APPENDIX A

Derivations Related to the Green's Function

In this appendix, it is shown how the spectral-domain Green's-function components of (3.56) to (3.63), for planarly multilayered substrates, can be derived from the spatial-domain form given by Chew [178]. The derivation is available in the literature [6], but is included here for completeness sake. Although this particular derivation is presented, the spectral-domain Green's-function components can of course also be derived in other ways. Furthermore, due to the presence of vertical electric current densities, some of the Green's-function components have to be integrated over the z and/or z' variables. In the text of Chapter 3, these are referred to as expanded Green's-function components. The expressions for these expanded Green's-function components will also be presented in this appendix as they are not, to the author's knowledge, available elsewhere.

A.1 SPECTRAL-DOMAIN GREEN'S-FUNCTION COMPONENTS

As shown by Chew [178], the spatial-domain dyadic Green's function for the planarly multilayered medium in Figure 3.4, can be expressed as

$$\bar{\mathbf{G}}(\mathbf{r}|\mathbf{r}') = \frac{-j}{8\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{k_{z(\ell')} k_\rho^2} \left[\bar{\mathbf{M}}(k_x, k_y, \mathbf{r}|\mathbf{r}') + \bar{\mathbf{N}}(k_x, k_y, \mathbf{r}|\mathbf{r}') \right] dk_x dk_y - \frac{1}{k_{(\ell')}^2} \delta(\mathbf{r} - \mathbf{r}') \, \hat{z}\hat{z}, \quad \text{(A.1)}$$

where

$$\bar{\mathbf{M}}(k_x, k_y, \mathbf{r}|\mathbf{r}') = \left(\nabla \times \hat{z}\right)\left(\nabla' \times \hat{z}\right) e^{-jk_x(x-x')-jk_y(y-y')} F_{(\ell)\pm}^{\mathrm{TE}}(z, z') \quad \text{(A.2)}$$

and

$$\bar{\mathbf{N}}(k_x,k_y,\mathbf{r}|\mathbf{r}') = \left[\frac{\nabla\times\nabla\times\hat{z}}{j\omega\varepsilon_{(\ell)}}\right]\left[\frac{\nabla'\times\nabla'\times\hat{z}}{-j\omega\mu_{(\ell')}}\right] e^{-jk_x(x-x')-jk_y(y-y')} F^{\mathrm{TM}}_{(\ell)\pm}(z,z'), \qquad \text{(A.3)}$$

with

$$\nabla = -jk_x\,\hat{x} - jk_y\,\hat{y} + \frac{\partial}{\partial z}\,\hat{z} \qquad \text{(A.4)}$$

and

$$\nabla' = jk_x\,\hat{x} + jk_y\,\hat{y} + \frac{\partial}{\partial z'}\,\hat{z}. \qquad \text{(A.5)}$$

By applying the two-dimensional Fourier transform of (3.18), to (A.1), the spectral-domain dyadic Green's function for a planarly multilayered medium can be expressed as [6]

$$\tilde{\bar{\mathbf{G}}}(k_x,k_y,z|z') = \frac{-\omega\mu_{(\ell')}}{2k_{z(\ell')}k_\rho^2}\left[\tilde{\bar{\mathbf{M}}}(-k_x,-k_y,z|z') + \tilde{\bar{\mathbf{N}}}(-k_x,-k_y,z|z')\right] - \frac{1}{j\omega\varepsilon_{(\ell')}}\,\delta(z-z')\,\hat{z}\hat{z}, \qquad \text{(A.6)}$$

where

$$\tilde{\bar{\mathbf{M}}}(k_x,k_y,z|z') = \left(\nabla\times\hat{z}\right)\left(\nabla'\times\hat{z}\right) F^{\mathrm{TE}}_{(\ell)\pm}(z,z') \qquad \text{(A.7)}$$

and

$$\tilde{\bar{\mathbf{N}}}(k_x,k_y,z|z') = \left(\frac{\nabla\times\nabla\times\hat{z}}{j\omega\varepsilon_{(\ell)}}\right)\left(\frac{\nabla'\times\nabla'\times\hat{z}}{-j\omega\mu_{(\ell')}}\right) F^{\mathrm{TM}}_{(\ell)\pm}(z,z'). \qquad \text{(A.8)}$$

The curl operations in the expression for $\tilde{\bar{\mathbf{M}}}$ can be carried out as

$$\nabla\times\hat{z} = -jk_x\,\hat{x} + jk_y\,\hat{y} \qquad \text{(A.9)}$$

and

$$\nabla'\times\hat{z} = jk_x\,\hat{x} - jk_y\,\hat{y}, \qquad \text{(A.10)}$$

resulting in

$$\left(\nabla\times\hat{z}\right)\left(\nabla'\times\hat{z}\right) = k_y^2\,\hat{x}\hat{x} + k_x^2\,\hat{y}\hat{y} - k_xk_y\,\hat{x}\hat{y} - k_xk_y\,\hat{y}\hat{x}. \qquad \text{(A.11)}$$

Similarly, the curl operations in the expression for $\tilde{\bar{\mathbf{N}}}$ can be carried out as

$$\nabla\times\nabla\times\hat{z} = -jk_x\frac{\partial}{\partial z}\,\hat{x} - jk_y\frac{\partial}{\partial z}\,\hat{y} + k_\rho^2\hat{z} \qquad \text{(A.12)}$$

and

$$\nabla'\times\nabla'\times\hat{z} = jk_x\frac{\partial}{\partial z'}\,\hat{x} + jk_y\frac{\partial}{\partial z'}\,\hat{y} + k_\rho^2\hat{z}, \qquad \text{(A.13)}$$

resulting in

$$\begin{aligned}\left(\nabla\times\nabla\times\hat{z}\right)\left(\nabla'\times\nabla\times\hat{z}\right) &= \frac{\partial^2}{\partial z\partial z'}\left(k_x^2\,\hat{x}\hat{x} + k_y^2\,\hat{y}\hat{y} + k_xk_y\,\hat{x}\hat{y} + k_xk_y\,\hat{y}\hat{x}\right)\\ &\quad - jk_\rho^2\frac{\partial}{\partial z}\left(k_x\,\hat{x}\hat{z} + jk_y\,\hat{y}\hat{z}\right) - jk_\rho^2\frac{\partial}{\partial z}\left(k_x\,\hat{x}\hat{z} + jk_y\,\hat{y}\hat{z}\right) + k_\rho^4\,\hat{z}\hat{z}. \qquad \text{(A.14)}\end{aligned}$$

Finally, by replacing the curl operators in (A.7) and (A.8) with their relevant expressions, it follows from (A.6) that the components of the spectral-domain dyadic Green's function can be expressed as

$$\tilde{\bar{\mathbf{G}}} = \hat{x}\,\tilde{G}_{xx}\,\hat{x} + \hat{x}\,\tilde{G}_{xy}\,\hat{y} + \hat{x}\,\tilde{G}_{xz}\,\hat{z} + \hat{y}\,\tilde{G}_{yx}\,\hat{x} + \hat{y}\,\tilde{G}_{yy}\,\hat{y} + \hat{y}\,\tilde{G}_{yz}\,\hat{z} + \hat{z}\,\tilde{G}_{zx}\,\hat{x} + \hat{z}\,\tilde{G}_{zy}\,\hat{y} + \hat{z}\,\tilde{G}_{zz}\,\hat{z}, \tag{A.15}$$

with

$$\tilde{G}_{xx}(k_x,k_y,z|z') = \frac{-\omega\mu_{(\ell')}}{2k_{z(\ell')}k_\rho^2}\left[k_y^2 F^{\mathrm{TE}}_{(\ell)\pm}(z,z') + \frac{k_x^2}{\omega^2\mu_{(\ell')}\varepsilon_{(\ell)}}\frac{\partial^2}{\partial z'\partial z}F^{\mathrm{TM}}_{(\ell)\pm}(z,z')\right], \tag{A.16}$$

$$\tilde{G}_{yy}(k_x,k_y,z|z') = \frac{-\omega\mu_{(\ell')}}{2k_{z(\ell')}k_\rho^2}\left[k_x^2 F^{\mathrm{TE}}_{(\ell)\pm}(z,z') + \frac{k_y^2}{\omega^2\mu_{(\ell')}\varepsilon_{(\ell)}}\frac{\partial^2}{\partial z'\partial z}F^{\mathrm{TM}}_{(\ell)\pm}(z,z')\right], \tag{A.17}$$

$$\begin{aligned}\tilde{G}_{xy}(k_x,k_y,z|z') &= \tilde{G}_{yx}(k_x,k_y,z|z')\\ &= \frac{-\omega\mu_{(\ell')}}{2k_{z(\ell')}k_\rho^2}\,k_xk_y\left[-F^{\mathrm{TE}}_{(\ell)\pm}(z,z') + \frac{1}{\omega^2\mu_{(\ell')}\varepsilon_{(\ell)}}\frac{\partial^2}{\partial z'\partial z}F^{\mathrm{TM}}_{(\ell)\pm}(z,z')\right],\end{aligned} \tag{A.18}$$

$$\tilde{G}_{xz}(k_x,k_y,z|z') = \frac{jk_x}{2k_{z(\ell')}\omega\varepsilon_{(\ell)}}\frac{\partial}{\partial z}F^{\mathrm{TM}}_{(\ell)\pm}(z,z'), \tag{A.19}$$

$$\tilde{G}_{yz}(k_x,k_y,z|z') = \frac{jk_y}{2k_{z(\ell')}\omega\varepsilon_{(\ell)}}\frac{\partial}{\partial z}F^{\mathrm{TM}}_{(\ell)\pm}(z,z'), \tag{A.20}$$

$$\tilde{G}_{zx}(k_x,k_y,z|z') = \frac{-jk_x}{2k_{z(\ell')}\omega\varepsilon_{(\ell)}}\frac{\partial}{\partial z'}F^{\mathrm{TM}}_{(\ell)\pm}(z,z'), \tag{A.21}$$

$$\tilde{G}_{zy}(k_x,k_y,z|z') = \frac{-jk_y}{2k_{z(\ell')}\omega\varepsilon_{(\ell)}}\frac{\partial}{\partial z'}F^{\mathrm{TM}}_{(\ell)\pm}(z,z') \tag{A.22}$$

and

$$\tilde{G}_{zz}(k_x,k_y,z|z') = \frac{-k_\rho^2}{2k_{z(\ell')}\omega\varepsilon_{(\ell)}}F^{\mathrm{TM}}_{(\ell)\pm}(z,z') - \frac{1}{j\omega\varepsilon_{(\ell')}}\delta(z-z'). \tag{A.23}$$

These are the expressions given in (3.56) to (3.63).

A.2 EXPANDED GREEN'S-FUNCTION COMPONENTS

The expanded Green's-function components include all those expressions where the product of the Green's function and the vertical current-density variation on the probes has to be integrated over z and/or z'. A single function, $f_n(z)$, can be used to represent the piecewise sinusoidal (PWS) parts of the basis and/or testing functions on the probes, as well as the probe part of the attachment modes. This function $f_n(z)$ can be expressed as

$$f_n(z) = \begin{cases} f^{PZ}_{nz}(z) = a\,e^{jk_Fz} + b\,e^{-jk_Fz}, & \text{for the probe basis/tesing functions}\\ f^{AZ}_{na}(z) = a\,e^{jk_Fz} + b\,e^{-jk_Fz}, & \text{for the probe part of the attachment modes,}\end{cases} \tag{A.24}$$

where

$$a = \begin{cases} \dfrac{-e^{-jk_F z_1}}{j2\sin[k_F(z_2 - z_1)]}, & nz \geq 1,\ z_{nz}^{PZ} \leq z \leq z_{nz}^{PZ} + \Delta z_{nz}^{PZ+} \\ \dfrac{e^{-jk_F z_1}}{j2\sin[k_F(z_2 - z_1)]}, & nz \geq 2,\ z_{nz}^{PZ} - \Delta z_{nz}^{PZ-} \leq z \leq z_{nz}^{PZ} \\ \dfrac{e^{-jk_F z_1}}{j2\sin[k_F(z_2 - z_1)]}, & z_{na}^{AZ} - \Delta z_{na}^{AZ-} \leq z \leq z_{na}^{AZ} \end{cases} \tag{A.25}$$

and

$$b = \begin{cases} \dfrac{e^{jk_F z_2}}{j2\sin[k_F(z_2 - z_1)]}, & nz \geq 1,\ z_{nz}^{PZ} \leq z \leq z_{nz}^{PZ} + \Delta z_{nz}^{PZ+} \\ \dfrac{-e^{jk_F z_2}}{j2\sin[k_F(z_2 - z_1)]}, & nz \geq 2,\ z_{nz}^{PZ} - \Delta z_{nz}^{PZ-} \leq z \leq z_{nz}^{PZ} \\ \dfrac{-e^{jk_F z_2}}{j2\sin[k_F(z_2 - z_1)]}, & z_{na}^{AZ} - \Delta z_{na}^{AZ-} \leq z \leq z_{na}^{AZ}, \end{cases} \tag{A.26}$$

with

$$z_1 = \begin{cases} z_{nz}^{PZ}, & nz \geq 1,\ z_{nz}^{PZ} \leq z \leq z_{nz}^{PZ} + \Delta z_{nz}^{PZ+} \\ z_{nz}^{PZ} - \Delta z_{nz}^{PZ-}, & nz \geq 2,\ z_{nz}^{PZ} - \Delta z_{nz}^{PZ-} \leq z \leq z_{nz}^{PZ} \\ z_{na}^{AZ} - \Delta z_{na}^{AZ-}, & z_{na}^{AZ} - \Delta z_{na}^{AZ-} \leq z \leq z_{na}^{AZ} \end{cases} \tag{A.27}$$

and

$$z_2 = \begin{cases} z_{nz}^{PZ} + \Delta z_{nz}^{PZ+}, & nz \geq 1,\ z_{nz}^{PZ} \leq z \leq z_{nz}^{PZ} + \Delta z_{nz}^{PZ+} \\ z_{nz}^{PZ}, & nz \geq 2,\ z_{nz}^{PZ} - \Delta z_{nz}^{PZ-} \leq z \leq z_{nz}^{PZ} \\ z_{na}^{AZ}, & z_{na}^{AZ} - \Delta z_{na}^{AZ-} \leq z \leq z_{na}^{AZ}. \end{cases} \tag{A.28}$$

The other parameters have already been defined in Sections 3.5.1, 3.6.1 and 3.6.2.

