STATIONARY MULTIVARIATE TIME SERIES ANALYSIS

by

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DEDICATION

I would like to dedicate this thesis to my mother, Katrien Malan.

CONTENTS

LIST OF SYMBOLS

$\boldsymbol{a}_t$: $k \times 1$ vector white noise process

$\hat{\boldsymbol{a}}_t$: $k \times 1$ residuals of the estimated model

$$\boldsymbol{A}_t : kp \times 1 = \begin{pmatrix} \boldsymbol{a}_t \\ \boldsymbol{0} \\ \vdots \\ \boldsymbol{0} \end{pmatrix} \quad \text{or} \quad k(p+q) \times 1 = \begin{pmatrix} \boldsymbol{a}_t \\ \boldsymbol{0} \\ \vdots \\ \boldsymbol{0} \\ \boldsymbol{a}_t \\ \boldsymbol{0} \\ \vdots \\ \boldsymbol{0} \end{pmatrix}$$

$\boldsymbol{A} : k \times T = \begin{pmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_T \end{pmatrix}$

$\boldsymbol{B} : k \times (kp+1) = \begin{pmatrix} \boldsymbol{c} & \boldsymbol{\Phi}_1 & \boldsymbol{\Phi}_2 & \cdots & \boldsymbol{\Phi}_p \end{pmatrix}$

$\hat{\boldsymbol{B}}$: least squares estimator of $\boldsymbol{B}$

$\boldsymbol{B}^* : k \times kp = \begin{pmatrix} \boldsymbol{\Phi}_1 & \boldsymbol{\Phi}_2 & \cdots & \boldsymbol{\Phi}_p \end{pmatrix}$

$\tilde{\boldsymbol{B}}^*$: maximum likelihood estimator of $\boldsymbol{B}^*$

$\boldsymbol{c}$: $k \times 1$ vector of constant terms

$\boldsymbol{C}_i$: $k \times k$ sample autocovariance matrix of $\{\boldsymbol{a}_t\}$

$\hat{\boldsymbol{C}}_i$: $k \times k$ residual autocovariance matrix

$\hat{\varepsilon}_t$: residuals of the estimated univariate model

$$\boldsymbol{F} : kp \times kp = \begin{pmatrix} \boldsymbol{\Phi}_1 & \boldsymbol{\Phi}_2 & \dots & \boldsymbol{\Phi}_{p-1} & \boldsymbol{\Phi}_p \\ \boldsymbol{I}_k & \boldsymbol{0} & \dots & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_k & \dots & \boldsymbol{0} & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \dots & \boldsymbol{I}_k & \boldsymbol{0} \end{pmatrix}$$

$\boldsymbol{\Gamma}(l): k \times k$ matrix of autocovariances at lag l

$\hat{\boldsymbol{\Gamma}}(l): k \times k$ sample autocovariance matrix at lag l

k : dimension of the multivariate time series

l : lag

L: lag operator

$$\boldsymbol{\xi}_t : kp \times 1 = \begin{pmatrix} \boldsymbol{y}_t - \boldsymbol{\mu} \\ \boldsymbol{y}_{t-1} - \boldsymbol{\mu} \\ \vdots \\ \boldsymbol{y}_{t-p+1} - \boldsymbol{\mu} \end{pmatrix}$$

$\boldsymbol{\mu} : k \times 1$ vector of means

$\boldsymbol{\mu}^* : kT \times 1 = \begin{pmatrix} \boldsymbol{\mu}' & \boldsymbol{\mu}' & \cdots & \boldsymbol{\mu}' \end{pmatrix}'$

$\tilde{\boldsymbol{\mu}}$: maximum likelihood estimator of $\boldsymbol{\mu}$

$\hat{\boldsymbol{\mu}}$: sample estimate of the process mean

p : autoregressive order

P : multivariate Portmanteau test statistic

P' modified multivariate Portmanteau test statistic

q : moving average order

$\boldsymbol{\Phi}_i : k \times k$ autoregressive coefficient matrix, $i = 1,2,\ldots p$

$\boldsymbol{\Phi} : k(p+q) \times k(p+q) = \begin{pmatrix} \boldsymbol{\Phi}_{11} & \boldsymbol{\Phi}_{12} \\ \boldsymbol{\Phi}_{21} & \boldsymbol{\Phi}_{22} \end{pmatrix}$ with

$$\boldsymbol{\Phi}_{11} : kp \times kp = \begin{pmatrix} \boldsymbol{\Phi}_1 & \cdots & \boldsymbol{\Phi}_{p-1} & \boldsymbol{\Phi}_p \\ \boldsymbol{I}_k & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \vdots & \ddots & \vdots & \vdots \\ \boldsymbol{0} & \cdots & \boldsymbol{I}_k & \boldsymbol{0} \end{pmatrix} \qquad \boldsymbol{\Phi}_{12} : kp \times kq = \begin{pmatrix} \boldsymbol{\Theta}_1 & \cdots & \boldsymbol{\Theta}_{q-1} & \boldsymbol{\Theta}_q \\ \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \vdots & \vdots & \vdots & \vdots \\ \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} \end{pmatrix}$$

$$\boldsymbol{\Phi}_{21} : kq \times kp = \boldsymbol{0} \qquad \boldsymbol{\Phi}_{22} : kq \times kq = \begin{pmatrix} \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{I}_k & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \vdots & \ddots & \vdots & \vdots \\ \boldsymbol{0} & \cdots & \boldsymbol{I}_k & \boldsymbol{0} \end{pmatrix}$$

$\boldsymbol{\Phi}_{pp}$: $k \times k$ partial autoregression matrix of lag p

$r_{mn,i}$: sample autocorrelation in row m, column n at lag i

R^2: multiple coefficient of determination

$\boldsymbol{R}_a$: $k \times k$ white noise correlation matrix

$\boldsymbol{R}_i$: $k \times k$ sample autocorrelation matrix of $\{\boldsymbol{a}_t\}$

$\hat{\boldsymbol{R}}_i$: $k \times k$ residual autocorrelation matrix

$\boldsymbol{R}_h^* = (\boldsymbol{R}_1 \quad \ldots \quad \boldsymbol{R}_h)$

$\hat{\boldsymbol{R}}_h^* = (\hat{\boldsymbol{R}}_1 \quad \ldots \quad \hat{\boldsymbol{R}}_h)$

$\boldsymbol{\rho}(l)$: $k \times k$ matrix of autocorrelations at lag l

$\hat{\boldsymbol{\rho}}(l)$: $k \times k$ sample autocorrelation matrix at lag l

$\rho_{mn,i}$: autocorrelation in row m, column n at lag i

$\boldsymbol{\Sigma}_a$: $k \times k$ white noise covariance matrix

$\hat{\boldsymbol{\Sigma}}_a$: unbiased estimator of $\boldsymbol{\Sigma}_a$

$\tilde{\boldsymbol{\Sigma}}_a$ maximum likelihood estimator of $\boldsymbol{\Sigma}_a$

$$\boldsymbol{\Sigma}_A : kp \times kp = \begin{pmatrix} \boldsymbol{\Sigma}_a & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \vdots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} \end{pmatrix} \text{ or } k(p+q) \times k(p+q) = \begin{pmatrix} \boldsymbol{\Sigma}_a & \boldsymbol{0} & \cdots & \boldsymbol{\Sigma}_a & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \boldsymbol{\Sigma}_a & \boldsymbol{0} & \cdots & \boldsymbol{\Sigma}_a & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} & \cdots & \boldsymbol{0} \end{pmatrix}$$

T : sample size

$\boldsymbol{\Theta}_i$: $k \times k$ moving average coefficient matrix, $i = 1,2,\ldots q$

$$\overline{\boldsymbol{\Theta}}_1 : kT \times k(T+1) = \begin{pmatrix} \boldsymbol{\Theta}_1 & \boldsymbol{I}_k & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Theta}_1 & \boldsymbol{I}_k & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{\Theta}_1 & \boldsymbol{I}_k \end{pmatrix}$$

$$\tilde{\boldsymbol{\Theta}}_1 : kT \times kT = \begin{pmatrix} \boldsymbol{I}_k & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{\Theta}_1 & \boldsymbol{I}_k & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{\Theta}_1 & \boldsymbol{I}_k \end{pmatrix}$$

$$\overline{\boldsymbol{\Theta}}_q : kT \times k(T+q) = \begin{pmatrix} \boldsymbol{\Theta}_q & \boldsymbol{\Theta}_{q-1} & \cdots & \boldsymbol{\Theta}_1 & \boldsymbol{I}_k & \boldsymbol{0} & \cdots & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Theta}_q & \ddots & \boldsymbol{\Theta}_2 & \boldsymbol{\Theta}_1 & \boldsymbol{I}_k & \cdots & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \cdots & \boldsymbol{\Theta}_q & \cdots & \cdots & \boldsymbol{\Theta}_1 & \boldsymbol{I}_k \end{pmatrix}$$

$$\tilde{\boldsymbol{\Theta}}_q : kT \times kT = \begin{pmatrix} \boldsymbol{I}_k & \boldsymbol{0} & \cdots & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{\Theta}_1 & \boldsymbol{I}_k & \cdots & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \vdots & \ddots & \ddots & & & \vdots & \vdots \\ \boldsymbol{\Theta}_q & \boldsymbol{\Theta}_{q-1} & \ddots & \ddots & & \vdots & \vdots \\ \boldsymbol{0} & \boldsymbol{\Theta}_q & & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & & \ddots & \ddots & \\ \boldsymbol{0} & \boldsymbol{0} & & \boldsymbol{\Theta}_q & & \boldsymbol{\Theta}_1 & \boldsymbol{I}_k \end{pmatrix}$$

$$\boldsymbol{U}_1 : kT \times kT = \begin{pmatrix} \boldsymbol{I}_k & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ -\boldsymbol{\Phi}_1 & \boldsymbol{I}_k & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & -\boldsymbol{\Phi}_1 & \boldsymbol{I}_k \end{pmatrix}$$

$\boldsymbol{V}^{\frac{1}{2}} : k \times k$ standard deviation matrix

$\hat{\boldsymbol{V}}^{\frac{1}{2}} : k \times k$ sample standard deviation matrix

$\boldsymbol{V}_a^{\frac{1}{2}} : k \times k$ diagonal matrix with the square root of the diagonal elements of $\boldsymbol{C}_0$

$$\boldsymbol{X} : kp \times T = \begin{pmatrix} \boldsymbol{y}_0 - \boldsymbol{\mu} & \boldsymbol{y}_1 - \boldsymbol{\mu} & \cdots & \boldsymbol{y}_{T-1} - \boldsymbol{\mu} \\ \boldsymbol{y}_{-1} - \boldsymbol{\mu} & \boldsymbol{y}_0 - \boldsymbol{\mu} & \cdots & \boldsymbol{y}_{T-2} - \boldsymbol{\mu} \\ \vdots & \vdots & \vdots & \vdots \\ \boldsymbol{y}_{-p+1} - \boldsymbol{\mu} & \boldsymbol{y}_{-p+2} - \boldsymbol{\mu} & \cdots & \boldsymbol{y}_{-p+T} - \boldsymbol{\mu} \end{pmatrix}$$

$$\boldsymbol{y}_t : k \times 1 = \begin{pmatrix} y_{1t} \\ y_{2t} \\ \vdots \\ y_{kt} \end{pmatrix}$$

$$\boldsymbol{Y} : k \times T = \begin{pmatrix} \boldsymbol{y}_1 & \boldsymbol{y}_2 & \cdots & \boldsymbol{y}_T \end{pmatrix} = \begin{pmatrix} y_{11} & y_{12} & \dots & y_{1T} \\ y_{21} & y_{22} & \dots & y_{2T} \\ \vdots & \vdots & \vdots & \vdots \\ y_{k1} & y_{k2} & \dots & y_{kT} \end{pmatrix}$$

$$\boldsymbol{Y}_t : k(p+q) \times 1 = \begin{pmatrix} \boldsymbol{y}_t \\ \boldsymbol{y}_{t-1} \\ \vdots \\ \boldsymbol{y}_{t-p+1} \\ \boldsymbol{a}_t \\ \boldsymbol{a}_{t-1} \\ \vdots \\ \boldsymbol{a}_{t-q+1} \end{pmatrix} \quad \text{or} \quad k(p+1) \times 1 = \begin{pmatrix} \boldsymbol{y}_t \\ \boldsymbol{y}_{t-1} \\ \vdots \\ \boldsymbol{y}_{t-p} \end{pmatrix}$$

$$\boldsymbol{Y}^0 : k \times T = \begin{pmatrix} \boldsymbol{y}_1 - \boldsymbol{\mu} & \boldsymbol{y}_2 - \boldsymbol{\mu} & \dots & \boldsymbol{y}_T - \boldsymbol{\mu} \end{pmatrix}$$

$$\boldsymbol{Z}_t : (kp+1) \times 1 = \begin{pmatrix} 1 \\ \boldsymbol{y}_t \\ \vdots \\ \boldsymbol{y}_{t-p+1} \end{pmatrix}$$

$$\boldsymbol{Z} : (kp+1) \times T = \begin{pmatrix} \boldsymbol{Z}_0 & \boldsymbol{Z}_1 & \cdots & \boldsymbol{Z}_{T-1} \end{pmatrix}$$

LIST OF ABBREVIATIONS

AAIC	Corrected Akaike information criterion
AIC	Akaike information criterion
AR	Autoregressive
ARCH	Autoregressive conditional heteroscedasticity
FPE	Final prediction error
GLP	General linear process
GLS	Generalised least squares
HQC / HQ	Hannan-Quinn criterion
IML	Interactive matrix language
JB	Jarque-Bera
LS	Least squares
MINIC	Minimum information criteria
ML	Maximum likelihood
MSE	Mean square error
SBC / SC	Schwarz Bayesian criterion
VAR	Vector autoregressive
VARMA	Vector autoregressive moving average
VARMAX	Vector autoregressive moving average processes with exogenous regressors
VMA	Vector moving average
SSE	Sum of squared differences of the observed and estimated values
SSR	Sum of squared differences of the estimated value and the mean
SST	Sum of squared differences of the observed value and the mean

CHAPTER 1

INTRODUCTION

1.1 INTRODUCTION AND BACKGROUND

In modern times the collection of data became such an easy process that we are able to gather data as frequently as we want, as well as on any number of variables. Since the availability of information is not a big concern nowadays, it only makes sense to analyse all related variables simultaneously to gain more insight on a specific variable. Thus, instead of observing a single time series, we rather observe several related time series. From this the need arises for multivariate time series analysis techniques.

During the early 1950s, the field of economics expressed the need to analyse more than one time series simultaneously. This sparked the beginning of multivariate time series analysis. Whittle (1953) derived the least square estimation equations for a nondeterministic stationary multiple process, while Bartlett & Rajalakshman (1953) were concerned with the goodness of fit of simultaneous autoregressive series. In 1957 Quenouille summarised the work up to that point, identified some gaps an addressed a few. Akaike (1969), Hannan (1970), Anderson (1984), up to the more recent Lütkepohl (1991), Hamilton (1994), Reinsel (1997), Lütkepohl (2005), are just some of the many that have studied and made contributions to the field of multivariate time series analysis.

