3.

CHAPTER 3

A FINITE SOURCE MULTI-SERVER INVENTORY SYSTEM WITH SERVICE FACILITY

3.1. INTRODUCTION

One implicit assumption made by many previous stochastic inventory models is that the item whose inventory is kept is made available to the customer immediately it is demanded. This is not generally true, however, as many items are delivered only after some work has been done on them. This is a particularly growing trend as many organisations are strategically shifting their production approach from a make-to-stock system to an assemble-to-order system. Such systems have longer lead time but maintain smaller inventory levels than the make-to-stock system. The implication of such increase in lead time on the level of service available to customers is an area that is now being actively researched by many authors.

Berman et al (1993) considered an inventory management system at a service facility which uses one item of inventory for each service provided. They assumed that both demand and service rates are deterministic and constant and queues can form only during stock outs. They determined optimal order quantity that minimizes the total cost rate. Berman and Kim (1999) analysed a problem in a stochastic environment where customers arrive at a service facility according to a Poisson process. The service times are exponentially distributed with mean inter-arrival time which is assumed to be larger than the mean service time. Under both the discounted and the average cost cases, the optimal policy of both the finite and infinite time horizon problem is a threshold ordering policy. A logically related model was studied by He et al. (1998), who analyzed a Markovian inventory - production system, in which demands are processed by a single machine in a batch of size one. Berman and Sapna (2000) studied an inventory control problem at a service facility which requires one item of the inventory. They assumed Poisson arrivals, arbitrarily distributed service times and zero lead times. They assumed that their the system has finite waiting room. Under a specified cost structure, the optimal ordering quantity that minimizes the long-run expected cost per unit time was derived. Schwarz et al. (2006) considered an inventory system with Poisson demand and exponentially distributed service time with deterministic and randomized ordering policies.

In all the above models the authors assumed that the service facility had a single server. But in many real life situations the service facility may provide more than one server so that more customers are handled at a time. Moreover if a customer's request cannot be processed for want of stock or free server he/she may prefer to leave the system and make an attempt at later time. The concept of having unserviced customers in an orbit and allowing them to retry for the service have been considered in queueing systems. A complete description of situations where queues with retrial customers arise can be found in Falin and Templeton (1997). A classified bibliography is given in Artalejo (1999). For more details on multi-server retrial queues see Anisimov and Artalejo (2001), Artalejo et al. (2001) and Chakravarthy and Dudin (2002).

Multi server inventory system with service facility was considered by Arivarignan et al (2008). They assumed a continuous review (s, S) perishable inventory system in which the customers arrive according to a Markovian arrival process. The service time, the lead time for the reorders and the life time of the items were assumed to be exponential. The customer who arrive during the stock-out period or all the items in the inventory are in service or all the servers are busy entered into the orbit of infinite size and these customers compete for their service after an exponentially distributed time interval. Using matrix geometric method, they derived the steady state probabilities and under a suitable cost structure, they calculated the long run total expected cost rate.

In this chapter, the focus is on the case in which the population of demanding customers under study is finite so that each individual customer generates his own flow of primary demand. The inventory system with finite source was received only a little attention. This concept was introduced by Sivakumar (2009). But the analysis of finite source retrial queue in continuous time have been considered by many authors, the interested reader see Falin and Templeton (1997), Artalejo (1998) and Falin and Artalejo (1998) Almasi et al., (2005) and Artalejo and Lopez-Herero (2007) and references therein. The chapter utilises the quasi-random distribution for the arrival process. A good reading on quasi-random distribution is Sharafali et al (2009).

The rest of the chapter is organized as follows. In the next section, the mathematical model and the notation used were described. The steady state analysis of the model is presented in section 3. In section 4, the various system performance measures in the steady state were derived. In the final section, the total expected cost rate in the steady state were calculated.

Notations :

$[A]_{i,j}$: element/sub-matrix at ith row, jth column of the matrix A.

$\mathbf{0}$: zero vector.

$\mathbf{I}$: identity matrix.

$e^T = (1,1,\dots,1)$.

$E_i^0 = \{0,1,\dots,i\}$.

$E_i^1 = \{1,2,\dots,i\}$.

$\delta_{ij} = \begin{cases} 1, & if j = i, \\ 0, & otherwise. \end{cases}$

$\bar{\delta}_{ij} = 1 - \delta_{ij}$.

3.2. MODEL DESCRIPTION

Consider a service facility which can stock a maximum of S units and c (≥ 1) identical servers. It is assumed that the arrival process of customers is quasi random with parameter α. The number of sources that generate the customers is assumed to be N. The customers demand a single item and the item is delivered to the customer after performing some service on the item. The service time is assumed to have exponential distribution. If a customer finds any one of the server is idle and at least one item is not in service, then he/she immediately accedes to the service. The customer who finds either all the servers are busy or all the items are in service enters the orbit of unsatisfied customers. These orbiting customers send requests at random time points for possible selection of their demands. The time intervals describing the repeated attempts are assumed to be independent and exponentially distributed with rate $\theta\bar{\delta}_{0j} + i\nu$, when there are i customers in orbit. The service times are independent exponential random variables with rate μ. As and when the on-hand inventory level drops to a prefixed level $s(\geq c)$, an order for $Q(= S - s > s)$ units is placed. The lead time distribution is exponential with parameter $\beta(> 0)$. The streams of arrival of customers, intervals separating successive repeated attempts, service times and lead times are assumed to be mutually independent.

3.3. ANALYSIS

Let $X(t), L(t)$ and $Y(t)$, respectively, denote the number of customers in the orbit, the on-hand inventory level (including those items that are in the service) and the number of busy servers at time t. From the assumptions made on the input and output processes, it may be verified that the stochastic process $\{(X(t), L(t), Y(t)), t \geq 0\}$ is a Markov process with the state space given by

$$\begin{aligned}\Omega = &\{(i,j,k); i \in E_{N-c}^0, j \in E_c^0, k \in E_j^0\} \cup \{(i,j,k); i \in E_{N-c}^0, j \in E_S \backslash E_c, k \in E_c^0\}\\ &\cup \{(i,j,k); i \in E_N \backslash E_{N-c}, j \in E_{N-i}^0, k \in E_j^0\}\\ &\cup \{(i,j,k); i \in E_N \backslash E_{N-c}, j \in E_S \backslash E_{N-i}, k \in E_{N-i}^0\}\end{aligned}$$

The infinitesimal generator of this process, defined by

$$P = (\ p((i,j,k),(l,m,n))\ ), \qquad (i,j,k),(l,m,n) \in E,$$

can be easily calculated and is given by

$$
\begin{cases}
(N-i-k)\alpha, & l=i, & i\in E^0_{N-c-1}, \\
& m=j, & j\in E_S, \\
& n=k+1, & k\in E^0_{\min(j-1,c-1)}, \\
& & or \\
& l=i, & i\in E_{N-1}\backslash E_{N-c-1}, \\
& m=j, & j\in E_S, \\
& n=k+1, & k=0,1,\dots,\min(j-1,N-i-1), \\
& & or \\
& l=i+1, & i\in E^0_{N-c-1}, \\
& m=j, & j\in E^0_S, \\
& n=k, & k=\min(j,c), \\
& & or \\
& l=i+1, & i\in E_{N-1}\backslash E_{N-c-1}, \\
& m=j, & j\in E^0_S, \\
& n=k, & k=\min(j,N-i), \\
\theta+i\nu, & l=i-1, & i\in E_{N-c-1}, \\
& m=j, & j\in E_S, \\
& n=k+1, & k=0,1,\dots,\min(j-1,c-1), \\
& & or \\
& l=i-1, & i\in E_N\backslash E_{N-c-1}, \\
& m=j, & j\in E_S, \\
& n=k+1, & k=0,1,\dots,\min(j-1,N-i-1),
\end{cases}
\tag{3.1}
$$

$$
\begin{cases}
\beta, & l=i, & i\in E_{N-c-1}, \\
& m=j+Q, & j\in E_S^0, \\
& n=k, & k=0,1,\dots,\min(j,c), \\
& & or \\
& l=i, & i\in E_N\backslash E_{N-c-1}, \\
& m=j, & j\in E_S^0, \\
& n=k, & k=0,1,\dots,\min(j,N-i), \\
\mu & l=i, & i\in E_{N-c-1}, \\
& m=j-1, & j\in E_S, \\
& n=k-1, & k=1,2,\dots,\min(j,c), \\
& & or \\
& l=i, & i\in E_N\backslash E_{N-c-1}, \\
& m=j-1, & j\in E_S, \\
& n=k-1, & k=1,2,\dots,\min(j,N-i), \\
-\big((N-i-k)\alpha+k\mu & l=i, & i\in E_{N-c-1}^0, \\
+h(s-j)\beta+\bar{\delta}_{i0}\bar{\delta}_{j0}(\theta+iv)\big), & m=j, & j\in E_S^0, \\
& n=k, & k=0,1,\dots,\min(j,c), \\
& & or \\
-\big((N-i-k)\alpha+k\mu & l=i, & i\in E_N\backslash E_{N-c-1}, \\
+h(s-j)\beta+\bar{\delta}_{j0}(\theta+iv)\big), & m=j, & j\in E_S^0, \\
& n=k, & k=0,1,\dots,\min(j,N-i), \\
0, & & otherwise.
\end{cases}
\tag{3.2}
$$

Define the following ordered sets

$$
\begin{aligned}
&For\ \ i=0,1,\dots,N-c, \\
&<i,j> \ = \ \begin{cases}((i,j,0),(i,j,1),\dots,(i,j,j)), & j=0,1,\dots,c, \\ ((i,j,0),(i,j,1),\dots,(i,j,c)), & j=c+1,c+2,\dots,S,\end{cases} \\
&For\ i=N-c+1,N-c+2,\dots,N, \\
&<i,j> \ = \ \begin{cases}((i,j,0),(i,j,1),\dots,(i,j,j)), & j=0,1,\dots,N-i, \\ ((i,j,0),(i,j,1),\dots,(i,j,N-i)), & j=N-i+1,N-i+2,\dots,S,\end{cases} \\
&<i> \ = \ (<i,0>,<i,1>,\dots,<i,S>), i=0,1,\dots,N.
\end{aligned}
\tag{3.3}
$$

Then the state space can be ordered as $(<0>,<1>,\dots,<N>)$.

The infinitesimal generator P of this process may be expressed conveniently as a block partitioned matrix with entries

$$[P]_{il} = \begin{cases} U_i, & l = i, & i = 0,1,\dots,N, \\ V_i, & l = i+1, & i = 0,1,\dots,N-1, \\ W_i, & l = i-1, & i = 1,2,\dots,N, \\ \mathbf{0}, & & otherwise. \end{cases} \tag{3.4}$$

More explicitly,

where

$$\begin{aligned}
&For\ \ i = 0,1,\dots,N-c-1,\\
&[V_i]_{jm} = \begin{cases} H_{ij}, & m = j, & j = 0,1,\dots,c-1, \\ H_{ic}, & m = j, & k = c,c+1,\dots,S, \\ \mathbf{0}, & & otherwise. \end{cases}\\
&For\ \ i = N-c, N-c+1,\dots,N-1,\\
&[V_i]_{jm} = \begin{cases} H_{ij}, & m = j, & j = 0,1,\dots,N-i-1, \\ \mathbf{0}, & & otherwise. \end{cases}\\
&For\ \ i = 0,1,\dots,N-c-1, j = 0,1,\dots,c\\
&[H_{ij}]_{kn} = \begin{cases} (N-i-k)\alpha, & n = k, & k = j, \\ 0, & & otherwise. \end{cases}\\
&For\ \ i = N-c, N-c+1,\dots,N-1, j = 0,1,\dots,N-i,\\
&[H_{ij}]_{kn} = \begin{cases} (N-i-k)\alpha, & n = k, & k = j, \\ 0, & & otherwise. \end{cases}\\
&For\ \ i = 1,2,\dots,N-c,\\
&[W_i]_{jm} = \begin{cases} M_{ij}, & m = j, & j = 1,2,\dots,c-1, \\ M_{ic}, & m = j, & j = c,c+1,\dots,S, \\ \mathbf{0}, & & otherwise. \end{cases}\\
&For\ \ i = N-c+1, N-c+2,\dots,N-1,\\
&[W_i]_{jm} = \begin{cases} M_{ij}, & m = j, & j = 1,2,\dots,N-i-1, \\ M_{i(N-i)}, & m = j, & j = N-i, N-i+1,\dots,S, \\ \mathbf{0}, & & otherwise. \end{cases}\\
&[W_N]_{jm} = \begin{cases} M_{i0}, & m = j, & j = 1,2,\dots,S, \\ \mathbf{0}, & & otherwise. \end{cases}\\
&For\ \ i = 1,2,\dots,N-c, j = 1,2,\dots,c,\\
&[M_{ij}]_{kn} = \begin{cases} \theta + iv, & n = k+1, & k = 0,1,\dots,j-1, \\ 0, & & otherwise. \end{cases}\\
&For\ \ i = N-c+1, N-c+2,\dots,N, j = 1,2,\dots,N-i+1,\\
&[M_{ij}]_{kn} = \begin{cases} \theta + iv, & n = k+1, & k = 0,1,\dots,j, \\ 0, & & otherwise. \end{cases}
\end{aligned} \tag{3.5}$$

