Part II

Risk-Neutral Valuation

Chapter 5

Risk-Neutral Valuation

5.1 Objectives[1]

The aim of this chapter is to provide essential background to continuous-time finance concepts and the standard risk-neutral valuation framework, which is the cornerstone of the Black-Scholes option pricing framework. The Black-Scholes framework is the benchmark pricing method for options. In this framework we assume constant volatility of stock returns which leads to the helpful property of a complete market model.

Empirical evidence shows that the constant volatility assumption is generally incorrect. The GARCH option pricing model discussed in chapters 6 and 7 is an attempt to include stochastic volatility into the option pricing framework, the price is that the market model is no longer complete. Although volatility is generally stochastic, it is important to know the risk-neutral valuation framework, since it is so widely used and because many of the concepts are used in incomplete market models.

In this chapter only the bare skeleton of the risk-neutral valuation framework is given. For more complete discussions see [25], [4], [32] or any of the many other similar books.

An introduction to continuous time stochastic calculus is given in section 5.2. The essential definitions of Brownian motion, martingales and Ito processes are given. The proofs of the Ito formula, absolute continuous measures and equivalent measures, the Radon-Nikodym theorem and Girsanov's theorem are excluded.

Continuous-time finance concepts are briefly discussed in section 5.3.

Section 5.4 is the core section of this chapter. The risk-neutral valuation framework is discussed under the assumption of constant volatility. Only the proofs vital for a better understanding of the model investigated in chapters 6 and 7 are proved. Special attention is paid to the concept of the market price of risk.

[1]Suggested reading: [4], [13], [17], [26] and [32].

5.2 Essentials of Continuous-time Stochastic Calculus

5.2.1 Brownian Motion

Definition 5.2.1 *Brownian motion, W_t, is a real-valued stochastic process satisfying the following conditions:*

1. *Continuous sample paths:* $t \to W_t \qquad P$ *a.s..*
2. *Stationary increments:* $W_{t+s} - W_t$ *has the same probability law for any* $t \in \mathbb{R}^+$ *varying and* $s \in \mathbb{R}^+$ *fixed.*
3. *Independent increments:* $W_{t+s} - W_t$ *is independent of*

   $$\mathcal{F}_t = \sigma(W_u, u \le t)$$

4. $W_0 = 0 \qquad P$ *a.s.*

The probability law mentioned in point 2, will throughout this dissertation be the Normal distribution with mean zero and variance s.

5.2.2 Martingales

Definition 5.2.2 *In discrete time: An adapted process,* $(M_t)_{t \in I}$*, where* I *is a countable index and* $E\,|M_t| < \infty$*, is called:*

1. *A martingale if*

   $$E(M_t \,|\, \mathcal{F}_s) = M_s \; Pa.s.$$

   for all $s, t \in I,\ s \le t.$

2. *A super-martingale if*

   $$E(M_t \,|\, \mathcal{F}_s) \le M_s \quad P \text{ a.s.}$$

   for all $s, t \in I,\ s \le t.$

Definition 5.2.3 *In continuous time: An adapted process,* $(M_t)_{t \in \mathbf{R}^+}$*, where* $\mathbb{R}^+$ *is the positive real numbers and,* $E\,|M_t| < \infty$ *is called:*

1. *A martingale if*

   $$E\left[M_t \,|\, \mathcal{F}_s\right] = M_s \quad P \text{ a.s.}$$

   for all $s, t \in I,\ s \le t.$

2. *A super-martingale if*

   $$E\left[M_t \,|\, \mathcal{F}_s\right] \le M_s \quad P \text{ a.s.}$$

   for all $s, t \in I,\ s \le t.$

5.2.3 Ito Process

Definition 5.2.4 *A stochastic process, X_t, is called an Ito process if it has a.s. continuous paths and*

$$X_t = X_0 + \int_0^T A(t,\omega)dt + \int_0^T B(t,\omega)dW_t \tag{5.1}$$

where $A(t,\omega)$ and $B(t,\omega)$ are $\mathcal{F}_t$ measurable,

$$\int_0^T |A(t,\omega)|\, dt < \infty \quad P \text{ a.s.}$$

and

$$E\left[\int_0^T B(t,\omega)^2 dt\right] < \infty \quad P \text{ a.s.}$$

X_t is also called the stock price process. In short hand notation

$$dX_t = A(t,\omega)dt + B(t,\omega)dW_t$$

Definition 5.2.5 *A stochastic process, S_t, follows a geometric Brownian motion if*

