Contribution to qualitative and constructive treatment of the heat equation with domain singularities

by

Pius Wiysanyuy Molo, Chin

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Declaration

I the undersigned, hereby declare that the thesis submitted herewith for the degree Philosophiae Doctor to the University of Pretoria contains my own, independent work and has not been submitted for any degree at any other University.

Signature:

Name: Pius Wiysanyuy Molo, Chin

Date: September 2011.

Dedication

This thesis is dedicated to the almighty God whose protective eyes have allow me to witness this glorious period of my existence. It is also in loving memory of my parents Mr/Mrs Chin Peter. My loving wife Chin-Molo nee Lange Maryann and my three girls, Chin-Molo Kinyuy, Asherinyuy and Bongnyang also form a bigger part of this dedication, as their prayers, love, patience and support led me to overcome this difficult task.

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Abstract

The diffusion phenomenon arises in several real-life situations in engineering, science and technology. Typical examples include heat flow, reaction diffusion, advection/convection-diffusion, chemotaxis, nonlocal mechanisms, models for animal dispersal and the spread of diseases.

Mathematically, diffusion problems are modeled by parabolic equations which are classically studied in the ideal framework of smooth domains. In this thesis, we focus on the model parabolic equation, which is defined by linear heat equation. This equation associated with an initial condition and the Dirichlet boundary condition is considered on a non-smooth domain namely a polygonal domain. In considering such a domain with edge singularities, one main difficulty arises: the variational solution is not smooth and this negatively impacts on the accuracy and performance of any classical numerical method. In this thesis, we clarify as optimally as possible the singular nature of the variational solution. More precisely, we show that the variational solution admits a decomposition into a regular part and a singular part, which captures the rough geometry of the domain. Furthermore, we show that the solution achieves global regularity in weighted Sobolev spaces in which the rough nature of the domain is once again suitably incorporated.

On the constructive side, the global regularity result is used to design and analyze an optimally convergent semi-discrete Finite Element Method (FEM) in which the mesh of the triangulation is adequately refined. Two types of fully discrete mesh refinement (FEM) are constructed. The first method is made of Fourier series discretization in time while the second method is the Non-standard Finite Difference (NSFD) discretization. It is shown that these fully discrete methods converge optimally in relevant norms, with the coupled NSFD and mesh refined FEM presenting the additional advantage of replicating the dynamics of the heat equation in the limit case of space independent equation.

The tool used throughout the thesis is the Laplace transform of vector-valued distributions, a topic on which we elaborated substantially in order to show that any (tempered)

vector-valued distribution can be approximated by a sequence of finite operators. Applied to the heat equation, the Laplace transform leads to a family of Helmholtz equations for a complex parameter $p \in \mathbb{C}$. This raises a second main challenge that we dealt with successfully by using another type of weighted Sobolev spaces. The said challenge is to obtain the solutions of the Helmholtz problems with a priori estimates with the same constant that is independent of the parameter p.

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Key Notation

$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$	Sets of natural numbers, (positive and negative) integers, real numbers and complex numbers.
$\mathbb{R}^2_+$	Half-plane $\{x = (x_1, x_2, ..., x_{n-1}) \in \mathbb{R}^2\};\{x_n > 0\}$.
$C^m(\Omega),\ m \geq 0$ integer	Space of m-times continuously differentiable real-valued functions on Ω.
$C^m_b(\Omega)$	Space of bounded continuous functions on Ω.
$\mathcal{D}(\Omega) \equiv C^\infty_0(\Omega)$	Space of infinitely differentiable real-valued functions in Ω with compact support contained in Ω.
$L^p(\Omega), 1 \leq p < \infty,\ \ \|\cdot\|_{0,p},$	Lebesgue space of classes of measurable real-valued functions f on Ω such that $x \rightarrowtail \|f(x)\|^p$ is integrable on Ω, with its natural norm.
$L^p_{loc}(\Omega),\ \ 1 \leq p < \infty$	Space of classes of measurable functions f on Ω such that $x \rightarrowtail \|f(x)\|^p$ is integrable on any compact set contained in Ω.
$\mathcal{S}(\mathbb{R}^n)$	Space of $C^\infty(\mathbb{R}^n)$ functions f which together with their derivatives are rapidly decreasing at infinity i.e. $\|x\|^k \|D^\alpha f(x)\| \to 0$ as $\|x\| \to \infty,\ \ \forall\, k \in \mathbb{N},\ \alpha \in \mathbb{N}^n$.
$\mathbb{O}_M(\mathbb{R}^n)$	Space of $C^\infty(\mathbb{R}^n)$ functions f which together with all their derivatives are slowly increasing at infinity $\forall \alpha \in \mathbb{N}^n,\ \exists K \in \mathbb{N}$ such that $\|x\|^{-k} \|D^\alpha f(x)\| \to 0$ as $\|x\| \to \infty$. The subscript M refers to the fact that $\mathbb{O}_M(\mathbb{R}^n)$ is a multiplicator of $\mathcal{S}'(\mathbb{R}^n)$ defined below.
$\mathcal{D}'(\Omega)$	Space of distributions on Ω.
$\mathcal{S}'(\mathbb{R}^n)$	Space of tempered distributions on $\mathbb{R}^n$.
$\mathcal{D}'_+(\mathbb{R})$ or $\mathcal{D}'_-(\mathbb{R})$	Space of distributions on $\mathbb{R}$ with support limited to the left or right.
$L_+(\mathbb{R})$	Space of distributions on $\mathbb{R}$ which have a Laplace transform.

