Chapter 4

Speed regulation

4.1 Introduction

In the previous chapter, optimal scheduling shows a better start for the design of a closed-loop controller for train handling. An LQR closed-loop controller is designed to verify the advantage of optimal scheduling. However, the closed-loop controller is based on full state feedback, which is not practical in train handling to measure all states. If partial states are measured, an observer can be designed to supplement the LQR controller. This is, however, not the approach employed in this chapter. Instead, output regulation with measurement feedback is adopted for the full ECP/iDP mode subject to the assumption that only speed measurement of locomotives is available: while optimality is retained in open loop control design, closed-loop control is done by employing a nonlinear system regulator theory.

In this chapter, the output regulation of nonlinear systems with measured output feedback is first formulated and solved for the global version and the local version. Then the result of the local version is applied to train control. The simulation result shows the feasibility of such a speed regulator with only measurement of the locomotives' speeds, in terms of its simplicity, cost-effectiveness and its implementation convenience.

4.2 Output regulation with measured output feedback

The output regulation problem in linear systems has been studied in [30, 31, 32]. The internal model principle is proposed in [30], enabling the conversion of output regulation problems into stabilization problems. The details on the solvability of the problem can be found in [31, 32].

The internal model principle is extended to nonlinear systems in [33], which shows that the error-driven controller of the output tracking necessarily incorporates the internal model of the exosystem. The conditions of the existence of regulators for nonlinear systems are detailed for different kinds of exosystems with bounded signals in [34, 35, 36]. The necessary and sufficient conditions are given in [37] for the local output regulation problem of nonlinear systems, which is the solvability of regulator equations. With an assumption added to the conditions in [37], the results in [37] have been improved in [38]. A differential vector space approach is used in [39] to develop solutions of state feedback for nonlinear systems with both bounded and unbounded exogenous signals. An approach for robust local output regulation problems is presented in [40] in a geometric insight. In [41], an output regulation problem of a class of single-input single-output (SISO) nonlinear systems is reformulated into an output feedback stabilization problem.

The robust version of output regulation problem of nonlinear systems with uncertain parameters is studied in [42]. Furthermore, the output regulation problem of nonlinear systems driven by linear, neutrally stable exosystems with uncertain parameters is presented in [72], in terms of the parallel connection of a robust stabilizer and an internal model, which has recently been in [73]. Recently, the concept of the steady-state generator has been advanced in [43] as well as that of the internal model candidate. Based on these dynamic systems, a framework for global output regulation of nonlinear systems with autonomous exosystems is proposed in [43] for bounded signals, in [44] for unbounded signals, and in [47] for nonlinear exosystems. The frameworks are in the form of output feedback or plus (partial) state feedback.

All the controller design approaches in these papers cannot be extended directly to the form of measurement (measured output) feedback. Measurement feedback is considered in this chapter instead of the output feedback, because generally the measurable output is different from the output to regulate. For example in this study, in the handling of heavy-haul trains, the outputs to be regulated are all the cars' speeds; however, only part of the speeds (for example, the first and last locomotives' speeds) can be practically measured. On the other hand, the measurable output covers the form of the output or output plus (partial) state, as considered in [44].

In the above papers, an important idea is to design an internal model to eliminate the effect of the unknown states of the exosystem. In the controllers of [37] for the local version of output regulation problem, the internal model is given together with the stabilizer directly. In the global version, it is proposed in [43] and [44] firstly to design an internal model candidate, and thus the solvability of the output regulation problem is transformed into the solvability of the stabilization problem. This is a very smart technique to deal with the output regulation problem. The internal model candidates incorporate the output (or plus [partial] state, dependent on the measurability of the state) and the input to design. It does not incorporate the information of the stabilizer, which will be known in the controller. In this chapter, another approach is proposed to solve output regulation problems. Similar to [43] and [44], an output regulation problem is transformed into a stabilization problem of a simplified system with the assumption that the states of the exosystem are known, and a stabilizer is designed for it. It will be shown that the existence of the stabilizer is sometimes necessary for the solvability of the output regulation problem. Then an internal model *with respect to the stabilizer* for the original system is constructed. Specifically, the internal model can incorporate the information of the stabilizer. This approach is more natural than the ones in [43] and [44], where the internal model is a prerequisite and first designed. It can be seen that the existence of the internal model is sometimes not necessary. However, the existence of the stabilizer is necessary when the output zeroing manifold is unique, which is the case in all the examples given in [43, 44] and examples 1, 2 and 4 in this chapter.

The definition of the output regulation problem of nonlinear systems in [44] is borrowed, but the feedback is in the form of measurement feedback. A stabilizer for the simplified system is firstly designed. Then with respect to the stabilizer, an internal model is constructed to estimate the exosystem states. If successful, the output regulation problem is solved. In this study, the exosystem may be linear or nonlinear, the signals of which may be bounded or unbounded. The results for both the global version and the local version of dynamic measurement feedback output regulation problem (DMFORP) are reported.

4.2.1 Problem formulation and preliminaries

In this chapter, as well as in the subsequent chapter, extensive use of the differential geometric concepts and notations will be made to show the application of two most recent nonlinear control techniques in the case of heavy haul trains. The prerequisite for these two chapters are [45] and [46], A. Isidori's classic books.

Consider a nonlinear system,

$$\begin{aligned} \dot{x} &= f(x,u,w), \\ \dot{w} &= s(w), \\ e &= h(x,w), \\ y_m &= h_m(x,w). \end{aligned} \tag{4.1}$$

The first equation describes the original system, with state $x \in X \subset R^n$, and input $u \in U \subset R^m$. The second one defines an exosystem, with state $w \in W$ with $W \in R^s$ a compact set. The exosystem models the class of disturbance and/or reference signals taken into consideration. The third one is the error equation. The fourth one is the measured output $y_m \in R^{p_m}$. The vector fields $f(x,u,w)$ and $s(w)$ are smooth, and the mappings $h(x,w)$ and $h_m(x,w)$ are smooth, too.

It is assumed that $f(0,0,0) = 0$, $s(0) = 0$, $h(0,0) = 0$, *i.e.*, the system (4.1) has the equilibrium state $col(x,w) = col(0,0)$ for $u = 0$ with zero track error $h(0,0) = 0$.

Before the problem is formulated, some concepts in [45, 46] are stated firstly.

A continuous function $f : [0,a) \to [0,+\infty)$ is a class-$\mathcal{K}$ function if $f(0) = 0$ and it is strictly increasing.

A continuous function $f : [0,+\infty) \to [0,+\infty)$is a class-$\mathcal{L}$ function if it is decreasing and

$$\lim_{t\to\infty} f(t) = 0.$$

The function $f(r,s)$ is a class-$\mathcal{K}L$ function if $f(\cdot,s) \in \mathcal{K}$ for all $s \geq 0$, and $f(r,\cdot) \in \mathcal{L}$ for all $r \geq 0$.

The equilibrium $x = 0$ of the system $\dot{x} = f(x)$ with $f(0) = 0$ is said to be globally (locally) asymptotically stable in the sense of $\mathcal{K}L$ functions if from any initial state $x_0 \in R^n$ (some initial state $x_0 \in X$ with $X \subset R^n$ through the origin), the solution of system satisfies $\|x\| \leq \varrho(\|x_0\|, t)$ for some class-$\mathcal{K}L$ functions $\varrho(\cdot,\cdot)$.

A system of $\dot{x} = f(x,u), y = h(x)$, with $f(0,0) = 0$ is said to be globally (locally) stabilizable if there exists a control law $\dot{z} = \eta(z,y), u = \vartheta(z,y)$ satisfying $\eta(0,0) = 0, \vartheta(0,h(0)) = 0$ so that the origin of the closed-loop system $\dot{x} = f(x,\vartheta(z,h(x))), \dot{z} = \eta(z,h(x))$ is globally (locally) asymptotically stable in the sense of $\mathcal{K}L$ functions.

The dynamic measurement feedback output regulation problem of the system (4.1) considered here is to design a controller, in the form of

$$\begin{aligned} u &= \vartheta(z, y_m), \\ \dot{z} &= \eta(z, y_m). \end{aligned} \tag{4.2}$$

An advantage of a controller in the form of (4.2) is that it depends only on the current values of the measured variables, instead of differentiated signals of the measured variables, as introduces noise, and stably filtered signals of the measured variables. As a result, the closed-loop system can be written as

$$\begin{aligned} \dot{x}_c &= f_c(x_c, w), \\ \dot{w} &= s(w), \\ e &= h_c(x_c, w), \\ y_m &= h_{m_c}(x_c, w), \end{aligned} \tag{4.3}$$

where $x_c = col(x,z)$, $f_c(x_c,w) = col(f(x,\vartheta(z,y_m),w), \eta(z,y_m))$, $h_c(x_c,w) = h(x,w)$, $h_{m_c}(x_c,w) = h_m(x,w)$.

The formulation of the output regulation problem involves two requirements from the closed-loop system (4.3). One is the asymptotical convergence to zero of the error output, that is,

P1
$$\lim_{t\to\infty} e(t) = \lim_{t\to\infty} h(x(t), w(t)) = 0.$$

This property reflects the objective of the output regulation problem.

The other one is the internal stability of the closed-loop system (4.3). The system (4.3) is said to be globally (locally) asymptotically stable, irrespective of w, when for all $w(0) \in W$ with W any (a) compact set (containing the origin), the subsystem, parameterized by w, of the first equation of the system (4.3) is globally (locally) asymptotically stable in the sense of $\mathcal{K}L$ functions.

When the local version of the output regulation problem is considered, assuming the exosystem is Poisson stable at the origin (neutrally stable), the internal stability is given in [37] by

P2 the equilibrium of the closed-loop system (4.3) with $w = 0$ is asymptotically stable (or exponentially stable).

Actually, this property guarantees the boundedness of the trajectories of the closed-loop system with sufficiently small exosystem signals. But once the signals of the exosystem are large, the boundedness may not be kept [44].

Example Consider the following system

$$\begin{aligned} \dot{x}_1 &= -x_1 + v, \\ \dot{x}_2 &= -x_2 + x_2 w, \\ \dot{w} &= 0, \\ e &= x_1 - w. \end{aligned} \tag{4.4}$$

The solution is

$$\begin{aligned} x_1 &= (x_1(0) - w)e^{-t} + w, \\ x_2 &= x_2(0)e^{(-1+w)t}, \\ e &= (x_1(0) - w)e^{-t}. \end{aligned} \tag{4.5}$$

It is easy to check that the condition P2 is satisfied, but when $w > 1$, the state x_2 approaches infinity. So P2 cannot be used for the global version of output regulation problem. The internal stability of a global version of output regulation problem is given in [43] as

P3 for all $w \in W$ with W any compact set, the trajectories of the closed-loop system (4.3) starting from any initial state x_{c_0} exist and are bounded for all $t > 0$.

This interpretation of internal stability is meaningful if the states of the exosystem are bounded. When they are unbounded, the condition is modified [44] as the following two properties:

P4 for all $w_0 \in W$, there exists a sufficiently smooth functions $\pi_c(w)$ with $\pi_c(0) = 0$ satisfying

$$\begin{aligned} \frac{\partial \pi_c(w)}{\partial w} s(w) &= f_c(\pi_c(w), w), \\ 0 &= h_c(\pi_c(w), w). \end{aligned} \tag{4.6}$$

P5 for all $w_0 \in W$, $x_c = \pi_c(w)$ of the closed-loop system of (4.3) is globally asymptotically stable, irrespective of w, in the sense of some class $\mathcal{K}L$ functions.