For planarly-orientated testing functions and vertically-orientated basis functions (or for calculating the far fields from vertically-orientated basis functions), the integrals

$$\tilde{G}_{xz}^{I}(k_x, k_y, z) = \int_{z'} \tilde{G}_{xz}(k_x, k_y, z|z') f_n(z')\, dz' \tag{A.29}$$

and

$$\tilde{G}_{yz}^{I}(k_x, k_y, z) = \int_{z'} \tilde{G}_{yz}(k_x, k_y, z|z') f_n(z')\, dz' \tag{A.30}$$

need to be evaluated. They can be expanded as

$$\tilde{G}_{xz}^{I}(k_x, k_y, z) = \begin{cases} \tilde{G}_{xz}^{I+}(k_x, k_y, z) & z \geq z_2' \\ \tilde{G}_{xz}^{I-}(k_x, k_y, z) & z \leq z_1' \end{cases} \tag{A.31}$$

and

$$\tilde{G}_{yz}^{I}(k_x, k_y, z) = \begin{cases} \tilde{G}_{yz}^{I+}(k_x, k_y, z) & z \geq z_2' \\ \tilde{G}_{yz}^{I-}(k_x, k_y, z) & z \leq z_1'. \end{cases} \tag{A.32}$$

By using the expressions in (A.24) to (A.28), the expression for $\tilde{G}_{xz}^{I+}$ can be derived as

$$\begin{aligned}
&\tilde{G}_{xz}^{I+}(k_x, k_y, z) \\
&= \int_{z_1'}^{z_2'} \tilde{G}_{xz}(k_x, k_y, z|z') f_n(z')\, dz', \quad z \geq z_2' \\
&= \frac{-jk_x}{2k_{z(\ell')}\omega\varepsilon_{(\ell)}} \int_{z_1'}^{z_2'} \frac{\partial}{\partial z} F_{(\ell)+}^{\mathrm{TM}}(z, z') \Big(a'\, e^{jk_F z'} + b'\, e^{-jk_F z'} \Big) dz' \\
&= \frac{k_x k_{z(\ell)} A_{(\ell)+}^{\mathrm{TM}}}{2k_{z(\ell')}\omega\varepsilon_{(\ell)}} \Big(-e^{-jk_{z(\ell)} z} + \check{R}_{(\ell),(\ell-1)}^{\mathrm{TM}}\, e^{-j2k_{z(\ell)} d_{(\ell-1)}}\, e^{jk_{z(\ell)} z} \Big) \\
&\cdot \Bigg(\frac{a'}{j[k_{z(\ell')} + k_F]} \Big\{ e^{j[k_{z(\ell')}+k_F] z_2'} - e^{j[k_{z(\ell')}+k_F] z_1'} \Big\} \\
&+ \frac{b'}{j[k_{z(\ell')} - k_F]} \Big\{ e^{j[k_{z(\ell')}-k_F] z_2'} - e^{j[k_{z(\ell')}-k_F] z_1'} \Big\} \\
&+ \frac{a' \check{R}_{(\ell'),(\ell'+1)}^{\mathrm{TM}}\, e^{j2k_{z(\ell')} d_{(\ell')}}}{j[-k_{z(\ell')} + k_F]} \Big\{ e^{j[-k_{z(\ell')}+k_F] z_2'} - e^{j[-k_{z(\ell')}+k_F] z_1'} \Big\} \\
&+ \frac{b' \check{R}_{(\ell'),(\ell'+1)}^{\mathrm{TM}}\, e^{j2k_{z(\ell')} d_{(\ell')}}}{j[-k_{z(\ell')} - k_F]} \Big\{ e^{j[-k_{z(\ell')}-k_F] z_2'} - e^{j[-k_{z(\ell')}-k_F] z_1'} \Big\} \Bigg).
\end{aligned} \tag{A.33}$$

It then also follows that $\tilde{G}_{yz}^{I+} = \tilde{G}_{xz}^{I+}\big|_{k_x = k_y}$. Furthermore, the expression for $\tilde{G}_{xz}^{I-}$ can be derived as

$$\begin{aligned}
&\tilde{G}_{xz}^{I-}(k_x, k_y, z) \\
&= \int_{z_1'}^{z_2'} \tilde{G}_{xz}(k_x, k_y, z|z') f_n(z')\, dz', \quad z \leq z_1' \\
&= \frac{-jk_x}{2k_{z(\ell')}\omega\varepsilon_{(\ell)}} \int_{z_1'}^{z_2'} \frac{\partial}{\partial z} F_{(\ell)-}^{\mathrm{TM}}(z, z') \Big(a'\, e^{jk_F z'} + b'\, e^{-jk_F z'} \Big) dz' \\
&= \frac{k_x k_{z(\ell)} A_{(\ell)-}^{\mathrm{TM}}}{2k_{z(\ell')}\omega\varepsilon_{(\ell)}} \Big(e^{jk_{z(\ell)} z} - \check{R}_{(\ell'),(\ell'+1)}^{\mathrm{TM}}\, e^{j2k_{z(\ell')} d_{(\ell')}}\, e^{-jk_{z(\ell)} z} \Big) \\
&\cdot \Bigg(\frac{a'}{j[-k_{z(\ell')} + k_F]} \Big\{ e^{j[-k_{z(\ell')}+k_F] z_2'} - e^{j[-k_{z(\ell')}+k_F] z_1'} \Big\} \\
&+ \frac{b'}{j[-k_{z(\ell')} - k_F]} \Big\{ e^{j[-k_{z(\ell')}-k_F] z_2'} - e^{j[-k_{z(\ell')}-k_F] z_1'} \Big\} \\
&+ \frac{a' \check{R}_{(\ell),(\ell-1)}^{\mathrm{TM}}\, e^{-j2k_{z(\ell-1)} d_{(\ell)}}}{j[k_{z(\ell')} + k_F]} \Big\{ e^{j[k_{z(\ell')}+k_F] z_2'} - e^{j[k_{z(\ell')}+k_F] z_1'} \Big\} \\
&+ \frac{b' \check{R}_{(\ell),(\ell-1)}^{\mathrm{TM}}\, e^{-j2k_{z(\ell-1)} d_{(\ell)}}}{j[k_{z(\ell')} - k_F]} \Big\{ e^{j[k_{z(\ell')}-k_F] z_2'} - e^{j[k_{z(\ell')}-k_F] z_1'} \Big\} \Bigg).
\end{aligned} \tag{A.34}$$

Once again, it then also follows that $\tilde{G}_{yz}^{I-} = \tilde{G}_{xz}^{I-}\big|_{k_x = k_y}$.

For vertically-orientated testing functions and planarly-orientated expansion functions, the integrals

$$\tilde{G}^I_{zx}(k_x, k_y, z') = \int_z f_m(z) \tilde{G}_{zx}(k_x, k_y, z|z')\, dz \tag{A.35}$$

and

$$\tilde{G}^I_{zy}(k_x, k_y, z') = \int_z f_m(z) \tilde{G}_{zy}(k_x, k_y, z|z')\, dz \tag{A.36}$$

need to be evaluated. They can be expanded as

$$\tilde{G}^I_{zx}(k_x, k_y, z) = \begin{cases} \tilde{G}^{I+}_{zx}(k_x, k_y, z) & z \geq z'_2 \\ \tilde{G}^{I-}_{zx}(k_x, k_y, z) & z \leq z'_1 \end{cases} \tag{A.37}$$

and

$$\tilde{G}^I_{zy}(k_x, k_y, z) = \begin{cases} \tilde{G}^{I+}_{zy}(k_x, k_y, z) & z \geq z'_2 \\ \tilde{G}^{I-}_{zy}(k_x, k_y, z) & z \leq z'_1. \end{cases} \tag{A.38}$$

By using the expressions in (A.24) to (A.28), the expression for $\tilde{G}^{I+}_{zx}$ can be derived as

$$\begin{aligned}
&\tilde{G}^{I+}_{zx}(k_x, k_y, z') \\
&= \int_{z_1}^{z_2} f_m(z) \tilde{G}_{zx}(k_x, k_y, z|z')\, dz, \quad z' \leq z_1 \\
&= \frac{j k_x}{2 k_{z(\ell')} \omega \varepsilon_{(\ell)}} \int_{z_1}^{z_2} \left(a\, e^{j k_F z} + b\, e^{-j k_F z} \right) \frac{\partial}{\partial z'} F^{\text{TM}}_{(\ell)+}(z, z')\, dz \\
&= \frac{-k_x A^{\text{TM}}_{(\ell)+}}{2 \omega \varepsilon_{(\ell)}} \left(e^{j k_{z(\ell')} z'} - \breve{R}^{\text{TM}}_{(\ell'),(\ell'+1)}\, e^{j 2 k_{z(\ell')} d_{(\ell')}}\, e^{-j k_{z(\ell')} z'} \right) \\
&\quad \cdot \left(\frac{a}{j[-k_{z(\ell)} + k_F]} \left\{ e^{j[-k_{z(\ell)} + k_F] z_2} - e^{j[-k_{z(\ell)} + k_F] z_1} \right\} \right. \\
&\quad + \frac{b}{j[-k_{z(\ell)} - k_F]} \left\{ e^{j[-k_{z(\ell)} - k_F] z_2} - e^{j[-k_{z(\ell)} - k_F] z_1} \right\} \\
&\quad + \frac{a \breve{R}^{\text{TM}}_{(\ell),(\ell-1)}\, e^{-j 2 k_{z(\ell-1)} d_{(\ell)}}}{j[k_{z(\ell)} + k_F]} \left\{ e^{j[k_{z(\ell)} + k_F] z_2} - e^{j[k_{z(\ell)} + k_F] z_1} \right\} \\
&\quad \left. + \frac{b \breve{R}^{\text{TM}}_{(\ell),(\ell-1)}\, e^{-j 2 k_{z(\ell-1)} d_{(\ell)}}}{j[k_{z(\ell)} - k_F]} \left\{ e^{j[k_{z(\ell)} - k_F] z_2} - e^{j[k_{z(\ell)} - k_F] z_1} \right\} \right).
\end{aligned} \tag{A.39}$$

It then also follows that $\tilde{G}^{I+}_{zy} = \tilde{G}^{I+}_{zx}\big|_{k_x = k_y}$. Furthermore, the expression for $\tilde{G}^{I-}_{zx}$ can be derived as

$$\begin{aligned}
&\tilde{G}^{I-}_{zx}(k_x, k_y, z') \\
&= \int_{z_1}^{z_2} f_m(z) \tilde{G}_{zx}(k_x, k_y, z|z')\, dz, \quad z' \geq z_2 \\
&= \frac{j k_x}{2 k_{z(\ell')} \omega \varepsilon_{(\ell)}} \int_{z_1}^{z_2} \left(a\, e^{j k_F z} + b\, e^{-j k_F z} \right) \frac{\partial}{\partial z'} F^{\text{TM}}_{(\ell)-}(z, z')\, dz
\end{aligned}$$

$$= \frac{-k_x A^{\mathrm{TM}}_{(\ell)-}}{2\omega\varepsilon_{(\ell)}} \left(-e^{-jk_{z(\ell')}z'} + \check{R}^{\mathrm{TM}}_{(\ell),(\ell-1)}\, e^{-j2k_{z(\ell)}d_{(\ell-1)}}\, e^{jk_{z(\ell')}z'} \right)$$
$$\cdot \left(\frac{a}{j[k_{z(\ell)}+k_F]} \left\{ e^{j[k_{z(\ell)}+k_F]z_2} - e^{j[k_{z(\ell)}+k_F]z_1} \right\} \right.$$
$$+ \frac{b}{j[k_{z(\ell)}-k_F]} \left\{ e^{j[k_{z(\ell)}-k_F]z_2} - e^{j[k_{z(\ell)}-k_F]z_1} \right\}$$
$$+ \frac{a\check{R}^{\mathrm{TM}}_{(\ell'),(\ell'+1)}\, e^{j2k_{z(\ell')}d_{(\ell')}}}{j[-k_{z(\ell)}+k_F]} \left\{ e^{j[-k_{z(\ell)}+k_F]z_2} - e^{j[-k_{z(\ell)}+k_F]z_1} \right\}$$
$$\left. + \frac{b\check{R}^{\mathrm{TM}}_{(\ell'),(\ell'+1)}\, e^{j2k_{z(\ell')}d_{(\ell')}}}{j[-k_{z(\ell)}-k_F]} \left\{ e^{j[-k_{z(\ell)}-k_F]z_2} - e^{j[-k_{z(\ell)}-k_F]z_1} \right\} \right). \quad \text{(A.40)}$$

Once again, it then also follows that $\tilde{G}^{I-}_{zy} = \tilde{G}^{I-}_{zx}\big|_{k_x=k_y}$.