$For\ i = 0,1, \dots, N-c,$

$$[U_i]_{jm} = \begin{cases} D_{ij}, & m = j, & j = 0,1, \dots, c-1, \\ D_{ic}, & m = j, & j = c, c+1, \dots, s, \\ D_{i(s+1)}, & m = j, & j = s+1, s+2, \dots, S, \\ F_{ij}, & m = j, & j = 1,2, \dots, c, \\ F_{i(c+1)}, & m = j, & j = c+1, c+2, \dots, S, \\ G_{ij}, & m = j+Q, & j = 0,1, \dots, c-1, \\ G_{ic}, & m = j+Q, & j = c, c+1, \dots, S, \\ \mathbf{0}, & & otherwise. \end{cases}$$

$For\ i = N-c+1, N-c+2, \dots, N-1,$

$$[U_i]_{jm} = \begin{cases} D_{ij}, & m = j, & j = 0,1, \dots, N-i-1, \\ D_{i(N-i)}, & m = j, & j = N-i, N-i+1, \dots, s, \\ D_{i(N-i+1)}, & m = j, & j = s+1, s+2, \dots, S, \\ F_{ij}, & m = j, & j = 1,2, \dots, N-i, \\ F_{i(N-i+1)}, & m = j, & j = N-i+1, N-i+2, \dots, S, \\ G_{ij}, & m = j+Q, & j = 0,1, \dots, N-i-1, \\ G_{i(N-i)}, & m = j+Q, & j = N-i, N-i+1, \dots, s, \\ \mathbf{0}, & & otherwise. \end{cases}$$

$For\ i = N,$

$$[U_i]_{jm} = \begin{cases} D_{ij}, & m = j, & j = 0, \\ D_{i1}, & m = j, & j = 1,2, \dots, s, \\ D_{i2}, & m = j, & j = s+1, s+2, \dots, S, \\ G_{i0}, & m = j+Q, & j = 0,1, \dots, s, \\ \mathbf{0}, & & otherwise. \end{cases}$$

$For\ i = 0,1, \dots, N, j = 0,1, \dots, \min(c, N-i),$

$$[G_{ij}]_{kn} = \begin{cases} \beta & n = k, \quad k = 0,1, \dots, j, \\ 0, & otherwise. \end{cases}$$

$For\ i = 0,1, \dots, N-c, j = 1,2, \dots, c,$

$$[F_{ij}]_{kn} = \begin{cases} k\mu & n = k-1, \quad k = 1,2, \dots, j, \\ 0, & otherwise. \end{cases}$$

$For\ i = 1,2, \dots, N-c,$

$$[F_{i(c+1)}]_{kn} = \begin{cases} k\mu & n = k-1, \quad k = 1,2, \dots, c, \\ 0, & otherwise. \end{cases}$$

$For\ i = N-c+1, 1, \dots, N, j = 1,2, \dots, N-i,$

$$[F_{ij}]_{kn} = \begin{cases} k\mu & n = k-1, \quad k = 1,2, \dots, j, \\ 0, & otherwise. \end{cases}$$

$For\ i = N-c+1, N-c+2, \dots, N-1,$

$$[F_{i(N-i+1)}]_{kn} = \begin{cases} k\mu, & n = k-1, \quad k = 1,2,\dots,N-i, \\ 0, & otherwise. \end{cases}$$

$$D_{00} = -(N\alpha + \beta),$$

$For\ j = 1,2,\dots,c,$

$$[D_{0j}]_{kn} = \begin{cases} -((N-k)\alpha + k\mu + \beta), & n = k, \quad k = 0,1,\dots,j, \\ (N-k)\alpha, & n = k+1, \quad k = 0,1,\dots,j-1, \\ 0 & otherwise. \end{cases}$$

$$[D_{0(c+1)}]_{kn} = \begin{cases} -((N-k)\alpha + k\mu), & n = k, \quad k = 0,1,\dots,c, \\ (N-k)\alpha, & n = k+1, \quad k = 0,1,\dots,c-1, \\ 0 & otherwise. \end{cases}$$

$For\ i = 1,2,\dots,N-c,$

$$D_{i0} = -((N-i)\alpha + \beta),$$

$For\ j = 1,2,\dots,c,$

$$[D_{ij}]_{kn} = \begin{cases} -((N-i-k)\alpha + k\mu + \beta + \bar{\delta}_{kj}(\theta + i\nu)), & n = k, \quad k = 0,1,\dots,j, \\ (N-k)\alpha, & n = k+1, \quad k = 0,1,\dots,j-1, \\ 0 & otherwise. \end{cases}$$

$$[D_{i(c+1)}]_{kn} = \begin{cases} -((N-i-k)\alpha + k\mu + \bar{\delta}_{kc}(\theta + i\nu)), & n = k, \quad k = 0,1,\dots,c, \\ (N-k)\alpha, & n = k+1, \quad k = 0,1,\dots,c-1, \\ 0 & otherwise. \end{cases}$$

$For\ i = N-c+1, N-c+2, \dots, N-1,$

$$D_{i0} = -((N-i)\alpha + \beta),$$

$For\ j = 1,2,\dots,N-i-2,$

$$[D_{ij}]_{kn} = \begin{cases} -((N-i-k)\alpha + k\mu + \beta + \bar{\delta}_{kj}(\theta + i\nu)), & n = k, \quad k = 0,1,\dots,j, \\ (N-i-k)\alpha, & n = k+1, \quad k = 0,1,\dots,j-1, \\ 0 & otherwise. \end{cases}$$

$$[D_{i(N-i-1)}]_{kn} = \begin{cases} -((N-i-k)\alpha + k\mu + \beta + \bar{\delta}_{kc}(\theta + i\nu)), & n = k, \quad k = 0,1,\dots,N-i-1, \\ (N-i-k)\alpha, & n = k+1, \quad k = 0,1,\dots,c-1, \\ 0 & otherwise. \end{cases}$$

$$[D_{i(N-i)}]_{kn} = \begin{cases} -((N-i-k)\alpha + k\mu + \bar{\delta}_{kc}(\theta + i\nu)), & n = k, \quad k = 0,1,\dots,N-i-1, \\ (N-i-k)\alpha, & n = k+1, \quad k = 0,1,\dots,c-1, \\ 0 & otherwise. \end{cases}$$

$$D_{N0} = -\beta,$$
$$D_{N1} = -((\theta + N\nu) + \beta),$$
$$D_{N2} = -(\theta + N\nu).$$

In table 3.1, the size of the sub matrices listed above were given.

Table 3.1: The submatrices and their size

Matrix	Size
$U_i, i = 0,1, \dots, N-c,$ $V_i, i = 0,1, \dots, N-c-1,$ $W_i, i = 1,2, \dots, N-c,$	$\frac{c(c+1)}{2} + (S-c+1)(c+1) \times \frac{c(c+1)}{2} + (S-c+1)(c+1)$
$U_i, i = N-c+1, N-c+2, \dots, N,$	$\frac{j(j+1)}{2} + (S-j+1)(j+1) \times \frac{j(j+1)}{2} + (S-j+1)(j+1),$ $j = N-i$
$V_i, i = N-c, N-c+1, \dots, N-1,$	$\frac{j(j+1)}{2} + (S-j+1)(j+1) \times \frac{(j+1)(j+2)}{2} + (S-j)(j+2),$ $j = N-i$
$W_i, i = N-c+1, N-c+2, \dots, N,$	$\frac{j(j-1)}{2} + (S-j+2)j \times \frac{j(j+1)}{2} + (S-j+1)(j+1),$ $j = N-i$
$H_{ij}, i = 0,1, \dots, N-c-1,$ $j = 0,1, \dots, c$	$(j+1) \times (j+1)$
$H_{ij}, i = N-c, N-c-1, \dots, N-1, j = 0,1, \dots, N-i$	$(j+1) \times (j+1)$
$M_{ij}, i = 0,1, \dots, N-c,$ $j = 1,2, \dots, c$	$(j+1) \times (j+1)$
$M_{ij}, i = N-c+1, N-c+2, \dots, N, j = 1,2, \dots, N-i+1$	$(j+1) \times (j+2)$
$G_{ij}, i = 0,1, \dots, N-c,$ $j = 0,1, \dots, c$	$(j+1) \times (c+1)$
$G_{ij}, i = N-c+1, N-c+2, \dots, N,$ $j = 0,1, \dots, N-i$	$(j+1) \times (N-i+1)$
$F_{ij}, i = 0,1, \dots, N-c,$ $j = 1,2, \dots, c$	$(j+1) \times j$

$F_{ij}, i = 0,1, \dots, N-c,$ $j = c+1,$	$(c+1) \times (c+1)$
$F_{ij}, i = N-c+1, N-c+2, \dots, N-1, j = 1,2, \dots, N-i$	$(j+1) \times j$
$F_{ij}, i = N-c+1, N-c+2, \dots, N-1, j = N-i+1$	$j \times j$
$D_{ij}, i = 0,1, \dots, N-c,$ $j = 0,1, \dots, c$	$(j+1) \times (j+1)$
$D_{ij}, i = 0,1, \dots, N-c,$ $j = c+1,$	$(c+1) \times (c+1)$
$D_{ij}, i = N-c+1, N-c+2, \dots, N, j = 0,1,2, \dots, N-i$	$(j+1) \times (j+1)$
$D_{ij}, i = N-c+1, N-c+2, \dots, N,\ j = N-i+1$	$j \times j$

3.3.1. Steady State Analysis

It can be seen from the structure of the infinitesimal generator P that the time-homogeneous Markov process $\{(X(t), L(t), Y(t)); t \geq 0\}$ on the finite state space E is irreducible. Hence the limiting distribution

$$\phi_{(i,j,k)} = \lim_{t \to \infty} Pr[X(t) = i, L(t) = j, Y(t) = k | X(0), L(0), Y(0)]$$

exists. Let

$$\phi_{(i,j)} = \begin{cases} (\phi_{(i,j,0)}, \phi_{(i,j,1)}, \dots, \phi_{(i,j,j)}), & j = 0,1, \dots, c, \quad i = 0,1, \dots, N-c, \\ (\phi_{(i,j,0)}, \phi_{(i,j,1)}, \dots, \phi_{(i,j,c)}), & j = c+1, c+2, \dots, S, \quad i = 0,1, \dots, N-c, \end{cases}$$

$$\phi_{(i,j)} = \begin{cases} (\phi_{(i,j,0)}, \phi_{(i,j,1)}, \dots, \phi_{(i,j,j)}), & j = 0,1, \dots, N-i, \\ & i = N-c+1, N-c+2, \dots, N, \\ (\phi_{(i,j,0)}, \phi_{(i,j,1)}, \dots, \phi_{(i,j,N-i)}), & j = N-i+1, N-i+2, \dots, S, \\ & i = N-c+1, N-c+2, \dots, N, \end{cases} \tag{3.6}$$

$$\phi_{(i)} = (\phi_{(i,0)}, \phi_{(i,1)}, \dots, \phi_{(i,S)}),$$
and
$$\Phi = (\phi_{(0)}, \phi_{(1)}, \dots, \phi_{(N)}).$$

Then the vector of limiting probabilities Φ satisfies

$$\Phi P = 0 \quad and \quad \Phi e = 1. \tag{3.7}$$

From the structure of P, it is seen that the Markov process under study falls into the class of birth and death process in a Markovian environment as discussed by Gaver et al. (1984). Hence using the same argument, the limiting probability vectors can be calculated. For the sake of completeness, the algorithm is provided here.

Algorithm :

Determine recursively the matrices

$$\begin{aligned} Z_0 &= U_0 \\ Z_i &= U_i + W_i(-Z_{i-1}^{-1})V_0, \quad i = 1,2,\dots,N. \end{aligned} \tag{3.8}$$

Compute recursively the vectors $\phi_{(i)}$ using

$$\phi_{(i)} = \phi_{(i+1)} W_{i+1}(-Z_i^{-1}), \quad i = N-1, N-2, \dots, 0, \tag{3.9}$$

Solve the system of equations

$$\phi_{(N)} Z_N = \mathbf{0} \tag{3.10}$$

and

$$\sum_{i=0}^{N} \phi_{(i)} e = 1. \tag{3.11}$$

From the system of equations (3.9) – (3.11), vector $\phi_{(N)}$ could be determined uniquely, up to a multiplicative constant.

