.12)} \\ &= \left\langle p\mathcal{F}\left(e^{-\xi t}T\right)(\eta),\ \ \ \varphi\right\rangle \\ &= \left\langle p\mathcal{L}\left(T\right)(p),\ \ \varphi\right\rangle. \end{aligned}$$

Hence by induction on $k \in \mathbb{N}$, we have $\left\langle \mathcal{L}\left(\frac{d^kT}{dt^k}\right)(p),\ \ \varphi\right\rangle = \left\langle p^k\mathcal{L}(T)(p),\ \varphi\right\rangle$, which means that

$$\mathcal{L}\left(\frac{d^kT}{dt^k}\right)(p) = p^k\mathcal{L}(T)(p) \ \text{ in } \ \mathcal{S}'(\mathbb{R}_\eta). \tag{2.5.16}$$

Our aim at this stage is to characterize Laplace transform of distributions in $L^2(0, +\infty)$. This is achievable first by considering the next definition.

Definition 2.5.9. ([67]) *(Hardy-Lebesgue Space)*

The Hardy-Lebesgue space denoted by $H^2(0)$ is defined as the set of functions $V : p \longrightarrow V(p)$ from the half complex plane

$$\mathbb{C}_+ = \{p = \xi + i\eta \in \mathbb{C}, \;\; \xi > 0\}$$

into the space $\mathbb{C}$ such that the following two conditions are satisfied:

1. *The function $V(p)$ is holomorphic for $\xi > 0$;*

2. *For each $\xi > 0$, the function $\eta \longrightarrow V(\xi + i\eta)$ is of class $L^2(-\infty, +\infty)$ such that*

$$\sup_{\xi>0}\left(\int_{-\infty}^{\infty} |V(\xi + i\eta)|^2 d\eta\right) < +\infty.$$

Proposition 2.5.10. ([67])
Let $v(t) \in L^2(0, +\infty)$. Then its Laplace transform $\widehat{v}(p)$ exists for $\xi \geq 0$ and $\widehat{v}(p) \in H^2(0)$.

Proof. Let $\xi \geq 0$. We denote by $\widetilde{v}(t)$ the extension of $v(t)$ by 0 outside $(0, +\infty)$ given in (2.5.2). Then, the function $t \in \mathbb{R} \longrightarrow e^{-\xi t}\widetilde{v}(t)$ is of class $L^2(-\infty, +\infty)$ and is therefore a tempered distribution. In other words $\xi \in I_{\widetilde{v}}$; in fact $[0, +\infty) \subset I_{\widetilde{v}}$ and thus $intI_{\widetilde{v}} \neq 0$. Thus, in view of Definition 2.5.8, $\widehat{v}(p)$ is well-defined for $p = \xi + i\eta$ with $\xi \geq 0$. The holomorphic property of $p \longrightarrow \widehat{v}(p)$ follows from Proposition 2.5.6 and Definition 2.5.8.

For condition 2 we have using the extension to L^2 of (2.5.3) and of the Parseval identity (2.5.8)

$$\begin{aligned}
\int_{-\infty}^{\infty} |\widehat{v}(\xi + i\eta)|^2 d\eta &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |\mathcal{F}(\widetilde{v}(t)e^{-t\xi})(\eta)|^2 d\eta \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |\widetilde{v}(t)e^{-t\xi}|^2 dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} |v(t)|^2 e^{-2t\xi} dt \\
&\leq \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} |v(t)|^2 \quad \text{since} \quad \xi > 0.
\end{aligned}$$

Hence

$$\sup_{\xi>0}\left(\int_{-\infty}^{\infty}|\widehat{v}(\xi+i\eta)|^2 d\eta\right) \leq \frac{1}{\sqrt{2\pi}}\int_0^{\infty}|v(t)|^2 dt < +\infty. \tag{2.5.17}$$

□

Theorem 2.5.11. ([67])(*Paley-Wiener Theorem)*

Let $V(\xi+i\eta) \in H^2(0)$. Then the boundary function $V(i\eta)$ of $V(\xi+i\eta)$ exists in $L^2(-\infty,+\infty)$ in the sense that

$$\lim_{\xi\to 0}\int_{-\infty}^{+\infty}|V(i\eta)-V(\xi+i\eta)|^2 d\eta = 0. \tag{2.5.18}$$

Furthermore, there exists a function $t \longrightarrow v(t)$ of class $L^2(-\infty,+\infty)$ such that $v(t)=0$ for $t<0$ and $V(\xi+i\eta)$ with $\xi>0$, is the Laplace transform of $v(t)$ at $p=\xi+i\eta$.

2.5.3 Laplace transform of vector-valued distributions

After the definition of the Laplace transform of scalar distributions, we extend this definition to vector-valued distributions. We denote by X the Hilbert space, with norm $\|\cdot\|_X$, in which the vector distributions take values.

Definition 2.5.12. ([22])

1. *We denote by $\mathcal{D}(X)$, the space of functions $t \to f(t)$ from $\mathbb{R}$ into X which are of class C^∞ and which have compact support.*
   $\mathcal{D}(X)$ is equipped with a pseudo-topology according to which a sequence (φ_j) converges to φ whenever we have the following conditions:
   - *there exists a compact set K of $\mathbb{R}$, such that*

     $$supp(\varphi_j) \subset K, \;\; \forall j \geq 1, \;\; supp(\varphi) \subset K$$

   - *$\varphi_j^{(n)}$ converges to $\varphi^{(n)}$ in X uniformly on K, for every $n \in \mathbb{N}$.*
2. *We denote by $\mathcal{D}_+(X)$ the subspace of $\mathcal{D}(X)$ consisting of vector-valued functions with support limited to the left i.e. contained in some $[\alpha,+\infty)$. The space $\mathcal{D}_+(X)$ is equipped*

with a pseudo-topology in which a sequence of functions $\varphi_j \in \mathcal{D}_+(X)$ converges to φ in $\mathcal{D}_+(X)$ if

- *the functions φ_j and φ are zero for $t_0 \leq t$, where t_0 is independent of j*
- *$\varphi_j^{(n)}$ converges uniformly to $\varphi^{(n)}$ in X over all compact set in $[\alpha, +\infty[$.*

Remark 2.5.13. *The corresponding space denoted by $\mathcal{D}_-(X)$ is the subspace of $\mathcal{D}(X)$ consisting of vector-valued functions with support limited to the right i.e. contained in some $(-\infty, \alpha])$. $\mathcal{D}_-(X)$ also has a pseudo-topology similar to the one in Definition 2.5.12(2).*
We recall that to avoid confusion, we will, whenever it is necessary, write $\mathbb{R}_t$ to emphasize that the argument of the functions $\varphi \in \mathcal{D}(X)$ is "t". We also would like to emphasize that, if $X = \mathbb{C}$ or $\mathbb{R}$, then the spaces described above will be written as follows:

$$\mathcal{D}(X) = \mathcal{D}, \;\; \mathcal{D}_+(X) = \mathcal{D}_+ \;\; \textit{and} \;\; \mathcal{D}_-(X) = \mathcal{D}_-$$

Definition 2.5.14. ([*22*])
We denote by $\mathcal{D}'(X)$ the space of distributions over $\mathbb{R}_t$, with values in X, defined by

$$\mathcal{D}'(X) := \mathcal{L}(\mathcal{D}; X)$$

where $\mathcal{L}(\mathcal{D}; X)$ is the space of continuous linear mapping from $\mathcal{D}$ into X.

The space $\mathcal{D}'(X)$ is equipped with the topology of uniform convergence over bounded subsets of $\mathcal{D}$. To emphasize on this, we denote $\mathcal{L}(\mathcal{D}, X)$ by $\mathcal{L}_\sigma(\mathcal{D}, X)$ where σ is the collection of bounded subsets of $\mathcal{D}$. Given the importance of this topology in what follows, we spend some space and time to make it more explicit. We do this by considering the following useful concepts of the space $\mathcal{D}(\mathbb{R})$ found in [15],[26], [27], [61] and [62].

Definition 2.5.15. ([*15*], [*26*])
Let $\mathbb{A} \subset \mathcal{D}(\mathbb{R})$. The subset $\mathbb{A}$ is said to be bounded if there exists a compact subset $K \subset \mathbb{R}$ such that

1. *$\forall\, \varphi \in \mathbb{A}$,*

$$supp(\varphi) \subset K, \tag{2.5.19}$$

2. $\forall m \in \mathbb{N}$, *there exists* $M_m > 0$, *such that*

$$\sup_{x\in\mathbb{R}} |\frac{d^p\varphi(x)}{dx^p}| \leq M_m, \ \forall \ \ p \leq m. \tag{2.5.20}$$

Instead of the pseudo-topology of $\mathcal{D}(\mathbb{R})$ given in Definition 2.3.1, we want now to specify Schwartz canonical topology of $\mathcal{D}(\mathbb{R})$. To this end, let us take $(K_n)_{n\geq 1}$ to be an increasing sequence of compact sets in $\mathbb{R}$ such that

$$\cup_n K_n = \mathbb{R}.$$

For each compact set K_n, we denote by $\mathcal{D}_{K_n}(\mathbb{R})$ the subspace of $\mathcal{D}(\mathbb{R})$ that consists of functions

$$\rho \in C_0^\infty(\mathbb{R}) \quad \text{such that} \quad supp(\rho) \subseteq K_n.$$

On each $\mathcal{D}_{K_n}(\mathbb{R})$, we introduce the sequence of semi-norms $(P_{K_{n,m}})_{m\geq 1}$ defined by

$$P_{K_{n,m}}(\rho) = \sup_{x\in K_n} \left| \frac{d^m}{dx^m}\rho(x) \right|.$$

By a standard procedure [26, 27], the sequence $(P_{K_{n,m}})_{m\geq 1}$ generates on $\mathcal{D}_{K_n}(\mathbb{R})$ a structure of locally convex topological vector space, with topology denoted by $\mathcal{T}_{K_n}$. From the same references, it is known that a fundamental system of neighborhoods of 0 for the topology $\mathcal{T}_{K_n}$ consists of the sets

$$V(m,\epsilon) := \left\{ \rho \in \mathcal{D}_{K_n}(\mathbb{R}) : \sup_{\substack{x\in K_n \\ 0\leq j\leq m}} \left| \frac{d^j}{dx^j}\rho(x) \right| \leq \epsilon \right\}, \ \ \epsilon > 0, \ \ m \in \mathbb{N}. \tag{2.5.21}$$

It is clear that

$$\mathcal{D}(\mathbb{R}) = \cup_{n=1}^\infty \mathcal{D}_{K_n}(\mathbb{R}). \tag{2.5.22}$$

The said Schwartz canonical topology $\mathcal{T}$ of $\mathcal{D}(\mathbb{R})$ is the inductive limit of the topologies $(\mathcal{T}_{K_n})_{n\geq 1}$. That is, $\mathcal{T}$ is the largest but not discrete locally convex topology on $\mathcal{D}(\mathbb{R})$ that makes all the embeddings $\mathcal{D}_{K_n}(\mathbb{R}) \hookrightarrow \mathcal{D}(\mathbb{R})$ continuous. Thus V is a convex neighborhood of 0 in $\mathcal{D}(\mathbb{R})$ if and only if $V \cap \mathcal{D}_{K_n}(\mathbb{R})$ is a neighborhood of 0 in $\mathcal{D}_{K_n}(\mathbb{R})$ for every n.

The topology $\mathcal{T}$ of $\mathcal{D}(\mathbb{R})$ is generated by a family of semi-norms obtained as follows from an increasing sequence of non-negative integers $(m_j)_{j\geq 0}$ where $m_j \to \infty$ as $j \to \infty$ and a decreasing sequence of positive real numbers $(\epsilon_j)_{j\geq 0}$ such that $\epsilon_j \to 0$:

$$N(\{m_j\}, \{\epsilon_j\})(\rho) := \sup_j \left(\sup_{\substack{|x|\geq j \\ 0\leq\alpha\leq m_j}} \frac{|\frac{d^\alpha \rho(x)}{dx^\alpha}|}{\epsilon_j} \right). \tag{2.5.23}$$

In line with (2.5.21) we introduce the set

$$V(\{m_j\}, \{\epsilon_j\}) := \left\{ \rho \in \mathcal{D}(\mathbb{R}) : \forall j\ |x| > j \text{ and } \ 0 \leq \alpha \leq m_j \quad \left| \frac{d^\alpha \rho(x)}{dx^\alpha} \right| \leq \epsilon_j, \right\}$$

which forms a fundamental system of neighborhoods of 0 in $\mathcal{D}(\mathbb{R})$ when $\{m_j\}$ and $\{\epsilon_j\}$ vary arbitrary.

Our next task is to be more explicit about the topology of $\mathcal{D}'(X)$ given in Definition 2.5.14. To this end, let Y be a locally convex topological vector space with topology generated in a standard way ([26, 27]) by a family of semi-norms

$$\mathcal{W}_I = \{q_\alpha,\ \alpha \in I\}.$$

We define $\mathcal{L}(\mathcal{D}, Y)$ as the space of linear continuous operators from $\mathcal{D}$ into Y. To understand the topology of $\mathcal{L}(\mathcal{D}, Y)$, we denote by σ the collection of all bounded subset of $\mathcal{D}(\mathbb{R})$ as defined in Definition 2.5.15. With each $\mathbb{A} \in \sigma$ and $\alpha \in I$, we associate a semi-norm $q_{\alpha,\mathbb{A}}$ on $\mathcal{L}(\mathcal{D}, Y)$ defined by

$$q_{\alpha,\mathbb{A}}(T) = \sup_{\rho\in\mathbb{A}} q_\alpha(T(\rho)).$$

The family of semi-norms

$$\mathcal{W}_{I,\sigma} = \{q_{\alpha,\mathbb{A}} : \alpha \in I,\ \ \mathbb{A} \in \sigma\} \tag{2.5.24}$$

defines on $\mathcal{L}(\mathcal{D}, Y)$ a locally convex (vector) topology called σ-topology. Thus again the notation $\mathcal{L}_\sigma(\mathcal{D}, Y)$.

Definition 2.5.16. *If $\mathcal{Y}_0$ denotes the collection of balanced neighborhoods of* 0 *for the topology of Y, then a fundamental system of neighborhood of* 0 *for the σ-topology of $\mathcal{L}(\mathcal{D}, Y)$ is given by*

$$\mathcal{B} = \{V(A, M) \subset \mathcal{L}(\mathcal{D}, Y) : \forall A \in \sigma_f, \ \forall M \in \mathcal{Y}_0\}$$

where σ_f is the collection of finite union of bounded set in σ and

$$V(A, M) = \{T \in \mathcal{L}(\mathcal{D}, Y) : T(A) \subset M\}.$$

We recall that all these concepts can be found in [26, 27]).

Proposition 2.5.17. *Let (T_j) be a sequence in $\mathcal{L}_\sigma(\mathcal{D}, Y)$ and let $T \in \mathcal{L}_\sigma(\mathcal{D}, Y)$ where the local convex topology of Y is generated by a filtered family $\mathcal{W} = \{q_\alpha, \ \alpha \in I\}$ of semi-norms. Then the following statements are equivalent:*

1. *The sequence (T_j) converges to T in $\mathcal{L}_\sigma(\mathcal{D}, Y)$. That is for any neighborhood V of* 0 *in $\mathcal{L}_\sigma(\mathcal{D}, Y)$, there exists an integer $j_0 = j_0(V)$ such that $T_j - T \in V$ whenever $j \geq j_0$.*

2. *The sequence (T_j) converges to T uniformly on any bounded subset $\mathbb{A} \in \sigma$. That is for any neighborhood $\mathcal{W}$ of* 0 *in Y and any $\mathbb{A} \in \sigma$, there exists $j_0 = j_0(\mathbb{A}, \mathcal{W})$ such that*

$$T_j(\rho) - T(\rho) \in \mathcal{W} \ \text{ for any } \ \rho \in \mathbb{A} \ \text{ whenever } \ j \geq j_0.$$

3. *For any $\alpha \in I$, and $\mathbb{A} \in \sigma$ the sequence of real-valued numbers*

$$q_\alpha(T_j(\rho) - T(\rho)) \ \text{ converges to } \ 0 \ \text{ uniformly on } \ \mathbb{A}.$$

Proof. To prove that (1) implies (2), let $\mathbb{A} \in \sigma$ and $\mathcal{W}$ be a neighborhood of 0 in Y. Then the set $V(\mathbb{A}, \mathcal{W})$ introduced in Definition 2.5.16 is a neighborhood of 0 in $\mathcal{L}_\sigma(\mathcal{D}, Y)$. Since by assumption (1), $T_j \to T$ in $\mathcal{L}_\sigma(\mathcal{D}, Y)$, there exists $j_0 = j_0(\mathbb{A}, \mathcal{W})$ such that $T_j - T \in V(\mathbb{A}, \mathcal{W})$ for $j \geq j_0$. By definition of $V(\mathbb{A}, \mathcal{W})$, we have $T_j(\rho) - T(\rho) \in \mathcal{W}, \ \rho \in \mathbb{A}$, for $j \geq j_0$. This proves (2).

Assume that (2) is true and let us prove (3). Fix $\epsilon > 0, \ \alpha \in I$ and $\mathbb{A} \in \sigma$ so that the set $\mathcal{W} = \{y \in Y; \ q_\alpha(\rho) < \epsilon\}$ in a neighborhood of 0 in Y. Using (2), we can find $j_0 = j_0(\epsilon, \alpha, \mathbb{A})$ such that $T_j(\rho) - T(\rho) \in \mathcal{W}$ for any $\rho \in \mathbb{A}$ and $j \geq j_0$. By definition of $\mathcal{W}$, we have $q_\alpha(T_j(\rho) - T(\rho)) < \epsilon$ for every $\rho \in \mathbb{A}$ whenever $j \geq j_0$ where j_0 does not depend on ρ. This proves (3).

To conclude, we assume that (3) holds and we want to prove (1). To this end let $\mathcal{V}$ be a neighborhood of 0 in $\mathcal{L}_\sigma(\mathcal{D}, Y)$. By the definition of the fundamental system of neighborhood of 0 given in Definition 2.5.16, there exist $\mathbb{A}_k \in \sigma, \ \ 1 \leq k \leq s$, and $\mathcal{W}$ a neighborhood of 0 in Y such that

$$V(\cup_{k=1}^{s} \mathbb{A}_k, \ \mathcal{W}) \subset \mathcal{V}.$$

It is easy to show that

$$V(\cup_{k=1}^{s} \mathbb{A}_k, \ \mathcal{W}) = \cap_{k=1}^{s} V(\mathbb{A}_k, \ \mathcal{W}).$$

On the other hand since the topology of Y is generated by the filtered family $\{q_\alpha, \ \alpha \in I\}$ of semi-norms there exists $\alpha_0 \in I$ and $\epsilon > 0$ such that the ball

$$\mathcal{W}_{\epsilon_0} := \{y \in Y; q_{\alpha_0}(y) < \epsilon_0\} \subset \mathcal{W}.$$

Applying the assumption in (3) to $\alpha_0, \ \epsilon_0$ and each $1 \leq k \leq s$, there exists an integer j_k such that $q_{\alpha_0}(T_j(\rho) - T(\rho)) < \epsilon_0$ for any $\rho \in A_k$ and $j \geq j_k$. Take $j_0 = j_1 + .. \cdot + j_k$. Then for $j \geq j_0$, we have $q_{\alpha_0}(T_j(\rho) - T(\rho)) < \epsilon_0$ for $\rho \in \cup_{k=1}^{s} \mathbb{A}_k$. This means that

$$T_j - T \in V(\cup_{k=1}^{s} \mathbb{A}_k, \ \mathcal{W}_{\epsilon_0}) \subset V(\cup_{k=1}^{s} \mathbb{A}_k, \ \mathcal{W}) \subset \mathcal{V}$$

for $j \geq j_0$. This proves (1). $\square$

Remark 2.5.18. *Proposition 2.5.17 motivates the fact that the σ-topology of $\mathcal{L}_\sigma(\mathcal{D}, Y)$ is also called the topology of uniform convergence on bounded subsets of $\mathcal{D}$.*

The material collected until now enable us to deal with the particular case of the space $\mathcal{L}(\mathcal{D}, \mathcal{D}) \equiv \mathcal{L}_\sigma(\mathcal{D}, \mathcal{D})$ where $Y = \mathcal{D}(\mathbb{R})$. With the family of semi-norms $N(\{m_j\}, \{\epsilon_j\})$ in (2.5.23) that generate the topology of $\mathcal{D}$, we associate the family of semi-norms $N_{\mathbb{A}}(\{m_j\}, \{\epsilon_j\})$, $\mathbb{A} \in \sigma$, on $\mathcal{L}(\mathcal{D}, \mathcal{D})$ defined by

$$N_{\mathbb{A}}(\{m_j\}, \{\epsilon_j\})(T) = \sup_{\rho \in \mathbb{A}} N(\{m_j\}, \{\epsilon_j\})(T(\rho)).$$

By the approach followed ealier in the general case, the family of semi-norms $N_{\mathbb{A}}(\{m_j\}, \{\epsilon_j\})$,

$\mathbb{A} \in \sigma$, generate the σ-topology of the space $\mathcal{L}(\mathcal{D}, \mathcal{D})$, which as shown in Proposition 2.5.17 and Remark 2.5.18 is the topology of uniform convergence on bounded subsets of $\mathcal{D}$.

With the above useful concepts on the space $\mathcal{D}(\mathbb{R})$, we return to the initial space $\mathcal{D}'(X)$. We also consider the notation $\mathcal{D}'_+(X)$ and $\mathcal{D}'_-(X)$ to represent the subspaces of $\mathcal{D}'(X)$ consisting of distributions with supports limited to the left and right, respectively:

$$\mathcal{D}'_+(X) := \mathcal{L}(\mathcal{D}_-; X), \quad \mathcal{D}'_-(X) := \mathcal{L}(\mathcal{D}_+; X).$$

The vector distributions in $\mathcal{D}'(X)$ have generally a very complex structure. That is why we approximate them by distributions that are relatively easy to work with. The first step is to define the tensor product of a distribution T with v.

Definition 2.5.19. ([*22*])
Given $T \in \mathcal{D}'$ and $v \in X$ we defined $T \otimes v \in \mathcal{D}'(X)$ the tensor product of T and v by

$$(T \otimes v)(\varphi) = \langle T, \varphi \rangle_{\mathcal{D}' \times \mathcal{D}} v, \quad \varphi \in \mathcal{D}(\mathbb{R}). \tag{2.5.25}$$

Definition 2.5.20. ([*27*])
A linear operator $T : \mathcal{D}(\mathbb{R}) \longmapsto X$ is a finite operator, if there exists $p_1, p_2, ..., p_n \in \mathcal{D}'(\mathbb{R})$ and $g_1, g_2, ..., g_n \in X$ such that

$$T(f) = \sum_{i=1}^{n} p_i(f) g_i. \tag{2.5.26}$$

More generally, we denote by $\mathcal{D}'(\mathbb{R}) \otimes X \equiv \mathcal{D}' \otimes X$ the subspace of $\mathcal{D}'(X)$ consisting of finite operators:

$$\mathcal{D}' \otimes X = \left\{ T \in \mathcal{D}'(X), \quad T = \sum_{j=1}^{n_T} T_j \otimes v_j, \;\; T_j \in \mathcal{D}', \;\; v_j \in X \right\}. \tag{2.5.27}$$

In the same way, we could define the subspaces of $\mathcal{D}'_+(X)$ and $\mathcal{D}'_-(X)$ denoted by $\mathcal{D}'_+ \otimes X$ and $\mathcal{D}'_- \otimes X$, respectively. We are now in a position to state the main theorem of this section, on which the definition and the properties of the Laplace transform of vector-valued distributions are based.

Theorem 2.5.21. (*[22]*)
The subspace $\mathcal{D}' \otimes X$ is dense in $\mathcal{D}'(X)$. Equally $\mathcal{D}'_+ \otimes X$ and $\mathcal{D}'_- \otimes X$ are dense in $\mathcal{D}'_+(X)$ and $\mathcal{D}'_-(X)$, respectively.

The proof of Theorem 2.5.21 is not straightforward. It will follow from a series of topological concepts of the space $\mathcal{D}(\mathbb{R})$ described after Definition 2.5.15 as well as on the results that we consider now.

Theorem 2.5.22. (*[61]*)
The space $\mathcal{D} \equiv \mathcal{D}(\mathbb{R})$ satisfies the strict approximation property. That is, the identity operator $I \in \mathcal{L}_\sigma(\mathcal{D}(\mathbb{R}), \mathcal{D}(\mathbb{R}))$ can be approximated in $\mathcal{L}_\sigma(\mathcal{D}(\mathbb{R}), \mathcal{D}(\mathbb{R}))$ by a sequence of finite operators.

Proof. Let $(\alpha_\nu)_{\nu\geq 1}$ be a sequence in $\mathcal{D}(\mathbb{R})$ such that the sequence $(\alpha_\nu^2)_{\nu\geq 1}$ is a partition of unity of $\mathbb{R}$ sub-ordinate to the open covering $(Q_\nu)_{\nu\geq 1}$ of $\mathbb{R}$ where $Q_\nu = (-\nu, \nu)$. Thus we have

$$\sum_{\nu\geq 1} \alpha_\nu^2(x) = 1 \;\forall x \in \mathbb{R}. \tag{2.5.28}$$

Let $\psi \in C^\infty(\mathbb{R})$ be such that $supp(\psi) \subset Q_\nu$ i.e. $\psi \in \mathcal{D}_{Q_\nu}(\mathbb{R})$.

We associate with ψ the unique periodic function $\widetilde{\psi}_\nu$ of period 2ν defined by

$$\widetilde{\psi}_\nu(x) = \psi(x) \text{ if } x \in Q_\nu. \tag{2.5.29}$$

We can therefore expand $\widetilde{\psi}_\nu$ in Fourier series

$$\widetilde{\psi}_\nu(x) = \sum_{l\in\mathbb{Z}} c_{l,\nu}(\psi) e^{-i\pi l x/\nu}. \tag{2.5.30}$$

By the properties of Fourier series, the linear functional

$$\psi \rightsquigarrow c_{l,\nu}(\psi) \tag{2.5.31}$$

is continuous in the following sense of the pseudo-topology of $\mathcal{D}_{Q_\nu}(\mathbb{R})$:

If a sequence $(\psi_j)_{j\geq 1}$ in $\mathcal{D}_{Q_\nu}(\mathbb{R})$ converges to zero i.e.

$$\forall m \in \mathbb{N} \quad \frac{d^m \psi_j}{dx^m} \text{ converges to 0 uniformly on } Q_\nu,$$

then the sequence of scalars $c_{l,\nu}(\psi_j)$ converges to 0 as $j \to +\infty$.

The next step is to construct a finite operator L_k for $k \in \mathbb{N}$. To this end let $\rho \in \mathcal{D}(\mathbb{R})$ be given. By the partition of unity property (2.5.28) and by the Fourier series expansion (2.5.30), we have consecutively the following for any $x \in \mathbb{R}$:

$$\begin{aligned}
\rho &= \sum_{\nu\geq 1} \alpha_\nu^2(x)\rho \\
&= \sum_{\nu\geq 1} \alpha_\nu(x)\alpha_\nu\rho \\
&= \sum_{\nu\geq 1} \alpha_\nu(x) \sum_{l\in\mathbb{Z}} c_{l,\nu}(\alpha_\nu\rho)e^{-i\pi lx/\nu} \\
&= \sum_{\nu\geq 1} \sum_{l\in\mathbb{Z}} \alpha_\nu(x)c_{l,\nu}(\alpha_\nu\rho)e^{-i\pi lx/\nu}.
\end{aligned}$$

From this, we construct a finite operator L_k by the following truncation process:

$$L_k\rho := \sum_{\substack{\nu\geq 1\\ |l|<k}} \alpha_\nu(x/2\nu)c_{l,\nu}(\alpha_\nu\rho)e^{-i\pi lx/\nu}. \tag{2.5.32}$$

In view of the continuity stated in (2.5.31), the L_k is continuous from $\mathcal{D}(\mathbb{R})$ into $\mathcal{D}(\mathbb{R})$.

We now show that, for a fixed $\rho \in \mathcal{D}(\mathbb{R})$, $L_k\rho$ converges to ρ in $\mathcal{D}(\mathbb{R})$ as $k \to \infty$. Since $supp(\rho)$ is compact, there exists $k_0 \geq 1$ such that $supp\rho \subset Q_{k_0}$ and

$$\alpha_\nu\rho \equiv 0 \text{ for all } \nu \geq k_0. \tag{2.5.33}$$

Thus (2.5.32) becomes

$$L_k\rho = \sum_{\substack{\nu<k_0\\ |l|<k}} \alpha_\nu(x/2\nu)c_{l,\nu}(\alpha_\nu\rho)e^{-i\pi lx/\nu}. \tag{2.5.34}$$

Clearly, from (2.5.33), $supp(L_k\rho) \subset supp(\rho) \cap Q_{k_0}$ for all $k \geq 1$. For $k \to +\infty$, the sequence $L_k\rho$ in (2.5.34) converges uniformly on $supp(\rho) \cap Q_{k_0}$ to

$$\begin{aligned}
\sum_{\substack{\nu<k_0\\ l\in\mathbb{Z}}} \alpha_\nu c_{l,\nu}(\alpha_\nu\rho)e^{-i\pi\frac{lx}{\nu}} &= \sum_{\nu<k_0} \alpha_\nu\widetilde{\alpha_\nu\rho}(x) \text{ by (2.5.30)} \\
&= \rho(x). \qquad (2.5.35)
\end{aligned}$$

The same thing applies by induction to the derivatives of $(L_k\rho)$.

Let now $\mathbb{A}$ be a bounded subset of $\mathcal{D}(\mathbb{R})$. By Definition 2.5.15, there exists a compact set $K \subset \mathbb{R}$ such that (2.5.19) and (2.5.20) hold. In view of (2.5.19), the argument used to prove (2.5.33) can be adapted to obtain the following: there exists $k_0 \geq 1$ such that

$$\alpha_\nu \rho \equiv 0 \;\; \forall \nu \geq k_0 \;\text{ and }\; \forall \rho \in \mathbb{A}. \tag{2.5.36}$$

Thus the sequence $L_k\rho$ converges to ρ uniformly on $\mathbb{A}$ and K in the sense that

$$\lim_{k\to\infty} \sup_{\substack{\rho\in\mathbb{A}\\ x\in K}} \left| \frac{d^m}{dx^m}\left[(L_k\rho)(x) - \rho(x)\right] \right| = 0 \;\; \forall\, m \in \mathbb{N}.$$

Thus L_k converges to the identity operator I in $\mathcal{L}_\sigma(\mathcal{D}(\mathbb{R}), \mathcal{D}(\mathbb{R}))$. □

Proof. (*Theorem* 2.5.21)
Let $T \in \mathcal{L}_\sigma(\mathcal{D}(\mathbb{R}), X)$. Let V_j defined by

$$V_j\rho = \sum_{k\leq n_j} c_{k,j}(\rho)\rho_{k,j} \;\text{ i.e }\; V_j = \sum_{k=1}^{n_j} c_{k,j} \otimes \rho_{k,j} \;\text{ with }\; c_{k,j} \in \mathcal{D}' \;\text{ and }\; \rho_{k,j} \in \mathcal{D}$$

be a sequence of finite operators that approximate I in $\mathcal{L}_\sigma(\mathcal{D}(\mathbb{R}), \mathcal{D}(\mathbb{R}))$ according to Theorem 2.5.22. By the continuity of T, the sequence of finite operators

$$T \circ V_j = \sum_{k=1}^{n_j} c_{k,j} \otimes T(\rho_{k,j}) \;\text{ converges to }\; T \text{ in }\; \mathcal{L}_\sigma(\mathcal{D}(\mathbb{R}), X).$$

This complete the proof. □

In what follows, we introduce another space of vector-valued distributions.

Definition 2.5.23. ([*22*])
We denote by $\mathcal{S}'(X)$ the space of tempered distributions over $\mathbb{R}_t$ with values in X, defined by

$$\mathcal{S}'(X) = \mathcal{L}(\mathcal{S}; X),$$

$\mathcal{S} \equiv \mathcal{S}(\mathbb{R})$ being equipped with the pseudo-topology given in Definition 2.3.6.

Remark 2.5.24. *The topologies of $\mathcal{S}(X)$ and $\mathcal{S}'(X)$ can be defined explicitly from appropriate family of semi-norms as we did for $\mathcal{D}'(X)$. For example the topology of $\mathcal{S}(\mathbb{R})$ is generated by the sequence of semi-norms*

$$d_{\alpha,\beta}(v) = \sup_{x\in\mathbb{R}} |x^{\alpha} \frac{d^{\beta}v(x)}{dx^{\beta}}| \ \ \alpha, \beta \in \mathbb{N}.$$

Note that a fundamental system of neighborhood of 0 for this topology is obtained in a standard way. Note also that the space $\mathcal{S}(\mathbb{R})$ is metrisable, through the metric

$$d(u, v) = \sum_{\alpha,\beta\geq 1} \frac{d_{\alpha,\beta}(u - v)}{1 + d_{\alpha,\beta}(u - v)},$$

in contrast to the space $\mathcal{D}(\mathbb{R})$.

