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Chapter 1

Introduction

Diffusion and heat flow processes occur extensively in science, engineering and in fact in real life situations. In the linear case, these processes are mathematically modeled by parabolic partial differential equations of the form

$$\frac{\partial u}{\partial t} - Lu = f \ \text{ in } \ \Omega \times (0, +\infty) \tag{1.0.1}$$

where

- $L = \sum_{|\alpha| \leq 2m} a_\alpha D^\alpha$ is a strongly elliptic operator of order $2m$ with constant coefficients,
- Ω is a domain of $\mathbb{R}^n$, with boundary $\partial\Omega \equiv \Gamma$.
- f is a given real-valued function.

The parabolic equation (1.0.1) coupled with suitable boundary and initial conditions has been extensively studied in the literature under some smoothness assumptions on the domain Ω. Following Lions and Magenes [38], the most popular assumption is to consider Ω to be a bounded open set with boundary Γ being a C^∞-manifold of dimension $n-1$, the set Ω being locally located at one side of Γ. In other words, $\bar{\Omega}$ is a compact manifold with boundary Γ of class C^∞.

Under this smoothness assumption, the famous qualitative result by Agmon, Douglis and Nirenberg [2] regarding elliptic problems can be stated as follows:
Let u in the Sobolev space $H_0^m(\Omega)$ be such that $Lu \in L^2(\Omega)$. Then u is optimally regular in the sense that

$$u \in H^{2m}(\Omega) \ \ \textit{and} \ \ \|u\|_{H^{2m}(\Omega)} \leq C\|Lu\|_{L^2(\Omega)}$$

for some constant $C > 0$ which does not depend on u.
In this smooth framework, similar results for parabolic and hyperbolic problems as well as further contributions to elliptic problems can be found in [38] and [39].

The qualitative analysis in the more difficult case when the domain Ω is non-smooth was considered relatively later. In this regard, the historical reference is Kondratiev [36] who investigated the singular behavior of solutions of elliptic equations in domains with conical and angular points. Since this seminal contribution of Kondratiev, there has been a surge of works on elliptic problems in non-smooth domains ranging from the case of operators of mathematical physics in simple two dimensional geometry (see [29], [31]) to more complicated cases that involve both conical and edge singularities (see [19], [30], [44], [49]). The specific two dimensional case of the parabolic problem (1.0.1) is investigated in the thesis [46]. Our work is mostly based on this thesis [46]. Given the level of generalization and complexity in [46], the purpose of our thesis is:

- To analyze and better understand the results obtained.
- To obtain results that are as explicit as possible;
- To visualize the impact of the rough geometry Ω in the result;
- To enrich and complete the theoretical study of [46] with reliable numerical methods in which the singularities of the continuous problem are relatively easily incorporated.

To achieve the above objectives, the setting of this thesis is made explicit as follows:

- The domain Ω is a polygon $(n = 2)$
- The operator L is taken to be

  $$Lu = -\Delta u + \lambda u, \quad \lambda \geq 0$$

  where

  $$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$$

  is the Laplace operator.

In other words, we are dealing with the following boundary value problem for the two dimensional heat/diffusion operator on a polygon:

$$\frac{\partial u}{\partial t} - \Delta u + \lambda u = f \ \text{ in } \ \Omega \times (0, +\infty)$$

$$u(x, 0) = 0, \ x \in \Omega$$

$$u(x, t) = 0, \ \ (x, t) \in \partial\Omega \times (0, +\infty).$$

There exist several methods for solving evolution problems (see for example [21]). In this work, we mainly employ the Laplace transform for the analytical part, whereas the finite element method (in space variable) coupled with the finite difference method or Fourier series method (in time variable) is used for the constructive part.

The use of the Laplace transform reduces the heat equation to a family of Helmholtz equations for a complex parameter $p \in \mathbb{C}$. The main challenge is to obtain for the solutions of the Helmholtz problems, a priori estimates with the same constant that is independent of the parameter p so that the inverse Laplace transform (i.e. the Paley-Wiener Theorem) is applicable to obtain solution of the heat equation in suitable function spaces. More precisely, our contribution can be outlined as follows:

1. We provide a comprehensive study of Laurent Schwartz extension of the Laplace transform to vector-valued distributions, which constitute a suitable framework for the heat equation;

2. We show that the family of solutions of the resulting Helmholtz equations satisfy the following properties:

   (a) the solutions belong to appropriate weighted Sobolev spaces and they depend continuously upon the family of the right hand side with the same constant that is independent of p;

   (b) the solutions admit decompositions into regular and singular parts, where the regular part (in usual Sobolev spaces) and the coefficients of the singular parts depend continuously upon the family of the right hand sides with independent constants;

3. We deduce from (2) above global regularity and singular decomposition results for the heat equation.

4. We design an optimally convergent mesh refinement finite element method for the Helmholtz equation, as a consequence of the regularity in (2) above.

5. We present two discrete methods for the heat equation. Firstly, we couple the Fourier series method (in the time variable) with the mesh refinement finite element method (FEM) (in the space variable). Secondly, we use the Non-standard finite difference (NSFD)method (in the time variable) in conjunction with the mesh refinement FEM (in the space variable).

   The idea of using the (NSFD) method for such problems is new. NSFD techniques introduced by Mickens [52] more than two decades ago have laid the foundation for designing methods that preserve the dynamics of the continuous differential models. In our context, the NSFD-FEM we obtained preserves some intrinsic properties of the solution of the heat equation.

The results of this thesis are published in the papers [14] and [13]. In view of our focus to better understand the complex issue of singularities, we deliberately spend a lot of time on some crucial details. This contributes to give a self-contained flavor to the thesis, which is essential given the amount of tools and deep concepts from various areas that are needed in this work. This also explains why despite the title of the thesis on the heat equation, much time and space are devoted to the Helmholtz problem, which is the backbone of the analysis of the heat equation.

As a matter of principle comments as to how our thesis fits in the literature are generally made throughout the text next to where the results are stated and proved. See for example Remark 4.3.3 regarding the literature on singularities.

We outline now chapter by chapter the content of the thesis. Chapter 2 is devoted to some basic tools mostly related to function spaces (e.g Sobolev spaces, etc) we need. A key aspect of this chapter is the analysis of the Laplace transform of vector-valued distributions, which requires from us to elaborate substantially on Laurent Schwartz's canonical topology of the space of test functions $\mathcal{D}$ in order to prove the density of the space of finite rank distributions into the space of vector-valued distributions [61].

Chapter 3 and 4 deal with the quantitative and qualitative analysis of the Dirichlet problem for the Helmholtz operator involving a parameter $p \in \mathbb{C}$. The quantitative analysis amounts to the well-posedness (with constant independent of p) of the problem in appropriate Sobolev spaces. The qualitative analysis takes care of two aspects. Firstly, in Chapter 3, we deal with the case when the domain is smooth and the Agmon, Douglis and Nirenberg [2] regularity results are presented. Secondly, in Chapter 4 when the domain is a polygon, the decomposition of solutions into regular and singular parts is investigated and this is exploited to establish the global regularity of the solutions into a weighted Sobolev space in such a way that the solutions depend continuously on the data with a constant independent of the parameter $p \in \mathbb{C}$.

The uniform (with respect to p) estimates obtained in the previous chapters combined with the Paley-Wiener theorem permit in Chapter 5 to establish for the heat equation, the existence of a unique variational solution, the tangential regularity (in the time variable) of the solution, the singular decomposition of the solution and its global regularity in vector-valued weighted Sobolev spaces.

Chapter 6 is reserved for numerical approximations of the heat equation. First, we design a semi-discrete (in time) mesh refinement finite element method which is optimally convergent. Next the time variable is discretized by the Fourier series method and the space variable by the mesh refinement FEM. This leads to a full discrete method which is optimally convergent in both the time and the space variables. Finally, we use an alternative approach of discretizing the time variable by the NSFD scheme while the mesh refinement FEM is used for the space variable. In addition to the optimal convergence, this NSFD-FEM procedure preserves some qualitative property of the continuous model of the solution such as the decay property in the limit case of space independent equation. These theoretical results are supported by numerical experiments.

Concluding remarks are gathered in chapter 7. They underline how this work fits in the literature and how it can be extended.

Chapter 2

Basic Tools

The study of boundary value problems such as the heat equation conventionally takes as its starting point the idea of function spaces in which the solution of the problem will be handled. For this reason, we will start this thesis with some introductory aspects of function spaces. The underlying domain on which the functions are defined is presented in section 2.1. The most prominent function spaces of interest in our study will be the spaces of continuous functions, Lebesgue space (section 2.2), Distributions (section 2.3) and Sobolev spaces (section 2.4). Some relevant results on Laplace transform (the second tool used in our study) are described in section 2.5.

2.1 The domain Ω

In what follows, we shall work with functions defined on a domain $\Omega \subset \mathbb{R}^2$, i.e., an open and connected set, with boundary denoted by $\partial\Omega$ or Γ. The domain Ω or its boundary Γ is supposed to satisfy some regularity conditions. Following Grisvard [29], our standard reference for function spaces, the regularity conditions can be grouped into the two categories.

The first category is to view $\Gamma \equiv \partial\Omega$ as being locally the graph of a function φ. The regularity of Γ is then described through the differentiability properties of φ. The precise definition reads as follows:

Definition 2.1.1. *We say that the boundary Γ is continuous (respectively, Lipschitz, m times continuously differentiable, etc.) if for every $x \in \Gamma$, there exist a neighborhood V of x in $\mathbb{R}^2$ and a new system of co-ordinates (y_1, y_2) such that,*

1. *V is a rectangle in the new co-ordinate system:*

$$V := \{y = (y_1, y_2) : -a_1 < y_1 < a_1, \ \ -a_2 < y_2 < a_2\},$$

2. *there exists a function $\varphi : (-a_1, a_1) \to \mathbb{R}$ which is continuous (respectively Lipschitz, m times continuously differentiable etc) and satisfies the following conditions:*

$$\begin{aligned} |\varphi(y_1)| &< \frac{a_2}{2} \ \textit{for every} \ y_1 \in V' := (-a_1, a_1), \\ \Omega \cap V &= \{y = (y_1, y_2) \in V : \quad y_2 < \varphi(y_1)\}, \\ \Gamma \cap V &= \{y = (y_1, y_2) \in V : \quad y_2 = \varphi(y_1)\}. \end{aligned}$$

More generally, Γ is called of class $\mathcal{H}$ when the above function φ is of class $\mathcal{H}$.

Definition 2.1.1 is illustrated in Figure 2.1

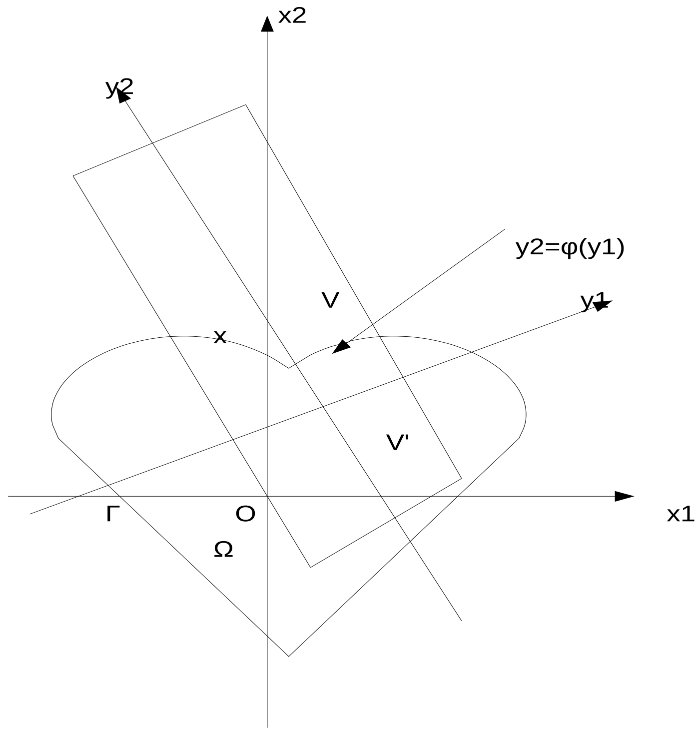

Figure 2.1: Lipschitz boundary Γ

Definition 2.1.1 implies that Ω is locally on one side of its boundary Γ. Indeed, it follows that $\Omega \cap V$ is below the graph of φ and $\Gamma \cap V$ is the graph. Consequently domains with cusps do not satisfy Definition 2.1.1.

The second category is to consider the closure $\overline{\Omega}$, of the domain Ω as a 2-dimensional manifold with the boundary imbedded in $\mathbb{R}^2$. The regularity assumptions are then added on the manifold.

Definition 2.1.2. *We say that $\overline{\Omega}$ is a 2-dimensional continuous (respectively, Lipschitz, m times continuously differentiable etc.) sub-manifold with boundary in $\mathbb{R}^2$, if for every $x \in \Gamma$ there exists a neighborhood V of x in $\mathbb{R}^2$ and a mapping T from V into $\mathbb{R}^2$ such that*

1. *T is injective,*
2. *T together with T^{-1} (defined on $T(V)$) are continuous (respectively, Lipschitz, m times continuously differentiable),*
3. *$\Omega \cap V = \{y \in \Omega : T_2(y) < 0\}$ where $T_2(y)$ denotes the 2th component of $T(y)$.*

Definition 2.1.2 is illustrated in Figure 2.2.