$$dS_t = S_t\mu(t,\omega)dt + S_t\sigma(t,\omega)dW_t$$

5.2.4 Ito Formula (in 1-Dimension)

Definition 5.2.6 *Let X_t be an Ito process as defined in equation (5.1). For the function*

$$f(t,x) \in C^2([0,\infty) \times \mathbb{R})$$

the Ito formula is given by

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dX_t)^2 \tag{5.2}$$

$$= (\frac{\partial f}{\partial t} + A\frac{\partial f}{\partial x} + \frac{1}{2}B^2\frac{\partial^2 f}{\partial x^2})dt + B\frac{\partial f}{\partial x}dW_t \tag{5.3}$$

In integral notation this is:

$$f_t = f_0 + \int_0^t (\frac{\partial f}{\partial t} + A\frac{\partial f}{\partial x} + \frac{1}{2}B^2\frac{\partial^2 f}{\partial x^2})dt + \int_0^t B\frac{\partial f}{\partial x}dW_t \tag{5.4}$$

5.2.5 Absolute Continuous

Definition 5.2.7 *In our probability space $(\Omega, \mathcal{F}, P)$, probability measure P_1 is said to be absolutely continuous with respect to P if*

$$P(A) = 0 \Rightarrow P_1(A) = 0$$

for all $A \in \mathcal{F}$. This is sometimes denoted by

$$P_1 << P$$

Theorem 5.2.8 *Probability measure P_1 is absolutely continuous with respect to P if and only if there exists an adapted random variable K such that*

$$P_1(A) = \int_A K(\omega)\, dP \tag{5.5}$$

Proof. *See Lamberton and Lapeyre [26].* ■

Definition 5.2.9 *The state price density is defined as*

$$\frac{dP_1}{dP}$$

thus from integral (5.5)

$$\frac{dP_1}{dP} = K$$

Definition 5.2.10 *In the probability space $(\Omega, \mathcal{F})$ two probability measures P_1 and P_2 are equivalent if*

$$P_1(A) = 0 \Leftrightarrow P_2(A) = 0$$

for all $A \in \mathcal{F}$.(See Lamberton and Lapeyre [26])

5.2.6 Radon-Nikodym

Theorem 5.2.11 *Let measure Q be absolutely continuous with respect to measure P. There then exists a random variable $\Lambda \geq 0$, such that*

$$E^P[\Lambda] = 1$$

and

$$Q(A) = \int_A dQ = \int_A \Lambda dP \tag{5.6}$$

for all $A \in \mathcal{F}$. Λ is $P-a.s.$ unique. Conversely, if there exists a random variable, Λ with the mentioned properties and Q is defined by equation 5.6, then Q is a probability measure and Q is absolutely continuous with respect to P.

Proof. See [25]. ■

5.2.7 Risk-neutral Probability Measure

Definition 5.2.12 *A probability measure, Q, is called a risk-neutral probability measure if*

1. *Q is equivalent to the "real world" measure P.*
2. $\frac{S_t}{B_t} = E^Q\left(\frac{S_{t+\tau}}{B_{t+\tau}}|\mathcal{F}_t\right)$ *for all $t, \tau \in \mathbb{R}^+$.*

In this definition, B_t is the deterministic price process of a risk-free asset, where

$$B_t = B_0 \exp\left(\int_0^t r(s)ds\right)$$

The variable $r(t)$ is the short rate.

5.2.8 Girsanov's Theorem in One Dimension

Girsanov's theorem is used to transform stochastic processes in terms of their drift parameters. In option pricing, Girsanov's theorem is used to find a probability measure under which the risk-free rate adjusted stock price process is a martingale.