Now given Y a locally convex topological space with topology generated by a family of semi-norms $W_I = \{q_{\alpha}, \alpha \in I\}$, the topology of the space $\mathcal{L}(\mathcal{S}, Y) \equiv \mathcal{S}'(Y)$ of linear continuous operators from $\mathcal{S}(\mathbb{R})$ into Y is generated by the family of semi-norms

$$W_{I,\sigma} = (q_{\alpha,\mathcal{A}})_{\alpha\in I,\mathcal{A}\in\sigma}$$

defined in a similar manner to (2.5.24).

In equation (2.5.27), we introduced the subspaces of $\mathcal{D}'(X)$ denoted by $\mathcal{D}' \otimes X$. In the same way, the subspace of $\mathcal{S}'(X)$ denoted by $\mathcal{S}' \otimes X$ will consists of finite operators:

$$\mathcal{S}' \otimes X = \left\{ T \in \mathcal{S}'(X), T = \sum_{j}^{n_T} T_j \otimes v_j, \ \ T_j \in \mathcal{S}'(\mathbb{R}), \ \ v_j \in X \right\}.$$

For $T \in \mathcal{S}' \otimes X$, we have

$$T(\varphi) = \sum_{j=1}^{n_T} \langle T_j, \varphi \rangle_{\mathcal{S}'\times\mathcal{S}} v_j \ \ \forall\varphi \in \mathcal{S}(\mathbb{R}). \tag{2.5.37}$$

We now state the result similar to Theorem 2.5.21.

Theorem 2.5.25. ([*22*])
The subspace $\mathcal{S}' \otimes X$ is dense in $\mathcal{S}'(X)$.

Proof. The proof of this theorem is analogous to that of Theorem 2.5.21. □

Definition 2.5.26. ([*22*])
Given a vector-valued distribution T in $\mathcal{S}' \otimes X$ with representation

$$T = \sum_{j=1}^{n_T} T_j \otimes v_j,$$

its Fourier transform denoted as in the scalar case, by $\mathcal{F}(T)$, is defined by

$$\mathcal{F}(T) = \sum_{j=1}^{n_T} \mathcal{F}(T_j) \otimes v_j. \tag{2.5.38}$$

For the Fourier transform of distributions as defined by (2.5.38), the analogous of the duality relation (2.5.11) is:

$$\text{for } \varphi \in \mathcal{S},\ \mathcal{F}(T)(\varphi) = T(\mathcal{F}(\varphi)) \tag{2.5.39}$$

Indeed, we have

$$\begin{aligned}
\mathcal{F}(T)(\varphi) &= \left(\sum_{j=1}^{n_T}(\mathcal{F}(T_j) \otimes v_j)\right)(\varphi) \ \text{ by (2.5.38)} \\
&= \sum_{j=1}^{n_T} \langle \mathcal{F}(T_j), \varphi \rangle \, v_j \\
&= \sum_{j=1}^{n_T} \langle T_j, \mathcal{F}(\varphi) \rangle \, v_j \ \text{ by (2.5.11)} \\
&= T(\mathcal{F}(\varphi)) \ \text{ by } \ (2.5.37).
\end{aligned}$$

Theorem 2.5.27. *The definition of the Fourier transform of T does not depend on its representation in Definition 2.5.26.*

Proof. Let $T \in \mathcal{S}' \otimes X$ be represented in two different ways:

$$T = \sum_{j=1}^{n_T} T_j \otimes v_j = \sum_{k=1}^{m_T} S_k \otimes u_k. \tag{2.5.40}$$

In view of (2.5.39) and (2.5.40) we have for $\varphi \in \mathcal{S}$

$$\begin{aligned}
\mathcal{F}(T)(\varphi) &= \left(\mathcal{F}(\sum_{j=1}^{n_T} T_j \otimes v_j)(\varphi)\right) \\
&= \sum_{j=1}^{n_T} \langle T_j, \mathcal{F}(\varphi)\rangle\, v_j \\
&= \sum_{j=1}^{m_T} \langle S_k, \mathcal{F}(\varphi)\rangle\, u_k \\
&= \sum_{j=1}^{m_T} \langle \mathcal{F}(S_k), \varphi\rangle\, u_k \\
&= \left(\mathcal{F}(\sum_{k=1}^{m_T} S_k \otimes v_k)(\varphi)\right). \qquad (2.5.41)
\end{aligned}$$

This proves the Theorem. □

We now proceed to extend the Fourier transform of the vector-valued distributions from the subspace $\mathcal{S}' \otimes X$ to the space of tempered vector-valued distributions $\mathcal{S}'(X)$; as a consequence of Theorem 2.5.25.

Theorem 2.5.28. *The Fourier transform $\mathcal{F}$ defined over $\mathcal{S}' \otimes X$ by (2.5.39) is uniquely extended by continuity into an isomorphism of $\mathcal{S}'(X)$ onto $\mathcal{S}'(X)$.*

Thus we have the following definition:

Definition 2.5.29. ([*22*])
Given a vector-valued distribution $T \in \mathcal{S}'(X)$, its Fourier transform denoted by $\mathcal{F}(T)$ is defined by

$$\mathcal{F}(T) = \lim_{j \to +\infty} \mathcal{F}(T_j),$$

where T_j is a sequence of finite operators in $\mathcal{S}' \otimes X$ that converges to T in $\mathcal{S}'(X)$.

The extension in Theorem 2.5.28 leads us to the connection of the Fourier transform of vector-valued distributions to the Laplace transform of vector-valued distributions. This connection is achieved by stating the analog of the set I_T introduced in the scalar case in equation (2.5.13).

Definition 2.5.30. ([22])
For $T \in \mathcal{D}'(X)$, we denote by I_T the subset of $\mathbb{R}$ given by

$$I_T = \left\{ \xi \in \mathbb{R} : \quad e^{-\xi t} T \in \mathcal{S}'(X) \right\}. \tag{2.5.42}$$

where $e^{-\xi t} T(\varphi) = T(e^{-\xi t}\varphi), \ \ \varphi \in \mathcal{S}$.

We state without proof the following result:

Proposition 2.5.31. ([22])
Let $T \in L_+(X)$ where $L_+(X)$ is the space of distributions on $\mathbb{R}$ with values in X which have a Laplace transform.

- *For all $\xi \in int(I_T)(\neq \emptyset)$, the Fourier transform of the distribution $e^{-\xi t}T$ is a function of $\mathbb{O}_M(X)$ where $\mathbb{O}_M(X)$ is the space of functions of class C^∞ with values in X which are "growing slowly in X" as are all their derivatives.*
- *The function $\mathcal{L}(T) : p \longrightarrow V(p) = \mathcal{F}(e^{-\xi t}T)(\eta)$ is holomorphic in the band $int(I_T) \times \mathbb{R}$ with values in X.*

In view of the Proposition 2.5.31, we can define the Laplace transform of vector-valued distribution as follows:

Definition 2.5.32. ([22])
Let $T \in \mathcal{D}'(X)$ be such that $int(I_T) \neq \emptyset$. The holomorphic function denoted by $\mathcal{L}(T) : p \longrightarrow \mathcal{L}(T)(p)$ and defined for $p \in int(I_T) \times \mathbb{R}$ by

$$\mathcal{L}(T)(p) := \mathcal{F}(e^{-\xi t}T)(\eta) \tag{2.5.43}$$

is called the Laplace transform of the vector-valued distribution $T \in \mathcal{D}'(X)$.

It should be noticed that for $T \in \mathcal{S}' \otimes X$ a finite operator with $intI_T \neq 0$, we have

$$\mathcal{L}\left(\frac{d^kT}{dt^k}\right) = p^k \mathcal{L}(T).$$

By the density result in Theorem 2.5.25, we have

Theorem 2.5.33. *For $T \in \mathcal{D}'_+(X)$ with $intI_T \neq 0$*

$$\mathcal{L}\left(\frac{d^kT}{dt^k}\right)(p) = p^k \mathcal{L}(T_j)(p). \tag{2.5.44}$$

After obtaining the Laplace transform of general vector-valued distributions, we restrict the analysis to vector-valued Lebesgue's space defined as follows:

Definition 2.5.34. ([*21*])
We denote by $L^2[(-\infty,+\infty);X]$ the space of (classes) of measurable functions $t \longrightarrow v(t)$ from $(-\infty,+\infty)$ into a Hilbert space X such that

$$\|v\|_{L^2[-\infty,+\infty;X]} = \left(\int_{-\infty}^{+\infty} \|v(t)\|_X^2 dt\right)^{\frac{1}{2}} < +\infty.$$

The Hardy-Lebesgue space $H^2(0)$ is extended to vector-valued functions as follows:

Definition 2.5.35. ([*21*]) *(Hardy-Lebesgue Space)*
Let X be a complex Hilbert space with norm denoted by $\|\cdot\|_X$. The Hardy-Lebesgue space denoted by $H^2[0;X]$ is defined as the set of vector-valued functions $V : p \longrightarrow V(p)$ from the half complex plane

$$\mathbb{C}_+ = \{p = \xi + i\eta \in \mathbb{C}, \;\; \xi \geq 0\},$$

into the space X such that the following two conditions are satisfied:

1. *The function $V(p)$ is holomorphic for $\xi > 0$,*
2. *For each $\xi > 0$, the vector-valued function $\eta \longrightarrow V(\xi+i\eta)$ is of class $L^2[(-\infty,+\infty);X]$ such that*

$$\sup_{\xi>0}\left(\int_{-\infty}^{\infty} \|V(\xi+i\eta)\|_X^2 d\eta\right) < +\infty.$$

Proposition 2.5.36. ([*21*])
Let $v(t) \in L^2[(0,+\infty);X]$. Then its Laplace transform $\widehat{v}(t)$ exists for $\xi \geq 0$ and $\widehat{v}(t) \in H^2[0;\ X]$.

Proof. The proof works word by word as that of the scalar case in Theorem 2.5.1 replacing everywhere the absolute value $|\cdot|$ by the Hilbert norm $\|\cdot\|_X$. □

The analogue of the Paley-Wiener theorem for vector-valued functions read as follows:

Theorem 2.5.37. ([*21*])*(Paley-Wiener Theorem)*
Let $V(p) \in H^2[0;X]$. For all $\xi > 0$, we put $v_\xi : \mathbb{R} \to X$ where

$$v_\xi(\eta) := V(\xi + i\eta).$$

Then, we have the following:

- *For $\xi \to 0$, the family of functions $v_\xi(\eta)$ converges in $L^2[(-\infty, +\infty); X]$ to some function $v_0 : \mathbb{R} \to X$ denoted by*

$$v_0(\eta) := V(i\eta),$$

and called the trace or boundary function of $V(\xi + i\eta)$;

- *There exists a $v(t) \in L^2[(-\infty, +\infty); X]$ such that $v(t) = 0$ for $t < 0$ and*

$$\mathcal{L}(v(t))(p) = \mathcal{F}\left(e^{-\xi t}\widetilde{v}(t)\right)(\eta) = v_0(\eta), \quad \text{for } \xi \geq 0 \tag{2.5.45}$$

where $\mathcal{L}$ and $\mathcal{F}$ are the Laplace and Fourier transforms of vector-valued distributions.

The final result that we shall use reads as follows:

Theorem 2.5.38. ([*20*], [*60*], [*64*])
The operator $-\Delta + p$, $p \in \mathbb{C}$, is analytic hypoelliptic. That is for any distribution $v \in \mathcal{D}'(\mathbb{R}^2)$, the fact that $(-\Delta + p)v$ is an analytic function on an open set of $\mathbb{R}^2$ implies that v is equally analytic on this open set.

Chapter 3

The Helmholtz problem in a smooth domain

In the preceding chapter, we built the theory of the Laplace transform of vector-valued distributions. We shall apply this theory to the heat equation in the next chapter. This will lead to the Helmholtz problem that will be considered in this chapter.

In section 3.1, we establish the well-posedness of the Helmholtz problem. In section 3.2, we examine the regularity of the solution of the Helmholtz problem in a smooth domain.

3.1 Well-posedness of the problem

We consider the following Dirichlet problem for the Helmholtz operator: given a complex number $p = \xi + i\eta$ and a complex-valued function g on Ω, find $w : \Omega \mapsto \mathbb{C}$, solution of

$$-\Delta w + p\, w = g \ \text{ in } \ \Omega \tag{3.1.1}$$

and

$$w = 0 \ \text{ on } \ \partial\Omega. \tag{3.1.2}$$

Here $\Omega \subset \mathbb{R}^2$ is a bounded domain. Despite the title of the chapter, we assume in this specific section that the boundary $\partial\Omega \equiv \Gamma$ is Lipschitz in the sense of Definition 2.1.1 because the results apply to the non-smooth case which is considered in the next chapter. Actual smoothness requirements on Γ will be made in the next section.

It is convenient to study problem (3.1.1)-(3.1.2) in the abstract setting of the following theorem ([40], [38]).

Theorem 3.1.1. *Let X be a Hilbert space with inner product and associated norm denoted by $(\cdot,\cdot)_X$ and $\|\cdot\|_X$ respectively. The conjugate dual of X is denoted by X' and its norm is $\|\cdot\|_{X'}$. Let $a(\cdot,\cdot)$ be a sesquilinear form, $l(\cdot)$ be a (conjugate) linear form on X. We make the following assumptions:*

1. *The linear form $l(\cdot)$ is continuous i.e. there exists a $M > 0$ such that,*

$$|l(v)| \leq M\|v\|_X, \quad \forall \ v \in X. \tag{3.1.3}$$

2. *The sesquilinear form $a(\cdot,\cdot)$ is continuous i.e there exists a constant $K > 0$*

$$|a(s,v)| \leq K||s\|_X \ \|v||_X \quad \forall \ \ s\, v \in X. \tag{3.1.4}$$

3. *The sesquilinear form $a(\cdot,\cdot)$ is X-elliptic or X-coercive i.e there exists a constant $\alpha > 0$ such that*

$$Re \ a(v,v) \geq \alpha\|v\|_X^2, \quad \forall \ v \in X. \tag{3.1.5}$$

Then the abstract variational problem of finding

$$s \in X \quad such\ that \ \ a(s,v) = l(v) \ \ \forall \ \ v \in X \tag{3.1.6}$$

is well-posed. In other words, there exists a unique $s \in X$, solution of (3.1.6) such that,

$$\|s\|_X \leq C\|l\|_{X'} \tag{3.1.7}$$

for some constant $C > 0$.

Proof. With the sesquilinear form $a(\cdot,\cdot)$, we associate the operator

$$A : X \longrightarrow X',$$

defined by

$$\langle Aw, v\rangle_{X'\times X} = a(w,v). \tag{3.1.8}$$

The variational problem (3.1.6) is then equivalent to the functional equation: find

$$s \in X \ \text{ such that } \ As = l \ \text{ in } \ X'. \tag{3.1.9}$$

It is clear from the sesquilinearity of $a(\cdot,\cdot)$ that A is linear. Likewise, A is bounded since the continuity in (3.1.4) of $a(\cdot,\cdot)$ yields

$$\|Aw\|_{X'} := \sup_{v \neq 0} \frac{|a(w,v)|}{\|v\|_X} \leq K\|w\|_X. \tag{3.1.10}$$

On the other hand, for $w \in X$, (3.1.5) and the boundedness of the form $Aw \in X'$ lead to

$$\begin{aligned} \alpha\|w\|_X^2 \leq Re\ a(w,w) &= Re\,\langle Aw, w\rangle \\ &\leq |\,\langle Aw, w\rangle\,| \\ &\leq \|Aw\|_{X'}\|w\|_X. \end{aligned}$$

Thus

$$\|Aw\|_{X'} \geq \alpha\|w\|_X \ \ \forall \ w \in X. \tag{3.1.11}$$

Let $A^\star \in B(X, X')$ be the adjoint operator of A. In the present context, it should be noted that,

$$\langle A^\star w, v\rangle_{X' \times X} = \overline{a(v,w)}. \tag{3.1.12}$$

Therefore, following the above argument that lead to (3.1.11), we obtain

$$\|A^\star w\|_{X'} \geq \alpha\|w\|_X \ \ \forall \ w \in X. \tag{3.1.13}$$

To prove the theorem, it is equivalent to show that the mapping $A : X \mapsto X'$ in the operator equation (3.1.9) is an isomorphism. We claim that the range $R(A)$ of A is dense in X'. Indeed, let φ in the bi-dual space X'' of X be such that

$$\varphi(Aw) = 0 \ \ \forall \ w \in X.$$

We show that $\varphi = 0$. The space X is reflexive, being a Hilbert space. Thus, there exists

$v \in X$ such that $\varphi = \mathbf{C}(v)$ where

$$\mathbf{C} : X \longrightarrow X''$$

is the canonical mapping of X to X''. Now for $w \in X$,

$$\begin{aligned} 0 &= \varphi(Aw) \quad \text{(by assumption)} \\ &= \mathbf{C}(v)(Aw) \\ &= \langle Aw, v \rangle \quad \text{(by definition of } \mathbf{C}) \\ &= < \overline{A^\star v, w} > \quad \text{by (3.1.8) and (3.1.12).} \end{aligned}$$

Hence $A^\star v = 0$. By (3.1.13), it follows that $v = 0$. Thus $\varphi = \mathbf{C}(v) = \mathbf{C}(0) = 0$.

We also claim that $R(A)$ is closed in X'. In fact, let (Aw_n) be a sequence in $R(A)$ such that

$$Aw_n \longrightarrow h \quad \text{in} \quad X' \quad \text{as} \quad n \longrightarrow \infty.$$

Then (Aw_n) is a Cauchy sequence in X'. By (3.1.11) and the linearity of A, we have

$$\alpha \|w_n - w_m\|_X \leq \|Aw_n - Aw_m\|_{X'},$$

which implies that (w_n) is a Cauchy sequence in X. Since X is complete, the sequence (w_n) converges to some $w \in X$. Continuity of the operator A leads to

$$Aw_n \longrightarrow Aw \quad \text{in} \quad X' \quad \text{as} \quad n \to \infty.$$

By uniqueness of limits, we have

$$h = Aw.$$

Hence $R(A)$ is closed. The density and the closedness of $R(A)$ in X' mean that the operator A is surjective. Since A is injective by (3.1.11), the operator A is bijective. The Banach open mapping theorem guarantees that A is an isomorphism. $\square$

Remark 3.1.2. *1. Theorem 3.1.1 can be proved by the Banach contraction mapping theorem. It is indeed possible to choose $\rho > 0$ such that the map*

$$v \longrightarrow v - \rho\tau(Aw - l);$$

is a contraction from X into X where

$$\tau : X' \longrightarrow X;$$

is the Riesz-representation operator (see [16]).

2. In the case when the sesquilinear form $a(\cdot,\cdot)$ is hermitian i.e.

$$a(w,v) = \overline{a(v,w)} \ \text{ so that } \ A = A^{\star},$$

Theorem 3.1.1 is the so-called Lax-Milgram lemma. Its proof is then a direct consequence of Riesz-representation theorem. In this case $a(\cdot,\cdot)$ defines an inner product on X the associated norm of which is equivalent to the norm $\|\cdot\|_X$. Note also that in this case, the variational problem (3.1.6) is equivalent to the minimization problem: find

$$s \in X \quad \text{such that} \quad J(s) = min_{v \in X} J(v) \tag{3.1.14}$$

where $J(v) := \frac{1}{2}a(v,v) - l(v)$ represents the total energy of the system under consideration. (See [16] for more details).

We want to put problem (3.1.1)-(3.1.2) in the general variational setting discussed in Theorem 3.1.1. The standard procedure to achieve this consists of four main steps described in [40]. To this end, we assume once and for all that, $g \in L^2(\Omega)$. We take $X = H_0^1(\Omega)$ and we define $a(\cdot,\cdot)$ and $l(\cdot)$ as follows:

$$a(w,v) := \int_\Omega \nabla w \nabla \bar{v} dx + \int_\Omega p w \bar{v} dx, \tag{3.1.15}$$

and

$$l(v) := \int_\Omega g \ \ \bar{v} dx. \tag{3.1.16}$$

We are therefore led to the following variational problem: find

$$w \in H_0^1(\Omega) \text{ such that } a(w,v) = l(v) \;\; \forall \;\; v \in H_0^1(\Omega). \tag{3.1.17}$$

Clearly, $a(\cdot,\cdot)$ is a sesquilinear form and $l(\cdot)$ is a conjugate or antilinear form. By the Cauchy-Schwarz inequality, the conjugate linear form in (3.1.16) is continuous on $H_0^1(\Omega)$ since

$$\begin{aligned} |l(v)| &\leq \left(\int_\Omega |g|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |v|^2 dx\right)^{\frac{1}{2}} \\ &\leq \|g\|_{0,\Omega}\|v\|_{1,\Omega}. \end{aligned} \tag{3.1.18}$$

Similarly, for $w, v \in H_0^1(\Omega)$, we have

$$\begin{aligned} |a(w,v)| &\leq \left(\int_\Omega |\nabla w|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla v|^2 dx\right)^{\frac{1}{2}} + |p| \left(\int_\Omega |w|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |v|^2 dx\right)^{\frac{1}{2}} \\ &\leq \|\nabla w\|_{0,\Omega}\|\nabla v\|_{0,\Omega} + |p|\|w\|_{0,\Omega}\|v\|_{0,\Omega} \\ &\leq (1+|p|)\|w\|_{1,\Omega}\|v\|_{1,\Omega} \end{aligned} \tag{3.1.19}$$

which show the continuity of the sesquilinear form. Regarding the H_0^1-ellipticity or H_0^1-coercivity of $a(\cdot,\cdot)$, we assume that

$$Re(p) = \xi \geq 0. \tag{3.1.20}$$

Under this assumption, we have for $w \in H_0^1(\Omega)$ and $Rep > 0$

$$\begin{aligned} Re\; a(w,w) &= \int_\Omega |\nabla w|^2 dx + Re(p)\int_\Omega |w|^2 dx \\ &\geq \min\{1, Re(p)\}\|w\|_{1,\Omega}. \end{aligned} \tag{3.1.21}$$

For $Re(p) = 0$, we have

$$Re\; a(w,w) \geq C\|w\|_{1,\Omega}^2, \tag{3.1.22}$$

by Poincaré Friedrichs inequality in Theorem 2.4.3. In summary, we have proved the following theorem:

Theorem 3.1.3. *Under the condition (3.1.20), the problem (3.1.17) is well-posed in $H_0^1(\Omega)$. More precisely, there exists a unique solution $w \in H_0^1(\Omega)$ of (3.1.17) and a constant K depending on p (except for $Re(p) = 0$) such that*

$$\|w\|_{1,\Omega} \leq K\|g\|_{0,\Omega}. \tag{3.1.23}$$

Notice that the constant K in (3.1.23) does indeed depend on p for $Re(p) > 0$ since, from (3.1.17) and (3.1.21) we have

$$\begin{aligned}
\min\{1, Re(p)\}\|w\|_{1,\Omega}^2 &\leq \int_\Omega |\nabla w|^2 dx + Re(p)\int_\Omega |w|^2 dx \\
&= Rea(w, w) \\
&= Re\int_\Omega g\overline{w}dx \\
&\leq \left(\int_\Omega |g|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |w|^2 dx\right)^{\frac{1}{2}} \quad \text{by Cauchy-Schwarz's inequality} \\
&\leq \|g\|_{0,\Omega}\|w\|_{1,\Omega}.
\end{aligned} \tag{3.1.24}$$

Thus

$$\|w\|_{1,\Omega} \leq \frac{1}{\min\{1, Re(p)\}}\|g\|_{0,\Omega} \ \text{ for } \ Re(p) > 0.$$

In the case when the unique solution w of (3.1.17) satisfies an estimate of the type (3.1.23) where the constant K does not depend on p, we will say that the problem (3.1.17) is uniformly well-posed. In order to achieve this, we work with weighted Sobolev spaces defined as follows:

Definition 3.1.4. *Given $\rho > 0$ and an integer $m \geq 0$, we denote by $H^m(\Omega, \rho)$, the Sobolev space $H^m(\Omega)$ equipped with the weighted norm*

$$\|s\|_{m,\,\Omega,\,\rho} := \sqrt{\int_\Omega \sum_{|\alpha|\leq m} \rho^{2(m-|\alpha|)}|D^\alpha s(x)|^2 dx}. \tag{3.1.25}$$

Proposition 3.1.5. *Let $\rho > 0$ be such that $\frac{x}{\rho} \in \Omega$ whenever $x \in \Omega$. Then on $H^m(\Omega)$, $m \geq 1$, integer, the weighted norm $\|\cdot\|_{m,\,\Omega,\,\rho}$ in Definition 3.1.4 is equivalent (with constants not* *depending on ρ) to the more economical weighted norm $|||\cdot|||_{m,\,\Omega,\,\rho}$ given by*

$$|||s|||^2_{m,\,\Omega,\,\rho} := \int_\Omega \left[\sum_{|\alpha|=m} |D^\alpha s(y)|^2 + \rho^{2m}|s(y)|^2 \right] dy. \tag{3.1.26}$$

Proof. Let us consider the change of variable

$$y = \frac{x}{\rho}, \text{ so that } dy = \rho^{-2}dx.$$

Given $s \in H^m(\Omega)$, we introduce the function $s_{\frac{1}{\rho}}$ given by

$$s_{\frac{1}{\rho}}(x) = s(\frac{x}{\rho}).$$

By the chain rule, we readily get

$$D_x^\alpha s_{\frac{1}{\rho}}(x) = \rho^{-|\alpha|} D_y s(y), \text{ for } |\alpha| \leq m.$$

This implies that we have

$$\rho^{1-m}\|s\|_{m,\,\Omega,\,\rho} = \|s_{\frac{1}{\rho}}\|_{m,\,\Omega} \text{ and } \rho^{1-m}|||s|||_{m,\,\Omega,\,\rho} = |||s_{\frac{1}{\rho}}|||_{m,\,\Omega} \tag{3.1.27}$$

where the economical norm $|||\cdot|||_{m,\,\Omega}$ is defined by

$$|||v|||^2_{m,\,\Omega} = \int_\Omega \left(\sum_{|\alpha|=m} |D^\alpha v(y)|^2 + |v(y)|^2 \right) dy. \tag{3.1.28}$$

But for Ω bounded (as in our case), the usual norm $\|\cdot\|_{m,\,\Omega}$ on $H^m(\Omega)$ is equivalent to $|||\cdot|||_{m,\Omega}$. (see Theorem 1.8 in [54]). This combined with (3.1.27) proves the proposition. □

Remark 3.1.6. *From Proposition 3.1.5, it follows that one can either work with the norm (3.1.25) or (3.1.26). The latter weighted norm is the one adopted in [19] and [46]. Note that the equivalence of norms stated in Proposition 3.1.5 holds for bounded domains. That is why in the case of G an infinite sector we will work with (3.1.25).*

Theorem 3.1.7. *Under the condition (3.1.20), the problem (3.1.17) is uniformly well-posed in the sense that its unique solution w obtained in Theorem 3.1.3 is such that*

$$\|w\|_{1,\Omega,1+|p|} \leq C\|g\|_{0,\Omega} \tag{3.1.29}$$

where $C > 0$ represents here and after in the thesis various constants that depend neither on p nor on other parameters such as the space step size $h = \Delta x$ and the time step size $k = \Delta t$ in the numerical part of the work.

Proof. We know from (3.1.15), (3.1.16) and (3.1.17) where v is replaced by the solution w that

$$\int_\Omega \left(|\nabla w|^2 + p|w|^2\right) dx = \int_\Omega g\bar{w} dx,$$

or

$$\int_\Omega |\nabla w|^2 dx + \xi \int_\Omega |w|^2 dx + i\eta \int_\Omega |w|^2 dx = \int_\Omega g\bar{w} dx. \tag{3.1.30}$$

Taking the real parts of each side of (3.1.30), we have in view of (3.1.20)

$$\int_\Omega \xi^2 |w|^2 dx \leq \int_\Omega |g|\ |\xi w| dx. \tag{3.1.31}$$

By Cauchy-Schwarz inequality, (3.1.31) leads to

$$\int_\Omega \xi^2 |w|^2 dx \leq \left(\int_\Omega |g|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega \xi^2 |w|^2 dx\right)^{\frac{1}{2}},$$

which implies that

$$\int_\Omega \xi^2 |w|^2 dx \leq \int_\Omega |g|^2 dx. \tag{3.1.32}$$

Similarly, considering the imaginary parts of both sides of (3.1.30) yields

$$\int_\Omega |\eta|^2 |w|^2 dx \leq \int_\Omega |g|^2 dx. \tag{3.1.33}$$

Finally from the real part of (3.1.30) using Cauchy-Schwarz inequality, we have

$$\int_\Omega |\nabla w|^2 dx \leq \left(\int_\Omega |g|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |w|^2 dx\right)^{\frac{1}{2}}$$

from where we have, in view of Poincaré Friedrichs inquality in Theorem 2.4.3

$$\int_\Omega \left(|\nabla w|^2 + |w|^2\right) dx \leq C \int_\Omega |\nabla w|^2 dx \leq C \left(\int_\Omega |g|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla w|^2 + |w|^2 dx\right)^{\frac{1}{2}}.$$

Thus

$$\int_\Omega \left(|\nabla w|^2 + |w|^2\right) dx \leq C \int_\Omega |g|^2 dx \tag{3.1.34}$$

Adding (3.1.32), (3.1.33) and (3.1.34), we have

$$\int_\Omega |\nabla w|^2 dx + (1+|p|)^2 \int_\Omega |w|^2 dx \leq 2(2+C) \int_\Omega |g|^2 dx, \tag{3.1.35}$$

in view of the identity

$$(1+|p|^2) \leq (1+|p|)^2 \leq 2(1+|p|^2). \tag{3.1.36}$$

Hence the theorem follows from (3.1.35). □

Remark 3.1.8. *The variational problem (3.1.17) solved in Theorem 3.1.3 is the distributional formulation of the Helmholtz problem (3.1.1)-(3.1.2) as explained below. Since the two sides of (3.1.17) are continuous on $H_0^1(\Omega)$ and $\mathcal{D}(\Omega)$ is dense in $H_0^1(\Omega)$, then the variational equation (3.1.17) is equivalent to the one obtained by replacing $v \in H_0^1(\Omega)$ with $v \in \mathcal{D}(\Omega)$. Furthermore, by the definition of the differentiation of distributions (Definition 2.3.8), (3.1.17) is equivalent to*

$$\langle -\Delta w + p\, w, \bar{v}\rangle_{\mathcal{D}'\times\mathcal{D}} = \langle g, \bar{v}\rangle \quad \textit{for all} \;\; v \in \mathcal{D}(\Omega). \tag{3.1.37}$$

Thus w is the solution of the distributional partial differential equation,

$$w \in H_0^1(\Omega),\; -\Delta w + p\, w = g \quad \textit{in} \;\; \mathcal{D}'(\Omega). \tag{3.1.38}$$

Remembering that $H_0^1(\Omega) = \{w \in H^1(\Omega), \gamma w = 0\}$ where γ is the trace operator and that

$g \in L^2(\Omega)$ with $L^2(\Omega)$ contained in $L^1_{loc}(\Omega)$, which is continuously embedded in $\mathcal{D}'(\Omega)$, we deduce from (3.1.38) that $w \in H^1_0(\Omega)$ is the solution of the problem

$$-\Delta w + p\, w = g \ \ a.e \ in \ \ \Omega, \quad \gamma w = 0.$$

Remark 3.1.9. *We consider the Helmholtz problem (3.1.1)-(3.1.2) when the condition (3.1.20) is not satisfied. Consider the linear operator $-\Delta$ acting from the subspace $E = \{v \in H^1_0(\Omega); -\Delta v \in L^2(\Omega)\}$ equipped with the topology of $L^2(\Omega)$ into $L^2(\Omega)$:*

$$-\Delta : E \subset L^2(\Omega) \to L^2(\Omega).$$

By Green formula, the operator $-\Delta$ is self-adjoint and positive. Furthermore, Theorem 3.1.3 and Rellich-Kondrachov Theorem 2.4.5 guarantee that the operator $-\Delta$ has a bounded compact inverse operator

$$(-\Delta)^{-1} : L^2(\Omega) \to E \hookrightarrow_c L^2(\Omega).$$

Consequently, Fredholm theory [67] guarantee that there exists a sequence (λ_j) of positive eigenvalues of $(-\Delta)^{-1}$ with associated eigenvectors w_j in $H^1_0(\Omega)$ such that $\lambda_j \to +\infty$ as $j \to +\infty$. Transposed to the operator $-\Delta$, we have $-\Delta w_j + \xi_j w_j = 0$ where $\xi_j = \frac{-1}{\lambda_j}$. Now if in (3.1.1) $p \neq \xi_j < 0$ for every j, then Fredholm theory guarantees that the Helmholtz equation (3.1.1)-(3.1.2) has a unique solution in $E \subset H^1_0(\Omega)$. However if $p = \xi_j < 0$ for some j, then Fredholm theory states that (3.1.1)-(3.1.2) has a solution (not unique) if and only if the right-hand side g is orthogonal in $L^2(\Omega)$ to any solution $z \in H^1_0(\Omega)$ of the homogeneous equation

$$-\Delta z + \xi_j z = 0.$$

Notice that for the Helmholtz problem considered on unbounded domains, the unique solutions can be achieved by imposing the so called Sommerfeld's radiation condition at infinity (see [20]).

3.2 Regularity of the solution in a smooth domain

After the study of the variational solution of the Helmholtz problem in section 3.1, we study in this section, the regularity of the solution of the said problem. We begin the section with the definition of the regularity of the solution.