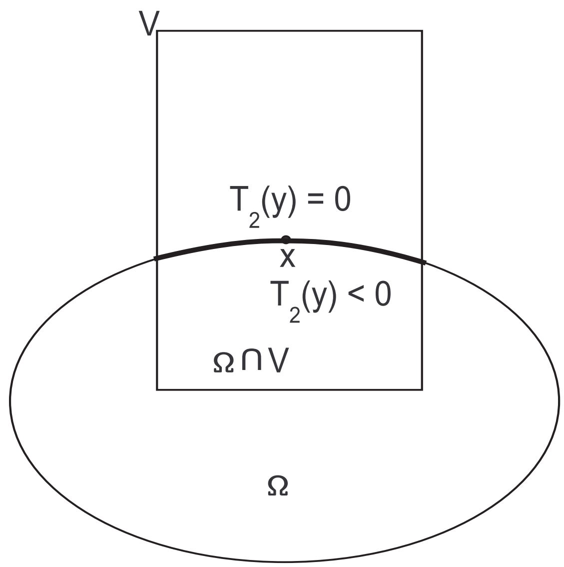

Figure 2.2: Local charts of the boundary Γ

As a result of condition (3) of Definition 2.1.2, the boundary Γ of Ω is defined locally by the equation $T_2(y) = 0$.

The comparison of Definition 2.1.1 and Definition 2.1.2 is an issue of interest. To this end, assuming that Definition 2.1.1 holds, let us define T by

$$T(y) = \{y_1, y_2 - \varphi(y_1)\}. \tag{2.1.1}$$

The function in (2.1.1) has its inverse given by

$$T^{-1}(z) = \{z_1, z_2 + \varphi(z_1)\}.$$

It is clear that T in (2.1.1) fulfils all the conditions in Definition 2.1.2 with the same amount of differentiability for T and T^{-1} as the function φ. In other words, Definition 2.1.1 implies Definition 2.1.2. However, the converse is partly true, namely when T is at least of class C^1. Indeed, assuming that Definition 2.1.2 holds,

$$T_2(y_1, y_2) = 0 \;\text{ for }\; (y_1, y_2) \in \Gamma \cap V. \tag{2.1.2}$$

Let $(y_1^\star, y_2^\star) \in \Gamma \cap V$ be such that $\frac{\partial T_2}{\partial y_2}(y_1^\star, y_2^\star) \neq 0$. Then by the implicit function theorem, there exists open neighborhoods $U \subset \mathbb{R}^2$ of $(y_1^\star, y_2^\star)$ and $V' \subset \mathbb{R}$ of $y_1^\star$ as well as C^1 function $\varphi : V' \to \mathbb{R}$ such that

$$(y_1, y_2) \in U \;\text{ solves }\; (2.1.2) \;\text{ if and if } y_2 = \varphi(y_1),\; y_1 \in V'.$$

The above constraint on the use of the implicit function theorem, motivates why we prefer Definition 2.1.1. In this regard, a typical example on which our thesis is based is given in the next result taken from [55].

Proposition 2.1.3. *A domain Ω with polygonal boundary Γ is Lipschitz in the sense of Definition 2.1.1.*

Proof. We take Ω to be the unit square represented by

$$\Omega = (-1, 1) \times (-1, 1)\,,$$

as illustrated in Figure 2.3.

Let $z \in \Gamma$ not be a vertex. We let the new co-ordinate system y_1, y_2 centered at z be such that the y_1-line coincides with the side of the square that contains z, while the y_2-line is perpendicular to the y_1-line (see Figure 2.4).We then take $V = [-\alpha + z_1, \alpha + z_1] \times [-\beta + z_2, \beta + z_2]$ in the new system and $\varphi(y_1) = 0$. It is clear that y_2 is of class C^∞. Next we consider the case when z is a vertex. In view of the symmetry of the square Ω, it is enough to restrict ourselves to the point $z = (1, 1)$. By Definition 2.1.1, we consider new co-ordinate system as follows, in in view of Figure 2.4. We pass to the co-ordinate (y_1, y_2) from (x_1, x_2) after performing a rotation through an angle of $\pi/4$ and a translation of $(3/4, 3/4)$. These transformations yield the follow equation

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 3/4 \\ 3/4 \end{bmatrix}.$$

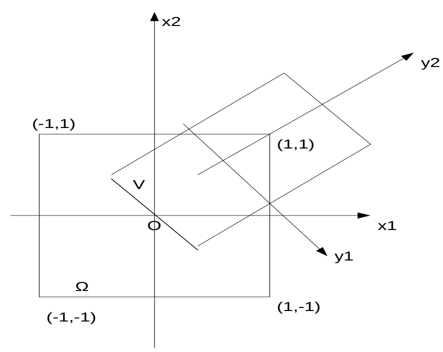

Figure 2.3: Polygon as Lipschitz domain (a)

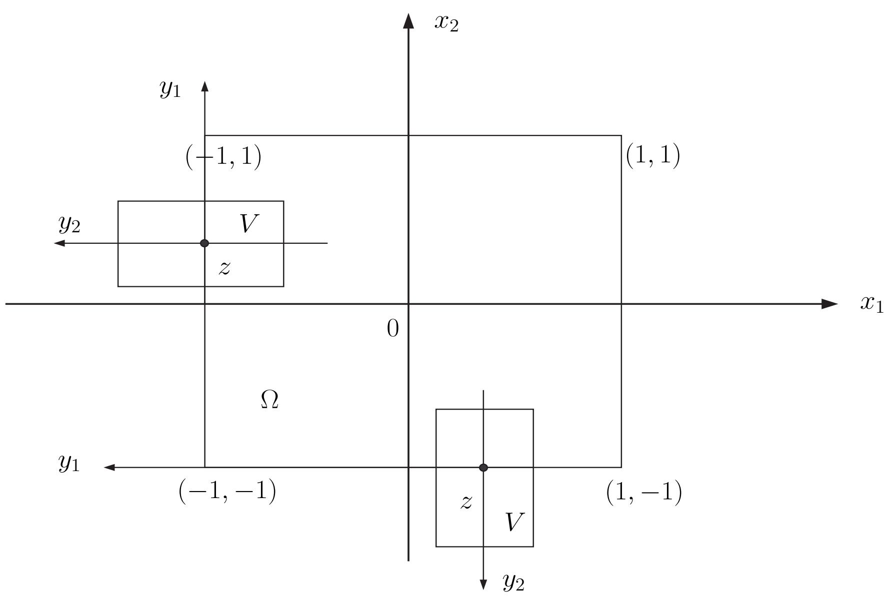

Figure 2.4: Polygon as Lipschitz domain (b)

In the new co-ordinate system, we take

$$V = \left(-\sqrt{2}/4, \sqrt{2}/4\right) \times \left(3\sqrt{2}/4, 3\sqrt{2}/4\right);$$

for a neighborhood of the point $(1,1)$. For the function φ, we consider

$$\varphi(y_1) = \sqrt{2}/4 - |y_1|.$$

With reference to Definition 2.1.1, we can check that

$$\begin{array}{rcl} |\varphi(y_1)| & < & \dfrac{3\sqrt{2}}{8} \text{ for every } y_1 \in V' = \left(-\sqrt{2}/4, \sqrt{2}/4\right), \\ \Omega \cap V & = & \{y = (y_1, y_2) \in V : \quad y_2 < \varphi(y_1)\}, \\ \Gamma \cap V & = & \{y = (y_1, y_2) \in V : \quad y_2 = \varphi(y_1)\}. \end{array}$$

For $y_1 \neq 0$ we have

$$\frac{\varphi(y_1) - \varphi(0)}{y_1} = \begin{cases} -1 & \text{if } y_1 \geq 0 \\ 1 & \text{if } y_1 \leq 0, \end{cases}$$

which implies that

$$|\varphi(y_1) - \varphi(0)| \leq |y_1|, \forall\, y_1 \in V'.$$

Now for arbitrary y_1 and y_1' in V', we have

$$|\varphi(y_1) - \varphi(y_1')| = |y_1 - y_1'|,$$

if the signs of y_1 and y_1' are the same. In the case where the signs are different we have

$$\begin{array}{rcl} |\varphi(y_1) - \varphi(y_1')| & \leq & |\varphi(y_1) - \varphi(0)| + |\varphi(0) - \varphi(y_1')| \\ & \leq & |y_1| + |y_1'| \\ & = & |y_1 - y_1'| \end{array}$$

We then have that

$$|\varphi(y_1) - \varphi(y_1')| \leq |y_1 - y_1'|, \;\; \forall \, y_1, y_1' \in V',$$

which conclude the proof. □

With Definition 2.1.1, we associate once and for all the notation below which will be used in future. For all $z \in \Gamma$ there exists a neighborhood V_z defined in a new co-ordinate system $x_z = (x_{1,z}, x_{2,z}) \equiv (x_1, x_2) \equiv x$ by

$$V_z = \{x = (x_1, x_2) : -a_{1,z} < x_1 < a_{1,z}, \; -a_{2,z} < x_2 < a_{2,z}\}.$$

Since the boundary Γ is compact, there exist $z_1, \; z_2, \; , ...z_k \in \Gamma$ such that $\Gamma \subset \cup_{j=1}^{k} V_j$, where $V_j \equiv V_{z,j}, \;\; a_{1,zj} \equiv a_{1,j}$ and $a_{2,zj} \equiv a_{2,j}$. In view of this notations, we can find an open set V_0 with $\bar{V}_0 \subset \Omega$ such that the family of open sets $V_j, j = 0, 1, 2, ...k$ is a covering of $\bar{\Omega}$.

Without loss of generality and following Necas [54], we can assume that for all $1 \leq j \leq k$

$$V_j = \{x = (x_1, x_2) :, \; -\alpha < x_1 < \alpha, \;\; -\beta < x_2 < \beta\}, \text{ for some } \alpha, \beta > 0$$

where we recall that (x_1, x_2) in the right hand side should be viewed as in the new co-ordinate system $(x_{1,j}, x_{2,j})$.Furthermore, we have the following regions of $\mathbb{R}^2$ demarcated by:

$$V_j^0 = \Gamma \cap V_j = \{(x_1, x_2) : x_2 = \varphi_j(x_1), \;\; -\alpha < x_1 < \alpha\},$$

$$V_j^+ = V_j \cap \Omega = \{(x_1, x_2) \in V_j : \varphi_j(x_1) - \beta < \varphi_j(x_1), \;\; -\alpha < x_1 < \alpha\},$$

$$V_j^- = V_j \cap (\mathbb{R}^2/\Omega) = \{(x_1, x_2) \in V_j : \varphi_j(x_1) + \beta > \varphi_j(x_1), \;\; -\alpha < x_1 < \alpha\}.$$

For a fixed $j, \;\; 1 \leq j \leq k$, we consider the T_j with its inverse T_j^{-1}

$$T_j : V_j \to Q \;\text{ and }\; T_j^{-1} : Q \to V_j,$$

defined by

$$T_j(x) \equiv T_j(x_1, x_2) = \left(\frac{x_1}{\alpha},\ \frac{\varphi_j(x_1) - x_2}{\beta} \right), \tag{2.1.3}$$

and

$$T_j^{-1}(y) \equiv T_j^{-1}(y_1, y_2) = (\alpha y_1,\ \varphi_j(\alpha y_1) - \beta y_2)\,. \tag{2.1.4}$$

where

$$Q = \{(y_1, y_2) : |y_1| < 1,\ \ |y_2| < 1\}\,,$$

is the unit square. The smoothness of T_j and T_j^{-1} is determined by that of the map φ_j in Definition 2.1.1. Furthermore, under the transformation T_j

$$V_j^+ \text{ becomes } Q_+ = \{(y_1, y_2) :\ |y_1| < 1,\ 0 < y_2 < 1\}\,,$$

$$V_j^- \text{ becomes } Q_- = \{(y_1, y_2) :\ |y_1| < 1,\ -1 < y_2 < 0\}\,,$$

$$V_j^0 \quad \text{becomes} \quad Q_0 = \{(y_1, 0) : \quad |y_1| < 1\}\,,$$

as seen in Figure 2.1.2.

With these notation in mind, there exist non-negative functions $\theta_j \in \mathcal{D}(V_j)$, $\theta_j \leq 1$, $0 \leq j \leq k$
satisfying

$$\forall x \in \bar{\Omega},\ \sum_{j=0}^{k} \theta_j(x) = 1 \text{ and } \forall x \in \Gamma,\ \sum_{j=1}^{k} \theta_j(x) = 1. \tag{2.1.5}$$

The family $(\theta_j)_{j=0}^k$ and $(\theta_j)_{j=1}^k$ are called C^∞-partition of unity on $\bar{\Omega}$ and Γ subordinated to the open coverings $(V_j)_{j=0}^k$ and $(V_j)_{j=1}^k$ respectively.