Multivariate time series analysis introduced a way to observe the relationship of variables over time, thus making use of all possible information. In the case of univariate time series one investigated the influence of the past values of a single time series on the future values of that specific time series. Now we can expand this to also look at the influence of other variables across time periods. This will ultimately improve the accuracy of the forecasts of an individual time series.

1.2 LAYOUT OF THE STUDY

This dissertation is intended to provide an overview of all the aspects involved in the model building process. This includes the identification of a possible model, the estimation thereof and establishing the goodness of fit of the selected model. The study is restricted to the class of stationary vector autoregressive moving average (VARMA) models.

Chapter 2 introduces the concept of stationarity and defines the different multivariate time series models, namely the vector autoregressive model (VAR), the vector moving average model (VMA) and the vector autoregressive moving average model (VARMA). The moments of these models are also derived under the assumption of stationarity. Chapter 3 is concerned with the estimation of VAR models. The least squares and maximum likelihood estimators are derived, and the importance of their asymptotic distributions is discussed. Deriving the likelihood function of VMA and VARMA models is the topic of Chapter 4. For the estimation of the coefficient matrices it is assumed that the order of the model is known, therefore Chapter 5 summarises some methods to determine the order of a possible model based on the observed multivariate time series. Once the order is identified and an appropriate model is estimated, the adequacy of the fitted model must be established. Chapter 6 deals with both multivariate and univariate diagnostic checks that can be utilised to assess the goodness of fit of the selected model. This chapter is concluded with some real data examples that illustrate the whole model building process.

CHAPTER 2

INTRODUCTION TO STATIONARY MULTIVARIATE TIME SERIES

2.1 INTRODUCTION

Multivariate time series analysis is a powerful tool for the analysis of data. The application is wide-spread from, for example, the medical field where the relationship between exercise and blood glucose can be modeled (Crabtree *et al*, 1990) to the engineering field where the process control effectiveness can be evaluated (De Vries & Wu, 1978).

This chapter serves as an introduction to some of the concepts, namely stationarity, invertibility, autocovariance and autocorrelation; and notation used in multivariate time series analysis. The notation is a generalisation of that introduced by Box & Jenkins (1970) for the univariate autoregressive moving average model. Jenkins & Alavi (1981), Newbold (1981) and Tiao & Box (1981) provide a thorough overview of the early developments in the field of multivariate time series analysis. In sections 2.3 to 2.5 the vector autoregressive, vector moving average and vector autoregressive moving average time series models are defined and their moments derived. Throughout the chapter examples will be used to illustrate some of the findings. The SAS programs as well as the *Mathematica*® calculations for these examples are available in appendices B and C, respectively.

2.2 NOTATION AND DEFINITIONS

Let the components of vector $\boldsymbol{y}_t$ represent k time series observed at time t,

$$\boldsymbol{y}_t = \begin{pmatrix} y_{1t} \\ y_{2t} \\ \vdots \\ y_{kt} \end{pmatrix} \text{ where } -\infty < t < \infty$$

If k time series is observed for a specific time period, say t = 1 to T, then the notation can be extended by using a $k \times T$ matrix:

$$\boldsymbol{Y}:k\times T=\begin{pmatrix}\boldsymbol{y}_1 & \boldsymbol{y}_2 & \cdots & \boldsymbol{y}_T\end{pmatrix}=\begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1T} \\ y_{21} & y_{22} & \cdots & y_{2T} \\ \vdots & \vdots & \vdots & \vdots \\ y_{k1} & y_{k2} & \cdots & y_{kT}\end{pmatrix} \tag{2.1}$$

where each row represents a univariate time series, and each column represents the observed measurements made on k variables at a specific point in time.

The process { $\boldsymbol{y}_t$ } is strictly or strongly stationary if the probability distribution of the random vectors $\left(\boldsymbol{y}_{t_1}, \boldsymbol{y}_{t_2}, \ldots, \boldsymbol{y}_{t_n}\right)$ and $\left(\boldsymbol{y}_{t_1+l}, \boldsymbol{y}_{t_2+l}, \ldots, \boldsymbol{y}_{t_n+l}\right)$ are the same for all $t_1, t_2, \ldots, t_n$, n and l. Therefore the probability distribution of a stationary vector process is independent of time. (Reinsel, 1997)

A weaker form of a stationary process, namely a covariance stationary process, can be defined as a process { $\boldsymbol{y}_t$ } that satisfies the following conditions:

(a) $E(\boldsymbol{y}_t)=\boldsymbol{\mu}$, constant for all values of t where $\boldsymbol{\mu}=\begin{pmatrix}\mu_1 & \mu_2 & \ldots & \mu_k\end{pmatrix}'$.

(b) The autocovariances, $\operatorname{cov}(\boldsymbol{y}_t, \boldsymbol{y}_{t-l})=\boldsymbol{\Gamma}(l)=E\left[(\boldsymbol{y}_t-\boldsymbol{\mu})(\boldsymbol{y}_{t-l}-\boldsymbol{\mu})'\right]$, do not depend on time t but just on the time period l that separates the two vectors.

Therefore, a process is weakly stationary if its first and second moments are time invariant. (Reinsel, 1997; Lütkepohl, 2005) In this text the term stationary will refer to covariance or weak stationarity.

The covariance and correlation between the i-th and the j-th components of the vector $\boldsymbol{y}_t$, at a specific lag, l, is denoted by

$$\gamma_{ij}(l)=\operatorname{cov}\left(y_{it}, y_{j,t-l}\right)=E\left(y_{it}-\mu_i\right)\left(y_{j,t-l}-\mu_j\right) \tag{2.2}$$

and

$$\rho_{ij}(l)=corr\left(y_{it}, y_{j,t-l}\right)=\frac{\gamma_{ij}(l)}{\left(\gamma_{ii}(0)\gamma_{jj}(0)\right)^{\frac{1}{2}}} \quad \text{where } \gamma_{ii}(0)=\operatorname{var}\left(y_{it}\right)$$

respectively.

In the univariate case we observed a single time series over a period of time and calculated the value of the covariance between observations at different lags, this resulted in a single value for each lag. In the multivariate case we need to calculate the covariance between the k different variables at varying lags, which results in a $k \times k$ matrix of cross-covariances (autocovariances) at lag l, which we denote by

$$\boldsymbol{\Gamma}(l) = E\left[(\boldsymbol{y}_t - \boldsymbol{\mu})(\boldsymbol{y}_{t-l} - \boldsymbol{\mu})'\right] = \begin{pmatrix} \gamma_{11}(l) & \gamma_{12}(l) & \dots & \gamma_{1k}(l) \\ \gamma_{21}(l) & \gamma_{22}(l) & \dots & \gamma_{2k}(l) \\ \vdots & \vdots & \vdots & \vdots \\ \gamma_{k1}(l) & \gamma_{k2}(l) & \dots & \gamma_{kk}(l) \end{pmatrix} \text{ for } -\infty < l < \infty \qquad (2.3)$$

The corresponding cross-correlation (autocorrelation) matrix at lag l is

$$\boldsymbol{\rho}(l) = \begin{pmatrix} \frac{\gamma_{11}(l)}{(\gamma_{11}(0)\gamma_{11}(0))^{\frac{1}{2}}} & \frac{\gamma_{12}(l)}{(\gamma_{11}(0)\gamma_{22}(0))^{\frac{1}{2}}} & \dots & \frac{\gamma_{1k}(l)}{(\gamma_{11}(0)\gamma_{kk}(0))^{\frac{1}{2}}} \\ \frac{\gamma_{21}(l)}{(\gamma_{11}(0)\gamma_{22}(0))^{\frac{1}{2}}} & \frac{\gamma_{22}(l)}{(\gamma_{22}(0)\gamma_{22}(0))^{\frac{1}{2}}} & \dots & \frac{\gamma_{2k}(l)}{(\gamma_{22}(0)\gamma_{kk}(0))^{\frac{1}{2}}} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\gamma_{k1}(l)}{(\gamma_{11}(0)\gamma_{kk}(0))^{\frac{1}{2}}} & \frac{\gamma_{k2}(l)}{(\gamma_{22}(0)\gamma_{kk}(0))^{\frac{1}{2}}} & \dots & \frac{\gamma_{kk}(l)}{(\gamma_{kk}(0)\gamma_{kk}(0))^{\frac{1}{2}}} \end{pmatrix}$$

Let $\boldsymbol{V}^{\frac{1}{2}}$ be the $k \times k$ standard deviation matrix defined as:

$$\boldsymbol{V}^{\frac{1}{2}} = \begin{pmatrix} \sqrt{\gamma_{11}(0)} & 0 & \dots & 0 \\ 0 & \sqrt{\gamma_{22}(0)} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sqrt{\gamma_{kk}(0)} \end{pmatrix}$$

then

$$\boldsymbol{\rho}(l) = \boldsymbol{V}^{-\frac{1}{2}}\boldsymbol{\Gamma}(l)\boldsymbol{V}^{-\frac{1}{2}} = \begin{pmatrix} \rho_{11}(l) & \rho_{12}(l) & \dots & \rho_{1k}(l) \\ \rho_{21}(l) & \rho_{22}(l) & \dots & \rho_{2k}(l) \\ \vdots & \vdots & \vdots & \vdots \\ \rho_{k1}(l) & \rho_{k2}(l) & \dots & \rho_{kk}(l) \end{pmatrix} \qquad (2.4)$$

In the scalar case $\gamma(l) = \gamma(-l)$ for $l = 0,1,2,\ldots$ for a stationary time series. When generalising to more dimensions, it can be shown that $\boldsymbol{\Gamma}(-l) = \boldsymbol{\Gamma}(l)'$ for $l = 1,2,\ldots$. The covariance between the i-th variable at time t and the j-th variable at time t - l, $\gamma_{ij}(l) = \mathrm{cov}(y_{it}, y_{j,t-l})$, is clearly not the same as the covariance between the j-th variable at time t and the i-th variable at time t - l, $\gamma_{ji}(l) = \mathrm{cov}(y_{jt}, y_{i,t-l})$. The autocovariances, $\gamma_{ij}(l)$, only depend on the difference in time, therefore we can replace t with t + l in (2.2), then

$$\begin{aligned}\gamma_{ij}(l) &= E(y_{it}-\mu_i)(y_{j,t-l}-\mu_j)\\ &= E(y_{i,t+l}-\mu_i)(y_{jt}-\mu_j)\\ &= E(y_{jt}-\mu_j)(y_{i,t+l}-\mu_i)\\ &= \gamma_{ji}(-l)\end{aligned}$$

therefore

$$\begin{aligned}\boldsymbol{\Gamma}(-l) &= \begin{pmatrix}\gamma_{11}(-l) & \gamma_{12}(-l) & \cdots & \gamma_{1k}(-l)\\ \gamma_{21}(-l) & \gamma_{22}(-l) & \cdots & \gamma_{2k}(-l)\\ \vdots & \vdots & \vdots & \vdots\\ \gamma_{k1}(-l) & \gamma_{k2}(-l) & \cdots & \gamma_{kk}(-l)\end{pmatrix}\\ &= \begin{pmatrix}\gamma_{11}(l) & \gamma_{21}(l) & \cdots & \gamma_{k1}(l)\\ \gamma_{12}(l) & \gamma_{22}(l) & \cdots & \gamma_{k2}(l)\\ \vdots & \vdots & \vdots & \vdots\\ \gamma_{1k}(l) & \gamma_{2k}(l) & \cdots & \gamma_{kk}(l)\end{pmatrix}\\ &= \boldsymbol{\Gamma}(l)' \end{aligned} \tag{2.5}$$

and similarly

$$\boldsymbol{\rho}(-l) = \boldsymbol{\rho}(l)' \tag{2.6}$$

2.3 VECTOR AUTOREGRESSIVE PROCESSES

The equation for modeling a univariate time series with an autoregressive model of order p (AR(p)) is $y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + ... + \phi_p y_{t-p} + a_t$ with $\{a_t\}$ a white noise time series. In the multivariate case this formula can be expanded to model the f-th time series by including the information provided by the k related time series processes. Thus

$$\begin{aligned} y_{ft} = c_f &+ \phi_{f1,1} y_{1,t-1} + \phi_{f2,1} y_{2,t-1} + ... + \phi_{fk,1} y_{k,t-1} + \\ &+ \phi_{f1,2} y_{1,t-2} + \phi_{f2,2} y_{2,t-2} + ... + \phi_{fk,2} y_{k,t-2} + \\ &+ ... + \\ &+ \phi_{f1,p} y_{1,t-p} + \phi_{f2,p} y_{2,t-p} + ... + \phi_{fk,p} y_{k,t-p} + a_{ft} \quad \text{for } f = 1,2,\ldots,k \end{aligned}$$

Take note that the first subscript of ϕ denotes the time series we model, the second denotes the related variable and the last indicates the lag. Thus, in matrix notation, the vector autoregressive model of order p (VAR(p)) is

$$\boldsymbol{y}_t = \begin{pmatrix} y_{1t} \\ y_{2t} \\ \vdots \\ y_{kt} \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix} + \begin{pmatrix} \phi_{11,1} & \phi_{12,1} & \cdots & \phi_{1k,1} \\ \phi_{21,1} & \phi_{22,1} & \cdots & \phi_{2k,1} \\ \vdots & \vdots & \vdots & \vdots \\ \phi_{k1,1} & \phi_{k2,1} & \cdots & \phi_{kk,1} \end{pmatrix} \begin{pmatrix} y_{1,t-1} \\ y_{2,t-1} \\ \vdots \\ y_{k,t-1} \end{pmatrix} + \ldots + \begin{pmatrix} \phi_{11,p} & \phi_{12,p} & \cdots & \phi_{1k,p} \\ \phi_{21,p} & \phi_{22,p} & \cdots & \phi_{2k,p} \\ \vdots & \vdots & \vdots & \vdots \\ \phi_{k1,p} & \phi_{k2,p} & \cdots & \phi_{kk,p} \end{pmatrix} \begin{pmatrix} y_{1,t-p} \\ y_{2,t-p} \\ \vdots \\ y_{k,t-p} \end{pmatrix} + \begin{pmatrix} a_{1t} \\ a_{2t} \\ \vdots \\ a_{kt} \end{pmatrix}$$

or

$$\boldsymbol{y}_t = \boldsymbol{c} + \boldsymbol{\Phi}_1 \boldsymbol{y}_{t-1} + \boldsymbol{\Phi}_2 \boldsymbol{y}_{t-2} + \ldots + \boldsymbol{\Phi}_p \boldsymbol{y}_{t-p} + \boldsymbol{a}_t$$

where

$\boldsymbol{y}_t : k \times 1$ random vector

$\boldsymbol{\Phi}_i : k \times k$ autoregressive coefficient matrix, $i = 1,2,\ldots p$

$\boldsymbol{c} : k \times 1$ vector of constant terms

$\boldsymbol{a}_t : k \times 1$ vector white noise process, which is defined as follows:

$$E(\boldsymbol{a}_t) = \boldsymbol{0}$$

$$E(\boldsymbol{a}_t \boldsymbol{a}_t') = \boldsymbol{\Sigma}_a = \begin{pmatrix} E(a_{1t}^2) & E(a_{1t}a_{2t}) & \cdots & E(a_{1t}a_{kt}) \\ E(a_{1t}a_{2t}) & E(a_{2t}^2) & \cdots & E(a_{2t}a_{kt}) \\ \vdots & \vdots & \vdots & \vdots \\ E(a_{1t}a_{kt}) & E(a_{2t}a_{kt}) & \cdots & E(a_{kt}^2) \end{pmatrix}, \tag{2.7}$$

a $k \times k$ symmetric, positive definite matrix, called the white noise covariance matrix, and

$E(\boldsymbol{a}_t \boldsymbol{a}_s') = \boldsymbol{0}$ for $t \neq s$, therefore uncorrelated across time.