For vertically-orientated testing functions and vertically-orientated expansion functions that do not overlap vertically, the integral

$$\tilde{G}^{I}_{zz}(k_x,k_y,z) = \int_{z'} \tilde{G}_{zz}(k_x,k_y,z|z') f_n(z')\, dz' \quad \text{(A.41)}$$

needs to be evaluated. It can be expanded as

$$\tilde{G}^{I}_{zz}(k_x,k_y,z) = \begin{cases} \tilde{G}^{I+}_{zz}(k_x,k_y,z) & z \geq z'_2 \\ \tilde{G}^{I-}_{zz}(k_x,k_y,z) & z \leq z'_1. \end{cases} \quad \text{(A.42)}$$

By using the expressions in (A.24) to (A.28), the expression for $\tilde{G}^{I+}_{zz}$ can be derived as

$$\tilde{G}^{I+}_{zz}(k_x,k_y,z)$$
$$= \int_{z'_1}^{z'_2} \tilde{G}_{zz}(k_x,k_y,z|z') f_n(z')\, dz', \quad z \geq z'_2$$
$$= \frac{-k_\rho^2}{2k_{z(\ell')}\omega\varepsilon_{(\ell)}} \int_{z'_1}^{z'_2} F^{\mathrm{TM}}_{(\ell)+}(z,z') \left(a'\, e^{jk_F z'} + b'\, e^{-jk_F z'} \right) dz'$$
$$- \frac{1}{j\omega\varepsilon_{(\ell')}} \int_{z'_1}^{z'_2} \delta(z-z') \left(a'\, e^{jk_F z'} + b'\, e^{-jk_F z'} \right) dz'$$
$$= \frac{-k_\rho^2 A^{\mathrm{TM}}_{(\ell)+}}{2k_{z(\ell')}\omega\varepsilon_{(\ell)}} \left(e^{-jk_{z(\ell)}z} + \check{R}^{\mathrm{TM}}_{(\ell),(\ell-1)}\, e^{-j2k_{z(\ell)}d_{(\ell-1)}}\, e^{jk_{z(\ell)}z} \right)$$
$$\cdot \left(\frac{a'}{j[k_{z(\ell')}+k_F]} \left\{ e^{j[k_{z(\ell')}+k_F]z'_2} - e^{j[k_{z(\ell')}+k_F]z'_1} \right\} \right.$$
$$+ \frac{b'}{j[k_{z(\ell')}-k_F]} \left\{ e^{j[k_{z(\ell')}-k_F]z'_2} - e^{j[k_{z(\ell')}-k_F]z'_1} \right\}$$
$$+ \frac{a'\check{R}^{\mathrm{TM}}_{(\ell'),(\ell'+1)}\, e^{j2k_{z(\ell')}d_{(\ell')}}}{j[-k_{z(\ell')}+k_F]} \left\{ e^{j[-k_{z(\ell')}+k_F]z'_2} - e^{j[-k_{z(\ell')}+k_F]z'_1} \right\}$$

$$+\frac{b'\check{R}^{\text{TM}}_{(\ell'),(\ell'+1)}\,e^{j2k_{z(\ell')}d_{(\ell')}}}{j[-k_{z(\ell')}-k_F]}\Big\{e^{j[-k_{z(\ell')}-k_F]z_2'}-e^{j[-k_{z(\ell')}-k_F]z_1'}\Big\}\Bigg). \qquad \text{(A.43)}$$

Also, the expression for $\tilde{G}^{I-}_{zz}$ can be derived as

$$\begin{aligned}
&\tilde{G}^{I-}_{zz}(k_x,k_y,z)\\
&=\int_{z_1'}^{z_2'}\tilde{G}_{xz}(k_x,k_y,z|z')f_n(z')\,dz', \quad z\le z_1'\\
&=\frac{-k_\rho^2}{2k_{z(\ell')}\omega\varepsilon_{(\ell)}}\int_{z_1'}^{z_2'}F^{\text{TM}}_{(\ell)-}(z,z')\Big(a'\,e^{jk_Fz'}+b'\,e^{-jk_Fz'}\Big)dz'\\
&\quad-\frac{1}{j\omega\varepsilon_{(\ell')}}\int_{z_1'}^{z_2'}\delta(z-z')\Big(a'\,e^{jk_Fz'}+b'\,e^{-jk_Fz'}\Big)dz'\\
&=\frac{-k_\rho^2A^{\text{TM}}_{(\ell)-}}{2k_{z(\ell')}\omega\varepsilon_{(\ell)}}\Big(e^{jk_{z(\ell)}z}+\check{R}^{\text{TM}}_{(\ell'),(\ell'+1)}\,e^{j2k_{z(\ell')}d_{(\ell')}}\,e^{-jk_{z(\ell)}z}\Big)\\
&\quad\cdot\Bigg(\frac{a'}{j[-k_{z(\ell')}+k_F]}\Big\{e^{j[-k_{z(\ell')}+k_F]z_2'}-e^{j[-k_{z(\ell')}+k_F]z_1'}\Big\}\\
&\quad+\frac{b'}{j[-k_{z(\ell')}-k_F]}\Big\{e^{j[-k_{z(\ell')}-k_F]z_2'}-e^{j[-k_{z(\ell')}-k_F]z_1'}\Big\}\\
&\quad+\frac{a'\check{R}^{\text{TM}}_{(\ell),(\ell-1)}\,e^{-j2k_{z(\ell-1)}d_{(\ell)}}}{j[k_{z(\ell')}+k_F]}\Big\{e^{j[k_{z(\ell')}+k_F]z_2'}-e^{j[k_{z(\ell')}+k_F]z_1'}\Big\}\\
&\quad+\frac{b'\check{R}^{\text{TM}}_{(\ell),(\ell-1)}\,e^{-j2k_{z(\ell-1)}d_{(\ell)}}}{j[k_{z(\ell')}-k_F]}\Big\{e^{j[k_{z(\ell')}-k_F]z_2'}-e^{j[k_{z(\ell')}-k_F]z_1'}\Big\}\Bigg).
\end{aligned} \qquad \text{(A.44)}$$

For vertically-orientated testing functions and vertically-orientated expansion functions that do overlap vertically, the integral

$$\tilde{G}^{II}_{zz}(k_x,k_y)=\int_z\int_{z'}f_m(z)\tilde{G}_{zz}(k_x,k_y,z|z')f_n(z')\,dz'dz \qquad \text{(A.45)}$$

needs to be evaluated. It can be expanded as

$$\tilde{G}^{II}_{zz}(k_x,k_y,z)=\begin{cases}\tilde{G}^{IZ+}_{zz}(k_x,k_y,z)+\tilde{G}^{IZ-}_{zz}(k_x,k_y,z), & z_1=z_1',\ z_2=z_2'\\ \tilde{G}^{II+}_{zz}(k_x,k_y,z), & z_1\ge z_2'\\ \tilde{G}^{II-}_{zz}(k_x,k_y,z), & z_2\le z_1'.\end{cases} \qquad \text{(A.46)}$$

By using the expressions in (A.24) to (A.28), the expression for $\tilde{G}^{IZ+}_{zz}$ can be derived as

$$\begin{aligned}
&\tilde{G}^{IZ+}_{zz}(k_x,k_y)\\
&=\int_{z_1}^{z_2}\int_{z_1'}^{z}f_m(z)\tilde{G}_{zz}(k_x,k_y,z|z')f_n(z')\,dz'dz, \quad z_1=z_1',\ z_2=z_2',\ z\ge z'
\end{aligned}$$

$$
\begin{aligned}
&= \frac{-k_\rho^2}{2k_{z(\ell')}\omega\varepsilon_{(\ell)}} \int_{z_1}^{z_2}\int_{z_1'}^{z}\left(a\,e^{jk_F z}+b\,e^{-jk_F z}\right)F_{(\ell)+}^{\mathrm{TM}}(z,z')\left(a'\,e^{jk_F z'}+b'\,e^{-jk_F z'}\right)dz'dz \\
&\quad - \frac{1}{j\omega\varepsilon_{(\ell')}} \int_{z_1}^{z_2}\int_{z_1'}^{z}\left(a\,e^{jk_F z}+b\,e^{-jk_F z}\right)\delta(z-z')\left(a'\,e^{jk_F z'}+b'\,e^{-jk_F z'}\right)dz'dz \\
&= \frac{-k_\rho^2 A_{(\ell)+}^{\mathrm{TM}}}{2k_{z(\ell')}\omega\varepsilon_{(\ell)}}\Bigg[\Bigg(\frac{-a'}{j[k_{z(\ell')}+k_F]}\,e^{j[k_{z(\ell')}+k_F]z_1'} - \frac{b'}{j[k_{z(\ell')}-k_F]}\,e^{j[k_{z(\ell')}-k_F]z_1'} \\
&\quad - \frac{a'\check{R}_{(\ell'),(\ell'+1)}^{\mathrm{TM}}\,e^{j2k_{z(\ell')}d_{(\ell')}}}{j[-k_{z(\ell')}+k_F]}\,e^{j[-k_{z(\ell')}+k_F]z_1'} - \frac{b'\check{R}_{(\ell'),(\ell'+1)}^{\mathrm{TM}}\,e^{j2k_{z(\ell')}d_{(\ell')}}}{j[-k_{z(\ell')}-k_F]}\,e^{j[-k_{z(\ell')}-k_F]z_1'}\Bigg) \\
&\quad \cdot\Bigg(\frac{a}{j[-k_{z(\ell)}+k_F]}\left\{e^{j[-k_{z(\ell)}+k_F]z_2}-e^{j[-k_{z(\ell)}+k_F]z_1}\right\} \\
&\quad + \frac{b}{j[-k_{z(\ell)}-k_F]}\left\{e^{j[-k_{z(\ell)}-k_F]z_2}-e^{j[-k_{z(\ell)}-k_F]z_1}\right\} \\
&\quad + \frac{a\check{R}_{(\ell),(\ell-1)}^{\mathrm{TM}}\,e^{-j2k_{z(\ell)}d_{(\ell-1)}}}{j[k_{z(\ell)}+k_F]}\left\{e^{j[k_{z(\ell)}+k_F]z_2}-e^{j[k_{z(\ell)}+k_F]z_1}\right\} \\
&\quad + \frac{b\check{R}_{(\ell),(\ell-1)}^{\mathrm{TM}}\,e^{-j2k_{z(\ell)}d_{(\ell-1)}}}{j[k_{z(\ell)}-k_F]}\left\{e^{j[k_{z(\ell)}-k_F]z_2}-e^{j[k_{z(\ell)}-k_F]z_1}\right\}\Bigg) \\
&\quad - \frac{aa'}{[k_{z(\ell')}+k_F][k_{z(\ell')}-k_{z(\ell)}+2k_F]}\left\{e^{j[k_{z(\ell')}-k_{z(\ell)}+2k_F]z_2}-e^{j[k_{z(\ell')}-k_{z(\ell)}+2k_F]z_1}\right\} \\
&\quad - \frac{ab'}{k_{z(\ell')}-k_F}\,j(z_2-z_1) \\
&\quad + \frac{aa'\check{R}_{(\ell'),(\ell'+1)}^{\mathrm{TM}}\,e^{j2k_{z(\ell')}d_{(\ell')}}}{[k_{z(\ell')}-k_F][-k_{z(\ell')}-k_{z(\ell)}+2k_F]}\left\{e^{j[-k_{z(\ell')}-k_{z(\ell)}+2k_F]z_2}-e^{j[-k_{z(\ell')}-k_{z(\ell)}+2k_F]z_1}\right\} \\
&\quad + \frac{ab'\check{R}_{(\ell'),(\ell'+1)}^{\mathrm{TM}}\,e^{j2k_{z(\ell')}d_{(\ell')}}}{[k_{z(\ell')}+k_F][-k_{z(\ell')}-k_{z(\ell)}]}\left\{e^{j[-k_{z(\ell')}-k_{z(\ell)}]z_2}-e^{j[-k_{z(\ell')}-k_{z(\ell)}]z_1}\right\} \\
&\quad - \frac{ba'}{k_{z(\ell')}+k_F}\,j(z_2-z_1) \\
&\quad - \frac{bb'}{[k_{z(\ell')}-k_F][k_{z(\ell')}-k_{z(\ell)}-2k_F]}\left\{e^{j[k_{z(\ell')}-k_{z(\ell)}-2k_F]z_2}-e^{j[k_{z(\ell')}-k_{z(\ell)}-2k_F]z_1}\right\} \\
&\quad + \frac{ba'\check{R}_{(\ell'),(\ell'+1)}^{\mathrm{TM}}\,e^{j2k_{z(\ell')}d_{(\ell')}}}{[k_{z(\ell')}-k_F][-k_{z(\ell')}-k_{z(\ell)}]}\left\{e^{j[-k_{z(\ell')}-k_{z(\ell)}]z_2}-e^{j[-k_{z(\ell')}-k_{z(\ell)}]z_1}\right\} \\
&\quad + \frac{bb'\check{R}_{(\ell'),(\ell'+1)}^{\mathrm{TM}}\,e^{j2k_{z(\ell')}d_{(\ell')}}}{[k_{z(\ell')}+k_F][-k_{z(\ell')}-k_{z(\ell)}-2k_F]}\left\{e^{j[-k_{z(\ell')}-k_{z(\ell)}-2k_F]z_2}-e^{j[-k_{z(\ell')}-k_{z(\ell)}-2k_F]z_1}\right\} \\
&\quad - \frac{aa'\check{R}_{(\ell),(\ell-1)}^{\mathrm{TM}}\,e^{-j2k_{z(\ell)}d_{(\ell-1)}}}{[k_{z(\ell')}+k_F][k_{z(\ell')}+k_{z(\ell)}+2k_F]}\left\{e^{j[k_{z(\ell')}+k_{z(\ell)}+2k_F]z_2}-e^{j[k_{z(\ell')}+k_{z(\ell)}+2k_F]z_1}\right\} \\
&\quad - \frac{ab'\check{R}_{(\ell),(\ell-1)}^{\mathrm{TM}}\,e^{-j2k_{z(\ell)}d_{(\ell-1)}}}{[k_{z(\ell')}-k_F][k_{z(\ell')}+k_{z(\ell)}]}\left\{e^{j[k_{z(\ell')}+k_{z(\ell)}]z_2}-e^{j[k_{z(\ell')}+k_{z(\ell)}]z_1}\right\} \\
&\quad + \frac{aa'\check{R}_{(\ell),(\ell-1)}^{\mathrm{TM}}\check{R}_{(\ell'),(\ell'+1)}^{\mathrm{TM}}\,e^{-j2k_{z(\ell)}d_{(\ell-1)}}\,e^{j2k_{z(\ell')}d_{(\ell')}}}{[k_{z(\ell')}-k_F][-k_{z(\ell')}+k_{z(\ell)}+2k_F]}\left\{e^{j[-k_{z(\ell')}+k_{z(\ell)}+2k_F]z_2}\right.
\end{aligned}
$$

