

Metrical aspects of the complexification of tensor products and tensor norms

by

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Summary

We study the relationship between real and complex tensor norms.

The theory of tensor norms on tensor products of Banach spaces, was developed, by A. Grothendieck, in his *Résumé de la théorie métrique des produits tensoriels topologiques* [3]. In this monograph he introduced a variety of ways to assign norms to tensor products of Banach spaces. As is usual in functional analysis, the real-scalar theory is very closely related to the complex-scalar theory. For example, there are, up to topological equivalence, fourteen “natural” tensor norms in each of the real-scalar and complex-scalar theories. This correspondence was remarked upon in the *Resumé*, but without proving any formal relationships, although hinting at a certain injective relationship between real and complex (topological) equivalence classes of tensor norms.

We make explicit connections between real and complex tensor norms in two different ways. This divides the dissertation into two parts.

In the first part, we consider the “complexifications” of real Banach spaces and find tensor norms and complexification procedures, so that the complexification of the tensor product, which is itself a Banach space, is isometrically isomorphic to the tensor product of the complexifications. We have results for the injective tensor norm as well as the projective tensor norm.

In the second part we look for isomorphic results rather than isometric. We show that one can define the complexification of real tensor norm in a natural way. The main result is that the complexification of real topological

equivalence classes that is induced by this definition, leads to an injective correspondence between the real and the complex tensor norm equivalence classes.

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Preface

As said in the summary, the goal of this thesis is to study the relation between real and complex tensor norms. Various formal relationships between certain real and complex tensor norms are presented.

Generally speaking, the theory that is discussed in the background chapter 1 is part of the known literature, but the results in chapters 2 and 3 are new, unless otherwise specified, usually by a reference. The main plan of chapter 3 is probably implicit in the *Résumé*.

We do not assume knowledge of complexification; an overview of this is given in chapter 1. We do however assume that the reader knows the basics of tensor norms, and do not explain in detail standard properties of the injective and projective tensor norms that we use, such as the isometry $C(K_1) \overset{\vee}{\otimes} C(K_2) \stackrel{1}{=} C(K_1 \times K_2)$ or the universal property of the projective norm; of course we usually give a reference when such results are used. Some references to the topic of tensor norms, as well as a brief summary of a few concepts, are given in appendix A. In chapter 3 knowledge of operator ideals will be useful, but the definitions that we need are given in this text.

The adjective “real” (resp. complex) preceding a concept such as operator ideal, or tensor norm, will always refer to the concept as it appears in the theory of real (resp. complex) Banach spaces. For example, the phrase “real projective norm” refers to the projective tensor norm of the theory of tensor products of real Banach spaces, while the phrase “complex projective norm” involves the projective tensor norm that is defined in the theory of tensor

norms on tensor products of complex Banach spaces. A real operator ideal is an ideal of operators between real Banach spaces, and so on.

We will use the notation \wedge for the projective tensor norm, and \vee for the injective tensor norm. (Many books use instead the symbols π and ϵ respectively.)

When α is a *tensor norm*, see appendix A, with X and Y Banach spaces, the notation $X \otimes_{\alpha} Y$ will denote the normed vector space $X \otimes Y$ endowed with the norm α , and the completion of this normed vector space will be denoted by $X \overset{\alpha}{\otimes} Y$.

We use $C_{\mathbb{C}}(K)$ to denote the space of complex valued continuous functions on a compact Hausdorff space K , while $C_{\mathbb{R}}(K)$ denotes the space of real valued continuous functions on K . The same conventions apply to the notations $L_{\mathbb{C}}^p(\mu)$ and $L_{\mathbb{R}}^p(\mu)$.

Chapter 1

Preliminaries on complexification

1.1 Complexification procedures

The field of complex numbers \mathbb{C} can be represented as the set of pairs of real numbers, with addition defined in the usual way, and complex scalar multiplication defined using $(a + ib)(x, y) := (ax - by, bx + ay)$. At the very least, this way of representing a complex vector space as the Cartesian square of some real vector space, extends to sequence spaces such as ℓ^p and function spaces such as $C(K)$, $L^p(\mu)$ etc. To consider the interaction of complexification with norms, we have to be a bit more formal.

Therefore we summarize, in this chapter, some definitions and results from Muñoz, Sarantopoulos and Tonge [4].

Let the notation $X \oplus Y$ denote the Cartesian product of vector spaces X and Y , with the vector space operations $(x, y) + (x', y') := (x + x', y + y')$ and $\lambda(x, y) := (\lambda x, \lambda y)$. Let X be a real vector space. Then $X \oplus X$ can be turned into a *complex* vector space, defining multiplication with a complex scalar, as mentioned above, by $(a + ib)(x, y) := (ax - by, bx + ay)$. This complex vector space is said to be the complexification of X , and denoted by $X_{\mathbb{C}}$.

(Our reference [4] defines $X_{\mathbb{C}}$ using conditions on the injection $X \hookrightarrow X_{\mathbb{C}}$, and then gives three complex isomorphic representations of $X_{\mathbb{C}}$. Two of these representations are the Cartesian square representation and the tensor product representation of the next paragraph.) It will be convenient to write $x + iy$ for the element $(x, y) \in X \oplus X$.

As mentioned, in addition to this “Cartesian square” representation of the complexification of a real vector space, we will also use a representation of $X_{\mathbb{C}}$ using tensor products. Let e_1 and e_2 denote the unit vectors of \mathbb{R}^2 . It is easy to see that $X_{\mathbb{C}}$ is isomorphic, as a complex vector space, to $X \otimes \mathbb{R}^2$, via the map $x + iy \mapsto x \otimes e_1 + y \otimes e_2$. (The reader unfamiliar with tensor products can find an introduction in the book [6]. See also appendix A for the definitions of terms such as cross-norm, tensor norm, finite generation property and so on; and further references.) This tensor product representation, of the complexification of a real vector space, will be seen to be the most convenient representation, if we want to give the complexification of a real Banach space a norm, so that it becomes a complex Banach space in addition to being a complex vector space.

So suppose X is a real Banach space. Now let α be cross-norm on $X \otimes \mathbb{R}^2$, where we put the usual Euclidean norm on \mathbb{R}^2 . Then we can use $\alpha(x \otimes e_1 + y \otimes e_2)$ as the norm of $x + iy$. The space $X \otimes \mathbb{R}^2$ thus endowed with a cross-norm, will be complete because the tensor product of a Banach space with a finite-dimensional space, is complete (that is, a Banach space) under any cross-norm. So the complexification of X under the above-mentioned norm can be identified with $X \overset{\alpha}{\otimes} \mathbb{R}^2$, endowed with complex scalar multiplication as above.

We have the property that this complexification norm of $x + i0 \in X_{\mathbb{C}}$ will equal $\|x\|_X$, because $\|x + i0\| = \alpha(x \otimes e_1 + 0 \otimes e_2) = \|x\|_X \|e_1\|_{\mathbb{R}^2} = \|x\|_X$. It will also satisfy the condition that $\|x + iy\| = \|x - iy\|$ [4, Proposition 9]. In fact, any “complexification procedure” that puts a norm on $X_{\mathbb{C}} = X \otimes \mathbb{R}^2$ satisfying these two conditions, arises from a *cross-norm* on the tensor product [4,

Proposition 8]. They are called *reasonable* complexification procedures.

The two Propositions just mentioned shows the close connection between complexification and cross-norms on tensor products. The Cartesian square representation does not seem to fit as well. For example, the functional $(x, y) \mapsto \sqrt{\|x\|_X^2 + \|y\|_X^2}$ is not even a norm on the complex vector space $X_{\mathbb{C}}$, unless X is a Hilbert space.

If α is the projective tensor norm, we will write X_{β} for the resulting complex Banach space. (The β alludes to the terminology “Bochnak complexification procedure” that is used for this case.) If α is the injective tensor norm, we will write X_{τ} for the resulting complex Banach space, and refer to this way of norming the complexified Banach space, as the Taylor complexification procedure.

1.2 More about the Taylor complexification procedure

The following two results describe elementary properties of the Taylor complexification procedure. (They are proved in [4].)