Definition 5.2.13 *A function $f(s,t) \in v(s,t)$ if*

$$f(t,\omega) : [0,\infty) \times \Omega \to \mathbb{R}$$

and the following holds:

1. $(t,\omega) \to f(t,\omega)$ *is $B \times \mathcal{F}$-measurable, where B is the Borel sets on* $[0,\infty)$
2. $f(t,\omega)$ *is adapted*
3. $E\left[\int_S^T f(t,\omega)^2 dt\right] < \infty$

Theorem 5.2.14 *Girsanov's theorem. Let $X_t \in \mathbb{R}$ be an Ito process, of the form*

$$dX_t = \beta(t,\omega) + \theta(t,\omega)dW_t$$

with $t \le T < \infty$. Suppose that there exist a $v(t,\omega)$-process $u(t,\omega) \in \mathbb{R}$ and $\alpha(t,\omega) \in \mathbb{R}$ such that

$$\theta(t,\omega)u(t,\omega) = \beta(t,\omega) - \alpha(t,\omega)$$

Since we are only looking at the one dimensional case

$$u(t,\omega) = \frac{(\beta(t,\omega) - \alpha(t,\omega))}{\theta(t,\omega)}$$

We further assume that

$$E^P\left[\exp\left(\frac{1}{2}\int_0^T u^2(s,\omega)\,ds\right)\right] < \infty \tag{5.7}$$

Let

$$M_t = \exp\left(-\int_0^t u(s,\omega)\,dW_t - \int_0^t u^2(s,\omega)\,ds\right) \tag{5.8}$$

and

$$dQ = M_T dP \tag{5.9}$$

We then have that

$$\tilde{W}_t = W_t + \int_0^t u(s,\omega)\,ds$$

is a Brownian motion with respect to Q. X_t in terms of $\tilde{W}_t$ is

$$dX_t = \alpha(t,\omega) + \theta(t,\omega)d\tilde{W}_t$$

M_t is a martingale.

Proof. *See Girsanov theorem II, Oksendal [27].* ■

Remark 5.2.15 *Result 5.9 is equivalent to*

$$E^Q[B] = E^P[BM_t]$$

for all Borel measurable sets B on $C[0,T]$.

5.3 Continuous-time Finance Essentials

This section contains a short summary of vital continuous-time finance concepts. For complete discussions on continuous-time finance see Bjork [4], Lamberton and Lapeyre [26] and Steele [32].

5.3.1 Self-financing

Definition 5.3.1 *A trading strategy is called self-financing if the value of the portfolio is due to the initial investment and gains and losses realized on the subsequent investments. This means that no funds are added or withdrawn from the portfolio.*

Theorem 5.3.2 *Let $\phi = \left(H_t^0, H_t\right)_{0\leq t\leq T}$ be an adapted process of portfolio weights satisfying*

$$\int_0^T \left|H_t^0\right| dt + \int_0^T H_t^2 dt < \infty \quad P\ a.s.$$

Then the discounted value of portfolio $V_t(\phi) = H_t^0\beta_t + H_tS_t$ namely, $\tilde{V}_t(\phi) = V_t(\phi)/\beta$ can be expressed for all $t \in [0,T]$ as

$$\tilde{V}_t(\phi) = V_0(\phi) + \int_0^t H_u d\tilde{S}_u \quad Q\ a.s.$$

if and only if ϕ is a self-financing strategy.

Proof. The product of $V_t(\phi)$ and with the bond process β yields

$$\begin{aligned} \frac{1}{\beta_t}V_t(\phi) &= V_0(\phi) + \int_0^t \frac{1}{\beta_t} dV_t(\phi) + \int_0^t V_s(\phi)\, d\frac{1}{\beta_t} + \left\langle V_t(\phi), \frac{1}{\beta_t}\right\rangle \\ &= V_0(\phi) + \int_0^t \frac{1}{\beta_t} dV_t(\phi) + \int_0^t V_s(\phi)\, d\frac{1}{\beta_t} \end{aligned}$$

since the process $\frac{1}{\beta_t}$ doesn't have a stochastic term. Since we can express $V_t(\phi)$ as