Definition 3.2.1. *Let w be the variational solution of (3.1.17) given by Theorem 3.1.3. Then the solution w is said to be regular, if $w \in H^2(\Omega)$ with*

$$\|w\|_{2,\Omega} \leq K\|g\|_{0,\Omega}, \tag{3.2.1}$$

for some constant $K > 0$ which depends on p and is independent of w. In other words, the linear operator $g \rightsquigarrow w$ is bounded from $L^2(\Omega)$ into $H^2(\Omega)$.The solution is uniformly regular if K does not depend on p.

Theorem 3.2.2. *We assume that the domain Ω has a boundary Γ of class C^2. Then the variational solution w of (3.1.17) is uniformly regular. More precisely, there exists a constant $C > 0$ independent of p such that*

$$\|w\|_{2,\,\Omega,\sqrt{1+|p|}} \leq C\|g\|_{0,\Omega}$$

The proof of Theorem 3.2.2 is presented in several auxiliary results stated below. Our presentation is based on [12].

Lemma 3.2.3. *We assume that $\Omega = \mathbb{R}^2$, $\ g \in L^2(\mathbb{R}^2)$ and $p \in \mathbb{C}$ with condition (3.1.20) satisfied.*
Then any variational solution, $w \in H^1(\mathbb{R}^2)$ of the problem

$$-\Delta w + p\,w = g \ \text{ in } \ \mathbb{R}^2 \tag{3.2.2}$$

is such that $w \in H^2(\mathbb{R}^2)$ and

$$\|w\|_{2,\mathbb{R}^2,\sqrt{|p|}} \leq 3\|g\|^2_{0,\mathbb{R}^2}. \tag{3.2.3}$$

Proof. First of all the variational solution $w \in H^1(\mathbb{R}^2)$ of the Helmholtz problem (3.2.2) satisfies the equation

$$\int_{\mathbb{R}^2} (\nabla w \nabla \overline{v} + p\, w\overline{v})dx = \int_{\mathbb{R}^2} g\,\overline{v}\,dx \quad \forall \ \ v \in H^1(\mathbb{R}^2). \tag{3.2.4}$$

Take $v = w$ in (3.2.4) to obtain

$$\int_{\mathbb{R}^2} \left(|\nabla w|^2 + p|w|^2\right) dx = \int_{\mathbb{R}^2} g\bar{w} dx$$

Taking separately the real and imaginal parts in this relation, we have for $p \neq 0$

$$\begin{aligned}
\int_{\mathbb{R}^2} \left(|\nabla w|^2 + |p||w|^2\right) dx &\leq \int_{\mathbb{R}^2} |g||w| dx \\
&\leq (\int_{\mathbb{R}^2} |g|^2 dx)^{\frac{1}{2}} (\int_{\mathbb{R}^2} |w|^2 dx)^{\frac{1}{2}} \text{ by Cauchy-Schwartz inequality} \\
&\leq \frac{1}{\sqrt{|p|}} (\int_{\mathbb{R}^2} |g|^2 dx)^{\frac{1}{2}} (\int_{\mathbb{R}^2} (|\nabla w|^2 + |p||w|^2 dx))^{\frac{1}{2}}
\end{aligned}$$

which implies that

$$\left(\int_{\mathbb{R}^2} \left(|\nabla w|^2 + |p||w|^2\right) dx\right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{|p|}} (\int_{\mathbb{R}^2} |g|^2 dx)^{\frac{1}{2}}.$$

Thus

$$\left(\int_{\mathbb{R}^2} \left(|p||\nabla w|^2 + |p|^2|w|^2\right) dx\right)^{\frac{1}{2}} \leq (\int_{\mathbb{R}^2} |g|^2 dx)^{\frac{1}{2}}. \tag{3.2.5}$$

We next use the technique of the difference quotient or the translation method due to Agmon, Douglis and Nirenberg [2]. Given a real-valued function v defined almost every where on $\mathbb{R}^2$ and given a vector $h \neq 0$ in $\mathbb{R}^2$, the difference quotient of v by h is denoted and defined by

$$(D_h v)(x) = \frac{(\tau_h v)(x) - v(x)}{|h|},$$

where $(\tau_h v)(x) = v(x+h)$ is the translation of v in the direction of h. Fix $h \neq 0$ in $\mathbb{R}^2$. Replacing v by $D_{-h}(D_h w)$ in (3.2.4) we have

$$\int_{\mathbb{R}^2} \left[\nabla w \nabla D_{-h}(D_h \overline{w}) + p\ w\ D_{-h}(D_h \overline{w})\right] dx = \int_{\mathbb{R}^2} g D_{-h}(D_h \overline{w}) dx. \tag{3.2.6}$$

In view of the property

$$\int_{\mathbb{R}^2} v D_{-h} \bar{S} dx = \int_{\mathbb{R}^2} (D_h v) \bar{S} dx, \quad \text{for} s \in H^1(\mathbb{R}^2)$$

we have from (3.2.6) that

$$\int_{\mathbb{R}^2} [|\nabla D_h w|^2 + p|D_h w|^2] dx = \int_{\mathbb{R}^2} g D_{-h}(D_h \bar{w}) dx.$$

Taking separately the real and imaginary parts in this identity, we obtain

$$\int_{\mathbb{R}^2} \left[|\nabla D_h w|^2 + p|D_h w|^2\right] dx \le 2\left|\int_{\mathbb{R}^2} g D_{-h}(D_h \overline{w}) dx\right|, \tag{3.2.7}$$

in view of the relation

$$(1/2)(|\xi| + |\eta|) \le |p| \le |\xi| + |\eta|. \tag{3.2.8}$$

Application of Cauchy-Schwarz inequality in (3.2.7) yields

$$\begin{aligned} \int_{\mathbb{R}^2} \left[|\nabla D_h w|^2 + |p||D_h w|^2\right] dx &\le 2\left(\int_{\mathbb{R}^2} |g|^2 dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} |D_{-h}(D_h \overline{w})|^2 dx\right)^{\frac{1}{2}} \\ &= 2\|g\|_{0,\mathbb{R}^2} \|D_{-h}(D_h w)\|_{0,\mathbb{R}^2}. \end{aligned} \tag{3.2.9}$$

At this stage, we use the following well-known property of $H^1(\mathbb{R}^2)$:

$$\|D_{-h} v\|_{0,\mathbb{R}^2} \le \|\nabla v\|_{0,\mathbb{R}^2}, \ \ \forall \ \ v \in H^1(\mathbb{R}^2). \tag{3.2.10}$$

Moreover a function $v \in L^2(\mathbb{R}^2)$ is of class $H^1(\mathbb{R}^2)$ if and only if there exists a constant $C > 0$ such that

$$\|D_h v\|_{0,\mathbb{R}^2} \le C, \ \ \forall \ \ 0 \ne h \in \mathbb{R}^2. \tag{3.2.11}$$

In this case we have

$$\|\nabla v\|_{0,\mathbb{R}^2} \le C. \tag{3.2.12}$$

Taking $v := D_h w \in H^1(\mathbb{R}^2)$ in (3.2.10), the relation (3.2.9) yields

$$\int_{\mathbb{R}^2} \left[|\nabla D_h w|^2 + |p||D_h w|^2\right] dx \ \ \le \ \ 2\|g\|_{0,\mathbb{R}^2} \|\nabla D_h w\|_{0,\mathbb{R}^2}. \tag{3.2.13}$$

Thus

$$\|\nabla D_h w\|_{0,\mathbb{R}^2} \leq 2\|g\|_{0,\mathbb{R}^2} \quad \text{or} \quad \|D_h \frac{\partial w}{\partial x_j}\|_{0,\mathbb{R}^2} \leq 2\|g\|_{0,\mathbb{R}^2} \ \text{ for } j = 1, 2. \tag{3.2.14}$$

In view of (3.2.11) and (3.2.12), we have from (3.2.14) that $\frac{\partial w}{\partial x_j} \in H^1(\mathbb{R}^2) \ \forall j$, with

$$\left\|\nabla \frac{\partial w}{\partial x_j}\right\|_{0,\mathbb{R}^2} \leq 2\|g\|_{0,\mathbb{R}^2} \ \ \forall j.$$

Therefore $\frac{\partial^2 w}{\partial x_i \partial x_j} \in L^2(\mathbb{R}^2)$ for $1 \leq i, j \leq 2$ and thus $w \in H^2(\mathbb{R}^2)$ such that

$$\left(\sum_{|\alpha|=2} \|D^\alpha w\|^2_{0,\mathbb{R}^2}\right)^{1/2} \leq 2\|g\|_{0,\mathbb{R}^2}. \tag{3.2.15}$$

Combining (3.2.5) with (3.2.15), we obtain (3.2.3). $\square$

Lemma 3.2.4. *Let $g \in L^2(\mathbb{R}^2_+)$ and $p \in \mathbb{C}$ such that condition (3.1.20) is satisfied. Then any variational solution $w \in H^1_0(\mathbb{R}^2_+)$ of the problem*

$$-\Delta w + pw = g \ \ in \ \mathbb{R}^2_+ \tag{3.2.16}$$

is such that $w \in H^2(\mathbb{R}^2_+)$ and

$$\|w\|_{2,\mathbb{R}^2_+,\sqrt{|p|}} \leq 6\|g\|_{0,\mathbb{R}^2_+} \tag{3.2.17}$$

Proof. The method as presented in the proof of Lemma 3.2.3 is still valid, but this time only in the tangential direction. In other words, we choose $0 \neq h \in \mathbb{R} \times \{0\}$, which means that h is parallel to the boundary $\partial\mathbb{R}^2_+$. We proceed by considering $w \in H^1_0(\mathbb{R}^2_+)$, the variational solution of (3.2.16). Thus

$$\int_{\mathbb{R}^2_+} (\nabla w \nabla \overline{v} + p\, w\overline{v})dx = \int_{\mathbb{R}^2_+} g, \ \overline{v}\, dx \ \forall v \in H^1_0(\mathbb{R}^2_+). \tag{3.2.18}$$

Arguing as in the proof of Lemma 3.2.3, we obtain the analogue of the inequality (3.2.9), which is

$$\int_{\mathbb{R}^2_+} \left[|\nabla D_h w|^2 + |p||D_h w|^2 \right] dx \le 2\|g\|_{0,\mathbb{R}^2_+} \|D_{-h}(D_h w)\|_{0,\mathbb{R}^2_+}. \tag{3.2.19}$$

Since $w \in H_0^1(\mathbb{R}^2_+)$, its extension $\widetilde{w}$ by zero outside $\mathbb{R}^2_+$ is such that $\widetilde{w} \in H^1(\mathbb{R}^2)$. Moreover, we have

$$D_h \widetilde{w} = \widetilde{D_h w} \;\text{ and }\; \nabla \widetilde{w} = \widetilde{\nabla w}.$$

This then leads to

$$\begin{aligned} \|D_{-h}(D_h w)\|_{0,\mathbb{R}^2_+} &= \|D_{-h}(D_h \widetilde{w})\|_{0,\mathbb{R}^2} \\ &\le \|\nabla D_h \widetilde{w}\|_{0,\mathbb{R}^2} \;\text{ by }\; (3.2.10) since \;\; D_h \widetilde{w} \in H^1(\mathbb{R}^2) \\ &= \|\nabla D_h w\|_{0,\mathbb{R}^2_+}. \end{aligned}$$

Using (3.2.19), we have

$$\int_{\mathbb{R}^2_+} \left[|\nabla D_h w|^2 + |p||D_h w|^2 \right] dx \le 2\|g\|_{0,\mathbb{R}^2_+} \|\nabla D_h w\|_{0,\mathbb{R}^2_+}.$$

from where we in turn have

$$\left(\int_{\mathbb{R}^2_+} \left[|\nabla D_h w|^2 + |p||D_h w|^2 \right] dx \right)^{\frac{1}{2}} \le 2\|g\|_{0,\mathbb{R}^2_+}, \tag{3.2.20}$$

and thus

$$\left(\int_{\mathbb{R}^2_+} |\frac{\partial}{\partial x_j} D_h w|^2 dx \right)^{\frac{1}{2}} \le 2\|g\|_{0,\mathbb{R}^2_+} \;\; \forall \;\; 1 \le j \le 2.$$

Letting h tend to zero, we obtain

$$\left(\int_{\mathbb{R}^2_+} \left| \frac{\partial^2 w}{\partial x_j \partial x_1} \right|^2 dx \right)^{\frac{1}{2}} \le 2\|g\|_{0,\mathbb{R}^2_+}, \;\; \text{for} \;\; 1 \le j \le 2. \tag{3.2.21}$$

In order to show that $\frac{\partial^2 w}{\partial x_2^2} \in L^2(\mathbb{R}^2_+)$ we go back to (3.2.16), which yields

$$-\frac{\partial^2 w}{\partial x_1^2} - \frac{\partial^2 w}{\partial x_2^2} + pw = g \text{ in } \mathbb{R}^2_+.$$

We then have by (3.2.21), the triangular inequality and by considering the variational formulation of (3.2.16) with $w \in H^1_0(\mathbb{R}^2_+)$ as test function

$$\begin{aligned} \|\frac{\partial^2 w}{\partial x_2^2}\|_{0,\mathbb{R}^2_+} &\leq \|g\|_{0,\mathbb{R}^2_+} + |p|\|w\|_{0,\mathbb{R}^2_+} + \|\frac{\partial^2 w}{\partial x_1^2}\|_{0,\mathbb{R}^2_+}; & (3.2.22)\\ &\leq 5\|g\|_{0,\mathbb{R}^2_+}. & (3.2.23)\end{aligned}$$

Combining (3.2.22) and (3.2.21) with the analogue of (3.2.5) for $\mathbb{R}^2_+$, which is valid by the same arguments we obtain Lemma 3.2.4. □

To come back to the set $\bar{\Omega}$ itself, we make use of its open covering $\{V_j\}_{j=0}^k$ constructed in chapter 2 (section 2.1) as well as of the C^∞-partition of unity $\{\theta_j\}_{j=0}^k$ given in formula (2.1.5). According to this formula, the solution $w \in H^1_0(\Omega)$ of (3.1.17) can be represented as

$$w = \sum_{j=0}^{k} \theta_j w \equiv \sum_{j=0}^{k} w_j. \tag{3.2.24}$$

We deal with the cases $j = 0$ and $1 \leq j \leq k$ differently in the next two results.

Lemma 3.2.5. *The variational solution $w \in H^1_0(\Omega)$ of the problem (3.1.1)-(3.1.2) is regular in the interior of Ω in the more precise sense that $\theta_0 w \in H^2(\Omega)$ and*

$$\|\theta_0 w\|_{2,\Omega,\sqrt{|p|}} \leq K\|g\|_{0,\Omega}, \tag{3.2.25}$$

where $K > 0$ is independent of p.

Proof. The function $\theta_0 w \in H^1_0(\Omega)$ because $\theta_0 \in \mathcal{D}(V_0)$ where $\bar{V}_0 \subset \Omega$. Thus $\widetilde{\theta_0 w} \in H^1(\mathbb{R}^2)$ such that

$$\begin{aligned} -\Delta(\widetilde{\theta_0 w}) + p\widetilde{\theta_0 w} &= \widetilde{\theta_0 g} - 2\nabla\widetilde{\theta}_0\nabla\widetilde{w} - (\Delta\widetilde{\theta}_0)\widetilde{w}\\ &=: g_0 \in L^2(\mathbb{R}^2). \end{aligned}$$

By Lemma 3.2.3, we have

$$\|\widetilde{\theta_0 w}\|_{2,\mathbb{R}^2,\sqrt{|p|}} \leq 3\|g_0\|_{0,\mathbb{R}^2}.$$

Thus

$$\|\theta_0 w\|_{2,\Omega,\sqrt{|p|}} \leq K(\|w\|_{1,\Omega} + \|g\|_{0,\Omega})$$

and

$$\|\theta_0 w\|_{2,\Omega,\sqrt{|p|}} \leq K\|g\|_{0,\Omega} \tag{3.2.26}$$

since $\|w\|_{1,\Omega} \leq K\|g\|_{0,\Omega}$ by Theorem 3.1.7 with K depending on p. □

Regarding the case when $1 \leq j \leq k$ in (3.2.24), we have the following result:

Lemma 3.2.6. *The variational solution $w \in H_0^1(\Omega)$ of (3.1.1)-(3.1.2) is regular near the boundary of Ω in the sense that $\theta_j w \in H^2(V_j^+)$, $\quad V_j^+ = V_j \cap \Omega$, and*

$$\|\theta_j w\|_{2,\Omega,\sqrt{|p|}} \leq K\|g\|_{0,\Omega},$$

where $K > 0$ is independent of p.

Proof. For a fixed $1 \leq j \leq k$, we have

$$-\Delta(\theta_j w) + p\theta_j w = \theta_j g - 2\nabla\theta_j \nabla w - (\Delta\theta_j)w := g_j \ \in L^2(V_j^+). \tag{3.2.27}$$

For simplicity, we use the notation $w_j = \theta_j w \in H_0^1(V_j^+)$. From (2.1.3), we use the C^2-diffeomorphism T_j that transforms $x \in V_j^+$ into $y = T_j(x) \in Q_+$ and we set

$$v_j(y) = w_j \circ T_j^{-1}(y) \in H_0^1(Q_+)$$

where T_j^{-1} is defined in (2.1.4). In short the idea of the rest of the proof is as follows: The equation (3.2.27) is transformed to the analogue in Q_+ of the form

$$L_j v_j + p\, v_j = f_j \ \in L^2(Q_+) \tag{3.2.28}$$

where L_j is a strongly elliptic operator of order 2. We then apply the analogue of Lemma 3.2.4 to problem (3.2.28) to obtain an estimate similar to (3.2.17). We come back to the desired estimate on V_j^+ by using the transformation

$$T_j : V_j^+ \longrightarrow Q_+.$$

The details are provided below. By the chain rule, we have

$$\begin{aligned}\frac{\partial v_j(y)}{\partial y_1} &= \sum_{k=1}^{2} \frac{\partial w_j(T_j^{-1}(y))}{\partial x_k}\frac{\partial x_k}{\partial y_1} = \frac{\partial w_j}{\partial x_1}\frac{\partial x_1}{\partial y_1} + \frac{\partial w_j}{\partial x_2}\frac{\partial x_2}{\partial y_1} \\ &= \alpha\frac{\partial w_j}{\partial x_1} + \alpha\varphi'(x_1)\frac{\partial w_j}{\partial x_2}\end{aligned}$$

because

$$T_j^{-1}(y) = (\alpha y_1,\ \varphi(\alpha y_1) - \beta y_2) \equiv (x_1,\ x_2) \ \text{ and } \ \frac{\partial x_1}{\partial y_1} = \alpha \ \text{ while } \ \frac{\partial x_2}{\partial y_1} = \alpha\varphi'(x_1).$$

Similarly

$$\frac{\partial v_j(y)}{\partial y_2} = \frac{\partial w_j}{\partial x_1}\frac{\partial x_1}{\partial y_2} + \frac{\partial w_j}{\partial x_2}\frac{\partial x_2}{\partial y_2} = -\beta\frac{\partial w_j}{\partial x_2}$$

since

$$\frac{\partial x_1}{\partial y_2} = 0 \ \text{ and } \ \frac{\partial x_2}{\partial y_2} = -\beta.$$

In the variational formulation of (3.2.27), the contribution of $-\Delta w_j$ is the following integral, which is transformed on Q_+ by change of variable: For ψ a test function, we have

$$\int_{V_j^+} \nabla w_j \nabla \psi dx = \int_{Q_+} \left[\begin{pmatrix} \frac{1}{\alpha} & \varphi'(x_1) \\ 0 & \frac{-1}{\beta} \end{pmatrix} \begin{pmatrix} \frac{\partial v_j}{\partial y_1} \\ \frac{\partial v_j}{\partial y_2} \end{pmatrix} \begin{pmatrix} \frac{1}{\alpha} & \varphi'(x_1) \\ 0 & \frac{-1}{\beta} \end{pmatrix} \begin{pmatrix} \frac{\partial \psi}{\partial y_1} \\ \frac{\partial \psi}{\partial y_2} \end{pmatrix} \right] \alpha\beta dy \tag{3.2.29}$$

Evaluating equation (3.2.29) leads to the following relation

$$\begin{aligned}&\int_{V_j^+} \nabla w_j \nabla \psi dx \\ &= \int_{Q_+} \left[\frac{1}{\alpha^2}\frac{\partial v_j}{\partial y_1}\frac{\partial \psi}{\partial y_1} + \frac{1}{\alpha\beta}\frac{\partial v_j}{\partial y_1}\frac{\partial \psi}{\partial y_2}\varphi'(x_1) + \frac{1}{\alpha\beta}\varphi'(x_1)\frac{\partial v_j}{\partial y_2}\frac{\partial \psi}{\partial y_1} + (\frac{1}{\beta^2}\varphi'(x_1)^2 + 1)\frac{\partial v_j}{\partial y_2}\frac{\partial \psi}{\partial y_2} \right] \alpha\beta dy.\end{aligned}$$

By Green formula the operator L_j is explicitly given by the following relation:

$$L_j := -\frac{\partial^2}{\partial y_1^2} - \frac{\partial}{\partial y_2}\left(\varphi'(x_1)\frac{\partial}{\partial y_1}\right) - \frac{\partial}{\partial y_1}\left(\varphi'(x_1)\frac{\partial}{\partial y_2}\right) - \frac{\partial}{\partial y_2}\left(1+(\varphi'(x_1))^2\frac{\partial}{\partial y_2}\right).$$

Lemma 3.2.7. *The operator*

$$L_j := -\frac{\partial^2}{\partial y_1^2} - \frac{\partial}{\partial y_2}\left(\varphi'(x_1)\frac{\partial}{\partial y_1}\right) - \frac{\partial}{\partial y_1}\left(\varphi'(x_1)\frac{\partial}{\partial y_2}\right) - \frac{\partial}{\partial y_2}\left(1+(\varphi'(x_1))^2\frac{\partial}{\partial y_2}\right)$$

is strongly uniformly elliptic in Q_+. That is, there exists a real number $\alpha > 0$ and a complex number γ such that

$$Re\left[-\gamma\left(\xi_1^2 + 2\xi_1\xi_2\varphi'(x_1) + (1+(\varphi'(x_1)))^2\xi_2^2\right)\right] \geq \alpha|\xi|^2, \ \ \forall \xi \in \mathbb{R}^2, \ \ y \in Q_+.$$

Proof. We take $\gamma = -1$ and $0 < \alpha < 1/2$. Then we have consecutively

$$\begin{aligned} Re\left[-\gamma\left(\xi_1^2 + 2\xi_1\xi_2\varphi'(x_1) + (1+(\varphi'(x_1)))^2\xi_2^2\right)\right] - \alpha|\xi|^2 &= \xi_1^2(1-\alpha) + \xi_2^2(1-\alpha+(\varphi'(x_1))^2) \\ &+ 2\rho'(x_1)\xi_1\xi_2 \\ &\geq 1/2\xi_1^2 + 1/2(\varphi'(x_1))^2\xi_2^2 + \varphi'(x_1)\xi_1\xi_2 \\ &= (\frac{\sqrt{2}}{2}\xi_1 + \frac{\sqrt{2}}{2}\varphi'(x_1)\xi_2)^2 \\ &\geq 0. \end{aligned}$$

Hence the proof of the Lemma. □

Applying the analogue of the Lemma 3.2.4 to (3.2.29) we obtain

$$\left(\sum_{|\alpha|=2} \|D^\alpha v_j\|^2_{0,Q_+}\right)^{\frac{1}{2}} \leq K_j\|f_j\|_{0,Q_+} \tag{3.2.30}$$

which is the analogue of (3.2.17) in Q_+. Making the change of variables $y = T_j(x)$ and $\theta_j w = v_j \circ T_j$, $g_j = f_j \circ T_j$ in (3.2.30) we obtain

$$\left(\sum_{|\alpha|=2} \|D^\alpha \theta_j w\|^2_{0,V_j^+}\right)^{\frac{1}{2}} \leq K_j\|g\|_{0,V_j^+} \tag{3.2.31}$$

together with

$$\|\theta_j w\|^2_{1,V_j^+} \leq K_j \|g\|_{0,V_j^+} \text{ (By Theorem 3.1.3)} \tag{3.2.32}$$

Adding (3.2.31) and (3.2.32) proves Lemma 3.2.6. □

Remark 3.2.8. *The underlying point in the proofs of Lemma 3.2.6 and 3.2.7 is that the ellipticity property is preserved by translation.*

Proof. of Theorem 3.2.2
We prove Theorem 3.2.2 by adding (3.2.26), (3.2.31) and (3.2.32) with (3.1.24) through $j = 0$ to $j = k$. □

Remark 3.2.9. *The inequality in the Theorem 3.2.2 is the particular case of some more general inequalities established in Agronovitch and Vishik [3].*

Chapter 4

The Helmholtz problem in a non-smooth domain

In the preceding chapter we study the regularity of the solution of Helmholtz problem in a smooth domain. In this chapter we study the same problem in the non-smooth domain specifically the polygonal domain. We begin the chapter with section 4.1 where we study the regularity of the solution of the Helmholtz problem far away from the corner. In section 4.2 and 4.3, we study the regularity of the solution of the problem at the corner for $p = 0$ and for $p \neq 0$ respectively. Finally, we show in section 4.4, that the solution of the Helmholtz problem attains its global regularity in a weighted Sobolev space $H^{2,\beta}(\Omega)$ to be defined.

4.1 Regularity far away from corners and reduction to a sector

The results of section 3.2 show that the solution of the Helmholtz problem is regular far away from the vertices (corners) of the polygonal domain. More precisely, we have the following result:

Theorem 4.1.1. *Let E be an open subset of the polygonal domain Ω such that the distance from E to the vertices of Γ is strictly positive. Then, the variational solution of the Helmholtz problem*

$$w \in H_0^1(\Omega), \quad -\Delta w + p\, w = g, \tag{4.1.1}$$

corresponding to $g \in L^2(\Omega)$, $Re(p) \geq 0$ *is such that*

$$w \in H^2(E).$$

Proof. We proceed by partition of unity as in section 3.2, observing that either $\bar{E} \cap \Gamma = \phi$ or $\bar{E} \cap \Gamma \neq \phi$. The first case corresponds to the interior regularity stated in Lemmas 3.2.3 and 3.2.5. The second case include the situation where the arc-length of $\bar{E} \cap \Gamma$ is positive, in which case $\bar{E} \cap \Gamma$ is locally represented as the graph of C^∞ functions. This then corresponds to the regularity near the boundary stated in Lemmas 3.2.4 and 3.2.6. □

In view of Theorem 4.1.1, the singular behavior of the solution of (4.1.1) is a local problem which is related to each corner. Thus we focus on one corner of Ω and assume for convenience that this corner is at the origin of $\mathbb{R}^2$. In the neighborhood of this corner, we assume that Ω coincides with the sector G defined by

$$G = \{(r\cos\theta, r\sin\theta); r > 0, 0 < \theta < \omega\}, \tag{4.1.2}$$

in the usual polar co-ordinate (r, θ) where ω is the size of the interior angle at the corner. It is further assumed that this is the only non-convex corner i.e $\omega > \pi$ of Ω as seen in Figure 4.1.

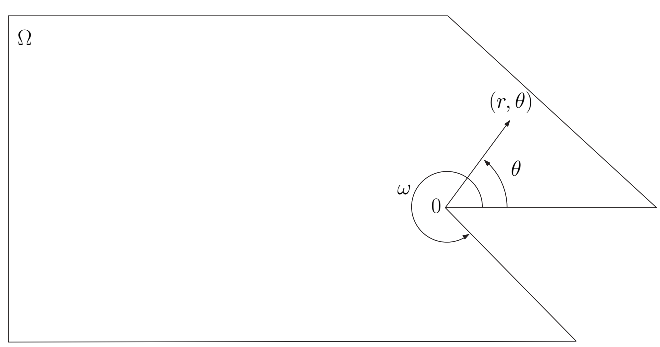

Figure 4.1: Model Polygonal domain

To be more specific on the local nature of the problem, we consider once and for all a cut-off function $\psi \equiv \psi(r) \in \mathcal{D}(\mathbb{R}^2)$ such that

$$\psi(r) = \begin{cases} 1 & \text{for} \quad 0 \leq r \leq \frac{r_0}{2} \\ \\ 0 & \text{for} \quad r \geq r_0, \end{cases} \tag{4.1.3}$$

where the number $r_0 > 0$ is so small that no other corner point of Ω lies in the disk $|x| < r_0$.

With $\widetilde{w} \in H_0^1(\mathbb{R}^2)$ being the extension of w by zero outside Ω, the solution of the local problem we will deal with is $\widetilde{w}\psi$. The right hand side is $\psi\widetilde{g} - \widetilde{w}\Delta\psi - 2\nabla\psi\nabla\widetilde{w}$. For simplicity, we write $\widetilde{w}\psi$ as w. Equally $\psi\widetilde{g} - \widetilde{w}\Delta\psi - 2\nabla\psi\nabla\widetilde{w}$ will be written as g. In summary, the local problem we deal with reads as follows: $w \in H_0^1(G)$ is solution of

$$-\Delta w + p\, w = g \in L^2(G) \tag{4.1.4}$$

where the involved functions have bounded supports in the following specific way:

$$w(r, \theta) = 0 \quad \text{for} \ \ r \geq r_0, \tag{4.1.5}$$

$$g(r, \theta) = 0 \quad \text{for} \ \ r \geq r_0. \tag{4.1.6}$$

Remark 4.1.2. *When there is no risk of confusion, a real-valued function v on the sector G will be written indistinctly by $v(x)$, $v(x_1, x_2)$, $v(r\sin\theta, r\cos\theta)$ or $v(r, \theta)$.*

By Hardy inequality [29], it follows that the local solution $w \in H_0^1(G)$ satisfies the inclusion

$$r^{|\alpha|-1} D^\alpha w \in L^2(G) \ \text{ for all } \ |\alpha| \leq 1. \tag{4.1.7}$$

This leads us to consider the so-called weighted Sobolev spaces introduced first by Kondratiev [36].

Definition 4.1.3. ([*29*], [*36*])
We denote by $P_2^k(G)$ the space of all distributions v 0n G such that,

$$r^{|\alpha|-k}D^\alpha v \in L^2(G) \ \textit{for all} \ |\alpha| \leq k,$$

where k is a non-negative integer. We equip $P_2^k(G)$ with the natural norm defined by

$$\|v\|^2_{P_2^k(G)} := \sum_{|\alpha|\leq k} \|r^{|\alpha|-k}D^\alpha v\|^2_{0,G}. \tag{4.1.8}$$

By using the chain rule and the change of variables in integrals via the Euler transformation

$$r = e^t, \tag{4.1.9}$$

the weighted Sobolev space on the sector G is linked to the usual Sobolev space on the strip $B = \mathbb{R} \times (0, \omega)$ as specified in the next Lemma.

Lemma 4.1.4. ([*29*])
Assume that $u \in P_2^k(G)$ with k a positive integer and define v by,

$$v(t, \theta) = u(e^t \cos\theta, e^t \sin\theta)e^{(-k+1)t}. \tag{4.1.10}$$

Then, $v(t, \theta) \in H^k(B)$.

4.2 Regularity and singularities when $p = 0$

We consider (4.1.4) in the particular case when $p = 0$. We are then dealing with the Dirichlet problem for the Laplace operator:

$$w \in H_0^1(G), -\Delta w = g \ \in L^2(G), \tag{4.2.1}$$

where w and g satisfy (4.1.5)-(4.1.6).

Theorem 4.2.1. *For the solution $w \in H_0^1(G)$ of the problem (4.2.1), we have the following singular decomposition : there exists a scalar A such that*

$$w_R := w - Ar^{\frac{\pi}{\omega}} \sin\frac{\pi}{\omega}\theta \ \ \in P_2^2(G) \cap H_0^1(G),$$

$$w_R^1 := w - A\psi(r)r^{\frac{\pi}{\omega}}\sin\frac{\pi}{\omega}\theta \quad \in H^2(G)\cap H_0^1(G),$$

and

$$\|w_R\|_{P_2^2(G)} + \|w_R^1\|_{2,G} + |A| \leq C\|g\|_{0,G}, \tag{4.2.2}$$

where $\psi \equiv \psi(r)$ *is the cut-off function in (4.1.3),* w_R *or* w_R^1 *is the regular part,* $r^{\frac{\pi}{\omega}}\sin\frac{\pi}{\omega}\theta$ *or* $\psi(r)r^{\frac{\pi}{\omega}}\sin\frac{\pi}{\omega}\theta$ *is the singular function and* A *is the coefficient of the singular function.*

The method used in proving Theorem 4.2.1 was developed by Kondratiev [36] and it demands a lot of theoretical knowledge. We shall essentially quote the important steps. For more details see for instance [29]. In polar co-ordinate, equation (4.2.1) takes the form

$$-\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) = g(r,\theta) \text{ in } G. \tag{4.2.3}$$

Now, we use the Euler transformation (4.1.9) and make a change of dependent variable

$$s(t,\theta) = w(e^t,\theta) = w(r,\theta). \tag{4.2.4}$$

Since

$$\frac{\partial w}{\partial r} = e^{-t}\frac{\partial s}{\partial t} \text{ and } \frac{\partial^2 w}{\partial r^2} = e^{-2t}\frac{\partial^2 s}{\partial t^2} - e^{-2t}\frac{\partial s}{\partial t},$$

(4.2.3) becomes

$$-\left(\frac{\partial^2 s}{\partial t^2} + \frac{\partial^2 s}{\partial \theta^2}\right) = e^{2t}g(t,\theta) \text{ in } B \tag{4.2.5}$$

with boundary conditions

$$s(t,\omega) = s(t,0) = 0, \tag{4.2.6}$$

where $s \in H_0^1(B)$ and $g(e^t\cos\theta, e^t\sin\theta)e^t \in L^2(B)$ in view of Lemma 4.1.4.