Theorem 1.2.1. [4, Proposition 10] *Let X be a real Banach space. Then, for any $x + iy \in X_{\tau}$,*

$$\|x + iy\|_{X_{\tau}} = \sup_{0 \leq \theta \leq 2\pi} \|x \cos \theta + y \sin \theta\|_X.$$

Proof. (In this proof, a and b are real numbers.) We get

$$\begin{aligned}
\|x + iy\|_{X_\tau} &= \|x \otimes e_1 + y \otimes e_2\|_{X \overset{\vee}{\otimes} \mathbb{R}^2} \\
&= \sup_{\phi \in B_{X^*}, \psi \in B_{(\mathbb{R}^2)^*}} |\phi(x)\psi(e_1) + \phi(y)\psi(e_2)| \\
&= \sup_{a^2+b^2=1, \phi \in B_{X^*}} |\phi(ax + by)| \\
&= \sup_{a^2+b^2=1} \|ax + by\|_X \\
&= \sup_{0 \leq \theta \leq 2\pi} \|x \cos \theta + y \sin \theta\|_X.
\end{aligned}$$

Here we used, in turn, one of the main descriptions of the injective tensor norm, the properties of $(\mathbb{R}^2)^*$, and duality. \square

Theorem 1.2.2. *Let K be a compact Hausdorff space. The complex Banach space $C_{\mathbb{R}}(K)_\tau$ is isometrically isomorphic to $C_{\mathbb{C}}(K)$.*

Proof. The mapping

$$C_{\mathbb{R}}(K)_\tau \rightarrow C_{\mathbb{C}}(K) : x + iy \mapsto x + iy$$

obviously has all the required algebraic properties. Furthermore, using reasoning similar to some steps of Theorem 1.2.1

$$\begin{aligned}
\|x + iy\|_{C_{\mathbb{R}}(K)_\tau} &= \sup_{\theta} \|x \cos \theta + y \sin \theta\|_{C_{\mathbb{R}}(K)} \\
&= \sup_{k \in K, \theta} |x(k) \cos \theta + y(k) \sin \theta| \\
&= \sup_{k \in K, a^2+b^2=1} |ax(k) + by(k)| \\
&= \sup_{k \in K} \sqrt{x(k)^2 + y(k)^2} \\
&= \sup_{k \in K} |(x + iy)(k)|.
\end{aligned}$$

\square

There is a similar relationship between – at least – Banach spaces of the form $\ell^1(\gamma)$, where γ is an arbitrary index set, and the Bochnak complexification procedure. This result will be mentioned and used shortly.

Many other things can be said about the Taylor complexification procedure, such as that it defines the smallest norm on $X_{\mathbb{C}}$ satisfying a certain relationship with $\|\cdot\|_X$; the reader can find it more in our main source in this chapter, reference [4].

In the next chapter we will consider the complexification of tensor products. Note that tensor products then appear in two distinct ways: there is the tensor product of two Banach spaces that we will consider, and then to define the complexification of that tensor product, the above-mentioned representation of a complexified Banach space as a tensor product with \mathbb{R}^2 is used.

Chapter 2

Complexification of the injective and projective tensor products of Banach spaces

(We presented most of the work of this chapter in reference [7] as well.)

In this chapter we show that there is a complexification procedure for the projective tensor norm, so that the complexification of the projective tensor product of any two Banach spaces is isometrically isomorphic to the tensor product of the complexifications. This gives a formal link between the real projective tensor norm and the complex projective tensor norm. (A general isometry between two sides, one defined in terms of the real projective norm, and the other one defined using the complex projective norm.) We prove an analogous result for the injective tensor norm.

Let X and Y be real Banach spaces. Then, of course, they are also real *vector* spaces. The algebraic basis of our approach to link real and complex tensor norms is the result that, on the vector space level, $(X \otimes Y)_{\mathbb{C}}$ is isomorphic to $X_{\mathbb{C}} \otimes Y_{\mathbb{C}}$. This is shown in the first section. In the second and third sections we see to what extent we can get an isometric result if one puts some real tensor norm on the left hand side, that is before taking

the complexification, and some complex tensor norm on the tensor product of complex Banach spaces on the right hand side.

2.1 The algebraic tensor product and complexification

As preparation for the proof, we recall from [1, Chapter 1] the definition of a tensor product of vector spaces. We use $Bil(X, Y; H)$ to denote the bilinear mappings from the Cartesian product of vector spaces $X \times Y$ to vector space H ; and $L(X, Y)$ to denote the linear operators between vector spaces X and Y .

Definition 2.1.1. *Let X, Y be two vector spaces over the same field of scalars. A pair (H, Φ_0) of a vector space H and a bilinear map $\Phi_0 \in Bil(X, Y; H)$ is called a tensor product of the pair (X, Y) if for each vector space G and each $\Phi \in Bil(X, Y; G)$ there is a unique $T \in L(H, G)$ with $\Phi = T\Phi_0$.*

$$\begin{array}{ccc}
 X \times Y & \xrightarrow{\Phi} & G \\
 \Phi_0 \downarrow & \dashrightarrow T & \\
 H & &
 \end{array}$$

We remind the reader that the tensor product exists for any pair of vector spaces X and Y , and that H is unique up to vector space isomorphism. Note also that the pair (H, Φ_0) is independent of the choice of the vector space G . The property of the tensor product that is stated by the definition will be referred to as the *universal mapping property*. It is standard practice to write $X \otimes Y$ for H and $x \otimes y$ for $\Phi_0(x, y)$. A book such as [1] can also be consulted for the proof of the following two facts, that we will need.

Firstly, if (H, Φ_0) is a tensor product of (X, Y) , then the linear span of the range of Φ_0 is H . As a consequence of this fact and the bilinearity of

\otimes , every $u \in X \otimes Y$ can be written as $u = \sum_{j=1}^n x_j \otimes y_j$ for some choice $(x_j)_{j=1}^n \subseteq X$ and $(y_j)_{j=1}^n \subseteq Y$.

Secondly, an element $u \in X \otimes Y$ with representation $u = \sum_{j=1}^n x_j \otimes y_j$, satisfies $u = 0$ if and only if for all linear functionals ϕ on X and ψ on Y , it holds that

$$\sum_{j=1}^n \phi(x_j)\psi(y_j) = 0.$$

Theorem 2.1.2. *Let X and Y be real vector spaces. The complex vector spaces $(X \otimes Y)_{\mathbb{C}}$ and $X_{\mathbb{C}} \otimes Y_{\mathbb{C}}$ are isomorphic.*

Proof. We apply Definition (2.1.1) to get a bilinear mapping $\Phi_0 : X_{\mathbb{C}} \times Y_{\mathbb{C}} \rightarrow X_{\mathbb{C}} \otimes Y_{\mathbb{C}}$ having the required universal mapping property. It will be convenient to use the notation $a \otimes_1 b := \Phi_0(a, b)$.

Applying Definition (2.1.1) again, but this time to the vector spaces X, Y there is a bilinear map, denoted by $\cdot \otimes_2 \cdot$ from $X \times Y$ to $X \otimes Y$ satisfying the universal mapping property.

The property of \otimes_1 says that for every bilinear $\Phi : X_{\mathbb{C}} \times Y_{\mathbb{C}} \rightarrow (X \otimes Y)_{\mathbb{C}}$ there is a unique linear map $T : X_{\mathbb{C}} \otimes Y_{\mathbb{C}}$ so that

$$\Phi(a, b) = T(a \otimes_1 b) \text{ for all } a \in X_{\mathbb{C}} \text{ and } b \in Y_{\mathbb{C}}. \quad (2.1)$$

Specifically, let Φ be the mapping

$$\Phi : X_{\mathbb{C}} \times Y_{\mathbb{C}} \rightarrow (X \otimes Y)_{\mathbb{C}} : (x + ix', y + iy') \mapsto (x \otimes_2 y - x' \otimes_2 y') + i(x' \otimes_2 y + x \otimes_2 y').$$

It is clearly bilinear. Furthermore, equation (2.1) tells us that $T((x + ix') \otimes_1 (y + iy')) = \Phi(x + ix', y + iy')$, where T is the induced linear mapping. So

$$T((x + ix') \otimes (y + iy')) = x \otimes_2 y - x' \otimes_2 y + i(x' \otimes_2 y + x \otimes_2 y').$$

Since T is linear, it follows that for any $u \in X_{\mathbb{C}} \otimes Y_{\mathbb{C}}$, say

$$u = \sum_{j=1}^n (x_j + ix'_j) \otimes_1 (y_j + iy'_j), \quad (2.2)$$

we have

$$Tu = \sum_{j=1}^n \left\{ (x_j \otimes_2 y_j - x'_j \otimes_2 y'_j) + i(x'_j \otimes_2 y_j + x_j \otimes_2 y'_j) \right\}. \quad (2.3)$$

To show that T is surjective, let $v = \sum_{j=1}^m x_k \otimes_2 y_j + i \sum_{j=1}^n \tilde{x}_j \otimes_2 \tilde{y}_j$ be any member of $(X \otimes Y)_{\mathbb{C}}$. Then $Tu = v$, where $u = \sum_{j=1}^{m+n} a_j \otimes_1 b_j \in X_{\mathbb{C}} \otimes Y_{\mathbb{C}}$, with

$$a_j := \begin{cases} x_j, & 1 \leq j \leq m \\ i\tilde{x}_{j-m}, & m+1 \leq j \leq m+n \end{cases}, \text{ and}$$

$$b_j := \begin{cases} y_j, & 1 \leq j \leq m \\ i\tilde{y}_{j-m}, & m+1 \leq j \leq m+n \end{cases}.$$

To show that T is injective, we assume that $Tu = 0$ and prove that $u = 0$: Now u is of the form of equation (2.2) and Tu of the form of equation (2.3) above.