3.4. SYSTEM PERFORMANCE MEASURES

In this section, some stationary performance measures of the system under study were derived. Using these measures, the total expected cost per unit time can be constructed.

3.4.1. Expected Inventory Level

Let ζ_I denote the expected inventory level in the steady state. Since ϕ_i is the steady state probability vector of i −th customer level with each component specifying a particular combination of the on-hand inventory level and the number of busy servers, the mean inventory level is given by

$$\begin{aligned}
\zeta_I \quad &= \quad \sum_{i=0}^{N} \sum_{j=1}^{S} \mathrm{j}\phi_{(i,j)}\mathbf{e} \\
&= \quad \sum_{i=0}^{N-c} \left(\sum_{j=1}^{c} \mathrm{j}\phi_{(i,j,k)} + \sum_{j=c+1}^{S} \sum_{k=0}^{c} \mathrm{j}\,\phi_{(i,j,k)}\right) \\
&\quad + \sum_{i=N-c+1}^{N-1} \left(\sum_{j=1}^{N-i} \sum_{k=0}^{j} \mathrm{j}\,\phi_{(i,j,k)} + \sum_{j=N-i+1}^{S} \sum_{k}^{N-i} \mathrm{j}\,\phi_{(i,j,k)}\right) \\
&\quad + \sum_{j=1}^{S} \mathrm{j}\phi_{(N,j,0)}.
\end{aligned} \tag{3.12}$$

3.4.2. Expected Reorder Rate

Let ζ_R denote the expected reorder rate in the steady state. A reorder is triggered when the inventory level drops to s. The steady state probability $\phi_{(i,s+1,k)}$ gives the rate at which $s+1$ is visited. After the inventory level reaches $s+1$, a service completion of any one of k servers if $k > 0$ takes the inventory level to s. This leads to

$$\zeta_R \quad = \quad \sum_{i=0}^{N-c} \sum_{k=1}^{c} k\mu\phi_{(i,s+1,k)} + \sum_{i=N-c+1}^{N-1} \sum_{k=1}^{N-i} k\mu\phi_{(i,s+1,k)} \tag{3.13}$$

3.4.3. Expected Customer Levels in the Orbit

Let ζ_O denote the expected number of customers in the orbit. Since ϕ_i is the steady state probability vector of i −th customer level with each component specifying a particular combination of the on-hand inventory level and the number of busy servers, the quantity $\phi_i\mathbf{e}$ gives the probability that the inventory level is i in the steady state. Hence, the expected customer level in the orbit is given by

$$\zeta_O \quad = \quad \sum_{i=1}^{N} i\phi_{(i)}\mathbf{e}. \tag{3.14}$$

3.4.4. Overall Rate of Retrials

Let ζ_{OR} denote the expectation of overall rate of retrials. This is given by

$$\zeta_{OR} \quad = \quad \sum_{i=1}^{N} (\theta + i\nu)\phi_{(i)}\mathbf{e}. \tag{3.15}$$

3.4.5. Successful Rate of Retrials

Let ζ_{SR} denote the expectation of successful rate of retrials. Note that a customer from the orbit enters into the service only when any one of the server is idle and at least one item is not in service. This lead to

$$\begin{aligned}\zeta_{SR} \quad = \quad & \sum_{i=1}^{N-c} \left(\sum_{j=1}^{c} \sum_{k=0}^{j-1} (\theta + i\nu)\phi_{(i,j,k)} + \sum_{j=c+1}^{S} \sum_{k=0}^{c-1} (\theta + i\nu)\phi_{(i,j,k)}\right) \\ & + \sum_{i=N-c+1}^{N-1} \left(\sum_{j=1}^{N-i} \sum_{k=0}^{j-1} (\theta + i\nu)\phi_{(i,j,k)} + \sum_{j=N-i}^{S} \sum_{k=0}^{N-i-1} (\theta + i\nu)\phi_{(i,j,k)}\right) \\ & + \sum_{j=1}^{S} (\theta + N\nu)\phi_{(N,j,0)}.\end{aligned} \tag{3.16}$$

3.4.6. Fraction of Successful Rate of Retrials

The fraction of successful rate of retrials ζ_{FSR} is given by

$$\zeta_{FSR} \quad = \quad \frac{\zeta_{SR}}{\zeta_{OR}}. \tag{3.17}$$

3.4.7. Number of Busy Servers

Let ζ_{BS} denote the expected number of busy servers in the steady state. Then ζ_{BS} is given by

$$\begin{aligned}\zeta_{BS} \quad = \quad & \sum_{i=0}^{N-c} \left(\sum_{j=1}^{c} \sum_{k=1}^{j} k\phi_{(i,j,k)} + \sum_{j=c+1}^{S} \sum_{k=1}^{c} k\phi_{(i,j,k)}\right) \\ & + \sum_{i=N-c+1}^{N-1} \left(\sum_{j=1}^{N-i} \sum_{k=1}^{j} k\phi_{(i,j,k)} + \sum_{j=N-i+1}^{S} \sum_{k=1}^{N-i} k\phi_{(i,j,k)}\right).\end{aligned} \tag{3.18}$$

3.4.8. Expected Number of Idle Servers

Let ζ_{IS} denote the expected number of idle servers in the steady state which is given by

$$\zeta_{IS} = c - \zeta_{BS} \tag{3.19}$$

3.5. TOTAL EXPECTED COST

The long-run expected cost rate for this model is defined to be

$$TC(S,s) = c_h\zeta_I + c_s\zeta_R + c_w\eta_O \tag{3.20}$$

where

c_h : The inventory carrying cost/unit/unit time.

c_s *:* The setup cost/order.

c_w *:* Waiting cost of a customer/unit time.

Substituting the values of ζ, we get the value of $TC(S,s)$.

Since the computation of the ϕ's are recursive, it is quite difficult to show the convexity of the total expected cost rate analytically.

3.6. CONCLUSION

In this chapter, a continuous review retrial inventory system with a finite source of customers and identical multiple servers in parallel was studied. The customers arrive according a quasi-random distribution. The customers demand unit item and the demanded items are delivered after performing some service which is distributed as exponential. The ordering policy is (s, S) policy, that is, once the inventory level drops to a prefixed level, say s, an order for $Q(= S - s)$ items would be placed. The lead times for the orders are assumed to have an exponential distribution. The arriving customer who finds all the servers are busy or all the items are in service joins an orbit of unsatisfied customers. The orbiting customers form a queue such that only a customer selected according to a certain rule can re-apply for service. The intervals separating two successive repeated attempts are exponentially distributed with rate $\theta + i\nu$, when the orbit has i customers $i \geq 1$. The joint probability distribution of the number of customer in the orbit, the number of busy servers and the inventory level is obtained in the steady state case. Various measures of stationary system performance are computed and the total expected cost per unit time is calculated.

4. [‡]

CHAPTER 4

TWO-COMMODITY PERISHABLE INVENTORY SYSTEM WITH BULK DEMAND FOR ONE COMMODITY

[‡] A modified version of this chapter has been published in the South African Journal of Industrial Engineering, Volume 21 N0 1, 2010

4.1. INTRODUCTION

One of the factors that contribute to the complexity of the present day inventory system is the multitude of items stocked and this necessitated the multi-commodity inventory systems. In dealing with such systems, in the earlier days, many models were proposed with independently established reorder points. But in situations where several products compete for limited storage space or share the same transport facility or are produced on (procured from) the same equipment (supplier) the above strategy overlooks the potential savings associated with joint ordering and, hence, will not be optimal. Thus, the coordinated approach, or what is known as joint replenishment, reduces the ordering and setup costs and allows the user to take advantage of quantity discounts, if any. Various models and references may be found in Miller (1971), Agarwal (1984), Silver (1974), Thomstone and Silver (1975), Kalpakam and Arivarignan (1993) and Srinivasan and Ravichandran (1994) and the references contained therein.

In continuous review inventory systems, Balintfy (1964) and Silver (1974) have considered a coordinated reordering policy which is represented by the triplet (S,c,s), where the three parameters S_i, c_i and s_i are specified for each item i with $s_i \le c_i \le S_i$, under the unit sized Poisson demand and constant lead time. In this policy, if the level of i-th commodity at any time is below s_i, an order is placed for $S_i - s_i$ items and at the same time, any other item $j(\neq i)$ with available inventory at or below its can-order level c_j, an order is placed so as to bring its level back to its maximum capacity S_j. Subsequently many articles have appeared with models involving the above policy and another article of interest is due to Federgruen, Groenevelt and Tijms (1984), which deals with the general case of compound Poisson demands and non-zero lead times.

The work on methods to solve the joint replenishment problem throughout the years has been extensive. Some further notable references include the publications of Fung and Ma (2001), Goyal (1973,1974,1988), Goyal and Satir (1989), Kaspi and Rosenblatt (1991), Nilsson et al. (2007), Nilsson and Silver (2008), Olsen (2005), Silver (1976), Van Eijs (1993), Viswanathan (1996,2002,2007) and Wildeman et al. (1997) and references therein.

Kalpakam and Arivarignan (1993) have introduced (s,S) policy with a single reorder level s defined in terms of the total number of items in the stock. This policy avoids separate ordering for each commodity and hence a single processing of orders for both commodities has some advantages in situation wherein procurement is made from the same supplies, items are produced on the same machine, or items have to be supplied by the same transport facility.

In the case of two-commodity inventory systems, Anbazhagan and Arivarignan (2000,2001a,2001b,2003) have proposed various ordering policies. Yadavalli et al. (2005b) have analyzed a model with joint ordering policy and variable order quantities. Sivakumar et al. (2005) have considered a two commodity substitutable inventory system in which the demanded items are delivered after a random time. Sivakumar et al. (2006) have considered a two commodity perishable inventory system with joint ordering policy.

There are some situations in which a single item is demanded for one commodity and multiple items are demanded for another commodity. For instance, a customer may buy a single razor or set of blades or both. Another example is the sales of DVD writer and set of DVDs. It may be noted that the seller would be placing a joint order for both commodities as these will be available from the same source. Moreover, a seller may not be willing to place orders frequently and may prefer to have one order to replenish his/her stock in a given cycle. These situations are modelled in this work by assuming demand processes that require single item for one commodity, multiple items for the other commodities or both commodities and by assuming a joint reorder for both commodities.

This paper is organized as follows: in section 2, the mathematical model and notations followed in the rest of the chapter were described. The steady state solution of the joint probability distribution for both commodities , the phase of the demand process and the phase of the lead time process is given in section 3. In section 4, the various measures of system performance in the steady state were derived and the total expected cost rate is calculated in section 5. Section 6 presents the cost analysis of the model using numerical examples.

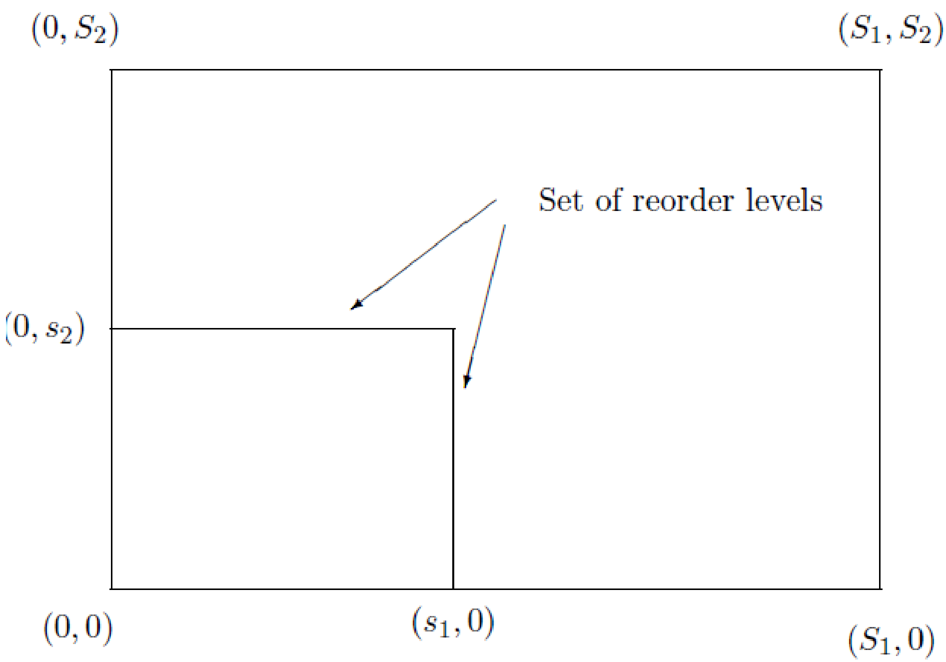

Figure 4.1: Space of Inventory levels

Notations

$\mathbf{0}$: zero matrix

I : an identity matrix

$$H(x) = \begin{cases} x & if \quad x > 0 \\ 0 & if \quad x \le 0 \end{cases}$$

E_i = $\{1,2,\ldots,i\}$

E_i^0 = $\{0,1,\ldots,i\}$

e = a column vector of ones.