$$V_t(\phi) = H_t^0\beta_t + H_tS_t$$

a change in $V_t(\phi)$ can be expressed by

$$dV_t(\phi) = H_t^0 d\beta_t + H_t dS_t$$

thus

$$\begin{aligned} &\frac{1}{\beta_t}V_t(\phi) \\ &= V_0(\phi) + \int_0^t \frac{1}{\beta_t}\left(H_t^0 d\beta_t + H_t dS_t\right) + \int_0^t \left(H_t^0\beta_t + H_tS_t\right) d\frac{1}{\beta_t} \\ &= V_0(\phi) + H_t^0\left(\int_0^t \frac{1}{\beta_t} d\beta_t + \beta_t d\frac{1}{\beta_t}\right) + H_t\left(\int_0^t \frac{1}{\beta_t} H_t dS_t + H_tS_t d\frac{1}{\beta_t}\right) \\ &= V_0(\phi) + H_t^0 d\frac{\beta_t}{\beta_t} + H_t d\frac{S_t}{\beta_t} \\ &= V_0(\phi) + H_t d\frac{S_t}{\beta_t} \end{aligned}$$

■

5.3.2 Admissible Trading Strategy

Definition 5.3.3 *A trading strategy is admissible if it is self-financing and if the corresponding discounted portfolio,* $\tilde{V}_t$ *is nonnegative and* $\sup_{t\in[0,T]} \tilde{V}_t$ *is square integrable under the risk-neutral probability measure* Q*.*

5.3.3 Attainable Claim

Definition 5.3.4 *A claim is attainable if there exists an admissible trading strategy replicating that claim.*

5.3.4 Arbitrage Opportunity

Definition 5.3.5 *An arbitrage opportunity is an admissible trading strategy, such that the value of the portfolio at initialization,* $V(0) = 0$ *and* $E[V(T)] > 0$.

5.3.5 Complete Market

The completeness of a market can be defined in terms of the risk-neutral probability measure or in terms of the attainability of a contingent claim.

Definition 5.3.6 *Under no arbitrage conditions, the market model is complete if and only if every contingent claim is attainable.*

Theorem 5.3.7 *The market model is complete if and only if there exists a unique risk-neutral probability measure.*

Proof. See Pliska [28]. ■

5.4 Risk-Neutral Valuation under Constant Volatility

The aim of this section is to introduce the notion of risk-neutral valuation. The process of risk-neutral valuation is as follows:

1. In section 5.4.1 a simple stock price process is evaluated. A solution to this process is found and its distribution is discussed. The solution is obtained by applying the Ito process.

2. The next step, in section 5.4.2, is to evaluate the discounted stock price process. We get the discounted stock price process by discounting the solution to the original process in step 1 and then utilizing the Ito formula in reverse order.

3. This new process still has a trend. The so-called risk-neutral measure and related Brownian process is derived with Girsanov's theorem in section 5.4.3.

4. A wide-class of options are priced under risk-neutral valuation in section 5.4.4.

5.4.1 The Stock Price Process

It is generally assumed that stock prices follow geometric Brownian motion, under the real world measure P,

$$dS_t = S_t \mu dt + S_t \sigma dW_t \tag{5.10}$$

where $\mu \in \mathbb{R}$ and $S_0, \sigma \in \mathbb{R}^+$, W_t is Brownian motion and the process is defined on $[0, T]$.

A solution, S_t, to this equation can be found with the help of Ito's formula. Let $f(t,x) = \ln(x)$. It follows from section 5.2.4 that $f(t,x) \in C^2([0,\infty) \times \mathbb{R})$. Fortunately, if we assume that $S_t \in \mathbb{R}^+$, we can define $f(t,x) \in C^2([0,\infty) \times \mathbb{R}^+)$. From (5.4) we have[2]

$$\begin{aligned} d\ln(S_t) &= \frac{1}{S_t} dS_t - \frac{1}{2}\frac{1}{S_t^2} dS_t^2 \\ &= \frac{1}{S_t}\left(S_t \mu dt + S_t \sigma dW_t\right) \\ &\quad -\frac{1}{2}\frac{1}{S_t^2}\left(S_t \mu dt + S_t \sigma dW_t\right)^2 \\ &= \mu dt + \sigma dW_t - \frac{1}{2}\sigma^2 dt \\ &= \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t \end{aligned}$$

which in integral notation is

$$\begin{aligned} \ln(S_t) &= \ln(S_0) + \int_0^t \left(\mu - \frac{1}{2}\sigma^2\right) du + \int_0^t \sigma dW_u \\ &= \ln(S_0) + \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t \end{aligned} \tag{5.11}$$

The solution, S_t, is

$$S_t = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right) \tag{5.12}$$

[2]In this chapter the drift μ, the variance σ and the risk-free interest rate r are all defined in terms of the same time period for instance 1 year.