$$
\begin{aligned}
&\quad - e^{j[-k_{z(\ell')}+k_{z(\ell)}+2k_F]z_1}\Big\} \\
&+ \frac{ab'\check{R}^{\mathrm{TM}}_{(\ell),(\ell-1)}\check{R}^{\mathrm{TM}}_{(\ell'),(\ell'+1)}\,e^{-j2k_{z(\ell)}d_{(\ell-1)}}\,e^{j2k_{z(\ell')}d_{(\ell')}}}{k_{z(\ell')}+k_F}\,j(z_2-z_1) \\
&- \frac{ba'\check{R}^{\mathrm{TM}}_{(\ell),(\ell-1)}\,e^{-j2k_{z(\ell)}d_{(\ell-1)}}}{[k_{z(\ell')}+k_F][k_{z(\ell')}+k_{z(\ell)}]}\Big\{e^{j[k_{z(\ell')}+k_{z(\ell)}]z_2} - e^{j[k_{z(\ell')}+k_{z(\ell)}]z_1}\Big\} \\
&- \frac{bb'\check{R}^{\mathrm{TM}}_{(\ell),(\ell-1)}\,e^{-j2k_{z(\ell)}d_{(\ell-1)}}}{[k_{z(\ell')}-k_F][k_{z(\ell')}+k_{z(\ell)}-2k_F]}\Big\{e^{j[k_{z(\ell')}+k_{z(\ell)}-2k_F]z_2} - e^{j[k_{z(\ell')}+k_{z(\ell)}-2k_F]z_1}\Big\} \\
&+ \frac{ba'\check{R}^{\mathrm{TM}}_{(\ell),(\ell-1)}\check{R}^{\mathrm{TM}}_{(\ell'),(\ell'+1)}\,e^{-j2k_{z(\ell)}d_{(\ell-1)}}\,e^{j2k_{z(\ell')}d_{(\ell')}}}{k_{z(\ell')}-k_F}\,j(z_2-z_1) \\
&+ \frac{bb'\check{R}^{\mathrm{TM}}_{(\ell),(\ell-1)}\check{R}^{\mathrm{TM}}_{(\ell'),(\ell'+1)}\,e^{-j2k_{z(\ell)}d_{(\ell-1)}}\,e^{j2k_{z(\ell')}d_{(\ell')}}}{[k_{z(\ell')}+k_F][-k_{z(\ell')}+k_{z(\ell)}-2k_F]}\Big\{e^{j[-k_{z(\ell')}+k_{z(\ell)}-2k_F]z_2} \\
&\quad - e^{j[-k_{z(\ell')}+k_{z(\ell)}-2k_F]z_1}\Big\}\Bigg]. \qquad \text{(A.47)}
\end{aligned}
$$

Also, the expression for $\tilde{G}^{IZ-}_{zz}$ can be derived as

$$
\begin{aligned}
&\tilde{G}^{IZ-}_{zz}(k_x,k_y) \\
&= \int_{z_1}^{z_2}\int_{z}^{z_2'} f_m(z)\tilde{G}_{zz}(k_x,k_y,z|z')f_n(z')\,dz'dz, \quad z_1=z_1',\ z_2=z_2',\ z\le z' \\
&= \frac{-k_\rho^2}{2k_{z(\ell')}\omega\varepsilon_{(\ell)}}\int_{z_1}^{z_2}\int_{z}^{z_2'}\Big(a\,e^{jk_Fz}+b\,e^{-jk_Fz}\Big)F^{\mathrm{TM}}_{(\ell)-}(z,z')\Big(a'\,e^{jk_Fz'}+b'\,e^{-jk_Fz'}\Big)\,dz'dz \\
&\quad - \frac{1}{j\omega\varepsilon_{(\ell')}}\int_{z_1}^{z_2}\int_{z}^{z_2'}\Big(a\,e^{jk_Fz}+b\,e^{-jk_Fz}\Big)\delta(z-z')\Big(a'\,e^{jk_Fz'}+b'\,e^{-jk_Fz'}\Big)\,dz'dz \\
&= \frac{-k_\rho^2A^{\mathrm{TM}}_{(\ell)-}}{2k_{z(\ell')}\omega\varepsilon_{(\ell)}}\Bigg[\Bigg(\frac{a'}{j[-k_{z(\ell')}+k_F]}\,e^{j[-k_{z(\ell')}+k_F]z_1'} + \frac{b'}{j[-k_{z(\ell')}-k_F]}\,e^{j[-k_{z(\ell')}-k_F]z_1'} \\
&\quad + \frac{a'\check{R}^{\mathrm{TM}}_{(\ell),(\ell-1)}\,e^{-j2k_{z(\ell)}d_{(\ell-1)}}}{j[k_{z(\ell')}+k_F]}\,e^{j[k_{z(\ell')}+k_F]z_1'} + \frac{b'\check{R}^{\mathrm{TM}}_{(\ell),(\ell-1)}\,e^{-j2k_{z(\ell)}d_{(\ell-1)}}}{j[k_{z(\ell')}-k_F]}\,e^{j[k_{z(\ell')}-k_F]z_1'}\Bigg) \\
&\quad \cdot\Bigg(\frac{a}{j[k_{z(\ell)}+k_F]}\Big\{e^{j[k_{z(\ell)}+k_F]z_2} - e^{j[k_{z(\ell)}+k_F]z_1}\Big\} \\
&\quad + \frac{b}{j[k_{z(\ell)}-k_F]}\Big\{e^{j[k_{z(\ell)}-k_F]z_2} - e^{j[k_{z(\ell)}-k_F]z_1}\Big\} \\
&\quad + \frac{a\check{R}^{\mathrm{TM}}_{(\ell'),(\ell'+1)}\,e^{j2k_{z(\ell')}d_{(\ell')}}}{j[-k_{z(\ell)}+k_F]}\Big\{e^{j[-k_{z(\ell)}+k_F]z_2} - e^{j[-k_{z(\ell)}+k_F]z_1}\Big\} \\
&\quad + \frac{b\check{R}^{\mathrm{TM}}_{(\ell'),(\ell'+1)}\,e^{j2k_{z(\ell')}d_{(\ell')}}}{j[-k_{z(\ell)}-k_F]}\Big\{e^{j[-k_{z(\ell)}-k_F]z_2} - e^{j[-k_{z(\ell)}-k_F]z_1}\Big\}\Bigg) \\
&\quad + \frac{aa'}{[-k_{z(\ell')}+k_F][-k_{z(\ell')}+k_{z(\ell)}+2k_F]}\Big\{e^{j[-k_{z(\ell')}+k_{z(\ell)}+2k_F]z_2} - e^{j[-k_{z(\ell')}+k_{z(\ell)}+2k_F]z_1}\Big\} \\
&\quad + \frac{ab'}{-k_{z(\ell')}-k_F}\,j(z_2-z_1)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{aa'\check{R}^{\mathrm{TM}}_{(\ell),(\ell-1)}\, e^{-j2k_{z(\ell)}d_{(\ell-1)}}}{[-k_{z(\ell')}-k_F][k_{z(\ell')}+k_{z(\ell)}+2k_F]}\left\{e^{j[k_{z(\ell')}+k_{z(\ell)}+2k_F]z_2} - e^{j[k_{z(\ell')}+k_{z(\ell)}+2k_F]z_1}\right\}\\
&- \frac{ab'\check{R}^{\mathrm{TM}}_{(\ell),(\ell-1)}\, e^{-j2k_{z(\ell)}d_{(\ell-1)}}}{[-k_{z(\ell')}+k_F][k_{z(\ell')}+k_{z(\ell)}]}\left\{e^{j[k_{z(\ell')}+k_{z(\ell)}]z_2} - e^{j[k_{z(\ell')}+k_{z(\ell)}]z_1}\right\}\\
&+ \frac{ba'}{-k_{z(\ell')}+k_F}\, j(z_2-z_1)\\
&+ \frac{bb'}{[-k_{z(\ell')}-k_F][-k_{z(\ell')}+k_{z(\ell)}-2k_F]}\left\{e^{j[-k_{z(\ell')}+k_{z(\ell)}-2k_F]z_2} - e^{j[-k_{z(\ell')}+k_{z(\ell)}-2k_F]z_1}\right\}\\
&- \frac{ba'\check{R}^{\mathrm{TM}}_{(\ell),(\ell-1)}\, e^{-j2k_{z(\ell)}d_{(\ell-1)}}}{[-k_{z(\ell')}-k_F][k_{z(\ell')}+k_{z(\ell)}]}\left\{e^{j[k_{z(\ell')}+k_{z(\ell)}]z_2} - e^{j[k_{z(\ell')}+k_{z(\ell)}]z_1}\right\}\\
&- \frac{bb'\check{R}^{\mathrm{TM}}_{(\ell),(\ell-1)}\, e^{-j2k_{z(\ell)}d_{(\ell-1)}}}{[-k_{z(\ell')}+k_F][k_{z(\ell')}+k_{z(\ell)}-2k_F]}\left\{e^{j[+k_{z(\ell')}+k_{z(\ell)}-2k_F]z_2} - e^{j[+k_{z(\ell')}+k_{z(\ell)}-2k_F]z_1}\right\}\\
&+ \frac{aa'\check{R}^{\mathrm{TM}}_{(\ell'),(\ell'+1)}\, e^{j2k_{z(\ell')}d_{(\ell')}}}{[-k_{z(\ell')}+k_F][-k_{z(\ell')}-k_{z(\ell)}+2k_F]}\left\{e^{j[-k_{z(\ell')}-k_{z(\ell)}+2k_F]z_2} - e^{j[-k_{z(\ell')}-k_{z(\ell)}+2k_F]z_1}\right\}\\
&+ \frac{ab'\check{R}^{\mathrm{TM}}_{(\ell'),(\ell'+1)}\, e^{j2k_{z(\ell')}d_{(\ell')}}}{[-k_{z(\ell')}-k_F][-k_{z(\ell')}-k_{z(\ell)}]}\left\{e^{j[-k_{z(\ell')}-k_{z(\ell)}]z_2} - e^{j[-k_{z(\ell')}-k_{z(\ell)}]z_1}\right\}\\
&- \frac{aa'\check{R}^{\mathrm{TM}}_{(\ell),(\ell-1)}\check{R}^{\mathrm{TM}}_{(\ell'),(\ell'+1)}\, e^{-j2k_{z(\ell)}d_{(\ell-1)}}\, e^{j2k_{z(\ell')}d_{(\ell')}}}{[-k_{z(\ell')}-k_F][k_{z(\ell')}-k_{z(\ell)}+2k_F]}\left\{e^{j[k_{z(\ell')}-k_{z(\ell)}+2k_F]z_2}\right.\\
&\left.\quad - e^{j[k_{z(\ell')}-k_{z(\ell)}+2k_F]z_1}\right\}\\
&- \frac{ab'\check{R}^{\mathrm{TM}}_{(\ell),(\ell-1)}\check{R}^{\mathrm{TM}}_{(\ell'),(\ell'+1)}\, e^{-j2k_{z(\ell)}d_{(\ell-1)}}\, e^{j2k_{z(\ell')}d_{(\ell')}}}{-k_{z(\ell')}+k_F}\, j(z_2-z_1)\\
&+ \frac{ba'\check{R}^{\mathrm{TM}}_{(\ell'),(\ell'+1)}\, e^{j2k_{z(\ell')}d_{(\ell')}}}{[-k_{z(\ell')}+k_F][-k_{z(\ell')}-k_{z(\ell)}]}\left\{e^{j[-k_{z(\ell')}-k_{z(\ell)}]z_2} - e^{j[-k_{z(\ell')}-k_{z(\ell)}]z_1}\right\}\\
&+ \frac{bb'\check{R}^{\mathrm{TM}}_{(\ell'),(\ell'+1)}\, e^{j2k_{z(\ell')}d_{(\ell')}}}{[-k_{z(\ell')}-k_F][-k_{z(\ell')}-k_{z(\ell)}-2k_F]}\left\{e^{j[-k_{z(\ell')}-k_{z(\ell)}-2k_F]z_2} - e^{j[-k_{z(\ell')}-k_{z(\ell)}-2k_F]z_1}\right\}\\
&- \frac{ba'\check{R}^{\mathrm{TM}}_{(\ell),(\ell-1)}\check{R}^{\mathrm{TM}}_{(\ell'),(\ell'+1)}\, e^{-j2k_{z(\ell)}d_{(\ell-1)}}\, e^{j2k_{z(\ell')}d_{(\ell')}}}{-k_{z(\ell')}-k_F}\, j(z_2-z_1)\\
&- \frac{bb'\check{R}^{\mathrm{TM}}_{(\ell),(\ell-1)}\check{R}^{\mathrm{TM}}_{(\ell'),(\ell'+1)}\, e^{-j2k_{z(\ell)}d_{(\ell-1)}}\, e^{j2k_{z(\ell')}d_{(\ell')}}}{[-k_{z(\ell')}+k_F][k_{z(\ell')}-k_{z(\ell)}-2k_F]}\left\{e^{j[k_{z(\ell')}-k_{z(\ell)}-2k_F]z_2}\right.\\
&\left.\quad - e^{j[k_{z(\ell')}-k_{z(\ell)}-2k_F]z_1}\right\}\Bigg]\\
&- \frac{1}{j\omega\varepsilon_{(\ell')}}\left[\frac{aa'}{j2k_F}\left(e^{j2k_F z_2}-e^{j2k_F z_1}\right) - \frac{bb'}{j2k_F}\left(e^{-j2k_F z_2}-e^{-j2k_F z_1}\right)\right.\\
&\left.\quad + (ab'+ba')(z_2-z_1)\right]. \qquad \text{(A.48)}
\end{aligned}
$$