The property P4 actually defines an invariant flow for the closed-loop system (4.3), on which the output error is exactly zero. The property P5 is about the asymptotical convergence of the invariant flow. From any initial state, the trajectories will converge to the invariant flow.

When the exosystem is neutrally stable, a closed-loop system satisfying properties P4 and P5 satisfies the properties P1 and P2.

If the P4 is fulfilled, then there exists an invariant manifold $x_c = \pi_c(w)$, on which the output is exactly zero. And for $\pi_c(0) = 0$, and from P5

$$\|x_c - \pi_c(w)\| \leq \varrho(\|x_c(0) - \pi_c(w_0)\|, t)$$

for some class-$\mathcal{K}L$ functions $\varrho(\cdot, \cdot)$, one has $\|x_c\| \leq \varrho(\|x_c(0)\|, t)$ when $w = 0$. Thus the asymptotical stability of P2 is satisfied. With the continuity of the output $h(x, w)$, $x_c \to \pi_c(w)$ as $t \to 0$, and w is bounded, P1 is satisfied. So the property P2 can be interpreted by

P4' for some compact set W through the origin, there exists sufficiently smooth functions $\pi_c(w)$ with $\pi_c(0) = 0$ satisfying the equations of (4.6).

P5' the equilibrium $x_c = \pi_c(w(t))$ of the closed-loop system (4.3) is locally asymptotically stable, irrespective of w, in the sense of $\mathcal{K}L$ functions.

It is pointed out that the asymptotical stability in the sense of $\mathcal{K}L$ functions is more general than the exponential stability. So the properties P4' and P5' are more general than the requirement P2 in [37].

The properties P4 and P5 are also sufficient conditions for the property P3 when the exosystem is Poisson stable. Because the $\pi_c(w)$ is bounded, and from the asymptotically stability, the state $x_c(t)$ is also bounded.

The global DMFORP is to find, if possible, a controller in a form (4.2) that allows the closed-loop system (4.3) to satisfy the properties P1, P4 and P5 for any compact set $W \subset R^s$.

The local DMFORP is to find, if possible, a controller in a form (4.2) that allows the closed-loop system (4.3) to satisfy P1, P4' and P5' for a compact set $W \subset R^s$ through the origin.

In the problem formulation, measurement feedback, instead of output feedback, is considered because the measured output is usually different from the output to regulate, and measurement feedback is more universal than output or plus (partial) state feedback. If measured output is the output error to regulate, then measurement feedback is output feedback. If it is just partial output, the feedback is in the form of partial output feedback, as in *example 1*. If the measurement includes the (partial) state, then the feedback includes (partial) state feedback. It is common for the measurement to be completely different from the output or/and state. When only measurement feedback can be used, such as in *example 2 and example 3*, the output regulation problem cannot be solved with output feedback or plus partial state feedback.

Remark 1 *The zeroing output is guaranteed by the property P1. The properties P4 and P5 usually imply P1. However, when the state of exosystem is unbounded, the output cannot be definitely zero sometimes, which can be seen in [44].*

Remark 2 *The exosystem considered in this study is given as $\dot{w} = s(w)$. In the global version of DMFORP, the exosystem may be Poisson stable or not, linear or nonlinear, bounded or unbounded. However, in the local version, the exosystem is required to be neutrally stable.*

Remark 3 *Even though the exosystem may be unbounded in global version, a constraint should be put on the exosystem. The state of the exosystem should not vary too quickly. This is because in that case it is difficult to design an observer to estimate or track the state of the exosystem $\dot{w} = s(w)$.*

4.2.2 Assumptions

In this chapter, one makes some assumptions. The first one is as follows:

A1 There exist sufficiently smooth functions $\pi(w)$ and $c(w)$ with $\pi(0) = 0$ and $c(0) = 0$, for all $w \in R^s$, satisfying,

$$\begin{aligned} \frac{\partial \pi(w)}{\partial w} s(w) &= f(\pi(w), c(w), w), \\ 0 &= h(\pi(w), w). \end{aligned} \tag{4.7}$$

Remark 4 *The above assumption is a standard one. It is a necessary condition for the solvability of the output regulation problem [37]. The equation* (4.7) *is called regulator equation.*

With this assumption, the original system can be written as the following (called *simplified system*) with the coordinate change $\bar{x} = x - \pi(w)$, $\bar{u} = u - c(w)$,

$$\begin{aligned} \dot{\bar{x}} &= \bar{f}(\bar{x}, \bar{u}, w), \\ y_m &= \bar{h}_m(\bar{x}, w), \end{aligned} \tag{4.8}$$

where $\bar{f}(\bar{x}, \bar{u}, w) = f(\bar{x} + \pi(w), c(w) + \bar{u}, w) - f(\pi(w), c(w), w)$, $\bar{h}_m(\bar{x}, w) = h_m(\bar{x} + \pi(w), w)$.

Furthermore, one makes another assumption.

A2 The above simplified system is globally stabilizable, irrespective of w, by a controller in the form of measurement feedback.

With this assumption, there exists a (dynamic) measurement feedback stabilizer in the form of

$$\begin{aligned} \dot{z}_1 &= \eta_1(z_1, y_m, w), \qquad \eta_1(0, \bar{h}_m(0, w), w) = 0, \\ \bar{u} &= \psi_1(z_1, y_m, w), \qquad \psi_1(0, \bar{h}_m(0, w), w) = 0, \end{aligned} \tag{4.9}$$

with which the equilibrium $col(\bar{x}, z_1) = col(0, 0)$ of the closed-loop simplified system composed of (4.8) and (4.9), for any $w \in W$, is globally asymptotically stable, irrespective of w, in the sense of $\mathcal{KL}$ functions.

Remark 5 *The assumption A2 is also necessary for the solvability of the output regulation problem on the zeroing output manifold $\{(x, w) \mid x = \pi(w), w \in W\}$. It is obvious that in this case, the state of the exosystem is assumed to be known. This is a middle case between full information feedback and output feedback [37], where the measurement is the output.*

The necessity is as follows:

Lemma 6 *If the global output regulation problem is solvable with a unique output zeroing manifold* $\{(x, w)|x = \pi(w), w \in W\}$*, then there must exist a stabilizer in the form of* (4.9) *such that for any* $w \in W$*, the equilibrium* $col(\bar{x}, z_1) = col(0, 0)$ *of the closed-loop simplified system is globally asymptotically stable, irrespective of* w*, in the sense of* $\mathcal{K}L$ *functions.*

Proof: Assume the condition P4 is satisfied, then there exists sufficiently smooth function $col(x, z) = \pi_c(w) = col(\pi(w), z^*(w))$ satisfying

$$\begin{aligned}
\frac{\partial \pi(w)}{\partial w} s(w) &= f(\pi(w), \vartheta(z^*(w), h_m(\pi(w), w)), w),\\
\frac{\partial z^*(w)}{\partial w} s(w) &= \eta(z^*(w), h_m(\pi(w), w), w)\\
0 &= h(\pi(w), w).
\end{aligned}$$

Choosing

$$\begin{aligned}
z_1 &= z - z^*(w),\\
\bar{x} &= x - \pi(w),\\
c(w) &= \vartheta(z^*(w), h_m(\pi(w), w)),\\
\psi_1(z_1, y_m, w) &= \vartheta(z_1 + z^*(w), y_m, w) - c(w),\\
\eta_1(z_1, y_m, w) &= \eta(z_1 + z^*(w), y_m) - \eta(z^*(w), h_m(\pi(w), w)),
\end{aligned}$$

one has the equation of (4.9) and the system (4.8).

Furthermore if the condition P5 is fulfilled, i.e, for all $w_0 \in W$, the trajectories of the closed-loop system (4.3) starting from any initial state $x_c(0)$ exist for all $t > 0$, and satisfying

$$\|x_c(t) - \pi_c(w(t))\| \le \varrho(\|x_c(0)) - \pi_c(w0)\|, t),$$

which can be written as $\|col(\bar{x}, z_1)\| \le \varrho(\|col(\bar{x}_0, z_{1_0})\|, t)$.

Thus the equilibrium $col(\bar{x}, z_1) = col(0, 0)$ of the closed-loop simplified system composed of (4.8) and (4.9), is globally asymptotically stable, irrespective of w, in the sense of $\mathcal{K}L$ functions.

Remark 7 *The above necessity is true when the original system has a unique output zeroing manifold. It is not difficult to give some conditions under which the output zeroing manifold is unique. For example, if the maximal output zeroing manifold of* (4.1) *takes the form* $N^* = \{x - \phi(w) = 0\}$ *for some smooth functions* $\phi(w)$*, then uniqueness is guaranteed. This, together with more general cases, are under separate study. It is nevertheless worth noting that all examples given in [43, 44] and* Example 1, 2 *and* 4 *given in this chapter admit unique output zeroing manifolds. When the original system has more than one output zeroing manifold, there are more than* *one corresponding simplified system. The existence of the stabilizer for one simplified system may be unnecessary for the solvability of DMFORP. However, if the DMFORP is solvable on a specific output zeroing manifold, there must exist a stabilizer for the corresponding simplified system. It is demonstrated in* Example 3.

For the observability of the states of the exosystem, one makes the following assumption.

A3 there exist mappings $\theta(w) : R^s \to R^d, \alpha : R^d \to R^d, \beta : R^d \to R^s$ satisfying

$$\begin{aligned} \frac{d\theta(w)}{dt} &= \alpha(\theta(w)), \\ w &= \beta(\theta(w)). \end{aligned} \tag{4.10}$$

Remark 8 *The dynamic system of* (4.10) *is very similar to the definition of the steady-state generator in [43]. The difference is that the output of* (4.10) *is exactly the state of the exosystem while it is the steady input or input plus (partial) state of the original system of* (4.1) *in [43] and [44]. With the output of this dynamic system, all the steady input and state of the original system can be generated. In this study it is therefore still called* steady-state generator.

When it comes to the local DMFORP, the first two assumptions are as follows:

A1' There exist sufficiently smooth functions $x = \pi(w)$, $u = c(w)$ with $\pi(0) = 0$, $c(0) = 0$, satisfying, $\forall w \in W$ with W a neighbourhood of the origin, regulator equation (4.7) is solved.

A2' The system (4.8) is locally stabilizable, irrespective of w, by a controller in the form of measurement feedback.

With the assumption A2', there exists a dynamic measurement feedback control law in the form of (4.9) with which the closed-loop simplified system composed of the simplified system (4.8) and the stabilizer (4.9) is locally asymptotically stable at $col(\bar{x}, z_1) = col(0, 0)$ for all $w \in W$.