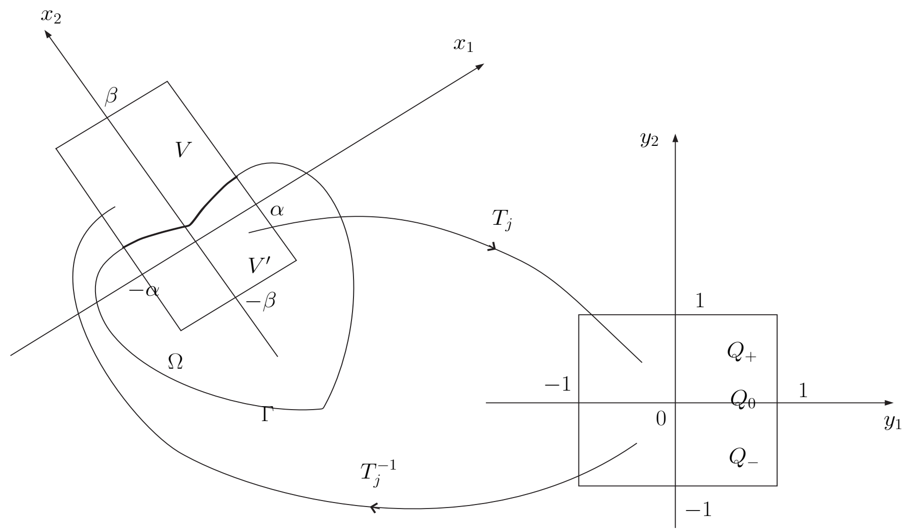

Figure 2.5: Boundary Γ of piecewise C^m class

2.2 Usual Function Spaces

With the domain Ω, we associate the following classical function spaces that we will use:

Definition 2.2.1. ([*40*])
Given an integer $m \geq 0$, we define

- $C^m(\Omega) = \{v : \Omega \to \mathbb{R};\ D^\alpha v$ *is continuous on* $\Omega\ \forall\ |\alpha| \leq m\}$. *This is the space of m times continuously differentiable functions on Ω.*

- $C_b^m(\Omega) := \{v \in C^m(\Omega),\ D^\alpha v$ *is bounded* $\forall\ |\alpha| \leq m\}$, $C_b^m(\Omega)$ *is a Banach space under the norm*

$$\|v\|_{m,\infty,\Omega} := \max_{|\alpha| \leq m} \sup_{x \in \Omega} |D^\alpha v(x)|. \tag{2.2.1}$$

- $C_0^m(\Omega) := \{v \in C^m(\Omega) : v$ *has a compact support contained in* $\Omega\}$.

- $\mathcal{D}(\Omega) \equiv C_0^\infty(\Omega) = \bigcap_{m\geq 0} C_0^m(\Omega)$. *This is the space of test functions, which consists of infinitely differentiable functions $v : \Omega \to \mathbb{R}$ with compact support in Ω.*
- $C^m(\bar{\Omega}) := \{v \in C^m(\Omega); \forall\, |\alpha| \leq m, x \to D^\alpha v(x)$ *is bounded and uniformly continuous on* $\Omega\}$.
- $C^{m,\theta}(\bar{\Omega}) := \left\{v \in C^m(\bar{\Omega});\ \exists\, C \geq 0 : |D^\alpha v(x) - D^\alpha v(y)| \leq C|x-y|^\theta\ \forall\, x, y \in \Omega\ \forall\, |\alpha| = m\right\}$ *is the Hölder space of order m and exponent $\theta \in (0,1]$.*

Definition 2.2.2. ([*40*])
Let $1 \leq p \leq +\infty$ be a real number. The Lebesgue space $L^p(\Omega)$ consists of classes of measurable functions v on Ω such that

$$\|v\|_{0,p,\Omega} = \begin{cases} V^1 < +\infty, & \textit{if}\, p < \infty \\ \\ V^{11} < +\infty, & \textit{if}\, p = \infty, \end{cases} \tag{2.2.2}$$

where

$$V^1 = \left(\int_\Omega |v(x)|^p dx\right)^{1/p}$$

and

$$V^{11} = ess \sup_{x\in\Omega} |v(x)| := \inf\left\{k \geq 0 : |v(x)| \leq k \ a.e\ on\, \Omega\right\}.$$

Notice that $L^1(\Omega)$ is the space of classes of measurable functions on Ω which are Lebesgue-integrable. Notice also that $L^p(\Omega)$ is a Banach space under the natural norm in (2.2.2) while $L^2(\Omega)$ is a Hilbert space for the inner product

$$(u,v)_{0,\Omega} := \int_\Omega u(x)v(x)dx. \tag{2.2.3}$$

Definition 2.2.3. *The space of locally integrable functions is denoted by $L^1_{loc}(\Omega)$ and defined by*

$$\begin{aligned} L^1_{loc}(\Omega) &:= \left\{v : \phi v \in L^1(\Omega),\ \forall\, \phi \in \mathcal{D}(\Omega)\right\} \\ &= \left\{v : v\chi_K \in L^1(\Omega),\ \forall\, K \subset \Omega,\ K \textit{ compact in}\, \mathbb{R}^2\right\}, \end{aligned}$$

where χ_K is the characteristic function of the set K.

Remark 2.2.4. *Spaces of functions* $C_0^\infty(\Omega)$ *and* $L^1_{loc}(\Omega)$ *are the smallest and the largest spaces of functions of interest in applications as depicted in Figure 2.6.*

$$\begin{array}{ccccccc}
C_0^\infty(\Omega) \subset C_0^m(\Omega) \subset L^p(\Omega) \subset & L^1_{loc}(\Omega) & \\
 & \cup & \\
C_b^m(\Omega) \subset & C^m(\Omega) & \\
 & \cup & \\
 & C^m(\bar{\Omega}) & .
\end{array}$$

Figure 2.6: Smallest and Largest spaces

2.3 Distributions

Functions in the smallest space $\mathcal{D}(\Omega)$ have many nice properties that functions in the largest space $L^1_{loc}(\Omega)$ fail to have. By duality on $\mathcal{D}(\Omega)$ we will construct a much larger space which contains $L^1_{loc}(\Omega)$ and possess the said nice properties in a weaker sense.

Definition 2.3.1. ([*40*])**(Pseudo-topology of** $\mathcal{D}(\Omega)$**)**
A sequence $(\varphi_n)_{n\geq 1}$ *in* $\mathcal{D}(\Omega)$ *converges to* $\varphi \in \mathcal{D}(\Omega)$ *if*

1. *There exists a compact set* K *of* $\mathbb{R}^2$ *such that* $K \subset \Omega, \quad supp(\varphi_n) \subset K, \ \forall n \geq 1, \ \ supp(\varphi) \subset K$*;*

2. *For every multi-index* $\alpha, (D^\alpha \varphi_n)$ *converges to* $(D^\alpha \varphi)$ *uniformly on* K*.*

We will elaborate a bit more on the topology of $\mathcal{D}(\Omega)$ in subsection 2.5.3 below.

Definition 2.3.2. ([*40*]) **(Pseudo-topology of** $L^1_{loc}(\Omega)$**)**
A sequence (v_n) *converges to* φ *in* $L^1_{loc}(\Omega)$ *if*

$$\forall \text{ compact } K \subset \Omega, \lim_{n\to\infty} \int_K |v_n - \varphi| dx = 0.$$

With all these structures we can then define distributions as follows:

Definition 2.3.3. ([*40*])

1. *By definition,* $\mathcal{D}'(\Omega)$ *the dual of* $\mathcal{D}(\Omega)$*, is the space of distributions on* Ω*. This means* $T \in \mathcal{D}'(\Omega)$ *if and only if the convergence to* 0 *in* $\mathcal{D}(\Omega)$ *of any sequence* (φ_n) *implies the linear convergence to* 0 *of the scalar sequence* $(\langle T, \varphi_n \rangle)$*. (The symbol* $\langle \cdot, \cdot \rangle_{\mathcal{D}' \times \mathcal{D}}$ *or* $\langle \cdot, \cdot \rangle$ *when there is no risk of confusion denotes the duality pairing between* $\mathcal{D}'(\Omega)$ *and* $\mathcal{D}(\Omega)$*).*

2. *A sequence* (T_n) *of distributions on* Ω *converges to* $0 \in \mathcal{D}'(\Omega)$ *if*
$\langle T_n, \varphi \rangle \to 0, \ \forall \varphi \in \mathcal{D}'(\Omega)$*.*

Remark 2.3.4. *The definition of convergent sequence* T_n *of distributions given in Definition 2.3.3 and used often in the literature is incomplete but sufficient in applications. The complete definition of this concept will be clarified when we equip* $\mathcal{D}'(\Omega)$ *with the topology of uniform convergence on bounded subsets of* $\mathcal{D}(\Omega)$ *(see Proposition 2.5.17).*

Another type of space of test functions which will be useful to us in the context of Fourier transform of distributions, is given in the next definition.

Definition 2.3.5. ([*61*])
Schwartz's space $\mathcal{S}(\mathbb{R})$ *of test functions consists of* C^∞ *functions which together with all their derivatives are rapidly decreasing at infinity. In other words* $\varphi \in \mathcal{S}(\mathbb{R})$ *if* $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ *is infinitely differentiable and for all integers* $m, n \geq 0$*, there exists a constant* $C_{m,n} \geq 0$ *such that*

$$\sup\{|x|^m |\frac{d^n \varphi}{dx^n}(x)| : x \in \mathbb{R}\} < C_{m,n}. \tag{2.3.1}$$

This is equivalent to $\varphi \in C^\infty(\mathbb{R})$ *and* $\lim_{|x| \to \infty} |x|^m \frac{d^n \varphi(x)}{dx^n} = 0 \ \ \forall \, m \in \mathbb{N}, \ \forall \, n \in \mathbb{N}$*.*

The space $\mathcal{S}(\mathbb{R})$ has the structure of a locally convex topological space when equipped with Schwartz canonical topology. In terms of this topology, we have the following definitions:

Definition 2.3.6. ([*40*])**(Pseudo-topology of $\mathcal{S}(\mathbb{R})$)**
A sequence (φ_j) *converges to* φ *in* $\mathcal{S}(\mathbb{R})$ *whenever*

$$\lim_{j \longrightarrow 0} \sup_{x \in \mathbb{R}} |x^m \left(\frac{d^n \varphi_j}{dx^n} - \frac{d^n \varphi}{dx^n} \right)(x)| = 0, \ \forall n \in \mathbb{N}, \ \forall m \in \mathbb{N}.$$

Definition 2.3.7. ([*40*])
By definition, the dual $\mathcal{S}'(\mathbb{R})$ of $\mathcal{S}(\mathbb{R})$ is the space of tempered distributions in $\mathbb{R}$. This means $T \in \mathcal{S}'(\mathbb{R})$ if and only if the convergence to 0 in $\mathcal{S}(\mathbb{R})$ of any sequence (φ_n) implies the convergence to 0 of numerical sequence $(\langle T, \varphi_n \rangle)$. (Again the symbol $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $\mathcal{S}'(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$.)

Definition 2.3.8. ([*40*])
Given a distribution $T \in \mathcal{D}'(\Omega)$, its derivative with respect to x_i, $1 \leq i \leq 2$ is the distribution denoted by $\frac{\partial T}{\partial x_i}$ and defined by

$$\forall \varphi \in \mathcal{D}(\Omega), \; < \frac{\partial T}{\partial x_i}, \varphi >= - < T, \frac{\partial \varphi}{\partial x_i} > .$$

In general, for a multi-index $\alpha \in \mathbb{N}^2$, the derivative of T of order α is the distribution $D^\alpha T$ defined by

$$\forall \varphi \in \mathcal{D}(\Omega), \; < D^\alpha T, \varphi >= (-1)^{|\alpha|} < T, D^\alpha \varphi > .$$

2.4 Sobolev Spaces

For a fixed parameter which is either the time variable t or a complex number p, the solutions of the heat and Helmholtz equations that we will consider in this thesis belong to the class of Sobolev spaces that we outline now. Our standard reference for Sobolev spaces is [29], though we add from time to time those references that we used most.

Definition 2.4.1. ([*29*])
Let $m \geq 0$ be an integer. The Sobolev space $H^m(\Omega)$ is defined by

$$H^m(\Omega) := \left\{ v \in \mathcal{D}'(\Omega) : D^\alpha v \in L^2(\Omega), \;\; \forall \, |\alpha| \leq m \right\}. \tag{2.4.1}$$

In other words, $H^m(\Omega)$ is the collection of all functions in $L^2(\Omega)$ such that all distributional derivatives up to order m are also in $L^2(\Omega)$.

We make $H^m(\Omega)$ a Hilbert space under the norm

$$\|v\|_{m,\Omega} := \left(\sum_{|\alpha| \leq m} \int_\Omega |D^\alpha v|^2 dx \right)^{1/2} \tag{2.4.2}$$

and the inner product

$$(w, v)_{m,\Omega} := \sum_{|\alpha| \leq m} \int_\Omega D^\alpha w D^\alpha v dx.$$

We denote by $|\cdot|_{m,\Omega}$ the semi-norm

$$|v|_{m,\Omega} := \left(\sum_{|\alpha|=m} \int_\Omega |D^\alpha v|^2 dx \right)^{1/2}. \tag{2.4.3}$$

Clearly the Sobolev space of order 0 i.e $H^0(\Omega) = L^2(\Omega)$. Unless $\Omega = \mathbb{R}^2$, or $m = 0$ the space $\mathcal{D}(\Omega)$ is not dense in $H^m(\Omega)$. For this reason we introduce the following subspace.

Definition 2.4.2. ([*29*])
We define the Sobolev subspace $H_0^m(\Omega)$ as the closure of $\mathcal{D}(\Omega)$ in the space $H^m(\Omega)$.