The expression for $\tilde{G}_{zz}^{II+}$ can be derived as

$$
\begin{aligned}
&\tilde{G}_{zz}^{II+}(k_x,k_y) \\
&= \int_{z_1}^{z_2}\int_{z_1'}^{z_2'} f_m(z)\tilde{G}_{zz}(k_x,k_y,z|z')f_n(z')\,dz'dz, \quad z_1 \geq z_2' \\
&= \frac{-k_\rho^2}{2k_{z(\ell')}\omega\varepsilon_{(\ell)}}\int_{z_1}^{z_2}\int_{z_1'}^{z_2'}\left(a\,e^{jk_F z}+b\,e^{-jk_F z}\right)F_{(\ell)+}^{\mathrm{TM}}(z,z')\left(a'\,e^{jk_F z'}+b'\,e^{-jk_F z'}\right)dz'dz \\
&\quad -\frac{1}{j\omega\varepsilon_{(\ell')}}\int_{z_1}^{z_2}\int_{z}^{z_2'}\left(a\,e^{jk_F z}+b\,e^{-jk_F z}\right)\delta(z-z')\left(a'\,e^{jk_F z'}+b'\,e^{-jk_F z'}\right)dz'dz \\
&= \frac{-k_\rho^2 A_{(\ell)+}^{\mathrm{TM}}}{2k_{z(\ell')}\omega\varepsilon_{(\ell)}}\left(\frac{a}{j[-k_{z(\ell)}+k_F]}\left\{e^{j[-k_{z(\ell)}+k_F]z_2}-e^{j[-k_{z(\ell)}+k_F]z_1}\right\}\right. \\
&\quad +\frac{b}{j[-k_{z(\ell)}-k_F]}\left\{e^{j[-k_{z(\ell)}-k_F]z_2}-e^{j[-k_{z(\ell)}-k_F]z_1}\right\} \\
&\quad +\frac{a\check{R}_{(\ell),(\ell-1)}^{\mathrm{TM}}\,e^{-j2k_{z(\ell)}d_{(\ell-1)}}}{j[k_{z(\ell)}+k_F]}\left\{e^{j[k_{z(\ell)}+k_F]z_2}-e^{j[k_{z(\ell)}+k_F]z_1}\right\} \\
&\quad \left.+\frac{b\check{R}_{(\ell),(\ell-1)}^{\mathrm{TM}}\,e^{-j2k_{z(\ell)}d_{(\ell-1)}}}{j[k_{z(\ell)}-k_F]}\left\{e^{j[k_{z(\ell)}-k_F]z_2}-e^{j[k_{z(\ell)}-k_F]z_1}\right\}\right) \\
&\quad \cdot\left(\frac{a'}{j[k_{z(\ell')}+k_F]}\left\{e^{j[k_{z(\ell')}+k_F]z_2'}-e^{j[k_{z(\ell')}+k_F]z_1'}\right\}\right. \\
&\quad +\frac{b'}{j[k_{z(\ell')}-k_F]}\left\{e^{j[k_{z(\ell')}-k_F]z_2'}-e^{j[k_{z(\ell')}-k_F]z_1'}\right\} \\
&\quad +\frac{a'\check{R}_{(\ell'),(\ell'+1)}^{\mathrm{TM}}\,e^{j2k_{z(\ell')}d_{(\ell')}}}{j[-k_{z(\ell')}+k_F]}\left\{e^{j[-k_{z(\ell')}+k_F]z_2'}-e^{j[-k_{z(\ell')}+k_F]z_1'}\right\} \\
&\quad \left.+\frac{b'\check{R}_{(\ell'),(\ell'+1)}^{\mathrm{TM}}\,e^{j2k_{z(\ell')}d_{(\ell')}}}{j[-k_{z(\ell')}-k_F]}\left\{e^{j[-k_{z(\ell')}-k_F]z_2'}-e^{j[-k_{z(\ell')}-k_F]z_1'}\right\}\right),
\end{aligned} \tag{A.49}
$$

while the expression for $\tilde{G}_{zz}^{II-}$ can be derived as

$$
\begin{aligned}
&\tilde{G}_{zz}^{II-}(k_x,k_y) \\
&= \int_{z_1}^{z_2}\int_{z_1'}^{z_2'} f_m(z)\tilde{G}_{zz}(k_x,k_y,z|z')f_n(z')\,dz'dz, \quad z_2 \leq z_1' \\
&= \frac{-k_\rho^2}{2k_{z(\ell')}\omega\varepsilon_{(\ell)}}\int_{z_1}^{z_2}\int_{z_1'}^{z_2'}\left(a\,e^{jk_F z}+b\,e^{-jk_F z}\right)F_{(\ell)-}^{\mathrm{TM}}(z,z')\left(a'\,e^{jk_F z'}+b'\,e^{-jk_F z'}\right)dz'dz \\
&\quad -\frac{1}{j\omega\varepsilon_{(\ell')}}\int_{z_1}^{z_2}\int_{z}^{z_2'}\left(a\,e^{jk_F z}+b\,e^{-jk_F z}\right)\delta(z-z')\left(a'\,e^{jk_F z'}+b'\,e^{-jk_F z'}\right)dz'dz \\
&= \frac{-k_\rho^2 A_{(\ell)-}^{\mathrm{TM}}}{2k_{z(\ell')}\omega\varepsilon_{(\ell)}}\left(\frac{a}{j[k_{z(\ell)}+k_F]}\left\{e^{j[k_{z(\ell)}+k_F]z_2}-e^{j[k_{z(\ell)}+k_F]z_1}\right\}\right. \\
&\quad +\frac{b}{j[k_{z(\ell)}-k_F]}\left\{e^{j[k_{z(\ell)}-k_F]z_2}-e^{j[k_{z(\ell)}-k_F]z_1}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{a\check{R}^{\mathrm{TM}}_{(\ell'),(\ell'+1)}\, e^{j2k_{z(\ell')}d_{(\ell')}}}{j[-k_{z(\ell)}+k_F]}\left\{e^{j[-k_{z(\ell)}+k_F]z_2} - e^{j[-k_{z(\ell)}+k_F]z_1}\right\} \\
&+ \frac{b\check{R}^{\mathrm{TM}}_{(\ell'),(\ell'+1)}\, e^{j2k_{z(\ell')}d_{(\ell')}}}{j[-k_{z(\ell)}-k_F]}\left\{e^{j[-k_{z(\ell)}-k_F]z_2} - e^{j[-k_{z(\ell)}-k_F]z_1}\right\}\Bigg) \\
&\cdot \Bigg(\frac{a'}{j[-k_{z(\ell')}+k_F]}\left\{e^{j[-k_{z(\ell')}+k_F]z'_2} - e^{j[-k_{z(\ell')}+k_F]z'_1}\right\} \\
&+ \frac{b'}{j[-k_{z(\ell')}-k_F]}\left\{e^{j[-k_{z(\ell')}-k_F]z'_2} - e^{j[-k_{z(\ell')}-k_F]z'_1}\right\} \\
&+ \frac{a'\check{R}^{\mathrm{TM}}_{(\ell),(\ell-1)}\, e^{-j2k_{z(\ell)}d_{(\ell-1)}}}{j[k_{z(\ell')}+k_F]}\left\{e^{j[k_{z(\ell')}+k_F]z'_2} - e^{j[k_{z(\ell')}+k_F]z'_1}\right\} \\
&+ \frac{b'\check{R}^{\mathrm{TM}}_{(\ell),(\ell-1)}\, e^{-j2k_{z(\ell)}d_{(\ell-1)}}}{j[k_{z(\ell')}-k_F]}\left\{e^{j[k_{z(\ell')}-k_F]z'_2} - e^{j[k_{z(\ell')}-k_F]z'_1}\right\}\Bigg).
\end{aligned}
\tag{A.50}
$$

APPENDIX B

Derivations Related to the Basis Functions

In this appendix, it is shown how the spatial-domain expression of the rectangular attachment mode can be derived. Although the derivation is available in the literature [6, 7], it is included here in a more compact form for completeness sake. It is also shown how the spectral-domain expressions for all of the basis functions can be derived. Please note that, in this appendix, a local (x, y, z) coordinate system is used for all the basis functions, instead of the (u, v, z) coordinate system as in Chapter 3. This is for easy comparison to most literature, where the standard (x, y, z) coordinate system is normally used.

B.1 SPATIAL REPRESENTATION OF THE RECTANGULAR ATTACHMENT MODE

Refer to Figure 3.10 for an illustration of the geometrical parameters associated with the rectangular attachment mode. The only differences here, are that the u and v axes are replaced by x and y axes respectively, and that the na subscript is dropped. In order to derive the expression for the rectangular attachment mode, the area between the patch and the ground plane can be viewed as a magnetic-wall cavity [6, 7, 129, 132]. Now, the electric field at (x, y) inside such a magnetic-wall cavity, excited by a uniform filament of current at the position $(\dot{x}^{AZ}, \dot{y}^{AZ})$, is found to be [6, 7]

$$\begin{aligned}\mathbf{E}^{AP}(x, y, \dot{x}^{AZ}, \dot{y}^{AZ}) &= E_z^{AP}(x, y, \dot{x}^{AZ}, \dot{y}^{AZ}) \\ &= \left\{ \frac{j\omega\mu_{(\text{eff})}}{LW} \sum_{\kappa=0}^{\infty} \sum_{i=0}^{\infty} \frac{\epsilon_\kappa \epsilon_i}{\varepsilon_{r(\text{eff})} k_0^2 - (\kappa\pi/L)^2 - (i\pi/W)^2} \right. \\ &\quad \cdot \cos\left[\frac{\kappa\pi}{L}\left(x + \frac{L}{2}\right)\right] \cos\left[\frac{\kappa\pi}{L}\left(\dot{x}^{AZ} + \frac{L}{2}\right)\right] \\ &\quad \left. \cdot \cos\left[\frac{i\pi}{W}\left(y + \frac{W}{2}\right)\right] \cos\left[\frac{i\pi}{W}\left(\dot{y}^{AZ} + \frac{W}{2}\right)\right] \right\} \hat{z},\end{aligned}$$

$$|x| \leq \frac{L}{2},\ |y| \leq \frac{W}{2},\ z \leq z^{AP}. \qquad \text{(B.1)}$$

It is assumed that the electric field is constant along the z direction. In (B.1), the Neumann numbers are given by

$$\epsilon_{\kappa,i} = \begin{cases} 1, & \kappa, i = 0 \\ 2, & \kappa, i > 0. \end{cases} \qquad \text{(B.2)}$$

Furthermore, $\varepsilon_{(\text{eff})}$ is the effective permittivity and $\mu_{(\text{eff})}$ the effective permeability of the layers below the patch.

The magnetic field inside the cavity can be derived from the electric field and is given by

$$\begin{aligned} \mathbf{H}^{AP}(x, y, \dot{x}^{AZ}, \dot{y}^{AZ}) &= \frac{-1}{j\omega\mu_{(\text{eff})}} \nabla \times \mathbf{E}^{AP}(x, y, \dot{x}^{AZ}, \dot{y}^{AZ}) \\ &= \frac{1}{j\omega\mu_{(\text{eff})}} \left(-\frac{\partial}{\partial y}\hat{x} + \frac{\partial}{\partial x}\hat{y} \right) E_z^{AP}(x, y, \dot{x}^{AZ}, \dot{y}^{AZ}), \\ &\qquad |x| \leq \frac{L}{2},\ |y| \leq \frac{W}{2},\ z \leq z^{AP}. \end{aligned} \qquad \text{(B.3)}$$

This in turn leads to the current density on the patch, which can be expressed as [6, 7]

$$\begin{aligned} \mathbf{J}^{AP}(x, y, \dot{x}^{AZ}, \dot{y}^{AZ}) &= -\hat{z} \times \mathbf{H}^{AP}(x, y, \dot{x}^{AZ}, \dot{y}^{AZ})|_{z=z^{AP}} \\ &= \frac{-1}{LW} \sum_{\kappa=0}^{\infty} \sum_{i=0}^{\infty} \frac{\epsilon_\kappa \epsilon_i}{\varepsilon_{r(\text{eff})} k_0^2 - (\kappa\pi/L)^2 - (i\pi/W)^2} \\ &\left(\left\{ \frac{\kappa\pi}{L} \sin\left[\frac{\kappa\pi}{L}\left(x + \frac{L}{2}\right)\right] \cos\left[\frac{\kappa\pi}{L}\left(\dot{x}^{AZ} + \frac{L}{2}\right)\right] \right.\right. \\ &\left. \cdot \cos\left[\frac{i\pi}{W}\left(y + \frac{W}{2}\right)\right] \cos\left[\frac{i\pi}{W}\left(\dot{y}^{AZ} + \frac{W}{2}\right)\right] \right\} \hat{x} \\ &+ \left\{ \frac{i\pi}{W} \cos\left[\frac{\kappa\pi}{L}\left(x + \frac{L}{2}\right)\right] \cos\left[\frac{\kappa\pi}{L}\left(\dot{x}^{AZ} + \frac{L}{2}\right)\right] \right. \\ &\left.\left. \cdot \sin\left[\frac{i\pi}{W}\left(y + \frac{W}{2}\right)\right] \cos\left[\frac{i\pi}{W}\left(\dot{y}^{AZ} + \frac{W}{2}\right)\right] \right\} \hat{y} \right), \\ &\qquad |x| \leq \frac{L}{2},\ |y| \leq \frac{W}{2},\ z = z^{AP}. \end{aligned} \qquad \text{(B.4)}$$