This model will be discussed in more detail in section 2.3.2. Let us consider the vector autoregressive model of order one, VAR(1).

2.3.1 Vector autoregressive model of order 1

In this section the vector autoregressive model of order 1 is considered. The stationarity condition is provided, the model is expressed in terms of a general linear model and the moments are derived. An explicit formula for establishing stationarity and determining the autocovariance matrix at lag 0 for a bivariate VAR(1) model is determined using computer algebra. The section is concluded with two numerical examples.

Definition

The vector autoregressive model of order 1, VAR(1), is given by

$$\boldsymbol{y}_t = \boldsymbol{c} + \boldsymbol{\Phi}_1 \boldsymbol{y}_{t-1} + \boldsymbol{a}_t \tag{2.8}$$

or in lag operator form

$$(\boldsymbol{I}_k - \boldsymbol{\Phi}_1 L)\boldsymbol{y}_t = \boldsymbol{c} + \boldsymbol{a}_t$$

where L is the lag operator, which operates on all the components of a vector, in this case $L^j \boldsymbol{y}_t = \boldsymbol{y}_{t-j}$, $j = \ldots,-1,0,1,2,\ldots$

Stationarity

If the eigenvalues of the autoregressive coefficient matrix of a VAR(1) process have modulus (see Appendix A4) less than one, it implies that $\{\boldsymbol{y}_t\}$ is a well-defined stochastic process. If this is the case we will say that the VAR(1) process is stable. This is not limited to VAR(1) processes, since VAR(p) and VARMA(p,q) processes also have a VAR(1) representation. The stability condition is also sometimes referred to as the stationarity condition, because stability implies stationarity. Time series with trends or seasonal patterns are examples of unstable processes. In what follows we will assume that the process is stable. (Lütkepohl, 2005)

General Linear Process (GLP)

The VAR(1) model can be rewritten by means of back substitution, thus

$$\begin{aligned}
\boldsymbol{y}_t &= \boldsymbol{c} + \boldsymbol{\Phi}_1 \boldsymbol{y}_{t-1} + \boldsymbol{a}_t \\
&= \boldsymbol{c} + \boldsymbol{\Phi}_1 (\boldsymbol{c} + \boldsymbol{\Phi}_1 \boldsymbol{y}_{t-2} + \boldsymbol{a}_{t-1}) + \boldsymbol{a}_t \\
&= \boldsymbol{c} + \boldsymbol{\Phi}_1 \boldsymbol{c} + \boldsymbol{\Phi}_1^2 \boldsymbol{y}_{t-2} + \boldsymbol{\Phi}_1 \boldsymbol{a}_{t-1} + \boldsymbol{a}_t \\
&= \boldsymbol{c} + \boldsymbol{\Phi}_1 \boldsymbol{c} + \boldsymbol{\Phi}_1^2 (\boldsymbol{c} + \boldsymbol{\Phi}_1 \boldsymbol{y}_{t-3} + \boldsymbol{a}_{t-2}) + \boldsymbol{\Phi}_1 \boldsymbol{a}_{t-1} + \boldsymbol{a}_t \\
&= \boldsymbol{c} + \boldsymbol{\Phi}_1 \boldsymbol{c} + \boldsymbol{\Phi}_1^2 \boldsymbol{c} + \boldsymbol{\Phi}_1^3 \boldsymbol{y}_{t-3} + \boldsymbol{\Phi}_1^2 \boldsymbol{a}_{t-2} + \boldsymbol{\Phi}_1 \boldsymbol{a}_{t-1} + \boldsymbol{a}_t \\
&= (\boldsymbol{I} + \boldsymbol{\Phi}_1 + \boldsymbol{\Phi}_1^2)\boldsymbol{c} + \boldsymbol{\Phi}_1^3 \boldsymbol{y}_{t-3} + \boldsymbol{\Phi}_1^2 \boldsymbol{a}_{t-2} + \boldsymbol{\Phi}_1 \boldsymbol{a}_{t-1} + \boldsymbol{a}_t \\
&= \ldots
\end{aligned}$$

after n substitutions this expands to

$$\boldsymbol{y}_t = (\boldsymbol{I} + \boldsymbol{\Phi}_1 + \boldsymbol{\Phi}_1^2 + \ldots + \boldsymbol{\Phi}_1^n)\boldsymbol{c} + \boldsymbol{\Phi}_1^{n+1} \boldsymbol{y}_{t-n-1} + \boldsymbol{\Phi}_1^n \boldsymbol{a}_{t-n} + \ldots + \boldsymbol{\Phi}_1 \boldsymbol{a}_{t-1} + \boldsymbol{a}_t \tag{2.9}$$

For this series to be stationary the effect of $\boldsymbol{y}_{t-n+1}$ on $\boldsymbol{y}_t$ must be negligible for large *n*, in other words $\boldsymbol{\Phi}_1^{\,m} \to \boldsymbol{0}$ as $m \to \infty$. Suppose that $\boldsymbol{\Phi}_1 : k \times k$ has $s \le k$ linearly independent eigenvectors. According to the Jordan decomposition a non-singular $(k \times k)$ matrix $\boldsymbol{P}$ exists such that

$$\boldsymbol{\Phi}_1 = \boldsymbol{P}\boldsymbol{J}\boldsymbol{P}^{-1}$$

with

$$\boldsymbol{J} : k \times k = \begin{pmatrix} \boldsymbol{\Lambda}_1 & & & \\ & \boldsymbol{\Lambda}_2 & & \\ & & \ddots & \\ & & & \boldsymbol{\Lambda}_s \end{pmatrix}$$

where $\boldsymbol{\Lambda}_i$ has the eigenvalue λ_i repeated on the main diagonal and unity repeated just above the main diagonal. Then

$$\boldsymbol{\Phi}_1^m = \left(\boldsymbol{P}\boldsymbol{J}\boldsymbol{P}^{-1}\right)^m = \boldsymbol{P}\boldsymbol{J}^m\boldsymbol{P}^{-1}$$

If the modulus of the eigenvalues of $\boldsymbol{\Phi}_1$ are less than one, then $\boldsymbol{\Phi}_1^m = \boldsymbol{P}\boldsymbol{J}^m\boldsymbol{P}^{-1} \to \boldsymbol{0}$ as $m \to \infty$. Therefore, for the VAR(1) model to be stationary, the modulus of the eigenvalues of $\boldsymbol{\Phi}_1$ need to be all less than one. (Hamilton, 1994; Lütkepohl, 2005) This is equivalent to the modulus of the roots of $\det(\boldsymbol{I} - \boldsymbol{\Phi}_1 z) = 0$ being greater than one. (Lütkepohl, 2005)

If $\boldsymbol{\Phi}_1^{\,n+1} \to \boldsymbol{0}$, it follows that (2.9) can be written as a pure vector moving average ($\mathrm{VMA}(\infty)$) process,

$$\boldsymbol{y}_t = \left(\boldsymbol{I} + \boldsymbol{\Phi}_1 + \boldsymbol{\Phi}_1^{\,2} + \ldots\right)\boldsymbol{c} + \boldsymbol{a}_t + \boldsymbol{\Phi}_1\boldsymbol{a}_{t-1} + \boldsymbol{\Phi}_1^{\,2}\boldsymbol{a}_{t-2} + \ldots \tag{2.10}$$

Moments

In the remainder of this section the moments of the VAR(1) model are derived. If a VAR(1) process is stationary, the mean ($\boldsymbol{\mu}$) is given by

$$\begin{aligned} E(\boldsymbol{y}_t) &= E(\boldsymbol{c} + \boldsymbol{\Phi}_1\boldsymbol{y}_{t-1} + \boldsymbol{a}_t) \\ &= \boldsymbol{c} + \boldsymbol{\Phi}_1 E(\boldsymbol{y}_{t-1}) + E(\boldsymbol{a}_t) \\ \boldsymbol{\mu} &= \boldsymbol{c} + \boldsymbol{\Phi}_1\boldsymbol{\mu} \\ (\boldsymbol{I} - \boldsymbol{\Phi}_1)\boldsymbol{\mu} &= \boldsymbol{c} \\ \boldsymbol{\mu} &= (\boldsymbol{I} - \boldsymbol{\Phi}_1)^{-1}\boldsymbol{c} \end{aligned} \tag{2.11}$$

In general, if the modulus of the eigenvalues of a matrix $\boldsymbol{A}$ are less than one, then $\det(\boldsymbol{I} - \boldsymbol{A}z) \neq 0$ for $z \leq 1$. The converse also holds. (Lütkepohl, 2005) From this property it follows that the inverse of $(\boldsymbol{I} - \boldsymbol{\Phi}_1)$ exists, since the assumption of stationarity implies that the modulus of the eigenvalues of $\boldsymbol{\Phi}_1$ are all less than one.

Another way of determining the mean of a VAR(1) process follows by taking expected values of the $\mathrm{VMA}(\infty)$ representation in (2.10) ,

$$\boldsymbol{\mu} = (\boldsymbol{I} + \boldsymbol{\Phi}_1 + \boldsymbol{\Phi}_1^{\ 2} + \ldots)\boldsymbol{c} \tag{2.12}$$

Suppose that $\{\boldsymbol{y}_t\}$ is a stationary VAR(1) process. The process $\{\boldsymbol{y}_t\}$ can be written in terms of the deviation from the mean,

$$(\boldsymbol{y}_t - \boldsymbol{\mu}) = \boldsymbol{\Phi}_1(\boldsymbol{y}_{t-1} - \boldsymbol{\mu}) + \boldsymbol{a}_t \tag{2.13}$$

where $E(\boldsymbol{y}_t) = E(\boldsymbol{y}_{t-1}) = \boldsymbol{\mu}$.

The matrix of autocovariances is determined by postmultiplying (2.13) by $(\boldsymbol{y}_{t-l} - \boldsymbol{\mu})'$ and taking the expected value,

$$\begin{aligned} E\left[(\boldsymbol{y}_t - \boldsymbol{\mu})(\boldsymbol{y}_{t-l} - \boldsymbol{\mu})'\right] &= E\left[\{\boldsymbol{\Phi}_1(\boldsymbol{y}_{t-1} - \boldsymbol{\mu}) + \boldsymbol{a}_t\}\{\boldsymbol{y}_{t-l} - \boldsymbol{\mu}\}'\right] \\ &= E\left[\{\boldsymbol{\Phi}_1(\boldsymbol{y}_{t-1} - \boldsymbol{\mu})\}\{\boldsymbol{y}_{t-l} - \boldsymbol{\mu}\}'\right] + E\left[\boldsymbol{a}_t\{\boldsymbol{y}_{t-l} - \boldsymbol{\mu}\}'\right] \end{aligned} \tag{2.14}$$

Thus for $l = 0$, the second term of (2.14) becomes

$$\begin{aligned} E\left[\boldsymbol{a}_t\{\boldsymbol{y}_{t-l} - \boldsymbol{\mu}\}'\right] &= E\left[\boldsymbol{a}_t\{\boldsymbol{y}_t - \boldsymbol{\mu}\}'\right] \\ &= E\left[\boldsymbol{a}_t\{\boldsymbol{\Phi}_1(\boldsymbol{y}_{t-1} - \boldsymbol{\mu}) + \boldsymbol{a}_t\}'\right] \\ &= E\left[\boldsymbol{a}_t(\boldsymbol{y}_{t-1} - \boldsymbol{\mu})'\boldsymbol{\Phi}_1'\right] + E[\boldsymbol{a}_t\boldsymbol{a}_t'] \\ &= E[\boldsymbol{a}_t\boldsymbol{a}_t'] \text{ since } E[\boldsymbol{a}_t\boldsymbol{y}_{t-1}'] = \boldsymbol{0} \\ &= \boldsymbol{\Sigma}_a \text{ , the white noise covariance matrix} \end{aligned}$$

and for $l > 0$

$$E\left[\boldsymbol{a}_t\{\boldsymbol{y}_{t-l} - \boldsymbol{\mu}\}'\right] = \boldsymbol{0}$$

since the innovation term, at time t, is not correlated with the value of the random variable at time $t-1, t-2, \ldots$.