4.2. MODEL DESCRIPTION

Consider a two-commodity perishable inventory system with the maximum capacity S_i units for i-th commodity $(i = 1,2)$. Assume that the demand for the first commodity is for single item and the demand for the second commodity is for bulk items. An arriving customer may demand only the first commodity or only the second commodity or both. The number of items demanded for the second commodity at any demand point is a random variable Y with probability function $p_k = Pr\{Y = k\}, \quad k = 1,2,3,\ldots$. The three type of demands for these two commodities occur according to a Markovian arrival process $\boldsymbol{MAP}$. The life time of each commodity is exponential with parameter $\gamma_i (i = 1,2)$ The reorder level for the i-th commodity is fixed at $s_i \ (1 \le s_i \le S_i)$ and the ordering quantity for the i-th commodity is $Q_i \ (= S_i - s_i > s_i + 1)$ items when both the inventory levels are less than or equal to their respective reorder levels. It is assumed that demands during stock-out period as well as unsatisfied demands are lost. The requirement $S_i - s_i > s_i + 1$, ensures that after a replenishment the inventory levels of both commodities will always be above the respective reorder levels. Otherwise, it may not be possible to place any reorder (according to this policy) which will lead to perpetual shortage. That is, if $L_i(t)$ represents inventory level of i-th commodity at time t, then a reorder is made when $L_1(t) \le s_1$ and $L_2(t) \le s_2$ (see figure 1). The time to deliver the items are assumed to be of phase (***PH***) type with representation (α, T) of order m_2. It can be noted that the phase type distribution is defined as the time until absorption in a finite state irreducible Markov chain with one absorbing state. The mean of the phase type distribution (α, T) is given by $\alpha(-T)^{-1}\mathbf{e}$. Let β denote the reciprocal of this mean. That is, $\beta = \left[\alpha(-T)^{-1}\mathbf{e}\right]^{-1}$ gives the rate of replenishment once an order is placed. Let T^0 be such that $T\mathbf{e} + T^0 = \mathbf{0}$.

For the description of the demand process, the description of ***MAP*** as given in Lucantoni (1991) was used. Consider a continuous-time Markov chain on the state space $1, 2, \ldots, m_1$. The demand process is constructively defined as follows. When the chain enters a state $i, 1 \le i \le m_1$, it stays for an exponential time with parameter θ_i. At the end of the sojourn time in state i, there are four possible transitions: with probabilities $a_{ij}, 1 \le j \le m_1$, the chain enters the state j when a demand for the first commodity occurs; with probabilities $b_{ij}, 1 \le j \le m_1$, the chain enters the state j when a demand for the second commodity occurs; with probabilities $c_{ij}, 1 \le j \le m_1$, the chain enters the state j when a demand for both commodities occurs; with probabilities $d_{ij}, 1 \le j \le m_1, i \ne j$, the transitions corresponds to no demand and the state of the chain is j. Note that the Markov chain can go from state i to state i only through a demand. Define the square matrices $D_k, k = 0,1,2,12$ of size $m_1 \times m_1$ by $[D_0]_{ii} = -\theta_i$ and $[D_0]_{ij} = \theta_i d_{ij}, i \neq j,$ $[D_1]_{ij} = \theta_i a_{ij}, [D_2]_{ij} = \theta_i b_{ij}$ and $[D_{12}]_{ij} = \theta_i c_{ij},$ $1 \leq i, j \leq m_1$. It is easily seen that $D = D_0 + D_1 + D_2 + D_{12}$ is an infinitesimal generator of a continuous-time Markov chain. It is assumed that D is irreducible and $D_0 e \neq 0$.

Let ζ be the stationary probability vector of the continuous-time Markov chain with generator D. That is, ζ is the unique probability vector satisfying

$$\zeta D = 0, \zeta e = 1.$$

Let η be the initial probability vector of the underlying Markov chain governing the $\boldsymbol{MAP}$. Then, by choosing η appropriately the time origin can be modelled to be

1. an arbitrary arrival point;
2. the end of an interval during which there are at least k arrivals;
3. the point at which the system is in specific state such as the busy period ends or busy period begins;

The important case is the one where one gets the stationary version of the $\boldsymbol{MAP}$ by $\eta = \zeta$. The constant $\lambda = \zeta(D_1 + D_2 + D_{12})e$, referred to as the **fundamental rate** gives the expected number of demands per unit of time in the stationary version of the $\boldsymbol{MAP}$. The quantities $\lambda_1 = \zeta D_1 e$, $\lambda_2 = \zeta D_2 e$ and $\lambda_{12} = \zeta D_{12} e$, give the arrival rate of demand for first commodity, second commodity and for both respectively. Note that $\lambda = \lambda_1 + \lambda_2 + \lambda_{12}$.

For further details on $\boldsymbol{MAP}$ and phase-type distributions and their usefulness in Stochastic modelling, the following are good references: Chapter 2 in Neuts (1994), Chapter 5 in Neuts (1989), Ramaswami (1981), Lucantoni (1991, 1993), Lucantoni et al. (1990), Latouche and Ramaswami (1999), Li and Li (1994), Lee and Jeon (2000) and Chakravarthy and Dudin (2003) and references therein for a detailed introduction of the $\boldsymbol{MAP}$ and phase-type distribution. Some recent reviews can be found in Neuts (1995) and Chakravarthy (2001).

Let $J_1(t)$ and $J_2(t)$, respectively, denote the phase of the demand process and the phase of the lead time process. Then the stochastic process $\{(L_1(t), L_2(t), J_1(t), J_2(t)), t \geq 0\}$ has the state space,

$$\Omega = \{(i_1, i_2, i_3, 0), i_1 \in E_{S_1} \setminus E_{s_1}, i_2 \in E_{S_2} \setminus E_{s_2}, i_3 \in E_{m_1}\}$$
$$\cup \{(i_1, i_2, i_3, 0), i_1 \in E_{S_1} \setminus E_{s_1}, i_2 \in E^0_{s_2}, i_3 \in E_{m_1}\}$$
$$\cup \{(i_1, i_2, i_3, 0), i_1 \in E^0_{s_1}, \in E_{S_2} \setminus E_{s_2}, i_3 \in E_{m_1}\}$$
$$\cup \{(i_1, i_2, i_3, i_4), i_1 \in E^0_{s_1}, i_2 \in E^0_{s_2}, i_3 \in E_{m_1}, i_4 \in E_{m_2}\}$$

From the assumptions made on the demand and the replenishment processes, it can be shown that $\{(L_1(t), L_2(t), J_1(t), J_2(t)), t \geq 0\}$ is a Markov process on the state space Ω. By ordering the sets of state space in lexicographic order, the infinitesimal generator of the Markov chain governing the system, in block partitioned form, is given by

$$[P]_{ij} = \begin{cases} A_i, & j = i, \quad\quad i = 0,1,\ldots,S_1, \\ B_i, & j = i-1, \quad i = 1,2,\ldots,S_1, \\ C, & j = i + Q_1, \quad i = 0,1,\ldots,s_1, \\ \mathbf{0}, & otherwise. \end{cases} \tag{4.1}$$

where

$$[C]_{ij} = \begin{cases} I_{m_1} \otimes T^0, & j = i + Q_2, \quad i = 0,1,\ldots,s_2, \\ \mathbf{0}, & otherwise. \end{cases} \tag{4.2}$$

For $k = s_1 + 2, s_1 + 3, \ldots, S_1,$

$$[B_k]_{ij} = \begin{cases} D_1 + k\gamma_1 I_{m_1}, & j = i, & i = 1,2,\ldots,S_2, \\ D_1 + D_{12} + k\gamma_1 I_{m_1}, & j = i, & i = 0, \\ p_{i-j} D_{12}, & j = 1,2,\ldots,i-1, & i = 2,3,\ldots,S_2, \\ p_i^{'} D_{12}, & j = 0, & i = 1,2,\ldots,S_2, \\ \mathbf{0}, & otherwise & \end{cases} \tag{4.3}$$

$$p_n^{'} = \sum_{i=n}^{\infty} p_i$$

For $k = s_1 + 1,$

$$[B_k]_{ij} = \begin{cases} D_1 + k\gamma_1 I_{m_1}, & j = i, & i = s_2+1, s_2+2,\ldots,S_2, \\ (D_1 + k\gamma_1 I_{m_1}) \otimes \alpha, & j = i, & i = 1,2,\ldots,s_2, \\ (D_1 + D_{12} + k\gamma_1 I_{m_1}) \otimes \alpha, & j = i, & i = 0, \\ p_{i-j} D_{12}, & j = s_2+1, s_2+2,\ldots,i-1, & i = s_2+2, s_2+3,\ldots,S_2, \\ p_{i-j} D_{12} \otimes \alpha, & j = 1,2,\ldots,s_2, & i = s_2+2, s_2+3,\ldots,S_2, \\ & or & \\ & j = 1,2,\ldots,i-1, & i = 2,3,\ldots,s_2+1, \\ p_i^{'} D_{12} \otimes \alpha, & j = 0, & i = 1,2,\ldots,S_2, \\ \mathbf{0}, & otherwise & \end{cases} \tag{4.4}$$

For $k = 1,2,\ldots,s_1$

$$[B_k]_{ij} = \begin{cases} D_1 + k\gamma_1 I_{m_1}, & j = i, & i = s_2+1, s_2+2,\ldots,S_2, \\ (D_1 + k\gamma_1 I_{m_1}) \otimes I_{m_2}, & j = i, & i = 1,2,\ldots,s_2, \\ (D_1 + D_{12} + k\gamma_1 I_{m_1}) \otimes I_{m_2}, & j = i, & i = 0, \\ p_{i-j} D_{12}, & j = s_2+1, s_2+2,\ldots,i-1, & i = s_2+2, s_2+3,\ldots,S_2, \\ p_{i-j} D_{12} \otimes \alpha, & j = 1,2,\ldots,s_2, & i = s_2+1, s_2+2,\ldots,S_2, \\ p_i^{'} D_{12} \otimes \alpha, & j = 0, & i = s_2+1, s_2+2,\ldots,S_2, \\ p_{i-j} D_{12} \otimes I_{m_2}, & j = 1,2,\ldots,i-1, & i = 2,3,\ldots,s_2, \\ p_i^{'} D_{12} \otimes I_{m_2}, & j = 0, & i = 1,2,\ldots,s_2, \\ \mathbf{0}, & otherwise. & \end{cases} \tag{4.5}$$

For $k = s_1 + 1, s_1 + 2, \dots, S_1$,

$$[A_k]_{ij} = \begin{cases} p_1 D_2 + k\gamma_2 I_{m_1}, & j = i-1, & i = 2,3,\dots,S_2, \\ p_{i-j} D_2, & j = 1,2,\dots,i-2, & i = 3,4,\dots,S_2, \\ p_i' D_2, & j = 0, & i = 1,2,\dots,S_2, \\ D_0 - (k\gamma_1 + i\gamma_1) I_{m_1} & j = i, & i = 1,2,\dots,S_2, \\ D_0 + D_2 - k\gamma_1 I_{m_1}, & j = i, & i = 0, \\ \mathbf{0}, & otherwise. & \end{cases} \qquad (4.6)$$