4.2.3 Solution of the global DMFORP

Definition 1: An internal model candidate with respect to the stabilizer (4.9) and the steady-state generator (4.10) is as (4.11) satisfying $\eta_2(0, \theta(w), h_m(\pi(w), w)) = \alpha(\theta(w))$.

$$\dot{z}_2 = \eta_2(z_1, z_2, y_m). \tag{4.11}$$

Remark 9 *This definition is consistent with the one in [43] and [44], which also requires* $\eta_2(0, \theta(w), h_m(\pi(w), w), w)) = \alpha(\theta(w))$ *when* $h_m(\pi(w), w) = e = 0$. *Here the concept is extended to the case of the measurement feedback form. Furthermore, this internal model candidate incorporates the information of the stabilizer, which is different from that in [43] and [44].*

The following system is called the *closed-loop stabilized system* in this chapter,

$$\begin{aligned} \dot{\bar{x}} &= f_{\bar{x}}(\bar{x}, z_1, \bar{z}_2, w), \\ \dot{z}_1 &= \eta_{z_1}(\bar{x}, z_1, \bar{z}_2, w), \\ \dot{\bar{z}}_2 &= \bar{\eta}_{\bar{z}_2}(\bar{x}, z_1, \bar{z}_2, w), \end{aligned} \tag{4.12}$$

where $f_{\bar{x}}(\bar{x}_1, z_1, \bar{z}_2, w) = f(\bar{x} + \pi(w), u_z, w) - f(\pi(w), c(w), w)$, $u_z = c(\beta(\bar{z}_2 + \theta(w))) + \psi_1(z_1, h_m(\bar{x} + \pi(w), w), \beta(\bar{z}_2 + \theta(w)))$, $\eta_{z_1}(\bar{x}, z_1, \bar{z}_2, w) = \eta_1(z_1, h_m(\bar{x} + \pi(w), w), \beta(\bar{z}_2 + \theta(w)))$, $\bar{\eta}_{\bar{z}_2}(\bar{x}, z_1, \bar{z}_2, w) = \eta_2(z_1, \bar{z}_2 + \theta(w), h_m(\bar{x} + \pi(w), w)) - \eta_2(0, \theta(w), h_m(\pi(w), w))$.

Definition 2: The internal model candidate (4.11) is called a global (local) internal model with respect to the stabilizer (4.9) and the steady-state generator (4.10) for the original system (4.1), if for any $w \in W$, the closed-loop stabilized system of (4.12) is globally (locally) asymptotically stable, irrespective of w, in the sense of $\mathcal{K}L$ functions $\varrho(\cdot, \cdot)$.

Remark 10 *An additional restriction is added to the internal model candidate. Thus the candidate becomes a "real one". The restriction gives an approach to modify the candidate to get a real internal model.*

In the original system, the output function $h(x, w)$ is globally Lipschitz with respect to x if there exists a class-$\mathcal{K}$ function l_0 such that for all $x^1, x^2 \in R^n, w \in R^s$,

$$\|h(x^1, w) - h(x^2, w)\| \leq l_0(\|x^1 - x^2\|). \tag{4.13}$$

Theorem 11 *If A1, A2, A3 hold, there exists a global internal model* (4.11) *with respect to the stabilizer* (4.9) *and the steady-state generator* (4.10)*, and the output function* $h(x, w)$ *is globally Lipschitz with respect to* x*, then the global DMFORP of the original system* (4.1) *is solvable.*

Proof Consider the following controller,

$$\begin{aligned} \dot{z}_1 &= \eta_1(z_1, y_m, \beta(z_2)), \\ \dot{z}_2 &= \eta_2(z_1, z_2, y_m), \\ u &= c(\beta(z_2)) + \psi_1(z_1, y_m, \beta(z_2)). \end{aligned} \tag{4.14}$$

Given any compact set W, the origin of the closed-loop stabilized system of (4.12) is globally asymptotically stable from initial state $(\bar{x}_0, z_{1_0}, \bar{z}_{2_0}, w_0)$. The state of the augmented system composed of the original system (4.1) and the controller (4.14) is $x_c = col(x, z_1, z_2)$, and the state of system (4.12) is $\bar{x}_c = col(\bar{x}, z_1, \bar{z}_2)$, with a relation

$$col(x, z_1, z_2) = col(\bar{x}, z_1, \bar{z}_2) + col(\pi(w), 0, \theta(w)).$$

It is easy to verify that $\pi_c(w) = col(\pi(w), 0, \theta(w))$ satisfies the equations (4.6) for the augmented system composed of the original system (4.1) and the controller (4.14). Thus the condition P4 is satisfied.

On the other hand, $x_c - \pi_c(w) = col(x, z_1, z_2) - col(\pi(w), 0, \theta(w)) = col(\bar{x}, z_1, \bar{z}_2) = \bar{x}_c$, and from the definition of the internal model, one has $\|\bar{x}_c(t)\| \leq \varrho(\|\bar{x}_c(0)\|, t)$, then

$$\|x_c(t) - \pi_c(w(t))\| \leq \varrho(\|x_c(0) - \pi_c(w_0)\|, t).$$

The condition P5 is satisfied.

With the Lipschitz condition, and $\bar{x}(t) \rightarrow 0$ as $t \rightarrow 0$,

$$\begin{aligned} \lim_{t\rightarrow\infty} \|e(t)\| &= \lim_{t\rightarrow\infty} h(\bar{x}(t) + \pi(w), w) - h(\pi(w), w) \\ &\leq \lim_{t\rightarrow\infty} l_0(\|\bar{x}\|) = 0, \end{aligned} \tag{4.15}$$

i.e., the requirement P1 of the asymptotical convergence to zero of the error output is satisfied.

Thus the global DMFORP is solvable.

Remark 12 *In the theorem, if the solution of* (4.12) *satisfies*

$$\lim_{t\rightarrow\infty} (h(\bar{x} + \pi(w), w) - h(\pi(w), w)) = 0 \tag{4.16}$$

for any $w \in W$, the Lipschitz condition is not necessary for the solvability of DMFORP.

Remark 13 *This theorem gives an approach to solve the output regulation problem. The first step is to design a stabilizer for the simplified system on the assumption that the states of the exosystem are known. Then an internal model candidate is constructed with respect to the stabilizer and a steady-state generator for the exosystem. If the internal model is found and the condition of* (4.16) *is satisfied, then the DMFORP is solved.*

Remark 14 *If w is not explicit in $\psi_1(z_1, y_m, w) + c(w)$ and the mapping $\eta_1(z_1, y_m, w)$, the internal model is unnecessarily constructed, and the dynamic measurement feedback output regulation problem is solved by the following controller.*

$$\begin{aligned} \dot{z}_1 &= \eta_1^0(z_1, y_m), \\ u &= \psi_1^0(z_1, y_m), \end{aligned} \tag{4.17}$$

where $\eta_1^0(z_1, y_m) = \eta_1(z_1, y_m, w)$, $\psi_1^0(z_1, y_m) = \psi_1(z_1, y_m, w) + c(w)$. *From the proof of the above proposition, one can see that the internal model is to estimate the functions in the stabilized system related to* w. *So in the steady-state generator definition, the output of the generator can be just the functions related to* w. *This is also consistent with the origin of the internal model candidate, which is to estimate the unknown parameters in the controller. The internal model can be thought of as an observer for the exosystem and its design can be thought of as an observer design for nonlinear systems, which is another problem. So the design techniques of the internal model are not detailed in this thesis.*

Proposition 15 *If the output regulation problem is solvable by the approach proposed in [43] and [44] with the measurement being the output or the output plus the (partial) state, it is also solvable with the above theorem.*

Proof In [43] and [44], output regulation problem of the following system

$$\begin{aligned} \dot{x} &= f(x, u, w), \\ e &= h(x, u, w), \end{aligned} \tag{4.18}$$

is solved by the controller

$$\begin{aligned} \dot{\eta} &= \gamma(\eta, x, u, e), \\ \dot{\xi} &= \zeta(\bar{x}_1, \cdots, \bar{x}_d, \xi, e), \\ u &= \beta_u(\eta) + k(\bar{x}_1, \cdots, \bar{x}_d, \xi, e), \end{aligned} \tag{4.19}$$

where the solution of the regulator equation is $x = \pi(w), u = c(w)$, and

$$\frac{d\theta(w)}{dt} = \alpha(\theta(w)),$$

$$col(\pi_1(w), \cdots, \pi_d(w), c(w)) = \beta(\theta(w)),$$

$$\beta_u = col(\beta_{d+1}, \cdots, \beta_{d+m}),$$

$$\gamma(\theta(w), \pi(w), c(w), 0) = \alpha(\theta(w)),$$

$$k(0, \cdots, 0, 0, 0) = 0, \ \zeta(0, \cdots, 0, 0, 0) = 0,$$

$$\bar{x}_i = x_i - \beta_i(\eta), \ i = 1, 2, \cdots, d.$$

With the approach proposed in this chapter, the stabilizer for the simplified system can be chosen as

$$\begin{aligned} \dot{\bar{\eta}} &= \bar{\gamma}(\bar{\eta}, x_1, \cdots, x_d, w), \\ \dot{\xi} &= \zeta(\bar{\bar{x}}_1, \cdots, \bar{\bar{x}}_d, \xi, e), \\ \bar{u} &= \beta_u(\eta) + k(\bar{\bar{x}}_1, \cdots, \bar{\bar{x}}_d, \xi, e) - c(w), \end{aligned} \tag{4.20}$$

where

$$\bar{\bar{x}}_i = x_i - \beta_i(\bar{\eta} + \theta(w)), \ i = 1, 2, \cdots, d,$$

$$\bar{\gamma}(\bar{\eta}, x_1, \cdots, x_d, w) = \gamma(\bar{\eta} + \theta(w), \bar{\bar{x}}_1, \cdots, \bar{\bar{x}}_d, \bar{u} + \alpha(\theta(w)), e) - \alpha(\theta(w)).$$

In the closed-loop stabilized system, only the function $\theta(w)$ is related to w. So the output of the internal model can be just the function $\theta(w)$. It is obvious that the dynamic system $\eta = \gamma(\eta, x, u, e)$ is naturally the internal model candidate. Actually this candidate is a "real one". This internal model is principally the same as part of the stabilizer. Combining the stabilizer and the internal model, the controller of (4.19) solves the global DMFORP.

All in all, if an output regulation problem is solvable by the approach proposed in [43] and [44], it must be solvable by the approach proposed in this chapter.

4.2.4 Solution of the local DMFORP

Similar to the global version, there is a theorem for the local DMFORP.

Theorem 16 *If A1', A2' and A3 hold, the exosystem is neutrally stable, and if there exists a local internal model with respect to a stabilizer* (4.9) *and a steady-state generator* (4.10)*, then the local DMFORP is solvable.*

Proof Consider the controller (4.14). The closed-loop stabilized system of (4.12) is locally asymptotically stable from the initial state $\bar{x}_0, z_{1_0}, \bar{z}_{2_0}, w_0$. The state of the augmented system composed of the original system (4.1) and the controller (4.14) is $x_c = col(x, z_1, z_2)$, and the state of the system (4.12) is $\bar{x}_c = col(\bar{x}, z_1, \bar{z}_2)$, with a relation

$$col(x, z_1, z_2) = col(\bar{x}, z_1, \bar{z}_2) + col(\pi(w), 0, \theta(w)).$$

It is easy to verify that $\pi_c(w) = col(\pi(w), 0, \theta(w))$ satisfies the equations (4.6) for the augmented system composed of the original system (4.1) and the controller (4.14). Thus the condition P4' is satisfied.