The matrix of autocovariances (2.14) for $l = 0$ is

$$\begin{aligned}
\boldsymbol{\Gamma}(0) &= E\left[(\boldsymbol{y}_t - \boldsymbol{\mu})(\boldsymbol{y}_t - \boldsymbol{\mu})'\right] \\
&= \boldsymbol{\Phi}_1 E\left[(\boldsymbol{y}_{t-1} - \boldsymbol{\mu})(\boldsymbol{y}_t - \boldsymbol{\mu})'\right] + E\left[\boldsymbol{a}_t \{\boldsymbol{y}_t - \boldsymbol{\mu}\}'\right] \\
&= \boldsymbol{\Phi}_1 \boldsymbol{\Gamma}(-1) + \boldsymbol{\Sigma}_a \\
&= \boldsymbol{\Phi}_1 \boldsymbol{\Gamma}(1)' + \boldsymbol{\Sigma}_a \quad \text{(from (2.5))} \qquad (2.15)
\end{aligned}$$

and for $l > 0$,

$$\begin{aligned}
\boldsymbol{\Gamma}(l) &= E\left[(\boldsymbol{y}_t - \boldsymbol{\mu})(\boldsymbol{y}_{t-l} - \boldsymbol{\mu})'\right] \\
&= \boldsymbol{\Phi}_1 E\left[(\boldsymbol{y}_{t-1} - \boldsymbol{\mu})(\boldsymbol{y}_{t-l} - \boldsymbol{\mu})'\right] + E\left[\boldsymbol{a}_t \{\boldsymbol{y}_{t-l} - \boldsymbol{\mu}\}'\right] \\
&= \boldsymbol{\Phi}_1 \boldsymbol{\Gamma}(l-1) + \boldsymbol{0} \\
&= \boldsymbol{\Phi}_1 \boldsymbol{\Gamma}(l-1) \qquad (2.16)
\end{aligned}$$

The equations used to calculate $\boldsymbol{\Gamma}(l)$ for $l \geq 0$ are known as the Yule-Walker equations. From these equations it follows that if $\boldsymbol{\Phi}_1$ and $\boldsymbol{\Gamma}(0)$ are known, the autocovariances at lag l, $\boldsymbol{\Gamma}(l)$, for $l > 0$ can be calculated recursively. $\boldsymbol{\Gamma}(0)$ can be determined by using the *vec* operator if $\boldsymbol{\Phi}_1$ and $\boldsymbol{\Sigma}_a$, the white noise covariance matrix, are known. The *vec* operator transforms a matrix into a column vector by stacking the columns of the matrix underneath each other. When simplifying (2.15) by applying (2.16)

$$\begin{aligned}
\boldsymbol{\Gamma}(0) &= \boldsymbol{\Phi}_1 \boldsymbol{\Gamma}(1)' + \boldsymbol{\Sigma}_a \\
&= \boldsymbol{\Phi}_1 \boldsymbol{\Gamma}(0) \boldsymbol{\Phi}_1' + \boldsymbol{\Sigma}_a \qquad (2.17)
\end{aligned}$$

then by using the properties of the *vec* operator (see Appendix A1)

$$\begin{aligned}
vec\boldsymbol{\Gamma}(0) &= vec\left(\boldsymbol{\Phi}_1 \boldsymbol{\Gamma}(0) \boldsymbol{\Phi}_1' + \boldsymbol{\Sigma}_a\right) \\
&= vec\left(\boldsymbol{\Phi}_1 \boldsymbol{\Gamma}(0) \boldsymbol{\Phi}_1'\right) + vec\boldsymbol{\Sigma}_a \qquad \text{using (A1.1)} \\
&= \left(\boldsymbol{\Phi}_1 \otimes \boldsymbol{\Phi}_1\right) vec\boldsymbol{\Gamma}(0) + vec\boldsymbol{\Sigma}_a \qquad \text{using (A1.3)} \\
\therefore vec\boldsymbol{\Gamma}(0) &= \left(\boldsymbol{I}_{k^2} - \boldsymbol{\Phi}_1 \otimes \boldsymbol{\Phi}_1\right)^{-1} vec\boldsymbol{\Sigma}_a \qquad (2.18)
\end{aligned}$$

The stationarity assumption implies that the modulus of the eigenvalues of $\boldsymbol{\Phi}_1$ are all less than one. From property (A2.5) of the Kronecker product (see Appendix A2) it follows that the eigenvalues of $\boldsymbol{\Phi}_1 \otimes \boldsymbol{\Phi}_1$ are just the product of the eigenvalues of $\boldsymbol{\Phi}_1$, therefore the modulus of the eigenvalues of $\boldsymbol{\Phi}_1 \otimes \boldsymbol{\Phi}_1$ are also less than one. This implies that the inverse, $\left(\boldsymbol{I}_{k^2} - \boldsymbol{\Phi}_1 \otimes \boldsymbol{\Phi}_1\right)^{-1}$, exists.

Explicit expression for $\boldsymbol{\Gamma}(0)$

Consider the bivariate VAR(1) model $\boldsymbol{y}_t = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} \boldsymbol{y}_{t-1} + \boldsymbol{a}_t$ with $\boldsymbol{\Sigma}_a = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}$.

Computer algebra was employed to derive explicit expressions for the roots of $\det\left(\boldsymbol{I}_2 - \boldsymbol{\Phi}_1 z\right) = 0$ in (2.18b) and $vec\boldsymbol{\Gamma}(0)$ in (2.18c). See Appendix C for the *Mathematica*® code.

The roots of $\det\left(\boldsymbol{I}_2 - \boldsymbol{\Phi}_1 z\right) = 0$ are

$$\left\{ \frac{\phi_{11} + \phi_{22} - \sqrt{\phi_{11}^2 + 4\,\phi_{12}\,\phi_{21} - 2\,\phi_{11}\,\phi_{22} + \phi_{22}^2}}{2\,(-\phi_{12}\,\phi_{21} + \phi_{11}\,\phi_{22})},\ \frac{\phi_{11} + \phi_{22} + \sqrt{\phi_{11}^2 + 4\,\phi_{12}\,\phi_{21} - 2\,\phi_{11}\,\phi_{22} + \phi_{22}^2}}{2\,(-\phi_{12}\,\phi_{21} + \phi_{11}\,\phi_{22})} \right\} \tag{2.18b}$$

The modulus of these roots must be greater than one for the VAR(1) process to be stationary.

The general formula for $vec\boldsymbol{\Gamma}(0)$ in (2.18) is

$$\begin{pmatrix} -\dfrac{-\sigma_{11}\left(-(-1+\phi_{22})(1+\phi_{22})(-1+\phi_{11}\phi_{22})+\phi_{12}\phi_{21}(1+\phi_{22}^2)\right)+\phi_{12}\left(\sigma_{22}\phi_{12}(1-\phi_{12}\phi_{21}+\phi_{11}\phi_{22})+2\sigma_{12}\left(\phi_{12}\phi_{21}\phi_{22}-\phi_{11}(-1+\phi_{22}^2)\right)\right)}{(-1+\phi_{12}\phi_{21}-\phi_{11}(-1+\phi_{22})+\phi_{22})(1+\phi_{12}\phi_{21}-\phi_{11}\phi_{22})(1-\phi_{12}\phi_{21}+\phi_{22}+\phi_{11}(1+\phi_{22}))} \\ -\dfrac{\sigma_{22}\phi_{12}\left(\phi_{11}\phi_{12}\phi_{21}-(-1+\phi_{11}^2)\phi_{22}\right)+\sigma_{11}\phi_{21}\left(\phi_{12}\phi_{21}\phi_{22}-\phi_{11}(-1+\phi_{22}^2)\right)+\sigma_{12}\left(1-\phi_{12}^2\phi_{21}^2-\phi_{22}^2+\phi_{11}^2(-1+\phi_{22}^2)\right)}{(-1+\phi_{12}\phi_{21}-\phi_{11}(-1+\phi_{22})+\phi_{22})(1+\phi_{12}\phi_{21}-\phi_{11}\phi_{22})(1-\phi_{12}\phi_{21}+\phi_{22}+\phi_{11}(1+\phi_{22}))} \\ -\dfrac{\sigma_{22}\phi_{12}\left(\phi_{11}\phi_{12}\phi_{21}-(-1+\phi_{11}^2)\phi_{22}\right)+\sigma_{11}\phi_{21}\left(\phi_{12}\phi_{21}\phi_{22}-\phi_{11}(-1+\phi_{22}^2)\right)+\sigma_{12}\left(1-\phi_{12}^2\phi_{21}^2-\phi_{22}^2+\phi_{11}^2(-1+\phi_{22}^2)\right)}{(-1+\phi_{12}\phi_{21}-\phi_{11}(-1+\phi_{22})+\phi_{22})(1+\phi_{12}\phi_{21}-\phi_{11}\phi_{22})(1-\phi_{12}\phi_{21}+\phi_{22}+\phi_{11}(1+\phi_{22}))} \\ -\dfrac{\phi_{21}\left(\sigma_{11}\phi_{21}(1-\phi_{12}\phi_{21}+\phi_{11}\phi_{22})+2\sigma_{12}\left(\phi_{11}\phi_{12}\phi_{21}-(-1+\phi_{11}^2)\phi_{22}\right)\right)+\sigma_{22}\left(1-\phi_{12}\phi_{21}+\phi_{11}\left(-\phi_{11}(1+\phi_{12}\phi_{21})+(-1+\phi_{11}^2)\phi_{22}\right)\right)}{(-1+\phi_{12}\phi_{21}-\phi_{11}(-1+\phi_{22})+\phi_{22})(1+\phi_{12}\phi_{21}-\phi_{11}\phi_{22})(1-\phi_{12}\phi_{21}+\phi_{22}+\phi_{11}(1+\phi_{22}))} \end{pmatrix} \tag{2.18c}$$

This method is very powerful, and the results can easily be programmed. It can also be extended to higher dimensions and higher order models. It is interesting, however, to note the extensiveness of the expressions, even for this low-dimensional case.

Two examples for illustrating the calculation of the autocovariance matrices of a VAR(1) model are given. The first one is numerical in nature, and illustrates the stationarity test, the calculation of $\boldsymbol{\Gamma}(0)$ in terms of the *vec* operator and the use of the Yule-Walker equations for the calculation of $\boldsymbol{\Gamma}(1)$ and $\boldsymbol{\Gamma}(2)$ for a two dimensional vector time series.

Example 2.2 provides an application of the explicit expressions derived in equations (2.18b) and (2.18c). A spreadsheet is constructed where one just has to enter the coefficient matrix and the white noise covariance matrix. Based on this information it will determine whether the model is stationary and then calculate $\boldsymbol{\Gamma}(0)$, $\boldsymbol{\Gamma}(1)$ and $\boldsymbol{\Gamma}(2)$.

Example 2.1[1*]

The numerical calculations for this example were performed with the IML module of SAS.

Consider the bivariate VAR(1) model $\boldsymbol{y}_t = \begin{pmatrix} 0.5 & 0.6 \\ 0.1 & 0.4 \end{pmatrix} \boldsymbol{y}_{t-1} + \boldsymbol{a}_t$ with $\boldsymbol{\Sigma}_a = \begin{pmatrix} 1.0 & 0.5 \\ 0.5 & 0.9 \end{pmatrix}$.

The eigenvalues of the autoregressive coefficient matrix are

$$\begin{vmatrix} 0.5-\lambda & 0.6 \\ 0.1 & 0.4-\lambda \end{vmatrix} = 0$$

$$(0.5-\lambda)(0.4-\lambda) - 0.06 = 0$$

$$0.14 - 0.9\lambda + \lambda^2 = 0$$

$$\therefore \lambda = 0.7 \text{ or } \lambda = 0.2$$

The model is stationary because the eigenvalues are less than one in absolute value. Another way to establish stationarity is that the roots of $\det(\boldsymbol{I}_2 - \boldsymbol{\Phi}_1 z) = 0$ must be greater than one in absolute value. In this example these roots are 1.429 and 5.

[1] Take note that these calculated values of $\boldsymbol{\Gamma}(l)$ are the transpose of those given by the VARMACOV CALL in SAS IML. This is due to the fact that SAS defines the autocovariances at lag l as $E\left[(\boldsymbol{y}_t - \boldsymbol{\mu})(\boldsymbol{y}_{t+l} - \boldsymbol{\mu})'\right]$. This corresponds to $\boldsymbol{\Gamma}(-l)$ according to (2.3), which is the same as the transpose of $\boldsymbol{\Gamma}(l)$ by using relation (2.5).

[*] The SAS program is provided in Appendix B page 125 and the *Mathematica*® calculations in Appendix C page 163.

The matrix of autocovariances at lag zero can be calculated by using (2.18).

$$vec\boldsymbol{\Gamma}(0) = \left(\boldsymbol{I}_4 - \boldsymbol{\Phi}_1 \otimes \boldsymbol{\Phi}_1\right)^{-1} vec\boldsymbol{\Sigma}_a$$

$$= \left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.25 & 0.30 & 0.30 & 0.36 \\ 0.05 & 0.20 & 0.06 & 0.24 \\ 0.05 & 0.06 & 0.20 & 0.24 \\ 0.01 & 0.04 & 0.04 & 0.16 \end{pmatrix}\right]^{-1} \begin{pmatrix} 1.0 \\ 0.5 \\ 0.5 \\ 0.9 \end{pmatrix}$$

$$= \begin{pmatrix} 2.941 \\ 1.273 \\ 1.273 \\ 1.228 \end{pmatrix}$$

$$\therefore \boldsymbol{\Gamma}(0) = \begin{pmatrix} 2.941 & 1.273 \\ 1.273 & 1.228 \end{pmatrix}$$

Since $\boldsymbol{\Gamma}(0)$ and $\boldsymbol{\Phi}_1$ are now known, $\boldsymbol{\Gamma}(l)$ for $l > 0$ can be calculated using the Yule-Walker equations derived in (2.16),

$$\boldsymbol{\Gamma}(1) = \boldsymbol{\Phi}_1 \boldsymbol{\Gamma}(0) = \begin{pmatrix} 2.234 & 1.373 \\ 0.803 & 0.618 \end{pmatrix},$$

$$\boldsymbol{\Gamma}(2) = \boldsymbol{\Phi}_1 \boldsymbol{\Gamma}(1) = \begin{pmatrix} 1.599 & 1.057 \\ 0.545 & 0.385 \end{pmatrix}, \ldots$$

Example 2.2

The Excel spreadsheet for establishing stationarity and calculating the autocovariance matrices based on the explicit formulae given in (2.18b) and (2.18c) for a VAR(1) model:

	A	B	C	D	E	F	G	H	I
1	**Enter the coefficient matrix and white noise covariance matrix of the VAR(1) model:**								
2									
3	$\boldsymbol{\Phi}$ =	0.5	0.6		$\boldsymbol{\Sigma}_a$ =	1	0.5		
4		0.1	0.4			0.5	0.9		
5									
6	**Calculations:**								
7									
8	ϕ_{11} =	0.5	σ_{11} =	1					
9	ϕ_{12} =	0.6	σ_{12} =	0.5					
10	ϕ_{21} =	0.1	σ_{21} =	0.5					
11	ϕ_{22} =	0.4	σ_{22} =	0.9					
12									
13	**The absolute value of the roots of** $\det\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \begin{pmatrix}\phi_{11} & \phi_{12}\\ \phi_{21} & \phi_{22}\end{pmatrix}z\right) = 0$								
14									
15	1.428571	5	The process is stationary if the roots are greater than one in absolute value						
16									
17	**The matrices of autocovariances are:**								
18	This only holds for a stationary process								
19									
20	$\Gamma(0)$ **is:**			$\Gamma(1)$ **is:**			$\Gamma(2)$ **is:**		
21	2.941	1.273		2.234	1.373		1.599	1.057	
22	1.273	1.228		0.803	0.618		0.545	0.385	

Calculation formulae:

A15:=IF(B8^2+4*B9*B10-2*B8*B11+B11^2>=0,ABS((B8+B11-SQRT(B8^2+4*B9*B10-2*B8*B11+B11^2))/(2*(-B9*B10+B8*B11))),SQRT(((B8+B11)/(2*(-B9*B10+B8*B11)))^2+(SQRT(-(B8^2+4*B9*B10-2*B8*B11+B11^2))/(2*(-B9*B10+B8*B11)))^2))

B15:=IF(B8^2+4*B9*B10-2*B8*B11+B11^2>=0,ABS((B8+B11+SQRT(B8^2+4*B9*B10-2*B8*B11+B11^2))/(2*(-B9*B10+B8*B11))),SQRT(((B8+B11)/(2*(-B9*B10+B8*B11)))^2+(SQRT(-(B8^2+4*B9*B10-2*B8*B11+B11^2))/(2*(-B9*B10+B8*B11)))^2))

A21:=-1*(-D8*(-1*(1+B11)*(1+B11)*(1+B8*B11)+B9*B10*(1+B11^2))+B9*(D11*B9*(1-B9*B10+B8*B11)+2*D9*(B9*B10*B11-B8*(-1+B11^2))))/((-1+B9*B10-B8*(-1+B11)+B11)*(1+B9*B10-B8*B11)*(1-B9*B10+B11+B8*(1+B11)))

```
A22 and B21:=-1*(D11*B9*(B8*B9*B10-(-1+B8^2)*B11)+D8*B10*(B9*B10*B11-B8*(-
        1+B11^2))+D9*(1-B9^2*B10^2-B11^2+B8^2*(-1+B11^2)))/((-1+B9*B10-
        B8*(-1+B11)+B11)*(1+B9*B10-B8*B11)*(1-B9*B10+B11+B8*(1+B11)))
B22:=-1*(B10*(D8*B10*(1-B9*B10+B8*B11)+2*D9*(B8*B9*B10-(-
    1+B8^2)*B11))+D11*(1-B9*B10+B8*(-B8*(1+B9*B10)+(-1+B8^2)*B11)))/((-
    1+B9*B10-B8*(-1+B11)+B11)*(1+B9*B10-B8*B11)*(1-B9*B10+B11+B8*(1+B11)))
D21:E22: =MMULT(B3:C4,A21:B22)
G21:H22 :=MMULT(B3:C4,D21:E22)
```

2.3.2 Vector autoregressive model of order *p*

In this section the vector autoregressive model of order *p* is defined, stationarity conditions provided and moments derived. The model is also represented as a VAR(1) model and as a vector moving average model of infinite order.