Taking the Fourier transform, the problem (4.2.5)-(4.2.6) becomes the following family of ordinary differential equation that depend on the parameter λ:

$$-\frac{d^2\widehat{s}(i\lambda,\theta)}{d\theta^2} + \lambda^2\widehat{s}(i\lambda,\theta) = \widehat{e^t g}(-\lambda_2 - 1 + i\lambda_1,\theta) \equiv \widehat{e^t g}(i\lambda - 1,\theta) \quad 0 < \theta < \omega \tag{4.2.7}$$

$$\widehat{s}(i\lambda, 0) = \widehat{s}(i\lambda, \omega) = 0. \tag{4.2.8}$$

Remark 4.2.2. *For a function $h : r \to h(r)$, the composition of the Euler transformation (4.1.9) and the Fourier transform is called the Mellin transform of h see [29]. Formally we have:*

$$(Mh)(\lambda) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} r^{-i\lambda-1} h(r) dr.$$

We apply Proposition 2.5.36 (corresponding to the scalar Theorem 2.5.1) to the L^2 vector-valued functions

$$t \in (-\infty, +\infty) \rightsquigarrow \frac{\partial^\beta s(t,\theta)}{\partial^{\beta_1} t \partial^{\beta_2} \theta}, \quad |\beta| \leq 1 \text{ and } t \in (-\infty, +\infty) \rightsquigarrow e^t g(t, \theta) \in L^2(-\infty, +\infty),$$

observing that the support of all these functions are contained in $I_\alpha = (-\infty, \alpha)$ where $\alpha = \ln r_0$.

We obtain that $\widehat{s}(i\lambda, \theta)$ is holomorphic in the region $\lambda_2 > 0$ and $\widehat{e^t g}(i\lambda - 1, \theta)$ is holomorphic in the region $\lambda_2 > -1$ such that the following estimates hold:

$$\sum_{j=0}^{1} \int_{-\infty}^{+\infty} \int_0^\omega |\lambda_1 + i\lambda_2|^{2j} |\frac{\partial^{1-j}\widehat{s}}{\partial\theta^{1-j}}(i\lambda, \theta)|^2 d\lambda_1 d\theta \leq r_0^{2\lambda_2} \sum_{|\beta|\leq 1} \int_{-\infty}^{+\infty} \int_0^\omega |\frac{\partial^\beta s(t,\theta)}{\partial t^{\beta_1} \partial\theta^{\beta_2}}|^2 dt d\theta$$

$$\int_{-\infty}^{+\infty} \int_0^\omega |\widehat{e^t g}(i\lambda - 1, \theta)|^2 d\lambda_1 d\theta \leq r_0^{2(\lambda_2+1)} \int_{-\infty}^{+\infty} \int_0^\omega |e^t g(t,\theta)|^2 dt d\theta.$$

In view of the above holomorphic property of $\widehat{s}(i\lambda, \theta)$ and $\widehat{e^t g}(i\lambda + 1, \theta)$, Theorem 2.5.38 implies that the solution $\widehat{s}(i\lambda, \theta)$ of (4.2.7)-(4.2.8) admits a meromorphic extension (which we denote in the same way) to the complex strip

$$-\infty < \lambda_1 < +\infty, \quad -1 < \lambda_2 < 0.$$

We want to say a bit more about this meromorphic extension. Firstly, the considerations in Remark 3.1.9 can be made more precise in this one-dimensional case. Indeed, it is well-known that the operator $u \rightsquigarrow -u''$ with boundary conditions $u(0) = u(\omega) = 0$ has the eigenvalues

$$\lambda_k^2 = (\frac{k\pi}{\omega})^2, \ \ k \in \mathbb{N}, \ \ k \neq 0,$$

with, for each k, the associated eigenvector

$$v_k = \sin \frac{k\pi}{\omega}\theta.$$

Now in the extension $\widehat{s}(i\lambda, \theta)$ of the solution, if we take

$$i\lambda = \sqrt{\lambda_k} = \frac{k\pi}{\omega}, \quad i.e.\ \lambda = \frac{-ik\pi}{\omega},$$

then it is clear that the only possible pole of the meromorphic function $\widehat{s}(i\lambda, \theta)$ in the strip $-\infty < \lambda_1 < +\infty, \quad -1 < \lambda_2 < 0$ is $\lambda = \frac{k\pi}{\omega}$. We distinguish two cases: if $\omega < \pi$, there is no pole in the said strip. However, there is indeed a unique pole in the non convex case $\omega > \pi$.

Secondly, we introduce the Green function $N \equiv N(i\lambda, \theta, \gamma)$ of the operator

$$v \in C^2(0, \omega) \rightsquigarrow -\frac{d^2 v}{d\theta^2} + \lambda^2 v, \quad \lambda = \frac{-ik\pi}{\omega}, \quad -1 < \lambda_2 < 0$$

with homogeneous Dirichlet boundary conditions $v(0) = v(\omega) = 0$.
By definition [66], the Green function satisfies the following properties:

1. The function $(\theta, \gamma) \rightsquigarrow N \equiv N(i\lambda, \theta, \gamma)$ is continuous on the square $(0, \omega) \times (0, \omega)$;

2. The partial derivatives $\frac{\partial N}{\partial \theta}$, $\frac{\partial^2 N}{\partial \theta^2}$ exist and are continuous on the triangles $0 \leq \theta \leq \gamma \leq \omega$ and $0 \leq \gamma \leq \theta \leq \omega$;

3. For each fixed $\gamma \in [0, \omega]$, $\frac{d^2 N}{\partial \theta^2} + \lambda^2 N = 0$ for $0 \leq \theta \leq \omega, \quad \theta \neq \gamma$;

4. On the diagonal $\theta = \gamma$, the first derivative makes a jump such that

$$\frac{\partial N(0^+, \theta)}{\partial \theta} - \frac{\partial N(0^-, \theta)}{\partial \theta} = -1 \text{ for } 0 < \theta < \omega;$$

5. $N(i\lambda, 0, \gamma) = N(i\lambda, \omega, \gamma) = 0$ for each $\gamma \in (0, \omega)$.

Following the classical procedure (see [66]), it can be shown that the Green function is given by the formula

$$N(i\lambda, \theta, \gamma) = \frac{-1}{\omega\lambda} \begin{cases} \gamma \ \sinh \lambda(\theta - \omega) & , \text{ if } \quad 0 \leq \gamma \leq \theta \leq \omega \\ \\ \theta \ \sinh \lambda(\theta - \omega) & , \text{ if } \quad 0 \leq \theta \leq \gamma \leq \omega. \end{cases} \tag{4.2.9}$$

Notice that

$$N(0,\ \theta,\ \gamma) = \begin{cases} \gamma\ (\theta - 1) & ,\ \text{for} \quad 0 \leq \gamma \leq \theta \leq \omega \\ \\ \theta\ (\gamma - 1) & ,\ \text{for} \quad 0 \leq \theta \leq \gamma \leq \omega \end{cases}$$

which is in agreement with the Green function given in Walter [66] and Gustafson [33]. In view of the expression of $N(i\lambda, \theta, \gamma)$, the solution of (4.2.7)-(4.2.8) admits the representation

$$\widehat{s}(i\lambda, \theta) = \int_0^\omega N(i\lambda, \theta, \gamma)\widehat{e^t g}(i\lambda - 1, \gamma)d\gamma; \text{ when } \lambda \neq \frac{-ik\pi}{\omega},\ \ \lambda_2 > -1. \tag{4.2.10}$$

The regularity of this extended solution of (4.2.7)-(4.2.8) is described in the next result.

Lemma 4.2.3. *There exist constants $C > 0$ and $K > 0$, such that*

$$\sum_{j=0}^{2} |\lambda_1|^{2-j} \|\widehat{s}(i\lambda, \theta)\|_{j,(0,\omega)} \leq C \|\widehat{e^t g}(i\lambda - 1, \cdot)\|_{0,(0,\omega)}, \ \ \textit{for}\ \ |\lambda_1| \geq K,\ \ -1 \leq \lambda_2 \leq 0.$$

Proof. For general problems, the proof of Lemma 4.2.3 is given in Grisvard [29] and Kondratiev [36]. For the case under consideration, the proof can be obtained explicitly either by using the Green function $N(i\lambda, \theta, \gamma)$ in (4.2.9) and the representation (4.2.10), which is valid or by simple arguments. We prefer the latter approach.

We assume that $\lambda_1 \geq K > 0$ for a constant to be determined shortly and we assume that $-1 \leq \lambda_2 \leq 0$. Then $\lambda \neq \frac{-ik\pi}{\omega}$. The arguments used below are similar to those that led to the proof of the inequality (3.1.29). Multiply both sides of (4.2.7) by $\lambda_1^2\bar{\widehat{s}}(i\lambda, \theta)$ and integrate by parts to obtain the following after using (4.2.8):

$$\int_0^\omega \left[\lambda_1^2 |\frac{d\widehat{s}(i\lambda, \theta)}{d\theta}|^2 + (\lambda_1^2 - \lambda_2^2 + 2i\lambda_1\lambda_2)|\widehat{s}(i\lambda, \theta)|^2\right] d\theta = \int_0^\omega \widehat{e^t g}(i\lambda - 1, \theta)\lambda_1^2\bar{\widehat{s}}(i\lambda, \theta)d\theta.$$

Using the real part of this identity and Cauchy-Schwarz inequality, we obtain:

$$\int_0^\omega \left[\lambda_1^2 |\frac{d\widehat{s}(i\lambda, \theta)}{d\theta}|^2 + \lambda_1^4 (1 - \frac{\lambda_2^2}{\lambda_1^2})|\widehat{s}(i\lambda, \theta)|^2\right] d\theta$$
$$\leq \left(\int_0^\omega |\widehat{e^t g}(i\lambda - 1, \theta)|^2 d\theta\right)^{\frac{1}{2}} \left(\int_0^\omega \lambda_1^4 |\widehat{s}(i\lambda, \theta)|^2 d\theta\right)^{\frac{1}{2}}. \tag{4.2.11}$$

Notice that $0 \leq \lambda_2^2 \leq 1$. We assume at this point in time that

$$K \geq 2, \quad \text{and} \quad \frac{\lambda_2^2}{\lambda_1^2} < \frac{1}{2} \quad \text{so that} \quad \frac{1}{2} < (1 - \frac{\lambda_2^2}{\lambda_1^2}).$$

Then, with $|\lambda_1| \geq K$ and so $|\lambda_1| \geq 2$, we obtain from (4.2.11)

$$\frac{1}{2}\int_0^\omega \left[\lambda_1^2 |\frac{d\widehat{s}(i\lambda,\theta)}{d\theta}|^2 + \lambda_1^4 |\widehat{s}(i\lambda,\theta)|^2\right] d\theta \leq$$
$$\left(\int_0^\omega |\widehat{e^t g}(i\lambda - 1,\theta)|^2 d\theta\right)^{\frac{1}{2}} \left(\int_0^\omega \left[\lambda_1^2 |\frac{d\widehat{s}(i\lambda,\theta)}{d\theta}|^2 + \lambda_1^4 |\widehat{s}(i\lambda,\theta)|^2\right] d\theta\right)^{\frac{1}{2}}.$$

Thus

$$\left(\int_0^\omega \left[\lambda_1^2 |\frac{d\widehat{s}(i\lambda,\theta)}{d\theta}|^2 + \lambda_1^4 |\widehat{s}(i\lambda,\theta)|^2\right] d\theta\right)^{\frac{1}{2}} \leq 2\left(\int_0^\omega |\widehat{e^t g}(i\lambda - 1,\theta)|^2 d\theta\right)^{\frac{1}{2}}. \tag{4.2.12}$$

On the other hand, from (4.2.7) we have

$$\begin{aligned}\left(\int_0^\omega |\frac{d^2\widehat{s}(i\lambda,\theta)}{d\theta^2}|^2 d\theta\right)^{\frac{1}{2}} \leq \quad & |\lambda|^2 \left(\int_0^\omega |\widehat{s}(i\lambda,\theta)|^2 d\theta\right)^{\frac{1}{2}} + \left(\int_0^\omega |\widehat{e^t g}(i\lambda - 1,\theta)|^2 d\theta\right)^{\frac{1}{2}} \\ \leq \quad & 2\lambda_1^2 \left(\int_0^\omega |\widehat{s}(i\lambda,\theta)|^2 d\theta\right)^{\frac{1}{2}} + \left(\int_0^\omega |\widehat{e^t g}(i\lambda - 1,\theta)|^2 d\theta\right)^{\frac{1}{2}}\end{aligned}$$

because

$$|\lambda|^2 = \lambda_1^2 + \lambda_2^2 \leq \lambda_1^2 + 4 \leq 2\lambda_1^2 \ \text{ and } \ |\lambda_1| \geq 2.$$

Using then (4.2.12), we have

$$\left(\int_0^\omega |\frac{d^2\widehat{s}(i\lambda,\theta)}{d\theta^2}|^2 d\theta\right)^{\frac{1}{2}} \leq 5\left(\int_0^\omega |\widehat{e^t g}(i\lambda - 1,\theta)|^2 d\theta\right)^{\frac{1}{2}} \tag{4.2.13}$$

Since $|\lambda_1| \geq 2$, it follows from (4.2.12) that

$$\left(\int_0^\omega \left[|\frac{d\widehat{s}(i\lambda,\theta)}{d\theta}|^2 + |\widehat{s}(i\lambda,\theta)|^2\right] d\theta\right)^{\frac{1}{2}} \leq 2\left(\int_0^\omega |\widehat{e^t g}(i\lambda - 1,\theta)|^2 d\theta\right)^{\frac{1}{2}}. \tag{4.2.14}$$

Taking the squares of (4.2.12), (4.2.13), (4.2.14) and adding these inequalities, we obtain the

Lemma 4.2.3 for the specific choice $K \geq 2$. □

Remark 4.2.4. *In terms of the weighted Sobolev space $H^m((0,\omega),\rho)$ introduced in Definition 3.1.4, the proof of Lemma 4.2.3 shows that*

$$\|\widehat{s}(i\lambda,\cdot)\|_{2,(0,\omega),|\lambda_1|} \leq C\|\widehat{e^t g}(i\lambda-1,\cdot)\|_{0,(0,\omega)} \ \ \text{for} \ \ |\lambda_1| \geq 2, \ -1 \leq \lambda_2 \leq 0.$$

Once again, this inequality is as mentioned in the proof of Theorem 3.2.2, a particular case of the results of Agranovitch and Vishik [3].

Corollary 4.2.5. *There exists a sequence (N_m) of integers such that*

$$N_m \geq K, \ \ \forall m \quad \lim_{m\to+\infty}\int_{-1}^{0}|\widehat{s}(\pm iN_m-\lambda_2,\theta)|d\lambda_2 = 0$$

for almost every $0 < \theta < \omega$.

Proof. From Lemma 4.2.3, we have

$$\int_0^{\omega}\int_{-1}^{0}|\widehat{s}(i\lambda_1-\lambda_2,\theta)|d\lambda_2 d\theta \leq \frac{C}{|\lambda_1|^2} \ \ \text{for} \ \ |\lambda_1| \geq K.$$

This implies that

$$\lim_{K\leq|N|\to+\infty}\int_0^{\omega}\int_{-1}^{0}|\widehat{s}(\pm iN-\lambda_2,\theta)|d\lambda_2 d\theta = 0.$$

By the fact that a Cauchy sequence in $L^P(0,\omega)$ admits a point-wise convergent subsequence (see Adams [1], Corollary 2.11), we can find a sequence $(N_m)_{m\geq 1}$ of integers which have the desired property. □

Proof. (Theorem 4.2.1)
At this point, we make use of the fact that the polygonal domain is non-convex, i.e $\omega > \pi$. This implies as observed earlier that no pole of the meromorphic function $\widehat{s}(i\lambda,\theta)$ or eigenvalue $\lambda = \frac{-i\pi}{\omega}$ of the problem (4.2.7)-(4.2.8) belongs to the line

$$\lambda_2 = -1.$$

Under this condition, the Plancherel-Parseval Theorem, implies that the function

$$\lambda_1 \rightsquigarrow \widehat{s}(i\lambda_1+1,\theta),$$

has the inverse Fourier transform

$$s_R(t,\theta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{i\lambda_1 t} \widehat{s}(i\lambda_1 + 1, \theta) d\lambda_1,$$

that belongs to the Sobolev space $H^2(B)$ such that

$$\|s_R\|_{2,B} \leq C\|e^t g\|_{0,B}. \tag{4.2.15}$$

Notice that the inverse Fourier transform of the function $\lambda_1 \rightsquigarrow \widehat{s}(i\lambda_1, \theta)$ i.e $\lambda_2 = 0$, given by

$$s(t,\theta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{i\lambda_1 t} \widehat{s}(i\lambda_1, \theta) d\lambda_1 \tag{4.2.16}$$

is of class $L^2(B)$ (in fact of class $H_0^1(B)$).

In order to link $s_R(t,\theta)$ to $s(t,\theta)$, we use the sequence (N_m) in the Corollary 4.2.5, observing that

$$\begin{aligned} s(t,\theta) &= \lim_{N_m\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-N_m}^{N_m} e^{i\lambda t} \widehat{s}(i\lambda, \theta) d\lambda \\ &= \lim_{N_m\to\infty} \frac{1}{\sqrt{2\pi}} \left[\int_{-N_m+0i}^{-N_m-i} + \int_{-N_m-i}^{N_m-i} + \int_{N_m-i}^{N_m+0i} + \int_{Q_m} \right] e^{i\lambda t} \widehat{s}(i\lambda, \theta) d\lambda \end{aligned} \tag{4.2.17}$$

where Q_m is the rectangle with vertices $-N_m + 0i, -N_m - i, N_m - i$ and $N_m + 0i$ illustrated in Figure 4.2.

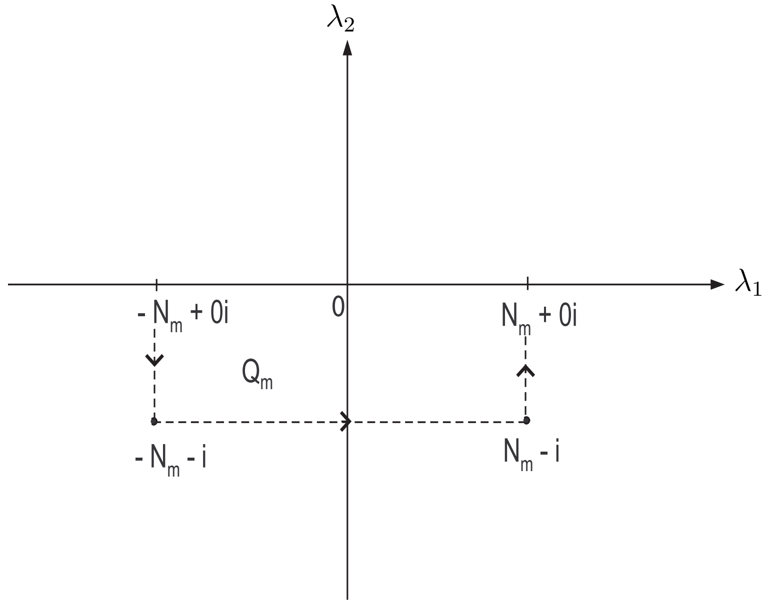

Figure 4.2: Application of the Residue theorem

By Corollary 4.2.5, we know that the limits corresponding to the first and the third integrals are zero. Recall that we are in the non-convex case for the sector G i.e. $\omega > \pi$. The only pole of $\widehat{s}(i\lambda, \theta)$ in the region Q_m being then $\frac{-i\pi}{\omega}$, the Laurent expansion of this function has the form

$$\widehat{s}(i\lambda, \theta) = \frac{P_1(\theta)}{\lambda + \frac{i\pi}{\omega}} + \alpha(\lambda, \theta), \tag{4.2.18}$$

with $\alpha(\lambda, \theta)$ being analytic. Applying to (4.2.18) the operator $u \rightsquigarrow u'' + \lambda^2 u$ with boundary conditions $u(0) = u(\omega) = 0$, it is easy to show in terms of the eigenvalues and associated eigenvectors of this operator that

$$P_1(\theta) = A_1 \sin \frac{\pi}{\omega}\theta, \text{ for some scalar } A_1. \tag{4.2.19}$$

By the Residue Theorem, the fourth integral in (4.2.17) is given by

$$\frac{1}{2\pi} \int_{Q_m} i\sqrt{2\pi}\, e^{i\lambda t} \widehat{s}(i\lambda, \theta) d\lambda = Res \left(i\sqrt{2\pi}\, e^{i\lambda t} \widehat{s}(i\lambda, \theta) \right)_{\lambda = \frac{-i\pi}{\omega}}.$$

Now considering the Taylor's expansion of $e^{i\lambda t}$ about $\lambda = \frac{-i\pi}{\omega}$ and the expression of $P_1(\theta)$ in

(4.2.19), we obtain

$$Res\left(i\sqrt{2\pi}\; e^{i\lambda t}\widehat{s}(i\lambda,\theta)\right)_{\lambda=\frac{-i\pi}{\omega}} = Ae^{\frac{\pi}{\omega}t}\sin\frac{\pi}{\omega}\theta.$$

Therefore (4.2.17) leads to

$$s(t,\theta) = s_R + A\; e^{\frac{\pi}{\omega}t}\sin\frac{\pi}{\omega}\theta. \tag{4.2.20}$$

where s_R satisfies (4.2.15).

In terms of the Euler transformation (4.1.9), the decomposition (4.2.20) becomes

$$w(r,\theta) = w_R(r,\theta) + A\; r^{\frac{\pi}{\omega}}\sin\frac{\pi}{\omega}\theta \tag{4.2.21}$$

where in view of (4.2.4), we have

$$w(r,\theta) \equiv w(e^t,\theta) = s(t,\theta) \;\text{ and }\; w_R(r,\theta) \equiv w_R(e^t,\theta) = s_R(t,\theta).$$

Furthermore, by a simple change of variables, we have (see Lemma 4.1.4) $w_R \in P_2^2(G) \cap H_0^1(G)$, with the inequality (4.2.15) becoming

$$\|w_R\|_{P_2^2(G)} \leq C\|g\|_{0,G}. \tag{4.2.22}$$

Finally, we use the cut-off function $\psi \equiv \psi(r)$ in (4.1.3) to rewrite (4.2.21) in the form

$$w(r,\theta) = w_R^1(r,\theta) + A\psi(r)r^{\frac{\pi}{\omega}}\sin\frac{\pi}{\omega}\theta \tag{4.2.23}$$

where

$$w_R^1(r,\theta) := (1-\psi(r))w(r,\theta) + \psi(r)w_R(r,\theta) \in H^2(G)\cap H_0^1(G)$$

such that

$$\|w_R^1\|_{2,G} \leq C\|g\|_{0,G} \tag{4.2.24}$$

because w is regular far away from the corner $(0,0)$ (see Theorem 4.1.1). Thus (4.2.23) and

(4.2.24) yield

$$|A|\|\psi(r)r^{\frac{\pi}{\omega}}\sin\frac{\pi}{\omega}\theta\|_{1,G} \leq \|w_R^1\|_{2,G} + \|w\|_{1,G} \leq \|g\|_{0,G}$$

from where we have

$$|A| \leq C\|g\|_{0,G}. \tag{4.2.25}$$

This completes the proof of Theorem 4.2.1. □

4.3 Regularity and singularities when $p \neq 0$

In the case $p \neq 0$, we proceed by first drawing a consequence of Theorem 4.2.1.

Corollary 4.3.1. *Let $K \subset \mathbb{C}$ be a compact set and let the complex parameter p with $Re(p) \geq 0$ vary in the set K. Then there exist a complex valued function $p \rightsquigarrow B_1(p)$ and a constant C not depending on p such that the solution of (4.1.4), (4.1.5) and (4.1.6) admits the singular representation*

$$w(x,p) = w_R^1(x,p) + B_1(p)\;\psi(r)r^{\frac{\pi}{\omega}}\sin\frac{\pi}{\omega}\theta \tag{4.3.1}$$

with regular part $w_R^1 \in H^2(G)\cap H_0^1(G)$ and coefficient $B_1(p)$ of the singular function satisfying the estimate

$$\|w_R^1\|_{2,G} + |B_1(p)| \leq C\|g\|_{0,G}. \tag{4.3.2}$$

Proof. The decomposition into regular part and singular function stated in Theorem 4.2.1 above means that the bounded linear map $-\Delta + p$ operating from $H^2 \cap H_0^1$ into L^2 has closed range with finite co-dimension 1 or that $-\Delta + p$ has index -1 (See [25], [38]). Notice that (4.3.1) is valid from Theorem 4.2.1 if we re-write (4.1.4) as $-\Delta w = g - p\,w$.

Applying $-\Delta + p$ to both sides of equation (4.3.1), we have

$$g = (-\Delta + p)w_R^1(\cdot,p) + B_1(p)(-\Delta + p)\psi(r)r^{\frac{\pi}{\omega}}\sin\frac{\pi}{\omega}\theta.$$

Now, letting $(-\Delta + p)w_R^1(\cdot,p) =: g_R$ and denoting by $\|(-\Delta + p)^{-1}\|$ the norm of the operator $(-\Delta + p)^{-1}$ from $L^2(G)$ into $H^2(G)$ with domain $D = \{(-\Delta + p)u:\; u \in H^2(G) \cap H^1(G)\}$, we have

$$\begin{aligned}
\|w_R^1(\cdot,p)\|_{2,G} &= \|(-\Delta+p)^{-1}\, g_R\|_{2,G} \\
&\leq \|((-\Delta+p))^{-1}\|\|g_R\|_{0,G} \\
&\leq C\|(-\Delta+p)^{-1}\|\|g\|_{0,G} \\
&\leq C\sup_{p\in K}\|((-\Delta+p))^{-1}\|\|g\|_{0,G} \\
&\leq C\|g\|_{0,G} \qquad (4.3.3)
\end{aligned}$$

because the coefficients of the operator $-\Delta+p$ are continuous and K is compact.

Furthermore, by (4.3.1), (4.3.3) and the analogue of (3.1.29), we have

$$|B_1(p)|\|\psi(r)r^{\frac{\pi}{\omega}}\sin\frac{\pi}{\omega}\theta\|_{1,G} \leq \|w\|_{1,G} + \|w_R^1\|_{2,G} \leq C\|g\|_{0,G},$$

which yields

$$|B_1(p)| \leq C\|g\|_{0,G}.$$

□

Theorem 4.3.2. *For $|p|$ large enough, there exist a regular function $w_R(x,p) \in H^2(G,\sqrt{|p|})$ and a complex valued-function $p \rightsquigarrow B_2(p)$ such that the solution of the problem (4.1.4), (4.1.5) and (4.1.6) admits the singular decomposition*

$$w(x,p) = w_R(x,p) + B_2(p)\ \psi(\sqrt{|p|}r)r^{\frac{\pi}{\omega}}\sin\frac{\pi}{\omega}\theta.$$

Furthermore, we have the estimate

$$\|w_R\|_{2,\,G,\,\sqrt{|p|}} + |B_2(p)||p|^{1-\frac{\pi}{\omega}} \leq C\|g\|_{0,G},$$

where we recall that here and after $C>0$ denotes various constants independent on p and the weighted norm $\|\cdot\|_{2,\,G,\,\sqrt{|p|}}$ is given in Definition 3.1.4.

Proof. We perform the change of variable $x \in G \to \rho x \in G$ where $\rho := \frac{1}{\sqrt{|p|}}$ and $\omega = \rho^2 p = \frac{p}{|p|}$. Problem (4.1.4) becomes

$$-\frac{1}{\rho^2}\left(\frac{\partial^2 w(\rho x)}{\partial x_1^2} + \frac{\partial^2 w(\rho x)}{\partial x_2^2}\right) + p\, w(\rho x) = g(\rho x)$$

or equivalently

$$(-\Delta + \omega)\, w_\rho(x) = h_\rho(x) \tag{4.3.4}$$

where $h_\rho(x) = \rho^2 g(\rho x)$ and $w_\rho(x) = w(\rho x)$. Since the complex parameter ω satisfies $|\omega| = 1$, Corollary 4.3.1 applies to (4.3.4). Thus w_ρ admits the singular decomposition

$$w_\rho = w^{R,\rho}(x,\omega) + \psi(r) B_1(\omega,\rho)\rho^{\frac{\pi}{\omega}} r^{\frac{\pi}{\omega}} \sin\frac{\pi}{\omega}\theta \tag{4.3.5}$$

or

$$w_\rho = w^{R,\rho}(x,\omega) + \psi(r) B_2(p) r^{\frac{\pi}{\omega}} \sin\frac{\pi}{\omega}\theta \tag{4.3.6}$$

where $B_1(\omega,\rho) = B_2(p)\rho^{\frac{\pi}{\omega}}$ with uniform estimate:

$$\|w^{R,\rho}\|_{2,G} + |B_2(p)|\rho^{\frac{\pi}{\omega}} \le C\|h_\rho\|_{0,G}. \tag{4.3.7}$$

Now from (4.3.7), we go back from the variable ρx in (4.3.4) to the initial variable x in (4.1.4) as follows: Put

$$w_R(x) = w^{R,\rho}\left(\frac{x}{\rho}\right) \text{ and } z = \frac{x}{\rho} \text{ so that } dz = \rho^{-2} dx,$$

$$\frac{\partial w_R}{\partial x_1}(x) = \frac{\partial w^{R,\rho}}{\partial z_1}(z)\frac{1}{\rho} \text{ and } \frac{\partial^2 w_R}{\partial x_1^2}(x) = \frac{\partial^2 w^{R,\rho}}{\partial z_1^2}(z)\frac{1}{\rho^2}$$

Thus we have

$$\begin{aligned}
\|w^{R,\rho}\|_{2,G}^2 &= \int_G (|w^{R,\rho}(z)|^2 + |\nabla w^{R,\rho}(z)|^2 + \sum_{|\alpha|=2} |D^\alpha w^{R,\rho}(z)|^2) dz \\
&= \int_G (|w_R(x)|^2 + \rho^2 |\nabla_x w_R(x)|^2 + \rho^4 \sum_{|\alpha|=2} |D_x^\alpha w_R(x)|^2) \rho^{-2} dx \\
&= \int_G (\rho^{-2}|w_R(x)|^2 + |\nabla_x w_R(x)|^2 + \rho^2 \sum_{|\alpha|=2} |D_x^\alpha w_R(x)|^2) dx \\
&= \rho^2 \int_G (\rho^{-4}|w_R(x)|^2 + \rho^{-2}|\nabla w_R(x)|^2 + \sum_{|\alpha|=2} |D^\alpha w_R(x)|^2) dx \\
&= |p|^{-1} \int_G (\sum_{|\alpha|=2} |D^\alpha w_R(x)|^2 + |p||\nabla w_R(x)|^2 + |p|^2 |w_R(x)|^2) dx \\
&= |p|^{-1} \|w_R\|_{2,G,\sqrt{|p|}}^2,
\end{aligned}$$

which implies that

$$\frac{1}{\rho^2}\|w^{R,\rho}\|_{2,G}^2 = \|w_R\|_{2,G,\frac{1}{\rho}}^2. \tag{4.3.8}$$

Similarly, the right hand side of (4.3.7) yields

$$\|h_\rho(\frac{x}{\rho})\|_{0,G}^2 = \int_G |h_\rho(z)|^2 dz = \int_G |\rho^2 g(\rho z)|^2 dz = \rho^2 \|g\|_{0,G}^2. \tag{4.3.9}$$

Using (4.3.7), (4.3.8) and (4.3.9) we have the desired estimate

$$\|w_R\|_{2,G,\sqrt{|p|}} + |B_2(p)||p|^{\frac{1}{2}-\frac{\pi}{2\omega}} \leq C\|g\|_{0,G},$$

together with the singular decomposition

$$w(x) = w_R(x) + B_2(p)\psi(\sqrt{|p|}r)r^{\frac{\pi}{\omega}} \sin\frac{\pi}{\omega}\theta.$$

$\square$

Remark 4.3.3. *The second part of Theorem 4.3.2 (case $|p|$ large) and its proof constitute a particular case of the deep results stated and proved in [19], [46] and [47] for general elliptic and parabolic problems with edge corners. An alternative approach is presented in [30]. The nature of the Helmholtz operator $-\Delta + pI$ makes the above proof simple and explicit in the* *following manner compared to a general operator of the form* $p + L(x, D_x)$ *investigated in the above mentioned references with* $L(x, D_x)$ *being a proper elliptic operator of order 2 with principal part frozen at the origin denoted by* $L_0(D_x)$. *In making the change of variable* $x \to \rho x$, *the analogue of (4.3.4) has the form*

$$M_\rho(x, D_x)w_\rho = h_\rho \tag{4.3.10}$$

where the operator M_ρ *tends to* $\omega + L_0$ *as* $\rho \to 0$.