On the other hand, $x_c - \pi_c(w) = col(x, z_1, z_2) - col(\pi(w), 0, \theta(w)) = col(\bar{x}, z_1, \bar{z}_2) = \bar{x}_c$, and from the definition of the local internal model, one has $\|\bar{x}_c(t)\| \le \varrho(\|\bar{x}_c(0)\|, t)$, then

$$\|x_c(t) - \pi_c(w(t))\| \le \varrho(\|x_c(0) - \pi_c(w_0)\|, t).$$

The condition P5' is satisfied.

With continuity of $h(x, w)$, $\bar{x}(t) \to 0$ as $t \to 0$, and with w bounded,

$$\lim_{t\to\infty} e(t) = \lim_{t\to\infty} h(\bar{x}(t) + \pi(w), w) = h(\pi(w), w) = 0,$$

that is, the requirement P1 is satisfied.

Thus the DMFORP is solved by the controller (4.14). If the state w of the exosystem is not explicit in the mapping $\eta_1(z_1, y_m, w)$ and the controller $u = \psi_1(z_1, y_m, w) + c(w)$, then the internal model need not be constructed.

For a nonlinear system (4.1), one has the notations:

$$A = \frac{\partial f(0,0,0)}{\partial x}, \quad S = \frac{\partial s}{\partial w}(0), \quad C_m = \frac{\partial h_m}{\partial x}(0,0), \quad P = \frac{\partial f}{\partial w}(0,0,0),$$
$$B = \frac{\partial f}{\partial u}(0,0,0), \quad C = \frac{\partial h}{\partial x}(0,0), \quad Q = \frac{\partial h}{\partial w}(0,0), \quad Q_m = \frac{\partial h_m}{\partial w}(0,0).$$

Proposition 17 *For* (4.1)*, if the exosystem is neutrally stable,* (A, B) *is controllable [28] and*

$$\left(\begin{bmatrix} A & P \\ 0 & S \end{bmatrix}, [C_m, Q_m]\right)$$

is detectable [28], furthermore the condition A1' is satisfied, then the local DMFORP is solvable.

Proof With the assumption A1', using the coordinate change $\bar{x} = x - \pi(w)$, $\bar{u} = u - c(w)$, the simplified system (4.8) can be stabilized by the following controller.

$$\begin{aligned} \dot{z}_{11} &= f(z_{11} + \pi(w), c(w) + \bar{u}, z_{12} + w) - f(\pi(w), c(w), w) \\ &\quad + G_1(h_m(\bar{x} + \pi(w), w) - h_m(z_{11} + \pi(w), z_{12} + w)), \\ \dot{z}_{12} &= s(z_{12} + w) - s(w) \\ &\quad + G_2(h_m(\bar{x} + \pi(w), w) - h_m(z_{11} + \pi(w), z_{12} + w)), \\ \bar{u} &= c(z_{12} + w) + H(z_{11} + \pi(w) - \pi(z_{12} + w)) - c(w), \end{aligned} \tag{4.21}$$

where $H,\ G_1,\ G_2$ are chosen such that

$$A + BH$$

and

$$\begin{bmatrix} A - G_1 C_m & P - G_1 Q_m \\ -G_2 C_m & S - G_2 Q_m \end{bmatrix}$$

are Hurwitz, which guarantees the following Jacobian matrix of the closed-loop simplified system is Hurwitz [28].

$$\begin{bmatrix} A & BH & BK \\ G_1 C_m & A + BH - G_1 C_m & P + BK - G_1 Q_m \\ G_2 C_m & -G_2 C_m & S - G_2 Q_m \end{bmatrix}, \tag{4.22}$$

where $K = (\frac{\partial c(w)}{\partial w} - H\frac{\partial \pi(w)}{\partial w})|_{w=0}$. Since the pair of (A, B) is controllable and the pair of $\left(\begin{bmatrix} A & P \\ 0 & S \end{bmatrix}, [C_m, Q_m]\right)$ is detectable, it is possible to choose $H,\ G_1,\ G_2$ satisfying the above conditions. With the Lyaponov theory, the Hurwitz Jacobian matrix means the system is exponentially stable, which leads to asymptotical stability in the sense of $\mathcal{K}L$ functions. So the above controller can be thought of as a stabilizer. The assumption A2' is satisfied.

Now the steady state generator is chosen as $\theta(w) = \begin{bmatrix} \pi(w) \\ w \end{bmatrix}$, $\beta(\theta(w)) = w$, and the internal model candidate $\dot{z}_2 = \eta_2(z_2, z_{11}, z_{12}, \bar{h}_m(\bar{x}, w))$ is constructed. If one chooses $z_2 = col(z_{21}, z_{22})$ satisfying $z_{21} = z_{11} + \pi(w),\ z_{22} = z_{12} + w$, then one gets

$$\begin{aligned} \dot{z}_{21} &= f(z_{11}, u_z, z_{21}) + G_1(y_m - h_m(z_{21}, z_{12})), \\ \dot{z}_{22} &= s(z_{22}) + G_2(y_m - h_m(z_{21}, z_{22})), \\ u_z &= c(z_{22}) + H(z_{21} - \pi(z_{12}), \end{aligned} \tag{4.23}$$

The stabilizer (4.21) is rewritten as (4.24).

$$\begin{aligned} \frac{d(z_{11} + \pi(w))}{dt} &= f(z_{11} + \pi(w), u, z_{12} + w) \\ &\quad + G_1(h_m(\bar{x} + \pi(w), w) - h_m(z_{11} + \pi(w), z_{12} + w)), \\ \frac{d(z_{12} + w)}{dt} &= s(z_{12} + w) \\ &\quad + G_2(h_m(\bar{x} + \pi(w), w) - h_m(z_{11} + \pi(w), z_{12} + w)), \\ u &= c(z_{12} + w) + H(z_{11} + \pi(w) - \pi(z_{12} + w)). \end{aligned} \tag{4.24}$$

With the internal model candidate (4.23), replacing the functions $z_{11}+\pi(w),\ z_{12}+w$ in the stabilizer (4.24) with $z_{21} = \bar{z}_{21} + \pi(w) = z_{11} + \pi(w)$ and $z_{22} = \bar{z}_{22} + w = z_{12} + w$, respectively, the closed-loop stabilized system can be written as

$$\begin{aligned} \dot{\bar{x}} &= f(\bar{x} + \pi(w), u, w) - f(\pi(w), c(w), w), \\ \dot{z}_{11} &= f(z_{11} + \pi(w), u, z_{12} + w) - f(\pi(w), c(w), w) \\ &\quad + G_1(y_m - h_m(z_{11} + \pi(w), z_{12} + w)), \\ \dot{z}_{12} &= s(z_{12} + w) - s(w) + G_2(y_m - h_m(z_{11} + \pi(w), z_{12} + w)), \\ u &= c(z_{12} + w) + H(z_{11} + \pi(w) - \pi(z_{12} + w), \end{aligned} \tag{4.25}$$

whose Jacobian matrix is the same as (4.22). That is, the internal model candidate (4.23) is a "real one".

It is noticed that in the closed-loop stabilized system (4.25), the dynamics of the internal model disappears because when one changes the coordinates of the internal model candidate, and replaces the functions of w in the stabilizer, the dynamics of the stabilizer is the same as the internal model candidate.

With the theorem 16, the local DMFORP is solved by the controller (4.23).

Remark 18 *From the proof, it can be seen that the problem is locally solved by the following controller,*

$$\begin{aligned}\dot{z}_{21} &= f(z_{11}) + g(z_{11})u_z + p(z_{11})z_{22} + G_1(y_m - h_m(z_{21}, z_{12})),\\ \dot{z}_{22} &= s(z_{22}) + G_2(y_m - h_m(z_{21}, z_{22})),\\ u_z &= c(z_{22}) + H(z_{21} - \pi(z_{12})),\end{aligned} \tag{4.26}$$

where H, G_1, G_2 *can simply be chosen such that*

$$A + BH$$

and

$$\begin{bmatrix} A - G_1C_m & P - G_1Q_m \\ -G_2C_m & S - G_2Q_m \end{bmatrix}$$

are Hurwitz, which guarantees the following Jacobean matrix of the closed-loop simplified system is Hurwitz,

$$\begin{bmatrix} A & BH & BK \\ G_1C_m & A + BH - G_1C_m & P + BK - G_1Q_m \\ G_2C_m & -G_2C_m & S - G_2Q_m \end{bmatrix},$$

where

$$K = (\frac{\partial c(w)}{\partial w} - H\frac{\partial \pi(w)}{\partial w})|_{w=0}.$$

In particular, when the state of the exosystem w *is known, for example,* $y_m = col(y_m^1,\ w)$*, the problem can be solved by*

$$\begin{aligned}\dot{z} &= f(z) + g(z)u + p(z)w + G_1(y_m^1 - h_m(z, w)),\\ u &= c(w) + H(z - \pi(w)),\end{aligned} \tag{4.27}$$

where G_1, H *are chosen such that* $A + BH$ *and* $A - G_1C_m^1$ *are Hurwitz* ($C_m^1 = \frac{\partial y_m^1(0,0)}{\partial x}$.)

Remark 19 *The parameters* K (H), G *can be chosen with different kinds of methods, such as with pole placement or a linear-quadratic state-feedback regulator approach. Although* K (H), G *are chosen with a linear system theory, the nonlinear regulator problem is solved by the controller* (4.26) *with them.*

If the measurement is the output, the above proposition leads to the theorem in [37]. However, Theorem 16 is not limited to processing the case of the proposition; as can be seen in *Example 4*, this theorem can process a nonlinear system whose closed-loop simplified system has a Jacobian matrix which is not Hurwitz but is stable in the sense of some class $\mathcal{KL}$ functions.

Consider a simplified system

$$\dot{\bar{x}} = \bar{f}(\bar{x}, \bar{u}, w)$$

can be stabilized by a stabilizer

$$\begin{aligned} \dot{z}_1 &= \eta_1(z_1, y_m, w), \\ \bar{u} &= \psi_1(z_1, y_m, w). \end{aligned} \tag{4.28}$$

The Jacobian matrix of the closed-loop simplified system at the origin is not Hurwitz, but it is still stable in the sense of some class $\mathcal{K}L$ functions.

Assuming an internal model candidate $\dot{z}_2 = \eta_2(z_1, z_2, y_m)$ with respect to the above stabilizer and a steady-state generator $\frac{d\theta(w)}{dt} = \alpha(\theta(w)),\ w = \beta(\theta(w))$ is a "real one", and with a coordinate change $\bar{z}_2 = z_2 - \theta(w)$, the closed-loop stabilized system can be written as

$$\begin{aligned} \dot{\bar{x}} &= \bar{f}(\bar{x}, \bar{u}, w), \\ \dot{z}_1 &= \eta_1(z_1, y_m, \beta(\bar{z}_2 + \theta(w))), \\ \dot{\bar{z}}_2 &= \eta_2(z_1, \bar{z}_2 + \theta(w), y_m) - \alpha(\theta(w)), \\ \bar{u} &= \psi_1(z_1, y_m, \beta(\bar{z}_2 + \theta(w))), \\ y_m &= \bar{h}_m(\bar{x}, w). \end{aligned} \tag{4.29}$$

When $\bar{z}_2 = 0$ in (4.29), it is the closed-loop simplified system, which is stable although its Jacobian matrix is not Hurwitz. So according to the reduction principle [45], it is possible to find a linear change of coordinates $col(p_1, p_2) = T col(\bar{x}, z_1) + K\bar{z}_2$, with T nonsingular, such that the closed-loop stabilized system (4.29) can be rewritten as

$$\begin{aligned} \dot{p}_1 &= F_1 p_1 + g_1(p_1, p_2, \bar{z}_2), \\ \dot{p}_2 &= F_{21} p_1 + F_2 p_2 + G_2 \bar{z}_2 + g_2(p_1, p_2, \bar{z}_2), \\ \dot{\bar{z}}_2 &= A_1 p_1 + A_2 p_2 + A\bar{z}_2 + g_3(p_1, p_2, \bar{z}_2), \end{aligned} \tag{4.30}$$

where F_1 has all the eigenvalues with zero real part, F_2 has all the eigenvalues with negative real part, and the functions g_1, g_2, g_3 vanish at $(p_1, p_2, \bar{z}_2) = (0, 0, 0)$ together with their first order derivatives.