Definition

The vector autoregressive model of order *p*, VAR(*p*), is given by

$$\boldsymbol{y}_t = \boldsymbol{c} + \boldsymbol{\Phi}_1 \boldsymbol{y}_{t-1} + \boldsymbol{\Phi}_2 \boldsymbol{y}_{t-2} + \ldots + \boldsymbol{\Phi}_p \boldsymbol{y}_{t-p} + \boldsymbol{a}_t \tag{2.19}$$

or in lag operator form

$$\left(\boldsymbol{I}_k - \boldsymbol{\Phi}_1 L - \boldsymbol{\Phi}_2 L^2 - \ldots - \boldsymbol{\Phi}_p L^p\right)\boldsymbol{y}_t = \boldsymbol{c} + \boldsymbol{a}_t \tag{2.20}$$

where $L^j \boldsymbol{y}_t = \boldsymbol{y}_{t-j}$

Stationarity

A VAR(*p*) process is stationary if the modulus (see Appendix A4) of the roots of $\det\left(\boldsymbol{I}_k - \boldsymbol{\Phi}_1 z - \boldsymbol{\Phi}_2 z^2 - \ldots - \boldsymbol{\Phi}_p z^p\right) = 0$ are all greater than one. (Hamilton, 1994)

The VAR(*p*) model can be written in the form of a VAR(1) model, which is given by

$$\xi_t = \boldsymbol{F}\xi_{t-1} + \boldsymbol{A}_t \tag{2.21}$$

where

$$\xi_t : kp \times 1 = \begin{pmatrix} \boldsymbol{y}_t - \boldsymbol{\mu} \\ \boldsymbol{y}_{t-1} - \boldsymbol{\mu} \\ \vdots \\ \boldsymbol{y}_{t-p+1} - \boldsymbol{\mu} \end{pmatrix}$$

$$\boldsymbol{F} : kp \times kp = \begin{pmatrix} \boldsymbol{\Phi}_1 & \boldsymbol{\Phi}_2 & \dots & \boldsymbol{\Phi}_{p-1} & \boldsymbol{\Phi}_p \\ \boldsymbol{I}_k & \boldsymbol{0} & \dots & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_k & \dots & \boldsymbol{0} & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \dots & \boldsymbol{I}_k & \boldsymbol{0} \end{pmatrix}$$

$$\boldsymbol{A}_t : kp \times 1 = \begin{pmatrix} \boldsymbol{a}_t \\ \boldsymbol{0} \\ \vdots \\ \boldsymbol{0} \end{pmatrix}$$

with $E(\boldsymbol{A}_t \boldsymbol{A}_t') : kp \times kp = \begin{pmatrix} \boldsymbol{\Sigma}_a & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \vdots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} \end{pmatrix} = \boldsymbol{\Sigma}_A$ and

$E(\boldsymbol{A}_t \boldsymbol{A}_s') = \boldsymbol{0}$ for $t \neq s$.

In the previous section we mentioned that a VAR(1) process is stationary if the eigenvalues of the coefficient matrix, $\boldsymbol{\Phi}_1$, have modulus less than one. Since the VAR(p) model can be represented as a VAR(1) model it follows that in order for the process to be stationary all the eigenvalues of $\boldsymbol{F}$ must have modulus less than one.

Moments

Assume that $\{\boldsymbol{y}_t\}$ is a stationary VAR(p) process. The VAR(p) model can be written in terms of the deviations from the mean

$$(\boldsymbol{y}_t - \boldsymbol{\mu}) = \boldsymbol{\Phi}_1(\boldsymbol{y}_{t-1} - \boldsymbol{\mu}) + \boldsymbol{\Phi}_2(\boldsymbol{y}_{t-2} - \boldsymbol{\mu}) + \ldots + \boldsymbol{\Phi}_p(\boldsymbol{y}_{t-p} - \boldsymbol{\mu}) + \boldsymbol{a}_t \tag{2.22}$$

To determine the Yule-Walker equations for $\boldsymbol{\Gamma}(0)$ and $\boldsymbol{\Gamma}(l), l > 0$ we need to postmultiply (2.22) with $(\boldsymbol{y}_{t-l} - \boldsymbol{\mu})'$ and take the expected value thereof, thus

$$E\left[(\boldsymbol{y}_t-\boldsymbol{\mu})(\boldsymbol{y}_{t-l}-\boldsymbol{\mu})'\right]=\boldsymbol{\Phi}_1 E\left[(\boldsymbol{y}_{t-1}-\boldsymbol{\mu})(\boldsymbol{y}_{t-l}-\boldsymbol{\mu})'\right]+\ldots+\boldsymbol{\Phi}_p E\left[(\boldsymbol{y}_{t-p}-\boldsymbol{\mu})(\boldsymbol{y}_{t-l}-\boldsymbol{\mu})'\right]$$
$$+E\left[\boldsymbol{a}_t(\boldsymbol{y}_{t-l}-\boldsymbol{\mu})'\right] \qquad (2.23)$$

The matrix of autocovariances (2.23) for $l=0$ is

$$\begin{aligned}\boldsymbol{\Gamma}(0)&=E\left[(\boldsymbol{y}_t-\boldsymbol{\mu})(\boldsymbol{y}_t-\boldsymbol{\mu})'\right]\\&=\boldsymbol{\Phi}_1 E\left[(\boldsymbol{y}_{t-1}-\boldsymbol{\mu})(\boldsymbol{y}_t-\boldsymbol{\mu})'\right]+\ldots+\boldsymbol{\Phi}_p E\left[(\boldsymbol{y}_{t-p}-\boldsymbol{\mu})(\boldsymbol{y}_t-\boldsymbol{\mu})'\right]+E\left[\boldsymbol{a}_t(\boldsymbol{y}_t-\boldsymbol{\mu})'\right]\\&=\boldsymbol{\Phi}_1\boldsymbol{\Gamma}(-1)+\ldots+\boldsymbol{\Phi}_p\boldsymbol{\Gamma}(-p)+\boldsymbol{\Sigma}_a\\&=\boldsymbol{\Phi}_1\boldsymbol{\Gamma}(1)'+\ldots+\boldsymbol{\Phi}_p\boldsymbol{\Gamma}(p)'+\boldsymbol{\Sigma}_a \quad \text{(from (2.5))}\end{aligned} \qquad (2.24)$$

and for $l>0$,

$$\boldsymbol{\Gamma}(l)=\boldsymbol{\Phi}_1\boldsymbol{\Gamma}(l-1)+\ldots+\boldsymbol{\Phi}_p\boldsymbol{\Gamma}(l-p) \qquad (2.25)$$

The Yule-Walker equations can be used to calculate $\boldsymbol{\Gamma}(l)$ recursively for $l \geq p$ if $\boldsymbol{\Phi}_1,\ldots,\boldsymbol{\Phi}_p$ and $\boldsymbol{\Gamma}(0)$ are known. The autocovariance matrices $\boldsymbol{\Gamma}(0),\ldots,\boldsymbol{\Gamma}(p-1)$ can be determined by using the VAR(1) representation of a VAR(*p*) process, as given in (2.21). From (2.17) it follows that

$$\boldsymbol{\Gamma}(0)^*=\boldsymbol{F}\boldsymbol{\Gamma}(0)^*\boldsymbol{F}'+\boldsymbol{\Sigma}_A \qquad (2.26)$$

where

$$\boldsymbol{\Gamma}(0)^*:kp\times kp=E(\xi_t\xi_t')=\begin{pmatrix}\boldsymbol{\Gamma}(0) & \boldsymbol{\Gamma}(1) & \cdots & \boldsymbol{\Gamma}(p-1)\\ \boldsymbol{\Gamma}(-1) & \boldsymbol{\Gamma}(0) & \cdots & \boldsymbol{\Gamma}(p-2)\\ \vdots & \vdots & \ddots & \vdots\\ \boldsymbol{\Gamma}(-p+1) & \boldsymbol{\Gamma}(-p+2) & & \boldsymbol{\Gamma}(0)\end{pmatrix}$$

In order to solve this we make use of the *vec* operator, therefore from (2.18),

$$vec\boldsymbol{\Gamma}(0)^*=\left(\boldsymbol{I}_{(kp)^2}-\boldsymbol{F}\otimes\boldsymbol{F}\right)^{-1}vec\boldsymbol{\Sigma}_A \qquad (2.27)$$

The following example is used to demonstrate the results of a VAR(2) model, by writing it as a VAR(1) model and using a similar approach as in Example 2.1.

Example 2.3[2*]

Consider the bivariate VAR(2) model $\boldsymbol{y}_t = \begin{pmatrix} -0.2 & 0.1 \\ 0.5 & 0.1 \end{pmatrix}\boldsymbol{y}_{t-1} + \begin{pmatrix} 0.8 & 0.5 \\ -0.4 & 0.5 \end{pmatrix}\boldsymbol{y}_{t-2} + \boldsymbol{a}_t$ with $\boldsymbol{\Sigma}_a = \begin{pmatrix} 1.0 & 0.5 \\ 0.5 & 0.9 \end{pmatrix}$.

The modulus of the roots of $\det\left(\boldsymbol{I}_2 - \boldsymbol{\Phi}_1 z - \boldsymbol{\Phi}_2 z^2\right) = 0$ are 1.072, 1.072, 1.160 and 1.25. They are all greater than one, implying stationarity. Another way to show that this process is stationary, is by considering the VAR(1) representation of the model. The eigenvalues of the autoregressive coefficient matrix of the VAR(1) representation must have modulus less than one. Using (2.21) the VAR(2) model can be rewritten as $\xi_t = \boldsymbol{F}\xi_{t-1} + \boldsymbol{A}_t$ where

$$\boldsymbol{F} = \begin{pmatrix} -0.2 & 0.1 & 0.8 & 0.5 \\ 0.5 & 0.1 & -0.4 & 0.5 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \text{ and } \boldsymbol{\Sigma}_A = \begin{pmatrix} 1 & 0.5 & 0 & 0 \\ 0.5 & 0.9 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The eigenvalues of $\boldsymbol{F}$ are 0.862, 0.8 and $-0.881 \pm 0.305i$ where $i = \sqrt{-1}$. The modulus of the eigenvalues are 0.862, 0.8, 0.933 and 0.933, respectively. The eigenvalues of $\boldsymbol{F}$ are the same as the roots of $\det\left(\boldsymbol{I}_2\lambda^2 - \boldsymbol{\Phi}_1\lambda - \boldsymbol{\Phi}_2\right) = 0$. This confirms that the VAR(2) process is stationary.

The autocovariance matrices, $\boldsymbol{\Gamma}(0)$ and $\boldsymbol{\Gamma}(1)$, can be determined by using the VAR(1) representation together with (2.27),

$$vec\boldsymbol{\Gamma}(0)^* = \left(\boldsymbol{I}_{16} - \boldsymbol{F} \otimes \boldsymbol{F}\right)^{-1} vec\boldsymbol{\Sigma}_A$$
$$= \begin{pmatrix} 6.4 & -0.1 & 0.6 & 2.8 & -0.1 & 5.6 & 4.4 & -2.5 & 0.6 & 4.4 & 6.4 & -0.1 & 2.8 & -2.5 & -0.1 & 5.6 \end{pmatrix}'$$

[2] As explained in Example 2.1, these calculated values of $\boldsymbol{\Gamma}(1), \boldsymbol{\Gamma}(2), \ldots$ are the transpose of those given by the `VARMACOV CALL` in `SAS IML`.

[*] The `SAS` program is provided in Appendix B page 125 and the *Mathematica*® calculations in Appendix C page 165.

$$\therefore \boldsymbol{\Gamma}(0)^* = \begin{pmatrix} \boldsymbol{\Gamma}(0) & \boldsymbol{\Gamma}(1) \\ \boldsymbol{\Gamma}(-1) & \boldsymbol{\Gamma}(0) \end{pmatrix} = \begin{pmatrix} 6.4 & -0.1 & 0.6 & 2.8 \\ -0.1 & 5.6 & 4.4 & -2.5 \\ 0.6 & 4.4 & 6.4 & -0.1 \\ 2.8 & -2.5 & -0.1 & 5.6 \end{pmatrix}$$

$$\therefore \boldsymbol{\Gamma}(0) = \begin{pmatrix} 6.4 & -0.1 \\ -0.1 & 5.6 \end{pmatrix} \text{ and } \boldsymbol{\Gamma}(1) = \begin{pmatrix} 0.6 & 2.8 \\ 4.4 & -2.5 \end{pmatrix}$$

By using $\boldsymbol{\Phi}_1, \boldsymbol{\Phi}_2, \boldsymbol{\Gamma}(0)$ and $\boldsymbol{\Gamma}(1)$, the Yule-Walker equations in (2.25) can be used to determine $\boldsymbol{\Gamma}(l),\ l \geq 2$, for example

$$\boldsymbol{\Gamma}(2) = \boldsymbol{\Phi}_1 \boldsymbol{\Gamma}(1) + \boldsymbol{\Phi}_2 \boldsymbol{\Gamma}(0) = \begin{pmatrix} 5.370 & 1.877 \\ -1.891 & 4.009 \end{pmatrix}.$$

After determining the autocovariances it is possible to obtain the autocorrelations, which is a measure independent of the unit of measurement used for the variables in the system. The autocorrelation matrix $\boldsymbol{\rho}(l)$ can be obtained by applying (2.4). In chapter 5 the pattern of the sample autocorrelation matrices at different lags will be utilised to identify a possible model.