For $k = 1,2,\dots,s_1$

$$[A_k]_{ij} = \begin{cases} p_1 D_2 + i\gamma_2 I_{m_1}, & j = i-1, & i = s_2+2, s_2+3, \dots, S_2, \\ (p_1 D_2 + i\gamma_2 I_{m_1}) \otimes \alpha, & j = i-1, & i = s_2+1, \\ (p_1 D_2 + i\gamma_2 I_{m_1}) \otimes I_{m_2}, & j = i-1, & i = 2,3,\dots,s_2, \\ p_{i-j} D_2, & j = s_2+1, s_2+2, \dots, i-2, & i = s_2+3, s_2+4, \dots, S_2, \\ p_{i-j} D_2 \otimes \alpha, & j = 1,2,\dots,i-2, & i = s_2+1, s_2+2, \\ & \quad or & \\ & j = 1,2,\dots,s_2, & i = s_2+3, s_2+4, \dots, S_2, \\ p_{i-j} D_2 \otimes I_{m_2}, & j = 1,2,\dots,i-2, & i = 3,4,\dots,s_2, \\ p_i' D_2 \otimes \alpha, & j = 0, & i = s_2+1, s_2+2, \dots, S_2, \\ p_i' D_2 \otimes I_{m_2}, & j = 0, & i = 1,2,\dots,s_2, \\ D_0 - (k\gamma_1 + i\gamma_2) I_{m_1}, & j = i, & i = s_2+1, s_2+2, \dots, S_2, \\ D_0 \oplus T - (k\gamma_1 + i\gamma_2) I_{m_1} \otimes I_{m_2}, & j = i, & i = 1,2,\dots,s_2, \\ (D_0 + D_2) \oplus T - k\gamma_1 I_{m_1} \otimes I_{m_2}, & j = i, & i = 0, \\ \mathbf{0}, & otherwise & \end{cases} \qquad (4.7)$$

For $k = 0$

$$[A_k]_{ij} = \begin{cases} p_1(D_2+D_{12})+i\gamma_2 I_{m_1}, & j=i-1, & i=s_2+2,s_2+3,\ldots,S_2,\\ (p_1(D_2+D_{12})+i\gamma_2 I_{m_1})\otimes\alpha, & j=i-1, & i=s_2+1,\\ (p_1(D_2+D_{12})+i\gamma_2 I_{m_1})\otimes I_{m_2}, & j=i-1, & i=2,3,\ldots,s_2,\\ p_{i-j}(D_2+D_{12}), & j=s_2+1,s_2+2,\ldots,i-2, & i=s_2+3,s_2+4,\ldots,S_2,\\ p_{i-j}(D_2+D_{12})\otimes\alpha, & j=1,2,\ldots,i-2, & i=s_2+1,s_2+2,\\ & \quad or & \\ & j=1,2,\ldots,s_2, & i=s_2+3,s_2+4,\ldots,S_2,\\ p_{i-j}(D_2+D_{12})\otimes I_{m_2}, & j=1,2,\ldots,i-2, & i=3,4,\ldots,s_2,\\ p_i'(D_2+D_{12})\otimes\alpha, & j=0, & i=s_2+1,s_2+2,\ldots,S_2,\\ p_i'(D_2+D_{12})\otimes I_{m_2}, & j=0, & i=1,2,\ldots,s_2,\\ D_0+D_1-(i\gamma_2)I_{m_1}, & j=i, & i=s_2+1,s_2+2,\ldots,S_2,\\ (D_0+D_1)\oplus T-(i\gamma_2)I_{m_1}\otimes I_{m_2}, & j=i, & i=1,2,\ldots,s_2,\\ D\oplus T, & j=i, & i=0,\\ \mathbf{0}, & otherwise & \end{cases} \tag{4.8}$$

It may be noted that the matrix C is of order $(Q_1m_1+(s_1+1)m_1m_2)\times(S_2+1)m_1$, the matrices $B_i, i=s_1+2,s_1+3,\ldots,S_1$, are of order $(S_2+1)m_1\times(S_2+1)m_1$, the matrix B_{s_1+1} is of order $(S_2+1)m_1\times(Q_1m_1+(s_1+1)m_1m_2)$, the matrices $B_i, i=1,2,\ldots,s_1$, are of order $(Q_1m_1+(s_1+1)m_1m_2)\times(Q_1m_1+(s_1+1)m_1m_2)$, the matrices $A_i, i=0,1,\ldots,s_1$ are of order $(Q_1m_1+(s_1+1)m_1m_2)\times(Q_1m_1+(s_1+1)m_1m_2)$, and the matrices $A_i, i=s_1+1,s_1+2\ldots,S_1$ are of order $(S_2+1)m_1\times(S_2+1)m_1$.

4.3. STEADY STATE ANALYSIS

It can be seen from the structure of P that the homogeneous Markov process $\{(L_1(t),L_2(t),J_1(t),J_2(t)),t\geq 0\}$ on the finite state space Ω is irreducible.

Hence, the limiting distribution $\phi_{(i,k,j_1,j_2)} =$

$$\lim_{t\to\infty} Pr\left[L_1(t)=i, L_2(t)=k, J_1(t)=j_1, J_2(t)=j_2 \mid L_1(0), L_2(0), J_1(0), J_2(0)\right]$$

exists. Let

$$\phi_{(i,k,j_1)} = \begin{cases} (\phi_{(i,k,j_1,1)}, \phi_{(i,k,j_1,2)}, \ldots, \phi_{(i,k,j_1,m_2)}), & (i,k,j_1)\in F_1, \\ (\phi_{(i,k,j_1,0)}), & (i,k,j_1)\in F_2, \end{cases}$$

where

$$F_1 = \left\{(i_1,i_2,i_3), i_1 \in E^0_{s_1}, i_2 \in E^0_{s_2}, i_3 \in E_{m_1}\right\}$$

$$F_2 = \left\{(i_1,i_2,i_3), i_1 \in E_{S_1} \setminus E_{s_1}, i_2 \in E_{S_2} \setminus E_{s_2}, i_3 \in E_{m_1}\right\}$$

$$\cup \left\{(i_1,i_2,i_3), i_1 \in E_{S_1} \setminus E_{s_1}, i_2 \in E^0_{s_2}, i_3 \in E_{m_1}\right\}$$

$$\cup \left\{(i_1,i_2,i_3), i_1 \in E^0_{s_1}, \in E_{S_2} \setminus E_{s_2}, i_3 \in E_{m_1}\right\}$$

$$\phi_{(i,k)} = \left(\phi_{(i,k,1)}, \phi_{(i,k,2)}, \ldots, \phi_{(i,k,m_1)}\right), k \in E_2, i \in E_1,$$

$$\phi^{(i)} = \begin{cases} (\phi_{(i,0)}, \phi_{(i,1)}, \ldots, \phi_{(i,S_2)}), & i \in E_1 \end{cases}$$

and

$$\Phi = \left(\Phi^{(0)}, \Phi^{(1)}, \ldots, \Phi^{(S_1)}\right).$$

Then the vector of limiting probabilities Φ satisfies

$$\Phi P = \mathbf{0} \quad and \quad \Phi \mathbf{e} = 1. \tag{4.9}$$

The first equation of the above yields the following set of equations:

$$\Phi^{(i+1)} B_{i+1} + \Phi^{(i)} A_i = 0, i = 0,1,\ldots,Q_1 - 1, \tag{4.10}$$

$$\Phi^{(i+1)} B_{i+1} + \Phi^{(i)} A_i + \Phi^{(i-Q_1)} C = 0, i = Q_1, \tag{4.11}$$

$$\Phi^{(i+1)}B_{i+1} + \Phi^{(i)}A_i + \Phi^{(i-Q_1)}C = 0, i = Q_1+1, Q_1+2, \ldots, S_1-1, \tag{4.12}$$

$$\Phi^{(i)}A_i + \Phi^{(i-Q_1)}C = 0, i = S_1. \tag{4.13}$$

The equations (except (4.11)) can be recursively solved to get

$$\Phi^{(i)} = \Phi^{(Q_1)}\theta_i, \quad i = 0,1,\ldots,S_1, \tag{4.14}$$

where

$$\theta_i = \begin{cases} (-1)^{Q_1-i} B_{Q_1}A_{Q_1-1}^{-1}B_{Q_1-1}\cdots B_{i+1}A_i^{-1}, \quad i = 0,1,\ldots,Q_1-1, \\ I, \quad i = Q_1, \\ (-1)^{2Q_1-i+1}\sum_{j=0}^{S-i}\left[\left(B_{Q_1}A_{Q_1-1}^{-1}B_{Q_1-1}\cdots B_{s_1+1-j}A_{s_1-j}^{-1}\right)CA_{S_1-j}^{-1} \right. \\ \qquad \left.\left(B_{S_1-j}A_{S_1-j-1}^{-1}B_{S_1-j-1}\cdots B_{i+1}A_i^{-1}\right)\right], \quad i = Q_1+1,\ldots,S_1. \end{cases} \tag{4.15}$$

Substituting the values of θ_i in equation (4.11) and in the normalizing condition thr following is obtained

$$\begin{aligned} \Phi^{(Q_1)}\Bigg[& (-1)^{Q_1}\sum_{j=0}^{s-1}\left[\left(B_{Q_1}A_{Q_1-1}^{-1}B_{Q_1-1}\cdots B_{s_1+1-j}A_{s_1-j}^{-1}\right)CA_{S_1-j}^{-1}\right. \\ & \left.\left(B_{S_1-j}A_{S_1-j-1}^{-1}B_{S_1-j-1}\cdots B_{Q_1+2}A_{Q_1+1}^{-1}\right)\right]B_{Q_1+1} + A_{Q_1} \\ & + (-1)^{Q_1}B_{Q_1}A_{Q_1-1}^{-1}B_{Q_1-1}\cdots B_1A_0^{-1}C\Bigg] = 0, \end{aligned} \tag{4.16}$$

and

$$\begin{aligned} \Phi^{(Q_1)}\Bigg[& \sum_{i=0}^{Q_1-1}\left((-1)^{Q_1-i}B_{Q_1}A_{Q_1-1}^{-1}B_{Q_1-1}\cdots B_{i+1}A_i^{-1}\right) + I \\ & + \sum_{i=Q_1+1}^{S_1}\Bigg((-1)^{2Q_1-i+1}\sum_{j=0}^{S-i}\left[\left(B_{Q_1}A_{Q_1-1}^{-1}B_{Q_1-1}\cdots B_{s_1+1-j}A_{s_1-j}^{-1}\right)CA_{S_1-j}^{-1}\right. \\ & \left.\left(B_{S_1-j}A_{S_1-j-1}^{-1}B_{S_1-j-1}\cdots B_{i+1}A_i^{-1}\right)\right]\Bigg)\boldsymbol{e}\Bigg] = 1 \end{aligned} \tag{4.17}$$

From the equation (4.16), the value of $\Phi^{(Q)}$ can be obtained up to a constant multiplication. This constant can be determined by substituting the value of $\Phi^{(Q)}$ in the equation (4.17). Substituting the value of $\Phi^{(Q)}$ in the equation (4.14) leads to the values of $\Phi^{(i)}, i = 0, 1, \ldots, S.$

4.4. SYSTEM PERFORMANCE MEASURES

In this section, some stationary performance measures of the system were derived. Using these measures, the total expected cost per unit time can be constructed.

4.4.1. Mean Inventory level

Let η_{I_k} denote the mean inventory level of $k-$ th commodity in the steady state $(k = 1,2)$. Since $\phi_{(i,j)}$ is the steady state probability vector for inventory level of first commodity i and the second commodity j, then

$$\eta_{I_1} = \sum_{i=1}^{S_1} \sum_{j=0}^{S_2} i \phi_{(i,j)} \mathbf{e}. \tag{4.18}$$

and

$$\eta_{I_2} = \sum_{i=0}^{S_1} \sum_{j=1}^{S_2} j \phi_{(i,j)} \mathbf{e}. \tag{4.19}$$

4.4.2. Mean Reorder Rate

A reorder for both commodities is made when the joint inventory level drops to either (s_1, s_2) or $(s_1, j), j < s_2$ or $(i, s_2), i < s_1$. Let η_R denote the mean reorder rate for both commodities in the steady state and it is given by

$$\eta_R = \sum_{k=0}^{s_1}\sum_{j=1}^{Q_2}\phi_{(k,s_2+j)}\sum_{u=j}^{\infty}p_u(D_2 \otimes \alpha)\mathbf{e} + \sum_{k=0}^{s_2}\phi_{(s_1+1,k)}(D_1 \otimes \alpha)\mathbf{e}$$

$$+\sum_{k=1}^{s_1+1}\sum_{j=1}^{Q_2}\phi_{(k,s_2+j)}\sum_{u=j}^{\infty}p_u(D_{12} \otimes \alpha)\mathbf{e} + \sum_{k=0}^{s_1}(s_2+1)\gamma_2\phi_{(k,s_2+1)}\mathbf{e} \tag{4.20}$$

$$+\sum_{k=0}^{s_2}(s_1+1)\gamma_1\phi_{(s_1+1,k)}\mathbf{e}.$$

4.4.3. Mean Shortage Rate

Let η_{Sh_i} denote the mean shortage rate of $i-$th type demand in the steady state $(i = 1,2,12)$. Then

$$\eta_{Sh_1} = \sum_{k=0}^{S_2}\phi_{(0,k)}D_1\mathbf{e}. \tag{4.21}$$

$$\eta_{Sh_2} = \sum_{i=0}^{S_1}\sum_{j=0}^{S_2}\phi_{(i,j)}\sum_{k=j+1}^{\infty}p_kD_2\mathbf{e}. \tag{4.22}$$

and

$$\eta_{Sh_{12}} = \left(\sum_{k=0}^{S_2}\phi_{(0,k)}D_{12}\mathbf{e} + \sum_{i=0}^{S_1}\sum_{j=0}^{S_2}\phi_{(i,j)}\sum_{k=j+1}^{\infty}p_kD_{12}\mathbf{e}\right). \tag{4.23}$$

4.4.4. Mean Failure Rate

Let the mean failure rate of commodity-i in the steady state be denoted by $\eta_{F_i},(i=1,2)$. A failure occurs when any one of the stocked items cease to work or perish. Since the rate of failure of a single item is γ_j for the commodity j, the rate at which any one of i items for $j-th$ commodity fails is given by $i\gamma_j,(j=1,2)$. When the process is in state (i,k,j_1,j_2), the rate of failure of any one of item of first commodity is given by $i\gamma_1$ (provided $i>0$) and the failure rate of any one item of second commodity is $k\gamma_2$ (provided $k>0$).