Thus by assuming that the stock price follows the geometric Brownian motion described in equation 5.10, we are also assuming that the stock price process is lognormally distributed. There are ample empirical evidence to support this assumption. This means that from equation 5.11

$$\ln(S_t) \sim N\left(\ln(S_0) + \left(\mu - \frac{1}{2}\sigma^2\right)t, \sigma^2 t\right)$$

5.4.2 The Discounted Stock Price Process

The next aim is to find a probability measure under which $\tilde{S}_t = S_t/B_t$ is a martingale, called the risk-neutral probability measure. The discounted process

$$\tilde{S}_t = S_0 \exp\left(\left(\mu - r - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right) \tag{5.13}$$

where $B_t = e^{rt}$ and r is the constant risk-free rate of interest.

To get the stochastic process driving $\tilde{S}_t = S_t e^{-rt}$, we again use Ito's formula

$$\begin{aligned}
df(t, S_t) &\equiv d\tilde{S}_t \\
&\equiv d\left(S_t e^{-rt}\right) \\
&= -rS_t e^{-rt}dt + e^{-rt}dS_t \\
&= -rS_t e^{-rt}dt + e^{-rt}\left(S_t \mu dt + S_t \sigma dW_t\right) \\
&= (\mu - r) S_t e^{-rt}dt + e^{-rt}S_t \sigma dW_t \\
&= (\mu - r)\tilde{S}_t dt + \tilde{S}_t \sigma dW_t
\end{aligned}$$

thus

$$d\tilde{S}_t = (\mu - r)\tilde{S}_t dt + \sigma \tilde{S}_t dW_t \tag{5.14}$$

In integral form this is

$$\tilde{S}_t = S_0 + \int_0^t \left(\mu - r - \frac{1}{2}\sigma^2\right) du + \int_0^t \sigma dW_u$$

5.4.3 Girsanov's Theorem Applied

It's clear that the process $\tilde{S}_t$ has a trend, $(\mu - r)\tilde{S}_t$. This trend causes $\tilde{S}_t$ not to be a P-martingale (a martingale under probability measure P).

The risk-neutral probability measure is found by employing Girsanov's theorem. By using the notation of the Girsanov theorem in section 5.2.8, we can define, for the process $\tilde{S}_t$,

$$u(t,\omega) = \frac{(\mu - r)\tilde{S}_t}{\sigma \tilde{S}_t} = \frac{(\mu - r)}{\sigma}.$$

Note that $\alpha(t,\omega) \equiv 0$ (in the sense of theorem 5.2.14) and $u(t,\omega) = u$ is a finite scalar since we assumed that σ is strictly positive. The result of this is that condition 5.7 is met and $u \in v(t,\omega)$.

M_t was defined in equation 5.8, as follows

$$M_t = \exp\left(-\int_0^t u(s,\omega)\,dW_t - \int_0^t u^2(s,\omega)\,ds\right)$$

In this case, for $u(t,\omega) = u$

$$M_t = \exp\left(-uW_t - u^2 t\right)$$

The new measure, the risk-neutral probability measure can be defined as

$$dQ = M_T dP$$

We can define a new process

$$\tilde{W}_t = ut + W_t$$

which is a Q- Brownian motion. The original process, $\tilde{S}_t$, in terms of $\tilde{W}_t$ is

$$d\tilde{S}_t = \sigma \tilde{S}_t d\tilde{W}_t \qquad (5.15)$$

Remark 5.4.1 *The scalar $u(t,s) = \frac{(\mu - r)}{\sigma}$ is also known as the market price of risk. If $\mu = r$ then the investor is called risk-neutral and $dP = dQ$. Under the measure Q we price instruments as if they are risk-neutral.*