Theorem 2.4.3. ([*29*])*(Poincaré-Friedrichs Inequality)*
Assume that Ω is bounded in one of the directions, say x_n. Then there exists a constant $C > 0$ depending upon Ω such that

$$\forall v \in H_0^1(\Omega),\ \|v\|_{0,\Omega} \leq C \|\frac{\partial v}{\partial x_n}\|_{0,\Omega}. \tag{2.4.4}$$

Consequently for $m \geq 1$ the semi-norm $|\cdot|_{m,\Omega}$ is a norm on $H_0^m(\Omega)$ equivalent to $\|\cdot\|_{m,\Omega}$. Occasionally, we will use the non-Hilbertian Sobolev space defined as follows:

Definition 2.4.4. 1. *For $1 \leq p < \infty$ the Sobolev space of integer order $m \geq 0$ is denoted $W^{m,p}(\Omega)$ and defined by*

$$W^{m,p}(\Omega) := \{v \in \mathcal{D}'(\Omega) : D^\alpha v \in L^p(\Omega),\ \forall\ |\alpha| \leq m\}. \tag{2.4.5}$$

It is clear that $W^{m,p}(\Omega)$ coincides with the Hilbertian Sobolev space $H^m(\Omega)$. However for $p \neq 2$ the space $W^{m,p}(\Omega)$ is a Banach space (not a Hilbert space) with the norm and semi-norm defined respectively by

$$\|v\|_{m,p,\Omega} = \begin{cases} \left(\sum_{|\alpha| \leq m} \int_\Omega |D^\alpha v(x)|^p dx \right)^{1/p} & \textit{if} \quad p < \infty \\ \max_{|\alpha| \leq m} ess \sup_{x \in \Omega} |D^\alpha v(x)| & \textit{otherwise} \end{cases}$$

and

$$|v|_{m,p,\Omega} = \begin{cases} \left(\sum_{|\alpha|=m} \int_\Omega |D^\alpha v(x)|^p dx\right)^{1/p} & \text{if} \quad p < \infty \\ \max_{|\alpha|=m} ess\sup_{x\in\Omega} |D^\alpha v(x)| & \text{otherwise.} \end{cases}$$

Theorem 2.4.5. ([29])*(Sobolev Continuous Embedding Theorem and Rellich Kondrachov Compact Embedding Theorem).*
Assume that a bounded open set Ω has boundary $\partial\Omega \equiv \Gamma$ which is Lipschitz if $m = 1$ or is of class C^m if $m > 1$. Consider the number $p^\star$ defined by

$$\frac{1}{p^\star} = \frac{1}{p} - \frac{m}{2}, \quad 1 < p < \infty, \quad m \geq 1.$$

1. *If $\frac{1}{p^\star} \geq 0$, i.e. $m \leq \frac{2}{p}$, then we have, for any $q \in [1, p^\star]$, the continuous embedding*

$$W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$$

which is compact in the particular case when $q \neq p^\star$;

2. *If $\frac{1}{p^\star} < 0$, i.e. $m > \frac{2}{p}$, we have the continuous and compact embedding*

$$W^{m,p}(\Omega) \hookrightarrow C^s(\bar{\Omega}).$$

where s is the non-negative integer satisfying $s \leq m - \frac{2}{p} < s + 1$.

Furthermore, if $m - \frac{2}{p}$ is not an integer, we have the continuous embedding

$$W^{m,p}(\Omega) \hookrightarrow C^{s,\theta}(\bar{\Omega})$$

where $\theta = m - \frac{2}{p} - s$ and $C^{s,\theta}(\bar{\Omega})$ the Hölder space equipped with the norm

$$\|v\|_{C^{s,\theta}(\bar{\Omega})} := \max_{|\alpha|\leq s} \sup_{x\in\Omega} |D^\alpha v(x)| + \max_{|\alpha|=s} \sup_{x\in\Omega} \frac{|D^\alpha v(x) - D^\alpha v(y)|}{|x-y|^\theta}$$

Remark 2.4.6. *Theorem 2.4.5 is valid in the one-dimensional case (i.e Ω is an interval) provided that 2 is replaced with 1 in the identity that defines $p^\star$.*

2.5 Laplace transform

The evolution equations that we study will be transformed into complex-parameter family of elliptic equations through the Laplace transform, which we outline in this section.

2.5.1 Laplace transform of functions

Given a test function $v \in \mathcal{D}(0, +\infty)$, its Laplace transform is denoted and defined by

$$(\mathcal{L}v)(p) \equiv \widehat{v}(p) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} e^{-pt} v(t) dt, \quad p = \xi + i\eta \in \mathbb{C}. \tag{2.5.1}$$

The connection of the Laplace transform with the Fourier transform is straight forward on extending the function $v \in \mathcal{D}(0, +\infty)$ to $\widetilde{v} \in \mathcal{D}(-\infty, +\infty)$ given by

$$\widetilde{v}(t) = \begin{cases} v(t) & \text{for} \quad t \geq 0 \\ \\ 0 & \text{for} \quad t < 0. \end{cases} \tag{2.5.2}$$

Indeed from (2.5.1), we have

$$\begin{aligned} \widehat{v}(p) &= \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} e^{-i\eta t} e^{-\xi t} v(t) dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i\eta t} e^{-\xi t} \widetilde{v}(t) dt. \end{aligned}$$

Thus

$$\widehat{v}(p) = \mathcal{F}\left(e^{-\xi t} \widetilde{v}(t)\right)(\eta), \tag{2.5.3}$$

where

$$\mathcal{F}(w)(\eta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i\eta t} w(t) dt, \tag{2.5.4}$$

is the Fourier transform of $w \in \mathcal{D}(-\infty, +\infty)$ and

$$w(t) = \mathcal{F}^{-1}\left(\mathcal{F}(w)\right)(t) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{+\infty} e^{i\eta t} \mathcal{F}(w)(\eta) d\eta \tag{2.5.5}$$

is the inverse Fourier transform of $\mathcal{F}(w)$.

Given a function $v \in \mathcal{D}(0, +\infty)$, it is easy to show by integration by parts that the Laplace transform of the derivative $\frac{d^k v}{dt^k}$ is given by the relation

$$\mathcal{L}\left(\frac{d^k v}{dt^k}\right)(p) = p^k \mathcal{L}(v)(p) \quad \text{for} \;\; k \in \mathbb{N}. \tag{2.5.6}$$

If another function $w \in \mathcal{D}(0, +\infty)$ is considered, we have for $\xi \in \mathbb{R}$ the Parseval identity

$$\int_0^{+\infty} v(t)w(t)e^{-2\xi t}dt = \int_{-\infty}^{+\infty} \widehat{v}(\xi + i\eta) \cdot \overline{\widehat{w}}(\xi + i\eta)d\eta, \tag{2.5.7}$$

which implies that the Laplace transform satisfies the relation

$$\left(\int_0^{+\infty} |v(t)e^{-\xi t}|^2 dt\right)^{\frac{1}{2}} = \left(\int_{-\infty}^{+\infty} |\widehat{v}(\xi + i\eta)|^2 d\eta\right)^{\frac{1}{2}}. \tag{2.5.8}$$

Furthermore, we have

$$\int_0^{+\infty} \widehat{v}(\xi + i\eta)w(\eta)d\eta = \int_0^{+\infty} e^{-\xi t}v(t)\overline{\mathcal{F}(w)}(t)dt. \tag{2.5.9}$$

It is clear that the Laplace transform of a function $v \in L^1(0, \infty)$ is well-defined by the integral (2.5.1) whenever the condition

$$\xi \geq 0, \tag{2.5.10}$$

is satisfied.

Theorem 2.5.1. *Let $g(t) \in L^2(-\infty, +\infty)$ have its support in the unbounded interval $I_\alpha = (-\infty, \alpha)$ or $I_\alpha = (\alpha, +\infty)$ where $\alpha \in \mathbb{R}$. Then the Laplace transform*

$$\widehat{g}(p) = \frac{1}{\sqrt{2\pi}} \int_{I_\alpha} e^{-pt} g(t)dt,$$

is defined for $Re(p) = \xi > 0$ if $I_\alpha = (\alpha, +\infty)$ or for $Re(p) = \xi < 0$ if $I_\alpha = (-\infty, \alpha)$.

Furthermore, $\widehat{g}(p)$ is a holomorphic function in the complex region

$$\mathbb{C}_\alpha = \begin{cases} p;\ \ \xi > 0 & if \quad I_\alpha = (\alpha, +\infty) \\ \\ p;\ \ \xi < 0 & if \quad I_\alpha = (-\infty, \alpha) \end{cases}$$

such that, for $p \in \mathbb{C}_\alpha$ with a fixed ξ, the function $\eta \to \widehat{g}(\xi + i\eta)$ is of class $L^2(-\infty, +\infty)$ and satisfies the relation

$$\int_{-\infty}^{+\infty} |\widehat{g}(\xi + i\eta)|^2 d\eta \leq e^{-2\xi\alpha} \int_{I_\alpha} |g(t)|^2 dt.$$

Proof. We prove the theorem for the case when $I_\alpha = (\alpha, +\infty)$, the situation $I_\alpha = (-\infty, \alpha)$ being analogue. We show that for $\xi > 0$, the function $t \to e^{-\xi t} g(t)$ is of class $L^1(\alpha, +\infty)$. Indeed, we have

$$\begin{aligned} |\widehat{g}(\xi + i\eta)| &= \frac{1}{\sqrt{2\pi}} \left| \int_\alpha^{+\infty} e^{-i\eta t} e^{-\xi t} g(t) dt \right| \\ &\leq \frac{1}{\sqrt{2\pi}} \int_\alpha^{+\infty} e^{-\xi t} |g(t)| dt \\ &\leq \frac{1}{\sqrt{2\pi}} \left(\int_\alpha^{+\infty} e^{-2\xi t} dt \right)^{\frac{1}{2}} \left(\int_\alpha^{+\infty} |g(t)|^2 dt \right)^{\frac{1}{2}}; \end{aligned}$$

where the previous inequality is due to Cauchy Schwarz inequality. This shows that $\widehat{g}(\xi + i\eta)$ is defined for $\xi > 0$ and also holomorphic by differentiation under the sum symbol.

On the other hand Plancherel-Parseval theorem yields for $\xi > 0$

$$\begin{aligned} \int_{-\infty}^{+\infty} |\widehat{g}(\xi + i\eta)|^2 d\eta &= \int_\alpha^{+\infty} |e^{-\xi t} g(t)|^2 dt \\ &= \int_\alpha^{+\infty} e^{-2\xi t} |g(t)|^2 dt \\ &\leq e^{-2\xi\alpha} \int_\alpha^{+\infty} |g(t)|^2 dt. \end{aligned}$$

□

2.5.2 Laplace transform of distributions

We want to define the Laplace transform of more general objects; namely, distributions in such a way that properties (2.5.6) and (2.5.8) remain valid. However, since the space $\mathcal{D}(\mathbb{R})$ is not invariant under the Fourier transform, we use Schwartz [61] space of test functions $\mathcal{S}(\mathbb{R})$ introduced in Definition 2.3.5. The estimate (2.3.1) guarantees that the Fourier transform of $\phi \in \mathcal{S}(\mathbb{R})$ is well-defined by the relation (2.5.4). More importantly, we have the following result.

Theorem 2.5.2. ([*21*])
The Fourier transform $\mathcal{F}$ is an isometric isomorphism, (with inverse $\mathcal{F}^{-1}$ given in (2.5.5)) from $\mathcal{S}(\mathbb{R})$ onto $\mathcal{S}(\mathbb{R})$ when $\mathcal{S}(\mathbb{R})$ is equipped with the $L^2(\mathbb{R})$-norm.

Motivated by the relations (2.5.3) and (2.5.9), we give the following definition:

Definition 2.5.3. ([*21*])
For a tempered distribution $T \in \mathcal{S}'(\mathbb{R})$, its Fourier transform is the tempered distribution denoted by $\mathcal{F}(T)$ and given by

$$\langle \mathcal{F}(T), \ \varphi \rangle_{\mathcal{S}'(\mathbb{R}_\eta)\times\mathcal{S}(\mathbb{R}_\eta)} = \langle T, \ \mathcal{F}(\varphi) \rangle_{\mathcal{S}'(\mathbb{R}_t)\times\mathcal{S}(\mathbb{R}_t)} \quad \forall \ \varphi \in \mathcal{S}(\mathbb{R}_\eta). \tag{2.5.11}$$

Here and after, the notation $\mathbb{R}_t$ means that distributions and test functions are considered with the argument t.