Therefore

$$\eta_{F_1} = \sum_{i=1}^{S_1}\sum_{k=0}^{S_2} i\gamma_1\phi_{(i,k)}\mathbf{e}. \tag{4.24}$$

and

$$\eta_{F_2} = \sum_{i=0}^{S_1}\sum_{k=1}^{S_2} k\gamma_2\phi_{(i,k)}\mathbf{e}. \tag{4.25}$$

4.5. COST ANALYSIS

The total expected cost per unit time (total expected cost rate) in the steady-state for this model is defined to be $TC(S_1,s_1,S_2,s_2)$

$$= c_{h_1}\eta_{I_1} + c_{h_2}\eta_{I_2} + c_s\eta_R + c_{sh_1}\eta_{Sh_1} + c_{sh_2}\eta_{Sh_2} + c_{sh_{12}}\eta_{Sh_{12}} + c_{f_1}\eta_{F_1} + c_{f_2}\zeta_{F_2} \tag{4.26}$$

where

c_{h_i} : The inventory carrying cost of i-th commodity per unit item per unit time

$(i = 1,2)$

c_s : Joint ordering cost per order.

c_{f_i} : The failure cost of i-th commodity per unit item per unit time $(i = 1,2)$.

c_{sh_i} : Shortage cost due to type i demand per unit time $(i = 1,2,12)$.

Since the total expected cost rate is known only implicitly, the analytical properties such as convexity of the total expected cost rate cannot be carried out in the present form. However the following numerical examples were presented to demonstrate the computability of the results derived in our work, and to illustrate the existence of local optima when the total cost function is treated as a function of only two variables.

4.6. ILLUSTRATIVE NUMERICAL EXAMPLES

As the total expected cost rate is obtained in a complex form, the convexity of the total expected cost rate cannot be studied by the analytical methods. Hence the use `simple' numerical search procedures to find the ``local'' optimal vales for any two of the decision variables $\{S_1, s_1, S_2, s_2\}$ by considering a small set of integer values for these variables. With a large number of numerical examples, it was found that the total cost rate per unit time in the long run is either convex function of both variables or an increasing function of any one variable.

The following five MAPs for arrival of demands are considered and it may be noted that these processes can be normalized to have a specific (given) demand rate λ when considered for arrival of demands.

1. **Exponential (Exp)**

$$H_0 = (-1) H_1 = (1)$$

2. **Erlang (Erl)**

$$H_0 = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \quad H_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

3. **Hyper-exponential (HExp)**

$$H_0 = \begin{pmatrix} -10 & 0 \\ 0 & -1 \end{pmatrix} \quad H_1 = \begin{pmatrix} 9 & 1 \\ 0.9 & 0.1 \end{pmatrix}$$

4. **MAP with Negative correlation (MNC)**

$$H_0 = \begin{pmatrix} -2 & 2 & 0 \\ 0 & -81 & 0 \\ 0 & 0 & -81 \end{pmatrix} \quad H_1 = \begin{pmatrix} 0 & 0 & 0 \\ 25.25 & 0 & 55.75 \\ 55.75 & 0 & 25.25 \end{pmatrix}$$

5. **MAP with Positive correlation (MPC)**

$$H_0 = \begin{pmatrix} -2 & 2 & 0 \\ 0 & -81 & 0 \\ 0 & 0 & -81 \end{pmatrix} \quad H_1 = \begin{pmatrix} 0 & 0 & 0 \\ 55.25 & 0 & 25.75 \\ 25.75 & 0 & 55.25 \end{pmatrix}$$

All the above $MAPs$ are qualitatively different in that they have different variance and correlation structures. The first three processes are special cases of renewal processes and the correlation between arrival times is 0. The demand process labelled as MNC

has correlated arrivals with correlation coefficient -0.1254 and the demands corresponding to the process labelled $\boldsymbol{MPC}$ has positive correlation coefficient 0.1213. Since Erlang has the least variance among the five arrival processes considered here, the ratios of the variances of the other four arrival processes, labelled as $\boldsymbol{Exp}, \boldsymbol{HExp}, \boldsymbol{MNC}$ and $\boldsymbol{MPC}$ above, with respect to the Erlang process are, 3.0, 15.1163, 8.1795, 8.1795, respectively. The ratios were given rather than the actual values since the variance depends on the arrival rate which is varied in the discussion.

For the lead time distribution, the following three $\boldsymbol{PH}$ distributions were considered. Again these processes can be normalized to have a specific (given) rate β when considered for replenishment.

1. **Exponential (Exp)**

$$\alpha = (1) T = (-1)$$

2. **Erlang (Erl)**

$$\alpha = (1,0,0,0) T = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

3. **Hyper-exponential (HExp)**

$$\alpha = (0.9,0.1) T = \begin{pmatrix} -10 & 0 \\ 0 & -1 \end{pmatrix}$$

Example 1: This example is to illustrate the effect of the demand rate λ, the lead time rate β, the five types of demand processes and the three types of lead time processes on the optimal values (S_1^*, S_2^*) and the optimal cost rate $TC(S_1^*, 2, S_2^*, 4)$. The following fixed values were assumed for the parameters and costs:

$D_0 = H_0, D_1 = 0.3H_1, D_2 = 0.4H_1, D_{12} = 0.3H_1, \gamma_1 = 0.8, \gamma_2 = 0.6, p_i = 0.6 * 0.4^{i-1}, i = 1,2,\ldots,$
$c_{h_1} = 0.05, c_{h_2} = 0.01, c_s = 10, c_{sh_1} = 0.8, c_{sh_2} = 1.5, c_{sh_{12}} = 1, c_{f_1} = 0.2, c_{f_2} = 0.2.$

Table 4.1 gives the optimum values, S_1^* and S_2^*, that minimize the total expected cost rate for each of the five $\boldsymbol{MAP}$s for arrivals of demands considered against each of the three $\boldsymbol{PH}$s for lead times. The associated total expected cost rate values are also given in the table. The lower entry in each cell gives the optimal expected cost rate and the upper entries are corresponding to S_1^* and S_2^*. The following observations were noticed from the table 1:

1. As λ increases the optimal total cost rate decreases for all the five demand processes and for all the three lead time processes. Similarly as β increases the optimal total cost rate decreases.
2. The optimal total expected cost rate has higher value for demand process having hyper-exponential distribution and has lower value for Erlang demand process.
3. The lead time distributed as Erlang has low optimal total cost rate except for $\boldsymbol{HExp}$ distributed demand process and $\boldsymbol{HExp}$ distributed lead time has high optimal total cost rate except for **HExp** distributed demand process. For $\boldsymbol{HExp}$ distributed demand process this observation reverse, i.e., $\boldsymbol{HExp}$ distributed lead time has low optimal total cost rate and $\boldsymbol{Erl}$ distributed lead time has high optimal total cost rate.

Example 2: This example serves to illustrate the effect of the arrival rate λ, the lead time rate β and the type of arrival and lead time processes on the optimal values (s_1^*, s_2^*) and optimal cost rate $TC(15, s_1^*, 30, s_2^*)$. The following fixed values were assumed for the parameters and cost:

$D_0 = H_0, D_1 = 0.3H_1, D_2 = 0.4H_1, D_{12} = 0.3H_1, \gamma_1 = 0.6, \gamma_2 = 0.5, p_i = 0.55 * 0.45^{i-1}, i = 1,2,\ldots,$
$ch_1 = 0.01, ch_2 = 0.01, c_s = 10, c_{sh_1} = 0.8, c_{sh_2} = 1.5, c_{sh_{12}} = 1, c_{f_1} = 0.2, c_{f_2} = 0.2.$

The optimum values, s_1^* and s_2^*, that minimizes the expected total cost for each of the five $\boldsymbol{MAP}$s for arrivals of demands considered against each of the three $\boldsymbol{PH}$s for lead times is given in the table 4.2. The associated total expected cost rate values are also given. The lower entry in each cell gives the optimal expected cost rate and the upper entries correspond to s_1^* and s_2^*. The key observations are summarized below.

1. As λ increases, the optimal total cost rate increases except for $\boldsymbol{Hexp}$ distributed demand process. For $\boldsymbol{Hexp}$ distributed demand process, the optimal total cost rate decreases as the demand rate λ increases.
2. When β increases, the optimal total cost rate increases for all combination of five arrival processes and three demands processes.
3. The optimal cost rate is high in the cases wherein the demand process is $\boldsymbol{Hexp}$ and it is low when the demand process is Erlang.
4. The optimal total cost rate is low when the lead time is $\boldsymbol{Erl}$ except for the $\boldsymbol{Hexp}$ distributed demand process. For $\boldsymbol{Hexp}$ distributed lead time the optimal total cost rate is high except for $\boldsymbol{Hexp}$ distributed demand process. For $\boldsymbol{HExp}$ distributed demand process this observation reverse., i.e., $\boldsymbol{Hexp}$ distributed lead time is associated with low optimal total cost rate and $\boldsymbol{Erl}$ is associated with high optimal total cost rate.

Table 4.1: Total expected cost rate as a function of (S_1, S_2)

Lead time distribution								
	β		10			15		
	λ		Exp	Erl	HExp	Exp	Erl	HExp
MAP demands distributions	6	Exp	(13,46)	(13,46)	(13,46)	(13,47)	(13,47)	(13,47)
			8.0236	8.0177	8.0277	8.2027	8.1999	8.2047
		Erl	(13,46)	(13,46)	(13,46)	(13,47)	(13,47)	(13,47)
			7.9967	7.9905	8.0009	8.1838	8.1809	8.1858
		HExp	(13,46)	(13,46)	(13,46)	(13,47)	(13,47)	(13,47)
			8.1568	8.1570	8.1567	8.2977	8.2978	8.2976
		MNC	(13,46)	(13,46)	(13,46)	(13,47)	(13,47)	(13,47)
			8.0736	8.0680	8.0774	8.2379	8.2352	8.2397
		MPC	(13,46)	(13,46)	(13,46)	(13,47)	(13,47)	(13,47)
			8.1267	8.1214	8.1303	8.2753	8.2727	8.2770
	8	Exp	(17,58)	(17,58)	(17,58)	(18,60)	(18,60)	(18,60)
			10.5224	10.5175	10.5258	10.8125	10.8101	10.8141
		Erl	(17,58)	(17,58)	(17,58)	(18,60)	(18,60)	(18,60)
			10.4959	10.4908	10.4994	10.7939	10.7915	10.7956
		HExp	(17,58)	(17,58)	(17,58)	(17,59)	(17,59)	(18,60)
			10.6604	10.6608	10.6601	10.9104	10.9106	10.9103
		MNC	(17,58)	(17,58)	(17,58)	(18,60)	(18,60)	(18,60)
			10.5717	10.5670	10.5749	10.8470	10.8448	10.8486
		MPC	(17,58)	(17,58)	(17,58)	(18,60)	(18,60)	(18,60)
			10.6260	10.6215	10.6291	10.8852	10.8830	10.8867

Table 4.2: Total expected cost rate as a function of (s_1, s_2)