By using the fact that [6, 7]

$$\sum_{\kappa=0}^{\infty} \frac{\epsilon_\kappa \cos(\kappa x)}{\kappa^2 - \alpha^2} (-1)^\kappa = \frac{-\pi \cos(\alpha x)}{\alpha \sin(\alpha\pi)}, \quad -\pi \leq x \leq \pi \qquad \text{(B.5)}$$

and

$$\sum_{\kappa=0}^{\infty} \frac{\epsilon_\kappa \kappa \sin(\kappa x)}{\kappa^2 - \alpha^2} (-1)^\kappa = \frac{-\pi \sin(\alpha x)}{\sin(\alpha \pi)}, \quad -\pi \le x \le \pi, \tag{B.6}$$

the expression in (B.4) can be simplified to yield [6, 7, 129, 132]

$$\mathbf{J}^{AP}(x, y, \dot{x}^{AZ}, \dot{y}^{AZ}) = \frac{-1}{2W} \sum_{i=0}^{\infty} \epsilon_i \cos\left[\frac{i\pi}{W}\left(\dot{y}^{AZ} + \frac{W}{2}\right)\right] \cdot \left\{ g_s(\beta, L, \dot{x}^{AZ}, x) f_c(i, W, y)\, \hat{x} + \left(\frac{i\pi}{W}\right) g_c(\beta, L, \dot{x}^{AZ}, x) f_s(i, W, y)\, \hat{y} \right\}, \quad |x| \le \frac{L}{2},\ |y| \le \frac{W}{2},\ z = z^{AP}, \tag{B.7}$$

where

$$g_s(\beta, L, \dot{x}^{AZ}, x) = \frac{\sin[\beta(x + \dot{x}^{AZ})] - \mathrm{sgn}(x - \dot{x}^{AZ}) \sin[\beta(L - |x - \dot{x}^{AZ}|)]}{\sin(\beta L)}, \tag{B.8}$$

$$g_c(\beta, L, \dot{x}^{AZ}, x) = \frac{\cos[\beta(x + \dot{x}^{AZ})] + \cos[\beta(L - |x - \dot{x}^{AZ}|)]}{\beta \sin(\beta L)}, \tag{B.9}$$

$$f_s(i, W, y) = \sin\left[\frac{i\pi}{W}\left(y + \frac{W}{2}\right)\right] \tag{B.10}$$

and

$$f_c(i, W, y) = \cos\left[\frac{i\pi}{W}\left(y + \frac{W}{2}\right)\right], \tag{B.11}$$

with

$$\mathrm{sgn}(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases} \tag{B.12}$$

and

$$\beta = \sqrt{\varepsilon_{r(\mathrm{eff})} k_0^2 - \left(\frac{i\pi}{W}\right)^2}. \tag{B.13}$$

In (B.13), $k_0 = 2\pi/\lambda_0$, where λ_0 is the free-space wavelength. Also, as has been mentioned, $\varepsilon_{r(\mathrm{eff})}$ is the effective relative permittivity of the layers below the patch. For two layers, it can be calculated as [65, 66, 179]

$$\varepsilon_{r(\mathrm{eff})} = \frac{\varepsilon_{r(1)} \varepsilon_{r(2)} \left[h_{(1)} + h_{(2)}\right]}{\varepsilon_{r(1)} h_{(2)} + \varepsilon_{r(2)} h_{(1)}} \left[1 - \tan\delta_{\varepsilon(1)} \tan\delta_{\varepsilon(2)}\right]. \tag{B.14}$$

The patch part of the attachment mode is then defined to be

$$\mathbf{f}^{AP}(x, y) = \frac{1}{2\pi} \int_0^{2\pi} \mathbf{J}^{AP}(x, y, \dot{x}^{AZ}, \dot{y}^{AZ})\, d\phi, \tag{B.15}$$

where

$$\dot{x}^{AZ} = x^{AZ} + a\cos(\phi) \tag{B.16}$$

and

$$\dot{y}^{AZ} = y^{AZ} + a\sin(\phi) \tag{B.17}$$

describe a position on the surface of the probe, while (x^{AZ}, y^{AZ}) is the position of the centre of the probe relative to the centre of the patch, and a is the radius of the probe.

B.2 SPECTRAL REPRESENTATION OF THE BASIS FUNCTIONS

It will now be shown how the spectral form of each basis function can be derived. However, before doing so, it is appropriate to first define some general coordinates and unit vectors.

In some cases, it is more convenient to work with polar (ρ, ϕ) coordinates rather than rectangular (x, y) coordinates. The relationship between them is given by

$$\rho = \sqrt{x^2 + y^2} \tag{B.18}$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right) \tag{B.19}$$

and

$$x = \rho\cos(\phi) \tag{B.20}$$

$$y = \rho\sin(\phi). \tag{B.21}$$

The radial unit vector associated with the polar coordinate system, is given by

$$\hat{\rho} = \frac{x}{\sqrt{x^2 + y^2}}\,\hat{x} + \frac{y}{\sqrt{x^2 + y^2}}\,\hat{y}. \tag{B.22}$$

Similarly, the wavenumbers that correspond with the two coordinate systems are related through

$$k_\rho = \sqrt{k_x^2 + k_y^2} \tag{B.23}$$

$$\varphi = \tan^{-1}\left(\frac{k_y}{k_x}\right) \tag{B.24}$$

and

$$k_x = k_\rho\cos(\varphi) \tag{B.25}$$

$$k_y = k_\rho\sin(\varphi). \tag{B.26}$$

The unit vector for the wavenumber associated with the radial direction, is given by

$$\hat{k}_\rho = \frac{k_x}{\sqrt{k_x^2 + k_y^2}}\,\hat{x} + \frac{k_y}{\sqrt{k_x^2 + k_y^2}}\,\hat{y}. \tag{B.27}$$

B.2.1 Piecewise-Sinusoidal Basis Functions on the Probes

Refer to Figure 3.7 for an illustration of the geometrical parameters associated with the probes. The only differences here, are that the u and v axes are replaced by x and y axes respectively, and that the np subscript is dropped. As was shown in Section 3.5.1, the basis functions on the probes can be expressed as

$$\mathbf{f}^{PZ}(x,y,z) = \frac{1}{2\pi a}\, f^{PZ}(z)\, \hat{z}\Big|_{x^2+y^2=a^2} \tag{B.28}$$

in the spatial domain.

The two-dimensional Fourier transform of this function in most easily evaluated by changing to polar variables. Due to the fact that ρ is limited to the surface of the probe, one can set

$$x = a\cos(\phi) \tag{B.29}$$

and

$$y = a\sin(\phi). \tag{B.30}$$

Then, by taking the two-dimensional Fourier transform of $\mathbf{f}^{PZ}(x,y,z)$, as defined in (3.18), it follows that

$$\begin{aligned}\tilde{\mathbf{f}}^{PZ}(k_x,k_y,z) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{f}^{PZ}(x,y,z)\, e^{-jk_x x}\, e^{-jk_y y}\, dxdy \\ &= \frac{1}{2\pi a}\, f^{PZ}(z)\left[\int_0^{2\pi} e^{-jk_\rho a\cos(\varphi)\cos(\phi) - jk_\rho a\sin(\varphi)\sin(\phi)}\, a\, d\phi\right]\hat{z} \\ &= \frac{1}{2\pi}\, f(z)^{PZ}\left[\int_0^{2\pi} e^{-jk_\rho a\cos(\varphi-\phi)}\, d\phi\right]\hat{z} \\ &= f(z)^{PZ} J_0(k_\rho a)\,\hat{z}.\end{aligned} \tag{B.31}$$

Here, $J_0(\cdot)$ is the Bessel function of the first kind of order 0.

B.2.2 Rooftop Basis Functions on the Rectangular Capacitor Patches

Refer to Figure 3.8 for an illustration of the geometrical parameters associated with the subdomain rooftop basis functions on the rectangular capacitor patches. The only differences here, are that the u and v axes are replaced by x and y axes respectively, and that the ns subscript is dropped. As was shown in Section 3.5.2, in the spatial domain, the rooftop basis functions can be expressed as

$$\mathbf{f}^{SX}(x,y) = \left(1 - \frac{|x|}{\Delta l}\right)\text{rect}\left(\frac{y}{\Delta w}\right)\hat{x}, \quad |x| \le \Delta l,\; z = z^S \tag{B.32}$$

for the x-directed functions and

$$\mathbf{f}^{SY}(x,y) = \left(1 - \frac{|y|}{\Delta w}\right)\text{rect}\left(\frac{x}{\Delta l}\right)\hat{y}, \quad |y| \le \Delta w,\; z = z^S \tag{B.33}$$

for the y-directed functions. In compact notation, (B.32) and (B.33) can be expressed as

$$\mathbf{f}^{SX}(x,y) = \Lambda(\Delta l, x)\Pi(\Delta w, y)\,\hat{x}, \quad |x| \leq \Delta l,\ z = z^S \tag{B.34}$$

and

$$\mathbf{f}^{SY}(x,y) = \Lambda(\Delta w, y)\Pi(\Delta l, x)\,\hat{y}, \quad |y| \leq \Delta w,\ z = z^S, \tag{B.35}$$

where

$$\Lambda(\Delta l, x) = \begin{cases} 1 - \dfrac{|x|}{\Delta l}, & |x| \leq \Delta l \\ 0, & \text{otherwise} \end{cases} \tag{B.36}$$

is the triangular function and

$$\Pi(\Delta l, x) = \text{rect}\left(\frac{x}{\Delta l}\right) = \begin{cases} 1, & |x| \leq \dfrac{\Delta l}{2} \\ 0, & \text{otherwise} \end{cases} \tag{B.37}$$

is the rectangular function.

The two-dimensional Fourier transforms of $\mathbf{f}^{SX}(x,y)$ and $\mathbf{f}^{SY}(x,y)$ can be reduced to the product of two one-dimensional Fourier transforms due to the fact that the variables in (B.32) and (B.33) can be separated. The spectral forms of the rooftop basis functions are then given by

$$\tilde{\mathbf{f}}^{SX}(k_x,k_y) = \tilde{\Lambda}(\Delta l, k_x)\tilde{\Pi}(\Delta w, k_y)\,\hat{x}, \quad |x| \leq \Delta l,\ z = z^S \tag{B.38}$$

and

$$\tilde{\mathbf{f}}^{SY}(k_x,k_y) = \tilde{\Lambda}(\Delta w, k_y)\tilde{\Pi}(\Delta l, k_x)\,\hat{y}, \quad |y| \leq \Delta w,\ z = z^S, \tag{B.39}$$

where the Fourier transforms, $\tilde{\Lambda}(\Delta l, k_x)$ and $\tilde{\Pi}(\Delta l, k_x)$, of the triangular and rectangular functions are well known [180]. They are given by

$$\tilde{\Lambda}(\Delta l, k_x) = \frac{4}{k_x^2 \Delta l}\sin^2\left(\frac{k_x \Delta l}{2}\right) \tag{B.40}$$

and

$$\tilde{\Pi}(\Delta l, k_x) = \frac{2}{k_x}\sin\left(\frac{k_x \Delta l}{2}\right) \tag{B.41}$$

respectively. This finally leads to

$$\tilde{\mathbf{f}}^{SX}(k_x,k_y) = \frac{8}{\Delta l k_x^2 k_y}\sin^2\left(\frac{k_x \Delta l}{2}\right)\sin\left(\frac{k_y \Delta w}{2}\right)\hat{x}, \quad z = z^S \tag{B.42}$$

and

$$\tilde{\mathbf{f}}^{SY}(k_x,k_y) = \frac{8}{\Delta w k_y^2 k_x}\sin^2\left(\frac{k_y \Delta w}{2}\right)\sin\left(\frac{k_x \Delta l}{2}\right)\hat{y}, \quad z = z^S. \tag{B.43}$$

B.2.3 Sinusoidal Basis Functions on the Resonant Patches

Refer to Figure 3.9 for an illustration of the geometrical parameters associated with the entire-domain sinusoidal basis functions on the rectangular resonant patches. The only differences here, are that the u and v axes are replaced by x and y axes respectively, and that the ne subscript is dropped. As was shown in Section 3.5.3, in the spatial domain, the sinusoidal basis functions can be expressed as

$$\mathbf{f}^{EX}(x,y) = \sin\left[\frac{p^{EX}\pi}{L}\left(x+\frac{L}{2}\right)\right]\cos\left[\frac{q^{EX}\pi}{W}\left(y+\frac{W}{2}\right)\right]\hat{x}, \quad |x|\leq\frac{L}{2},\ |y|\leq\frac{W}{2},\ z=z^E \quad \text{(B.44)}$$

for the x-directed functions and as

$$\mathbf{f}^{EY}(x,y) = \sin\left[\frac{p^{EY}\pi}{W}\left(y+\frac{W}{2}\right)\right]\cos\left[\frac{q^{EY}\pi}{L}\left(x+\frac{L}{2}\right)\right]\hat{y}, \quad |x|\leq\frac{L}{2},\ |y|\leq\frac{W}{2},\ z=z^E \quad \text{(B.45)}$$

for the y-directed functions. In compact notation, (B.44) and (B.45) can be expressed as

$$\mathbf{f}^{EX}(x,y) = f_s(p^{EX},L,x)f_c(q^{EX},W,y)\,\hat{x}, \quad |x|\leq\frac{L}{2},\ |y|\leq\frac{W}{2},\ z=z^E \quad \text{(B.46)}$$

and

$$\mathbf{f}^{EY}(x,y) = f_s(p^{EY},W,y)f_c(q^{EY},L,x)\,\hat{y}, \quad |x|\leq\frac{L}{2},\ |y|\leq\frac{W}{2},\ z=z^E, \quad \text{(B.47)}$$

where $f_s(p,L,x)$ and $f_c(p,L,x)$ have already been defined in (B.10) and (B.11).