When $\bar{z}_2 = 0$ in (4.30), the first two equations are as follows.

$$\begin{aligned} \dot{p}_1 &= F_1 p_1 + g_1(p_1, p_2, 0), \\ \dot{p}_2 &= F_{21} p_1 + F_2 p_2 + g_2(p_1, p_2, 0), \end{aligned} \tag{4.31}$$

which is the transformation of the closed-loop simplified system with the linear change of coordinates $col(p_1, p_2) = T col(\bar{x}, z_1)$. For the origin $col(\bar{x}, z_1) = col(0, 0)$ of the closed-loop simplified system is locally asymptotically stable in the sense of $\mathcal{K}L$ functions, the origin $col(p_1, p_2) = col(0, 0)$ of the (4.31) is also locally asymptotically stable in the sense of $\mathcal{K}L$ functions. So according to the reduction principle, there must exist a center manifold $p_2 = \pi_2(p_1)$ for the system (4.31) and the origin $x = 0$ of the reduced system $\dot{x} = F_1 x + g_1(x, \pi_2(x), 0)$ is necessarily locally asymptotically stable in the sense of $\mathcal{K}L$ functions.

If $A_1 z_1 + A_2 \pi_2(z_1) + g_3(z_1, \pi_2(z_1), 0) = 0$ and $\begin{bmatrix} F_2 & G_2 \\ A_2 & A \end{bmatrix}$ is Hurwitz, then $p_2 = \pi_2(p_1),\ \bar{z}_2 = 0$ is a center manifold for the system (4.30) and the reduced system is the same as that of the system (4.31), the origin of which is locally asymptotically stable in the sense of $\mathcal{KL}$ functions. According to the reduction principle, $col(p_1, p_2, \bar{z}_2) = (0, 0, 0)$ of the system (4.30) is locally asymptotically stable in the sense of $\mathcal{KL}$ functions. In addition, T is nonsingular in the linear change of coordinates, the origin $col(\bar{x}, z_1, \bar{z}_2) = col(0, 0, 0)$ of the closed-loop stabilized system is locally asymptotically stable in the sense of $\mathcal{KL}$ functions. According to the definition of the internal model, the internal model candidate is a "real one", that is, the aforementioned assumption is true. According to Theorem 16, the local DMFORP is solved.

The above analysis gives an approach to check if the observer (internal model candidate) is a "real one" in the local DMFORP.

Remark 20 *The above condition is sufficient for the solvability of the local DMFORP.*

4.2.5 Examples

Four examples are employed to show the application cases of the proposed theories. Example 1 will show the solution of DMFORP with partial output feedback (static stabilizer and measured output being partial of the regulated output), Example 2 will show the solution of DMFORP with measured output feedback (static stabilizer and measured output being different from the regulated output) and Example 3 will show the solution of DMFORP with measured output feedback (dynamic stabilizer and measured output being different from the regulated output). These three examples are application cases of Theorem 10 while Example 4 is an application case of Theorem 15.

Example 1 Here, consider the system [44],

$$\begin{aligned}
\dot{x}_1 &= x_1 + u_1,\\
\dot{x}_2 &= -x_2 + 1.2(x_1 - w) + 0.3u_2,\\
\dot{w} &= 2w,\\
e_1 &= x_1 - w,\\
e_2 &= x_1x_2 - 0.2w^2,\\
y_m &= e_1 = x_1 - w.
\end{aligned} \tag{4.32}$$

The solution of the regulator equation is $x = \pi(w) = \begin{bmatrix} w \\ 0.2w \end{bmatrix}$, $u = c(w) = \begin{bmatrix} w \\ 2w \end{bmatrix}$. One can get the following simplified system with $\bar{x} = x - \pi(w)$, $\bar{u} = u - c(w)$,

$$\begin{aligned}
\dot{\bar{x}}_1 &= \bar{x}_1 + \bar{u},\\
\dot{\bar{x}}_2 &= -\bar{x}_2 + 1.2\bar{x}_1 + 0.3\bar{u}_2.
\end{aligned} \tag{4.33}$$

It is obvious that the above linear system is globally asymptotically stable if one chooses the stabilizer $\bar{u}_1 = -k_1\bar{x}_1 = -k_1 y_m,\ \bar{u}_2 = 0,\ k_1 > 1$.

One considers a steady-state generator with $\theta(w) = w, \beta(\theta(w)) = \theta(w)$, and an internal model candidate with respect to the stabilizer in the form of $\dot{z}_2 = ky_m + 2z_2 = k\bar{x}_1 + 2z_2, k \in R$. Then one can get the following closed-loop stabilized system with $\bar{z}_2 = z_2 - \theta(w)$,

$$\begin{aligned}\dot{\bar{x}}_1 &= (1-k_1)x_1 + \bar{z}_2,\\ \dot{\bar{x}}_2 &= 1.2\bar{x}_1 - \bar{x}_2 + 0.6\bar{z}_2,\\ \dot{\bar{z}}_2 &= k\bar{x}_1 + 2\bar{z}_2\end{aligned} \tag{4.34}$$

If $k_1,\ k$ is chosen such that $k_1 - 3 > 0,\ k + 2(k_1 - 1) < 0$, for example, $k_1 = 7,\ k = -15$, the above system is globally asymptotically stable and thus $\dot{z}_2 = ky_m + 2z_2$ is an internal model with respect to the stabilizer. Then the output regulation problem of (4.32) is solved by

$$\begin{aligned}\dot{z}_2 &= ky_m + 2z_2,\\ u_1 &= -k_1 y_m + z_2,\\ u_2 &= 2z_2.\end{aligned} \tag{4.35}$$

Remark 21 *This example is in the form of (partial) output feedback. And the stabilizer is designed for a linear system. Some parameters in the stabilizer can be tuned together with some parameters in the internal model such that the origin of the closed-loop stabilized system is globally asymptotically stable in the sense of some class-$\mathcal{KL}$ functions.*

Example 2 Consider the following system with some changes in the above example,

$$\begin{aligned}\dot{x}_1 &= x_1 + u_1,\\ \dot{x}_2 &= -x_2 + 1.2(x_1 - w) + 0.3u_2,\\ \dot{w} &= 2w,\\ e_1 &= x_1 - w,\\ e_2 &= x_1x_2 - 0.2w^2,\\ y_m &= x_1 - w/2.\end{aligned} \tag{4.36}$$

The solution of the regulator equation and the simplified system are the same as in *Example 1*. But the stabilizer is in the form of $\bar{u}_1 = -k_1\bar{x}_1 = -k_1(y_m - w/2),\ \bar{u}_2 = 0,\ k_1 > 1$.

One considers a steady-state generator with $\theta(w) = w, \beta(\theta(w)) = \theta(w)$, and an internal model candidate with respect to the stabilizer in the form of $\dot{z}_2 = ky_m - kz_2/2 + 2z_2 = k\bar{x}_1 - kz_2/2 + kw/2 + 2z_2, k \in R$. Then one can get the following closed-loop stabilized system with $\bar{z}_2 = z_2 - \theta(w)$,

$$\begin{aligned}\dot{\bar{x}}_1 &= (1-k_1)x_1 + (1+k_1/2)\bar{z}_2,\\ \dot{\bar{x}}_2 &= 1.2\bar{x}_1 - \bar{x}_2 + 0.6\bar{z}_2,\\ \dot{\bar{z}}_2 &= k\bar{x}_1 + (2-k/2)\bar{z}_2.\end{aligned} \tag{4.37}$$

If $k_1,\ k$ is chosen such that $k_1 > 1,\ k_1 - 3 + k/2 > 0,\ -3k/2 - 2(k_1 - 1) < 0$, for example, $k_1 = 10, k = -13$, the above system is globally asymptotically stable. Then the output regulation problem of (4.36) is solved by

$$\begin{aligned}\dot{z}_2 &= ky_m + (2+k/2)z_2,\\ u_1 &= -k_1 y_m + (1+k_1/2)z_2,\\ u_2 &= 2z_2.\end{aligned} \tag{4.38}$$

Remark 22 *From the above examples, one can see that not all stabilizers have the corresponding internal model such that the closed-loop stabilized system is globally asymptotically stable. It should be pointed out that all the examples in [44] can be dealt with in the framework proposed in this chapter with stabilizers in the form of static feedback. Next, a system that cannot be stabilized by a static measurement feedback controller is considered.*

Example 3 Consider the following system

$$\begin{aligned}\dot{x}_1 &= -2x_1 - x_2 + u + 3w,\\ \dot{x}_2 &= x_1 + 3x_2 - 3w,\\ \dot{w} &= w,\\ e &= x_1x_2 - w^2,\\ y_m &= x_2.\end{aligned} \tag{4.39}$$

A solution of the regulator equation is $x = \pi(w) = col(w, w), u = c(w) = w$. One can get the following simplified system with $\bar{x} = x - \pi(w),\ \bar{u} = u - c(w)$,

$$\begin{aligned}\dot{\bar{x}}_1 &= -2\bar{x}_1 - \bar{x}_2 + \bar{u},\\ \dot{\bar{x}}_2 &= \bar{x}_1 + 3\bar{x}_2,\end{aligned} \tag{4.40}$$

with measurement $y_m = \bar{x}_2 + w$. This system cannot be stabilized by a static feedback controller even if w is known. However, it can be globally stabilized by a dynamic feedback controller,

$$\begin{aligned}\dot{z}_{11} &= -(y_m - w) - 8z_{11} - 29z_{12} &&= -\bar{x}_2 - 8z_{11} - 29z_{12},\\ \dot{z}_{12} &= 9.5(y_m - w) + z_{11} - 6.5z_{12} &&= 9.5\bar{x}_2 + z_{11} - 6.5z_{12},\\ \bar{u} &= -6z_{11} - 29z_{12},\end{aligned} \tag{4.41}$$

Consider a steady-state generator with $\theta(w) = w, \beta(\theta(w)) = \theta(w)$, and an internal model candidate $\dot{z}_2 = k_1(y_m - z_{12} - z_2) + z_2 = k_1\bar{x}_2 - k_1 z_{12} - k_1(z_2 - w) + z_2, k_1 \in R$. Then one can get the closed-loop stabilized system with $\bar{z}_2 = z_2 - \theta(w)$,

$$\begin{aligned}
\dot{\bar{x}}_1 &= -2\bar{x}_1 - \bar{x}_2 - 6z_{11} - 29z_{12} + \bar{z}_2, \\
\dot{\bar{x}}_2 &= \bar{x}_1 + 3\bar{x}_2, \\
\dot{z}_{11} &= -(y_m - z_2) - 8z_{11} - 29z_{12} && = -\bar{x}_2 - 8z_{11} - 29z_{12} + \bar{z}_2, \\
\dot{z}_{12} &= 9.5(y_m - z_2) + z_{11} - 6.5z_{12} && = 9.5\bar{x}_2 + z_{11} - 6.5z_{12} - 9.5\bar{z}_2, \\
\dot{\bar{z}}_2 &= k_1\bar{x}_2 - k_1 z_{12} + (1 - k_1)\bar{z}_2.
\end{aligned} \tag{4.42}$$