The analogue of (4.3.6) is neither explicit nor does it give a uniform estimate of the form (4.3.7). Such an estimate is achieved provided that a perturbation argument together with the convergence of M_ρ *to* $\omega + L_0$ *is used. On the contrary, for the Helmholtz operator,* M_ρ *is reduced to the constant operator* $-\Delta + \omega$.

So far, the analysis of the regularity and the singularity of the solution of problem (4.1.1) was done in two local steps: far away from vertices (Theorem 4.1.1) and near each vertex (Theorem 4.2.1, Corollary 4.3.1 and Theorem 4.3.2). We now combine these steps to obtain the following global result on Ω.

Theorem 4.3.4. *There exists a positive number* $\delta_0 > 0$ *such that the solution of the problem (4.1.1) admits the singular decomposition*

$$w(x, p) = w_R^1(x, p) + B_1(p)\psi(r)r^{\frac{\pi}{\omega}} \sin \frac{\pi}{\omega}\theta$$

with regular part $w_R^1(x, p) \in H^2(\Omega)$ *and coefficients of singularity* $B_1(p) \in \mathbb{C}$ *satisfying the estimate*

$$\|w_R^1\|_{2,\Omega} + |B_1(p)| \leq C\|g\|_{0,\Omega}$$

for $|p| \leq \delta_0$. *Furthermore, the singular decomposition becomes*

$$w(x, p) = w_R^2(x, p) + B_2(p)\psi(r\sqrt{|p|})r^{\frac{\pi}{\omega}} \sin \frac{\pi}{\omega}\theta$$

where $w_R^2(x, p) \in H^2(\Omega, \sqrt{|p|})$ *and*

$$\|w_R^2\|_{2,\Omega,\sqrt{|p|}} + |B_2(p)||p|^{\frac{1}{2}-\frac{\pi}{2\omega}} \leq C\|g\|_{0,\Omega}$$

for $|p| > \delta_0$.

Proof. Notice that Ω was assumed to have only one non-convex vertex, which is localized through the cut-off function $\psi = \psi(r)$ used before.

The solution w of (4.1.1) can then be written as

$$w(x,p) = (1-\psi)w(x,p) + \psi\, w(x,p) \ \text{ on } \ \Omega.$$

Corollary 4.3.1 and Theorem 4.3.2 guarantee the existence of $\delta_0 > 0$ such that the singular decompositions and the estimates in these two results apply to the local solution ψw of (4.1.4) with right-hand side

$$\psi\, g - w\Delta\psi - 2\nabla w \nabla\psi.$$

More precisely, for $|p| \leq \delta_0$, we have

$$w(x,p) = (1-\psi)w(x,p) + w_R^1(x,p) + B_1(p)\psi(r) r^{\frac{\pi}{\omega}} \sin\frac{\pi}{\omega}\theta \tag{4.3.11}$$

with

$$\|w_R^1\|_{2,\Omega} + |B_1(p)| \leq C\|\psi\, g - w\Delta\psi - 2\nabla w\nabla\psi\|_{0,\Omega}.$$

The desired regular part for w is

$$w_R^{1,1} := (1-\psi)w + w_R^1$$

which is indeed of class $H^2(\Omega)$ due to the regularity far away from the vertex that guarantees that $(1-\psi)w \in H^2(\Omega)$. Then with

$$\Omega_{r_0} := \{x \in \Omega; r_0/2 < |x| = r \leq r_0\}$$

we have

$$\begin{aligned}
\|w_R^{1,1}\|_{2,\Omega} + |B_1(p)| &\leq \|(1-\psi)w\|_{2,\Omega} + \|w_R^1\|_{2,\Omega} + |B_1(p)| \\
&\leq C\|w\|_{2,\Omega_{r_0}} + C\|\psi g - w\Delta\psi - 2\nabla w\nabla\psi\|_{0,\Omega} \\
&\leq C\|w\|_{2,\Omega_{r_0}} + C\|g\|_{0,\Omega} + C\|w\|_{1,\Omega_{r_0}} \\
&\leq C\|g\|_{0,\Omega},
\end{aligned}$$

by the regularity of the solution far away from the origin and specifically on Ω_{r_0}. Notice that the various constants C above do not depend on p because p moves in the the compact set $\bar{B}(0, \delta_0)$.

Regarding the case when $|p| > \delta_0$, the singular decomposition to be used in place of (4.3.11) is

$$w(x,p) = (1-\psi)w(x,p) + w_R^2(x,p) + B_2(p)\psi(r\sqrt{|p|})r^{\frac{\pi}{\omega}} \sin \frac{\pi}{\omega}\theta$$

with

$$\|w_R^2\|_{2,\Omega,\sqrt{|p|}} + |B_2(p)||p|^{\frac{1}{2}-\frac{\pi}{2\omega}} \leq C\|\psi g - w\Delta\psi - 2\nabla w \nabla\psi\|_{0,\Omega}.$$

Take $w_R^{2,2} := (1-\psi)w + w_R^2 \in H^2(\Omega)$ as the regular part. In view of the analogue of the Theorem 3.2.2 we have

$$\|(1-\psi)w\|_{2,\Omega,\sqrt{|p|}} \leq C\|g\|_{0,\Omega}.$$

Therefore we have as in the previous case

$$\|w_R^{2,2}\|_{2,\Omega,\sqrt{|p|}} + |B_2(p)||p|^{\frac{1}{2}-\frac{\pi}{2\omega}} \leq C\|g\|_{0,\Omega}.$$

□

4.4 Global regularity of the solution

We devote this section to show that the solution of the Helmholtz problem is regular in a weighted Sobolev space. This result is fundamental to our study as the constructive analysis to come is based on it. The weighted Sobolev space in question is defined as follows:

Definition 4.4.1. *For β a non-negative real number, we denote by $H^{2,\beta}(\Omega)$ the space of all distributions $v \in H^1(\Omega)$ such that*

$$r^\beta D^\alpha v \in L^2(\Omega) \ \ \forall\, \alpha \text{ such that } |\alpha| = 2$$

where $r \equiv r(x) = d(x,\ vertices)$ is the distance to the vertices of the domain Ω.

The weighted Sobolev space $H^{2,\beta}(\Omega)$ is equipped with its natural Hilbert structure given by the inner product

$$(w,v)_{H^{2,\beta}(\Omega)} = (w,v)_{1,\Omega} + \sum_{|\alpha|=2} \int_\Omega r^\beta D^\alpha w . D^\alpha v dx.$$

The norm of the space $H^{2,\beta}(\Omega)$ is written $\|\cdot\|_{H^{2,\beta}(\Omega)}$ while the following is simply a semi-norm:

$$|v|_{H^{2,\beta}(\Omega)} := \left(\sum_{|\alpha|=2} \int_\Omega |r^\beta D^\alpha v|^2 dx \right)^{\frac{1}{2}}.$$

Remark 4.4.2. *The usual Sobolev space $H^2(\Omega)$ is continuously embedded in the weighted Sobolev space $H^{2,\beta}(\Omega)$:*

$$H^2(\Omega) \hookrightarrow H^{2,\beta}(\Omega).$$

Indeed, this is obvious for $\beta = 0$ since $H^2(\Omega) = H^{2,0}(\Omega)$.

For $\beta > 0$ and for $v \in H^2(\Omega)$, we have

$$\begin{aligned} \int_\Omega \left(|v|^2 + |\nabla v|^2 + \sum_{|\alpha|=2} |D^\alpha v|^2 \right) dx &= \int_\Omega \left(|v|^2 + |\nabla v|^2 + \sum_{|\alpha|=2} r^{2\beta} |D^\alpha v|^2 r^{-2\beta} \right) dx \\ &\geq C \int_\Omega \left(|v|^2 + |\nabla v|^2 + \sum_{|\alpha|=2} r^{2\beta} |D^\alpha v|^2 \right) dx \end{aligned}$$

where $C = \min \left\{ 1, \left(\frac{1}{diameter(\Omega)} \right)^{2\beta} \right\}$, observing that $\sup_{x \in \bar{\Omega}} d(x, vertices) \leq \; diameter(\Omega)$.

Theorem 4.4.3. *The space $H^{2,\beta}(\Omega)$ is continuously and compactly embedded in $C^0(\bar{\Omega})$ for $0 \leq \beta < 1$:*

$$H^{2,\beta}(\Omega) \hookrightarrow_c C^0(\bar{\Omega}).$$

Furthermore, the embedding of $H^{2,\beta}(\Omega)$ into $H^1(\Omega)$ is compact: $H^{2,\beta}(\Omega) \hookrightarrow_c H^1(\Omega)$

Proof. The case when $\beta = 0$ is well-known because $H^{2,0}(\Omega) = H^2(\Omega)$ (Sobolev and Rellich-Kondrachov embeddings, Theorem 2.4.5). So we assume that $\beta > 0$. Let v be in $H^{2,\beta}(\Omega)$ so that

$$v \in L^p(\Omega),\ \forall\, p \in [1,+\infty) \text{ and } D^\alpha v = (r^\beta D^\alpha v).r^{-\beta}\ \forall\ 1 \leq |\alpha| \leq 2. \tag{4.4.1}$$

The first inclusion in (4.4.1) is due to the fact that $v \in H^1(\Omega)$, which is embedded in $L^p(\Omega)\ \ \forall\, p \in [1,+\infty)$ by Theorem 2.4.5. We want to show that $D^\alpha v \in L^p(\Omega)\ \ 1 \leq |\alpha| \leq 2$ for some $p > 1$ and $p < 2$. Take $q_1 = \frac{2}{p}$ with conjugate $q_2 = \frac{2}{2-p}$ i.e. $\frac{1}{q_1} + \frac{1}{q_2} = 1$. Then $r^{-\beta p}$ is of class $L^{q_2}(\Omega)$ iff $1 \leq p < \frac{2}{1+\beta}$. By Hölder's inequality, we deduce from (4.4.1) and the choice of $p,\ q_1$ and q_2 that

$$\begin{aligned} \int_\Omega |D^\alpha v|^p dx &\leq \left(\int_\Omega (|r^\beta D^\alpha v|^p)^{q_1} dx\right)^{\frac{1}{q_1}} \left(\int_\Omega r^{-\beta p q_2} dx\right)^{\frac{1}{q_2}} \\ &= \left(\int_\Omega |r^\beta D^\alpha v|^2 dx\right)^{\frac{p}{2}} \left(\int_\Omega r^{\frac{-2\beta p}{2-p}} r dr\right)^{\frac{2-p}{2}} \\ &\leq C\|v\|^p_{H^{2,\beta}(\Omega)}. \end{aligned}$$

Notice that if $|\alpha| = 0$ and $\beta = 0$ in (4.4.1), we could show in a similar manner that $v \in L^p(\Omega)$ for the specific choice of p made above. Thus $H^{2,\beta}(\Omega) \hookrightarrow W^{2,p}(\Omega)$. But by the Sobolev and Rellich Kondrachov imbeddings, Theorem 2.4.5, the Sobolev space $W^{2,p}(\Omega)$ is continuously and compactly embedded into $C^0(\bar{\Omega})$ and $H^1(\Omega)$, respectively. This proves the first and the second claims and hence the proof of the Theorem is completed. $\square$

We are now in a position to state one of our main contributions that will have an impact on the heat equation and on its numerical approximation in the next chapter. This result is announced in [14] and [13].

Theorem 4.4.4. *Assume that $0 < \beta < 1 - \frac{\pi}{\omega}$. Then the solution w of the Helmholtz problem (4.1.1) is of class $H^{2,\beta}(\Omega)$ such that the following estimate holds for some constant $C > 0$ independent on p:*

$$\|w\|_{H^{2,\beta}(\Omega)} \leq C\|g\|_{0,\Omega}.$$

Proof. The existence of a number $\delta_0 > 0$ in Theorem 4.3.4 is the rephrasing of the requirement that $|p|$ is large enough in Theorem 4.3.2. From Remark 4.4.2, we have for the regular part in Theorem 4.3.4:

$$\|w_R^1\|_{H^{2,\beta}(\Omega)} \leq C\|w_R^1\|_{2,\Omega} \leq C\|g\|_{0,\Omega}$$

$$\|w_R^2\|_{H^{2,\beta}(\Omega,|p|)} \leq C\|w_R^2\|_{2,\Omega,|p|} \leq C\|g\|_{0,\Omega}$$

where the weighted norm on $H^{2,\beta}(\Omega, \sqrt{|p|})$ is defined in a similar manner as that of $H^2(\Omega, \sqrt{|p|})$ of Definition 3.1.4 by

$$\|v\|^2_{H^{2,\beta}(\Omega,\sqrt{|p|})} = \int_\Omega \left(|p|^2|v|^2 + |p||\nabla v|^2 + \sum_{|\alpha|=2} |r^\beta D^\alpha v|^2 \right) dx. \tag{4.4.2}$$

Regarding the singular part, we proceed as follows. Firstly, the function $\psi(r)r^{\frac{\pi}{\omega}} \sin \frac{\pi}{\omega}\theta$ belongs to $H^{2,\beta}(\Omega)$ because near the non-convex corner $(0,0)$, $r^\beta D^\alpha \psi(r) r^{\frac{\pi}{\omega}} \sin \frac{\pi}{\omega}\theta$ with $|\alpha| = 2$, behaves like $r^{\beta+\frac{\pi}{\omega}-2}$ which is of class $L^2(\Omega)$ in view of the condition $0 < \beta < 1 - \frac{\pi}{\omega}$.
Thus for $|p| \leq \delta_0$, the estimate for $|B_1(p)|$ in Theorem 4.3.4 yields

$$\|\psi(r)B_1(p)r^{\frac{\pi}{\omega}} \sin \frac{\pi}{\omega}\theta\|_{H^{2,\beta}(\Omega)} \leq C\|g\|_{0,\Omega}.$$

For $|p| > \delta_0$, the same argument as above shows that $\psi(r|p|)r^{\frac{\pi}{\omega}} \sin \frac{\pi}{\omega}\theta$ is of class $H^{2,\beta}(\Omega)$. Now the estimate for $B_2(p)$ in Theorem 4.3.4 leads to

$$\begin{aligned} \|\psi(r\sqrt{|p|})B_2(p)r^{\frac{\pi}{\omega}} \sin \frac{\pi}{\omega}\theta\|_{H^{2,\beta}(\Omega)} &\leq C|B_2(p)| \\ &\leq C|p|^{\frac{\pi}{2\omega}-\frac{1}{2}}\|g\|_{0,\Omega} \\ &\leq C(\delta_0)^{\frac{\pi}{2\omega}-\frac{1}{2}}\|g\|_{0,\Omega}. \end{aligned}$$

□

Remark 4.4.5. *The underlying point of our investigation is that the linear operator*

$$g(.,p) \in L^2(\Omega) \rightsquigarrow w(.,p) \in H^{2,\beta}(\Omega)$$

is bounded with norm independent on $p \in \mathbb{C}$ satisfying (3.1.20). Theorem 4.4.4 is proved in Grisvard [29] in the particular case when $p = 0$. This originates from the study by Raugel [58], [59] of the regularity in the general case where g is in the Sobolev space $H^m(\Omega)$, $m > 0$. In this case weighted Sobolev spaces $H^{m+2,\beta}(\Omega)$ of higher order are essential as demonstrated by Raugel.

Remark 4.4.6. *In this thesis we used three types of weighted Sobolev spaces which play completely different roles:*

- *The weighted Sobolev space* $H^m(\Omega, \rho)$ *(cf. Definition 3.1.4 and Proposition 3.1.5), which is exactly the usual Sobolev space* $H^m(\Omega)$ *equipped with a weighted norm. The space* $H^m(\Omega, \rho)$ *arises generally when the (partial) Fourier transform with respect to* t *is applied to functions* $(t, x) \to v(x, t)$ *in the usual Sobolev space* $H^m(\Omega \times \mathbb{R})$*. In fact the norm* $\|v\|_{m,\Omega\times\mathbb{R}}$ *is equivalent to* $\left(\int_{\mathbb{R}} \|(\mathcal{F}v)(\eta)\|^2_{m,\Omega,1+|\eta|} d\eta\right)^{\frac{1}{2}}$ *see Dauge [19].*
- *The Kondratiev weighted Sobolev space* $P_2^k(G)$ *(cf Definition 4.1.3) serves to investigate the regularity and the singularity for an elliptic problem localized in a sector* G*. The space* $P_2^k(G)$ *is not equal to the usual Sobolev space* $H^k(G)$*. However, it is related to* H^k *through Lemma 4.1.4 and we have* $P_2^k(G) \subset H^k_{loc}(G)$ *i.e.* $v \in P_2^k(G) \Rightarrow \rho v \in H^k(G)\ \forall \rho \in \mathcal{D}(G)$.
- *The weighted Sobolev space* $H^{2,\beta}(\Omega)$ *is a replacement for* $H^2(\Omega)$ *for the global regularity of the solutions.*

Chapter 5

The heat equation

The material collected in the previous chapters enable us now to study the heat equation

$$\frac{\partial u}{\partial t} - \Delta u = f \text{ in } \Omega \tag{5.0.1}$$

appended with the initial condition

$$u(x, 0) = 0, \quad x \in \Omega \tag{5.0.2}$$

and the boundary condition

$$u(x, t) = 0 \text{ on } \partial\Omega \times (0, +\infty). \tag{5.0.3}$$

The investigation deals with two main steps. The first step is the quantitative and some particular qualitative analysis (section 5.1). The second step deals exclusively with the qualitative analysis regarding the regularity and the corner singularity of the solution (section 5.2).

5.1 Well-posedness and tangential regularity

In the heat equation and generally in parabolic problems, the time variable "t" plays a special role compared to the space variable "x". We will reflect the different roles of these variables by separating them as follows in the Sobolev spaces where the solution lives. For a function $v : (x, t) \in \Omega \times (0, +\infty) \to v(x, t) \in \mathbb{R}$, we write $v(t) \equiv v(\cdot, t) : \Omega \to \mathbb{R}$, $\quad v(t)(x) = v(x, t)$ and $v(x) \equiv v(x, \cdot) : (0, +\infty) \to \mathbb{R}$, $\quad v(x)(t) = v(x, t)$ when the variables t and x are fixed, respectively. The definitions and the comments below can be found in Lions and Magenes [39] though Ω is a polygon in our case.

Definition 5.1.1. *Given two integers $r \geq 0$ and $s \geq 0$, we denote by $H^{r,s}(\Omega \times (0, +\infty))$ the anisotropic Sobolev space defined by*

$$H^{r,s}(\Omega \times (0, +\infty)) := L^2\left((0, +\infty), H^r(\Omega)\right) \cap H^s\left((0, +\infty), L^2(\Omega)\right)$$

and equipped with the Hilbert structure via the norm

$$\|v\|_{H^{r,s}(\Omega\times(0,+\infty))} := \left[\int_0^{+\infty} \left(\|v(\cdot,t)\|^2_{r,\Omega} + \sum_{j=0}^{s} \|\frac{\partial^j v(\cdot,t)}{\partial t^j}\|^2_{0,\Omega}\right) dt\right]^{\frac{1}{2}}.$$

Remark 5.1.2. *Notice that $H^{0,0}(\Omega \times (0, +\infty)) = L^2(\Omega \times (0, +\infty)) = L^2((0, +\infty),\ L^2(\Omega))$. The following subspaces of $H^{r,s}(\Omega \times (0, +\infty))$ will be used from time to time:*

- $$H^{r,s}_{0,}\left(\Omega \times (0, +\infty)\right) := L^2\left((0, +\infty), H^r_0(\Omega)\right) \cap H^s\left((0, +\infty), L^2(\Omega)\right);$$

  This is characterized as the closure in $H^{r,s}(\Omega \times (0, +\infty))$ of the subspace of functions which are equal to zero in a neighborhood of the set $\Gamma \times (0, +\infty)$;

- $$H^{r,s}_{,0}(\Omega \times (0, +\infty)) := L^2\left((0, +\infty), H^r(\Omega)\right) \cap H^s_0\left((0, +\infty), L^2(\Omega)\right),$$

  which is also the closure in $H^{r,s}(\Omega \times (0, +\infty))$ of the subspace of functions that are equal to zero near $t = 0$ and $t = \infty$;

- $$H^{r,s}_{0,0}(\Omega \times (0, +\infty)) := H^{r,s}_{0,}\left(\Omega \times (0, +\infty)\right) \cap H^{r,s}_{,0}\left(\Omega \times (0, +\infty)\right),$$

  which is the closure in $H^{r,s}(\Omega\times(0, +\infty))$ of the space $\mathcal{D}(\Omega\times(0, +\infty))$ of test functions;

- $$\widetilde{H}^{r,s}(\Omega \times (0, +\infty) := L^2\left((0, +\infty), H^r(\Omega)\right) \cap \widetilde{H}^s\left((0, +\infty), L^2(\Omega)\right),$$

  where $\widetilde{H}^s\left((0, +\infty), L^2(\Omega)\right)$ is the space of functions $v \in H^s\left((0, +\infty), L^2(\Omega)\right)$ such that their extension $\widetilde{v}$ by zero outside $(0, +\infty)$ belong to $H^s(\mathbb{R}, L^2(\Omega))$. Notice that

  $$H^{r,s}_{,0}(\Omega \times (0, +\infty)) \subset \widetilde{H}^{r,s}(\Omega \times (0, +\infty)).$$

In what follows in this section, we consider the well-posedness of the boundary value problem associated with the heat operator and the tangential regularity of its solution. Though these results are classical, we give the proofs in detail for convenience. (See Lions and Magenes [38]).

Theorem 5.1.3. *Under the assumption $f \in L^2\left[(0,+\infty), L^2(\Omega)\right] \equiv L^2((0,+\infty)\times\Omega)$, there exists a unique variational solution*

$$u \in \widetilde{H}^{1,1}_{0,}\left(\Omega \times (0,+\infty)\right) \tag{5.1.1}$$

of the heat equation (5.0.1)-(5.0.3) such that

$$\|u\|_{\widetilde{H}^{1,1}_{0,}(\Omega\times(0,+\infty))} \leq C\|f\|_{0,\Omega\times(0,+\infty)}. \tag{5.1.2}$$

In other words, the solution

$$u \in L^2\left[(0,+\infty), H^1_0(\Omega)\right] \tag{5.1.3}$$

satisfying

$$\|u\|_{L^2\left[(0,+\infty),H^1_0(\Omega)\right]} \leq C\|f\|_{0,\Omega\times(0,+\infty)}. \tag{5.1.4}$$

is also tangentially regular in the sense that

$$u \in \widetilde{H}^1\left[(0,+\infty), L^2(\Omega)\right]$$

which is the optimal differentiability smoothness in the time variable "t", such that

$$\|u\|_{\widetilde{H}^1(0,+\infty),L^2(\Omega)} \leq C\|f\|_{0,\Omega\times(0,+\infty)}. \tag{5.1.5}$$

Proof. The fact that $f \in L^2\left[(0,+\infty), L^2(\Omega)\right]$ implies that f is a vector-valued distribution, $f \in \mathcal{D}'(L^2(\Omega))$, such that, for $\xi \geq 0$, $e^{-\xi t}\widetilde{f} \in \mathcal{S}'(L^2(\Omega))$ is a vector-valued tempered distribution. (see Definition 2.5.14 and 2.5.23). Therefore, it is natural to look for a solution u of (5.0.1)-(5.0.3) which is a vector-valued distributions $u \in \mathcal{D}'(L^2(\Omega))$. We proceed by necessary conditions and assume that a solution $u \in \mathcal{D}'(L^2(\Omega))$ exists such that for $p = \xi + i\eta \quad \xi \geq 0$, we have $e^{-\xi t}u \in \mathcal{S}'(L^2(\Omega))$.

Since $f \in L^2\left[\Omega \times (0,+\infty)\right] = L^2\left[(0,+\infty), L^2(\Omega)\right]$, Proposition 2.5.36 implies that its

Laplace transform $\widehat{f}(\cdot,p)$ exists for $Re(p) \geq 0$, with more precisely $\widehat{f}(\cdot,p)$ belonging to the Hardy-Lebesgue space: $\widehat{f}(p) \in H^2\,[0; L^2(\Omega)]$.

For the class of solutions we are interested in, Definition 2.5.32 guarantees the existence of the Laplace transform $\widehat{u}(p)$. Therefore, taking the Laplace transform of the distributional equation (5.0.1)-(5.0.3) leads to the Helmholtz problem

$$-\Delta\widehat{u} + p\,\widehat{u} = \widehat{f} \ \text{ in } \ \Omega \tag{5.1.6}$$

$$\widehat{u} = 0 \ \text{ on } \ \partial\Omega. \tag{5.1.7}$$

We now make use of the results established in chapters 3 and 4 about the Helmholtz problem.

Firstly, since $\widehat{f}(p) \in L^2(\Omega)$ for $Re(p) \geq 0$, Theorem 3.1.7 guarantees that there exists a unique variational solution

$$\widehat{u} \in H_0^1(\Omega, 1 + |p|)$$

of (5.1.6)-(5.1.7) satisfying the relation

$$\|\widehat{u}(p)\|_{1,\Omega,1+|p|} \leq C\|\widehat{f}(p)\|_{0,\Omega}^2 \tag{5.1.8}$$

From (5.1.8) and the fact that $\widehat{f}(p) \in H^2\,[0; L^2(\Omega)]$, we deduce

$$\begin{aligned} \sup_{\xi>0}\left(\int_{-\infty}^{\infty} \|\widehat{u}(\cdot,\xi+i\eta)\|_{1,\Omega,1+|p|}^2 d\eta\right) &\leq C\sup_{\xi>0}\left(\int_{-\infty}^{\infty} \|\widehat{f}(\cdot,\xi+i\eta)\|_{0,\Omega}^2 d\eta\right) \\ &< +\infty. \end{aligned} \tag{5.1.9}$$

Secondly, the function $p \rightsquigarrow \widehat{u}(p)$ is holomorphic in the complex region $Re(p) \geq 0$ since $\widehat{f}(p)$ enjoys this property and the operator $-\Delta + p$, is analytic hypo-elliptic (Theorem 2.5.38). Thus

$$\widehat{u}(p) \in H^2\left[0; H_0^1(\Omega)\right].$$

In the third step, we use the conclusion of the second step, which enables us to apply the Paley-Wiener theorem (Theorem 2.5.37): there exists a function

$$v \in L^2\left[(-\infty,+\infty); H_0^1(\Omega)\right], \ \text{ with } \ v(\cdot,t) \ = 0 \ \text{ for } \ t < 0$$

and

$$\widehat{v}(p) = \widehat{u}(p) \quad \text{for} \quad \xi = Re(p) \geq 0.$$

By injectivity of the Laplace transform, we have $u = v$. This proves (5.1.3).

In the fourth step, we take $p = i\eta$ in (5.1.8) and integrate both sides, to obtain

$$\int_{-\infty}^{\infty} \left[\|\nabla \widehat{u}(i\eta)\|_{0,\Omega}^2 + (1+|\eta|)^2 \|\widehat{u}(i\eta)\|_{0,\Omega}^2\right] d\eta \leq C \int_{-\infty}^{\infty} \|\hat{f}(i\eta)\|_{0,\Omega}^2 d\eta. \tag{5.1.10}$$

Using the Plancherel-Parseval theorem, the relation (5.1.10) leads to

$$\int_{-\infty}^{\infty} \left[\|\nabla u(t)\|_{0,\Omega}^2 + \|u(t)\|_{0,\Omega}^2 + \|\frac{\partial u(t)}{\partial t}\|_{0,\Omega}^2\right] dt \leq C \int_{-\infty}^{\infty} \|f(t)\|_{0,\Omega}^2 dt. \tag{5.1.11}$$

The relation (5.1.11) implies in particular that

$$u \in H^1\left[(-\infty, +\infty); L^2(\Omega)\right].$$

By the Sobolev embedding theorem, valid for vector-valued Sobolev spaces, the space

$$H^1\left[(-\infty, +\infty); L^2(\Omega)\right]$$

is continuously embedding in $C^0\left[(-\infty, +\infty); L^2(\Omega)\right]$. Therefore $u(0) = 0$ because $u(t) = 0$ for $t < 0$. Consequently u satisfies the inclusion (5.1.1) and the relation (5.1.11) leads to (5.1.2), (5.1.4), (5.1.5).

Conversely, if we do not start from a solution $u \in \mathcal{D}'(L^2(\Omega))$ such that $e^{-\xi t}u \in \mathcal{S}'(L^2(\Omega))$ for $\xi \geq 0$, we consider the Helmholtz problem in (5.1.6)-(5.1.7) where $\widehat{u}$ is unknown. All the arguments following (5.1.7) remain valid and lead to the existence of a unique solution satisfying (5.1.1)-(5.1.5). The theorem is proved. $\square$

Remark 5.1.4. *It can be shown that $u \in \widetilde{H}_{0,}^{1,1}(\Omega \times (0, +\infty))$ obtained in Theorem 5.1.3 is the only function of this class such that, for $t > 0$,*

$$\int_{\Omega} \left[\frac{\partial u}{\partial t}(x,t)v(x) + \nabla_x u(x,t) \nabla_x v(x)\right] dx = \int_{\Omega} f(x,t)v(x)dx, \quad \forall v \in H_0^1(\Omega). \tag{5.1.12}$$

Equation (5.1.12) is the variational formulation of the heat problem (5.0.1)-(5.0.3). For the *study of the variational problem (5.1.12), we refer the reader to [37].*

5.2 Regularity and singularities of the solution

In section 5.2, we assumed that the domain Ω has a boundary $\partial\Omega \equiv \Gamma$ of class C^2 in the sense of Definition 2.1.1. For the problem (5.1.6)-(5.1.7), the relation (5.1.8) combined with Theorem 3.2.2 regarding the regularity of the solution of this problem implies that we have the estimate

$$\sum_{|\alpha|=2} \|D^\alpha \widehat{u}(i\eta)\|^2_{0,\Omega} + \|\nabla \widehat{u}(i\eta)\|^2_{0,\Omega} + \|\widehat{u}(i\eta)\|^2_{0,\Omega} \leq C\|\widehat{f}(i\eta)\|^2_{0,\Omega}. \tag{5.2.1}$$

By the Plancherel-Parseval theorem and the fact that $u(t) = 0$ for $t \leq 0$, we have

$$\int_0^{+\infty} \left[\sum_{|\alpha|=2} \|D^\alpha u(t)\|^2_{0,\Omega} + \|\nabla u(t)\|^2_{0,\Omega} + \|u(t)\|^2_{0,\Omega}\right] dt \leq C \int_0^{+\infty} \|f(t)\|^2_{0,\Omega} dt. \tag{5.2.2}$$

Consequently, we have proved the following regularity result:

Theorem 5.2.1. *Under the assumption that the domain Ω has a boundary of class C^2, the solution u of the heat equation obtained in Theorem 5.1.3 is regular in the sense that*

$$u \in \widetilde{H}^{2,1}(\Omega \times (0, +\infty))$$

such that

$$\|u\|_{\widetilde{H}^{2,1}(\Omega\times(0,+\infty))} \leq C\|f\|_{0,\Omega\times(0,+\infty)}.$$

The non-smooth case addresses the study of the regularity and singularity of the solution of the heat equation specifically in the polygonal domain. The result reads as follows:

Theorem 5.2.2. *Let Ω be a bounded open polygonal subset of $\mathbb{R}^2$ with only one non-convex vertex of interior angle $\omega > \pi$, $f \in L^2(\Omega \times (0, +\infty))$ and u the solution of the heat equation given in Theorem 5.1.3. Then there holds the singular decomposition*

$$u = u_R + [K *_t \phi(r,t)]\, r^{\frac{\pi}{\omega}} \sin\frac{\pi}{\omega}\theta \tag{5.2.3}$$

where:

- *The function*
$$u_R \in \widetilde{H}^{1,1}_{0,}\left(\Omega \times (0,+\infty)\right) \cap \widetilde{H}^{2,1}\left(\Omega \times (0,+\infty)\right)$$
is the regular part;

- *The function*
$$K \in \widetilde{H}^{\frac{1}{2}-\frac{\pi}{2\omega}}(0,+\infty)$$
is the "coefficient" of singularity;

- *The function $\phi(r,t)$ is a regularizing kernel family to be specified shortly in the proof;*

- *The symbol $*_t$ represents the convolution in the variable t.*

Moreover, we have the estimate

$$\|u_R\|_{\widetilde{H}^{2,1}(\Omega\times(0,+\infty))} + \|K\|_{\widetilde{H}^{\frac{1}{2}-\frac{\pi}{2\omega}}(0,+\infty)} \leq C\|f\|_{0,\Omega\times(0,+\infty)}. \tag{5.2.4}$$

Proof. By performing the Laplace transform of vector-valued distributions, (5.0.1)-(5.0.3) becomes the Helmholtz problem (4.1.1) or (5.1.6)-(5.1.7) where $w(p) = \widehat{u}(p)$ and $g(p) = \widehat{f}(p)$. We use extensively the notation in Theorem 4.3.4. Let $\delta_0 > 0$ be as in Theorem 4.3.4 where we take $\xi = 0$ i.e. $p = i\eta$. From this theorem we define

$$w_R(i\eta) := \begin{cases} w_R^1(i\eta) & \text{if} \quad |\eta| \leq \delta_0 \\ \\ w_R^2(i\eta) & \text{if} \quad |\eta| > \delta_0, \end{cases}$$

$$B(i\eta) := \begin{cases} B_1(i\eta) & \text{if} \quad |\eta| \leq \delta_0 \\ \\ B_2(i\eta) & \text{if} \quad |\eta| > \delta_0, \end{cases}$$

and

$$M(r,i\eta) := \begin{cases} \psi(r) & \text{if} \quad |\eta| \leq \delta_0 \\ \\ \psi(r\sqrt{|\eta|}) & \text{if} \quad |\eta| > \delta_0. \end{cases}$$

Here and after, for the purpose of Remark 5.2.3 below, the cut-off function ψ is considered to be slightly different from the previous one in (4.1.3) in the sense that $\psi \equiv \psi(r) \in C_0^\infty(\mathbb{R})$ is an even function satisfying $\psi(r) = 1$ if $r \leq \delta_0$ and $\int_{\mathbb{R}_t} \mathcal{F}^{-1}\{\psi(\sqrt{|\eta|})\}(t)dt = 1$, where $\mathcal{F}^{-1}$ is the inverse Fourier transform.