The two-dimensional Fourier transforms of $\mathbf{f}^{EX}(x,y)$ and $\mathbf{f}^{EY}(x,y)$ can be reduced to the product of two one-dimensional Fourier transforms, due to the fact that the variables in (B.46) and (B.47) can be separated. The spectral forms of the sinusoidal basis functions are then given by

$$\tilde{\mathbf{f}}^{EX}(k_x,k_y) = \tilde{f}_s(p^{EX},L,k_x)\tilde{f}_c(q^{EX},W,k_y)\,\hat{x}, \quad z=z^E \quad \text{(B.48)}$$

and

$$\tilde{\mathbf{f}}^{EY}(k_x,k_y) = \tilde{f}_s(p^{EY},W,k_y)\tilde{f}_c(q^{EY},L,k_x)\,\hat{y}, \quad z=z^E, \quad \text{(B.49)}$$

where the Fourier transforms, $\tilde{f}_s(p,L,k_x)$ and $\tilde{f}_c(p,L,k_x)$, of $f_s(p,L,x)$ and $f_c(p,L,x)$ can easily be derived. They are given by

$$\tilde{f}_s(p,L,k_x) = \frac{p\pi/L}{k_x^2-(p\pi/L)^2}\left[(-1)^p e^{-jk_xL/2} - e^{jk_xL/2}\right] \quad \text{(B.50)}$$

and

$$\tilde{f}_c(p, L, k_x) = \frac{jk_x}{k_x^2 - (p\pi/L)^2}\left[(-1)^p e^{-jk_xL/2} - e^{jk_xL/2}\right] \quad \text{(B.51)}$$

respectively. This finally leads to

$$\begin{aligned}\tilde{\mathbf{f}}^{EX}(k_x, k_y) = {} & \frac{p^{EX}\pi/L}{k_x^2 - (p^{EX}\pi/L)^2}\left[(-1)^{p^{EX}} e^{-jk_xL/2} - e^{jk_xL/2}\right] \\ & \cdot \frac{jk_y}{k_y^2 - (q^{EX}\pi/W)^2}\left[(-1)^{q^{EX}} e^{-jk_yW/2} - e^{jk_yW/2}\right]\hat{x}, \quad z = z^E \quad \text{(B.52)}\end{aligned}$$

and

$$\begin{aligned}\tilde{\mathbf{f}}^{EY}(k_x, k_y) = {} & \frac{p^{EY}\pi/W}{k_y^2 - (p^{EY}\pi/W)^2}\left[(-1)^{p^{EY}} e^{-jk_yW/2} - e^{jk_yW/2}\right] \\ & \cdot \frac{jk_x}{k_x^2 - (q^{EY}\pi/L)^2}\left[(-1)^{q^{EY}} e^{-jk_xL/2} - e^{jk_xL/2}\right]\hat{y}, \quad z = z^E. \quad \text{(B.53)}\end{aligned}$$

B.2.4 Rectangular Attachment Mode

It is possible to reduce the two-dimensional Fourier transform of $\mathbf{J}^{AP}(x, y, \dot{x}^{AZ}, \dot{y}^{AZ})$ to one-dimensional Fourier transforms, due to the fact that the variables in (B.7) can be separated. The spectral form of the patch part of the rectangular attachment mode is then given by [6, 7, 129, 132]

$$\begin{aligned}\tilde{\mathbf{J}}^{AP}(k_x, k_y, \dot{x}^{AZ}, \dot{y}^{AZ}) = {} & \frac{-1}{2W}\sum_{i=0}^{\infty}\epsilon_i \cos\left[\frac{i\pi}{W}\left(\dot{y}^{AZ} + \frac{W}{2}\right)\right] \\ & \cdot \left\{\tilde{g}_s(\beta, L, \dot{x}^{AZ}, k_x)\tilde{f}_c(i, W, k_y)\,\hat{x} + \left(\frac{i\pi}{W}\right)\tilde{g}_c(\beta, L, \dot{x}^{AZ}, k_x)\tilde{f}_s(i, W, k_y)\,\hat{y}\right\}, \\ & z = z^{AP}, \quad \text{(B.54)}\end{aligned}$$

where

$$\begin{aligned}\tilde{g}_s(\beta, L, \dot{x}^{AZ}, k_x) = {} & \frac{-j2}{(k_x^2 - \beta^2)\sin(\beta L)}\left(\beta\left\{e^{j\beta\dot{x}^{AZ}}\sin\left[\frac{L(k_x - \beta)}{2}\right]\right.\right. \\ & \left.\left. + e^{-j\beta\dot{x}^{AZ}}\sin\left[\frac{L(k_x + \beta)}{2}\right]\right\} - k_x e^{-jk_x\dot{x}^{AZ}}\sin(\beta L)\right) \\ = {} & \frac{-j2}{(k_x^2 - \beta^2)}\left[jk_x e^{-jk_x\dot{x}^{AZ}} + j\frac{\sin(k_xL/2)\beta\cos(\beta\dot{x}^{AZ})}{\sin(\beta L/2)}\right. \\ & \left. - \frac{\cos(k_xL/2)\beta\sin(\beta\dot{x}^{AZ})}{\cos(\beta L/2)}\right] \quad \text{(B.55)}\end{aligned}$$

and

$$\tilde{g}_c(\beta, L, \dot{x}^{AZ}, k_x) = \frac{2}{(k_x^2 - \beta^2)\beta\sin(\beta L)}\left(k_x\left\{e^{j\beta\dot{x}^{AZ}}\sin\left[\frac{L(k_x - \beta)}{2}\right]\right.\right.$$

$$+ e^{-j\beta \dot{x}^{AZ}} \sin\left[\frac{L(k_x+\beta)}{2}\right]\Bigg\} - \beta\, e^{-jk_x\dot{x}^{AZ}} \sin(\beta L)\Bigg)$$

$$= \frac{2}{(k_x^2-\beta^2)}\left[-e^{-jk_x\dot{x}^{AZ}} + \frac{\sin(k_xL/2)k_x\cos(\beta\dot{x}^{AZ})}{\beta\sin(\beta L/2)} + j\frac{\cos(k_xL/2)k_x\sin(\beta\dot{x}^{AZ})}{\beta\cos(\beta L/2)}\right] \tag{B.56}$$

are the Fourier transforms of $g_s(\beta, L, \dot{x}^{AZ}, x)$ and $g_c(\beta, L, \dot{x}^{AZ}, x)$ in (B.8) and (B.9) respectively. These expressions appear very intimidating, but can be derived from standard Fourier transforms. The expressions for the functions $\tilde{f}_s(i, W, k_y)$ and $\tilde{f}_c(i, W, k_y)$ have already been given in (B.50) and (B.51) respectively.

Substitution of all the relevant expressions into (B.54), leads to

$$\begin{aligned}
&\tilde{\mathbf{J}}^{AP}(k_x, k_y, \dot{x}^{AZ}, \dot{y}^{AZ}) \\
&= \frac{-1}{W}\sum_{i=0}^{\infty} \epsilon_i \cos\left[\frac{i\pi}{W}\left(\dot{y}^{AZ}+\frac{W}{2}\right)\right]\frac{(-1)^i e^{-jk_yW/2} - e^{jk_yW/2}}{(k_x^2-\beta^2)[k_y^2-(i\pi/W)^2]} \\
&\cdot\Bigg\{jk_y\left[jk_x\, e^{-jk_x\dot{x}^{AZ}} + j\frac{\sin(k_xL/2)\beta\cos(\beta\dot{x}^{AZ})}{\sin(\beta L/2)} - \frac{\cos(k_xL/2)\beta\sin(\beta\dot{x}^{AZ})}{\cos(\beta L/2)}\right]\hat{x} \\
&+\left(\frac{i\pi}{W}\right)^2\left[-e^{-jk_x\dot{x}^{AZ}} + \frac{\sin(k_xL/2)k_x\cos(\beta\dot{x}^{AZ})}{\beta\sin(\beta L/2)} + j\frac{\cos(k_xL/2)k_x\sin(\beta\dot{x}^{AZ})}{\beta\cos(\beta L/2)}\right]\hat{y}\Bigg\}, \\
&\qquad z = z^{AP}.
\end{aligned} \tag{B.57}$$

The expression in (B.57) can now be separated into a regular component, which is a fast convergent series, and a singular component, which is a slowly convergent series. The spectral form of $\mathbf{J}^{AP}(x, y, \dot{x}^{AZ}, \dot{y}^{AZ})$ can therefore be expressed as [6, 7]

$$\tilde{\mathbf{J}}^{AP}(k_x, k_y, \dot{x}^{AZ}, \dot{y}^{AZ}) = \tilde{\mathbf{J}}^{AP,\text{reg}}(k_x, k_y, \dot{x}^{AZ}, \dot{y}^{AZ}) + \tilde{\mathbf{J}}^{AP,\text{sing}}(k_x, k_y, \dot{x}^{AZ}, \dot{y}^{AZ}), \tag{B.58}$$

where

$$\begin{aligned}
&\tilde{\mathbf{J}}^{AP,\text{reg}}(k_x, k_y, \dot{x}^{AZ}, \dot{y}^{AZ}) \\
&= \frac{-1}{W}\sum_{i=0}^{\infty} \epsilon_i \cos\left[\frac{i\pi}{W}\left(\dot{y}^{AZ}+\frac{W}{2}\right)\right]\frac{(-1)^i e^{-jk_yW/2} - e^{jk_yW/2}}{(k_x^2-\beta^2)[k_y^2-(i\pi/W)^2]} \\
&\cdot\left[k_y\,\hat{x} + k_x\frac{(i\pi/W)^2}{\beta^2}\,\hat{y}\right]\left[\frac{\beta\cos(\beta\dot{x}^{AZ})\sin(k_xL/2)}{\sin(\beta L/2)} - j\frac{\beta\sin(\beta\dot{x}^{AZ})\cos(k_xL/2)}{\cos(\beta L/2)}\right], \\
&\qquad z = z^{AP}
\end{aligned} \tag{B.59}$$

and

$$\begin{aligned}
&\tilde{\mathbf{J}}^{AP,\text{sing}}(k_x, k_y, \dot{x}^{AZ}, \dot{y}^{AZ}) \\
&= \frac{1}{W}\, e^{-jk_x\dot{x}^{AZ}} \sum_{i=0}^{\infty} \epsilon_i \cos\left[\frac{i\pi}{W}\left(\dot{y}^{AZ} + \frac{W}{2}\right)\right] \frac{(-1)^i e^{-jk_yW/2} - e^{jk_yW/2}}{(k_x^2-\beta^2)[k_y^2-(i\pi/W)^2]} \left[k_xk_y\,\hat{x} + \left(\frac{i\pi}{W}\right)^2 \hat{y}\right] \\
&= \frac{e^{-jk_x\dot{x}^{AZ}}}{k_y^2-k_\alpha^2}\left\{k_xk_y[A(k_\alpha)-A(k_y)]\,\hat{x} + [k_\alpha^2 A(k_\alpha) - k_y^2 A(k_y)]\,\hat{y}\right\}, \quad z = z^{AP}.
\end{aligned} \tag{B.60}$$

In the final expression of (B.60), the substitutions

$$k_\alpha^2 = \varepsilon_{r(\text{eff})} k_0^2 - k_x^2, \tag{B.61}$$

$$k_y^2 - k_\alpha^2 = -\left[\varepsilon_{r(\text{eff})}k_0^2 - k_x^2 - k_y^2\right] = -k_{z(\text{eff})}^2 \tag{B.62}$$

and

$$A(\chi) = \frac{W}{\pi^2}\sum_{i=0}^{\infty} \epsilon_i \cos\left[\frac{i\pi}{W}\left(\dot{y}^{AZ} + \frac{W}{2}\right)\right] \frac{(-1)^i e^{-jk_yW/2} - e^{jk_yW/2}}{i^2 - (\chi W/\pi)^2} \tag{B.63}$$

have been made.