Remark 2.5.4. *Note that Definition 2.5.3 does not make sense for an arbitrary distribution $T \in \mathcal{D}'(\mathbb{R})$ in view of the fact that $\mathcal{F}(\varphi) \notin \mathcal{D}(\mathbb{R})$ for $\varphi \in \mathcal{D}(\mathbb{R})$. Therefore we had to use the largest space of test functions $\mathcal{S}(\mathbb{R})$ into which $\mathcal{D}(\mathbb{R})$ is densely and continuously embedded in order for Definition 2.5.3 to work for the small space $\mathcal{S}'(\mathbb{R})$ of tempered distributions which is densely and continuously embedded in $\mathcal{D}'(\mathbb{R})$.*

One of the important properties of the Fourier transform of distributions we shall need in this study is the Fourier transform of the derivative with respect to the time t. For $T \in \mathcal{S}'(\mathbb{R})$ and any non-negative integer n, we have

$$\mathcal{F}\frac{d^nT}{dt^n} = (i\eta)^n \mathcal{F}(T) \quad \in \mathcal{S}'(\mathbb{R}_\eta). \tag{2.5.12}$$

Indeed if $\varphi \in \mathcal{S}(\mathbb{R}_\eta)$, we have

$$\begin{aligned}
\left\langle \mathcal{F}(\frac{d^nT}{dt^n})(\eta),\ \varphi \right\rangle_{\mathcal{S}'(\mathbb{R}_\eta)\times\mathcal{S}(\mathbb{R}_\eta)} &= \left\langle \frac{d^nT}{dt^n},\ \mathcal{F}(\varphi) \right\rangle_{\mathcal{S}'(\mathbb{R}_t)\times\mathcal{S}(\mathbb{R}_t)}, \quad \text{by (2.5.11)} \\
&= (-1)^n \left\langle T,\ \frac{d^n}{dt^n}(\mathcal{F}(\varphi)) \right\rangle_{\mathcal{S}'(\mathbb{R}_t)\times\mathcal{S}(\mathbb{R}_t)}, \quad \text{by Definition 2.3.8, which} \\
&\qquad \text{is the same for tempered distributions} \\
&= (-1)^n \left\langle T,\ (i\eta)^n \mathcal{F}(\varphi) \right\rangle_{\mathcal{S}'(\mathbb{R}_t)\times\mathcal{S}(\mathbb{R}_t)}, \quad \text{by the properties of Fourier} \\
&\qquad \text{transform of usual functions} \\
&= \left\langle (i\eta)^n \mathcal{F}(T),\ \varphi \right\rangle_{\mathcal{S}'(\mathbb{R}_\eta)\times\mathcal{S}(\mathbb{R}_\eta)} \quad \text{by (2.5.11).}
\end{aligned}$$

With the above in mind, we are led to study the subspace $\mathcal{D}'_+(\mathbb{R})$ of $\mathcal{D}'(\mathbb{R})$ consisting of distributions T with support limited to the left. i.e. $supp(T) \subset [\alpha, +\infty) \quad \alpha \in \mathbb{R}$. Notice that distributions $\mathcal{D}'_+(\mathbb{R})$ are tested against functions φ in the space $\mathcal{D}_-(\mathbb{R})$ where $\varphi \in \mathcal{D}(\mathbb{R})$ is such that $supp(\varphi) \subset (-\infty, \beta], \quad \beta \in \mathbb{R}$. (The spaces $\mathcal{D}'_-(\mathbb{R})$ and $\mathcal{D}_+(\mathbb{R})$ are defined analogously). For $T \in \mathcal{D}'_+(\mathbb{R})$ we want to connect its Laplace transform to the Fourier transform of distributions via the analogue (2.5.3) and (2.5.11). To investigate this connection, we consider an important set introduced in [22].

Definition 2.5.5. ([22])
With a distribution $T \in \mathcal{D}'(\mathbb{R}_t)$, we associate the set I_T of real numbers given by

$$I_T = \left\{ \xi \in \mathbb{R} : \quad e^{-\xi t} T \in \mathcal{S}'(\mathbb{R}) \right\}. \tag{2.5.13}$$

The properties of the set I_T are summarized in the following result:

Proposition 2.5.6. ([22])

1. *For $T \in \mathcal{D}'(\mathbb{R})$, I_T is a convex set which may be empty;*
2. *If $T \in \mathcal{D}'_+(\mathbb{R})$ and if $I_T \neq \emptyset$, then $I_T = \mathbb{R}$ or $[\xi_0, +\infty)$ with $\xi_0 \in \mathbb{R}$.*

The next proposition specifies some useful properties of tempered distributions associated with $T \in \mathcal{D}'(\mathbb{R})$ and I_T.

Proposition 2.5.7. ([22]).
Let $T \in \mathcal{D}'(\mathbb{R})$. Denote by $int(I_T)$ the interior of I_T and suppose that it is non-empty. Then:

1. *For all $\xi \in int(I_T)$ the Fourier transform $\mathcal{F}\left(e^{-\xi t}T\right)(\eta)$ of the distribution $e^{-\xi t}T$ is a function of $\mathbb{O}_M$ where $\mathbb{O}_M$ is the space of C^∞ functions which together with all their derivatives are slowly increasing at infinity. That is, $v \in \mathbb{O}_M \Leftrightarrow v \in C^\infty(\mathbb{R}), \forall j \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that*

$$\lim_{|x|\to\infty} |x|^{-N}|v^{(j)}(x)| = 0,$$

2. *The function*

$$p = \xi + i\eta \to \mathcal{F}\left(e^{-\xi t}T\right)(\eta)$$

is holomorphic in the band $int(I_T) \times \mathbb{R}$.

In view of Proposition, 2.5.6 and 2.5.7, we can define the Laplace transform of a distribution as follows:

Definition 2.5.8. ([22]).
Let $T \in \mathcal{D}'(\mathbb{R})$ be such that $int(I_T) \neq \emptyset$. The holomorphic function denoted by $\mathcal{L}(T) : p \longrightarrow \mathcal{L}(T)(p)$ and defined for $p \in int(I_T) \times \mathbb{R}$ by

$$\widehat{T}(p) \equiv \mathcal{L}(T)(p) := \mathcal{F}\left(e^{-\xi t}T\right)(\eta) \tag{2.5.14}$$

is called the Laplace transform of the distribution $T \in \mathcal{D}'(\mathbb{R})$.

As mentioned earlier, the properties (2.5.6) and (2.5.9) are valid in this general setting of Definition 2.5.8 as shown below. For $T \in \mathcal{D}'(\mathbb{R})$ with $I_T \neq \emptyset$, and $\varphi \in \mathcal{S}(\mathbb{R})$, we have

$$\begin{aligned}
\langle \mathcal{L}(T)(p), \varphi \rangle_{\mathcal{S}'(\mathbb{R}_\eta)\times\mathcal{S}(\mathbb{R}_\eta)} &= \left\langle \mathcal{F}(e^{-\xi t}T)(\eta), \varphi \right\rangle_{\mathcal{S}'(\mathbb{R}_\eta)\times\mathcal{S}(\mathbb{R}_\eta)} \text{ by (2.5.14)} \\
&= \left\langle e^{-\xi t}T, \mathcal{F}(\varphi) \right\rangle_{\mathcal{S}'(\mathbb{R}_t)\times\mathcal{S}(\mathbb{R}_t)} \text{ by (2.5.11)} \\
&= \langle T, \mathcal{L}(\varphi)(p) \rangle_{\mathcal{S}'(\mathbb{R}_t)\times\mathcal{S}(\mathbb{R}_t)} .
\end{aligned}$$

This is the analogue of (2.5.9). On the other hand, by (2.5.11)

$$\begin{aligned}\left\langle \mathcal{L}\left(\frac{d^kT}{dt^k}\right)(p),\ \ \varphi\right\rangle &= \left\langle \mathcal{F}\left(e^{-\xi t}\frac{d^kT}{dt^k}\right)(\eta),\ \varphi\right\rangle_{\mathcal{S}'(\mathbb{R}_\eta)\times\mathcal{S}(\mathbb{R}_\eta)} \\ &= \left\langle e^{-\xi t}\frac{d^kT}{dt^k},\ \ \mathcal{F}(\varphi)\right\rangle_{\mathcal{S}'(\mathbb{R}_t)\times\mathcal{S}(\mathbb{R}_t)}. \end{aligned} \tag{2.5.15}$$

Since

$$\frac{d}{dt}\left(e^{-\xi t}T\right) = -\xi e^{-\xi t}T + e^{-\xi t}\frac{dT}{dt},$$

then (2.5.15), for $k = 1$, yields

$$\begin{aligned}\left\langle \mathcal{L}\left(\frac{dT}{dt}\right)(p),\ \varphi\right\rangle &= \left\langle \xi e^{-\xi t}T + \frac{d}{dt}\left(e^{-\xi t}T\right),\ \mathcal{F}(\varphi)\right\rangle_{\mathcal{S}'(\mathbb{R}_t)\times\mathcal{S}(\mathbb{R}_t)} \\ &= \left\langle \xi\mathcal{F}\left(e^{-\xi t}T\right)(\eta) + \mathcal{F}\left(\frac{d}{dt}\left(e^{-\xi t}T\right)\right)(\eta),\ \ \varphi\right\rangle \\ &= \left\langle \xi\mathcal{F}\left(e^{-\xi t}T\right)(\eta) + i\eta\mathcal{F}\left(e^{-\xi t}T\right)(\eta),\quad \varphi\right\rangle \ \text{ by (2.5.12)} \\ &= \left\langle p\mathcal{F}\left(e^{-\xi t}T\right)(\eta),\quad \varphi\right\rangle \\ &= \left\langle p\mathcal{L}\left(T\right)(p),\ \ \varphi\right\rangle. \end{aligned}$$

Hence by induction on $k \in \mathbb{N}$, we have $\left\langle \mathcal{L}\left(\frac{d^kT}{dt^k}\right)(p),\ \ \varphi\right\rangle = \left\langle p^k\mathcal{L}(T)(p),\ \varphi\right\rangle$, which means that

$$\mathcal{L}\left(\frac{d^kT}{dt^k}\right)(p) = p^k\mathcal{L}(T)(p) \ \text{ in } \ \mathcal{S}'(\mathbb{R}_\eta). \tag{2.5.16}$$

Our aim at this stage is to characterize Laplace transform of distributions in $L^2(0, +\infty)$. This is achievable first by considering the next definition.

Definition 2.5.9. ([*67*]) *(Hardy-Lebesgue Space)*

The Hardy-Lebesgue space denoted by $H^2(0)$ is defined as the set of functions $V : p \longrightarrow V(p)$ from the half complex plane

$$\mathbb{C}_+ = \{p = \xi + i\eta \in \mathbb{C}, \;\; \xi > 0\}$$

into the space $\mathbb{C}$ such that the following two conditions are satisfied:

1. *The function $V(p)$ is holomorphic for $\xi > 0$;*

2. *For each $\xi > 0$, the function $\eta \longrightarrow V(\xi + i\eta)$ is of class $L^2(-\infty, +\infty)$ such that*

$$\sup_{\xi>0} \left(\int_{-\infty}^{\infty} |V(\xi + i\eta)|^2 d\eta \right) < +\infty.$$

Proposition 2.5.10. ([*67*])
Let $v(t) \in L^2(0, +\infty)$. Then its Laplace transform $\widehat{v}(p)$ exists for $\xi \geq 0$ and $\widehat{v}(p) \in H^2(0)$.

Proof. Let $\xi \geq 0$. We denote by $\widetilde{v}(t)$ the extension of $v(t)$ by 0 outside $(0, +\infty)$ given in (2.5.2). Then, the function $t \in \mathbb{R} \longrightarrow e^{-\xi t}\widetilde{v}(t)$ is of class $L^2(-\infty, +\infty)$ and is therefore a tempered distribution. In other words $\xi \in I_{\widetilde{v}}$; in fact $[0, +\infty) \subset I_{\widetilde{v}}$ and thus $intI_{\widetilde{v}} \neq 0$. Thus, in view of Definition 2.5.8, $\widehat{v}(p)$ is well-defined for $p = \xi + i\eta$ with $\xi \geq 0$. The holomorphic property of $p \longrightarrow \widehat{v}(p)$ follows from Proposition 2.5.6 and Definition 2.5.8.

For condition 2 we have using the extension to L^2 of (2.5.3) and of the Parseval identity (2.5.8)

$$\begin{aligned}
\int_{-\infty}^{\infty} |\widehat{v}(\xi + i\eta)|^2 d\eta &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |\mathcal{F}(\widetilde{v}(t)e^{-t\xi})(\eta)|^2 d\eta \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |\widetilde{v}(t)e^{-t\xi}|^2 dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} |v(t)|^2 e^{-2t\xi} dt \\
&\leq \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} |v(t)|^2 \quad \text{since} \quad \xi > 0.
\end{aligned}$$

Hence

$$\sup_{\xi>0}\left(\int_{-\infty}^{\infty}|\widehat{v}(\xi+i\eta)|^2d\eta\right)\leq\frac{1}{\sqrt{2\pi}}\int_0^{\infty}|v(t)|^2dt<+\infty. \tag{2.5.17}$$

□

Theorem 2.5.11. ([67])*(Paley-Wiener Theorem)*

Let $V(\xi+i\eta)\in H^2(0)$. Then the boundary function $V(i\eta)$ of $V(\xi+i\eta)$ exists in $L^2(-\infty,+\infty)$ in the sense that

$$\lim_{\xi\to 0}\int_{-\infty}^{+\infty}|V(i\eta)-V(\xi+i\eta)|^2d\eta=0. \tag{2.5.18}$$

Furthermore, there exists a function $t\longrightarrow v(t)$ of class $L^2(-\infty,+\infty)$ such that $v(t)=0$ for $t<0$ and $V(\xi+i\eta)$ with $\xi>0$, is the Laplace transform of $v(t)$ at $p=\xi+i\eta$.

2.5.3 Laplace transform of vector-valued distributions

After the definition of the Laplace transform of scalar distributions, we extend this definition to vector-valued distributions. We denote by X the Hilbert space, with norm $\|\cdot\|_X$, in which the vector distributions take values.