$L^p\left[(-\infty,+\infty);X\right]$	Lebesgue space of functions on $\mathbb{R}$ with values in X, where X is here and after either a Hilbert with inner product $(\cdot,\cdot)_X$ or Banach space with norm $\|\cdot\|_X$, X' being the dual of X.
$\mathcal{D}'(X) \equiv \mathcal{L}(\mathcal{D}(\mathbb{R}),\, X)$ or $\mathcal{S}'(X) = \mathcal{L}(\mathcal{S}(\mathbb{R},\, X)$	Spaces of distributions or tempered distributions on $\mathbb{R}$ with values in X.
$H^m(\Omega),\ \|\cdot\|_{m,\Omega},\ \lvert\cdot\rvert_{m,\Omega}$	Sobolev space of non-negative integer of order m, with its natural Hilbert norm and semi-norm.
$W^{m,p}(\Omega);\ 1 \leq p < \infty$ $\|\cdot\|_{m,p,\Omega}, \lvert\cdot\rvert_{m,p,\Omega}$	The general Sobolev space of order m, with its natural Banach norm and semi-norm.
$H_0^m(\Omega)$	Closure of $\mathcal{D}(\Omega)$ in $H^m(\Omega)$.
$H^{-m}(\Omega)$	The dual space of $H_0^m(\Omega)$.
$\mathcal{L}(v)(p) \equiv \widehat{v}(p)$	Laplace transform of the function or distribution v at the point $p = \xi + i\eta$.
$\mathcal{F}(w)(\eta)$	Fourier transform of the function or distribution w at the point $\eta \in \mathbb{R}$.
$\mathcal{F}^{-1}(w)(t)$	Inverse Fourier transform of the function or distribution w at the point $t \in \mathbb{R}$.
$\langle\cdot,\cdot\rangle$	Duality pairing between $\mathcal{S}'(\mathbb{R}^n$ and $\mathcal{S}(\mathbb{R}^n)$ or $\mathcal{D}'(\Omega)$ and $\mathcal{D}(\Omega)$.
$v *_t w$	The convolution product of v and w over the argument t.
I_T	Set $I_T := \left\{\xi \in \mathbb{R} : e^{-\xi t}T \in \mathcal{S}'(\mathbb{R})\right\}$ for $T \in \mathcal{D}'(\mathbb{R}_t)$ where $\mathbb{R}_t$ means that the distributions are taken with argument t.
$H^2(O)$ and $H^2\left[O,X\right]$	Hardy-Lebesgue scalar and vector-valued spaces.
$E \otimes Y$	Tensor product of the spaces E and Y.
$\mathcal{D}_K(\mathbb{R})$,	The subspace of $\mathcal{D}_K(\mathbb{R})$ consisting of functions with compact support in K.
$(P_{K,m})\ m \geq 1$	A sequence of semi-norms on $\mathcal{D}_K(\mathbb{R})$ defined by $P_{K,m}(\rho) = \sup_{x\in K}\left\lvert\frac{d^m\rho(x)}{dx^m}\right\rvert$
$V(m,\epsilon)$	A fundamental system of neighborhoods of the origin 0 for the topology of $\mathcal{D}_K(\mathbb{R})$.
$V(\{m_j\},\{\epsilon_j\})$	A fundamental system of neighborhoods of the origin 0 for the topology of $\mathcal{D}(\mathbb{R})$, where the sequences $\{m_j\}$ and $\{\epsilon_j\}$ vary arbitrarily.
$N(\{m_j\},\{\epsilon_j\})$	A family of semi-norms that generates the topology of $\mathcal{D}(\mathbb{R})$.