VMA(∞) **representation**

A stationary VAR(p) process can be represented in the form of a VMA(∞) process. This representation is key in deriving certain theoretical concepts. Furthermore, the dynamics of a model is summarised in the coefficient matrices. The dynamic multiplier $\frac{\partial \boldsymbol{y}_{t+j}}{\partial \boldsymbol{a}_t'}$ gives the effect on $\boldsymbol{y}_{t+j}$ of a one-unit increase in $\boldsymbol{a}_t$.

By means of back substitution (2.21) becomes

$$\begin{aligned} \boldsymbol{\xi}_t &= \boldsymbol{F}\boldsymbol{\xi}_{t-1} + \boldsymbol{A}_t \\ &= \boldsymbol{F}(\boldsymbol{F}\boldsymbol{\xi}_{t-2} + \boldsymbol{A}_{t-1}) + \boldsymbol{A}_t \\ &= \boldsymbol{F}^2\boldsymbol{\xi}_{t-2} + \boldsymbol{F}\boldsymbol{A}_{t-1} + \boldsymbol{A}_t \\ &= \boldsymbol{F}^2(\boldsymbol{F}\boldsymbol{\xi}_{t-3} + \boldsymbol{A}_{t-2}) + \boldsymbol{F}\boldsymbol{A}_{t-1} + \boldsymbol{A}_t \\ &= \boldsymbol{F}^3\boldsymbol{\xi}_{t-3} + \boldsymbol{F}^2\boldsymbol{A}_{t-2} + \boldsymbol{F}\boldsymbol{A}_{t-1} + \boldsymbol{A}_t \\ &= \ldots \end{aligned}$$

after n substitutions this expands to

$$\boldsymbol{\xi}_t = \boldsymbol{F}^{n+1}\boldsymbol{\xi}_{t-n-1} + \boldsymbol{F}^n\boldsymbol{A}_{t-n} + \boldsymbol{F}^{n-1}\boldsymbol{A}_{t-n+1} + \ldots + \boldsymbol{F}\boldsymbol{A}_{t-1} + \boldsymbol{A}_t$$

The first k rows of ξ_t are

$$\boldsymbol{y}_t - \boldsymbol{\mu} = \boldsymbol{a}_t + \left(\boldsymbol{F}^1\right)_{11}\boldsymbol{a}_{t-1} + \left(\boldsymbol{F}^2\right)_{11}\boldsymbol{a}_{t-2} + \ldots + \left(\boldsymbol{F}^n\right)_{11}\boldsymbol{a}_{t-n} + \boldsymbol{F}^{n+1}\xi_{t-n-1}$$

where $\left(\boldsymbol{F}^1\right)_{11}$ is the row 1 column 1 submatrix of $\boldsymbol{F}^1$. From the stationarity assumption it follows that $\boldsymbol{F}^{n+1} \to \boldsymbol{0}$ as $n \to \infty$, therefore

$$\begin{aligned} \boldsymbol{y}_t &= \boldsymbol{\mu} + \boldsymbol{a}_t + \boldsymbol{\Psi}_1\boldsymbol{a}_{t-1} + \boldsymbol{\Psi}_2\boldsymbol{a}_{t-2} + \ldots \\ &= \boldsymbol{\mu} + \boldsymbol{\Psi}(L)\boldsymbol{a}_t \end{aligned} \tag{2.28}$$

where $\boldsymbol{\Psi}(L) = \boldsymbol{I} + \boldsymbol{\Psi}_1 L + \boldsymbol{\Psi}_2 L^2 + \ldots$ with $\boldsymbol{\Psi}_1 = \left(\boldsymbol{F}^1\right)_{11}, \boldsymbol{\Psi}_2 = \left(\boldsymbol{F}^2\right)_{11}, \ldots$

The moving average coefficient matrices, $\boldsymbol{\Psi}_j$, can be calculated by writing (2.20) in terms of deviations from the mean form,

$$\begin{aligned} &\left(\boldsymbol{I}_k - \boldsymbol{\Phi}_1 L - \boldsymbol{\Phi}_2 L^2 - \ldots - \boldsymbol{\Phi}_p L^p\right)\left(\boldsymbol{y}_t - \boldsymbol{\mu}\right) = \boldsymbol{a}_t \\ &\boldsymbol{\Phi}(L)\left(\boldsymbol{y}_t - \boldsymbol{\mu}\right) = \boldsymbol{a}_t \end{aligned} \tag{2.29}$$

where $\boldsymbol{\Phi}(L) = \left(\boldsymbol{I}_k - \boldsymbol{\Phi}_1 L - \boldsymbol{\Phi}_2 L^2 - \ldots - \boldsymbol{\Phi}_p L^p\right)$.

Then, by operating both sides of (2.29) with $\boldsymbol{\Psi}(L)$,

$$\boldsymbol{\Psi}(L)\boldsymbol{\Phi}(L)\left(\boldsymbol{y}_t - \boldsymbol{\mu}\right) = \boldsymbol{\Psi}(L)\boldsymbol{a}_t$$

but from (2.28),

$$\left(\boldsymbol{y}_t - \boldsymbol{\mu}\right) = \boldsymbol{\Psi}(L)\boldsymbol{a}_t$$

therefore

$$\boldsymbol{\Psi}(L)\boldsymbol{\Phi}(L) = \boldsymbol{I}_k = \boldsymbol{\Phi}(L)^{-1}\boldsymbol{\Phi}(L) \tag{2.30}$$

$$\therefore \boldsymbol{\Psi}(L) = \left[\boldsymbol{\Phi}(L)\right]^{-1}$$

To obtain the coefficient matrices of the $\mathrm{VMA}(\infty)$ representation we make use of (2.30),

$$\left(\boldsymbol{I}_k - \boldsymbol{\Phi}_1 L - \boldsymbol{\Phi}_2 L^2 - \ldots - \boldsymbol{\Phi}_p L^p\right)\left(\boldsymbol{I}_k + \boldsymbol{\Psi}_1 L + \boldsymbol{\Psi}_2 L^2 + \ldots\right) = \boldsymbol{I}_k$$

Grouping the coefficients of L^j and setting them equal to zero,

$$\begin{aligned} &\boldsymbol{\Psi}_1 - \boldsymbol{\Phi}_1 = \boldsymbol{0} && \therefore \boldsymbol{\Psi}_1 = \boldsymbol{\Phi}_1 \\ &\boldsymbol{\Psi}_2 - \boldsymbol{\Phi}_2 - \boldsymbol{\Phi}_1\boldsymbol{\Psi}_1 = \boldsymbol{0} && \therefore \boldsymbol{\Psi}_2 = \boldsymbol{\Phi}_1\boldsymbol{\Psi}_1 + \boldsymbol{\Phi}_2 \\ &\boldsymbol{\Psi}_3 - \boldsymbol{\Phi}_3 - \boldsymbol{\Phi}_1\boldsymbol{\Psi}_2 - \boldsymbol{\Phi}_2\boldsymbol{\Psi}_1 = \boldsymbol{0} && \therefore \boldsymbol{\Psi}_3 = \boldsymbol{\Phi}_1\boldsymbol{\Psi}_2 + \boldsymbol{\Phi}_2\boldsymbol{\Psi}_1 + \boldsymbol{\Phi}_3 \\ & && \quad\vdots \\ & && \boldsymbol{\Psi}_j = \boldsymbol{\Phi}_1\boldsymbol{\Psi}_{j-1} + \boldsymbol{\Phi}_2\boldsymbol{\Psi}_{j-2} + \ldots + \boldsymbol{\Phi}_j\boldsymbol{\Psi}_0 \end{aligned}$$

where $\boldsymbol{\Psi}_0 = \boldsymbol{I}_k$.

In general, the stationary VAR(p) process can be written as a VMA(∞) process, $\boldsymbol{y}_t = \boldsymbol{\mu} + \boldsymbol{\Psi}(L)\boldsymbol{a}_t$ where $\boldsymbol{\Psi}(L) = \boldsymbol{I} + \boldsymbol{\Psi}_1 L + \boldsymbol{\Psi}_2 L^2 + \dots$. The VMA$(\infty)$ coefficient matrices are

$$\boldsymbol{\Psi}_j = \sum_{i=1}^{j} \boldsymbol{\Phi}_i \boldsymbol{\Psi}_{j-i} \text{ and } \boldsymbol{\Phi}_j = \boldsymbol{0} \quad \text{for } j > p \tag{2.31}$$

Consider a stationary VAR(2) model. The VMA(∞) coefficient matrices according to (2.31) are

$$\begin{aligned}
&\boldsymbol{\Psi}_0 = \boldsymbol{I}_k \\
&\boldsymbol{\Psi}_1 = \boldsymbol{\Phi}_1\boldsymbol{\Psi}_0 = \boldsymbol{\Phi}_1 \\
&\boldsymbol{\Psi}_2 = \boldsymbol{\Phi}_1\boldsymbol{\Psi}_1 + \boldsymbol{\Phi}_2\boldsymbol{\Psi}_0 = \boldsymbol{\Phi}_1^{\,2} + \boldsymbol{\Phi}_2 \\
&\boldsymbol{\Psi}_3 = \boldsymbol{\Phi}_1\boldsymbol{\Psi}_2 + \boldsymbol{\Phi}_2\boldsymbol{\Psi}_1 + \boldsymbol{\Phi}_3\boldsymbol{\Psi}_0 = \boldsymbol{\Phi}_1\left(\boldsymbol{\Phi}_1^{\,2} + \boldsymbol{\Phi}_2\right) + \boldsymbol{\Phi}_2\boldsymbol{\Phi}_1 \text{ since } \boldsymbol{\Phi}_j = \boldsymbol{0} \text{ for } j > 2 \\
&\vdots \\
&\boldsymbol{\Psi}_i = \boldsymbol{\Phi}_1\boldsymbol{\Psi}_{i-1} + \boldsymbol{\Phi}_2\boldsymbol{\Psi}_{i-2} \quad \text{for } i = 2,3,\dots \\
&\therefore \boldsymbol{y}_t = \boldsymbol{\mu} + \boldsymbol{a}_t + \boldsymbol{\Phi}_1\boldsymbol{a}_{t-1} + \left(\boldsymbol{\Phi}_1^{\,2} + \boldsymbol{\Phi}_2\right)\boldsymbol{a}_{t-2} + \dots
\end{aligned}$$

This is the same as obtained by using back substitution,

$$\begin{aligned}
\boldsymbol{y}_t &= \boldsymbol{c} + \boldsymbol{\Phi}_1\boldsymbol{y}_{t-1} + \boldsymbol{\Phi}_2\boldsymbol{y}_{t-2} + \boldsymbol{a}_t \\
&= \boldsymbol{c} + \boldsymbol{\Phi}_1\left(\boldsymbol{c} + \boldsymbol{\Phi}_1\boldsymbol{y}_{t-2} + \boldsymbol{\Phi}_2\boldsymbol{y}_{t-3} + \boldsymbol{a}_{t-1}\right) + \boldsymbol{\Phi}_2\left(\boldsymbol{c} + \boldsymbol{\Phi}_1\boldsymbol{y}_{t-3} + \boldsymbol{\Phi}_2\boldsymbol{y}_{t-4} + \boldsymbol{a}_{t-2}\right) + \boldsymbol{a}_t \\
&= \left(\boldsymbol{I}_k + \boldsymbol{\Phi}_1 + \boldsymbol{\Phi}_2\right)\boldsymbol{c} + \boldsymbol{a}_t + \boldsymbol{\Phi}_1\boldsymbol{a}_{t-1} + \boldsymbol{\Phi}_2\boldsymbol{a}_{t-2} + \boldsymbol{\Phi}_1^2\boldsymbol{y}_{t-2} + \left(\boldsymbol{\Phi}_1\boldsymbol{\Phi}_2 + \boldsymbol{\Phi}_2\boldsymbol{\Phi}_1\right)\boldsymbol{y}_{t-3} + \boldsymbol{\Phi}_2^2\boldsymbol{y}_{t-4} \\
&= \left(\boldsymbol{I}_k + \boldsymbol{\Phi}_1 + \boldsymbol{\Phi}_2\right)\boldsymbol{c} + \boldsymbol{a}_t + \boldsymbol{\Phi}_1\boldsymbol{a}_{t-1} + \boldsymbol{\Phi}_2\boldsymbol{a}_{t-2} + \boldsymbol{\Phi}_1^2\left(\boldsymbol{c} + \boldsymbol{\Phi}_1\boldsymbol{y}_{t-3} + \boldsymbol{\Phi}_2\boldsymbol{y}_{t-4} + \boldsymbol{a}_{t-2}\right) \\
&\quad + \left(\boldsymbol{\Phi}_1\boldsymbol{\Phi}_2 + \boldsymbol{\Phi}_2\boldsymbol{\Phi}_1\right)\left(\boldsymbol{c} + \boldsymbol{\Phi}_1\boldsymbol{y}_{t-4} + \boldsymbol{\Phi}_2\boldsymbol{y}_{t-5} + \boldsymbol{a}_{t-3}\right) + \boldsymbol{\Phi}_2^2\left(\boldsymbol{c} + \boldsymbol{\Phi}_1\boldsymbol{y}_{t-5} + \boldsymbol{\Phi}_2\boldsymbol{y}_{t-6} + \boldsymbol{a}_{t-4}\right) \\
&= \left(\boldsymbol{I}_k + \boldsymbol{\Phi}_1 + \boldsymbol{\Phi}_2 + \boldsymbol{\Phi}_1^2 + \boldsymbol{\Phi}_1\boldsymbol{\Phi}_2 + \boldsymbol{\Phi}_2\boldsymbol{\Phi}_1 + \boldsymbol{\Phi}_2^2\right)\boldsymbol{c} + \boldsymbol{a}_t + \boldsymbol{\Phi}_1\boldsymbol{a}_{t-1} + \left(\boldsymbol{\Phi}_1^2 + \boldsymbol{\Phi}_2\right)\boldsymbol{a}_{t-2} + \left(\boldsymbol{\Phi}_1\boldsymbol{\Phi}_2 + \boldsymbol{\Phi}_2\boldsymbol{\Phi}_1\right)\boldsymbol{a}_{t-3} \\
&\quad + \boldsymbol{\Phi}_2^2\boldsymbol{a}_{t-4} + \boldsymbol{\Phi}_1^3\boldsymbol{y}_{t-3} + \left(2\boldsymbol{\Phi}_1^2\boldsymbol{\Phi}_2 + \boldsymbol{\Phi}_2\boldsymbol{\Phi}_1^2\right)\boldsymbol{y}_{t-4} + \left(\boldsymbol{\Phi}_1\boldsymbol{\Phi}_2^2 + 2\boldsymbol{\Phi}_2^2\boldsymbol{\Phi}_1\right)\boldsymbol{y}_{t-5} + \boldsymbol{\Phi}_2^3\boldsymbol{y}_{t-6} \\
&= \dots \\
&= \boldsymbol{\mu} + \boldsymbol{a}_t + \boldsymbol{\Phi}_1\boldsymbol{a}_{t-1} + \left(\boldsymbol{\Phi}_1^{\,2} + \boldsymbol{\Phi}_2\right)\boldsymbol{a}_{t-2} + \dots
\end{aligned}$$

2.4 VECTOR MOVING AVERAGE PROCESSES

In this section the vector moving average model of order q is defined and the moments derived. Explicit expressions for the autocovariance matrix at lag l is provided for the simplest case, namely a bivariate VMA(1) model. The conditions for stationarity and invertibility are provided and it is shown that an invertible model can be represented as a vector autoregressive model of infinite order.