Definition 2.5.12. ([22])

1. *We denote by $\mathcal{D}(X)$, the space of functions $t\to f(t)$ from $\mathbb{R}$ into X which are of class C^∞ and which have compact support.*
   $\mathcal{D}(X)$ is equipped with a pseudo-topology according to which a sequence (φ_j) converges to φ whenever we have the following conditions:
   - *there exists a compact set K of $\mathbb{R}$, such that*
     $$supp(\varphi_j)\subset K,\ \ \forall j\geq 1,\ \ supp(\varphi)\subset K$$
   - *$\varphi_j^{(n)}$ converges to $\varphi^{(n)}$ in X uniformly on K, for every $n\in\mathbb{N}$.*
2. *We denote by $\mathcal{D}_+(X)$ the subspace of $\mathcal{D}(X)$ consisting of vector-valued functions with support limited to the left i.e. contained in some $[\alpha,+\infty)$. The space $\mathcal{D}_+(X)$ is equipped*

with a pseudo-topology in which a sequence of functions $\varphi_j \in \mathcal{D}_+(X)$ converges to φ in $\mathcal{D}_+(X)$ if

- *the functions φ_j and φ are zero for $t_0 \leq t$, where t_0 is independent of j*
- *$\varphi_j^{(n)}$ converges uniformly to $\varphi^{(n)}$ in X over all compact set in $[\alpha, +\infty[$.*

Remark 2.5.13. *The corresponding space denoted by $\mathcal{D}_-(X)$ is the subspace of $\mathcal{D}(X)$ consisting of vector-valued functions with support limited to the right i.e. contained in some $(-\infty, \alpha])$. $\mathcal{D}_-(X)$ also has a pseudo-topology similar to the one in Definition 2.5.12(2).*
We recall that to avoid confusion, we will, whenever it is necessary, write $\mathbb{R}_t$ to emphasize that the argument of the functions $\varphi \in \mathcal{D}(X)$ is "t". We also would like to emphasize that, if $X = \mathbb{C}$ or $\mathbb{R}$, then the spaces described above will be written as follows:

$$\mathcal{D}(X) = \mathcal{D}, \ \ \mathcal{D}_+(X) = \mathcal{D}_+ \ \ \textit{and} \ \ \mathcal{D}_-(X) = \mathcal{D}_-$$

Definition 2.5.14. ([*22*])
We denote by $\mathcal{D}'(X)$ the space of distributions over $\mathbb{R}_t$, with values in X, defined by

$$\mathcal{D}'(X) := \mathcal{L}(\mathcal{D}; X)$$

where $\mathcal{L}(\mathcal{D}; X)$ is the space of continuous linear mapping from $\mathcal{D}$ into X.

The space $\mathcal{D}'(X)$ is equipped with the topology of uniform convergence over bounded subsets of $\mathcal{D}$. To emphasize on this, we denote $\mathcal{L}(\mathcal{D}, X)$ by $\mathcal{L}_\sigma(\mathcal{D}, X)$ where σ is the collection of bounded subsets of $\mathcal{D}$. Given the importance of this topology in what follows, we spend some space and time to make it more explicit. We do this by considering the following useful concepts of the space $\mathcal{D}(\mathbb{R})$ found in [15],[26], [27], [61] and [62].

Definition 2.5.15. ([*15*], [*26*])
Let $\mathbb{A} \subset \mathcal{D}(\mathbb{R})$. The subset $\mathbb{A}$ is said to be bounded if there exists a compact subset $K \subset \mathbb{R}$ such that

1. $\forall \, \varphi \in \mathbb{A}$,

$$supp(\varphi) \subset K, \tag{2.5.19}$$

2. $\forall m \in \mathbb{N}$, *there exists* $M_m > 0$, *such that*

$$\sup_{x\in\mathbb{R}} \left|\frac{d^p\varphi(x)}{dx^p}\right| \leq M_m, \ \forall \ p \leq m. \tag{2.5.20}$$

Instead of the pseudo-topology of $\mathcal{D}(\mathbb{R})$ given in Definition 2.3.1, we want now to specify Schwartz canonical topology of $\mathcal{D}(\mathbb{R})$. To this end, let us take $(K_n)_{n\geq 1}$ to be an increasing sequence of compact sets in $\mathbb{R}$ such that

$$\cup_n K_n = \mathbb{R}.$$

For each compact set K_n, we denote by $\mathcal{D}_{K_n}(\mathbb{R})$ the subspace of $\mathcal{D}(\mathbb{R})$ that consists of functions

$$\rho \in C_0^\infty(\mathbb{R}) \quad \text{such that} \quad supp(\rho) \subseteq K_n.$$

On each $\mathcal{D}_{K_n}(\mathbb{R})$, we introduce the sequence of semi-norms $(P_{K_{n,m}})_{m\geq 1}$ defined by

$$P_{K_{n,m}}(\rho) = \sup_{x\in K_n}\left|\frac{d^m}{dx^m}\rho(x)\right|.$$

By a standard procedure [26, 27], the sequence $(P_{K_{n,m}})_{m\geq 1}$ generates on $\mathcal{D}_{K_n}(\mathbb{R})$ a structure of locally convex topological vector space, with topology denoted by $\mathcal{T}_{K_n}$. From the same references, it is known that a fundamental system of neighborhoods of 0 for the topology $\mathcal{T}_{K_n}$ consists of the sets

$$V(m,\epsilon) := \left\{\rho \in \mathcal{D}_{K_n}(\mathbb{R}) : \sup_{\substack{x\in K_n \\ 0\leq j\leq m}} \left|\frac{d^j}{dx^j}\rho(x)\right| \leq \epsilon\right\}, \ \ \epsilon > 0, \ \ m \in \mathbb{N}. \tag{2.5.21}$$

It is clear that

$$\mathcal{D}(\mathbb{R}) = \cup_{n=1}^\infty \mathcal{D}_{K_n}(\mathbb{R}). \tag{2.5.22}$$

The said Schwartz canonical topology $\mathcal{T}$ of $\mathcal{D}(\mathbb{R})$ is the inductive limit of the topologies $(\mathcal{T}_{K_n})_{n\geq 1}$. That is, $\mathcal{T}$ is the largest but not discrete locally convex topology on $\mathcal{D}(\mathbb{R})$ that makes all the embeddings $\mathcal{D}_{K_n}(\mathbb{R}) \hookrightarrow \mathcal{D}(\mathbb{R})$ continuous. Thus V is a convex neighborhood of 0 in $\mathcal{D}(\mathbb{R})$ if and only if $V \cap \mathcal{D}_{K_n}(\mathbb{R})$ is a neighborhood of 0 in $\mathcal{D}_{K_n}(\mathbb{R})$ for every n.

The topology $\mathcal{T}$ of $\mathcal{D}(\mathbb{R})$ is generated by a family of semi-norms obtained as follows from an increasing sequence of non-negative integers $(m_j)_{j\geq 0}$ where $m_j \to \infty$ as $j \to \infty$ and a decreasing sequence of positive real numbers $(\epsilon_j)_{j\geq 0}$ such that $\epsilon_j \to 0$:

$$N(\{m_j\},\{\epsilon_j\})(\rho) := \sup_j \left(\sup_{\substack{|x|\geq j \\ 0\leq \alpha \leq m_j}} \frac{|\frac{d^\alpha \rho(x)}{dx^\alpha}|}{\epsilon_j} \right). \tag{2.5.23}$$

In line with (2.5.21) we introduce the set

$$V(\{m_j\},\{\epsilon_j\}) := \left\{ \rho \in \mathcal{D}(\mathbb{R}) : \forall j \ |x| > j \text{ and } \ 0 \leq \alpha \leq m_j \quad \left| \frac{d^\alpha \rho(x)}{dx^\alpha} \right| \leq \epsilon_j, \right\}$$

which forms a fundamental system of neighborhoods of 0 in $\mathcal{D}(\mathbb{R})$ when $\{m_j\}$ and $\{\epsilon_j\}$ vary arbitrary.

Our next task is to be more explicit about the topology of $\mathcal{D}'(X)$ given in Definition 2.5.14. To this end, let Y be a locally convex topological vector space with topology generated in a standard way ([26, 27]) by a family of semi-norms

$$\mathcal{W}_I = \{q_\alpha, \ \alpha \in I\}.$$

We define $\mathcal{L}(\mathcal{D}, Y)$ as the space of linear continuous operators from $\mathcal{D}$ into Y. To understand the topology of $\mathcal{L}(\mathcal{D}, Y)$, we denote by σ the collection of all bounded subset of $\mathcal{D}(\mathbb{R})$ as defined in Definition 2.5.15. With each $\mathbb{A} \in \sigma$ and $\alpha \in I$, we associate a semi-norm $q_{\alpha,\mathbb{A}}$ on $\mathcal{L}(\mathcal{D}, Y)$ defined by

$$q_{\alpha,\mathbb{A}}(T) = \sup_{\rho\in\mathbb{A}} q_\alpha(T(\rho)).$$

The family of semi-norms

$$\mathcal{W}_{I,\sigma} = \{q_{\alpha,\mathbb{A}} : \alpha \in I, \ \ \mathbb{A} \in \sigma\} \tag{2.5.24}$$

defines on $\mathcal{L}(\mathcal{D}, Y)$ a locally convex (vector) topology called σ-topology. Thus again the notation $\mathcal{L}_\sigma(\mathcal{D}, Y)$.

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Definition 2.5.16. *If $\mathcal{Y}_0$ denotes the collection of balanced neighborhoods of 0 for the topology of Y, then a fundamental system of neighborhood of 0 for the σ-topology of $\mathcal{L}(\mathcal{D}, Y)$ is given by*

$$\mathcal{B} = \{V(A, M) \subset \mathcal{L}(\mathcal{D}, Y) : \forall A \in \sigma_f, \ \forall M \in \mathcal{Y}_0\}$$

where σ_f is the collection of finite union of bounded set in σ and

$$V(A, M) = \{T \in \mathcal{L}(\mathcal{D}, Y) : T(A) \subset M\}.$$

We recall that all these concepts can be found in [26, 27]).

Proposition 2.5.17. *Let (T_j) be a sequence in $\mathcal{L}_\sigma(\mathcal{D}, Y)$ and let $T \in \mathcal{L}_\sigma(\mathcal{D}, Y)$ where the local convex topology of Y is generated by a filtered family $\mathcal{W} = \{q_\alpha, \ \alpha \in I\}$ of semi-norms. Then the following statements are equivalent:*

1. *The sequence (T_j) converges to T in $\mathcal{L}_\sigma(\mathcal{D}, Y)$. That is for any neighborhood V of 0 in $\mathcal{L}_\sigma(\mathcal{D}, Y)$, there exists an integer $j_0 = j_0(V)$ such that $T_j - T \in V$ whenever $j \geq j_0$.*

2. *The sequence (T_j) converges to T uniformly on any bounded subset $\mathbb{A} \in \sigma$. That is for any neighborhood $\mathcal{W}$ of 0 in Y and any $\mathbb{A} \in \sigma$, there exists $j_0 = j_0(\mathbb{A}, \mathcal{W})$ such that*

$$T_j(\rho) - T(\rho) \in \mathcal{W} \ \text{ for any } \ \rho \in \mathbb{A} \ \text{ whenever } \ j \geq j_0.$$

3. *For any $\alpha \in I$, and $\mathbb{A} \in \sigma$ the sequence of real-valued numbers*

$$q_\alpha(T_j(\rho) - T(\rho)) \ \text{ converges to } \ 0 \ \text{ uniformly on } \ \mathbb{A}.$$

Proof. To prove that (1) implies (2), let $\mathbb{A} \in \sigma$ and $\mathcal{W}$ be a neighborhood of 0 in Y. Then the set $V(\mathbb{A}, \mathcal{W})$ introduced in Definition 2.5.16 is a neighborhood of 0 in $\mathcal{L}_\sigma(\mathcal{D}, Y)$. Since by assumption (1), $T_j \to T$ in $\mathcal{L}_\sigma(\mathcal{D}, Y)$, there exists $j_0 = j_0(\mathbb{A}, \mathcal{W})$ such that $T_j - T \in V(\mathbb{A}, \mathcal{W})$ for $j \geq j_0$. By definition of $V(\mathbb{A}, \mathcal{W})$, we have $T_j(\rho) - T(\rho) \in \mathcal{W}, \ \rho \in \mathbb{A}$, for $j \geq j_0$. This proves (2).

Assume that (2) is true and let us prove (3). Fix $\epsilon > 0, \ \alpha \in I$ and $\mathbb{A} \in \sigma$ so that the set $\mathcal{W} = \{y \in Y; \ q_\alpha(\rho) < \epsilon\}$ in a neighborhood of 0 in Y. Using (2), we can find $j_0 = j_0(\epsilon, \alpha, \mathbb{A})$ such that $T_j(\rho) - T(\rho) \in \mathcal{W}$ for any $\rho \in \mathbb{A}$ and $j \geq j_0$. By definition of $\mathcal{W}$, we have $q_\alpha(T_j(\rho) - T(\rho)) < \epsilon$ for every $\rho \in \mathbb{A}$ whenever $j \geq j_0$ where j_0 does not depend on ρ. This proves (3).