5.4.4 Pricing Options under Constant Volatility

Theorem 5.4.2 *The option price at time t defined by a nonnegative, $\mathcal{F}_t$-measurable random variable h such that*

$$E^Q\left[h^2\right] < \infty$$

is replicable and its value at time t is given by

$$V_t = e^{-r(T-t)} E^Q\left[h \mid \mathcal{F}_t\right] \qquad (5.16)$$

Proof. Lets *assume* there exists an admissible trading strategy $\phi = (H_t^0, H_t)_{t\in[0,T]}$ *replicating* the option. The value of the replicating portfolio at time t is

$$V_t = H_t^0 \beta_t + H_t S_t$$

The discounted value of the process at time t is

$$\begin{aligned}\tilde{V}_t &= e^{-rt} V_t \\ &= H_t^0 + H_t \tilde{S}_t\end{aligned}$$

Since no new funds are added or removed from the replicating portfolio, the portfolio is self-financing, by theorem 5.3.2 we can write the portfolio as

$$\tilde{V}_t(\phi) = V_0(\phi) + \int_0^t H_u d\tilde{S}_u$$

by equation 5.15 we can write

$$\tilde{V}_t(\phi) = V_0(\phi) + \int_0^t H_u \sigma \tilde{S}_u d\tilde{W}_u$$

By the assumption of an admissible trading strategy we have by theorem 5.3.3 proved that $\sup_{t\in[0,T]} \tilde{V}_t^2$ is square integrable. It can then be proven (see Lamberton and Lapeyre [26]) that if

$$E^Q\left[\sup_{t\in[0,T]} \tilde{V}_t^2\right] < \infty$$

then

$$E^Q\left[\int_0^t \left(H_u \sigma \tilde{S}_u\right)^2 du\right] < \infty \tag{5.17}$$

Further, there exists a unique continuous mapping from the class of adapted processes with property 5.17 to the space of continuous $\mathcal{F}_t$ martingales on $[0,T]$. We thus have that

$$\tilde{V}_t = E^Q\left[\tilde{V}_T \mid \mathcal{F}_t\right]$$

and hence

$$V_t = E^Q\left[e^{-r(T-t)} h \mid \mathcal{F}_t\right] \tag{5.18}$$

which is a square-integrable martingale.

We have assumed that there exists a portfolio replicating the option, an admissible trading strategy can easily be found by the use of the martingale representation theorem (see Lamberton and Lapeyre [26]). By the martingale representation theorem there exists a square integrable martingale under Q with respect to $\mathcal{F}_t$ such that for every $0 \leq t \leq T$,

$$M_t = E^Q\left[e^{-rT}h \mid \mathcal{F}_t\right]$$

and that any such martingale is a stochastic integral with respect to $\tilde{W}$, such that

$$E^Q\left[e^{-rT}h \mid \mathcal{F}_t\right] = M_0 + \int_0^t \eta_u d\tilde{W}_u$$

where η_t is adapted to $\mathcal{F}_t$ and

$$E^Q\left[\int_0^T (\eta_s)^2\, ds\right] < \infty.$$

By letting $H_0 = M_t - H_t\tilde{S}_t$ and $H_t = \eta_t/\left(\sigma\tilde{S}_t\right)$ we have found a self-financing trading strategy. ■

5.4.5 The Black-Scholes Formula and Implied Volatility

The Black-Scholes formula for a European put option is a solution to equation 5.16 when

$$h = (X - S_T)_+$$

Black and Scholes (1973) and Merton (1973) proved that this as a solution to the Black-Scholes partial differential equation (pde). A martingale proof was later discovered. For the derivation of the pde proof for this formula see Black and Scholes [5], for a martingale proofs see Lamberton and Lapeyre [26] and Steele [32]. The Black-Scholes formula for a European put option at time t is

$$P^{BS} = e^{-r(T-t)}KN\left(-d_2\right) - S_tN\left(-d_1\right)$$

where

$$d_1 = \frac{\ln\left(S_0/X\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

and

$$d_2 = d_1 - \sigma\sqrt{T}$$

In this formula K is the strike price of the option and $N(\cdot)$ is the cumulative normal distribution. The risk-free interest rate r and the variance σ^2 are both annualized.

Volatility is the only parameter of the Black-Scholes formula that isn't directly observable. *Implied volatility*, σ, is the solution to the following problem

$$\min_{\sigma} \left| P^{BS}(\sigma) - P \right|$$

where $P^{BS}(\sigma)$ is the estimate of the put option as a function of implied volatility and P is the market value of the put option at time t.