Definition

The vector moving average model of order q, VMA(q), is given by

$$\boldsymbol{y}_t = \boldsymbol{\mu} + \boldsymbol{a}_t + \boldsymbol{\Theta}_1 \boldsymbol{a}_{t-1} + \boldsymbol{\Theta}_2 \boldsymbol{a}_{t-2} + \ldots + \boldsymbol{\Theta}_q \boldsymbol{a}_{t-q} \tag{2.32}$$

or in lag operator form

$$\boldsymbol{y}_t = \boldsymbol{\mu} + \left(\boldsymbol{I}_k + \boldsymbol{\Theta}_1 L + \boldsymbol{\Theta}_2 L^2 + \ldots + \boldsymbol{\Theta}_q L^q\right)\boldsymbol{a}_t \tag{2.33}$$

where

$\boldsymbol{y}_t$: $k \times 1$ random vector

$\boldsymbol{\Theta}_i$: $k \times k$ moving average coefficient matrix, $i = 1,2,\ldots q$

$\boldsymbol{\mu}$: $k \times 1$ vector of means

$\boldsymbol{a}_t$: $k \times 1$ vector white noise process which is defined as follows:

$E(\boldsymbol{a}_t) = \boldsymbol{0}$

$E(\boldsymbol{a}_t \boldsymbol{a}_t') = \boldsymbol{\Sigma}_a$, white noise covariance matrix

$E(\boldsymbol{a}_t \boldsymbol{a}_s') = \boldsymbol{0}$ for $t \neq s$, uncorrelated across time

$L^j \boldsymbol{y}_t = \boldsymbol{y}_{t-j}$

Moments

The mean of a VMA(q) process is denoted by $E(\boldsymbol{y}_t) = \boldsymbol{\mu}$, and the autocovariances at lag l, $l \geq 0$, is

$$\begin{aligned}
\boldsymbol{\Gamma}(0) &= E\left[(\boldsymbol{y}_t - \boldsymbol{\mu})(\boldsymbol{y}_t - \boldsymbol{\mu})'\right] \\
&= E\left[(\boldsymbol{a}_t + \boldsymbol{\Theta}_1 \boldsymbol{a}_{t-1} + \boldsymbol{\Theta}_2 \boldsymbol{a}_{t-2} + \ldots + \boldsymbol{\Theta}_q \boldsymbol{a}_{t-q})(\boldsymbol{a}_t + \boldsymbol{\Theta}_1 \boldsymbol{a}_{t-1} + \boldsymbol{\Theta}_2 \boldsymbol{a}_{t-2} + \ldots + \boldsymbol{\Theta}_q \boldsymbol{a}_{t-q})'\right] \\
&= E(\boldsymbol{a}_t \boldsymbol{a}_t') + \boldsymbol{\Theta}_1 E(\boldsymbol{a}_{t-1} \boldsymbol{a}_{t-1}')\boldsymbol{\Theta}_1' + \ldots + \boldsymbol{\Theta}_q E(\boldsymbol{a}_{t-q} \boldsymbol{a}_{t-q}')\boldsymbol{\Theta}_q' \\
&= \boldsymbol{\Sigma}_a + \boldsymbol{\Theta}_1 \boldsymbol{\Sigma}_a \boldsymbol{\Theta}_1' + \ldots + \boldsymbol{\Theta}_q \boldsymbol{\Sigma}_a \boldsymbol{\Theta}_q'
\end{aligned} \tag{2.34}$$

$$\begin{aligned}
\boldsymbol{\Gamma}(l) &= E\left[(\boldsymbol{y}_t - \boldsymbol{\mu})(\boldsymbol{y}_{t-l} - \boldsymbol{\mu})'\right] \\
&= E\left[(\boldsymbol{a}_t + \boldsymbol{\Theta}_1 \boldsymbol{a}_{t-1} + \boldsymbol{\Theta}_2 \boldsymbol{a}_{t-2} + \ldots + \boldsymbol{\Theta}_q \boldsymbol{a}_{t-q})(\boldsymbol{a}_{t-l} + \boldsymbol{\Theta}_1 \boldsymbol{a}_{t-l-1} + \boldsymbol{\Theta}_2 \boldsymbol{a}_{t-l-2} + \ldots + \boldsymbol{\Theta}_q \boldsymbol{a}_{t-l-q})'\right] \\
&= \boldsymbol{\Theta}_l E(\boldsymbol{a}_{t-l} \boldsymbol{a}_{t-l}') + \boldsymbol{\Theta}_{l+1} E(\boldsymbol{a}_{t-l-1} \boldsymbol{a}_{t-l-1}')\boldsymbol{\Theta}_1' + \ldots + \boldsymbol{\Theta}_q E(\boldsymbol{a}_{t-q} \boldsymbol{a}_{t-q}')\boldsymbol{\Theta}_{q-l}' \\
&= \begin{cases} \boldsymbol{\Theta}_l \boldsymbol{\Sigma}_a + \boldsymbol{\Theta}_{l+1} \boldsymbol{\Sigma}_a \boldsymbol{\Theta}_1' + \ldots + \boldsymbol{\Theta}_q \boldsymbol{\Sigma}_a \boldsymbol{\Theta}_{q-l}' & \text{for } l = 1,2,\ldots q \\ \boldsymbol{0} & \text{for } l > q \end{cases}
\end{aligned} \tag{2.35}$$

The autocovariances, $\boldsymbol{\Gamma}(l)$, for $l<0$ can be determined by making use of the relationship derived in (2.5), namely that $\boldsymbol{\Gamma}(-l)=\boldsymbol{\Gamma}(l)'$.

Explicit expression for $\boldsymbol{\Gamma}(l)$

Using (2.34) and (2.35), it is possible to obtain formulae for the autocovariance matrices, in terms of the coefficient matrices and white noise covariance matrix, for VMA models of different dimensions and orders.

Considers, as an example, the bivariate VMA(1) model, $\boldsymbol{y}_t=\boldsymbol{a}_t+\begin{pmatrix}\theta_{11} & \theta_{12}\\ \theta_{21} & \theta_{22}\end{pmatrix}\boldsymbol{a}_{t-1}$ with $\boldsymbol{\Sigma}_a=\begin{pmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{12} & \sigma_{22}\end{pmatrix}$, which is always stationary, but only invertible if the modulus of the roots of $\det(\boldsymbol{I}_2+\boldsymbol{\Theta}_1 z)=0$ are greater than one. Stationarity and invertibility will be discussed after Example 2.5. The roots can be expressed in terms of the elements of the coefficient matrix by employing computer algebra. See Appendix C for the *Mathematica*® code. These roots are

$$\left\{\frac{-\theta_{11}-\theta_{22}-\sqrt{\theta_{11}^2+4\,\theta_{12}\,\theta_{21}-2\,\theta_{11}\,\theta_{22}+\theta_{22}^2}}{2\,(-\theta_{12}\,\theta_{21}+\theta_{11}\,\theta_{22})},\ \frac{-\theta_{11}-\theta_{22}+\sqrt{\theta_{11}^2+4\,\theta_{12}\,\theta_{21}-2\,\theta_{11}\,\theta_{22}+\theta_{22}^2}}{2\,(-\theta_{12}\,\theta_{21}+\theta_{11}\,\theta_{22})}\right\} \tag{2.35b}$$

The explicit expressions for the autocovariance matrices at lag 0 (2.34) and lag 1 (2.35) are given by

$$\boldsymbol{\Gamma}(0)=\boldsymbol{\Sigma}_a+\boldsymbol{\Theta}_1\boldsymbol{\Sigma}_a\boldsymbol{\Theta}_1'=$$

$$\begin{pmatrix}(1+\theta_{11}^2)\,\sigma_{11}+\theta_{12}\,(2\,\theta_{11}\,\sigma_{12}+\theta_{12}\,\sigma_{22}) & \sigma_{12}+\theta_{21}\,(\theta_{11}\,\sigma_{11}+\theta_{12}\,\sigma_{12})+\theta_{22}\,(\theta_{11}\,\sigma_{12}+\theta_{12}\,\sigma_{22})\\ \sigma_{12}+\theta_{11}\,(\theta_{21}\,\sigma_{11}+\theta_{22}\,\sigma_{12})+\theta_{12}\,(\theta_{21}\,\sigma_{12}+\theta_{22}\,\sigma_{22}) & \theta_{21}^2\,\sigma_{11}+2\,\theta_{21}\,\theta_{22}\,\sigma_{12}+(1+\theta_{22}^2)\,\sigma_{22}\end{pmatrix} \tag{2.35c}$$

and

$$\boldsymbol{\Gamma}(1)=\boldsymbol{\Theta}_1\boldsymbol{\Sigma}_a=\begin{pmatrix}\theta_{11}\,\sigma_{11}+\theta_{12}\,\sigma_{12} & \theta_{11}\,\sigma_{12}+\theta_{12}\,\sigma_{22}\\ \theta_{21}\,\sigma_{11}+\theta_{22}\,\sigma_{12} & \theta_{21}\,\sigma_{12}+\theta_{22}\,\sigma_{22}\end{pmatrix} \tag{2.35d}$$

From lag 2 onwards the autocovariance matrices are all equal to zero.

To ease the computational aspect, the explicit expressions given in equations (2.35b) to (2.35d) can be programmed in an Excel spreadsheet. The spreadsheet was designed to calculate the autocovariances once the coefficient matrix and the white noise covariance matrix have been entered. This is illustrated in Example 2.4.

Example 2.4

The Excel spreadsheet for establishing invertibility and calculating the autocovariance matrices based on the explicit formulae given in (2.35b) to (2.35d) for a VMA(1) model:

	A	B	C	D	E	F	G	H	I
1	**Enter the coefficient matrix and white noise covariance matrix of the VMA(1) model:**								
2									
3	$\boldsymbol{\Theta}$ =	0.7	-0.4		$\boldsymbol{\Sigma}_a$ =	1	0.5		
4		0.2	0.3			0.5	0.9		
5									
6	**Calculations:**								
7									
8	Θ_{11} =	0.7	σ_{11} =	1					
9	Θ_{12} =	-0.4	σ_{12} =	0.5					
10	Θ_{21} =	0.2	σ_{21} =	0.5					
11	Θ_{22} =	0.3	σ_{22} =	0.9					
12									
13	**The absolute value of the roots of** $\det\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+\begin{pmatrix}\theta_{11} & \theta_{12}\\ \theta_{21} & \theta_{22}\end{pmatrix}z\right)=0$								
14									
15	1.856953	1.856953	The process is invertible if the roots are greater than one in absolute value						
16									
17	**The matrices of autocovariances are:**								
18									
19	$\boldsymbol{\Gamma}(\boldsymbol{0})$ **is:**			$\boldsymbol{\Gamma}(\boldsymbol{1})$ **is:**					
20	1.354	0.597		0.500	-0.010				
21	0.597	1.081		0.350	0.370				

Calculation formulae:

A15:=IF(B8^2+4*B9*B10-2*B8*B11+B11^2>=0,ABS((-B8-B11-SQRT(B8^2+4*B9*B10-2*B8*B11+B11^2))/(2*(-B9*B10+B8*B11))),SQRT(((-B8-B11)/(2*(-B9*B10+B8*B11)))^2+(SQRT(-(B8^2+4*B9*B10-2*B8*B11+B11^2))/(2*(-B9*B10+B8*B11)))^2))

B15:=IF(B8^2+4*B9*B10-2*B8*B11+B11^2>=0,ABS((-B8-B11+SQRT(B8^2+4*B9*B10-2*B8*B11+B11^2))/(2*(-B9*B10+B8*B11))),SQRT(((-B8-B11)/(2*(-B9*B10+B8*B11)))^2+(SQRT(-(B8^2+4*B9*B10-2*B8*B11+B11^2))/(2*(-B9*B10+B8*B11)))^2))

A20:=(1+B8^2)*D8+B9*(2*B8*D9+B9*D11)

A21 and B20:=D9+B10*(B8*D8+B9*D9)+B11*(B8*D9+B9*D11)

B21:=B10^2*D8+2*B10*B11*D9+(1+B11^2)*D11

D20:=B8*D8+B9*D9

D21:=B10*D8+B11*D9

E20:=B8*D9+B9*D11

E21:=B10*D9+B11*D11

The following example provides a numerical application of the calculation of the autocovariance matrices at different lags and it illustrates two equivalent forms of the invertibility test. The concept of invertibility is the topic of the next paragraph.

Example 2.5[3*]

Consider the VMA(2) model, $\boldsymbol{y}_t = \boldsymbol{a}_t + \begin{pmatrix} 0.2 & 0.1 \\ 0.1 & 0.4 \end{pmatrix} \boldsymbol{a}_{t-1} + \begin{pmatrix} 0.4 & 0 \\ 0.6 & 0.1 \end{pmatrix} \boldsymbol{a}_{t-2}$ with $\boldsymbol{\Sigma}_a = \begin{pmatrix} 1.0 & 0.5 \\ 0.5 & 0.9 \end{pmatrix}$.

The autocovariances at different lags according to (2.34) and (2.35) are

$$\boldsymbol{\Gamma}(0) = \boldsymbol{\Sigma}_a + \boldsymbol{\Theta}_1 \boldsymbol{\Sigma}_a \boldsymbol{\Theta}_1' + \boldsymbol{\Theta}_2 \boldsymbol{\Sigma}_a \boldsymbol{\Theta}_2' = \begin{pmatrix} 1.229 & 0.861 \\ 0.861 & 1.523 \end{pmatrix}$$

$$\boldsymbol{\Gamma}(1) = \boldsymbol{\Theta}_1 \boldsymbol{\Sigma}_a + \boldsymbol{\Theta}_2 \boldsymbol{\Sigma}_a \boldsymbol{\Theta}_1' = \begin{pmatrix} 0.350 & 0.310 \\ 0.469 & 0.631 \end{pmatrix}$$

$$\boldsymbol{\Gamma}(2) = \boldsymbol{\Theta}_2 \boldsymbol{\Sigma}_a = \begin{pmatrix} 0.40 & 0.20 \\ 0.65 & 0.39 \end{pmatrix}$$

[3] Take note that SAS defines a VMA model with a negative sign in front of the moving average coefficient matrices, therefore to obtain the same answers as above we need to put a negative sign in front of theta specified in the VARMACOV CALL in SAS IML. Also, as explained in example 2.1, the calculated values given above are the transposed of those obtained using this SAS function.