To conclude, we assume that (3) holds and we want to prove (1). To this end let $\mathcal{V}$ be a neighborhood of 0 in $\mathcal{L}_\sigma(\mathcal{D}, Y)$. By the definition of the fundamental system of neighborhood of 0 given in Definition 2.5.16, there exist $\mathbb{A}_k \in \sigma$, $1 \leq k \leq s$, and $\mathcal{W}$ a neighborhood of 0 in Y such that

$$V(\cup_{k=1}^{s}\mathbb{A}_k,\ \mathcal{W}) \subset \mathcal{V}.$$

It is easy to show that

$$V(\cup_{k=1}^{s}\mathbb{A}_k,\ \mathcal{W}) = \cap_{k=1}^{s} V(\mathbb{A}_k,\ \mathcal{W}).$$

On the other hand since the topology of Y is generated by the filtered family $\{q_\alpha,\ \alpha \in I\}$ of semi-norms there exists $\alpha_0 \in I$ and $\epsilon > 0$ such that the ball

$$\mathcal{W}_{\epsilon_0} := \{y \in Y; q_{\alpha_0}(y) < \epsilon_0\} \subset \mathcal{W}.$$

Applying the assumption in (3) to α_0, ϵ_0 and each $1 \leq k \leq s$, there exists an integer j_k such that $q_{\alpha_0}(T_j(\rho) - T(\rho)) < \epsilon_0$ for any $\rho \in A_k$ and $j \geq j_k$. Take $j_0 = j_1 + .. \cdot + j_k$. Then for $j \geq j_0$, we have $q_{\alpha_0}(T_j(\rho) - T(\rho)) < \epsilon_0$ for $\rho \in \cup_{k=1}^{s}\mathbb{A}_k$. This means that

$$T_j - T \in V(\cup_{k=1}^{s}\mathbb{A}_k,\ \mathcal{W}_{\epsilon_0}) \subset V(\cup_{k=1}^{s}\mathbb{A}_k,\ \mathcal{W}) \subset \mathcal{V}$$

for $j \geq j_0$. This proves (1). □

Remark 2.5.18. *Proposition 2.5.17 motivates the fact that the σ-topology of $\mathcal{L}_\sigma(\mathcal{D}, Y)$ is also called the topology of uniform convergence on bounded subsets of $\mathcal{D}$.*

The material collected until now enable us to deal with the particular case of the space $\mathcal{L}(\mathcal{D}, \mathcal{D}) \equiv \mathcal{L}_\sigma(\mathcal{D}, \mathcal{D})$ where $Y = \mathcal{D}(\mathbb{R})$. With the family of semi-norms $N(\{m_j\}, \{\epsilon_j\})$ in (2.5.23) that generate the topology of $\mathcal{D}$, we associate the family of semi-norms $N_{\mathbb{A}}(\{m_j\}, \{\epsilon_j\})$, $\mathbb{A} \in \sigma$, on $\mathcal{L}(\mathcal{D}, \mathcal{D})$ defined by

$$N_{\mathbb{A}}(\{m_j\}, \{\epsilon_j\})(T) = \sup_{\rho \in \mathbb{A}} N(\{m_j\}, \{\epsilon_j\})(T(\rho)).$$

By the approach followed ealier in the general case, the family of semi-norms $N_{\mathbb{A}}(\{m_j\}, \{\epsilon_j\})$,

$\mathbb{A} \in \sigma$, generate the σ-topology of the space $\mathcal{L}(\mathcal{D}, \mathcal{D})$, which as shown in Proposition 2.5.17 and Remark 2.5.18 is the topology of uniform convergence on bounded subsets of $\mathcal{D}$.

With the above useful concepts on the space $\mathcal{D}(\mathbb{R})$, we return to the initial space $\mathcal{D}'(X)$. We also consider the notation $\mathcal{D}'_+(X)$ and $\mathcal{D}'_-(X)$ to represent the subspaces of $\mathcal{D}'(X)$ consisting of distributions with supports limited to the left and right, respectively:

$$\mathcal{D}'_+(X) := \mathcal{L}(\mathcal{D}_-; X), \quad \mathcal{D}'_-(X) := \mathcal{L}(\mathcal{D}_+; X).$$

The vector distributions in $\mathcal{D}'(X)$ have generally a very complex structure. That is why we approximate them by distributions that are relatively easy to work with. The first step is to define the tensor product of a distribution T with v.

Definition 2.5.19. ([*22*])
Given $T \in \mathcal{D}'$ and $v \in X$ we defined $T \otimes v \in \mathcal{D}'(X)$ the tensor product of T and v by

$$(T \otimes v)(\varphi) = \langle T, \varphi \rangle_{\mathcal{D}' \times \mathcal{D}} \, v, \quad \varphi \in \mathcal{D}(\mathbb{R}). \tag{2.5.25}$$

Definition 2.5.20. ([*27*])
A linear operator $T : \mathcal{D}(\mathbb{R}) \longmapsto X$ is a finite operator, if there exists $p_1, p_2, ..., p_n \in \mathcal{D}'(\mathbb{R})$ and $g_1, g_2, ..., g_n \in X$ such that

$$T(f) = \sum_{i=1}^{n} p_i(f) g_i. \tag{2.5.26}$$

More generally, we denote by $\mathcal{D}'(\mathbb{R}) \otimes X \equiv \mathcal{D}' \otimes X$ the subspace of $\mathcal{D}'(X)$ consisting of finite operators:

$$\mathcal{D}' \otimes X = \left\{ T \in \mathcal{D}'(X), \quad T = \sum_{j=1}^{n_T} T_j \otimes v_j, \; T_j \in \mathcal{D}', \; v_j \in X \right\}. \tag{2.5.27}$$

In the same way, we could define the subspaces of $\mathcal{D}'_+(X)$ and $\mathcal{D}'_-(X)$ denoted by $\mathcal{D}'_+ \otimes X$ and $\mathcal{D}'_- \otimes X$, respectively. We are now in a position to state the main theorem of this section, on which the definition and the properties of the Laplace transform of vector-valued distributions are based.

Theorem 2.5.21. ([*22*])
The subspace $\mathcal{D}' \otimes X$ is dense in $\mathcal{D}'(X)$. Equally $\mathcal{D}'_+ \otimes X$ and $\mathcal{D}'_- \otimes X$ are dense in $\mathcal{D}'_+(X)$ and $\mathcal{D}'_-(X)$, respectively.

The proof of Theorem 2.5.21 is not straightforward. It will follow from a series of topological concepts of the space $\mathcal{D}(\mathbb{R})$ described after Definition 2.5.15 as well as on the results that we consider now.

Theorem 2.5.22. ([*61*])
The space $\mathcal{D} \equiv \mathcal{D}(\mathbb{R})$ satisfies the strict approximation property. That is, the identity operator $I \in \mathcal{L}_\sigma(\mathcal{D}(\mathbb{R}), \mathcal{D}(\mathbb{R}))$ can be approximated in $\mathcal{L}_\sigma(\mathcal{D}(\mathbb{R}), \mathcal{D}(\mathbb{R}))$ by a sequence of finite operators.

Proof. Let $(\alpha_\nu)_{\nu\geq 1}$ be a sequence in $\mathcal{D}(\mathbb{R})$ such that the sequence $(\alpha_\nu^2)_{\nu\geq 1}$ is a partition of unity of $\mathbb{R}$ sub-ordinate to the open covering $(Q_\nu)_{\nu\geq 1}$ of $\mathbb{R}$ where $Q_\nu = (-\nu, \nu)$. Thus we have

$$\sum_{\nu\geq 1} \alpha_\nu^2(x) = 1 \;\forall x \in \mathbb{R}. \tag{2.5.28}$$

Let $\psi \in C^\infty(\mathbb{R})$ be such that $supp(\psi) \subset Q_\nu$ i.e. $\psi \in \mathcal{D}_{Q_\nu}(\mathbb{R})$.

We associate with ψ the unique periodic function $\widetilde{\psi}_\nu$ of period 2ν defined by

$$\widetilde{\psi}_\nu(x) = \psi(x) \text{ if } x \in Q_\nu. \tag{2.5.29}$$

We can therefore expand $\widetilde{\psi}_\nu$ in Fourier series

$$\widetilde{\psi}_\nu(x) = \sum_{l\in\mathbb{Z}} c_{l,\nu}(\psi) e^{-i\pi l x/\nu}. \tag{2.5.30}$$

By the properties of Fourier series, the linear functional

$$\psi \rightsquigarrow c_{l,\nu}(\psi) \tag{2.5.31}$$

is continuous in the following sense of the pseudo-topology of $\mathcal{D}_{Q_\nu}(\mathbb{R})$:

If a sequence $(\psi_j)_{j\geq 1}$ in $\mathcal{D}_{Q_\nu}(\mathbb{R})$ converges to zero i.e.

$$\forall m \in \mathbb{N} \quad \frac{d^m \psi_j}{dx^m} \text{ converges to 0 uniformly on } Q_\nu,$$

then the sequence of scalars $c_{l,\nu}(\psi_j)$ converges to 0 as $j \to +\infty$.

The next step is to construct a finite operator L_k for $k \in \mathbb{N}$. To this end let $\rho \in \mathcal{D}(\mathbb{R})$ be given. By the partition of unity property (2.5.28) and by the Fourier series expansion (2.5.30), we have consecutively the following for any $x \in \mathbb{R}$:

$$\begin{aligned}\rho &= \sum_{\nu\geq 1} \alpha_\nu^2(x)\rho \\ &= \sum_{\nu\geq 1} \alpha_\nu(x)\alpha_\nu\rho \\ &= \sum_{\nu\geq 1} \alpha_\nu(x) \sum_{l\in\mathbb{Z}} c_{l,\nu}(\alpha_\nu\rho)e^{-i\pi lx/\nu} \\ &= \sum_{\nu\geq 1}\sum_{l\in\mathbb{Z}} \alpha_\nu(x)c_{l,\nu}(\alpha_\nu\rho)e^{-i\pi lx/\nu}.\end{aligned}$$

From this, we construct a finite operator L_k by the following truncation process:

$$L_k\rho := \sum_{\substack{\nu\geq 1\\ |l|<k}} \alpha_\nu(x/2\nu)c_{l,\nu}(\alpha_\nu\rho)e^{-i\pi lx/\nu}. \tag{2.5.32}$$

In view of the continuity stated in (2.5.31), the L_k is continuous from $\mathcal{D}(\mathbb{R})$ into $\mathcal{D}(\mathbb{R})$.

We now show that, for a fixed $\rho \in \mathcal{D}(\mathbb{R})$, $L_k\rho$ converges to ρ in $\mathcal{D}(\mathbb{R})$ as $k \to \infty$. Since $supp(\rho)$ is compact, there exists $k_0 \geq 1$ such that $supp\rho \subset Q_{k_0}$ and

$$\alpha_\nu\rho \equiv 0 \ \text{ for all } \ \nu \geq k_0. \tag{2.5.33}$$

Thus (2.5.32) becomes

$$L_k\rho = \sum_{\substack{\nu<k_0\\ |l|<k}} \alpha_\nu(x/2\nu)c_{l,\nu}(\alpha_\nu\rho)e^{-i\pi lx/\nu}. \tag{2.5.34}$$

Clearly, from (2.5.33), $supp(L_k\rho) \subset supp(\rho) \cap Q_{k_0}$ for all $k \geq 1$. For $k \to +\infty$, the sequence $L_k\rho$ in (2.5.34) converges uniformly on $supp(\rho) \cap Q_{k_0}$ to

$$\begin{aligned}\sum_{\substack{\nu<k_0\\ l\in\mathbb{Z}}} \alpha_\nu c_{l,\nu}(\alpha_\nu\rho)e^{-i\pi\frac{lx}{\nu}} &= \sum_{\nu<k_0} \alpha_\nu\widetilde{\alpha_\nu\rho}(x) \ \text{ by } \ (2.5.30) \\ &= \rho(x).\end{aligned} \tag{2.5.35}$$

The same thing applies by induction to the derivatives of $(L_k\rho)$.

Let now $\mathbb{A}$ be a bounded subset of $\mathcal{D}(\mathbb{R})$. By Definition 2.5.15, there exists a compact set $K \subset \mathbb{R}$ such that (2.5.19) and (2.5.20) hold. In view of (2.5.19), the argument used to prove (2.5.33) can be adapted to obtain the following: there exists $k_0 \geq 1$ such that

$$\alpha_\nu \rho \equiv 0 \;\; \forall \nu \geq k_0 \;\text{ and }\; \forall \rho \in \mathbb{A}. \tag{2.5.36}$$

Thus the sequence $L_k\rho$ converges to ρ uniformly on $\mathbb{A}$ and K in the sense that

$$\lim_{k\to\infty} \sup_{\substack{\rho\in\mathbb{A} \\ x\in K}} \left| \frac{d^m}{dx^m}\left[(L_k\rho)(x) - \rho(x)\right] \right| = 0 \;\; \forall\, m \in \mathbb{N}.$$

Thus L_k converges to the identity operator I in $\mathcal{L}_\sigma(\mathcal{D}(\mathbb{R}), \mathcal{D}(\mathbb{R}))$. □

Proof. (*Theorem* 2.5.21)
Let $T \in \mathcal{L}_\sigma(\mathcal{D}(\mathbb{R}), X)$. Let V_j defined by

$$V_j\rho = \sum_{k\leq n_j} c_{k,j}(\rho)\rho_{k,j} \;\text{ i.e }\; V_j = \sum_{k=1}^{n_j} c_{k,j} \otimes \rho_{k,j} \;\text{ with }\; c_{k,j} \in \mathcal{D}' \;\text{ and }\; \rho_{k,j} \in \mathcal{D}$$

be a sequence of finite operators that approximate I in $\mathcal{L}_\sigma(\mathcal{D}(\mathbb{R}), \mathcal{D}(\mathbb{R}))$ according to Theorem 2.5.22. By the continuity of T, the sequence of finite operators

$$T \circ V_j = \sum_{k=1}^{n_j} c_{k,j} \otimes T(\rho_{k,j}) \;\text{ converges to }\; T \text{ in }\; \mathcal{L}_\sigma(\mathcal{D}(\mathbb{R}), X).$$

This complete the proof. □

In what follows, we introduce another space of vector-valued distributions.