Now observe that, since $Tu = 0$, we have for any linear functionals ϕ on X and ψ on Y that

$$\sum_{j=1}^n \left\{ (\phi(x_j)\psi(y_j) - \phi(x'_j)\psi(y'_j)) \right\} = 0, \quad (2.4)$$

$$\text{and } \sum_{j=1}^n \left\{ \phi(x'_j)\psi(y_j) + \phi(x_j)\psi(y'_j) \right\} = 0. \quad (2.5)$$

To prove that $u = 0$, let ξ be any given linear functional on $X_{\mathbb{C}}$ and η any linear functional on $Y_{\mathbb{C}}$. We write $\xi = \xi_r + i\xi_c := \mathcal{R}e \{ \xi \} + i\mathcal{I}m \{ \xi \}$ and $\eta = \eta_r + i\eta_c := \mathcal{R}e \{ \eta \} + i\mathcal{I}m \{ \eta \}$. Then ξ_r, ξ_c are linear functionals on X

and η_r, η_c are linear functionals on Y . Now

$$\begin{aligned}
& \sum_{j=1}^n \xi(x_j + ix'_j) \eta(y_j + iy'_j) \\
&= \sum_{j=1}^n \left\{ \xi_r(x_j) \eta_r(y_j) - \xi_r(x'_j) \eta_r(y'_j) \right\} + i \sum_{j=1}^n \left\{ \xi_r(x_j) \eta_c(y'_j) - \xi_r(x'_j) \eta_c(y'_j) \right\} \\
& \quad + i \sum_{j=1}^n \left\{ \xi_r(x_j) \eta_r(y'_j) + \xi_r(x'_j) \eta_c(y'_j) \right\} - \sum_{j=1}^n \left\{ \xi_r(x_j) \eta_c(y'_j) + \xi_r(x'_j) \eta_c(y'_j) \right\} \\
& \quad + i \sum_{j=1}^n \left\{ \xi_c(x_j) \eta_r(y_j) - \xi_c(x'_j) \eta_c(y_j) \right\} - \sum_{j=1}^n \left\{ \xi_c(x_j) \eta_r(y'_j) + \xi_c(x'_j) \eta_r(y_j) \right\} \\
& \quad - \sum_{j=1}^n \left\{ \xi_c(x_j) \eta_c(y_j) - \xi_c(x'_j) \eta_c(y'_j) \right\} - i \sum_{j=1}^n \left\{ \xi_c(x_j) \eta_c(y'_j) + \xi_c(x'_j) \eta_c(y'_j) \right\}.
\end{aligned}$$

Applying equations (2.4) or (2.5) to each of these summations, yields that $\sum_{j=1}^n \xi(x_j + ix'_j) \eta(y_j + iy'_j) = 0$, for arbitrary linear functionals ξ on $X_{\mathbb{C}}$ and η on $Y_{\mathbb{C}}$. Therefore $u = 0$.

We have shown that $T : X_{\mathbb{C}} \otimes Y_{\mathbb{C}} \rightarrow (X \otimes Y)_{\mathbb{C}}$ is a vector space isomorphism. \square

2.2 Complexification of the projective tensor product

In this section we will show an isometric relationship between $(X \hat{\otimes} Y)_{\beta}$ and $X_{\beta} \hat{\otimes} Y_{\beta}$, for any two real Banach spaces X and Y . Here $(\cdot)_{\beta}$ denotes the Bochnak complexification procedure that was described in chapter . Note that the space $(X \hat{\otimes} Y)_{\beta}$ involves the projective tensor norm of the real-scalar theory, while the space $X_{\beta} \hat{\otimes} Y_{\beta}$ involves the projective tensor norm of the complex-scalar theory.

Notation

We will use the notation $\ell_{\mathbb{R}}^1(K)$ for the Banach space of families $(x_k)_{k \in K}$ of real numbers for which $\sum_{k \in K} |x_k| < \infty$. The notation $\ell_{\mathbb{C}}^1(K)$ will denote the analogous space of families of complex numbers. If the choice of scalar is clear from the context, or does not matter, we will simply write $\ell^1(K)$. We remind the reader that K need not be countable, but of course for $x = (x_k) \in \ell^1(K)$, x_k will only be non-zero for a countable number of indices k .

Let's also use the notation, for any fixed x in some Banach space X , and with the index set $K = B_X$, of $\gamma_k^x = \begin{cases} \|x\| & \text{if } k = \frac{x}{\|x\|} \\ 0 & \text{otherwise} \end{cases}$. This defines a member $\gamma^x := (\gamma_k^x)_{k \in K}$ of $\ell^1(B_X)$.

Some preliminary results

For easy reference, we state a well known result from Banach space theory. It is discussed, for example, in [1, Appendix A].

Theorem 2.2.1. *For every Banach space X there is a metric surjection $q_X : \ell^1(B_X) \rightarrow X : (\lambda_x) \mapsto \sum_{x \in B_X} \lambda_x x$.*

Note that, for any $x_0 \in X$, we have that $q_X(\gamma^{x_0}) = x_0$.

The following result is implicitly proved in [4].

Theorem 2.2.2. *The mapping $(\ell_{\mathbb{R}}^1(K))_{\beta} \rightarrow \ell_{\mathbb{C}}^1(K) : (x_k)_{k \in K} + i(y_k)_{k \in K} \mapsto (x_k + iy_k)_{k \in K}$ is an isometric isomorphism.*

The (algebraic) tensor product can be defined by an *universal mapping property*, which states the existence of a unique linear map $T : X \otimes Y \rightarrow Z$ that linearizes a bilinear map $\phi : X \times Y \rightarrow Z$. When one puts a norm on the tensor product, it is natural to ask whether $\|T\| = \|\phi\|$. The projective tensor norm achieves this: (This theorem can be found, for example, in [2, Theorem 1.1.8] or in [6, Theorem 2.9].)

Theorem 2.2.3. For any Banach spaces X, Y and Z , the space $\mathcal{L}(X \hat{\otimes} Y; Z)$ of all bounded linear operators from $X \hat{\otimes} Y$ to Z is isometrically isomorphic to the space $\mathcal{B}(X, Y; Z)$ of all bounded bilinear transformations taking $X \times Y$ to Z . The natural correspondence establishing this isometric isomorphism is given by

$$v \in \mathcal{L}(X \hat{\otimes} Y; Z) \iff \phi \in \mathcal{B}(X, Y; Z)$$

via $v(x \otimes y) = \phi(x, y)$.

The next observation should be known, but we couldn't find a reference. It plays a key role in the proof of our main result.

Lemma 2.2.4. Let $Q_1 : W \rightarrow X$ and $Q_2 : W \rightarrow Y$ be two metric surjections between Banach spaces, and $T : X \rightarrow Y$ be such that $TQ_1 = Q_2$. Then $\|T\| \leq 1$.

$$\begin{array}{ccc} W & \xrightarrow{Q_2} & Y \\ Q_1 \downarrow & & \nearrow T \\ X & & \end{array}$$

Proof. Let $x \in X$ and $\epsilon > 0$ be given. Then there is a $w \in W$ so that $x = Q_1 w$, and $\|w\| \leq \|x\| + \epsilon$, since Q_1 is a metric surjection. Thus $\|Tx\| = \|TQ_1 w\| = \|Q_2 w\| \leq \|Q_2\| \|w\| \leq \|x\| + \epsilon$ because metric surjections have norm one. Since $\epsilon > 0$ is arbitrary, $\|T\| \leq 1$. \square

Theorem 2.2.5. Suppose $Q : X \rightarrow Y$ is a metric surjection between Banach spaces. Then $Q \oplus Q : X_\beta \rightarrow Y_\beta : x + i\tilde{x} \mapsto Qx + iQ\tilde{x}$ is a metric surjection.

Proof. The projective property of the tensor norm \wedge allows us to deduce from the fact that $Q : X \rightarrow Y$ is a metric surjection, that $Q \otimes id_{\mathbb{R}^2} : X \otimes \mathbb{R}^2 \rightarrow Y \otimes \mathbb{R}^2$ is a metric surjection.

But $(Q \otimes id_{\mathbb{R}^2})(x \otimes e_1 + \tilde{x} \otimes e_2) = Qx \otimes e_1 + Q\tilde{x} \otimes e_2 = \Phi((Qx, Q\tilde{x}))$, where Φ is the canonical isomorphism between $X \oplus X$ and $X \hat{\otimes} \mathbb{R}^2$. In this way the translation – from the tensor representation of $x + i\tilde{x}$ to the (x, \tilde{x})

representation – of the fact that $Q \otimes id_{\mathbb{R}^2}$ is a metric surjection, is that $Q \oplus Q$ is a metric surjection. \square

The following theorem is well known, but we couldn't find a reference to it in the form that is needed. So we derive it from [5, Theorem 7.2.3], which is described in the proof.