σ	The collection of all bounded subsets $\mathbb{A}$ of $\mathcal{D}(\mathbb{R})$.
$\mathcal{W}_I = \{q_\alpha\}_{\alpha \in I}$	Family of semi-norms that generate the topology of a locally convex topological vector space Y.
$q_{\alpha,A}$	Semi-norm defined on $\mathcal{L}(\mathcal{D}(\mathbb{R}), Y))$ with $\alpha \in I$.
$\mathcal{W}_{I,\sigma} = \{q_{\alpha,\mathbb{A}}\}_{q_\alpha \in \mathcal{W}_I,\, \mathbb{A} \in \sigma}$	A family of semi-norms $q_{\alpha,A}$ that generate the σ-topology $\mathcal{L}(\mathcal{D}(\mathbb{R}), Y))$ with the topology of uniform convergence on bounded subsets.
$\mathcal{L}_\sigma(\mathcal{D}(\mathbb{R}), Y))$	The space $\mathcal{L}(\mathcal{D}(\mathbb{R}))$ equipped with the σ-topology.
σ_f	The collection of finite union of bounded set σ.
$\mathbb{B} = \{V(A, M), \mathbb{A} \in \sigma_f\}$	A fundamental system of neighborhoods of the origin 0 for the σ-topology of $\mathcal{L}(\mathcal{D}(\mathbb{R}), Y)$.
$N_A(\{m_j\}, \{\epsilon_j\})$	A family of semi-norms that generate the σ-topology $\mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{D}(\mathbb{R}))$.
V_j	The rectangle $[-\alpha, \alpha] \times [-\beta, \beta]$ in a new co-ordinate system $(x = x_{1,j}, x_{2,j})$.
V_j^+	The set $\{(x_1, x_2) \in \Omega :\ -\beta < x_2 < \varphi_j(x_1),\ \ -\alpha < x_1 < \alpha\}$.
V_j^-	The set $\{(x_1, x_2) \in \Omega :\ \beta > x_2 > \varphi_j(x_1),\ -\alpha < x_1 < \alpha\}$.
V_j^0	The set $\{(x_1, x_2) \in \Gamma : x_2 = \varphi_j(x_1),\ \ -\alpha < x_1 < \alpha\}$.
Q	The unit square described by $\{(y_1, y_2) : \lvert y_1\rvert < 1, \lvert y_2\rvert < 1\}$.
Q_+	Positive half of the unit square i.e the set consisting of $(y_1, y_2) \in Q$ such that $y_2 > 0$.
Q_-	Negative half of the unit square i.e the set consisting of $(y_1, y_2) \in Q$ such that $y_2 < 0$.
Q_0	Intersection of the unit square Q with the horizontal line $y_2 = 0$.
G	A sector described in polar co-ordinates (r, θ) centered at a vertex of Γ the origin of the plane such that $G = \{(r\cos\theta, r\sin\theta) : r > 0,\ 0 < \theta < \omega\}$.

$P_2^k(G)$	Kondratiev weighted Sobolev space of all distributions v in G such that $r^{\vert\alpha\vert-k} D^\alpha v \in L^2(G)\ \forall\ \vert\alpha\vert \leq k$ where k is a non-negative integer with its natural norm $\Vert \cdot \Vert_{P_2^k(G)}$.
$H^{2,\beta}(\Omega)$	Weighted Sobolev space of all distributions $w \in H^1(\Omega)$ such that $r^\beta D^\alpha w \in L^2(\Omega)\quad \forall\ \alpha$ such that $\vert\alpha\vert = 2$ with its natural norm $\Vert \cdot \Vert_{H^{2,\beta}(\Omega)}$.
$\widetilde{H}^m\left[(0,+\infty); L^2(\Omega)\right]$	Space of functions $v \in H^m\left[(0,+\infty); L^2(\Omega)\right]$ such that the extension $\widetilde{v}$ by zero outside $(0,+\infty)$ belong to $H^m\left[(-\infty,+\infty);\ L^2(\Omega)\right]$ with its natural norm $\Vert \cdot \Vert_{\widetilde{H}^m[(0,+\infty);\ L^2(\Omega)]}$.