If $k_1 = -3$, the above system is globally asymptotically stable and the internal model candidate is a “real one”. So the output regulation problem is solved by

$$\begin{aligned}
\dot{z}_{11} &= -(y_m - z_2) - 8z_{11} - 29z_{12}, \\
\dot{z}_{12} &= 9.5(y_m - z_2) + z_{11} - 6.5z_{12}, \\
\dot{z}_2 &= -3(y_m - z_{12} - z_2) + z_2, \\
u &= -6z_{11} - 29z_{12} + z_2.
\end{aligned} \tag{4.43}$$

Another solution of the regulator equation is $x = \pi(w) = col(2w, 0.5w)$, $u = c(w) = 3.5w$. With a similar approach, one can get another solution of the output regulation problem as follows,

$$\begin{aligned}
\dot{z}_{11} &= -(y_m - 0.5z_2) - 8z_{11} - 29z_{12}, \\
\dot{z}_{12} &= 9.5(y_m - 0.5z_2) + z_{11} - 6.5z_{12}, \\
\dot{z}_2 &= -6(y_m - z_{12} - 0.5z_2) + z_2, \\
u &= -6z_{11} - 29z_{12} + 3.5z_2.
\end{aligned} \tag{4.44}$$

From this example, one can see that the DMFORP may be solved on different manifolds.

Remark 23 *Not all stabilizers have the corresponding internal model such that the closed-loop stabilized system is globally asymptotically stable. It should be pointed out that all the examples in [44] can be processed by the framework proposed in this chapter with stabilizers in the form of static feedback. The output regulation problem in this example cannot be solved by a static measurement feedback even if the state w of the exosystem is known. When a dynamic measurement feedback controller is considered, it is possible to find an internal model to incorporate the exosystem. In the example, the design of the stabilizer is not detailed, for there are a number of methods to go about it. However, it really concerns the example of how to apply the idea of this chapter to solve dynamic measurement feedback output regulation problems.*

Example 4 Consider the following system,

$$\begin{aligned}\dot{x}_1 &= -x_1^3,\\ \dot{x}_2 &= x_1(x_2 - w) + x_2 + u + 2w,\\ \dot{w} &= 0,\\ e &= x_2 - w,\\ y_m &= x_2.\end{aligned} \tag{4.45}$$

The unique solution of the regulator equation is $\pi(w) = col(0, w)$, $c(w) = -3w$. A1' holds. With $\bar{x} = x - \pi(w), \bar{u} = u - c(w)$, the simplified system is

$$\begin{aligned}\dot{\bar{x}}_1 &= -\bar{x}_1^3,\\ \dot{\bar{x}}_2 &= \bar{x}_2 + \bar{x}_1\bar{x}_2 + \bar{u}.\end{aligned} \tag{4.46}$$

It can be stabilized by a static controller $\bar{u} = -k_1 y_m + k_1 w, k_1 > 1$. A2' holds. Consider a steady-state generator $\theta(w) = w, \alpha(\theta(w)) = 0, \beta(w) = w$, and an internal model candidate $\dot{z}_2 = k_2 y_m - k_2 z_2 = k_2\bar{x}_2 - k_2(z_2 - w)$. With $\bar{z}_2 = z_2 - w$, the stabilized closed-loop system is

$$\begin{aligned}\dot{\bar{x}}_1 &= -\bar{x}_1^3,\\ \dot{\bar{x}}_2 &= (1 - k_1)\bar{x}_2 + (k_1 - 3)\bar{z}_2 + \bar{x}_1\bar{x}_2,\\ \dot{\bar{z}}_2 &= k_2\bar{x}_2 - k_2\bar{z}_2.\end{aligned} \tag{4.47}$$

The above system is actually locally asymptotically stable in the sense of $\mathcal{KL}$ functions when $k_1 > 1, k_2 > 0$. The internal model is a "real one". According to Theorem 16, the local DMFORP is solvable. With w in the stabilizer $u = \bar{u} + c(w) = (-k_1 y_m + k_1 w) - 3w$ replaced by z_2, the controller is as follows,

$$\begin{aligned}\dot{z}_2 &= k_2 y_m - k_2 z_2, && k_2 > 0,\\ u &= -k_1 y_m + (k_1 - 3)z_2, && k_1 > 1.\end{aligned} \tag{4.48}$$

4.3 Speed regulation

In the previous chapter, it is shown that optimal scheduling can improve the performance of the closed-loop controller, and that the 2-2 strategy, the ECP/iDP mode, is the best of all the strategies.

In this section, one considers the application of output regulation of nonlinear systems with measured output feedback to the control of heavy haul trains. Optimal scheduling is still based on "trading off" the equilibria. Thus the balance between energy consumption and in-train forces is still maintained. For closed-loop control, speed regulation is imposed. This approach to design is practically feasible and manageable, and by its nature, is also easily integrable with human drivers. Instead of the linear system theory, a nonlinear system theory is adopted so that without a linear approximation philosophy, the control is closer to reality. Another advantage of the approach is the assumption that only the locomotives' speeds are available for measurement.

4.3.1 Application of output regulation in heavy haul trains

The mathematical model of train is repeated as follows,

$$\begin{aligned} m_s \dot{v}_s &= u_s + f_{in_{s-1}} - f_{in_s} - f_{a_s}, \quad s = 1, 2, \cdots, n, \\ \dot{x}_j &= v_j - v_{j+1}, \quad j = 1, 2, \cdots, n-1, \end{aligned} \tag{4.49}$$

For the model of a train (4.49), some changes are required to be made for the application of output regulation. On the one hand, the origin is not an equilibrium of the system (4.49). On the other hand, there are many trajectories to annihilate the output (to regulate the output to the reference). However, for train handling, the choice of trajectories involves the balance between energy consumption and in-train forces. So in the application scheme, a quadratic programming algorithm is firstly applied to calculate the equilibrium of the system (4.49) with the reference speed held. Then, based on the equilibrium, a difference system between state of the origin system (4.49) and the equilibrium is formed, which can be stabilized with output regulation in the form of measurement feedback.

Optimal scheduling is referred to in section 3.4.2, where the equilibrium calculation is a quadratic programming problem. The performance function, considering the "trade-off" between the in-train forces and the energy consumption, is as (3.15).

In open loop control, the dynamic process in the train is ignored and the train is assumed to be in its steady state with the reference speed maintained, that is,

$$\begin{aligned} \frac{dv_i}{dt} &= 0, \quad i = 1, 2, \cdots, n, \\ \frac{dx_j}{dt} &= 0, \quad i = 1, 2, \cdots, n-1. \end{aligned} \tag{4.50}$$

Applying (4.50) to (4.49), one has

$$u_s + f_{in_{s-1}} - f_{in_s} - f_{a_s} = 0, \qquad s = 1, 2, \cdots, n. \tag{4.51}$$

In practical operations, u_i and f_{in_i} have some constraints.

$$\begin{aligned} \underline{U}_i \le u_i \le \overline{U}_i, &\qquad i = 1, 2, \cdots, n; \\ \underline{F}_{in_j} \le f_{in_j} \le \overline{F}_{in_j}, &\quad j = 1, 2, \cdots, n-1, \end{aligned} \tag{4.52}$$

where $\underline{U}_i, \overline{U}_i$ are the upper and the lower constraints for the ith input, and $\underline{F}_{in_j}, \overline{F}_{in_j}$ are the upper and lower constraints for the jth in-train force, respectively. For wagons,

$\overline{U_i} = 0$ and the values of $\underline{U_i}$ depend on the braking capacities of the wagons. For locomotives, the constraints $\underline{U}_i, \overline{U}_i$ depend on the locomotives' capacities in traction efforts and the running states. The constraints $\underline{F}_{in_j}, \overline{F}_{in_j}$ are limited because of the requirement of safe operation and low maintenance cost.

Thus optimal scheduling is a standard quadratic programming (QP) problem with objective function (3.15), equality constraints (4.51) and inequality constraints (4.52). The input operation limits are not considered. When the inputs are applied to the model, an anti-windup technique, detailed later, is applied.

With the above scheduling, the equilibrium can be denoted as $f^0_{in_j}(x^0_j)$, $v^0_i(v_r)$, u^0_i, $j = 1, 2, \cdots, n-1$, $i = 1, 2, \cdots, n$, which are the in-train forces (static displacement of coupler), the velocities (reference velocity) and the efforts of the cars. Then one can rewrite the train model as:

$$\begin{aligned} &\delta \dot{v}_s = (\delta u_s + \delta f_{in_{s-1}} - \delta f_{in_s} - \delta f a_s)/m_s, \qquad s = 1, \cdots, n, \\ &\delta \dot{x}_j = \delta v_j - \delta v_{j+1}, \qquad j = 1, \cdots, n-1, \end{aligned} \tag{4.53}$$

where $\delta v_s = v_s - v^0_s = v_s - v_r$, $\delta u_s = u_s - u^0_s$, $\delta f_{in_s} = f_{in_s} - f^0_{in_s}$, $\delta x_j = x_j - x^0_j$.

Thus in the controller design, the system (4.53) can be rewritten as

$$\dot{X} = f(X) + g(X)U, \tag{4.54}$$

where

$$X = col(\delta v_1, \cdots, \delta v_n, \delta x_1, \cdots, \delta x_{n-1});$$
$$U = col(\delta u_1, \cdots, \delta u_n);$$
$$f_i(X) = \frac{1}{m_i}(k_{i-1}X_{n+i-1} - k_i X_{n+i}) - (c_{i_1} + 2c_{i_2}v_r)X_i - c_{i_2}X_i^2, \; i = 1, 2 \cdots, n,$$
$$f_{n+i}(X) = X_i - X_{i+1}, \; i = 1, 2, \cdots, n-1;$$
$$g(x) = \begin{bmatrix} \mathrm{diag}(\frac{1}{m_1}, \cdots, \frac{1}{m_n}) \\ 0_{(n-1)\times n} \end{bmatrix}.$$

The outputs to regulate are the cars' speeds, *i.e.*, assuming the reference speed is w_1, which is to be designed later, $e_i = v_i - w_1 = X_i + v_r - w_1$. The measured output is part of the cars' speeds, *i.e.*, $y_m = C_m(X + v_r)$, where $C_m = (C_{ij})_{p_m \times (2n-1)}$ and all the entries of the row vectors of C_m are zeros, only except one of the first n ones, which is one. For example, $C_m = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{bmatrix}$ if only the first two cars' speeds are measured.

Notice that the measured output is different from the output error.