Lead time distribution									
	β		10				15		
	λ		Exp	Erl	HExp		Exp	Erl	HExp
MAP Demands Distri-butions	6	Exp	(4,4)	(4,4)	(4,4)		(4,4)	(4,4)	(4,4)
			7.2328	7.2286	7.2356		7.3624	7.3604	7.3638
		Erl	(4,4)	(4,4)	(4,4)		(4,4)	(4,4)	(4,4)
			7.2080	7.2037	7.2111		7.3450	7.3429	7.3464
		HExp	(4,4)	(4,4)	(4,4)		(4,4)	(4,4)	(4,4)
			7.3579	7.3598	7.3567		7.4518	7.4527	7.4512
		MNC	(4,4)	(4,4)	(4,4)		(4,4)	(4,4)	(4,4)
			7.2787	7.2748	7.2814		7.3947	7.3929	7.3960
		MPC	(4,4)	(4,4)	(4,4)		(4,4)	(4,4)	(4,4)
			7.3282	7.3245	7.3307		7.4296	7.4279	7.4309
	8	Exp	(3,4)	(3,4)	(3,4)		(3,4)	(3,4)	(3,4)
			9.4935	9.4902	9.4957		9.7144	9.7129	9.7155
		Erl	(3,4)	(3,4)	(3,4)		(3,4)	(3,4)	(3,4)
			9.4694	9.4660	9.4717		9.6977	9.6961	9.6988
		HExp	(3,4)	(3,4)	(3,4)		(3,4)	(3,4)	(3,4)
			9.6194	9.6211	9.6183		9.8030	9.8038	9.8025
		MNC	(3,4)	(3,4)	(3,4)		(3,4)	(3,4)	(3,4)
			9.5381	9.5351	9.5402		9.7455	9.7441	9.7465
		MPC	(3,4)	(3,4)	(3,4)		(3,4)	(3,4)	(3,4)
			9.5876	9.5847	9.5895		9.7799	9.7786	9.7809

Example 3: Next, the impact of c_{f_1} and c_{f_2} on the total expected cost rate was considered. For this, the following values were considered for the parameters and costs: $D_0 = H_0, D_1 = 0.3H_1, D_2 = 0.4H_1, D_{12} = 0.3H_1, \lambda = 8, \beta = 0.5, \gamma_1 = 0.6, \gamma_2 = 0.5, p_i = 0.55 * 0.45^{i-1}$, $i = 1,2,\ldots, c_{h_1} = 0.01, c_{h_2} = 0.01, c_s = 10, c_{sh_1} = 0.8, c_{sh_2} = 1.5, c_{sh_{12}} = 1.$

The graphs of the total expected cost rate as a function of c_{f_1} and c_{f_2} were plotted for the three lead time processes and the five demand processes in figures 4.2 – 4.6. In all the figures the lead time distributions $\boldsymbol{Exp}, \boldsymbol{Erl}$ and $\boldsymbol{HExp}$ are coloured as blue, black and red respectively. The following were noted:

- In all the five arrival processes, as c_{f_1} and c_{f_2} increase simultaneously, the total expected cost rate increases. But the increasing rate for c_{f_2} is high compared to c_{f_1}.

- The Erlang lead time process is associated with low total expected cost rate and for the hyper exponential lead time process case the total expected cost rate is high.

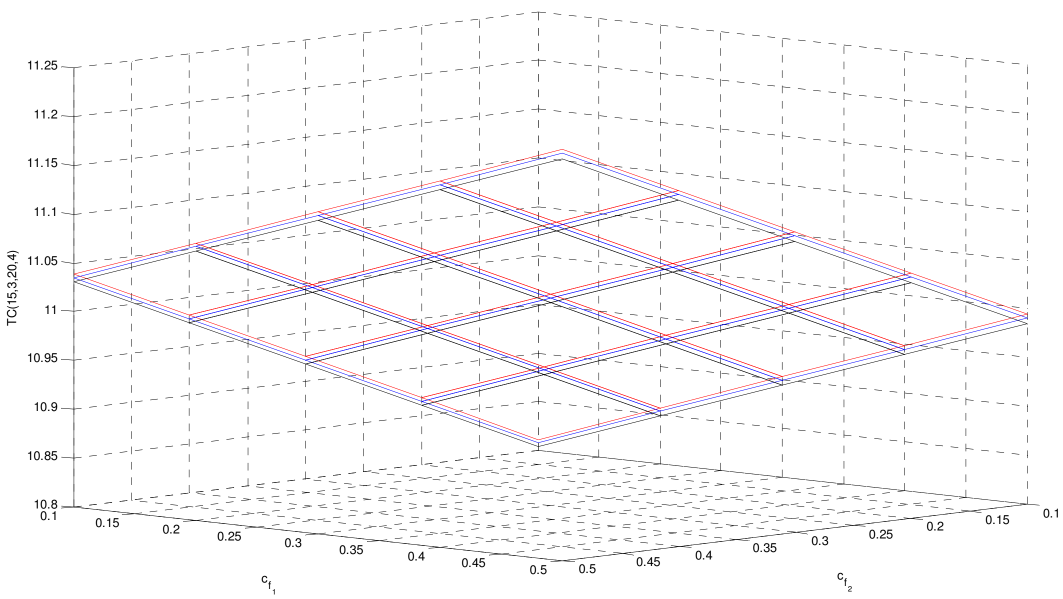

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Figure 4.2: $\boldsymbol{Exp}$ demand process

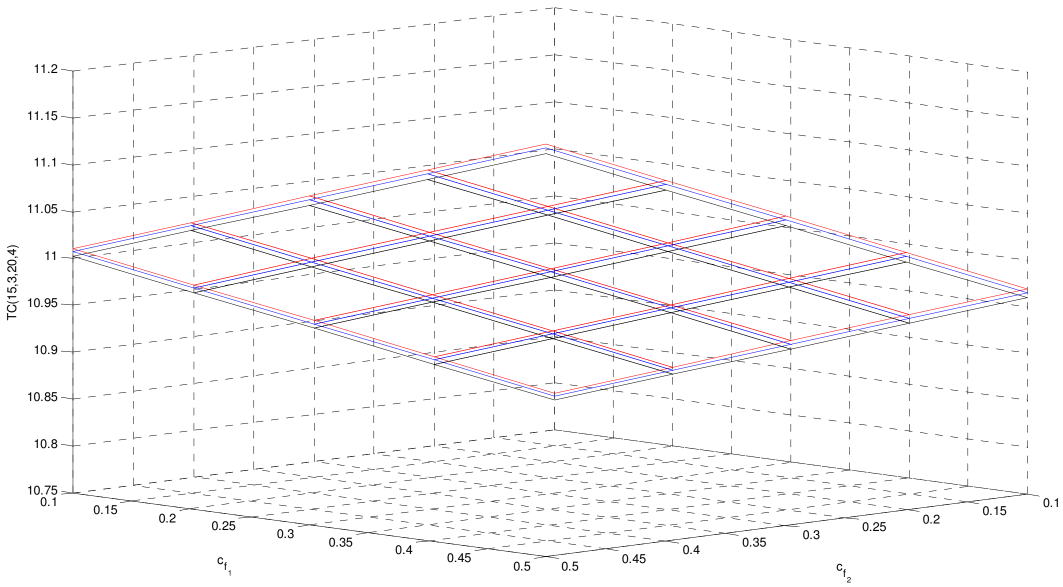

Figure 4.3: Erl demand process

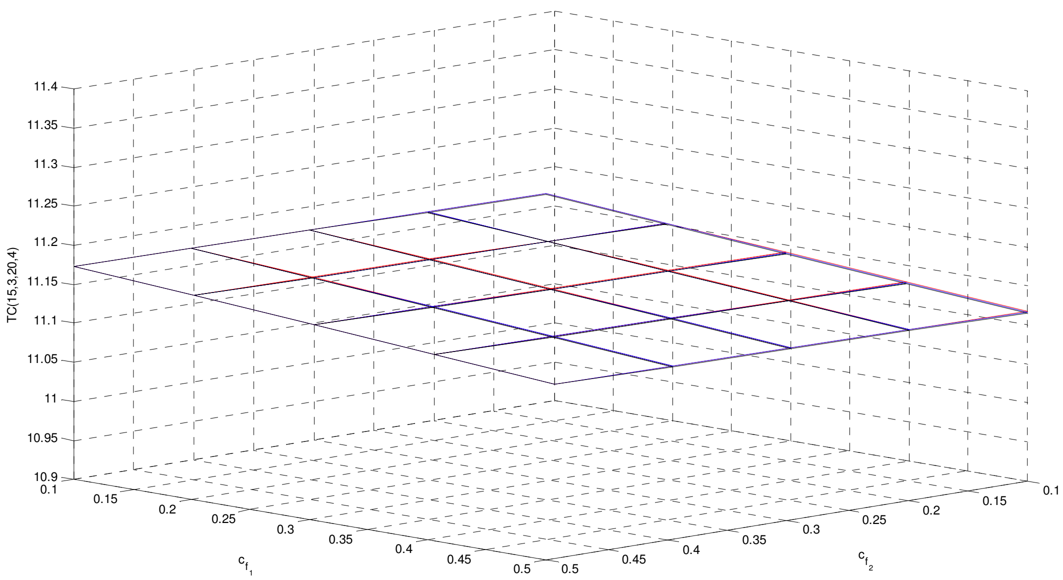

Figure 4.4: $HExp$ demand process

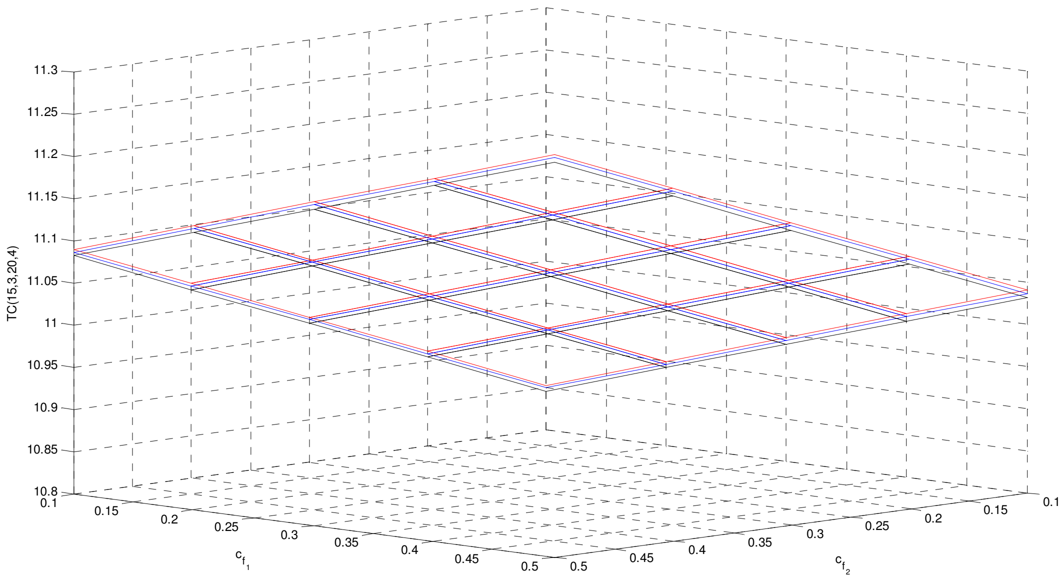

Figure 4.5: MNC demand process

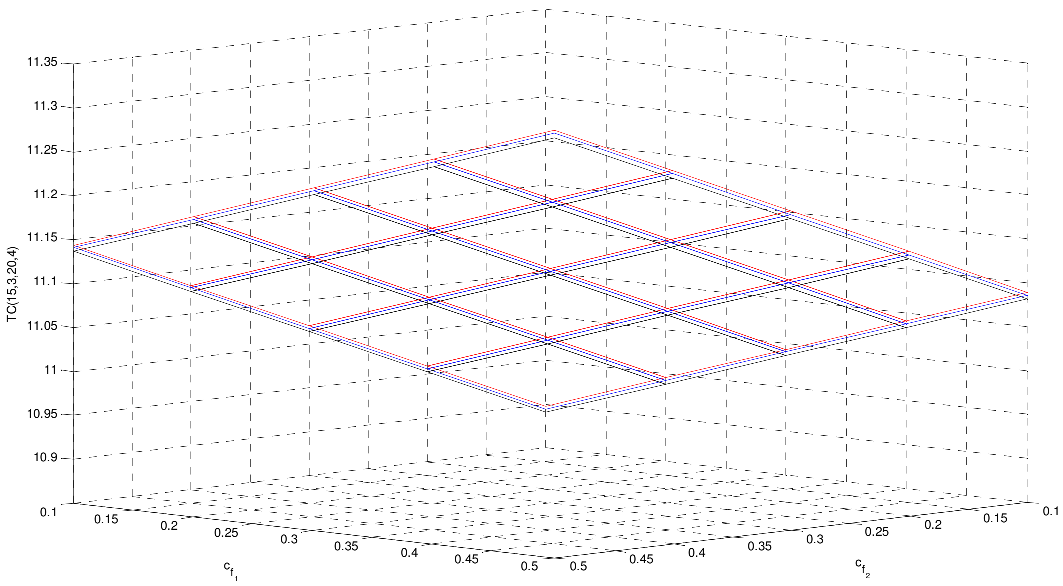

Figure 4.6: MPC demand process

Example 4: In the final example, the impact of c_{h_1} and c_{h_2} on the total expected cost rate was shown. The following values were considered for the parameters and costs: $D_0 = H_0, D_1 = 0.3H1, D_2 = 0.4H1, D_{12} = 0.3H1, \lambda = 15, \beta = 2, \gamma_1 = 0.8, \gamma_2 = 0.4, p_i = 0.6 * 0.4^{i-1}$, $i = 1,2,\ldots, c_s = 10, c_{sh_1} = 0.8, c_{sh_2} = 1.5, c_{sh_{12}} = 1, c_{f_1} = 0.2, c_{f_2} = 0.2.$

The graphs of the total expected cost rate as a function of c_{f_1} and c_{f_2} were plotted for the three lead time processes and the five demand processes in figures 4.7 – 4.11. In all the figures the plots for the lead time distributions $\boldsymbol{Exp}$**,** $\boldsymbol{Erl}$ and $\boldsymbol{HExp}$ are coloured as blue, black and red respectively. The following were observed:

- In all the five arrival processes, as c_{h_1} and c_{h_2} increase, the total expected cost rate increases. But the increasing rate for c_{h_2} is high compared to that of c_{h_1}.