Theorem 2.2.6. *Let the scalar field, reals or complex numbers, be fixed. Let Γ and Θ be any two sets. Then $\ell^1(\Gamma) \hat{\otimes} \ell^1(\Theta)$ and $\ell^1(\Gamma \times \Theta)$ are isometrically isomorphic.*

Proof. It will be convenient to use the notation – for the purposes of this proof only – of $\ell^1_\Gamma(Y)$ to denote the Banach space of families $(\lambda_x)_{x \in \Gamma}$ of elements of the Banach space Y , indexed by Γ , for which $\|(\lambda_x)_{x \in \Gamma}\| := \sum_{x \in \Gamma} \|\lambda_x\|_Y$ is finite. In the case where Y is the space of scalars, we will simply write ℓ^1_Γ for $\ell^1_\Gamma(Y)$. With this notation, [5, Theorem 7.2.3] shows that

$$S : \ell^1_\Gamma \hat{\otimes} \ell^1_\Theta \rightarrow \ell^1_\Gamma(\ell^1_\Theta)$$

$$\sum_{i=1}^{\infty} (\kappa_{e,i})_{e \in \Gamma} \otimes (\lambda_{f,i})_{f \in \Theta} \mapsto \left(\left(\sum_{i=1}^{\infty} \kappa_{e,i} \lambda_{f,i} \right)_{f \in \Theta} \right)_{e \in \Gamma}$$

is an isometric isomorphism. (Recall that the projective tensor norm has the property that any $u \in X \hat{\otimes} Y$ is of the form $u = \sum_{i=1}^{\infty} x_i \otimes y_i$, where $x_i \in X$, $y_i \in Y$.) And we define

$$T : \ell^1_\Gamma(\ell^1_\Theta) \rightarrow \ell^1_{\Gamma \times \Theta}$$

$$\left(\left(\sum_{i=1}^{\infty} \kappa_{e,i} \lambda_{f,i} \right)_{f \in \Theta} \right)_{e \in \Gamma} \mapsto \left(\sum_{i=1}^{\infty} \kappa_{e,i} \lambda_{f,i} \right)_{e \in \Gamma, f \in \Theta}.$$

Then

$$\begin{aligned} \left\| \sum_{i=1}^{\infty} \kappa_{e,i} \lambda_{f,i} \right\|_{\ell_{\Gamma \times \Theta}^1} &= \sum_{e \in \Gamma, f \in \Theta} \left| \sum_{i=1}^{\infty} \kappa_{e,i} \lambda_{f,i} \right| \\ &= \sum_{e \in \Gamma} \left(\sum_{f \in \Theta} \left| \sum_{i=1}^{\infty} \kappa_{e,i} \lambda_{f,i} \right| \right) = \left\| \left(\sum_{i=1}^{\infty} \kappa_{e,i} \lambda_{f,i} \right)_{e \in \Gamma, f \in \Theta} \right\|_{\ell_{\Gamma}^1(\ell_{\Theta}^1)}, \end{aligned}$$

so T is isometric as well. Obviously T is a (linear) isomorphism, thus TS is an isometric isomorphism, as was to be proved. \square

Main results

Lemma 2.2.7. *Let K and K' be any two sets. Then there is an isometric isomorphism $\Phi : \ell_{\mathbb{R}}^1(K)_{\beta} \hat{\otimes} \ell_{\mathbb{R}}^1(K')_{\beta} \rightarrow (\ell_{\mathbb{R}}^1(K) \hat{\otimes} \ell_{\mathbb{R}}^1(K'))_{\beta}$, such that*

$$\begin{aligned} &\Phi \left([(\lambda_k)_{k \in K} + i(\tilde{\lambda}_k)_{k \in K}] \otimes [(\mu_m)_{m \in K'} + i(\tilde{\mu}_m)_{m \in K'}] \right) \\ &= [(\lambda_k)_{k \in K} \otimes (\mu_m)_{m \in K'} - (\tilde{\lambda}_k)_{k \in K} \otimes (\tilde{\mu}_m)_{m \in K'}] \\ &\quad + i[(\lambda_k)_{k \in K} \otimes (\tilde{\mu}_m)_{m \in K'} + (\tilde{\lambda}_k)_{k \in K} \otimes (\mu_m)_{m \in K'}]. \end{aligned}$$

Proof. We define Φ to be the following composition of canonical maps, each of which will be specified below,

$$\begin{aligned} &(\ell_{\mathbb{R}}^1(K))_{\beta} \hat{\otimes} (\ell_{\mathbb{R}}^1(K'))_{\beta} \\ &\rightarrow \ell_{\mathbb{C}}^1(K) \hat{\otimes} \ell_{\mathbb{C}}^1(K') \\ &\rightarrow \ell_{\mathbb{C}}^1(K \times K') \\ &\rightarrow (\ell_{\mathbb{R}}^1(K \times K'))_{\beta} \\ &\rightarrow (\ell_{\mathbb{R}}^1(K) \hat{\otimes} \ell_{\mathbb{R}}^1(K'))_{\beta}. \end{aligned}$$

Here the first mapping is $\phi \otimes \phi$, where $\phi : \ell_{\mathbb{R}}^1(K)_{\beta} \rightarrow \ell_{\mathbb{C}}^1(K)$ is the isometric isomorphism of Theorem (2.2.2). Since ϕ is an isometric isomorphism, $\phi \otimes \phi$ is an isometric isomorphism, and it maps $[(\lambda_k)_{k \in K} + i(\tilde{\lambda}_k)_{k \in K}] \otimes [(\mu_m)_{m \in K'} + i(\tilde{\mu}_m)_{m \in K'}]$ to $[(\lambda_k + i\tilde{\lambda}_k)_{k \in K}] \otimes [(\mu_m + i\tilde{\mu}_m)_{m \in K'}]$.

The second mapping, that is from $\ell_{\mathbb{C}}^1(K) \hat{\otimes} \ell_{\mathbb{C}}^1(K')$ to $\ell_{\mathbb{C}}^1(K \times K')$, is defined to be the isometric isomorphism of Theorem (2.2.6). The isomorphism that is used in the proof of that theorem, maps

$$\begin{aligned} & [(\lambda_k + i\tilde{\lambda}_k)_{k \in K}] \otimes [(\mu_m + i\tilde{\mu}_m)_{m \in K'}] \\ \text{to } & [(\lambda_k + i\tilde{\lambda}_k) \cdot (\mu_m + i\tilde{\mu}_m)]_{k \in K, m \in K'} \\ & = [(\lambda_k \mu_m - \tilde{\lambda}_k \tilde{\mu}_m) + i(\lambda_k \tilde{\mu}_m + \tilde{\lambda}_k \mu_m)]_{k \in K, m \in K'}. \end{aligned}$$

The third mapping is the inverse of the isometric isomorphism described by Theorem (2.2.2).

The fourth mapping is defined to be

$$\psi \oplus \psi : (\ell_{\mathbb{R}}^1(K \times K'))_{\beta} \rightarrow (\ell_{\mathbb{R}}^1(K) \hat{\otimes} \ell_{\mathbb{R}}^1(K'))_{\beta},$$

where $\psi : \ell_{\mathbb{R}}^1(K \times K') \rightarrow \ell_{\mathbb{R}}^1(K) \hat{\otimes} \ell_{\mathbb{R}}^1(K')$ is the inverse of the isometric isomorphism that is described in Theorem (2.2.6). (Of course we use the real-scalar variant of that theorem.) Thus $\psi \oplus \psi$ will map

$$\begin{aligned} & [\lambda_k \mu_m - \tilde{\lambda}_k \tilde{\mu}_m]_{k \in K, m \in K'} + i[\lambda_k \tilde{\mu}_m + \tilde{\lambda}_k \mu_m]_{k \in K, m \in K'} \\ \text{to } & [(\lambda_k)_{k \in K} \otimes (\mu_m)_{m \in K'} - (\tilde{\lambda}_k)_{k \in K} \otimes (\tilde{\mu}_m)_{m \in K'}] \\ & + i[(\lambda_k)_{k \in K} \otimes (\tilde{\mu}_m)_{m \in K'} + (\tilde{\lambda}_k)_{k \in K} \otimes (\mu_m)_{m \in K'}]. \end{aligned}$$

Therefore Φ , being the composition of (complex-linear) isometric isomorphisms, is itself an isometric isomorphism. \square

Theorem 2.2.8. *Let X and Y be real Banach spaces. Then $X_{\beta} \hat{\otimes} Y_{\beta}$ is isometrically isomorphic to $(X \hat{\otimes} Y)_{\beta}$.*

Proof. Recall the metric surjection of Theorem (2.2.1), $q_X : \ell_{\mathbb{R}}^1(B_X) \rightarrow X : (\lambda_x) \mapsto \sum_{x \in B_X} \lambda_x x$. We use the notation q_Y for the analogous metric surjection $\ell^1(B_Y) \rightarrow Y$.