Since the function $\eta \leadsto \sqrt{1+|\eta|}$ is equivalent to the function $\eta \leadsto \sqrt{|\eta|}$ for $|\eta| > \delta_0$ and to the constant function $\eta \to 1$ for $|\eta| \leq \delta_0$, then the two parts of Theorem 4.3.4 can be combined as

$$w(i\eta) = w_R(i\eta) + B(i\eta)M(r, i\eta)r^{\frac{\pi}{\omega}} \sin \frac{\pi}{\omega}\theta \tag{5.2.5}$$

with

$$\|w_R(i\eta)\|_{2,\,\Omega,\,\sqrt{1+|\eta|}} + |B(i\eta)|(1+|\eta|)^{\frac{1}{2}-\frac{\pi}{2\omega}} \leq C\|g(i\eta)\|_{0,\Omega}. \tag{5.2.6}$$

Notice that the estimate (5.2.6) is valid if $i\eta$ is replaced by $p = \xi + i\eta$ with $\xi \geq 0$ in the reasoning above. This shows that in terms of the Hardy-Lebesgue space $w_R(p) \in H^2(0, L^2(\Omega))$ and $B(p) \in H^2(0)$. Denote by $u_R(t), K(t)$ and $\phi(r,t)$ the inverse Fourier transform of $w_R(i\eta), B(\eta)$ and $M(r, i\eta)$, respectively. From (5.2.6), the Plancherel-Parseval theorem and Paley-Wiener Theorem 2.5.37 yield

$$u_R \in \widetilde{H}^{2,1}\left((\Omega \times (0 + \infty)\right) \quad \text{and} \quad K \in \widetilde{H}^{\frac{1}{2}-\frac{\pi}{2\omega}}(0, +\infty)$$

with the decomposition (5.2.3) as well as the estimate (5.2.4). $\square$

Remark 5.2.3. *The function*

$$\phi(r,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{it\eta} M(r, i\eta) d\eta$$

is a regularizing kernel family in the following sense ([46] Lemma 2.20, [19] and [54]). If $K(t) \in H^s(\mathbb{R})$, then:

- *$\phi(r,t) *_t K(t) \in C^\infty(\mathbb{R})$ such that $\phi(r,t) *_t K(t) \in C_0^\infty(\mathbb{R})$ if $K(t)$ has a compact support,*
- *$\phi(r,t) *_t K(t)$ converges to $K(t)$ in $H^s(\mathbb{R})$ as $r \to 0$.*

An alternative proof of Theorem 5.2.2 can be found in [30] and [31] where the kernel $\phi(r,t)$ is replaced by $\phi(r,t) = \frac{1}{\sqrt{t}} e^{\frac{-r^2}{4t}}$ $t > 0$.

For the function $w(p) \equiv \widehat{u}(p)$, which is the solution of the problem (5.1.6)-(5.1.7) and which admits the singular decomposition (5.2.5) and (5.2.6), Theorem 4.4.4 applies. Thus $\widehat{u}(p) \in H^{2,\beta}(\Omega)$, for $0 < \beta < 1 - \frac{\pi}{\omega}$, such that

$$\|\widehat{u}(i\eta)\|_{H^{2,\beta}(\Omega)} \leq C\|\widehat{f}(i\eta)\|_{0,\Omega}. \tag{5.2.7}$$

Applying the Plancherel-Parseval Theorem, combined with the tangential regularity in Theorem 5.1.3, we obtain the following global regularity result for the heat equation, which as mentioned earlier, is one of our main contributions in which the numerical approach is based. The result was announced in [14] and [13].

Theorem 5.2.4. *Let Ω and f be as in Theorem 5.2.2. For $0 < \beta < 1 - \frac{\pi}{\omega}$ and the solution u in Theorem 5.1.3, we have the inclusion*

$$u \in \widetilde{H}^{2(\beta),1}\left(\Omega \times (0, +\infty)\right) \cap \widetilde{H}^{1,1}_{0,}\left(\Omega \times (0, +\infty)\right)$$

such that

$$\|u\|_{\widetilde{H}^{2(\beta),1}(\Omega\times(0,+\infty))} \leq C\|f\|_{0,\Omega\times(0,+\infty)},$$

where

$$\widetilde{H}^{2(\beta),1}\left(\Omega \times (0, +\infty)\right) := L^2\left((0, +\infty), H^{2,\beta}(\Omega)\right) \cap \widetilde{H}^1\left((0, +\infty), L^2(\Omega)\right).$$

Remark 5.2.5. *More tangential regularity can be achieved on the solution u by assuming such regularity on the datum f. More precisely, if $f \in \widetilde{H}^s\left[(0, +\infty); L^2(\Omega)\right]$, $s \geq 0$ an integer, then*

$$u \in \widetilde{H}^{s+1}\left[(0, +\infty); L^2(\Omega)\right] \cap L^2\left[(0, +\infty); H^{2,\beta}(\Omega) \cap H^1_0(\Omega)\right].$$

In [39] and [46] the datum is taken such that $f \in \widetilde{H}^{s-1,\,\frac{s-1}{2}}\left(\Omega \times (0, +\infty)\right)$ in order to have $u \in \widetilde{H}^{1,\,\frac{s+1}{2}}\left(\Omega \times (0, +\infty)\right)$ and $u_R \in \widetilde{H}^{s+1,\,\frac{s+1}{2}}\left(\Omega \times (0, +\infty)\right)$ for the regular part in Theorem 5.2.2. In this case, we need to consider a weighted Sobolev space $H^{s+1,\beta}(\Omega)$ of higher order like in [42] for the global regularity.

Remark 5.2.6. *If Ω is convex i.e. $\omega < \pi$ in Theorem 5.2.2, then we take $\beta = 0$ in Theorem 5.2.4, which means that u has the classical optimal smoothness property.*

Chapter 6

Some numerical approximations

In the previous chapters, we obtained the solution $u \in L^2\left[(0,+\infty); H_0^1(\Omega)\right]$ of the heat equation (5.0.1)-(5.0.3) as the inverse Fourier transform of the variational solution $\widehat{u} \equiv \widehat{u}(p)$ of the Helmholtz problem (5.1.6)-(5.1.7), which satisfies: $\widehat{u} \in H_0^1(\Omega)$

$$\int_\Omega [\nabla\widehat{u}\ \nabla\bar{v} + p\ \widehat{u}\ \bar{v}]\,dx = \int_\Omega \widehat{f}\ \bar{v}\ dx,\ \ p = \xi + i\eta,\ \xi \geq 0,\ \ \forall v \in H_0^1(\Omega). \tag{6.0.1}$$

In this chapter, we consider the discrete counterpart of this procedure. More precisely, to the discrete solution of (6.0.1), we apply the inverse Fourier transform to generate an approximate solution of the heat equation. This is done in three steps each of which deals specifically with two cases: smooth and non-smooth solutions. The first step (section 6.1) is a semi-discrete method where the finite element method is used in the space variable, while the time variable remains continuous. The second step (section 6.2) is a fully discrete method with Fourier discretization in time and finite element approximation in space. For the next step (section 6.3), the finite element approximation in space is maintained while the standard and non-standard finite difference methods are used in the time variable. The last part, (section 6.4) provides numerical experiments.

6.1 Semi-discrete finite element method

We assume that Ω in (6.0.1) is a polygon. Throughout this section, we assume further that the polygon Ω is convex. Let $(\mathcal{T}_h)_{h>0}$ be a regular family of triangulations of $\bar{\Omega}$ consisting of compatible triangles T with exterior diameter $h_T \leq h$ and interior diameter ρ_T. Thus there exists a constant $\sigma > 0$ such that

$$\frac{h_T}{\rho_T} \leq \sigma, \quad \forall\, T \in \cup_{h>0} \mathcal{T}_h \tag{6.1.1}$$

or equivalently, there exists $\theta_0 > 0$, such that

$$\theta_T \;\geq \theta_0, \;\forall\, T \;\in \cup_{h>0} \mathcal{T}_h \tag{6.1.2}$$

where θ is the smallest angle of the triangle T. With each $\mathcal{T}_h$, we associate the finite element space V_h of continuous piecewise linear functions that are zero on the boundary:

$$V_h := \left\{ v_h \in C^0(\bar{\Omega});\; v_h|_{\partial\Omega} = 0,\; v_h|_T \in P_1,\; \forall T \in \mathcal{T}_h \right\} \tag{6.1.3}$$

where P_1 is the space of polynomials of degree less than or equal to 1. It is well-known that V_h is a finite-dimensional subspace of the Sobolev space $H_0^1(\Omega)$.

The finite element method (FEM) for the problem (6.0.1) reads as follows: find $\widehat{u}_h \equiv \widehat{u}_h(p) \in V_h$, solution of

$$\int_\Omega [\nabla \widehat{u}_h \; \nabla \bar{v}_h + p \; \widehat{u}_h \; \bar{v}_h] \, dx = \int_\Omega \widehat{f} \; \bar{v}_h \; dx, \; \forall v_h \in V_h. \tag{6.1.4}$$

Our standard references for all concepts concerning the classical finite element method are [16], [57].

By the generalized Lax-Milgram lemma (Theorem 3.1.1), there exists a unique solution $\widehat{u}_h \in V_h$ to (6.1.4). As in the continuous case (Theorem 3.1.7), this discrete solution satisfies the estimate

$$\|\widehat{u}_h\|^2_{1,\Omega,1+\sqrt{|p|}} := \|\nabla \widehat{u}_h\|^2_{0,\Omega} + (1 + \sqrt{|p|})^2 \|\widehat{u}_h\|^2_{0,\Omega} \leq C \|\widehat{f}\|^2_{0,\Omega}, \tag{6.1.5}$$

where we recall that $C > 0$ represent, here and after, various constants that are independent of the involved arguments and parameters (e.g Fourier arguments, step sizes, etc).

It should be noted that each finite element (T, P_T, Σ_T), where $P_T = P_1(T)$ and $\Sigma_T = \{$ vertices of $T\}$, is affine-equivalent to the reference finite element $(\widetilde{T}, \widetilde{P}, \widetilde{\Sigma})$ where $\widetilde{T}$ is the unit triangle with vertices $\widetilde{\Sigma} = \{\widetilde{a}_1 = (0,0),\; \widetilde{a}_2 = (1,0),\; \widetilde{a}_3 = (0,1)\}, \quad \widetilde{P} = P_1(\widetilde{T})$. This means that for any $T \in \mathcal{T}_h$, there exists an invertible affine mapping

$$F_T : \widetilde{x} \in \mathbb{R}^2 \rightsquigarrow x = F_T(\widetilde{x}) = B_T \widetilde{x} + b_T \in \mathbb{R}^2 \tag{6.1.6}$$

such that

$$T = F_T(\widetilde{T}), \ \ \Sigma_T = F_T(\widetilde{\Sigma}) \ \text{ and } \ P_T = \{p = \widetilde{p} \circ F_T^{-1}, \ \ \widetilde{p} \in \widetilde{P}\}.$$

We shall constantly use the notation

$$\widetilde{v} = v \circ F_T \ \text{ and } \ v = \widetilde{v} \circ F_T^{-1} \tag{6.1.7}$$

relating a function $v : x \in T \rightsquigarrow v(x) \in \mathbb{R}$ and the associated function $\widetilde{v} : \widetilde{x} \in \widetilde{T} \rightsquigarrow \widetilde{v}(\widetilde{x}) \in \mathbb{R}$ when considering the affine equivalent finite elements (T, P_T, Σ_T) and $(\widetilde{T}, \widetilde{P}, \widetilde{\Sigma})$. For such functions, we have $v \in H^m(T)$ if and only if $\widetilde{v} \in H^m(\widetilde{T})$ and there hold the estimates

$$|v|_{m,T} \leq C\|B_T^{-1}\|^m |detB_T|^{\frac{1}{2}} |\widetilde{v}|_{m,\widetilde{T}}, \tag{6.1.8}$$

and

$$|\widetilde{v}|_{m,\widetilde{T}} \leq C\|B_T\|^m |detB_T|^{-\frac{1}{2}} |v|_{m,T}, \tag{6.1.9}$$

where the Euclidean norms of the involved matrices are bounded as follows:

$$\|B_T^{-1}\| \leq \frac{\sqrt{2}}{\rho_T} \quad \text{and} \quad \|B_T\| \leq \sqrt{2} h_T. \tag{6.1.10}$$

By Céa's Lemma (Theorem 2.4.1 in Ciarlet [16]) we have the a priori estimate

$$\|\widehat{u} - \widehat{u}_h\|^2_{1,\, \Omega,\, 1+\sqrt{|p|}} \leq C \inf_{v_h \in V_h} \|\widehat{u} - v_h\|^2_{1,\, \Omega,\, 1+\sqrt{|p|}}. \tag{6.1.11}$$

In what follows, Π_h and Π_T denote suitable global and local interpolation operators that satisfy the relation

$$(\Pi_h v)|_T = \Pi_T v_T \quad \forall T \in \mathcal{T}_h. \tag{6.1.12}$$

Typically, we consider these to be the Lagrange interpolation operator when the argument v is of class $C^0(\bar{\Omega})$. When the domain of the operator consists of non-smooth functions such as those in the space $H^1(\Omega)$, we work with Π_h and Π_T as Clément's regularization operator ([16], [17] [28]). Using the latter operator and Theorem A4 in [28] or Exercise 3.2.3 in [16], we have

$$\inf_{v_h \in V_h} \|\widehat{u} - v_h\|_{0,\Omega}^2 \leq \|\widehat{u} - \Pi_h \widehat{u}\|_{0,\Omega}^2 \leq Ch^2 |\widehat{u}|_{1,\Omega}^2. \tag{6.1.13}$$

Since Ω is convex, the solution $\widehat{u}$ is of class $H^2(\Omega)$, which by Sobolev embedding theorem (Theorem 2.4.5) is embedded in $C^0(\bar{\Omega})$ and so the Lagrange interpolation operator is used. Therefore estimating $\inf_{v_h \in V_h} \|\nabla\widehat{u} - \nabla v_h\|_{0,\Omega}^2$ is reduced to estimating the local interpolation errors $\|\nabla\widehat{u} - \Pi_T \widehat{u}\|_{0,T}^2$ because

$$\|\nabla\widehat{u} - \nabla\Pi_h \widehat{u}\|_{0,\Omega}^2 = \sum_{T \in \tau_h} \|\nabla\widehat{u} - \nabla\Pi_T \widehat{u}\|_{0,T}^2. \tag{6.1.14}$$

We have the following result:

Lemma 6.1.1.

$$|\widehat{u} - \Pi_T \widehat{u}|_{m,T}^2 \leq C h_T^{4-2m} |\widehat{u}|_{2,T}^2, \quad 0 \leq m \leq 1.$$

Proof. The proof of this classical result is reproduced here because the argument will help us to adjust the non-smooth case. We have

$$|\widehat{u} - \Pi_T \widehat{u}|_{m,T}^2 \leq C \|B_T^{-1}\|^{2m} |det B_T| |\widetilde{\widehat{u} - \Pi_T \widehat{u}}|_{m,\widetilde{T}}^2 \text{ by (6.1.8)}. \tag{6.1.15}$$

Now for any polynomial $\widetilde{p} \in P_1(\widetilde{T})$, we have $\Pi_{\widetilde{T}} \widetilde{p} = \widetilde{p}$. Thus, we have

$$\begin{aligned} |\widetilde{\widehat{u} - \Pi_T \widehat{u}}|_{m,\widetilde{T}}^2 &= |\widetilde{\widehat{u}} - \Pi_T \widetilde{\widehat{u}}|_{m,\widetilde{T}}^2 \\ &= |(I - \Pi_{\widetilde{T}})(\widetilde{\widehat{u}} + \widetilde{p})|_{m,\widetilde{T}}^2 \\ &\leq \|I - \Pi_{\widetilde{T}}\|_{\mathcal{L}(H^2(\widetilde{T}), H^m(\widetilde{T}))}^2 \|\widetilde{\widehat{u}} + \widetilde{p}\|_{H^2(\widetilde{T})}^2 \end{aligned}$$

The last inequality is true because the linear operator $\Pi_{\widetilde{T}} : H^2(\widetilde{T}) \to H^m(\widetilde{T})$ is bounded for $0 \leq m \leq 1$. This implies that

$$|\widetilde{\widehat{u} - \Pi_T \widehat{u}}|_{m,\widetilde{T}}^2 \leq C \inf_{\widetilde{p} \in P_1(\widetilde{T})} \|\widetilde{\widehat{u}} + \widetilde{p}\|_{2,\widetilde{T}}^2. \tag{6.1.16}$$

But the norm of the quotient space $\frac{H^2(\widetilde{T})}{P_1(\widetilde{T})}$ is equivalent to the associated semi-norm. This yields

$$\begin{aligned} |\widetilde{\widehat{u} - \Pi_T \widehat{u}}|^2_{m,\widetilde{T}} &\leq C|\widetilde{\widehat{u}}|^2_{2,\widetilde{T}} \\ &\leq C\|B_T\|^4 |det B_T|^{-1} |\widehat{u}|^2_{2,T}, \end{aligned} \tag{6.1.17}$$

owing to (6.1.9). Due to (6.1.15) and (6.1.17), we obtain

$$|\widehat{u} - \Pi_T \widehat{u}|^2_{m,T} \leq C\|B_T^{-1}\|^{2m} \|B_T\|^4 |\widehat{u}|^2_{2,T}. \tag{6.1.18}$$

Now making use of (6.1.10) and the regularity (6.1.1) of the triangulation, we obtain from (6.1.18) the desired estimate in the Lemma. □

As a consequence of Lemma 6.1.1 as well as of the inequalities (6.1.5), (6.1.11) and (6.1.13) we have proved the estimate

$$\begin{aligned} \|\widehat{u} - \widehat{u}_h\|^2_{1,\Omega,1+\sqrt{|p|}} &\leq Ch^2 \{|\widehat{u}|^2_{2,\Omega} + (1+\sqrt{|p|})^2 |\widehat{u}|^2_{1,\Omega}\} \\ &\leq Ch^2 \|\widehat{u}\|^2_{2,\Omega,1+\sqrt{|p|}} \\ &\leq Ch^2 \|\widehat{f}\|^2_{0,\Omega}, \end{aligned} \tag{6.1.19}$$

the latter inequality being obtained similarly to Theorem 3.2.2. Notice that the Aubin-Nitsche duality argument (cf. Theorem 3.2.4 in [16]) yields the estimate

$$\|\widehat{u} - \widehat{u}_h\|^2_{0,\Omega} \leq Ch^4 \|\widehat{f}\|^2_{0,\Omega}. \tag{6.1.20}$$

Using Plancherel-Parseval theorem and the inverse Fourier transform (which works because the various constants C are independent of the Fourier argument), we have the following result:

Theorem 6.1.2. *Assume that the polygon Ω is convex. Then the semi-discrete solution*

$$u_h(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{it\eta} \widehat{u}_h(i\eta) d\eta$$

of the heat equation (5.0.1)-(5.0.3) is convergent, with optimal error estimate

$$\|u - u_h\|^2_{0,(\Omega\times(0,+\infty))} + h^2 \|u - u_h\|^2_{L^2[(0,+\infty),\, H^1(\Omega)]} \leq C\, h^4 \|f\|^2_{0,(\Omega\times(0,+\infty))}.$$

To deal with the case $\beta > 0$, we need the analogue of the arguments used in the classical case $\beta = 0$. The first result is related to that in [23] and reads as follows:

Lemma 6.1.3. *On the quotient space* $\frac{H^{2,\beta}(\widetilde{T})}{P_1(\widetilde{T})}$, *the semi-norm,*

$$|\dot{v}|_{H^{2,\beta}(\widetilde{T})} = \sqrt{\sum_{|\alpha|=2} \|r^\beta D^\alpha v\|^2_{0,\widetilde{T}}} \qquad v \in \dot{v},$$

is a norm equivalent to the usual norm

$$\|\dot{v}\|_{\frac{H^{2,\beta}(\widetilde{T})}{P_1(\widetilde{T})}} = \inf_{p \in P_1(\widetilde{T})} \|v + p\|_{H^{2,\beta}(\widetilde{T})}. \tag{6.1.21}$$

Proof. Let $\{p_1, p_2, p_3\}$ be an orthonormal basis of the space $P_1(\widetilde{T})$ with respect to the inner product of $H^{2,\beta}(\widetilde{T})$. For any $p \in P_1(\widetilde{T})$, we have

$$p = \sum_{i=1}^{3} (p;\ p_i)_{H^{2,\beta}(\widetilde{T})}\, p_i. \tag{6.1.22}$$

Firstly, we prove that there exists a constant $C > 0$ such that for any $v \in H^{2,\beta}(\widetilde{T})$, we have

$$\|v\|^2_{H^{2,\beta}(\widetilde{T})} \leq C \left[\sum_{|\alpha|=2} \|r^\beta D^\alpha v\|^2_{0,\widetilde{T}} + \sum_{i=1}^{3} |(v;\ p_i)_{H^{2,\beta}(\widetilde{T})}|^2 \right]. \tag{6.1.23}$$

Assume by contradiction that (6.1.23) is not true. Then for any integer n, there exists $v_n \in H^{2,\beta}(\widetilde{T})$ such that

$$\|v_n\|_{H^{2,\beta}(\widetilde{T})} = 1 \tag{6.1.24}$$

and

$$\sum_{|\alpha|=2} \|r^\beta D^\alpha v_n\|^2_{0,\widetilde{T}} + \sum_{i=1}^{3} |(v_n;\ p_i)_{H^{2,\beta}(\widetilde{T})}|^2 < \frac{1}{n}. \tag{6.1.25}$$

By the compactness of the embedding $H^{2,\beta}(\widetilde{T}) \hookrightarrow H^1(\widetilde{T})$ (Theorem 4.4.3) and (6.1.24), there exists a subsequence (v_{nj}) of (v_n) such that (v_{nj}) is convergent in $H^1(\widetilde{T})$, while (6.1.25) implies that the sequence $\left(r^\beta D^\alpha v_{nj}\right)$, $|\alpha| = 2$, converges to zero in $L^2(\widetilde{T})$.

These two facts imply that (v_{nj}) is a Cauchy-sequence in $H^{2,\beta}(\widetilde{T})$ and it converges therefore to some $v \in H^{2,\beta}(\widetilde{T})$, which in view of (6.1.24) and (6.1.25) satisfies

$$\|v\|_{H^{2,\beta}(\widetilde{T})} = 1, \tag{6.1.26}$$

$$(v,\ p_i)_{H^{2,\beta}(\widetilde{T})} = 0, \tag{6.1.27}$$

$$\|r^\beta D^\alpha v\|_{0,\widetilde{T}} = 0 \text{ for } |\alpha| = 2 \text{ and } v \in P_1(\widetilde{T}). \tag{6.1.28}$$

Using (6.1.22) and (6.1.27), we have $v = 0$ which is a contradiction to (6.1.26). The estimate (6.1.23) is therefore proved.

On the other hand, for $v \in H^{2,\beta}(\widetilde{T})$, let $q \in P_1(\widetilde{T})$ be such that

$$(v + q,\ p_i)_{H^{2,\beta}(\widetilde{T})} = 0 \ \text{ for } i = 1, 2, 3.$$

The inequality (6.1.23) applied to $v + q$ yields

$$\inf_{p \in P_1(\widetilde{T})} \|v + p\|_{H^{2,\beta}(\widetilde{T})} \le \|v + q\|_{H^{2,\beta}(\widetilde{T})} \le C \sqrt{\sum_{|\alpha|=2} \|r^\beta D^\alpha v\|_{L^2(\widetilde{T})}}.$$

This proves the equivalence of the norms. $\square$

The second result reads as follows:

Lemma 6.1.4. *$\widetilde{v} \in H^{2,\beta}(\widetilde{T})$ if and only if $v \in H^{2,\beta}(T)$ with*

$$|\widetilde{v}|^2_{H^{2,\beta}(\widetilde{T})} \le C \|B_T\|^4 \|B_T^{-1}\|^{2\beta} |detB_T|^{-1} |v|^2_{H^{2,\beta}(T)}$$

and

$$|v|^2_{H^{2,\beta}(T)} \le C \|B_T^{-1}\|^4 \|B_T\|^{2\beta} |detB_T| |\widetilde{v}|^2_{H^{2,\beta}(\widetilde{T})}.$$

Proof. If $\widetilde{v} \in H^{2,\beta}(\widetilde{T})$, then by Definition 4.4.1 and setting from (6.1.7)

$v(x) = (\widetilde{v} \circ F_T^{-1})(x)$, $\widetilde{v}(\widetilde{x}) = v(F_T(\widetilde{x}))$, we have

$$\begin{aligned}
|\widetilde{v}|^2_{H^{2,\beta}(\widetilde{T})} &= \sum_{|\alpha|=2} \int_{\widetilde{T}} |r^\beta(\widetilde{x}) D^\alpha(\widetilde{v}(\widetilde{x}))|^2 d\widetilde{x} \\
&= \sum_{|\alpha|=2} \int_T |r^\beta(F_T^{-1}(x)) D^\alpha v(F_T(\widetilde{x}))|^2 |detB_T|^{-1} dx \\
&\leq \|B_T\|^4 |detB_T|^{-1} \|B_T^{-1}\|^{2\beta} \sum_{|\alpha|=2} \int_T |r^\beta(x) D^\alpha v(x)|^2 dx.
\end{aligned}$$

The last inequality is due to the fact that $\dfrac{\partial^2 v(F_T(\widetilde{x}))}{\partial \widetilde{x}_i \partial \widetilde{x}_j} = \displaystyle\sum_{p=1,\, l=1}^{2} \frac{\partial^2 v(x)}{\partial x_p \partial x_l} B_T^{l,i} B_T^{p,j}$ by the chain rule and for any vertex a of T, and $x \in \mathbb{R}^2$ we have from (6.1.6)

$$\|F_T^{-1}(x) - F_T^{-1}(a)\| = \|B_T^{-1}(x-a)\| \leq \|B_T^{-1}\| \|x-a\|$$

and

$$r(F_T^{-1}(x)) \leq \|B_T^{-1}\| r(x).$$

Thus

$$|\widetilde{v}|^2_{H^{2,\beta}(\widetilde{T})} \leq \|B_T\|^4 |detB_T|^{-1} \|B_T^{-1}\|^{2\beta} |v|^2_{H^{2,\beta}(T)}. \tag{6.1.29}$$

If on the other hand, $v \in H^{2,\beta}(T)$ then in a similar way, setting $\widetilde{v}(\widetilde{x}) = v(F_T(\widetilde{x}))$ we have

$$\begin{aligned}
|v|^2_{H^{2,\beta}(T)} &= \sum_{|\alpha|=2} \int_T |r^\beta(x) D^\alpha v(x)|^2 dx \\
&= \sum_{|\alpha|=2} \int_T |r^\beta(F_T(\widetilde{x})) D^\alpha \widetilde{v}(F_T^{-1}(x))|^2 |detB_T| d\widetilde{x} \\
&\leq C\|B_T^{-1}\|^4 |detB_T| \sum_{|\alpha|=2} \int_{\widetilde{T}} |r^\beta(F_T(\widetilde{x})) D^\alpha \widetilde{v}(\widetilde{x})|^2 d\widetilde{x}
\end{aligned}$$

because $\dfrac{\partial^2 \widetilde{v}(F_T^{-1}(x))}{\partial x_i \, \partial x_j} = \displaystyle\sum_{p=1,l=1}^{2} \frac{\partial^2 \widetilde{v}(\widetilde{x})}{\partial \widetilde{x}_p \, \partial \widetilde{x}_l} (B_T^{-1})^{l,i} (B_T^{-1})^{p,j}$ by chain rule.

Since as above $r(F_T(\widetilde{x})) \leq \|B_T\| r(\widetilde{x})$, we then have

$$|v|^2_{H^{2,\beta}(T)} \leq C\|B_T^{-1}\|^4\|B_T\|^{2\beta}|detB_T||\widetilde{v}|^2_{H^{2,\beta}(\widetilde{T})}, \tag{6.1.30}$$

which completes the proof of the Lemma. □

We are now in a position to deal with the case when $\beta > 0$. Indeed, following the argument of the classical case, that led to (6.1.16), we have

$$|\widetilde{\widehat{u} - \Pi_T\widehat{u}}|^2_{m,\widetilde{T}} \leq C \inf_{p\in P_1(\widetilde{T})} \|\widetilde{\widehat{u}} + p\|^2_{H^{2.\beta}(\widetilde{T})}.$$

Then Lemma 6.1.3 implies that

$$|\widetilde{\widehat{u} - \Pi_T\widehat{u}}|^2_{m,\widetilde{T}} \leq C|\widetilde{\widehat{u}}|^2_{H^{2,\beta}(\widetilde{T})}. \tag{6.1.31}$$

Although we are in the non-smooth case, we still use the Lagrange interpolation operator because the solution belongs to the space $H^{2,\beta}(\Omega)$ which is embedded in $C^0(\bar{\Omega})$ (cf. Theorem 4.4.3). The right hand side of (6.1.31) is dealt with by using Lemma 6.1.4, which yields

$$|\widetilde{\widehat{u}}|^2_{H^{2,\beta}(\widetilde{T})} \leq C\|B_T\|^4\|B_T^{-1}\|^{2\beta}|detB_T|^{-1}|\widehat{u}|^2_{H^{2,\beta}(T)}. \tag{6.1.32}$$

Combining (6.1.15), (6.1.31), (6.1.32) and (6.1.10) yield

$$\begin{aligned} |\widehat{u} - \Pi_T\widehat{u}|^2_{m,T} &\leq C\|B_T^{-1}\|^{2m+2\beta}\|B_T\|^4|\widehat{u}|^2_{H^{2,\beta}(T)} \\ &\leq C\rho_T^{-2m-2\beta}h_T^4|\widehat{u}|^2_{H^{2,\beta}(T)}. \end{aligned}$$

Using the regularity of the triangulation (6.1.1), we end up with

$$|\widehat{u} - \Pi_T\widehat{u}|^2_{m,T} \leq Ch_T^{4-2m-2\beta}|\widehat{u}|^2_{H^{2,\beta}(T)}. \tag{6.1.33}$$

The analysis covered so far is valid for the case when the critical vertex $(0,0)$ which is responsible for the singularity belongs to T. In the case when $(0,0) \notin T$, we have $\widehat{u} \in H^2(T)$.

Therefore for $0 \leq m \leq 1$ we have

$$\begin{aligned} |\widehat{u} - \Pi_T \widehat{u}|^2_{m,T} &\leq Ch_T^{4-2m} \sum_{|\alpha|=2} \int_T |D^\alpha \widehat{u}|^2 dx \quad \text{by Lemma 6.1.1} \\ &= Ch_T^{4-2m} \sum_{|\alpha|=2} \int_T |r^\beta(x) D^\alpha \widehat{u}|^2 r^{-2\beta}(x) dx \\ &\leq Ch_T^{4-2m} \sup_{x\in T} r^{-2\beta}(x) |\widehat{u}|^2_{H^{2,\beta}(T)}. \end{aligned} \tag{6.1.34}$$

At this stage, we require the triangulation $(\mathcal{T}_h)$ to satisfy the mesh requirement conditions:

$$h_T \leq \begin{cases} Ch^{\frac{1}{1-\beta}}, & \text{if } (0,0) \in T \\ \\ Ch \inf_{x\in T} r^\beta(x), & \text{if } (0,0) \notin T, \end{cases} \tag{6.1.35}$$

In view of (6.1.11), (6.1.13) and (6.1.14) which are valid for the non-smooth case, the relations (6.1.33), (6.1.34) and (6.1.35) imply that

$$\begin{aligned} \|\widehat{u} - \widehat{u}_h\|^2_{1,\,\Omega,\,1+\sqrt{|p|}} &\leq Ch^2 \left\{ |\widehat{u}|^2_{H^{2,\beta}(\Omega)} + (1+\sqrt{|p|})^2 |\widehat{u}|^2_{1,\Omega} \right\} \\ &\leq Ch^2 \|\widehat{u}\|^2_{H^{2,\beta}(\Omega,\,1+\sqrt{|p|})} \\ &\leq Ch^2 \|\widehat{f}\|^2_{0,\Omega} \end{aligned}$$

where the norm of the weighted Sobolev space $H^{2,\beta}(\Omega, \rho)$ is defined in (4.4.2). It should be noted that the inequality

$$\|\widehat{u}\|_{H^{2,\beta}(\Omega,\,1+\sqrt{|p|})} \leq C \|\widehat{f}\|_{0,\Omega}$$

used here can be deduced from the proof of Theorem 4.4.4 where this weighted Sobolev space appeared for the first time. Using the Plancherel-Parseval Theorem and the inverse Fourier transform together with the Aubin-Nitsche duality argument yield the following result.