Definition 2.5.23. ([*22*])
We denote by $\mathcal{S}'(X)$ the space of tempered distributions over $\mathbb{R}_t$ with values in X, defined by

$$\mathcal{S}'(X) = \mathcal{L}(\mathcal{S}; X),$$

$\mathcal{S} \equiv \mathcal{S}(\mathbb{R})$ being equipped with the pseudo-topology given in Definition 2.3.6.

Remark 2.5.24. *The topologies of $\mathcal{S}(X)$ and $\mathcal{S}'(X)$ can be defined explicitly from appropriate family of semi-norms as we did for $\mathcal{D}'(X)$. For example the topology of $\mathcal{S}(\mathbb{R})$ is generated by the sequence of semi-norms*

$$d_{\alpha,\beta}(v) = \sup_{x\in\mathbb{R}} |x^{\alpha}\frac{d^{\beta}v(x)}{dx^{\beta}}| \ \ \alpha, \beta \in \mathbb{N}.$$

Note that a fundamental system of neighborhood of 0 for this topology is obtained in a standard way. Note also that the space $\mathcal{S}(\mathbb{R})$ is metrisable, through the metric

$$d(u, v) = \sum_{\alpha,\beta\geq 1} \frac{d_{\alpha,\beta}(u - v)}{1 + d_{\alpha,\beta}(u - v)},$$

in contrast to the space $\mathcal{D}(\mathbb{R})$.

Now given Y a locally convex topological space with topology generated by a family of semi-norms $W_I = \{q_\alpha, \alpha \in I\}$, the topology of the space $\mathcal{L}(\mathcal{S}, Y) \equiv \mathcal{S}'(Y)$ of linear continuous operators from $\mathcal{S}(\mathbb{R})$ into Y is generated by the family of semi-norms

$$W_{I,\sigma} = (q_{\alpha,\mathcal{A}})_{\alpha\in I,\mathcal{A}\in\sigma}$$

defined in a similar manner to (2.5.24).

In equation (2.5.27), we introduced the subspaces of $\mathcal{D}'(X)$ denoted by $\mathcal{D}' \otimes X$. In the same way, the subspace of $\mathcal{S}'(X)$ denoted by $\mathcal{S}' \otimes X$ will consists of finite operators:

$$\mathcal{S}' \otimes X = \left\{ T \in \mathcal{S}'(X), T = \sum_{j}^{n_T} T_j \otimes v_j, \ \ T_j \in \mathcal{S}'(\mathbb{R}), \ \ v_j \in X \right\}.$$

For $T \in \mathcal{S}' \otimes X$, we have

$$T(\varphi) = \sum_{j=1}^{n_T} \langle T_j, \varphi\rangle_{\mathcal{S}'\times\mathcal{S}} v_j \ \ \forall\varphi \in \mathcal{S}(\mathbb{R}). \tag{2.5.37}$$

We now state the result similar to Theorem 2.5.21.

Theorem 2.5.25. ([*22*])
The subspace $\mathcal{S}' \otimes X$ is dense in $\mathcal{S}'(X)$.

Proof. The proof of this theorem is analogous to that of Theorem 2.5.21. ☐

Definition 2.5.26. ([*22*])
Given a vector-valued distribution T in $\mathcal{S}' \otimes X$ with representation

$$T = \sum_{j=1}^{n_T} T_j \otimes v_j,$$

its Fourier transform denoted as in the scalar case, by $\mathcal{F}(T)$, is defined by

$$\mathcal{F}(T) = \sum_{j=1}^{n_T} \mathcal{F}(T_j) \otimes v_j. \tag{2.5.38}$$

For the Fourier transform of distributions as defined by (2.5.38), the analogous of the duality relation (2.5.11) is:

$$\text{for } \varphi \in \mathcal{S},\ \mathcal{F}(T)(\varphi) = T(\mathcal{F}(\varphi)) \tag{2.5.39}$$

Indeed, we have

$$\begin{aligned}
\mathcal{F}(T)(\varphi) &= \left(\sum_{j=1}^{n_T} (\mathcal{F}(T_j) \otimes v_j)\right)(\varphi) \text{ by (2.5.38)} \\
&= \sum_{j=1}^{n_T} \langle \mathcal{F}(T_j), \varphi \rangle\, v_j \\
&= \sum_{j=1}^{n_T} \langle T_j, \mathcal{F}(\varphi) \rangle\, v_j \text{ by (2.5.11)} \\
&= T(\mathcal{F}(\varphi)) \text{ by (2.5.37).}
\end{aligned}$$

Theorem 2.5.27. *The definition of the Fourier transform of T does not depend on its representation in Definition 2.5.26.*

Proof. Let $T \in \mathcal{S}' \otimes X$ be represented in two different ways:

$$T = \sum_{j=1}^{n_T} T_j \otimes v_j = \sum_{k=1}^{m_T} S_k \otimes u_k. \tag{2.5.40}$$

In view of (2.5.39) and (2.5.40) we have for $\varphi \in \mathcal{S}$

$$\begin{aligned}
\mathcal{F}(T)(\varphi) &= \left(\mathcal{F}(\sum_{j=1}^{n_T} T_j \otimes v_j)(\varphi)\right) \\
&= \sum_{j=1}^{n_T} \langle T_j, \mathcal{F}(\varphi)\rangle\, v_j \\
&= \sum_{j=1}^{m_T} \langle S_k, \mathcal{F}(\varphi)\rangle\, u_k \\
&= \sum_{j=1}^{m_T} \langle \mathcal{F}(S_k), \varphi\rangle\, u_k \\
&= \left(\mathcal{F}(\sum_{k=1}^{m_T} S_k \otimes v_k)(\varphi)\right). \qquad (2.5.41)
\end{aligned}$$

This proves the Theorem. □

We now proceed to extend the Fourier transform of the vector-valued distributions from the subspace $\mathcal{S}' \otimes X$ to the space of tempered vector-valued distributions $\mathcal{S}'(X)$; as a consequence of Theorem 2.5.25.

Theorem 2.5.28. *The Fourier transform $\mathcal{F}$ defined over $\mathcal{S}' \otimes X$ by (2.5.39) is uniquely extended by continuity into an isomorphism of $\mathcal{S}'(X)$ onto $\mathcal{S}'(X)$.*

Thus we have the following definition:

Definition 2.5.29. ([*22*])
Given a vector-valued distribution $T \in \mathcal{S}'(X)$, its Fourier transform denoted by $\mathcal{F}(T)$ is defined by

$$\mathcal{F}(T) = \lim_{j \to +\infty} \mathcal{F}(T_j),$$

where T_j is a sequence of finite operators in $\mathcal{S}' \otimes X$ that converges to T in $\mathcal{S}'(X)$.

The extension in Theorem 2.5.28 leads us to the connection of the Fourier transform of vector-valued distributions to the Laplace transform of vector-valued distributions. This connection is achieved by stating the analog of the set I_T introduced in the scalar case in equation (2.5.13).

Definition 2.5.30. ([22])
For $T \in \mathcal{D}'(X)$, we denote by I_T the subset of $\mathbb{R}$ given by

$$I_T = \left\{\xi \in \mathbb{R} : \quad e^{-\xi t}T \in \mathcal{S}'(X)\right\}. \tag{2.5.42}$$

where $e^{-\xi t}T(\varphi) = T(e^{-\xi t}\varphi), \ \ \varphi \in \mathcal{S}$.

We state without proof the following result:

Proposition 2.5.31. ([22])
Let $T \in L_+(X)$ where $L_+(X)$ is the space of distributions on $\mathbb{R}$ with values in X which have a Laplace transform.

- *For all $\xi \in int(I_T)(\neq \emptyset)$, the Fourier transform of the distribution $e^{-\xi t}T$ is a function of $\mathbb{O}_M(X)$ where $\mathbb{O}_M(X)$ is the space of functions of class C^∞ with values in X which are "growing slowly in X" as are all their derivatives.*
- *The function $\mathcal{L}(T) : p \longrightarrow V(p) = \mathcal{F}(e^{-\xi t}T)(\eta)$ is holomorphic in the band $int(I_T) \times \mathbb{R}$ with values in X.*

In view of the Proposition 2.5.31, we can define the Laplace transform of vector-valued distribution as follows:

Definition 2.5.32. ([22])
Let $T \in \mathcal{D}'(X)$ be such that $int(I_T) \neq \emptyset$. The holomorphic function denoted by $\mathcal{L}(T) : p \longrightarrow \mathcal{L}(T)(p)$ and defined for $p \in int(I_T) \times \mathbb{R}$ by

$$\mathcal{L}(T)(p) := \mathcal{F}(e^{-\xi t}T)(\eta) \tag{2.5.43}$$

is called the Laplace transform of the vector-valued distribution $T \in \mathcal{D}'(X)$.

It should be noticed that for $T \in \mathcal{S}' \otimes X$ a finite operator with $intI_T \neq 0$, we have

$$\mathcal{L}\left(\frac{d^kT}{dt^k}\right) = p^k\mathcal{L}(T).$$

By the density result in Theorem 2.5.25, we have

Theorem 2.5.33. *For $T \in \mathcal{D}'_+(X)$ with $intI_T \neq 0$*

$$\mathcal{L}\left(\frac{d^kT}{dt^k}\right)(p) = p^k\mathcal{L}(T_j)(p). \tag{2.5.44}$$

After obtaining the Laplace transform of general vector-valued distributions, we restrict the analysis to vector-valued Lebesgue's space defined as follows:

Definition 2.5.34. ([*21*])
We denote by $L^2[(-\infty,+\infty);X]$ *the space of (classes) of measurable functions* $t \longrightarrow v(t)$ *from* $(-\infty,+\infty)$ *into a Hilbert space* X *such that*

$$\|v\|_{L^2[-\infty,+\infty;X]} = \left(\int_{-\infty}^{+\infty} \|v(t)\|_X^2 dt\right)^{\frac{1}{2}} < +\infty.$$

The Hardy-Lebesgue space $H^2(0)$ is extended to vector-valued functions as follows:

Definition 2.5.35. ([*21*]) *(Hardy-Lebesgue Space)*
Let X *be a complex Hilbert space with norm denoted by* $\|\cdot\|_X$. *The Hardy-Lebesgue space denoted by* $H^2[0;X]$ *is defined as the set of vector-valued functions* $V : p \longrightarrow V(p)$ *from the half complex plane*

$$\mathbb{C}_+ = \{p = \xi + i\eta \in \mathbb{C},\ \ \xi \geq 0\},$$

into the space X *such that the following two conditions are satisfied:*

1. *The function* $V(p)$ *is holomorphic for* $\xi > 0$,
2. *For each* $\xi > 0$, *the vector-valued function* $\eta \longrightarrow V(\xi+i\eta)$ *is of class* $L^2[(-\infty,+\infty);X]$ *such that*

$$\sup_{\xi>0}\left(\int_{-\infty}^{\infty} \|V(\xi+i\eta)\|_X^2 d\eta\right) < +\infty.$$

Proposition 2.5.36. ([*21*])
Let $v(t) \in L^2[(0,+\infty);X]$. *Then its Laplace transform* $\widehat{v}(t)$ *exists for* $\xi \geq 0$ *and* $\widehat{v}(t) \in H^2[0;\ X]$.

Proof. The proof works word by word as that of the scalar case in Theorem 2.5.1 replacing everywhere the absolute value $|\cdot|$ by the Hilbert norm $\|\cdot\|_X$. $\square$

The analogue of the Paley-Wiener theorem for vector-valued functions read as follows:

Theorem 2.5.37. ([*21*])*(Paley-Wiener Theorem)*
Let $V(p) \in H^2\left[0; X\right]$*. For all* $\xi > 0$*, we put* $v_\xi : \mathbb{R} \to X$ *where*

$$v_\xi(\eta) := V(\xi + i\eta).$$

Then, we have the following:

- *For* $\xi \to 0$*, the family of functions* $v_\xi(\eta)$ *converges in* $L^2\left[(-\infty, +\infty); X\right]$ *to some function* $v_0 : \mathbb{R} \to X$ *denoted by*

  $$v_0(\eta) := V(i\eta),$$

  and called the trace or boundary function of $V(\xi + i\eta)$*;*

- *There exists a* $v(t) \in L^2\left[(-\infty, +\infty); X\right]$ *such that* $v(t) = 0$ *for* $t < 0$ *and*

  $$\mathcal{L}(v(t))(p) = \mathcal{F}\left(e^{-\xi t}\widetilde{v}(t)\right)(\eta) = v_0(\eta), \quad \text{for } \xi \geq 0 \tag{2.5.45}$$

  where $\mathcal{L}$ *and* $\mathcal{F}$ *are the Laplace and Fourier transforms of vector-valued distributions.*

The final result that we shall use reads as follows:

Theorem 2.5.38. ([*20*], [*60*], [*64*])
The operator $-\Delta + p$*,* $p \in \mathbb{C}$*, is analytic hypoelliptic. That is for any distribution* $v \in \mathcal{D}'(\mathbb{R}^2)$*, the fact that* $(-\Delta + p)v$ *is an analytic function on an open set of* $\mathbb{R}^2$ *implies that* v *is equally analytic on this open set.*