By Theorem (2.2.5), $q_X \oplus q_X : \ell_{\mathbb{R}}^1(B_X)_{\beta} \rightarrow X_{\beta}$ is a metric surjection.

Similarly, $q_Y \oplus q_Y : \ell_{\mathbb{R}}^1(B_Y)_\beta \rightarrow Y_\beta$ is a metric surjection as well. By the projective property of the projective tensor norm, the operator

$$(q_X \oplus q_X) \otimes (q_Y \oplus q_Y) : \ell_{\mathbb{R}}^1(B_X)_\beta \hat{\otimes} \ell_{\mathbb{R}}^1(B_Y)_\beta \rightarrow X_\beta \hat{\otimes} Y_\beta$$

is a metric surjection. We'll refer to this operator as Q_1 .

By the projective property of the projective tensor norm again, the operator $q_X \otimes q_Y : \ell_{\mathbb{R}}^1(B_X) \hat{\otimes} \ell_{\mathbb{R}}^1(B_Y) \rightarrow X \hat{\otimes} Y$ is a metric surjection too. Using Theorem (2.2.5) again,

$$(q_X \otimes q_Y) \oplus (q_X \otimes q_Y) : (\ell_{\mathbb{R}}^1(B_X) \hat{\otimes} \ell_{\mathbb{R}}^1(B_Y))_\beta \rightarrow (X \hat{\otimes} Y)_\beta$$

is then a metric surjection.

Recall now the map $\Phi : \ell_{\mathbb{R}}^1(K)_\beta \hat{\otimes} \ell_{\mathbb{R}}^1(K')_\beta \rightarrow (\ell_{\mathbb{R}}^1(K) \hat{\otimes} \ell_{\mathbb{R}}^1(K'))_\beta$ of Lemma (2.2.7). We set $K = B_X$ and $K' = B_Y$. It will be convenient to denote by Q_2 the composition $((q_X \otimes q_Y) \oplus (q_X \otimes q_Y)) \circ \Phi : \ell_{\mathbb{R}}^1(B_X)_\beta \hat{\otimes} \ell_{\mathbb{R}}^1(B_Y)_\beta \rightarrow (X \hat{\otimes} Y)_\beta$.

$$\begin{array}{ccc} \ell_{\mathbb{R}}^1(B_X)_\beta \hat{\otimes} \ell_{\mathbb{R}}^1(B_Y)_\beta & \xrightarrow{Q_2} & (X \hat{\otimes} Y)_\beta \\ \downarrow Q_1 & \nearrow T & \\ X_\beta \hat{\otimes} Y_\beta & & \end{array}$$

We define a linear map $T : X_\beta \hat{\otimes} Y_\beta \rightarrow (X \hat{\otimes} Y)_\beta$ in the following way: Suppose $B : X_\beta \times Y_\beta \rightarrow (X \hat{\otimes} Y)_\beta$ is the bilinear map $(x + i\tilde{x}, y + i\tilde{y}) \mapsto (x \otimes y - \tilde{x} \otimes \tilde{y}) + i(x \otimes \tilde{y} + \tilde{x} \otimes y)$. Then B is bounded. In fact, use of the triangle inequality, and that $\|u + i0\|_{E_\beta} = \|u\|_E$ holds for any member u of any Banach space E , shows that $\|B\| \leq 4$. The universal mapping property of the projective tensor norm, Theorem 2.2.3, guarantees the existence of a boundend linear map $T : X_\beta \hat{\otimes} Y_\beta \rightarrow (X \hat{\otimes} Y)_\beta$ so that

$$T((x + i\tilde{x}) \otimes (y + i\tilde{y})) = (x \otimes y - \tilde{x} \otimes \tilde{y}) + i(x \otimes \tilde{y} + \tilde{x} \otimes y).$$

We claim that $TQ_1 = Q_2$. Indeed, an elementary (that is, rank-one) tensor $v \in \ell_{\mathbb{R}}^1(B_X)_\beta \hat{\otimes} \ell_{\mathbb{R}}^1(B_Y)_\beta$ is of the form $v = [(\lambda_x)_{x \in B_X} + i(\tilde{\lambda}_x)_{x \in B_X}] \otimes [(\mu_y)_{y \in B_Y} + i(\tilde{\mu}_y)_{y \in B_Y}]$. The projective norm allows us then to represent any $u \in \ell_{\mathbb{R}}^1(B_X)_\beta \hat{\otimes} \ell_{\mathbb{R}}^1(B_Y)_\beta$ as

$$u = \sum_{j=1}^{\infty} [(\lambda_{x,j})_{x \in B_X} + i(\tilde{\lambda}_{x,j})_{x \in B_X}] \otimes [(\mu_{y,j})_{y \in B_Y} + i(\tilde{\mu}_{y,j})_{y \in B_Y}],$$

for some sequences of ℓ^1 -families $(\lambda_{x,j})_{x \in B_X}, (\tilde{\lambda}_{x,j})_{x \in B_X} \in \ell_{\mathbb{R}}^1(B_X)$ and $(\mu_{y,j})_{y \in B_Y}, (\tilde{\mu}_{y,j})_{y \in B_Y} \in \ell_{\mathbb{R}}^1(B_Y)$. Now Q_1 is continuous, so maps convergent series to convergent series. Therefore $Q_1 u = \sum_j [q_X((\lambda_{x,j})_{x \in B_X}) + iq_X((\tilde{\lambda}_{x,j})_{x \in B_X})] \otimes [q_Y((\mu_{y,j})_{y \in B_Y}) + iq_Y((\tilde{\mu}_{y,j})_{y \in B_Y})]$. Thus we get that

$$\begin{aligned} TQ_1 u = \sum_j \left\{ [q_X((\lambda_{x,j})_{x \in B_X}) \otimes q_Y((\mu_{y,j})_{y \in B_Y}) - q_X((\tilde{\lambda}_{x,j})_{x \in B_X}) \otimes q_Y((\tilde{\mu}_{y,j})_{y \in B_Y})] \right. \\ \left. + i[q_X((\lambda_{x,j})_{x \in B_X}) \otimes q_Y((\tilde{\mu}_{y,j})_{y \in B_Y}) + q_X((\tilde{\lambda}_{x,j})_{x \in B_X}) \otimes q_Y((\mu_{y,j})_{y \in B_Y})] \right\}. \end{aligned}$$

It is now routine to check, with the help of Lemma 2.2.7, that $Q_2 u$ is the same thing, so

$$TQ_1 = Q_2. \quad (2.6)$$

Thus, Lemma 2.2.4 can be applied. This shows that $\|T\| \leq 1$.

Now we want to show that T is invertible. Let $u \in X_\beta \hat{\otimes} Y_\beta$ so that $Tu = 0$, say $u = \sum_{k=1}^{\infty} (x_k + i\tilde{x}_k) \hat{\otimes} (y_k + i\tilde{y}_k)$. To prove what is needed to show that T is invertible, namely to prove that $u = 0$, let $\phi \in (X_\beta \hat{\otimes} Y_\beta)^*$. Then, using the universal property of the projective norm again there is a unique bounded bilinear map $D : X_\beta \times Y_\beta \rightarrow \mathbb{C}$ so that $D(x_k + i\tilde{x}_k, y_k + \tilde{y}_k) = \phi((x_k + i\tilde{x}_k) \otimes (y_k + \tilde{y}_k))$.

Next, consider the functional

$$\psi : (X \hat{\otimes} Y)_\beta \rightarrow \mathbb{C} : \sum_{k=1}^{\infty} (a_k + i\tilde{a}_k \otimes (b_k + i\tilde{b}_k) \mapsto \sum_{k=1}^{\infty} D(a_k + i\tilde{a}_k, b_k + i\tilde{b}_k).$$

The fact that D is bounded implies that the series on the RHS is convergent.

Now $\psi Tu = \phi u$. But $Tu = 0$, so $\phi u = 0$. Since ϕ is an arbitrary element of the dual, $u = 0$. Since u is arbitrary, T is invertible.

It follows from the bounded inverse theorem that T^{-1} is bounded.

So from equation (2.6), we can deduce that $Q_1 = T^{-1}Q_2$. Thus Lemma 2.2.4 can be applied again, this time to, in order, Q_2 , Q_1 and T^{-1} . We get that $\|T^{-1}\| \leq 1$.

Since $\|T\| \leq 1$ and $\|T^{-1}\| \leq 1$, we conclude that T is isometric.

We have shown that $T : X_\beta \hat{\otimes} Y_\beta \rightarrow (X \hat{\otimes} Y)_\beta$ is an isometric isomorphism.

□

2.3 Complexification of the injective tensor product

In this section we show that the main result of the previous section has an analogue when the projective norm is replaced with the injective norm, and the Bocnak complexification procedure is replaced with the Taylor complexification procedure that was discussed in chapter 1.