By using the fact that in general

$$\sum_{i=0}^{\infty} \frac{\epsilon_i \cos(ix)}{i^2-\alpha^2} = \sum_{i=0}^{\infty} \frac{\epsilon_i \cos[i(\pi-x)]}{i^2-\alpha^2}(-1)^i = \frac{-\pi\cos[\alpha(\pi-x)]}{\alpha\sin(\alpha\pi)}, \quad 0 \le x \le 2\pi, \tag{B.64}$$

$A(\chi)$ can be summed up in close form, leading to

$$\begin{aligned}
A(\chi) &= \frac{-1}{\chi}\left[\cos\left(\chi\dot{y}^{AZ} + \frac{\chi W}{2}\right)\frac{e^{-jk_yW/2}}{\sin(\chi W)} - \cos\left(\chi\dot{y}^{AZ} - \frac{\chi W}{2}\right)\frac{e^{jk_yW/2}}{\sin(\chi W)}\right] \\
&= \frac{j}{\chi}\left[\frac{\cos(\chi\dot{y}^{AZ})\sin(k_yW/2)}{\sin(\chi W/2)} - j\frac{\sin(\chi\dot{y}^{AZ})\cos(k_yW/2)}{\cos(\chi W/2)}\right].
\end{aligned} \tag{B.65}$$

If this expression for $A(\chi)$ is inserted into (B.60), $\tilde{\mathbf{J}}^{AP,\text{sing}}(k_x, k_y, \dot{x}^{AZ}, \dot{y}^{AZ})$ can be expressed as

$$\begin{aligned}
\tilde{\mathbf{J}}^{AP,\text{sing}}(k_x, k_y, \dot{x}^{AZ}, \dot{y}^{AZ}) = {}& \frac{j(k_x\,\hat{x} + k_y\,\hat{y})}{k_{z(\text{eff})}^2}\, e^{-j(k_x\dot{x}^{AZ} + k_y\dot{y}^{AZ})} \\
&- \frac{k_xk_y\,\hat{x} + k_\alpha^2\,\hat{y}}{k_{z(\text{eff})}^2}\, A(k_\alpha)\, e^{-jk_x\dot{x}^{AZ}}, \quad z = z^{AP}.
\end{aligned} \tag{B.66}$$

Following from (B.15), the spectral form of the rectangular attachment mode is given by

$$\tilde{\mathbf{f}}^{AP}(k_x, k_y) = \frac{1}{2\pi}\int_0^{2\pi} \tilde{\mathbf{J}}^{AP}(k_x, k_y, z, \dot{x}^{AZ}, \dot{y}^{AZ})\, d\phi. \tag{B.67}$$

Now, by realising that [6, 7]

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-j(k_x\dot{x}^{AZ}+k_y\dot{y}^{AZ})}\,d\phi = J_0(k_\rho a)\,e^{-j(k_x x^{AZ}+k_y y^{AZ})}, \tag{B.68}$$

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-jk_x\dot{x}^{AZ}}\cos(k_\alpha\dot{y}^{AZ})\,d\phi = J_0(\sqrt{\varepsilon_{r(\text{eff})}}k_0 a)\,e^{-jk_x x^{AZ}}\cos(k_\alpha y^{AZ}) \tag{B.69}$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-jk_x\dot{x}^{AZ}}\sin(k_\alpha\dot{y}^{AZ})\,d\phi = J_0(\sqrt{\varepsilon_{r(\text{eff})}}k_0 a)\,e^{-jk_x x^{AZ}}\sin(k_\alpha y^{AZ}), \tag{B.70}$$

the spectral form of the rectangular attachment mode is finally given by

$$\begin{aligned}\tilde{\mathbf{f}}^{AP}(k_x,k_y) = {} & \frac{j(k_x\,\hat{x}+k_y\,\hat{y})}{k^2_{z(\text{eff})}}\,J_0(k_\rho a)\,e^{-j(k_x x^{AZ}+k_y y^{AZ})} \\ & - \frac{(k_x k_y\,\hat{x}+k_\alpha^2\,\hat{y})}{k^2_{z(\text{eff})}}\,A(k_\alpha)J_0(\sqrt{\varepsilon_{r(\text{eff})}}k_0 a)\,e^{-jk_x x^{AZ}} \\ & - \frac{1}{W}\sum_{i=0}^{\infty}\epsilon_i\cos\left[\frac{i\pi}{W}\left(y^{AZ}+\frac{W}{2}\right)\right]\frac{(-1)^i e^{-jk_yW/2}-e^{jk_yW/2}}{[k_y^2-(i\pi/W)^2][k_x^2-\beta^2]} \\ & \cdot\left[k_y\,\hat{x}+k_x\frac{(i\pi/W)^2}{\beta^2}\,\hat{y}\right]\left[\frac{\beta\cos(\beta x^{AZ})\sin(k_xL/2)}{\sin(\beta L/2)}\right. \\ & \left. - j\frac{\beta\sin(\beta x^{AZ})\cos(k_xL/2)}{\cos(\beta L/2)}\right]J_0(\sqrt{\varepsilon_{r(\text{eff})}}k_0 a), \quad z = z^{AP},\end{aligned} \tag{B.71}$$

with

$$A(k_\alpha) = \frac{j}{k_\alpha}\left[\frac{\cos(k_\alpha y^{AZ})\sin(k_yW/2)}{\sin(k_\alpha W/2)} - j\frac{\sin(k_\alpha y^{AZ})\cos(k_yW/2)}{\cos(k_\alpha W/2)}\right]. \tag{B.72}$$

B.2.5 Circular Attachment Mode

Refer to Figure 3.11 for an illustration of the geometrical parameters associated with the circular attachment mode. The only differences here, are that the u and v axes are replaced by x and y axes respectively, and that the na subscript is dropped. As was shown in Section 3.6.2, in the spatial domain, the patch part of the circular attachment mode can be expressed as

$$\mathbf{f}^{AP}(x,y) = \begin{cases} \dfrac{-\rho}{2\pi b^2}\,\hat{\rho}, & 0\le\rho<a,\ z=z^{AP} \\ \left(\dfrac{-\rho}{2\pi b^2}+\dfrac{1}{2\pi\rho}\right)\hat{\rho}, & a\le\rho\le b,\ z=z^{AP} \\ 0, & \rho>b,\ z=z^{AP}. \end{cases} \tag{B.73}$$

Now, for a general function $\mathbf{f}$, which is only a function of ρ, the two-dimensional Fourier transform can be simplified as [127]

$$\tilde{\mathbf{f}}(k_x, k_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{f}(x,y)\, e^{-jk_x x}\, e^{-jk_y y}\, dxdy$$
$$= \left[-j2\pi \int_0^{\infty} J_1(k_\rho \rho) f(\rho)\rho\, d\rho\right] \hat{k}_\rho, \quad \mathbf{f}(x,y) = f(\rho)\,\hat{\rho}, \tag{B.74}$$

where $J_1(\cdot)$ is the Bessel function of the first kind of order 1 (in general, $J_i(\cdot)$ is the Bessel function of the first kind of order i). This implies that the spectral form of the circular attachment mode can be written as

$$\tilde{\mathbf{f}}^{AP}(k_x, k_y) = -j2\pi \Bigg[\underbrace{\int_0^b J_1(k_\rho \rho)\frac{-\rho^2}{2\pi b^2}\, d\rho}_{\Upsilon_1(k_x,k_y)} + \underbrace{\int_a^b J_1(k_\rho \rho)\frac{1}{2\pi}\, d\rho}_{\Upsilon_2(k_x,k_y)}\Bigg] \hat{k}_\rho, \quad z = z^{AP}. \tag{B.75}$$

By using the integral properties of Bessel functions [73], it can be shown that

$$\int_0^{\alpha} \rho^2 J_1(k_\rho \rho)\, d\rho = \frac{\alpha^2}{k_\rho} J_2(k_\rho \alpha). \tag{B.76}$$

It is also known that [211]

$$\int_0^{\alpha} J_1(k_\rho \rho)\, d\rho = \frac{1}{k_\rho}\Big[1 - J_0(k_\rho \alpha)\Big]. \tag{B.77}$$

With these relations, the integrals $\Upsilon_1(k_x, k_y)$ and $\Upsilon_2(k_x, k_y)$ can be evaluated as

$$\Upsilon_1(k_x, k_y) = \int_0^b J_1(k_\rho \rho)\frac{-\rho^2}{2\pi b^2}\, d\rho$$
$$= \frac{-1}{2\pi b^2}\int_0^b \rho^2 J_1(k_\rho \rho)\, d\rho$$
$$= \frac{-1}{2\pi b^2}\left[\frac{\rho^2}{k_\rho} J_2(k_\rho \rho)\right]\Bigg|_0^b$$
$$= \frac{-1}{2\pi k_\rho} J_2(k_\rho b) \tag{B.78}$$

and

$$\Upsilon_2(k_x, k_y) = \int_a^b J_1(k_\rho \rho)\frac{1}{2\pi}\, d\rho$$
$$= \frac{1}{2\pi}\int_a^b J_1(k_\rho \rho)\, d\rho$$
$$= \frac{1}{2\pi k_\rho}\Big[1 - J_0(k_\rho \rho)\Big]\Bigg|_a^b$$
$$= \frac{1}{2\pi k_\rho}\Big[J_0(k_\rho a) - J_0(k_\rho b)\Big] \tag{B.79}$$

respectively.

By using the integrals in (B.78) and (B.79), the spectral form of the circular attachment mode can be expressed as

$$\tilde{\mathbf{f}}^{AP}(k_x, k_y) = \left[\frac{jJ_2(k_\rho b)}{k_\rho} - \frac{jJ_0(k_\rho a)}{k_\rho} + \frac{jJ_0(k_\rho b)}{k_\rho}\right]\left[\frac{k_x}{k_\rho}\,\hat{x} + \frac{k_y}{k_\rho}\,\hat{y}\right], \quad z = z^{AP}, \tag{B.80}$$

which can further be simplified by using the property of the Bessel function that [182, 212]

$$k_\rho \rho J_{i-1}(k_\rho \rho) + k_\rho \rho J_{i+1}(k_\rho \rho) = 2iJ_i(k_\rho \rho). \tag{B.81}$$

Finally, the spectral form of the circular attachment mode can then be expressed as

$$\tilde{\mathbf{f}}^{AP}(k_x, k_y) = \left[\frac{j2J_1(k_\rho b)}{k_\rho^2 b} - \frac{jJ_0(k_\rho a)}{k_\rho}\right]\left[\frac{k_x}{k_\rho}\,\hat{x} + \frac{k_y}{k_\rho}\,\hat{y}\right], \quad z = z^{AP}. \tag{B.82}$$

B.2.6 Higher-Order Circular Attachment Mode

Refer to Figure 3.11 for an illustration of the geometrical parameters associated with the higher-order circular attachment mode. The only differences here, are that the u and v axes are replaced by x and y axes respectively, and that the na subscript is dropped. As was shown in Section 3.6.3, in the spatial domain, the patch part of the higher-order circular attachment mode can be expressed as

$$\mathbf{f}^{AP}(x, y) = \begin{cases} \dfrac{-\rho^3}{2\pi b^4}\,\hat{\rho}, & 0 \le \rho < a,\ z = z^{AP} \\ \left(\dfrac{-\rho^3}{2\pi b^4} + \dfrac{1}{2\pi\rho}\right)\hat{\rho}, & a \le \rho \le b,\ z = z^{AP} \\ 0, & \rho > b,\ z = z^{AP}. \end{cases} \tag{B.83}$$

In a similar way to that shown in Section B.2.5, the spectral form of the higher-order circular attachment mode can be written as

$$\tilde{\mathbf{f}}^{AP}(k_x, k_y) = -j2\pi\Bigg[\underbrace{\int_0^b J_1(k_\rho\rho)\frac{-\rho^4}{2\pi b^4}\,d\rho}_{\Upsilon_3(k_x,k_y)} + \underbrace{\int_a^b J_1(k_\rho\rho)\frac{1}{2\pi}\,d\rho}_{\Upsilon_2(k_x,k_y)}\Bigg]\hat{k}_\rho, \quad z = z^{AP}. \tag{B.84}$$

By using the integral properties of Bessel functions [73], it can be shown that

$$\int_0^\alpha \rho^4 J_1(k_\rho\rho)\,d\rho = \frac{-\alpha^4}{k_\rho}\,J_4(k_\rho\alpha) + \frac{4\alpha^3}{k_\rho^2}\,J_3(k_\rho\alpha). \tag{B.85}$$

This can be used to evaluate the integral $\Upsilon_3(k_x, k_y)$ as

$$\begin{aligned}\Upsilon_3(k_x, k_y) &= \int_0^b J_1(k_\rho \rho) \frac{-\rho^4}{2\pi b^4} \, d\rho \\ &= \frac{-1}{2\pi b^4} \int_0^b \rho^4 J_1(k_\rho \rho) \, d\rho \\ &= \frac{-1}{2\pi b^4} \left[\frac{-\rho^4}{k_\rho} J_4(k_\rho \rho) + \frac{4\rho^3}{k_\rho^2} J_3(k_\rho \rho) \right] \Bigg|_0^b \\ &= \frac{1}{2\pi k_\rho} J_4(k_\rho b) - \frac{4}{2\pi k_\rho^2 b} J_3(k_\rho b). \end{aligned} \tag{B.86}$$

The integral $\Upsilon_2(k_x, k_y)$ has already been evaluated in Section B.2.5 and is given by (B.79).

By using the integrals in (B.79) and (B.86), the spectral form of the higher-order circular attachment mode can be expressed as

$$\tilde{\mathbf{f}}^{AP}(k_x, k_y) = \left[\frac{-jJ_4(k_\rho b)}{k_\rho} + \frac{j4J_3(k_\rho b)}{k_\rho^2 b} - \frac{jJ_0(k_\rho a)}{k_\rho} + \frac{jJ_0(k_\rho b)}{k_\rho} \right] \left[\frac{k_x}{k_\rho} \hat{x} + \frac{k_y}{k_\rho} \hat{y} \right], \quad z = z^{AP}. \tag{B.87}$$

Then, by making use of the property of the Bessel function in (B.81), the spectral-domain form of the higher-order circular attachment mode can further be simplified to

$$\tilde{\mathbf{f}}^{AP}(k_x, k_y) = \left[\frac{-j16J_1(k_\rho b)}{k_\rho^4 b^3} + \frac{j4J_1(k_\rho b)}{k_\rho^2 b} + \frac{j8J_0(k_\rho b)}{k_\rho^2 b^2} - \frac{jJ_0(k_\rho a)}{k_\rho} \right] \left[\frac{k_x}{k_\rho} \hat{x} + \frac{k_y}{k_\rho} \hat{y} \right], \quad z = z^{AP}. \tag{B.88}$$