- For all the demand process, the Erlang lead time process has low total expected cost rate and hyper exponential lead time process has high total expected cost rate.

- The difference between the total expected cost rate for any two lead time process is high except for $\boldsymbol{HExp}$ demand process. For the $\boldsymbol{HExp}$ demand process, the difference between the total expected cost rate for any two lead time process is low.

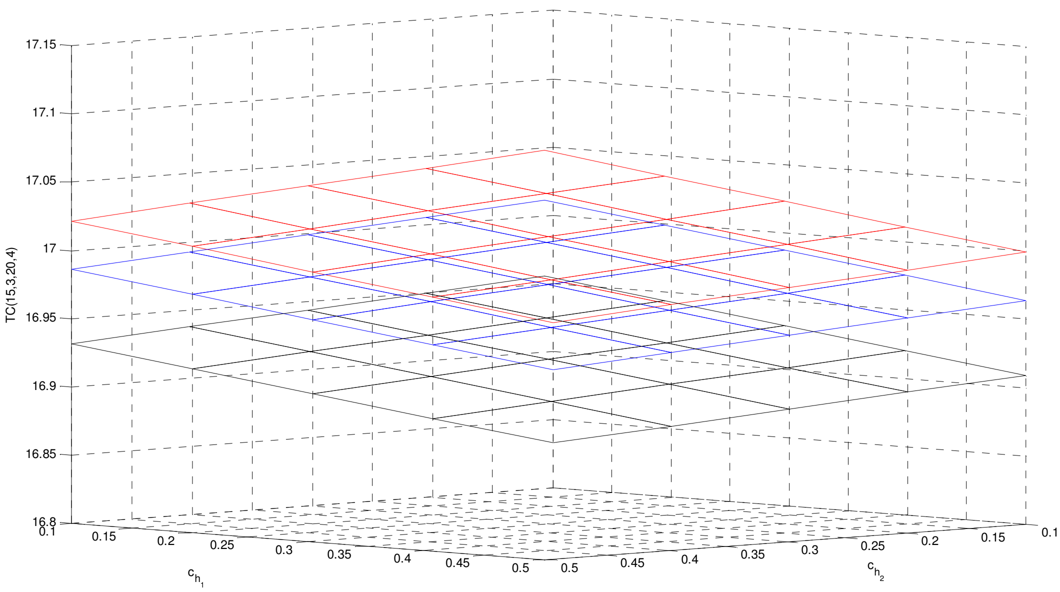

Figure 4.7: $\boldsymbol{Exp}$ **demand process**

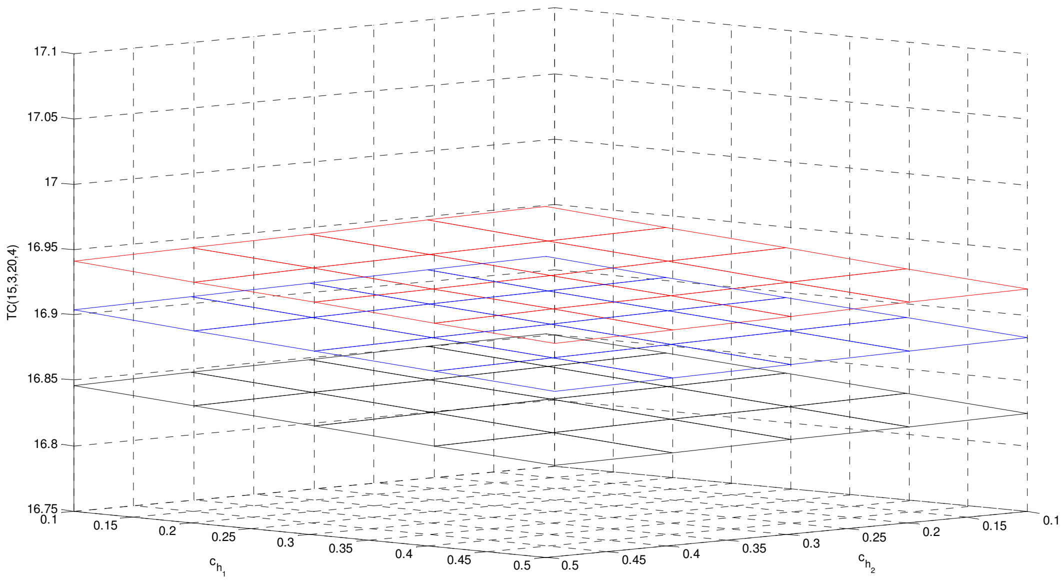

Figure 4.8: Erl **demand process**

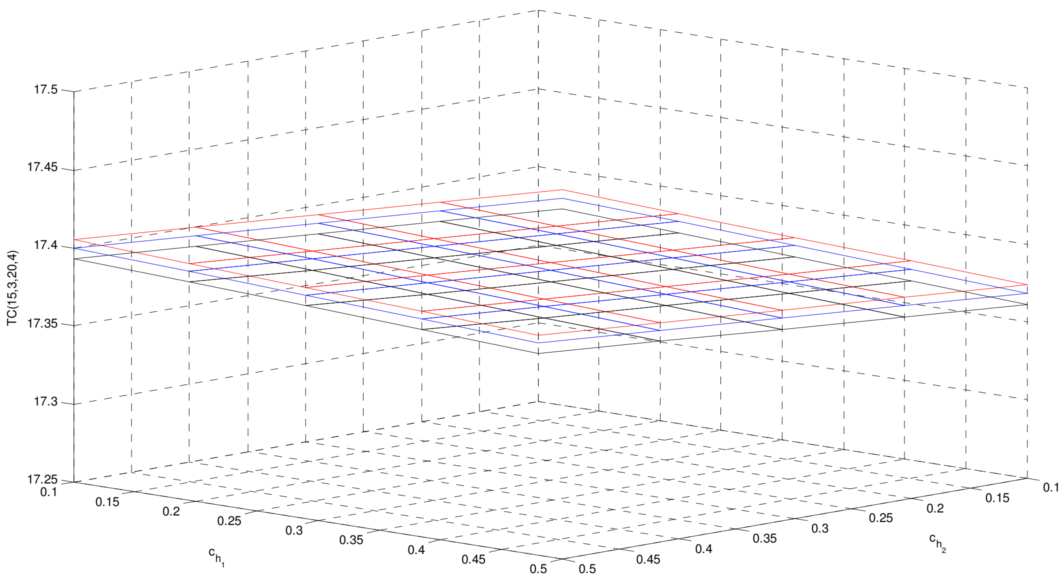

Figure 4.9: $HExp$ **demand process**

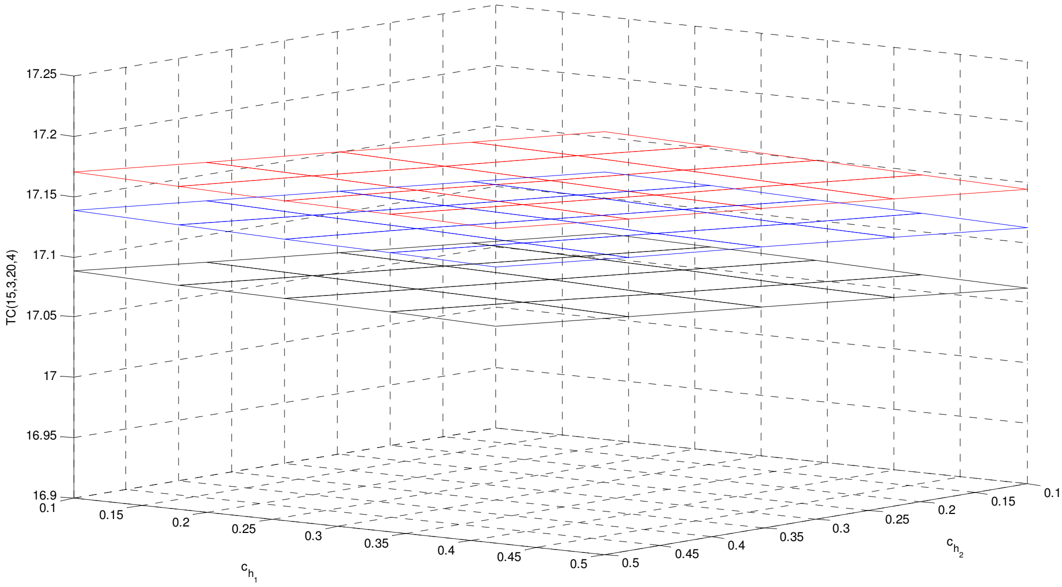

Figure 4:10.: MNC demand process

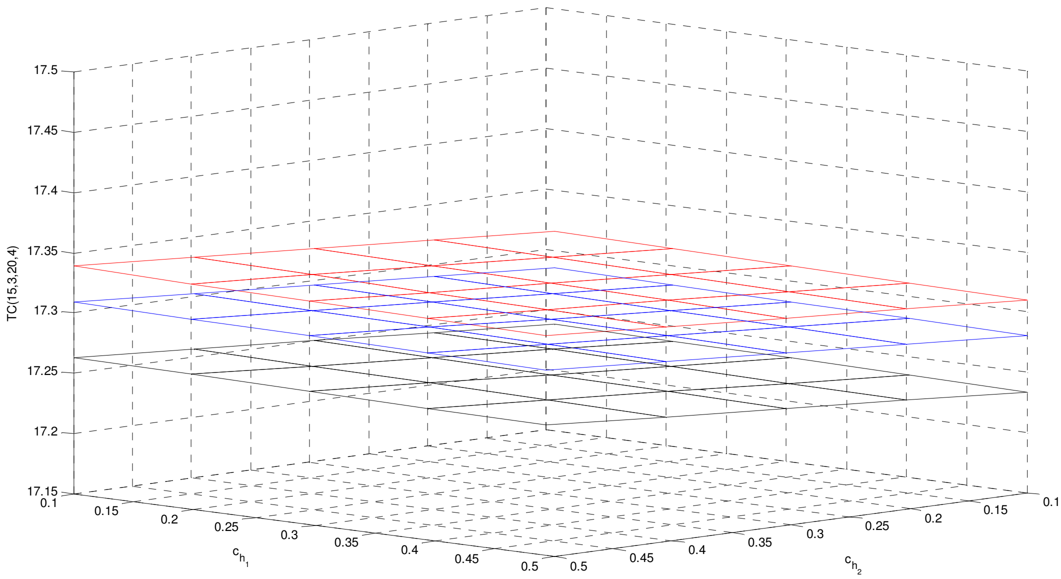

Figure 4.11: MPC demand process

4.7. CONCLUSION

The existing work on two-commodity continuous review inventory system have been extended by introducing the perishability for both commodities, Markov Arrival Process for demand time points and phase type distribution for lead time. It was also assumed that one of the commodities may accept bulk demands. Steady state solutions for the joint distribution of inventory levels have been provided. Under suitable cost structure, the total expected cost rate in steady state have been constructed. To demonstrate the computability of results derived here, ample numerical illustrations have been provided. The effect of the parameters and costs on the total expected cost rate have also been numerically analyzed.