Let X, Y be real Banach spaces. Since $X_{\mathbb{C}} \otimes Y_{\mathbb{C}}$ is isomorphic, as a complex vector space, to $(X \otimes Y)_{\mathbb{C}}$, the normed vector space $((X_{\mathbb{C}}, \tau) \otimes ((Y_{\mathbb{C}}, \tau), \vee)$ is isomorphic, as a vector space, to $((X \otimes Y, \vee), \tau)$. (Here $(X_{\mathbb{C}}, \tau)$ means the vector space $X_{\mathbb{C}}$ with the above-mentioned Taylor complexification norm, we mention again that it is a complete normed space under the complexification norm; $(\square \otimes \square, \vee)$ means $\square \otimes \square$ endowed with the injective tensor norm, it is

not necessarily complete; and so on.) The next result will show that they are also isomorphic, in fact isometrically isomorphic, as normed vector spaces.

Theorem 2.3.1. *Let X and Y be real Banach spaces. Let $u \in X_\tau \otimes_\vee Y_\tau$. Then $\|u\|_{X_\tau \otimes_\vee Y_\tau} = \|u\|_{(X \otimes_\vee Y)_\tau}$.*

Proof. On the one hand - defining $Z_X := \{\hat{x}(\cdot) | x \in X\}$ as the usual isometric imbedding of X into a subspace of $C(B_{X^*})$ and defining Z_Y similarly - we have $X \stackrel{1}{=} Z_X$ and $Y \stackrel{1}{=} Z_Y$. (Isometric isomorphisms.)

By injectivity of \otimes_\vee , we find $Z_X \otimes_\vee Z_Y$ is a subspace of $C(B_{X^*}) \otimes_\vee C(B_{Y^*})$ and thus isometrically isomorphic to a subspace of $C(B_{X^*} \times B_{Y^*})$.

Now the injectivity of the Taylor complexification procedure (see [4, p. 14]) can be applied to get

$$\begin{aligned} (Z_X \otimes_\vee Z_Y)_\tau &\stackrel{1}{=} \text{a subspace of } C(\hat{B}_{X^*} \times \hat{B}_{Y^*})_\tau \\ &\stackrel{1}{=} \text{a subspace of } C_{\mathbb{C}}(B_{X^*} \times B_{Y^*}). \end{aligned}$$

Following all these isometries, we get that

$$\begin{aligned} B : (X \otimes_\vee Y)_\tau &\rightarrow C_{\mathbb{C}}(B_{X^*} \times B_{Y^*}) \\ \sum_{j=1}^n (e_j \otimes f_j + i(g_j \otimes h_j)) &\mapsto \sum_{j=1}^n (\hat{e}_j(\cdot) \hat{f}_j(\cdot) + i(\hat{g}_j(\cdot) \hat{h}_j(\cdot))) \end{aligned}$$

is isometric.

On the other hand $X \stackrel{1}{=} Z_X$ implies $X_\tau \stackrel{1}{=} (Z_X)_\tau$, same with Y , and now the injectivity of the Taylor procedure and then the injectivity of \otimes_\vee yield

$$\begin{aligned} X_\tau \otimes_\vee Y_\tau &\stackrel{1}{=} (Z_X)_\tau \otimes_\vee (Z_Y)_\tau \\ &\stackrel{1}{=} \text{a subspace of } C(B_{X^*})_\tau \otimes_\vee C(B_{Y^*})_\tau \\ &\stackrel{1}{=} \text{a subspace of } C_{\mathbb{C}}(B_{X^*}) \otimes_\vee C_{\mathbb{C}}(B_{Y^*}) \\ &\stackrel{1}{=} \text{a subspace of } C_{\mathbb{C}}(B_{X^*} \times B_{Y^*}). \end{aligned}$$

The canonical isometries combined in the above set of statements imply that the mapping

$$D : X_\tau \otimes_\vee Y_\tau \rightarrow C_{\mathbb{C}}(B_{X^*} \times B_{Y^*})$$

$$\sum_{j=1}^n (a_j + ib_j) \otimes (c_j + id_j) \mapsto \sum_{j=1}^n (\hat{a}_j(\cdot) + \hat{b}_j(\cdot))(\hat{c}_j(\cdot) + i\hat{d}_j(\cdot))$$

is isometric.

Let $u \in X_\tau \otimes_\vee Y_\tau = (X \otimes_\vee Y)_\tau$ be given. It is easy to verify that both B and D are complex-linear mappings, and that for any $x \otimes y$ in the vector space generating set $X \otimes Y$ it holds that $B(x \otimes y) = \hat{x}(\cdot)\hat{y}(\cdot) = D(x \otimes y)$; therefore, $B(u) = D(u)$.

Now because $B(u) = D(u)$ and both B and D are isometries, we get that $\|u\|_{X_\tau \otimes_\vee Y_\tau} = \|D(u)\| = \|B(u)\| = \|u\|_{(X \otimes_\vee Y)_\tau}$. \square

Since $((X_{\mathbb{C}}, \tau) \otimes ((Y_{\mathbb{C}}, \tau), \vee)$ is isomorphic, as a normed vector space, to $((X \otimes Y, \vee), \tau)$, we can take completions to get the following corollary.

Corollary 2.3.2. *Let X, Y be real Banach spaces. The spaces $(X \overset{\vee}{\otimes} Y)_\tau$ and $X_\tau \overset{\vee}{\otimes} Y_\tau$ are isometrically isomorphic.*

This corollary is analogous to Theorem 2.2.8.

Chapter 3

An injective relationship between the real and the complex tensor norm equivalence classes

How do the real tensor norms fit in with the complex tensor norms? Does every real tensor norm have a complex counterpart? Or are there “fewer” complex norms, perhaps because the conditions in the complex-scalar world are more stringent? All the examples of real tensor norms that one thinks of, such as the projective tensor norm, or the hilbertian tensor norm, have complex analogues. This chapter shows that that is to be expected, at least on the level of topological equivalence classes of (finitely generated) tensor norms.

The motivation for this chapter is the few paragraphs of the *Résumé* [3, p 19] that A. Grothendieck uses to discuss the relation between real and complex tensor norms.

He starts by saying that the relation on the isometric level is not as simple as it appears, because the integral norm of even the identity operator on

a finite-dimensional Banach space depends on whether one uses the real or complex scalars. Grothendieck then asserts that the situation is much clearer when one considers the topological equivalence classes of tensor norms. (Remember that a “tensor norm” such as for example the injective tensor norm, is actually a family of norms in the conventional sense, the family being indexed by the collection of tensor products of Banach spaces. Two tensor norms are equivalent when they are equivalent on every tensor product of Banach spaces. By an “infinite L^p -sum” argument, it turns out that this happens if and only if there are equivalence constants that are independent of the Banach spaces that appear in the tensor product to which the norms are applied.) Then, to continue paraphrasing Grothendieck, there will be an injective correspondence, between the family of real α -, and the family of complex tensor norm equivalence classes. This correspondence can be constructed using classes of α -integral operators.

Grothendieck doesn’t prove this statement, and the reader should keep in mind that there are very few of the statements that appear in the *Résumé* are given explicit proofs. (See, however the book [2].) In this chapter we construct such a correspondence. On the one hand our construction uses results of complexification [4], which appeared decades after the *Résumé*. On the other hand, we use finite-dimensional normed operator ideals, which can be shown to be in one-to-one correspondence with Grothendieck’s α -integral operators. Thus, the construction as presented in this chapter is probably the one that Grothendieck had in mind.

The plan, broadly speaking, is this (definitions will be given in the main text): We start with a real tensor norm α . We can associate to it in a canonical way a unique real finite-dimensional operator ideal. Then we show that one can complexify any real finite-dimensional operator ideal. Using the one-to-one correspondence between finite-dimensional operator ideals and tensor norms, this leads to a complex tensor norm. Using a similar route via finite-dimensional operator ideals, one can not only complexify a real tensor

norm, but also define the real version of a complex tensor norm. We then show that complexifying a real tensor norm and then taking the real part gives a real tensor norm that is always equivalent to the original real tensor norm. Then, it is not difficult to deduce that complexifying a topological equivalence class gives a well-defined complex topological equivalence class, in an injective way.

3.1 Finite dimensional operator ideal norms and tensor norms

(This section makes heavy use of definitions from the research monograph [1], especially section 17 on the representation of maximal operator ideals.)

Definition 3.1.1. [1, Section 9.1] *An operator ideal \mathcal{A} is a subclass of the class \mathcal{L} of all continuous linear operators between Banach spaces such that for all Banach spaces E and F its components*

$$\mathcal{A}(E, F) := \mathcal{L}(E, F) \cap \mathcal{A}$$

satisfy

1. $\mathcal{A}(E, F)$ is a linear subspace of $\mathcal{L}(E, F)$ which contains the finite rank operators.
2. If $S \in \mathcal{A}(E_0, F_0)$, $R \in \mathcal{L}(E, E_0)$ and $T \in \mathcal{L}(F_0, F)$ then the composition TSR is in $\mathcal{A}(E, F)$.