The linearized system of (4.54) has system matrixes

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

$$A_{11} = -\mathrm{diag}(c_{1_1} + c_{2_1} v_r, \cdots, c_{1_n} + c_{2_n} v_r),$$

$$A_{12} = \begin{bmatrix} -\frac{k_1}{m_1} & 0 & \cdots & 0 & 0 \\ \frac{k_1}{m_2} & -\frac{k_2}{m_2} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \frac{k_{n-2}}{m_{n-1}} & -\frac{k_{n-1}}{m_{n-1}} \\ 0 & \cdots & 0 & 0 & \frac{k_{n-1}}{m_n} \end{bmatrix},$$

$$A_{21} = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \end{bmatrix},$$

$$A_{22} = 0_{(n-1)\times(n-1)},$$

$$B = \begin{bmatrix} \mathrm{diag}(\frac{1}{m_1}, \cdots, \frac{1}{m_n}) \\ 0_{(n-1)\times n} \end{bmatrix}.$$

It can be verified that the above linearized pair (A, B) is controllable and (A, C_m) is observable with the *PBH* criterion in [28]. First one verifies the controllability.

$$\begin{aligned} [\lambda I - A \mid B] &= \begin{bmatrix} \lambda I_{n\times n} - A_{11} & -A_{12} & \mathrm{diag}(\frac{1}{m_i}) \\ -A_{21} & \lambda I_{(n-1)\times(n-1)} & 0_{(n-1)\times n} \end{bmatrix} \\ &\sim \begin{bmatrix} \lambda I_{n\times n} - A_{11} & -A_{12} & I_{n\times n} \\ -A_{21} & \lambda I_{(n-1)\times(n-1)} & 0_{(n-1)\times n} \end{bmatrix} \\ &\sim \begin{bmatrix} 0_{n-1\times n-1} & 0 & 0_{n\times(n-1)} & I_{n\times n} \\ I_{(n-1)\times(n-1)} & 0 & \lambda I_{(n-1)\times(n-1)} & 0_{(n-1)\times n} \end{bmatrix}, \end{aligned}$$

from which one can get rank $([\lambda I - A \mid B]) = 2n - 1$, and the pair (A, B) is controllable according to the *PBH* criterion.

When it comes to observability, if the first or the last car's speed is measured, the pair (A, C_m) is observable. Assuming, for example, the first car's speed is available, one has

$$\begin{bmatrix} A - \lambda I \\ C_m \end{bmatrix} = \begin{bmatrix} A_{11} - \lambda I_{n\times n} & A_{12} \\ A_{21} & -\lambda I_{(n-1)\times(n-1)} \\ 1 \quad 0 \quad \cdots \quad 0 & 0_{1\times(n-1)} \end{bmatrix} \sim \begin{bmatrix} I_{(2n-1)\times(2n-1)} \\ 0_{1\times(2n-1)} \end{bmatrix},$$

from which one knows that rank $\left(\begin{bmatrix} A - \lambda I \\ C_m \end{bmatrix}\right) = 2n - 1$, and the pair (A, C_m) is observable according to the *PBH* criterion. Actually the first car of a train is usually a locomotive, which is often a leader and whose speed is available. So the above assumption does not lose generality.

4.3.2 Trajectory design of heavy haul trains

As described, only the speed maintenance phase, speed acceleration and speed deceleration phases are discussed in this study. The cars' speeds are the subject of regulation.

To apply Theorem 16 into train control, the trajectory of the reference speed should satisfy the condition of neutral stability. It can be designed as

$$\begin{aligned} \dot{w}_1 &= a w_2, \\ \dot{w}_2 &= -a(w_1 - w_3), \\ \dot{w}_3 &= 0, \end{aligned} \tag{4.55}$$

whose solution is

$$\begin{aligned} w_1 &= w_3(0) + A\sin(at + \phi_0), \\ w_2 &= A\cos(at + \phi_0), \\ w_3 &= w_3(0), \end{aligned} \tag{4.56}$$

where A and ϕ_0 are determined by the initial conditions $(w_1(0), w_2(0), w_3(0))$.

Within the cruise phase, the initial conditions are chosen as

$$(w_1(0), w_2(0), w_3(0)) = (v_r, 0, v_r),$$

where v_r is the cruise speed.

Assuming the reference speed before acceleration/deceleration is v_{r_1} and the reference speed after acceleration/deceleration is v_{r_2}, then the initial conditions are chosen such that $w_3(0) = v_{r_1}, \phi_0 = 0, A = \sqrt{2}(v_{r_2} - v_{r_1})$.

The variable a in (4.55) is chosen considering the acceleration limit a_r or deceleration limit a_c of the train, which is determined by the effort capacity of the train. In simulation, $a = \frac{a_r}{A}$ within the acceleration phase and $a = \frac{a_c}{A}$ within the deceleration phase. For example, one chooses $a_r = 0.07$ m/s^2, $a_c = -0.2$ m/s^2, $\phi_0 = 0$, and the time interval $T_1 = \frac{\pi}{4a}$ as acceleration/deceleration phase. The modified speed file according to the speed profile is shown in Fig. 4.1.

The coefficient matrix of (4.55) is constant and its eigenvalues obviously lie on the imaginary axis, so the above designed trajectories are neutrally stable.

4.3.3 Speed regulation controller design

From the above designed trajectories, the conditions in Proposition 17 are satisfied if the regulator equations (4.7) are solved. Actually, one can verify that $X = \pi(w) =$

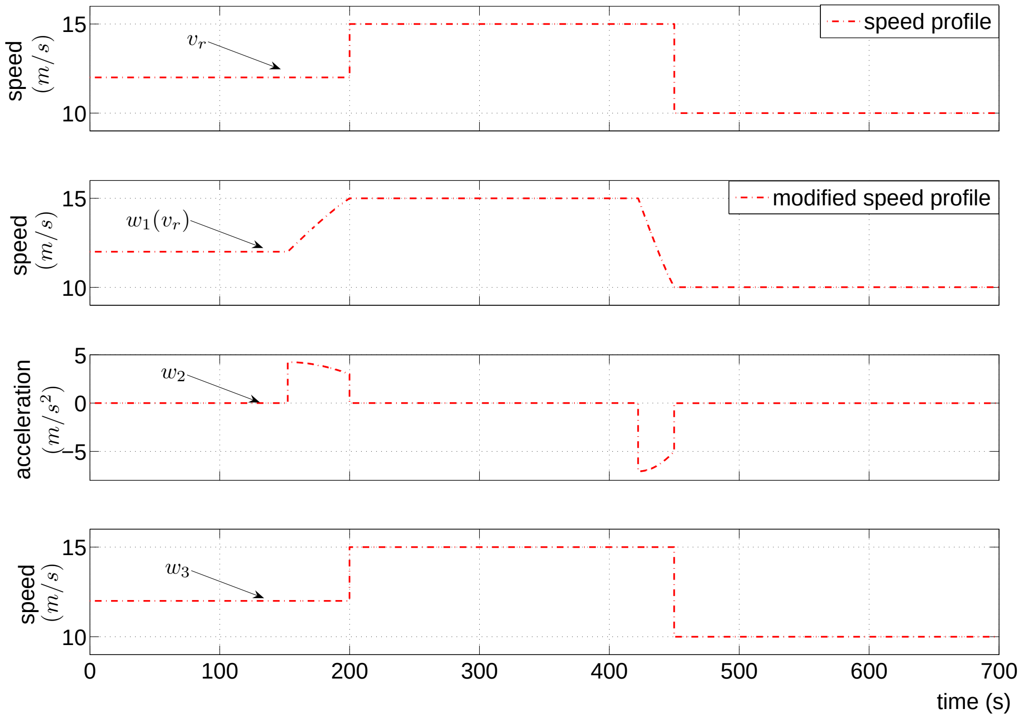

Figure 4.1: Modified speed profile

$(w_1 - w_3)[(1_{1\times n}, 0_{1\times(n-1)}]^T, U = c(w) = w_2 B_1^{-1} \cdot 1_{n\times 1} - B_1^{-1} f^1(\pi(w))$, where f^1 is the first n entries of f and B_1 is the first n rows of B, is a solution of (4.7).

According to Remark 18, the output regulating controller with measurement feedback is

$$\begin{aligned} \dot{z} &= f(z) + g(z)U + G_1(y_m - C_m z), \\ U &= c(w) + K(z - \pi(w)), \end{aligned} \tag{4.57}$$

where G_1, K are chosen such that $A + BK$ and $A - G_1 C_m$ are Hurwitz.

Based on the optimal scheduling and the output regulating controller, the complete closed-loop controller is

$$u = U + u^0. \tag{4.58}$$

In simulation, one chooses K with a linear quadratic algorithm in [15], where the performance function is

$$\delta J = \int (X'QX + U'RU)\, dt = \int \left(\sum_{i=0}^{n-1} K_f^o \delta x_i^2 + \sum_{i=0}^{n} K_e \delta u_i^2 + \sum_{i=0}^{n} K_v^o \delta v_i^2 \right) dt,$$

in which the variables K_f^o, K_e, K_v^o are the weights for in-train forces, energy consumption and velocity tracking, respectively. The different choices of the values of the weights lead to speed emphasized control, in-train force emphasized control and energy consumption emphasized control, respectively.

The parameter G is also obtained by a quadratic programming algorithm in simulation where the weights for all the entries are equal.

These choices of K and G are consistent with Remark 19.

4.3.4 Simulation of speed regulation

Simulation setting

The simulation setting and parameters are the same as those in the previous chapter except that the deceleration limit is -0.2 m/s^2.

The weights for in-train forces, energy and velocity are $K_f,\ K_e,\ K_v$, respectively, and $K_f^o = 3\times 10^8 K_f, K_v^o = 5\times 10^6 K_v$, which leads to the same quantities of the items of the in-train forces, speed and input in (4.3.3) when $\delta x = 0.01$ m$, \delta v = 0.1$ m/s$^2, \delta u =$ 200 N with $K_f = K_e = K_v$.

The acceleration limit a_r is 0.07 m/s^2. This value is calculated on the assumption that the train is running on a flat track and all the traction power of the locomotives is used to accelerate. The maximum acceleration can be $760 \times 2/(252 \times 2 + 417 \times 50) = 0.07118$ m/s^2. The absolute value of the deceleration a_c is more than that of the acceleration.

The observer is designed on the assumption that the front and rear locomotive group speeds are available. Since the exosystem is designed and its state is known, the observer is just to estimate the running state of the train model (deviations of the cars' speeds and the displacements of the couplers). The initial states of the observer are set to be zeros.

The observer is designed based on the difference system (4.53), which is related to w_3 of the exosystem. When w_3 is changed, the observer needs some time to track the state of the difference system. So in the control design, when w_3 is changed, the closed-loop controller is disabled (*i.e.*, only open loop scheduling is used) for some interval, during which the observer will track more closely to the state of the difference system. In simulation the interval is assumed to be a distance interval, whose length is equal to $15 \times w_3$.

Simulation result

A simulation result is shown in Fig. 4.2 with the optimal parameters $K_f = 1, K_v = 1, K_e = 1$.