Theorem 6.1.5. *Assume that the triangulations are refined according to (6.1.35). Then the semi-discrete solution*

$$u_h(t) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{it\eta} \widehat{u}_h(i\eta) d\eta$$

is such that

$$\|u-u_h\|^2_{L^2[\Omega\times(0,+\infty)]}+h^2\|u-u_h\|^2_{L^2[(0,+\infty),\,H^1(\Omega)]\cap H^{\frac{1}{2}}[(0,+\infty),L^2(\Omega)]}\le C\,h^4\|f\|^2_{L^2[\Omega\times(0,+\infty)]}.$$

Remark 6.1.6. *The Mesh Refinement Method (MRM) (6.1.3), (6.1.4) and (6.1.35) was introduced by Babuska [8]. An alternative approach to it is the so-called Singular Function Method (SFM) introduced initially by Strang and Fix [63]. The SFM consists in replacing* V_h *in (6.1.3) by the family of augemented finite element spaces* $V_h^+(p)$, $p=i\eta$, *defined by*

$$V_h^+(p)=V_h\oplus\quad span\left\{M(r,p)r^{\frac{\pi}{\omega}}\sin\frac{\pi}{\omega}\theta\right\}$$

where $M(r,p)r^{\frac{\pi}{\omega}}\sin\frac{\pi}{\omega}\theta$ *is the singular function given in (5.2.5) for the Helmholtz equation. The SFM for problems with edge singularities is investigated in [43] and [44]. Further contributions on the MRM and SFM can be found in [10].*

6.2 Fourier finite element method

From the practical point of view, the semi-discrete finite element method in the previous section must be coupled with some discretization in the time-variable t, so that we have a fully discrete method. In this section, we use for the time variable t, the Fourier series method, which is the backbone of many modern techniques such as the spectral method and the wavelets method; see for instance [9, 11, 41, 51].

The Fourier-Finite Element method presented here is along the lines of [34] and has been extensively used in the literature for elliptic problems. (See for instance [35, 44, 41, 50]). Here, we implement this method for the heat equation, which is a parabolic problem.

The starting point is to consider the Fourier series of the solution $u(x,t)$ and of the datum $f(x,t)$ for the heat equation (5.0.1)-(5.0.3). More precisely, for $x\in\Omega$ and $t\in(0,2\pi)$, we have the expansions

$$u(x,t)=\sum_{k\in\mathbb{Z}}e^{ikt}u_k(x)\ \text{ and }\ f(x,t)=\sum_{k\in\mathbb{Z}}e^{ikt}f_k(x)\tag{6.2.1}$$

which mean that

$$\lim_{N\to+\infty}\|u-u^N\|_{L^2[(0,2\pi),H^1(\Omega)]}=0=\lim_{N\to+\infty}\|f-f^N\|_{L^2[(0,2\pi),L^2(\Omega)]}.\tag{6.2.2}$$

Here, for $N \in \mathbb{N}$,

$$u^N(x,t) := \sum_{|k|\leq N} e^{ikt} u_k(x) \ \text{ and } \ f^N(x,t) := \sum_{|k|\leq N} e^{ikt} f_k(x) \tag{6.2.3}$$

are truncated Fourier series, whereas each $u_k,\ k \in \mathbb{Z}$, is the unique variational solution (see Theorems 3.1.3 and 3.1.7) of the Helmholtz problem (3.1.1)-(3.1.2) with right side f_k and $p = ik$.

The following estimate between Fourier series and truncated Fourier series is useful (the beneath properties are available in [11]).

Lemma 6.2.1.

$$\|u - u^N\|_{0,(\Omega\times(0,2\pi))} \leq C\ N^{-1}\|u\|_{H^1[(0,2\pi),L^2(\Omega)]} \leq C\ N^{-1}\|f\|_{0,(\Omega\times(0,2\pi))}.$$

Furthermore, we have

$$\|u - u^N\|_{L^2((0,2\pi),H^1(\Omega))} \leq C\ N^{-1}\|u\|_{H^1((0,2\pi),\ H^1(\Omega))}$$

whenever $u \in H^1\left((0,2\pi),\ H^1(\Omega)\right)$.

Proof. We have

$$\begin{aligned}
\|u - u^N\|^2_{L^2[(0,2\pi),L^2(\Omega)]} &= \|\sum_{|k|\geq N} e^{ikt} u_k(x)\|^2_{L^2[(0,2\pi),L^2(\Omega)]} \text{ by } (6.2.1) - (6.2.3) \\
&\leq \frac{1}{N^2}\|\sum_{|k|\geq N} e^{ikt} iku_k(x)\|^2_{L^2[(0,2\pi),L^2(\Omega)]} \\
&\leq \frac{C}{N^2}\sum_{|k|\geq N}\|iku_k\|^2_{0,\Omega} \text{ by Plancherel-Parseval theorem} \\
&\leq \frac{C}{N^2}\sum_{k\in\mathbb{Z}}\|iku_k\|^2_{0,\Omega} \\
&\leq \frac{C}{N^2}\|u\|^2_{H^1[(0,2\pi),L^2(\Omega)]} \\
&\leq \frac{C}{N^2}\|f\|_{L^2((0,2\pi),\ L^2(\Omega))} \ \text{ by } (5.1.4).
\end{aligned}$$

The second but the last inequality is due to Plancherel-Parseval Theorem and the fact that the solution has the tangential regularity $H^1\left[(0,2\pi), L^2(\Omega)\right]$. $\square$

Fix $N \in \mathbb{N}$. For each $k \in \mathbb{Z}$ with $|k| \leq N$, let $u_{k,h} \in V_h$ be the unique solution of the finite element method (6.1.3)-(6.1.4) in which $p = ik$. Notice that $u_{k,h}$ is an approximation of the solution u_k of the Helmholtz problem (3.1.1)-(3.1.2) with $\widehat{f}$ inline of g.

The fully discrete solution of interest to us is $u_h^N(x,t)$ defined as follows:

$$u_h^N(x,t) := \sum_{|k|\leq N} e^{ikt} u_{k,h}(x), \quad x \in \Omega, \quad t \in (0, 2\pi). \tag{6.2.4}$$

The quality of the discrete solutions u_h^N is described in the following result.

Theorem 6.2.2. *The discrete solution u_h^N converges to the exact solution u in $L^2\left[(0,2\pi), H^1(\Omega)\right]$ as $N \to +\infty$ and $h \to 0$.*

Proof. The theorem is proved without making use of any smoothness property in the x-variable of the exact solution u. Let $\epsilon > 0$ be given. By the convergence of the Fourier expansion (6.2.2), there exists $N_0 \in \mathbb{N}$ such that for $N \geq N_0$ we have

$$\|u - u^N\|^2_{L^2[(0,2\pi),H^1(\Omega)]} < \frac{\epsilon^2}{2}. \tag{6.2.5}$$

On the other hand, we have for each $N \in \mathbb{N}$

$$\begin{aligned}
\|u^N - u_h^N\|^2_{L^2[(0,2\pi),H^1(\Omega)]} &= \sum_{|k|\leq N} \|u_k - u_{k.h}\|^2_{1,\Omega} \text{ by Plancherel-Parseval Theorem} \\
&\leq C \sum_{|k|\leq N} \inf_{v_h \in V_h} \|u_k - v_h\|^2_{1,\Omega,\sqrt{|k|}} \text{ by Cea's Lemma} \\
&\leq C \sum_{|k|\leq N} \|u_k - \Pi_h v_k\|_{1,\Omega,\sqrt{|k|}}
\end{aligned}$$

for any $v \in \mathcal{D}(\Omega \times (0,2\pi))$ such that v_k are Fourier coefficients of v. Thus, using triangular inequality, interpolation theory in Sobolev spaces and Plancherel-Parseval Theorem, we have

$$\begin{aligned}
\|u^N - u_h^N\|^2_{L^2[(0,2\pi),H^1(\Omega)]} &\leq C \sum_{|k|\leq N} \left\{ \|u_k - v_k\|^2_{1,\Omega,\sqrt{|k|}} + \|v_k - \Pi_h v_k\|^2_{1,\Omega,\sqrt{|k|}} \right\} \\
&\leq C \left\{ \|u - v\|^2_{L^2[(0,2\pi),H^1(\Omega)]} + h^2 \|v\|^2_{L^2[(0,2\pi),H^2(\Omega)]} \right\}.
\end{aligned}$$

Since $\mathcal{D}(\Omega \times (0,2\pi))$ is dense in $L^2\left[(0,2\pi), H_0^1(\Omega)\right]$, we can choose

$$v \in \mathcal{D}(\Omega \times (0,2\pi)) \quad \text{such that} \quad \|v - u\|_{L^2[(0,2\pi),H^1(\Omega)]} < \frac{\epsilon}{2\sqrt{C}}.$$

This implies that for every N

$$\|u^N - u_h^N\|^2_{L^2[(0,2\pi),H^1(\Omega)]} \leq C\left(\frac{\epsilon^2}{4C} + h^2\|v\|^2_{L^2[(0,2\pi),H^2(\Omega)]}\right).$$

Furthermore, there exists $h_0 > 0$ such that for $h \leq h_0$ we have

$$h^2\|v\|^2_{L^2[(0,2\pi),H^2(\Omega)]} < \frac{\epsilon^2}{4C}$$

and thus

$$\|u^N - u_h^N\|^2_{L^2[(0,2\pi),H^1(\Omega)]} < \frac{\epsilon^2}{2} \quad \text{for} \;\; h \leq h_0 \;\; \text{and for any} \;\; N.$$

Combining this with (6.2.5) and the triangle inequality, we have

$$\|u - u_h^N\|^2_{L^2[(0,2\pi),H^1(\Omega)]} < \epsilon^2 \;\; \text{for} \;\; N \geq N_0 \;\; \text{and} \;\; h \leq h_0.$$

$\square$

Further qualities of the discrete solution u_h^N are specified in the next result.

Theorem 6.2.3. *If the polygon Ω is convex, there holds the error estimate*

$$\|u_h^N - u\|_{0,\Omega\times(0,2\pi)} \leq C\left(h^2 + N^{-1}\right) \tag{6.2.6}$$

for the coupled Fourier series method (6.2.4) and classical FEM (6.1.3)-(6.1.4). When Ω is not convex, the same error estimate holds provided that the triangulations meet the mesh refinement conditions (6.1.35). Moreover, in the two cases, we have the error estimate

$$\|u_h^N - u\|_{L^2[(0,2\pi),H^1(\Omega)]} \leq C\left(h + N^{-1}\right) \tag{6.2.7}$$

whenever u has the tangential regularity $u \in H^1\left[(0,2\pi), H^1(\Omega)\right]$.

Proof. The proof is done in two parts: the convex and non-convex cases. We start with the first result by using the triangular inequality on the error as follows:

$$\|u - u_h^N\|^2_{0,\Omega\times(0,2\pi)} \leq \|u - u^N\|^2_{0,\Omega\times(0,2\pi)} + \|u^N - u_h^N\|^2_{0,\Omega\times(0,2\pi)}.$$

Using Lemma 6.2.1 on the first term and the Plancherel-Parseval theorem on the other term we have

$$\begin{aligned}\|u - u_h^N\|^2_{0,\Omega\times(0,2\pi)} \leq \quad & C\left\{N^{-2}\|u\|^2_{H^1[(0,2\pi),\, L^2(\Omega)]} + \sum_{|k|\leq N}\|u_k - u_{k,h}\|^2_{0,\Omega}\right\} \\ \leq \quad & C\left\{N^{-2}\|f\|^2_{L^2[(0,2\pi),\, L^2(\Omega)]} + \sum_{|k|\leq N}\|u_k - u_{k,h}\|^2_{0,\Omega}\right\}. \qquad (6.2.8)\end{aligned}$$

By Aubin-Nitsche duality argument, we have since $u_k \in H^2(\Omega, \sqrt{|k|})$, that

$$\begin{aligned}\sum_{|k|\leq N}\|u_k - u_{k,h}\|^2_{0,\Omega} &\leq Ch^4\sum_{|k|\leq N}\|u_k\|^2_{2,\Omega,\sqrt{|k|}} \\ &\leq Ch^4\sum_{|k|\leq N}\|f_k\|^2_{0,\Omega}.\end{aligned}$$

This yields

$$\sum_{|k|\leq N}\|u_k - u_{k,h}\|^2_{0,\Omega} \leq Ch^4\|f\|^2_{0,\Omega\times(0,2\pi)}. \qquad (6.2.9)$$

The proof for the convex case is now followed from (6.2.8)-(6.2.9).

For the non-convex case, the same method works provided that after (6.2.8), we use the inclusion $u_k \in H^{2,\beta}(\Omega, \sqrt{|k|})$ and the mesh refinement conditions (6.1.35) instead of $u_k \in H^2(\Omega, \sqrt{|k|})$.

The proof of the second part is based on the second estimate in Lemma 6.2.1. Using this estimate, the method of proof is the same. Basically from,

$$\|u - u_h^N\|^2_{L^2[(0,2\pi),\, H^1(\Omega)]} \leq \|u - u^N\|^2_{L^2[(0,2\pi),\, H^1(\Omega)]} + \|u^N - u_h^N\|^2_{L^2[(0,2\pi),\, H^1(\Omega)]},$$

we use the estimates

$$\|u - u^N\|^2_{L^2[(0,2\pi),\, H^1(\Omega)]} \leq CN^{-2}\|u\|^2_{H^1[(0,2\pi),\, H^1(\Omega)]}.$$

and

$$
\begin{aligned}
\|u^N - u_h^N\|^2_{L^2[(0,2\pi),\, H^1(\Omega)]} &\leq \sum_{|k|\leq N} \|u_k - u_{k,h}\|^2_{1,\Omega,\sqrt{|k|}} \\
&\leq Ch^2 \sum_{|k|\leq N} \|u_k\|^2_{2,\Omega,\sqrt{|k|}} \\
&\leq Ch^2 \|f\|^2_{0,\Omega\times(0,2\pi)}.
\end{aligned}
$$

In view of the Plancherel-Parseval Theorem, we deduce that

$$
\|u - u_h^N\|_{L^2[(0,2\pi),\, H^1(\Omega)]} \quad \leq \quad CN^{-1}\|u\|_{H^1((0,2\pi),\, H^1(\Omega))} + Ch\|f\|_{0,\Omega\times(0,2\pi)}
$$

The case when Ω is non-convex is dealt with similarly, on replacing the inclusion $u_k \in H^2(\Omega, \sqrt{|k|})$ with $u_k \in H^{2,\beta}(\Omega, \sqrt{|k|})$ and using the mesh refinement conditions (6.1.35). □

6.3 Coupled non-standard finite difference and finite element methods

Unlike section 6.2, where the time variable was discretized by Fourier series, we now discretize it using the non-standard Finite Difference (NSFD) method. The NSFD approach was initiated more than two decades ago by Mickens [52] as a powerful tool that replicates the dynamics of the differential system under consideration. Major contributions to the mathematics foundation of the NSFD method are due to Anguelov and Lubuma [5, 6, 7] (see [56] for and overview). Since then, the NSFD method has been extensively applied to many concrete problems in engineering and science (see for example [32], [53]).

To understand the relevance of the NSFD method in this thesis, we consider the heat equation in the following specific form:

$$
\frac{\partial u}{\partial t} - \Delta u + \lambda u = f, \quad \lambda > 0, \quad \text{on} \;\; \Omega \times (0, +\infty) \tag{6.3.1}
$$

$$
u = 0 \;\; \text{on} \;\; \partial\Omega \times (0, +\infty) \tag{6.3.2}
$$

$$
u(x, 0) = u^0(x), \quad \text{for} \;\; x \in \Omega. \tag{6.3.3}
$$

An appropriate trace theorem (see [39]) can reduce (6.3.1)-(6.3.3) to our standard model (5.0.1)-(5.0.3).

When $f = 0$, the space independent case of (6.3.1)-(6.3.3) is the decay ordinary differential equation

$$\frac{\partial u}{\partial t} = -\lambda u, \tag{6.3.4}$$

$$u(0) = u^0, \tag{6.3.5}$$

which has the unique solution

$$u(t) = u^0 e^{-\lambda t}. \tag{6.3.6}$$

Let $t_k := k\triangle t,\ k = 0, 1, 2, \cdots$ be the discrete time variable with $\triangle t$ representing the time step size. At the time $t = t_{k+1}$, the solution

$$u(t_{k+1}) = u^0 e^{-\lambda t_{k+1}}, \tag{6.3.7}$$

given by (6.3.6) can be written as $u(t_k)e^{-\lambda\triangle t}$ or

$$e^{\lambda\triangle t} u(t_{k+1}) = u(t_k) \tag{6.3.8}$$

in view of the semi-group property of solutions of ordinary differential equations. By adding and subtracting $u(t_{k+1})$ from (6.3.8), we obtain the following equivalent formulation of (6.3.7) where the notation $u^k := u(t_k)$ is used:

$$\frac{u^{k+1} - u^k}{\frac{e^{\lambda\triangle t} - 1}{\lambda}} + \lambda u^{k+1} = 0. \tag{6.3.9}$$

By definition, (6.3.9) is called the exact scheme for the decay equation (6.3.4) (see Mickens [52]). The terminology is self-explanatory: at the time $t = t_k$, the difference equation (6.3.9) has the same general solution as the differential equation (6.3.4).

Clearly, the exact scheme (6.3.9) is dynamically consistent with any property of the initial value problem (6.3.4)-(6.3.5) irrespective of the value of the step size $\triangle t$. In particular the discrete scheme (6.3.9) replicates the positivity and the decay to zero which are the main features of the solution (6.3.6) of (6.3.4)-(6.3.5).

Equation (6.3.9) is a typical non-standard finite difference scheme in the following sense (cf [4]):

Definition 6.3.1. *A difference equation*

$$u^{k+1} = g(u^k, u^{k+1})$$

for approximating a differential equation

$$\frac{du}{dt} = g(u)$$

is called a non-standard finite difference method, if at least one of the following conditions is met:

1. *In the first order discrete derivative*

$$\frac{u^{k+1} - u^k}{\Delta t}$$

the traditional denominator Δt is replaced by a positive function $\phi(\Delta t)$ satisfying the property

$$\phi(\Delta t) = \Delta t + O((\Delta t)^2) \quad \text{as} \;\; \Delta t \to 0. \tag{6.3.10}$$

2. *Non-local term in $g(u)$ are approximated in a non-local way, i.e. by a suitable function of several points of the mesh.*

Remark 6.3.2. *Condition 2 in the Definition 6.3.1 is not necessary in our case since we are dealing with a linear problem. However, the condition is very useful in non-linear problems.*

For more on non-standard finite difference schemes, we refer the reader to [45, 56] and edited volumes [32] and [53].

Our aim is to design for (6.3.1)-(6.3.3) a fully discrete method, which will preserve the properties in the limit case of space independent equation and $f = 0$. To this end, we approximate (6.3.1)-(6.3.3) by coupling the FEM in space and the NSFD scheme in time as follows: With the initial guess $u_h^0 := \Pi_h u^0 \in V_h$ via the interpolation operator Π_h, let $(u_h^k)_{k\geq 1}$

be the sequence in the finite element space V_h defined recursively as unique solution of

$$\int_\Omega \left[\frac{u_h^k - u_h^{k-1}}{\frac{e^{\lambda \Delta t}-1}{\lambda}} \, v_h + \nabla u_h^k \, \nabla v_h + \lambda u_h^k \, v_h \right] dx = \int_\Omega f(t_k) \, v_h \, dx \;\; \forall \, v_h \in V_h, \tag{6.3.11}$$

$$u_h^0 = \Pi_h u^0, \;\; k = 1, 2, 3, ... \tag{6.3.12}$$

The idea of coupling the NSFD method with the FEM and their implementation presented here are new. The results are published in [14] and [13].

Let $\phi(\Delta t)$ denote the function $\frac{e^{\lambda \Delta t}-1}{\lambda}$ that satisfies (6.3.10). It is clear that (6.3.11) can be written as follows for any $v_h \in V_h$:

$$\begin{aligned}\left(u_h^k, v_h\right)_{0,\Omega} + \phi(\Delta t) \left(\nabla u_h^k, \nabla v_h\right)_{0,\Omega} + \lambda \phi(\Delta t) \left(u_h^k, v_h\right)_{0,\Omega} = & \;\; \phi(\Delta t) \left(f(t_h), v_h\right)_{0,\Omega} \\ + & \;\; \left(u_h^{k-1}, v_h\right)_{0,\Omega}.\end{aligned} \tag{6.3.13}$$

Equation (6.3.13) will be considered in conjunction with the continuous relation below, which in view of (5.1.12) is the variational formulation of (6.3.1)-(6.3.3): $u \in H_{0,}^{1,1}(\Omega \times (0, +\infty))$ satisfying (6.3.3) is the unique solution of

$$\left(\frac{\partial u(t)}{\partial t}, v\right)_{0,\Omega} + \left(\nabla_x u(t), \nabla_x v\right)_{0,\Omega} + \lambda \left(u(t), v\right)_{0,\Omega} = \left(f(t), v\right)_{0,\Omega}, \;\; t > 0, \; \forall v \in H_0^1(\Omega). \tag{6.3.14}$$

We let p_h be the elliptic or Ritz projection onto V_h defined with respect to the energy inner product

$$(\nabla v, \; \nabla w)_{0,\Omega} + \lambda \, (v \; w)_{0,\Omega}$$

associated with the elliptic problem, which is the following stationary problem of (6.3.1)-(6.3.3):

$$-\Delta u + \lambda u = f \text{ in } \Omega \tag{6.3.15}$$

$$u = 0 \text{ on } \partial\Omega. \tag{6.3.16}$$

More precisely, for $u \in H_0^1(\Omega)$, its Ritz projection $p_h u \in V_h$ is the unique solution, for all $v_h \in V_h$, of the problem

$$(\nabla p_h u, \nabla v_h)_{0,\Omega} + \lambda (p_h u, v_h)_{0,\Omega} = (\nabla u, \nabla v_h)_{0,\Omega} + \lambda (u, v_h)_{0,\Omega} . \tag{6.3.17}$$

Thus $p_h u$ is the finite element approximation of the solution of the elliptic problem (6.3.15)-(6.3.16). This Ritz projection is used to rewrite the global error in the form below, which is convenient in what follows:

$$u_h^k - u(t_k) = \left(u_h^k - p_h u(t_k)\right) + (p_h u(t_k) - u(t_k)) \equiv \theta^k + \rho^k. \tag{6.3.18}$$

With these highlights, we have the following result:

Theorem 6.3.3. *Let the polygon Ω be convex. We assume that u^0 and u, are smoother to the extent that $u^0 \in H^2(\Omega)$ and $u \in H^2\left((0, +\infty), H^2(\Omega)\right)$. Fix a time $t^\star$ that can be written in several ways as $t^\star = k\Delta t$. Then, there exists a constant $C^\star \equiv C(t^\star)$, depending on $t^\star$ and there holds the error estimate*

$$\|u_h^k - u(t_k)\|_{0,\Omega} \leq C^\star(\Delta t + h^2),$$

for the coupled NSFD method and classical FEM (6.3.11)-(6.3.12). When Ω is not convex, the same error estimate holds provided that $H^2(\Omega)$ is replaced with $H^{2,\beta}(\Omega)$, $0 < \beta < 1 - \frac{\pi}{\omega}$, in the regularity assumption of u with however $u^0 \in H^2(\Omega)$ and the triangulations, meeting the mesh refinement conditions (6.1.35).

Under the assumptions of the two cases above, we have the error estimate

$$\|u_h^k - u(t_k)\|_{1,\Omega} \leq C^\star(\sqrt{\Delta t} + h),$$

whenever h is proportional to $\sqrt{\Delta t}$.

Proof. The proof in the case when Ω is convex follows from the arguments in Thomée [65], which work because $u(t_k) \in H^2(\Omega)$ in this case. In what follows, we adapt and give details to these arguments of [65] for the non-convex case. If Ω is not convex, then $u(t_k) \in H^{2,\beta}(\Omega)$ and from the interpolation theory discussed in section 5.1, we have, under the mesh refinement conditions (6.1.35),

$$\begin{aligned} \|\rho^k\|_{0,\Omega} &\leq Ch^2 \|u(t_k)\|_{H^{2,\beta}(\Omega)} \\ &\leq Ch^2 \left[\|u^0\|_{2,\Omega} + \int_0^{t_k} \|\frac{\partial u}{\partial s}\|_{H^{2,\beta}(\Omega)} ds \right] \end{aligned} \tag{6.3.19}$$

since $u(t_k) = u^0 + \int_0^{t_k} \frac{\partial u}{\partial s} ds, \quad u^0 \in H^2(\Omega)$ and $u \in H^1\left[(0, +\infty),\ H^{2,\beta}(\Omega)\right]$.

Given a sequence $(\gamma^k)_{k\geq 1}$ in $H_0^1(\Omega)$, we denote by $\frac{\bar{\partial}\gamma^k}{\partial t}$ the non-standard backward finite difference of γ^k defined by

$$\frac{\bar{\partial}\gamma^k}{\partial t} = \frac{\gamma^k - \gamma^{k-1}}{\phi(\Delta t)}. \tag{6.3.20}$$

Fix $v_h \in V_h$ and consider the sequence (θ^k) in $H_0^1(\Omega)$ defined in (6.3.18). Having the discrete and continuous variational problems (6.3.11) or (6.3.13) and (6.3.14) in mind, we have:

$$\begin{aligned} &\left(\frac{\bar{\partial}\theta^k}{\partial t}, v_h\right)_{0,\Omega} + \left(\nabla\theta^k, \nabla v_h\right)_{0,\Omega} + \lambda\left(\theta^k, v_h\right)_{0,\Omega} \\ &\qquad = \left(\frac{\bar{\partial}(u_h^k - p_h u(t_k))}{\partial t}, v_h\right)_{0,\Omega} + \left(\nabla(u_h^k - p_h u(t_k)), \nabla v_h\right)_{0,\Omega} \\ &\qquad +\lambda\left((u_h^k - p_h u(t_k)), v_h\right)_{0,\Omega}, \text{by (6.3.18)} \\ &\qquad = -\left(p_h \frac{\bar{\partial}u(t_{t_k})}{\partial t}, v_h\right)_{0,\Omega} + (f(t_k), v_k)_{0,\Omega} - (\nabla u(t_k), \nabla v_h)_{0,\Omega} \\ &\qquad -\lambda\left(u(t_k), v_h\right)_{0,\Omega}, \text{by} \quad (6.3.17) \\ &\qquad = -\left(p_h \frac{\bar{\partial}u(t_k)}{\partial t}, v_h\right)_{0,\Omega} + \left(\frac{\partial u(t_k)}{\partial t},\ v_h\right)_{0,\Omega}, \quad \text{by} \quad (6.3.14) \\ &\qquad = \left((I - p_h)\frac{\bar{\partial}u(t_k)}{\partial t}, v_h\right)_{0,\Omega} + \left(\frac{\partial u(t_k)}{\partial t} - \frac{\bar{\partial}u(t_k)}{\partial t}, v_h\right)_{0,\Omega} \\ &\qquad \equiv \left(w^k, v_h\right)_{0,\Omega} \\ &\qquad \equiv \left(w_1^k, v_h\right)_{0,\Omega} + \left(w_2^k, v_h\right)_{0,\Omega} \end{aligned} \tag{6.3.21}$$

where $w_1^k = (I - p_h)\frac{\bar{\partial}u(t_k)}{\partial t}$ and $w_2^k = \frac{\partial u(t_k)}{\partial t} - \frac{\bar{\partial}u(t_k)}{\partial t}$.

If $v_h = \theta^k$ in (6.3.21), we have

$$\left(\frac{\bar{\partial}\theta^k}{\partial t}, \theta^k\right)_{0,\Omega} + \left(\nabla\theta^k, \nabla\theta^k\right)_{0,\Omega} + \lambda\left(\theta^k, \theta^k\right)_{0,\Omega} = \left(w^k, \theta^k\right)_{0,\Omega}. \tag{6.3.22}$$

Using (6.3.20), we obtain

$$\begin{aligned}\left(\frac{\bar{\partial}\theta^k}{\partial t}, \theta^k\right)_{0,\Omega} &= \phi^{-1}(\Delta t)\left(\theta^k - \theta^{k-1}, \theta^k\right)_{0,\Omega} \\ &= \phi^{-1}(\Delta t)\left(\theta^k, \theta^k\right)_{0,\Omega} - \phi^{-1}(\Delta t)\left(\theta^{k-1}, \theta^k\right)_{0,\Omega}, \\ &= \phi^{-1}(\Delta t)\|\theta^k\|_{0,\Omega}^2 - \phi^{-1}(\Delta t)\left(\theta^{k-1}, \theta^k\right)_{0,\Omega},\end{aligned}$$

which combined with (6.3.22) yields

$$\phi^{-1}(\Delta t)\left[\|\theta^k\|_{0,\Omega}^2 - \left(\theta^{k-1}, \theta^k\right)_{0,\Omega}\right] \leq \left(w^k, \theta^k\right)_{0,\Omega}. \tag{6.3.23}$$

Using Cauchy-Schwarz inequality we have

$$\|\theta^k\|_{0,\Omega}^2 \leq \phi(\Delta t)\|w^k\|_{0,\Omega}\|\theta^k\|_{0,\Omega} + \|\theta^{k-1}\|_{0,\Omega}\|\theta^k\|_{0,\Omega}$$

and thus

$$\|\theta^k\|_{0,\Omega} \leq \phi(\Delta t)\|w^k\|_{0,\Omega} + \|\theta^{k-1}\|_{0,\Omega}. \tag{6.3.24}$$

By mathematical induction, (6.3.24) becomes

$$\|\theta^k\|_{0,\Omega} \leq \|\theta^0\|_{0,\Omega} + \phi(\Delta t)\sum_{j=1}^{k}\|w_1^j\|_{0,\Omega} + \phi(\Delta t)\sum_{j=1}^{k}\|w_2^j\|_{0,\Omega}. \tag{6.3.25}$$

Notice that

$$\begin{aligned}\|\theta^0\|_{0,\Omega} &= \|u_h^0 - p_h u^0\|_{0,\Omega} \\ &= \|\Pi_h u^0 - p_h u^0\|_{0,\Omega} \quad \text{by (6.3.12)} \\ &\leq \|u^0 - \Pi_h u^0\|_{0,\Omega} + \|u^0 - p_h u^0\|_{0,\Omega} \\ &\leq Ch^2\|u^0\|_{2,\Omega} \text{ since } u^0 \in H^2(\Omega).\end{aligned} \tag{6.3.26}$$

A bound for $\phi(\Delta t)\sum_j^k \|w_1^j\|_{0,\Omega}$ is obtained by using (6.3.20) and (6.3.21) as follows:

$$\begin{aligned} w_1^j &= (I-p_h)\,\frac{\bar{\partial} u(t_j)}{\partial t} \\ &= (I-p_h)\,\phi^{-1}(\Delta t)\,(u(t_j-u(t_{j-1}))) \\ &= (I-p_h)\,\phi^{-1}(\Delta t)\int_{t_{j-1}}^{t_j}\frac{\partial u}{\partial s}ds. \end{aligned}$$

Thus we have

$$\begin{aligned} \phi(\Delta t)\sum_j^k \|w_1^j\|_{0,\Omega} &\leq \sum_j^k \int_{t_{j-1}}^{t_j} \|(I-p_h)\frac{\partial u}{\partial s}\|_{0,\Omega} ds \\ &\leq Ch^2 \int_{t_0}^{t_k} \|\frac{\partial u}{\partial s}\|_{H^{2,\beta}(\Omega)} ds \\ &\leq Ch^2 \text{ since } u\in H^1\left((0,+\infty),H^{2,\beta}(\Omega)\right). \end{aligned} \tag{6.3.27}$$

On the other hand, a bound for $\phi(\Delta t)\sum_j^k \|w_2^j\|_{0,\Omega}$ is obtained using (6.3.21) as follows:

$$\begin{aligned} w_2^j &= \frac{\bar{\partial} u(t_j)}{\partial t} - \frac{\partial u(t_j)}{\partial t} \\ &= \phi^{-1}(\Delta t)(u(t_j)-u(t_{j-1})) - \frac{\partial u(t_j)}{\partial t}. \end{aligned}$$

This implies that

$$\begin{aligned} \phi(\Delta t)\sum_j^k w_2^j &= u(t_j)-u(t_{j-1}) - \Delta t\frac{\partial u(t_j)}{\partial t} + \Delta t\frac{\partial u(t_j)}{\partial t} - \phi(\Delta t)\frac{\partial u(t_j)}{\partial t} \\ &= (u(t_j)-u(t_{j-1})) - \Delta t\frac{\partial u(t_j)}{\partial t} + (\Delta t-\phi(\Delta t))\frac{\partial u(t_j)}{\partial t} \\ &= -\int_{t_{j-1}}^{t_j}(s-t_{j-1})\frac{\partial^2 u(s)}{\partial s^2}ds + (\Delta t-\phi(\Delta t))\frac{\partial u(t_j)}{\partial t}. \end{aligned}$$

by Taylor theorem with integral expression of the remainder term.