Definition 3.1.2. *A normed operator ideal (\mathcal{A}, A) is an operator ideal \mathcal{A} together with a function $A : \mathcal{A} \rightarrow [0, \infty)$ such that*

1. $A|_{\mathcal{A}(E, F)}$ is a norm for all Banach spaces E and F .
2. $A(id_{\mathbb{K}}) = 1$.

3. If $S \in \mathcal{A}(E_0, F_0)$, $T \in \mathcal{L}(F_0, F)$ and $R \in \mathcal{L}(E, E_0)$, then the composition satisfies $A(TSR) \leq \|T\|A(S)\|R\|$.

If all the components $\mathcal{A}(E, F)$ are complete with respect to A then (\mathcal{A}, A) is called a Banach operator ideal.

For our purposes we only need to consider the ideal norm A , and then only need to know its value for finite dimensional Banach spaces. A finite dimensional operator ideal norm (abbreviated FDOI norm) is a norm that satisfies all the conditions of the definition above when all the Banach spaces concerned are finite-dimensional. There are several procedures to extend FDOI norms to Banach operator ideals. One of these procedures produces the so-called *maximal* Banach operator ideals [1, section 17.2]. (This turns out to be the ideals of α -integral operators. The point is that both tensor norms and maximal Banach operator ideals are uniquely determined by their behaviour on finite-dimensional Banach spaces.)

What is the relationship with tensor norms?

On the one hand, if (\mathcal{A}, A) is a normed operator ideal, then given a tensor $u = \sum_{k=1}^n x_k \otimes y_k \in X \otimes Y$ for finite-dimensional Banach spaces X, Y we can associate to it an operator $T_u : X^* \rightarrow Y$. Namely

$$T_u(x^*) := \sum_{k=1}^n x^*(x_k)y_k.$$

Then it can be shown that the definition of α by $\alpha(u) := A(T_u)$ satisfies all the properties of a tensor norm on finite-dimensional spaces. And since a tensor norm is finitely generated, α can be extended to a tensor norm.

On the other hand, let α be a tensor norm. Then, for finite-dimensional Banach spaces X, Y a norm A on $\mathcal{L}(X, Y)$ can be defined by

$$A(T : X \rightarrow Y) := \alpha(u_T),$$

where $u_T \in X^* \otimes Y$ is the tensor that is associated to T in the vector space

isomorphism $X^* \otimes Y = \mathcal{L}(X, Y)$. (Valid at least when X and Y are finite-dimensional vector spaces.)

It is not very difficult to show that in this way a one-to-one correspondence between (finitely generated) tensor norms α and maximal Banach operator ideals (\mathcal{A}, A) – and hence FDOI norms A – is found.

In what follows we will only be concerned with the FDOI norms.

3.2 Complexification of real finite dimensional operator ideals

Let N be a real FDOI norm. Let $T : X \rightarrow Y$ be a complex operator. Then T is also a real operator, by forgetting about multiplication by complex scalars. In the same way, the operator $A : X \rightarrow X : x \mapsto ix$ can be considered as a real operator on X_0 , where X_0 is the notation for X considered as a real Banach space. (Similarly Y_0 denotes the space Y considered as a real Banach space.)

When we have to indicate the Banach space that we are working with, we will write A_X for the operator $X \rightarrow X : x \mapsto ix$, considered a operator between the real spaces X_0 and Y_0 . Note that $TA_X = A_Y T$.

The following definition is reminiscent of the Taylor complexification procedure for real Banach spaces.

Definition 3.2.1. *Let N be a real FDOI norm, and $T : X \rightarrow Y$ a complex operator. Then we define*

$$N_c(T) := \frac{1}{\rho_N} \sup_{0 \leq \theta \leq 2\pi} \{N(T_\theta)\},$$

where $T_\theta : X_0 \rightarrow Y_0 : x \mapsto \cos \theta T x + \sin \theta T A x$, and

$$\rho_N := \sup_{0 \leq \theta \leq 2\pi} N(\cos \theta id_X + \sin \theta A).$$

Theorem 3.2.2. *If N is a real FDOI norm, then N_c is a complex FDOI norm.*

Proof. We first prove that N_c is a norm, when complex Banach spaces X and Y are fixed. That $N_c(S + T) \leq N_c(S) + N_c(T)$, follows from a standard property of sup's. The other properties are also easy to check. We prove the property $N_c(\lambda T) = |\lambda|N_c(T)$. So let $\lambda = \lambda_r + i\lambda_c \in \mathbb{C}$ be given. We will use the identity $\sup_{0 \leq \theta \leq 2\pi} \{a \cos \theta + b \sin \theta\} = \sqrt{a^2 + b^2}$ for reals a and b . Then

$$\begin{aligned}
N_c(\lambda T) &= \frac{1}{\rho_N} \sup_{\theta} N((\lambda_r + \lambda_c A)(\cos \theta T + \sin \theta T A)) \\
&= \frac{1}{\rho_N} \sup_{\theta} N((\lambda_r T + \lambda_c T A) \cos \theta + (\lambda_r T A - \lambda_c T) \sin \theta) \\
&= \frac{1}{\rho_N} \sup_{\theta, \phi \in (\mathcal{N}, \mathcal{N})^*} |\phi(\lambda_r T + \lambda_c T A) \cos \theta + \phi(\lambda_r T A - \lambda_c T) \sin \theta| \\
&= \frac{1}{\rho_N} \sup_{\phi} \sqrt{(\lambda_r \phi(T) + \lambda_c \phi(T A))^2 + (\lambda_r \phi(T A) - \lambda_c \phi(T))^2} \\
&= \frac{1}{\rho_N} \sqrt{\lambda_r^2 + \lambda_c^2} \sup_{\phi} \sqrt{\phi(T)^2 + \phi(T A)^2} \\
&= |\lambda| \frac{1}{\rho_N} \sup_{\theta, \phi} \{\phi(T) \cos \theta + \phi(T A) \sin \theta\} \\
&= |\lambda| N_c(T).
\end{aligned}$$

Thus N_c defines a norm whenever complex finite-dimensional Banach spaces X, Y are fixed. To prove the ideal properties, note that

$$N_c(id_X) = \frac{1}{\rho_N} \sup_{\theta} N(\cos \theta id_X + \sin \theta A) = \frac{1}{\rho_N} \rho_N = 1.$$

Finally, let $S : Y \rightarrow Z$ and $U : W \rightarrow Z$ be any complex operators. Then

$$\begin{aligned}
N_c(STU) &= \sup_{\theta} N(\cos\theta STU + \sin\theta STUA_Z) \\
&= \sup_{\theta} N(ST(\cos\theta id_X + \sin\theta A_X)U) \\
&\leq \|S\| \|U\| \sup_{\theta} N(T(\cos\theta id_X + \sin\theta A_X)),
\end{aligned}$$

using the ideal properties of N . Now, in principle we have to be careful whether the norm $\|S\|$ refers to the real or the complex operator norm. (That is, whether S is considered as a real or a complex operator.) It can be checked that it does not matter. So we have shown that $N_c(STU) \leq \|S\| \|U\| N_c(T)$. \square

3.3 The real version of a complex finite-dimensional operator ideal norm

Let a complex FDOI norm C be given. Suppose $T : X \rightarrow Y$ is a real operator. Then T can be extended to the complexification

$$\tilde{T} : X_{\mathbb{C}} \rightarrow Y_{\mathbb{C}} : x + i\tilde{x} \mapsto Tx + iT\tilde{x}.$$

(This extension is the unique complex-linear extension.) We will need to know how the norm of \tilde{L} is related to the norm of L . (Of course, it depends on how the Banach spaces X and Y are complexified, too.) This has been studied in the reference cited below.

Definition 3.3.1. [4, p. 7] *A natural complexification procedure ν is a way of defining a reasonable complexification norm $\|\cdot\|_{\nu}$ on the complexification \tilde{E} of any real Banach space which has the property that if E, F are real Banach spaces and $L \in \mathcal{L}(E, F)$, then the complex-linear extension $\tilde{L} : (\tilde{E}, \|\cdot\|_{\nu}) \rightarrow (\tilde{F}, \|\cdot\|_{\nu})$ has the same norm as L .*

Theorem 3.3.2. [4, Proposition 9] *Let E be a real Banach space. If $\tilde{E} = E \otimes_{\alpha} \mathbb{R}^2$ where α is a tensor norm, then it is a natural complexification of E .*

(The reference uses ℓ_2^2 instead of \mathbb{R}^2 , meaning two-dimensional Hilbert space, but this is not a real difference because we assume that \mathbb{R}^2 has the Euclidean norm.)