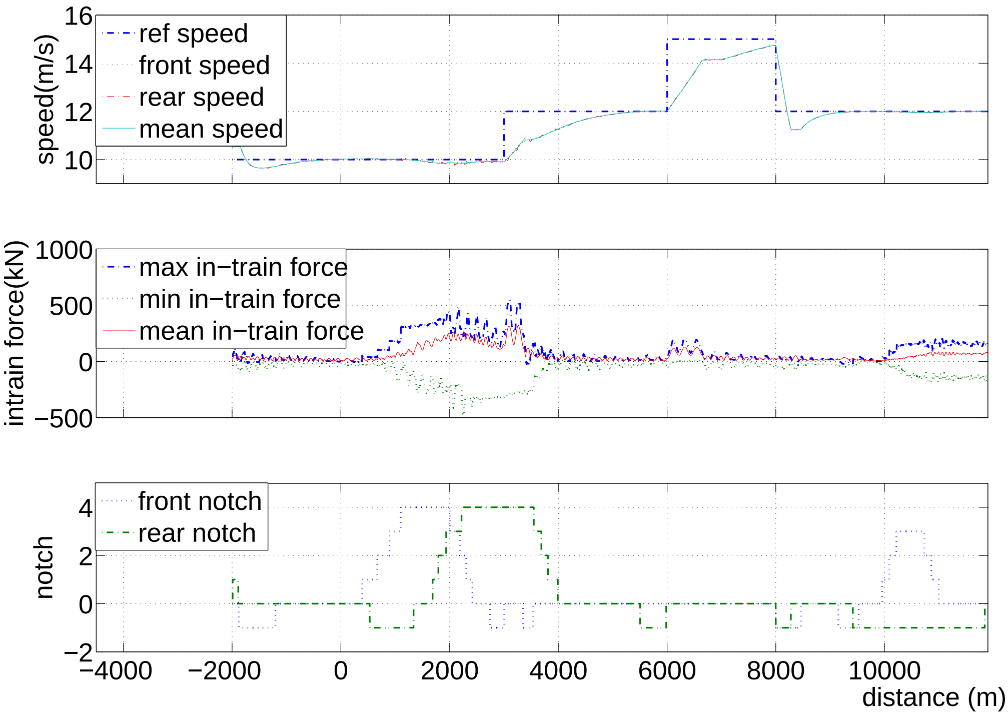

Figure 4.2: Output regulation with measurement feedback

In comparing the results shown in the above figure with Fig. 3.10 and Fig. 3.14, it can be seen that the oscillation is most obvious in open loop scheduling, while it is least in the output regulation with measurement feedback. The steady state error exists in open loop scheduling while it is smaller in optimal control with state feedback and output regulation with measurement feedback. However, it tracks the reference speed more quickly with state feedback than with measurement feedback. This is because of the application of the observer in the latter, which needs some time to track the state of the train. Coincidentally for the same reason, the in-train forces in Fig. 4.2 are smaller than those of the other two in the steady state. This is because the slower response of the observer leads to more gentle output.

Table 4.1 shows the simulation results of the state feedback controllers $S_i, i = 1, 2, 3, 4$ advanced in Section 3.5 and measurement feedback controllers $M_i, i = 1, 2, 3, 4$ proposed in this chapter with different tuning parameters. The indices 1, 2, 3, 4 denote the different sets of parameters $(K_e, K_f, K_v) = (1, 1, 1)$, $(K_e, K_f, K_v) = (1, 1, 10)$, $(K_e, K_f, K_v) = (1, 10, 1)$, $(K_e, K_f, K_v) = (100, 1, 1)$. $|\delta\bar{v}|$ is the absolute value of the

Table 4.1: Performance comparison

	$\lvert\delta\bar{v}\rvert$(m/s)			$\overline{\lvert f_{in}\rvert}$(kN)			E
	max	mean	std	max	mean	std	(MJ)
S1	3.0182	0.3166	0.48	454.50	97.40	86.44	16,528
S2	3.0225	0.2443	0.50	408.70	74.07	76.34	16,524
S3	3.0090	0.3667	0.47	405.70	70.77	78.04	15,007
S4	3.2470	0.4918	0.47	297.27	78.90	63.27	13,422
M1	2.9827	0.3250	0.53	322.02	56.49	63.54	12,400
M2	2.9801	0.2969	0.53	329.39	54.28	65.28	12,713
M3	2.9692	0.3290	0.52	329.00	56.74	64.14	12,570
M4	3.6094	0.8942	0.62	405.34	98.41	73.50	10,493

difference between the reference velocity and the mean value of all the cars' velocities at a specific point. $\overline{|f_{in}|}$ is the mean value of the absolute values of all the couplers' in-train forces at a specific point. The items max, mean and std are the maximum value, mean value and standard deviation of the statistical variable.

These data reflect the working of the optimization parameters. From Table 4.1, it can be seen that more energy is consumed in the optimal controller with state feedback than in output regulating controller with measurement feedback, no matter which group of the optimal parameters is chosen. This is because the optimal controllers of state feedback are sensitive to the state deviation from the equilibrium, and the energy optimization is local, thus the locomotives' traction efforts and the cars' braking are more frequent, which leads to the consumption of more energy.

For speed tracking, the optimal controller with state feedback is a little better than the output regulating controller with measurement feedback, and for in-train forces, the former is worse than the latter. This confirms the above result, by comparing Fig. 3.14 and Fig. 4.2.

In the above chapter, it is said that the length of the track does not affect the result in this thesis. To show this, the above speed regulator will be simulated on a longer track (27 km), as indicated in Fig. 4.3. The simulation result is reflected in Fig. 4.4.

From Fig. 4.4, it can be seen that the train tracks the reference speed well except within the distance from 18 km to 21 km, where the train speed is much lower than the reference speed. This is because the train is passing over a hill during this period, which can be seen from Fig. 4.3. Because of the operational constraint (the time delay between two notch changes), the train has enough traction power to maintain or be closer to the reference speed.

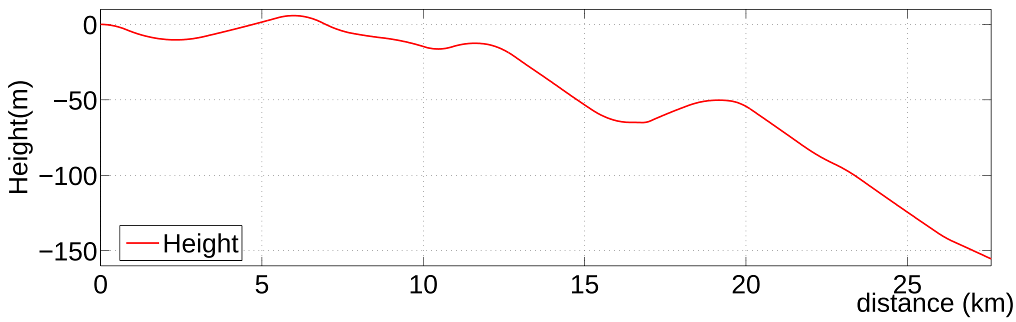

Figure 4.3: A longer track profile

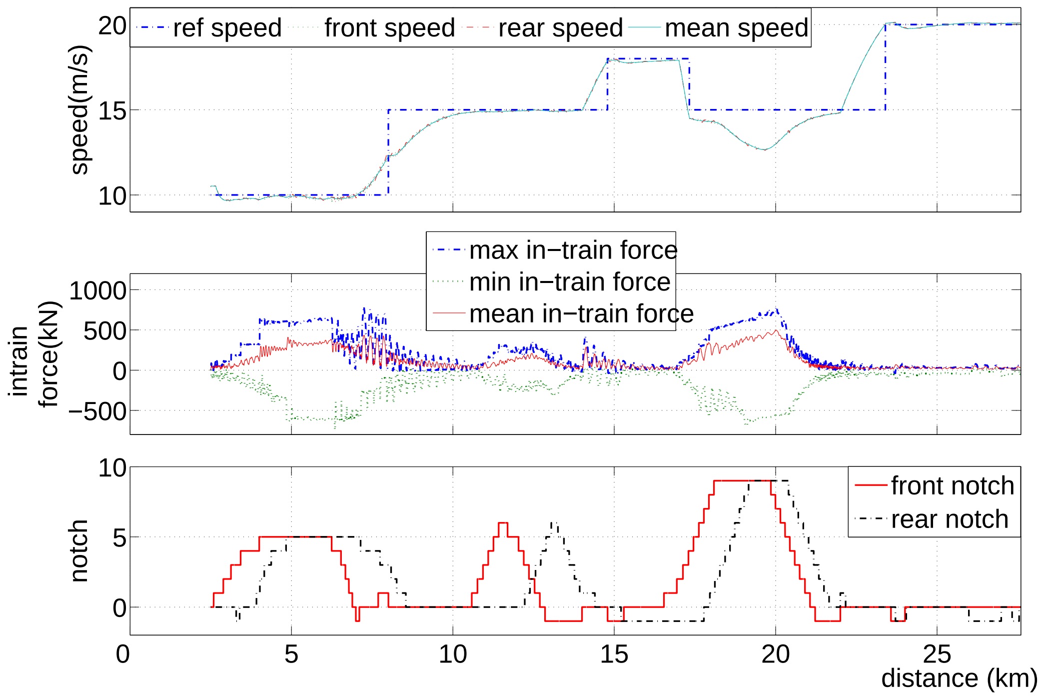

Figure 4.4: Speed regulation on a longer track profile

4.4 Conclusion

In this chapter, a framework is introduced to solve the output regulation problem using measurement feedback.

The measurement feedback is considered because the measurement can cover the output and/or (partial) state, even some measurable output different from the output and state, that is, it is more general.

This framework can also incorporate different kinds of exosystems with bounded signal or unbounded signal, Poisson stable or not. Some assumptions in this chapter are necessary.

Similar to [43] and [44], the solvability of the output regulation problem is transformed to the solvability of the corresponding stabilization problem. The difference is that in this chapter a stabilizer is firstly designed assuming the states of the exosystem are known, and then an internal model is designed with respect to this stabilizer and a steady-state generator. The internal model is in nature an observer of the state of the exosystem. The existence of the stabilizer is sometimes a necessary condition for the solvability of the output problem. The properties of the internal model, in which the state of the stabilizer and the measurement of the original system can appear, are also given.

It should be pointed out that not all stabilizers have the corresponding internal models. Sometimes the parameters in the stabilizer and the internal model candidate need to be tuned. The design techniques of the internal model, in essence an observer for nonlinear systems, are not detailed, nor are the design techniques of the stabilizer, which are out of the scope of this study.

The application of output regulation of nonlinear systems with measured output feedback to the control of heavy haul trains is investigated in section 4.3. The optimal scheduling of the open loop controller is still based on "trading off" the equilibria. Thus the balance between energy consumption and in-train forces is still maintained. For closed-loop control, speed regulation is imposed. This approach to design is practically feasible and manageable, and by its nature, is also easily integrable with human drivers, because the human drivers drive the train according to the train's speed.

Instead of the linear system theory, a nonlinear system theory is adopted so that without a linear approximation philosophy, the control is closer to the reality. Another advantage of the approach is the assumption that only the locomotives' speeds are available for measurement.

A controller of speed regulator is designed based on the result mentioned in the first part of this chapter. The conditions of the application are verified. In the controller design, Optimal scheduling is retained in the control of output regulation. It is noted that when the difference system is changed with the variation of the reference speed, the state of the observer is changed suddenly and sufficient time should be given to the observer to track the state of the difference system, and thus the control of the output regulation is disabled during this period. Simulation shows the feasibility of the output regulating controller with only measurement of the locomotive speeds, in terms of its simplicity, cost-effectiveness and its implementation convenience.