Taking the norm in $L^2(\Omega)$ and summing both sides of the equation, we have

$$\begin{aligned}
\phi(\Delta t)\sum_{j=1}^{k}\|w_2^j\|_{0,\Omega} \leq & \sum_{j=1}^{k}\|\int_{t_{j-1}}^{t_j}(s-t_{j-1})\frac{\partial^2 u(s)}{\partial s^2}ds\|_{0,\Omega} + C(\Delta t)^2\sum_{j=1}^{k}\|\frac{\partial u(t_j)}{\partial t}\|_{0,\Omega} \text{ by (6.3.10)} \\
\leq & \Delta t\int_0^{t_k}\|\frac{\partial^2 u(s)}{\partial s^2}\|_{0,\Omega}^2 ds + C(\Delta t)^2 k\sup_{1\leq j\leq k}\|\frac{\partial u(t)}{\partial t}\|_{0,\Omega} \\
\leq & \Delta t\left(\int_0^{t_k}\|\frac{\partial^2 u(s)}{\partial s^2}\|_{0,\Omega}^2 ds + Ct_k\|\frac{\partial u(s)}{\partial s}\|_{H^1((0,+\infty),L^2(\Omega))}\right) \text{ because} \\
& t^\star \equiv t_k = k\Delta t \text{ and } u\in H^2\left((0,+\infty),L^2(\Omega)\right) \text{ with } H^1\left((0,+\infty),L^2(\Omega)\right) \\
& \text{being continuously embedded in } C^0\left((0,+\infty),L^2(\Omega)\right), \\
\leq & C(t^\star)\Delta t. \qquad (6.3.28)
\end{aligned}$$

Combining (6.3.25), (6.3.26), (6.3.27) and (6.3.28) we have

$$\|\theta^k\|_{0,\Omega} \leq C(t^\star)\left(\Delta t + h^2\right). \qquad (6.3.29)$$

Hence, in view of (6.3.19) and (6.3.29), we obtain the required estimate

$$\|u_h^k - u(t_k)\|_{0,\Omega} \leq C(t^\star)\left(\Delta t + h^2\right). \qquad (6.3.30)$$

This proves the first part of the theorem.

The second part of the Theorem, is proved thanks to the relation (6.3.18) as follows:

$$\begin{aligned}
\|\nabla(u_h^k - u(t_k))\|_{0,\Omega} &\leq \|\nabla(u_h^k - p_h u(t_k))\|_{0,\Omega} + \|\nabla(p_h u(t_k) - u(t_k))\|_{0,\Omega} \\
&= \|\nabla\theta^k\|_{0,\Omega} + \|\nabla\rho^k\|_{0,\Omega}. \qquad (6.3.31)
\end{aligned}$$

Again, we give details for the non-convex case only, the convex case being mere classical due to the $H^2(\Omega)$ smoothness of the solution at every time $t > 0$. For Ω non-convex, we immediately bound $\nabla\rho^k$ by interpolation theory in section 5.1 as follows:

$$\|\nabla\rho^k\|_{0,\Omega} = \|\nabla(p_h u(t_k) - u(t_k))\|_{0,\Omega} \leq Ch\|u(t_k)\|_{H^{2,\beta}(\Omega)}. \qquad (6.3.32)$$

Letting $v_h = \theta^k$ in (6.3.21), we bound $\nabla\theta^k$ as follows:

$$\begin{aligned}
\|\nabla\theta^k\|_{0,\Omega}^2 &\leq \left(w^k, \theta^k\right)_{0,\Omega} - \left(\frac{\bar{\partial}\theta^k}{\partial t}, \theta^k\right)_{0,\Omega} \\
&= \left(w^k - \frac{\bar{\partial}\theta^k}{\partial t}, \theta^k\right)_{0,\Omega} \\
&= \left(w^k, \theta^k\right)_{0,\Omega} - \frac{\left(\theta^k, \theta^k\right)_{0,\Omega}}{\phi(\Delta t)} + \frac{\left(\theta^{k-1}, \theta^k\right)_{0,\Omega}}{\phi(\Delta t)} \\
&\leq \left(w^k, \theta^k\right)_{0,\Omega} + \frac{\left(\theta^{k-1}, \theta^k\right)_{0,\Omega}}{\phi(\Delta t)}.
\end{aligned}$$

When Cauchy-Schwarz inequality is applied, we obtain

$$\begin{aligned}
\phi(\Delta t)\|\nabla\theta^k\|_{0,\Omega}^2 &\leq \phi(\Delta t)\|w^k\|_{0,\Omega}\|\theta^k\|_{0,\Omega} + \|\theta^{k-1}\|_{0,\Omega}\|\theta^k\|_{0,\Omega} \\
&= \left(\phi(\Delta t)\|w^k\|_{0,\Omega} + \|\theta^{k-1}\|_{0,\Omega}\right)\|\theta^k\|_{0,\Omega} \\
&\leq \left(\phi(\Delta t)\|w^k\|_{0,\Omega} + \|\theta^{k-1}\|_{0,\Omega}\right)^2 \quad \text{by (6.3.24).}
\end{aligned}$$

Using (6.3.29), (6.3.27) and (6.3.28), we have

$$\phi(\Delta t)\|\nabla\theta^k\|_{0,\Omega}^2 \leq C^\star\left(h^2 + \Delta t\right)^2.$$

In the previous inequality, we let Δt be proportional to h^2 (i.e $h = C\sqrt{\Delta t}$), we divide both sides of the inequality by $\sqrt{\phi(\Delta t)}$ to obtain

$$\begin{aligned}
\|\nabla\theta^k\|_{0,\Omega} &\leq C^\star\left(h \cdot \frac{h}{\sqrt{\phi(\Delta t)}} + \sqrt{\Delta t}\sqrt{\frac{\Delta t}{\phi(\Delta t)}}\right) \\
&\leq C^\star(Ch + \sqrt{\Delta t})\sqrt{\frac{\Delta t}{\phi(\Delta t)}} \\
&\leq C^\star\left(h + \sqrt{\Delta t}\right) \quad \text{in view of (6.3.10).} \qquad (6.3.33)
\end{aligned}$$

Combining (6.3.32), (6.3.33) together with Poincaré Friedrichs inequality (2.4.3), we have

$$\|u_h^k - u(t_k)\|_{1,\Omega} \leq C^\star\left(h + \sqrt{\Delta t}\right). \qquad (6.3.34)$$

This completes the proof of the theorem. □

By construction of the coupled NSFD scheme and FEM, we readily have the following qualitative stability result, which gives an indication on the relevance of this coupling.

Theorem 6.3.4. *The discrete method (6.3.11)-(6.3.12) reduces to a numerical procedure for the space independent limit case of the boundary value problem (6.3.1)-(6.3.3). The latter method corresponds to the exact scheme (6.3.9) of the decay equation (6.3.4)-(6.3.5) when* $f \equiv 0$.

Remark 6.3.5. *In line with Theorem 6.3.4, the following comments are in order to understand the good performance of the NSFD method in the numerical experiment in the next section. The convergence (6.3.34) in* H^1 *norm implies that there exists a subsequence of* u_h^k *still denoted in the same way such that* u_h^k *converges point-wise to* u *as* $h \to 0$ *and* $k \to +\infty$ *(see [1], Corollary 2.11). Assume that* $\Delta u = 0$ *near a point* $a \in \Omega$*. Now if* v_h *in (6.3.11) is chosen in such a way that its support containing the point* a*, is very small and* $v_h = 1$ *near* a*, then we can use the approximation*

$$\int_\Omega g v_h dx = g(a)K \;\; \textit{where} \;\; K \;\; \textit{is the measure of the} \;\; supp(v_h).$$

Using this approximation in (6.3.11), it follows that $u_h^k(a)$ *is a discrete solution of the ordinary differential equation associated with (6.3.1) and (6.3.3) when we fix* $x = a$*. Of course* $u_h^k(a)$ *is the solution of the exact scheme (6.3.9) if we also have* $f(a,t) = 0$*.*

More generally, the above reasoning could be used without considering a subsequence of u_h^k. Indeed, the practical implementation of the method (6.3.11) amounts to considering what Strang and Fix [63] call "variational crimes". That is using numerical integration in (6.3.11). In this regard, assume that a is the barycenter of a fixed triangle T of triangulation of $\bar{\Omega}$ and let us assume as above that $\Delta u = 0$ near the point a. We take v_h in (6.3.11) having its support in such that $v_h = 1$ near a and we use the approximation

$$\int_\Omega g v_h dx = g(a) \;\text{ measure } (T).$$

We then proceed as before to conclude that $u_h^k(a)$ is a discrete solution of the associated ordinary differential equation.

6.4 Numerical experiment

This section is devoted to demonstrate computationally the optimal convergence of some of the numerical schemes presented in the preceding sections of this chapter.

Before we proceed with the numerical experiments that support the theory, we want to show that triangulations $(\mathcal{T}_h)$ of the polygonal domain $\bar{\Omega}$ that are refined according to the condition (6.1.35) exist in practice. To this end, we follow the procedure proposed by Raugel [59] and summarized in [30].

More precisely, observing that the vertex that is responsible for singularities is placed at the origin $(0,0)$, we consider the following steps:

1. Divide the polygon Ω into big triangles;
2. Divide each side of each of the big triangles that has no vertex at $(0,0)$ into $n = \frac{1}{h}$ subsegments of equal length and proceed, following the usual triangulation technique (See Ciarlet [16], Raviart and Thomas [57]);
3. Divide each of the big triangles that has a vertex at $(0,0)$, according to the ratios

$$\left(\frac{i}{n}\right)^{\frac{1}{1-\beta}}, \quad 1 \leq i \leq n,$$

   along the sides that ends at $(0,0)$; divide the third side in the usual way and proceed as usual.

Figure 6.1 illustrates this case, for $n = 4$, with one of the sides that ends at the vertex $(0,0)$ lying on the $0x_1$ axis.

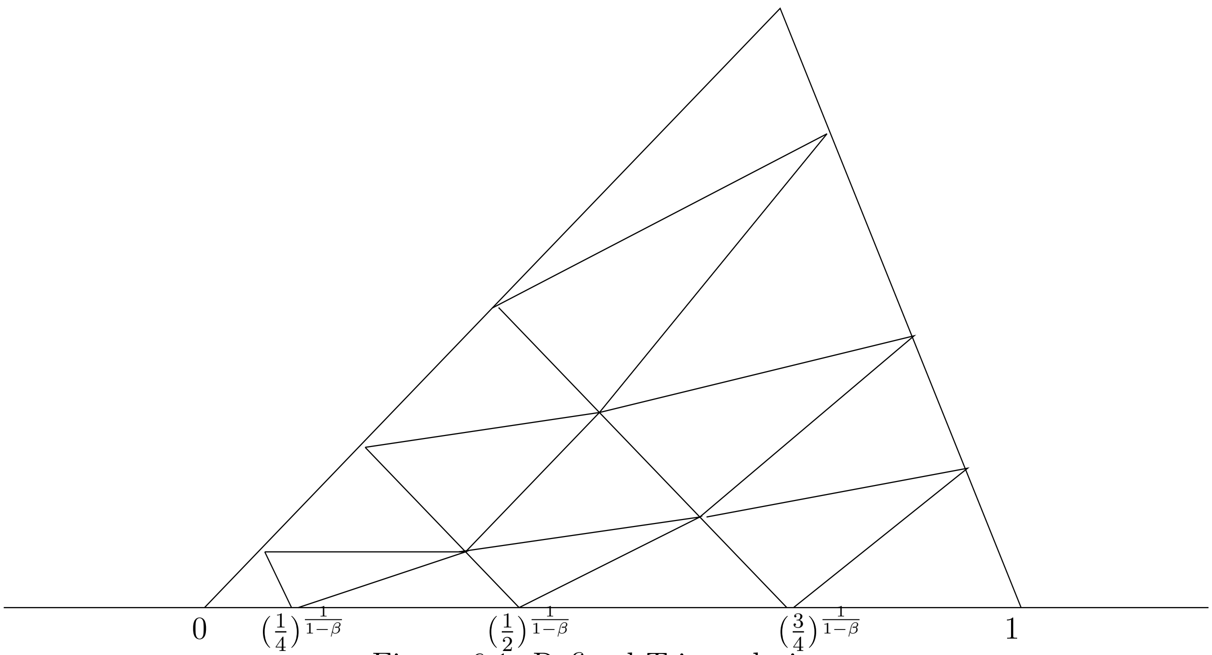

Figure 6.1: Refined Triangulation

The mesh refinement conditions (6.1.35) in this case reduce to

$$h_i \leq \begin{cases} C\left(\frac{1}{n}\right)^{\frac{1}{1-\beta}}, & \text{if} \quad i = 0 \\ \\ C\frac{1}{n}\inf_{T_i} r^{\beta}, & \text{if} \quad i \neq 0 \end{cases} \tag{6.4.1}$$

where

$$h_i = \left(\frac{i+1}{n}\right)^{\frac{1}{1-\beta}} - \left(\frac{i}{n}\right)^{\frac{1}{1-\beta}} \quad \text{and} \quad h = \frac{1}{n}$$

and $C > 0$ is a constant independent of n.

Let us prove (6.4.1). The proof for $i = 0$ is obvious by the definition of h_i.
In the case when $i \neq 0$, we have

$$\begin{aligned} h_i &= \frac{1}{1-\beta}(\xi)^{\frac{\beta}{1-\beta}}\frac{1}{n}, \quad \text{with} \quad \frac{i}{n} < \xi < \frac{i+1}{n}, \quad \text{by the Mean-Value Theorem.} \\ &\leq \frac{1}{1-\beta}\left(\frac{i+1}{n}\right)^{\frac{\beta}{1-\beta}}\frac{1}{n} \quad \text{since} \quad 0 < \beta < 1. \end{aligned}$$

On the other hand, we have

$$\begin{aligned}\frac{\left(\frac{i+1}{n}\right)^{\frac{\beta}{1-\beta}}}{\left(\frac{i}{n}\right)^{\frac{\beta}{1-\beta}}} &= \left(\frac{i+1}{i}\right)^{\frac{\beta}{1-\beta}}\\ &= \left(1+\frac{1}{i}\right)^{\frac{\beta}{1-\beta}}\\ &\leq (2)^{\frac{\beta}{1-\beta}} \text{ because } i \geq 1.\end{aligned}$$

Therefore

$$h_i \leq C(\frac{i}{n})^{\frac{\beta}{1-\beta}}\frac{1}{n} \text{ and } h_i \leq C\frac{1}{n}\inf_{T_i} r^{\beta}.$$

This proves (6.4.1).

After this justification of the existence of the mesh refined triangulations, we proceed by considering Ω to be an L-shaped domain as shown in Figure 6.2. This consists of the re-entrant angle $\omega = \frac{3\pi}{2}$ that is responsible for singularities at the origin of the plane. The

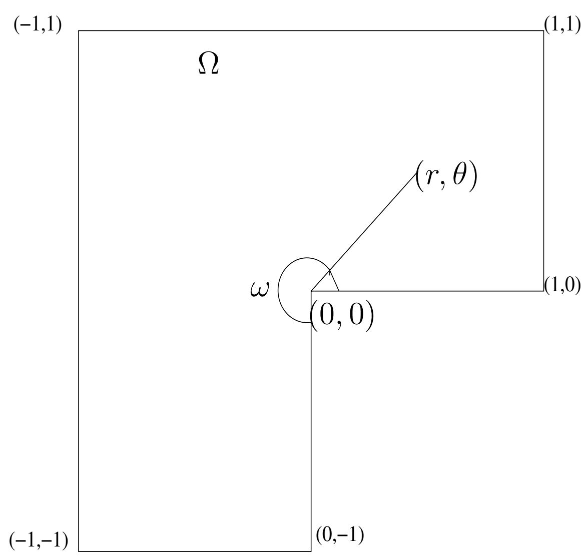

Figure 6.2: L-shaped domain

right-hand side f of equation (6.3.1) is taken in such a way that

$$u(x,t) = te^{-t}\psi(r)r^{\frac{2}{3}}\sin\frac{2}{3}\theta. \tag{6.4.2}$$

is the exact solution of the problem (6.3.1)-(6.3.3) where $\psi(r)$ is a smooth cut-off function such that $\psi = 1$ for $r \leq 1/4$ and $\psi = 0$ for $r \geq 1/2$. We use a uniform mesh for $\beta = 0$ and a refined mesh for $\beta = 1/3$ on the method (6.3.11)-(6.3.12). A similar construction is done when the denominator of the first term of (6.3.11) is replaced by Δt. The pictures resulting from these techniques are illustrated in Figures 6.3 and 6.4 for $n = 10$.

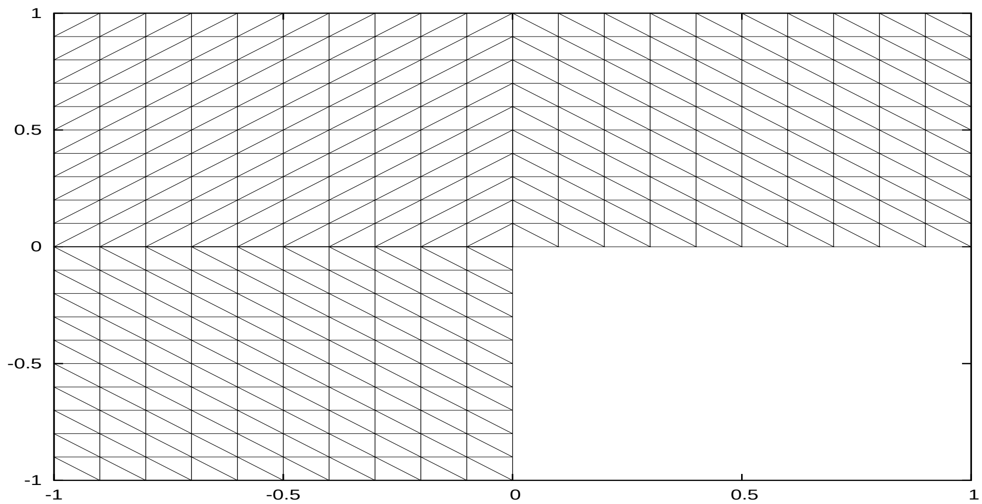

Figure 6.3: Uniform mesh for $n = 10$

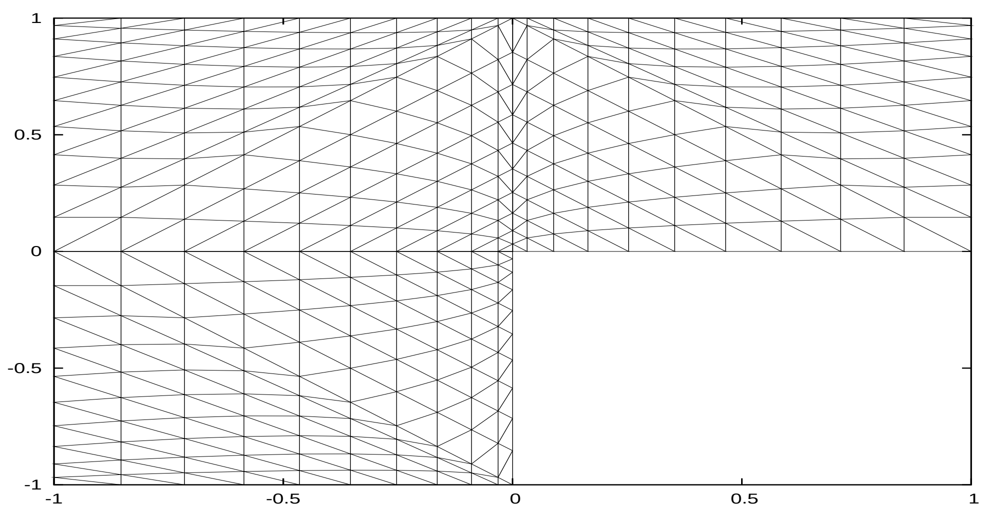

Figure 6.4: Refined mesh for $n = 10$

In Figure 6.3, the domain Ω is filled with a uniform mesh of identical triangles in the classical manner. This is followed by Figure 6.4, where the domain Ω is refined following the procedure of Raugel [59] presented above and illustrated in Figure 6.1.

For our numerical experiments, we take $n = 10, 50, 100, 125$. The refinement parameter β is taken to be $\beta = 0$ for a uniform mesh and $\beta = 1/3$ for a refined mesh. A similar approach to this choice of n values was done for the Laplace equation in a polygon in [24]. We approach the numerical solution to the problem using two techniques. The first technique is by coupling the standard finite difference method (SFDM) and finite element method. The second technique is by combining the non-standard finite difference and the finite element methods. In both cases we keep once and for all, the time fixed.

For the numerical solution obtained by coupling the SFDM and FEM, the error $\|u-u_h\|_{1,\Omega}$ was computed. Table 6.1 shows the rates of convergence for the uniform mesh ($\beta = 0$) and the refined mesh ($\beta = 1/3$). Figure 6.5 shows in logarithm scale the slope of the curves that correspond to the approximate rates of convergence, which are 0.27 (poor) for the uniform and 0.8123 for the refined mesh.

Similarly, for the numerical solution obtained by combining the NSFD method and FEM, the error $\|u-u_h\|_{0,\Omega}$ was computed. Table 6.2 shows the rates of convergence for the uniform mesh and the refined mesh whereas Figure 6.6 shows in logarithm scale that the approximate rates are 0.5 (poor) and 1.95 for the uniform mesh and the refined mesh, respectively.

We have therefore proved computationally that the refined mesh provides better (optimal) rates of convergence than the classical uniform mesh.

Table 6.1: Error in the H^1-norm for both uniform and refined meshes

n	**Uniform Mesh**	**Refined Mesh**
	$\|\|u - u_h\|\|_{1,\Omega}$	$\|\|u - u_h\|\|_{1,\Omega}$
10	3.0854E-3	2.9221E-3
50	1.2875E-3	5.8442E-4
100	9.0016E-4	2.9110E-4
125	8.1009E-4	2.3288E-4

Table 6.2: Error in the L^2-norm for both uniform and refined meshes

n	**Uniform Mesh**	**Refined Mesh**
	$\|\|u - u_h\|\|_{0,\Omega}$	$\|\|u - u_h\|\|_{0,\Omega}$
10	1.2469E-3	1.3411E-6
50	2.4939E-4	5.5372E-8
100	1.3027E-4	1.3860E-8
125	1.0457E-4	8.8717E-9

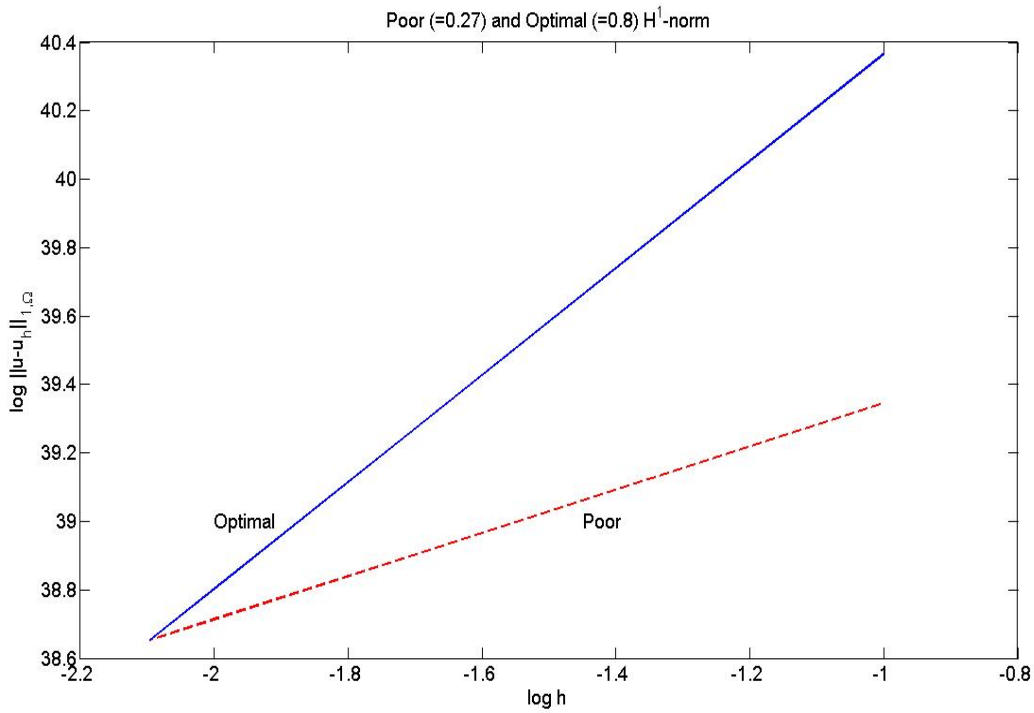

Figure 6.5: Rate of convergence for H^1-norm

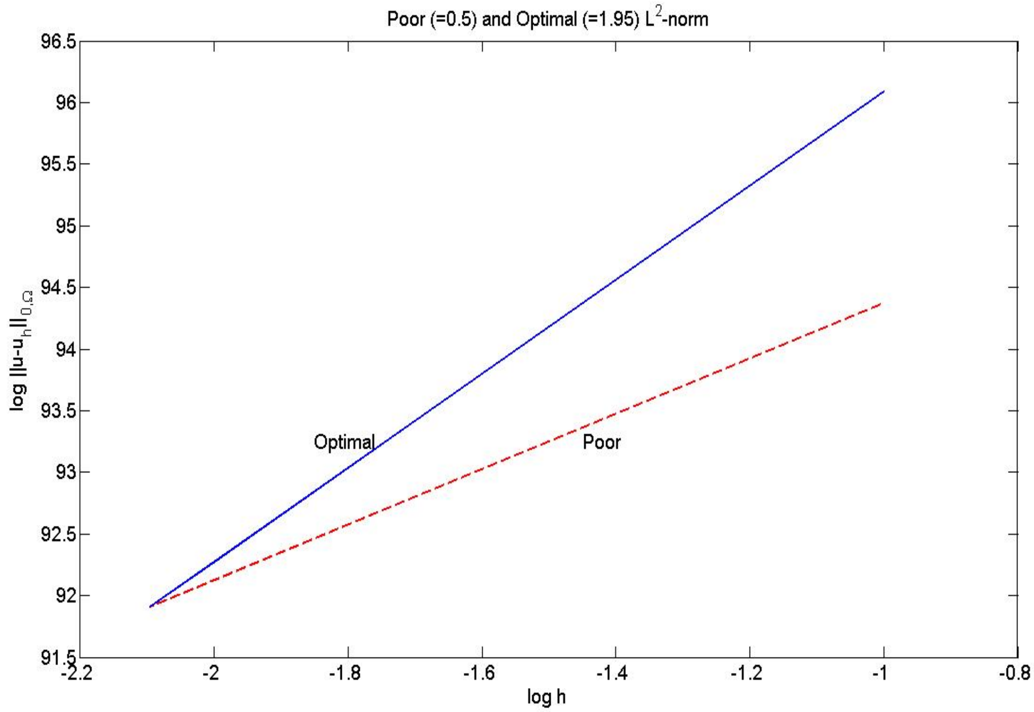

Figure 6.6: Rate of convergence for L^2-norm

We conclude this section by studying the impact and the power of the non-standard finite difference method. Let f in (6.3.1) be such that

$$u(x,t) = \alpha e^{-\lambda t}\psi(r)r^{2/3}\sin 2/3\theta \tag{6.4.3}$$

is the solution of (6.3.1)-(6.3.3) for the parameters λ and α. We fix once and for all, $x = (-0.0316, 0.0554)$, $\lambda = 3$ and $\alpha = \pm 0.5$. Since $|x| \leq \frac{1}{4}$, then $u(t) \equiv u(x,t)$ is a solution of the decay equation (6.3.4)-(6.3.6); $u(t)$ is plotted against the time on Fig 6.7(a) and (b). For the same fixed x, Fig 6.7(c) and (d) depict $u_h^k \equiv u_h^k(x)$ obtained from the NSFDM-FEM (6.3.1) as well as from the classical finite difference method with $\Delta t = 0.5$. For the latter method, there is no restriction on the value of Δt since it is implicit [65]. The figures speak for themselves.

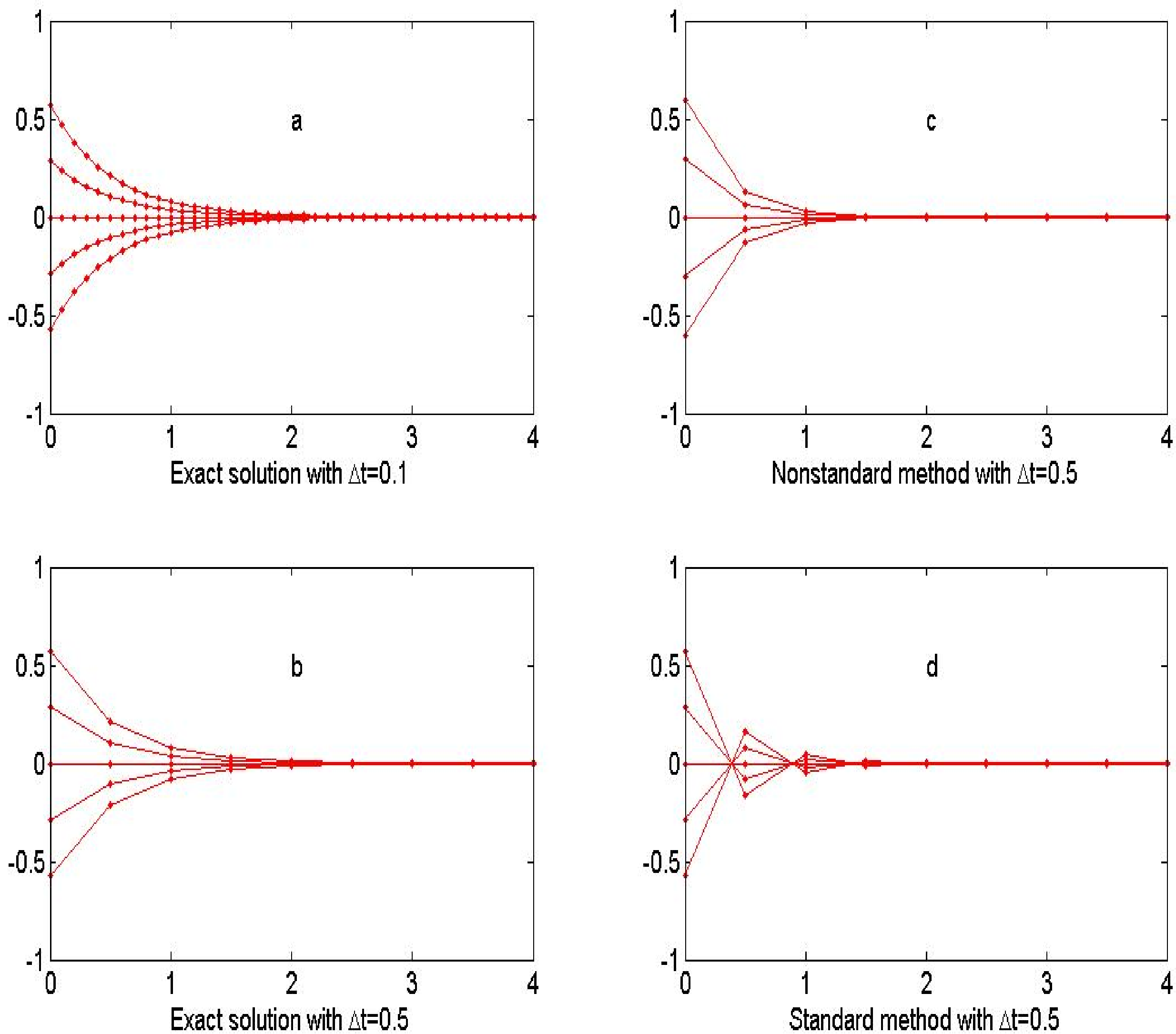

Figure 6.7: Impact of non-standard and Standard approaches

Chapter 7

Conclusion

This work was initially motivated by the Ph.D thesis of Maghnouji [46] where the singularities of a parabolic equation for a strongly elliptic operator on a polygonal domain are studied.

The initial aim was to provide the numerical analysis counterpart of [46]. However, given the complexity and level of generality of [46], we opted to work with the heat equation in order to better understand the difficulties and to obtain explicit results in which the geometry of the domain is clearly reflected. Some results for the heat equation are obtained in Grisvard [29] and [31]; but the approach used here is different as we are mostly concerned with the Laplace transform of vector-valued distributions.

The main results we obtained can be summarized as follows:

- We established the singular decomposition of the solution of the heat equation with an explicit representation of the singular part;
- We established the tangential regularity of the solution in the time variable;
- We showed that the solution is globally regular in a weighted Sobolev space in which the weight depends on the corners of the domain Ω;
- The mesh size being suitably refined in the triangulations of the space domain $\bar{\Omega}$, we implemented two optimally convergent numerical methods: the coupled Fourier-finite element method and the coupled Nonstandard finite difference method-finite element method. The latter method has the advantage of replicating some intrinsic properties of the exact solution.

Possible extensions of this thesis that we will consider in future include:

- The numerical study of parabolic problems in the general framework of [46]. For elliptic problems, this is done for instance in [43].
- The study of the heat equation with more regular right hand side. This would require the introduction and better understanding of anisotropic Sobolev spaces as in [46].
- The extension of the study to domains with both edge and vertex singularities such as polyhedrons. This is done in [42, 44] for elliptic problems as well as in [18, 19], [48, 49].
- Extension to nonlinear reaction diffusion equations and construction of suitable numerical methods. Reliable NSFD schemes in this case were considered in [4].

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