Because the Taylor complexification procedure is of the form as required above, with $\alpha = \vee$, the Taylor procedure is a natural complexification procedure. (In fact, it is for this chapter not important which natural complexification procedure to use on the Banach spaces, because all the reasonable – see section 1.1 – complexification norms are topologically equivalent [4, Proposition 3].)

So we have that $\|\tilde{T} : X_{\tau} \rightarrow Y_{\tau}\| = \|T : X \rightarrow Y\|$; and whenever we use the notation \tilde{T} , we imply that we are using the Taylor complexification procedure.

Definition 3.3.3. *Let C be any complex FDOI norm, $T : X \rightarrow Y$ any real operator, and $\tilde{T} : X_{\tau} \rightarrow Y_{\tau}$ its extension to the complexifications of X and Y . Then we define*

$$C_r(T) = C(\tilde{T}).$$

It is easy to see that C_r , when specialized to $\mathcal{L}(X, Y)$, is a norm. Is it a real FDOI norm? We check that

$$C_r(id_X) = C(id_{X_{\tau}}) = 1$$

using the fact that C is a FDOI norm.

Furthermore, using the definition of C_r , the ideal property of C and the fact that the Taylor complexification procedure is a natural complexification

procedure, we get

$$\begin{aligned}
C_r(RST) &= C(\widetilde{RST}) \\
&= C(\widetilde{R}\widetilde{S}\widetilde{T}) \\
&\leq \|\widetilde{R}\|C(\widetilde{S})\|\widetilde{T}\| \\
&= \|R\|C_r(S)\|T|.
\end{aligned}$$

Thus, if C is a complex FDOI norm, then C_r is a real FDOI norm.

3.4 Preservation of topological equivalence

Lemma 3.4.1. *Let N be a real FDOI norm. Then $N_{cr} \leq \frac{4}{\rho_N}N \leq 4N_{cr}$. That is, for any real finite-rank operator T ,*

$$(N_c)_r(T) \leq \frac{4}{\rho_N}N(T) \leq 4(N_c)_r(T).$$

Proof. Let $T : X \rightarrow Y$ be a real finite-rank operator.

Since we are dealing with complexifications and then taking real parts again, we use notation, as follows. We write $\widetilde{T} : X_\tau \rightarrow Y_\tau$, for the unique complex-linear extension of a an operator between real Banach spaces to an operator between the complexifications. And given a complex operator $S : W \rightarrow Z : w \mapsto S(w)$, we write S_0 for the operator $W_0 \rightarrow Z_0 : w \mapsto S(w)$ where W_0 is the real Banach space obtained by considering W as a real Banach space (forgetting about complex scalar multiplication). (Similarly Z_0 is Z considered as a real Banach space.)

To prove the first inequality, we write

$$\begin{aligned}
N_{cr}(T) &= N_c(\tilde{T}) \\
&= \frac{1}{\rho_N} \sup_{\theta} \{N(x \mapsto \cos \theta \tilde{T}x + \sin \theta \tilde{T}Ax)\} \\
&\leq \frac{1}{\rho_N} \sup_{\theta} \{N(x \mapsto \cos \theta \tilde{T}x)\} + \frac{1}{\rho_N} \sup_{\theta} \{N(x \mapsto \sin \theta \tilde{T}Ax)\} \\
&= \frac{1}{\rho_N} N(\tilde{T}_0) + \frac{1}{\rho_N} N(\tilde{T}_0 A) \\
&\leq \frac{1}{\rho_N} N(\tilde{T}_0) + \frac{1}{\rho_N} N(\tilde{T}_0) \|A\|
\end{aligned}$$

due to ideal properties,

$$\begin{aligned}
&= \frac{2}{\rho_N} N(\tilde{T}_0) \quad \text{since } \|A\| = 1 \\
&\leq \frac{4}{\rho_N} N(T),
\end{aligned}$$

using in the last step ideal properties and the fact that \tilde{T} can be expressed as $T \oplus T : X \oplus X \rightarrow Y \oplus Y$.

To prove the second inequality, note that

$$\begin{aligned}
N_{cr}(T) &\geq \frac{1}{\rho_N} N(x \mapsto \tilde{T}x) \\
&= \frac{1}{\rho_N} N(x \mapsto Tx) \\
&= \frac{1}{\rho_N} N(T).
\end{aligned}$$

□

Let us use the notation $N \sim M$ to indicate that FDOI norms N and M are equivalent.

Let N, M be real FDOI norms. It is easy to check that $N \sim M \Rightarrow N_c \sim M_c \Rightarrow N_{cr} \sim M_{cr}$. However, it follows from what we have just shown in Lemma 3.4.1 that $N \sim N_{cr}$ and $M \sim M_{cr}$.

So we deduce ...

Theorem 3.4.2. *Let N and M be any two FDOI norms. Then*

$$N \sim M \Leftrightarrow N_c \sim M_c.$$

Proof. $N \sim M \Rightarrow N_c \sim M_c \Rightarrow N_{cr} \sim M_{cr} \Rightarrow N \sim M_{cr} \Rightarrow N \sim M.$ □

We can use the terminology that the *complexification of an equivalence class* is the class generated by the complexification of any one of its members. We have shown that the complexification of tensor norm equivalence classes have the injective property, namely that it is impossible for two different equivalence classes to have the same complexification. This is what we wanted to show in this chapter.

This leads to the question whether this correspondence between real and complex equivalence classes is not also onto. The analogue of this question, in the setting of (not necessarily normed) operator ideals, has been answered in the negative ([8], [9]). We do not see a reason why the answer for tensor norms cannot be negative too.

Appendix A

Tensor norms

The main purpose of this appendix is to collect some definitions about tensor norms, for the sake of fixing terminology. It is highly unlikely that a reader unfamiliar with tensor norms will grasp these concepts from reading this appendix alone, since we do not try to put the definitions into context. The original reference is Grothendieck's *Résumé* [3]. A recent exposition including detailed proofs of virtually all the results of the *Résumé*, can be found in the book [2]. An extensive research monograph on this topic, as well as the interplay with operator ideals, is [1]. We also refer to the introduction to tensor products of, and tensor norms on, Banach spaces [6].

Let X, Y be Banach spaces, both over the same scalar field \mathbb{R} or \mathbb{C} . Recall that every $u \in X \otimes Y$ has a representation $u = \sum_{j=1}^n x_j \otimes y_j$ for some $\{x_j\}_{j=1}^n \subseteq X$ and $\{y_j\}_{j=1}^n \subseteq Y$. Let us consider a norm $\alpha_{X,Y}$ on $X \otimes Y$. We require that $\alpha_{X,Y}(x \otimes y) = \|x\|_X \|y\|_Y$ for any $x \in X$ and $y \in Y$. We also require that for every $x^* \in X^*$ and $y^* \in Y^*$, the linear functional $x^* \otimes y^* : \sum_{j=1}^n x_j \otimes y_j \mapsto \sum_{j=1}^n x^*(x_j) y^*(y_j)$ on $X \otimes Y$ is bounded, with $\|x^* \otimes y^*\| = \|x^*\|_{X^*} \|y^*\|_{Y^*}$.

Norms on tensor products satisfying these two requirements are called *reasonable crossnorms*.

It is customary to drop the superscripts, and to write $\alpha(u)$ or $\|u\|_\alpha$, for

$\alpha_{X,Y}(u)$. The normed space $(X \otimes Y, \|\cdot\|_\alpha)$ is not necessarily complete, and we denote its completion by $X \overset{\alpha}{\otimes} Y$.

Let $R : X \rightarrow V$ and $S : Y \rightarrow W$ be bounded linear operators between Banach spaces. Then, by continuity, there is a unique bounded linear map $R \otimes S : X \overset{\alpha}{\otimes} Y \rightarrow V \overset{\alpha}{\otimes} W$ so that $(R \otimes S)(\sum_{j=1}^n x_j \otimes y_j) = \sum_{j=1}^n (Rx_j) \otimes (Sy_j)$, for any choice of $\{x_j\}_{j=1}^n \subseteq X$ and $\{y_j\}_{j=1}^n \subseteq Y$. (In other words, we define $(R \otimes S)(x \otimes y) = Rx \otimes Ry$ and extend the domain of $R \otimes S$ by linearity and continuity to $X \otimes Y$.)

A *uniform* crossnorm is an assignment, to each pair X, Y of Banach spaces a reasonable crossnorm $\alpha_{X,Y}$, so that this assignment satisfies that for any $R \otimes S : X \otimes Y \rightarrow V \otimes W$, it holds that $\|R \otimes S\| = \|R\| \|S\|$.

A *tensor norm* is a uniform crossnorm that is finitely generated, in the sense that for every $u \in X \otimes Y$, we have

$$\alpha_{X,Y}(u) = \inf\{\alpha_{M,N}(u) : u \in M \otimes N, \dim M, \dim N < \infty\}.$$

All the norms on tensor products that we consider in this thesis, such as the injective and projective norms, are tensor norms.

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