[*] The SAS program is provided in Appendix B page 126 and the *Mathematica*® calculations in Appendix C page 169.

$\boldsymbol{\Gamma}(l)$ for $l > 2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

The roots of $\det\left(\boldsymbol{I}_2 + \boldsymbol{\Theta}_1 z + \boldsymbol{\Theta}_2 z^2\right) = 0$ are $-0.987 \pm 2.942i$ and $-0.513 \pm 1.528i$ with modulus 3.103, 3.103, 1.611 and 1.611, respectively. These are greater than one, which implies that the model is invertible. The condition that the modulus of the roots of $\det\left(\boldsymbol{I}_2 + \boldsymbol{\Theta}_1 z + \boldsymbol{\Theta}_2 z^2\right) = 0$ must be greater than one is equivalent to the modulus of the roots of $\det\left(\boldsymbol{I}_2 \lambda^2 - \boldsymbol{\Theta}_1 \lambda - \boldsymbol{\Theta}_2\right) = 0$ being less than one. The latter are $0.471, 0.229, 0.425$ and 0.875.

Stationarity and Invertibility

Neither the vector of means nor the autocovariance matrices depend on time, implying that all VMA(q) processes are stationary. In Section 2.3.2 it was shown that a VAR(p) process can be expressed as a $\text{VMA}(\infty)$ process, only if the stationariaty condition was met, in other words when the modulus of the roots of $\det\left(\boldsymbol{I}_k - \boldsymbol{\Phi}_1 z - \boldsymbol{\Phi}_2 z^2 - \ldots - \boldsymbol{\Phi}_p z^p\right) = 0$ were all greater than one. The next paragraph represents a VMA(q) process in the form of a $\text{VAR}(\infty)$ process. This is only possible when the modulus of the roots of $\det\left(\boldsymbol{I}_k + \boldsymbol{\Theta}_1 z + \boldsymbol{\Theta}_2 z^2 + \ldots + \boldsymbol{\Theta}_q z^q\right) = 0$ are all greater than one. A VMA(q) process that satisfies this condition is called invertible.

An invertible VMA(q) process can be written as a $\text{VAR}(\infty)$ process, namely $\boldsymbol{\Pi}(L)\left(\boldsymbol{y}_t - \boldsymbol{\mu}\right) = \boldsymbol{a}_t$, since

$$\begin{aligned} \boldsymbol{y}_t - \boldsymbol{\mu} &= \left(\boldsymbol{I}_k + \boldsymbol{\Theta}_1 L + \boldsymbol{\Theta}_2 L^2 + \ldots + \boldsymbol{\Theta}_q L^q\right)\boldsymbol{a}_t \\ \boldsymbol{y}_t - \boldsymbol{\mu} &= \boldsymbol{\Theta}(L)\boldsymbol{a}_t \end{aligned} \tag{2.36}$$

where $\boldsymbol{\Theta}(L) = \left(\boldsymbol{I}_k + \boldsymbol{\Theta}_1 L + \boldsymbol{\Theta}_2 L^2 + \ldots + \boldsymbol{\Theta}_q L^q\right)$.

Then, by operating on both sides of (2.36) with $\boldsymbol{\Pi}(L)$,

$$\boldsymbol{\Pi}(L)\left(\boldsymbol{y}_t - \boldsymbol{\mu}\right) = \boldsymbol{\Pi}(L)\boldsymbol{\Theta}(L)\boldsymbol{a}_t$$

but the $\text{VAR}(\infty)$ representation is given by $\boldsymbol{\Pi}(L)\left(\boldsymbol{y}_t - \boldsymbol{\mu}\right) = \boldsymbol{a}_t$, therefore

$$\boldsymbol{\Pi}(L)\boldsymbol{\Theta}(L) = \boldsymbol{I}_k = \boldsymbol{\Theta}(L)^{-1}\boldsymbol{\Theta}(L) \tag{2.37}$$

$$\therefore \boldsymbol{\Pi}(L) = \left[\boldsymbol{\Theta}(L)\right]^{-1}$$

Note that the inverse operator, $[\boldsymbol{\Theta}(L)]^{-1}$, will exist only if the process is invertible.

To obtain the coefficients of the $\text{VAR}(\infty)$ representation we make use of (2.37),

$$(\boldsymbol{I}_k + \boldsymbol{\Theta}_1 L + \boldsymbol{\Theta}_2 L^2 + \ldots + \boldsymbol{\Theta}_q L^q)(\boldsymbol{I}_k - \boldsymbol{\Pi}_1 L - \boldsymbol{\Pi}_2 L^2 - \ldots) = \boldsymbol{I}_k$$

Grouping the coefficients of L^j and setting them equal to zero,

$$\begin{aligned}
&\boldsymbol{\Theta}_1 - \boldsymbol{\Pi}_1 = \boldsymbol{0} && \therefore \boldsymbol{\Pi}_1 = \boldsymbol{\Theta}_1 \\
&\boldsymbol{\Theta}_2 - \boldsymbol{\Pi}_2 - \boldsymbol{\Theta}_1\boldsymbol{\Pi}_1 = \boldsymbol{0} && \therefore \boldsymbol{\Pi}_2 = \boldsymbol{\Theta}_2 - \boldsymbol{\Theta}_1\boldsymbol{\Pi}_1 \\
&\boldsymbol{\Theta}_3 - \boldsymbol{\Pi}_3 - \boldsymbol{\Theta}_1\boldsymbol{\Pi}_2 - \boldsymbol{\Theta}_2\boldsymbol{\Pi}_1 = \boldsymbol{0} && \therefore \boldsymbol{\Pi}_3 = \boldsymbol{\Theta}_3 - \boldsymbol{\Theta}_1\boldsymbol{\Pi}_2 - \boldsymbol{\Theta}_2\boldsymbol{\Pi}_1 \\
& && \vdots \\
& && \boldsymbol{\Pi}_j = \boldsymbol{\Theta}_j - \boldsymbol{\Theta}_1\boldsymbol{\Pi}_{j-1} - \ldots - \boldsymbol{\Theta}_{j-1}\boldsymbol{\Pi}_1
\end{aligned}$$

where $\boldsymbol{\Theta}_j = 0$ for $j > q$.

In general, the invertible VMA(q) process can be written as a $\text{VAR}(\infty)$ process, $\boldsymbol{\Pi}(L)(\boldsymbol{y}_t - \boldsymbol{\mu}) = \boldsymbol{a}_t$ where $\boldsymbol{\Pi}(L) = \boldsymbol{I}_k - \boldsymbol{\Pi}_1 L - \boldsymbol{\Pi}_2 L^2 - \ldots$. The $\text{VAR}(\infty)$ coefficient matrices are

$$\begin{aligned}
\boldsymbol{\Pi}_1 &= \boldsymbol{\Theta}_1 \\
\boldsymbol{\Pi}_j &= \boldsymbol{\Theta}_j - \sum_{i=1}^{j-1} \boldsymbol{\Theta}_i \boldsymbol{\Pi}_{j-i} \qquad \text{for } j = 2,3,\ldots
\end{aligned} \tag{2.38}$$

Consider an invertible VMA(1) model, $\boldsymbol{y}_t = \boldsymbol{\mu} + \boldsymbol{a}_t + \boldsymbol{\Theta}_1 \boldsymbol{a}_{t-1}$. According to (2.38) the $\text{VAR}(\infty)$ representation is given by

$$\begin{aligned}
\boldsymbol{a}_t &= \boldsymbol{\Pi}(L)(\boldsymbol{y}_t - \boldsymbol{\mu}) \\
&= (\boldsymbol{I}_k - \boldsymbol{\Pi}_1 L - \boldsymbol{\Pi}_2 L^2 - \ldots)(\boldsymbol{y}_t - \boldsymbol{\mu}) \\
&= (\boldsymbol{y}_t - \boldsymbol{\mu}) - \boldsymbol{\Pi}_1(\boldsymbol{y}_{t-1} - \boldsymbol{\mu}) - \boldsymbol{\Pi}_2(\boldsymbol{y}_{t-2} - \boldsymbol{\mu}) - \ldots
\end{aligned}$$

with

$$\begin{aligned}
\boldsymbol{\Pi}_1 &= \boldsymbol{\Theta}_1 \\
\boldsymbol{\Pi}_2 &= \boldsymbol{\Theta}_2 - \sum_{i=1}^{2-1} \boldsymbol{\Theta}_i \boldsymbol{\Pi}_{2-i} = \boldsymbol{\Theta}_2 - \boldsymbol{\Theta}_1\boldsymbol{\Pi}_1 = \boldsymbol{0} - \boldsymbol{\Theta}_1\boldsymbol{\Pi}_1 = -\boldsymbol{\Theta}_1^2 \\
&\ldots
\end{aligned}$$

$$\therefore \boldsymbol{a}_t = (\boldsymbol{y}_t - \boldsymbol{\mu}) - \boldsymbol{\Theta}_1(\boldsymbol{y}_{t-1} - \boldsymbol{\mu}) + \boldsymbol{\Theta}_1^2(\boldsymbol{y}_{t-2} - \boldsymbol{\mu}) - \ldots$$

This is the same as obtained by recursive back substitution,

$$
\begin{aligned}
\boldsymbol{a}_t &= (\boldsymbol{y}_t - \boldsymbol{\mu}) - \boldsymbol{\Theta}_1 \boldsymbol{a}_{t-1} \\
&= (\boldsymbol{y}_t - \boldsymbol{\mu}) - \boldsymbol{\Theta}_1 [(\boldsymbol{y}_{t-1} - \boldsymbol{\mu}) - \boldsymbol{\Theta}_1 \boldsymbol{a}_{t-2}] \\
&= (\boldsymbol{y}_t - \boldsymbol{\mu}) - \boldsymbol{\Theta}_1 (\boldsymbol{y}_{t-1} - \boldsymbol{\mu}) + \boldsymbol{\Theta}_1^2 \boldsymbol{a}_{t-2} \\
&= (\boldsymbol{y}_t - \boldsymbol{\mu}) - \boldsymbol{\Theta}_1 (\boldsymbol{y}_{t-1} - \boldsymbol{\mu}) + \boldsymbol{\Theta}_1^2 [(\boldsymbol{y}_{t-2} - \boldsymbol{\mu}) - \boldsymbol{\Theta}_1 \boldsymbol{a}_{t-3}] \\
&= (\boldsymbol{y}_t - \boldsymbol{\mu}) - \boldsymbol{\Theta}_1 (\boldsymbol{y}_{t-1} - \boldsymbol{\mu}) + \boldsymbol{\Theta}_1^2 (\boldsymbol{y}_{t-2} - \boldsymbol{\mu}) - \boldsymbol{\Theta}_1^3 \boldsymbol{a}_{t-3} \\
&= \cdots \\
&= (\boldsymbol{y}_t - \boldsymbol{\mu}) - \boldsymbol{\Theta}_1 (\boldsymbol{y}_{t-1} - \boldsymbol{\mu}) + \boldsymbol{\Theta}_1^2 (\boldsymbol{y}_{t-2} - \boldsymbol{\mu}) - \boldsymbol{\Theta}_1^3 (\boldsymbol{y}_{t-3} - \boldsymbol{\mu}) + \ldots
\end{aligned}
$$

2.5 VECTOR AUTOREGRESSIVE MOVING AVERAGE PROCESSES

In this section the vector autoregressive moving average (VARMA) processes are considered. The model is defined, the stationarity and invertiblility conditions are provided and the moments are derived. In order to obtain the autocovariance matrices it is also necessary to express the VARMA model as a VAR(1) model. Take note that the VAR and VMA processes discussed in previous sections are special cases of the VARMA process.

Definition

The vector autoregressive moving average model of orders p and q, VARMA(p,q), is a combination of the VAR(p) and VMA(q) processes. The model is

$$\boldsymbol{y}_t = \boldsymbol{c} + \boldsymbol{\Phi}_1 \boldsymbol{y}_{t-1} + \boldsymbol{\Phi}_2 \boldsymbol{y}_{t-2} + \ldots + \boldsymbol{\Phi}_p \boldsymbol{y}_{t-p} + \boldsymbol{a}_t + \boldsymbol{\Theta}_1 \boldsymbol{a}_{t-1} + \boldsymbol{\Theta}_2 \boldsymbol{a}_{t-2} + \ldots + \boldsymbol{\Theta}_q \boldsymbol{a}_{t-q} \tag{2.39}$$

or in lag operator form

$$
\begin{aligned}
&(\boldsymbol{I}_k - \boldsymbol{\Phi}_1 L - \boldsymbol{\Phi}_2 L^2 - \ldots - \boldsymbol{\Phi}_p L^p)\boldsymbol{y}_t = \boldsymbol{c} + (\boldsymbol{I}_k + \boldsymbol{\Theta}_1 L + \boldsymbol{\Theta}_2 L^2 + \ldots + \boldsymbol{\Theta}_q L^q)\boldsymbol{a}_t \\
&\boldsymbol{\Phi}(L)\boldsymbol{y}_t = \boldsymbol{c} + \boldsymbol{\Theta}(L)\boldsymbol{a}_t
\end{aligned}
\tag{2.40}
$$

where

$\boldsymbol{y}_t : k \times 1$ random vector

$\boldsymbol{\Phi}_i : k \times k$ autoregressive coefficient matrix, $i = 1,2,\ldots p$

$\boldsymbol{\Theta}_i : k \times k$ moving average coefficient matrix, $i = 1,2,\ldots q$

$\boldsymbol{c} : k \times 1$ vector of constant terms

$\boldsymbol{a}_t : k \times 1$ white noise process which is defined as follows:

$E(\boldsymbol{a}_t) = \boldsymbol{0}$

$E(\boldsymbol{a}_t \boldsymbol{a}_t') = \boldsymbol{\Sigma}_a$, white noise covariance matrix

$$E(\boldsymbol{a}_t\boldsymbol{a}_s') = \boldsymbol{0} \text{ for } t \neq s \text{, uncorrelated across time.}$$

$$L^j \boldsymbol{y}_t = \boldsymbol{y}_{t-j}$$

Stationarity and Invertibility

The process is stationary if the modulus of the roots of $\det\left(\boldsymbol{I}_k - \boldsymbol{\Phi}_1 z - \boldsymbol{\Phi}_2 z^2 - \ldots - \boldsymbol{\Phi}_p z^p\right) = 0$ are all greater than one and invertible if the modulus of the roots of $\det\left(\boldsymbol{I}_k + \boldsymbol{\Theta}_1 z + \boldsymbol{\Theta}_2 z^2 + \ldots + \boldsymbol{\Theta}_q z^q\right) = 0$ are all greater than one.

In what follows the moments of the VARMA(p,q) process will be derived. Without loss of generality it will be assumed that $\{\boldsymbol{y}_t\}$ is a stationary VARMA(p,q) process with zero mean. This implies that the constant $\boldsymbol{c}$ in (2.39) is equal to zero.

Moments

In order to obtain the matrix of autocovariances at lag l we need to postmultiply the zero mean VARMA(p,q) model by $\boldsymbol{y}_{t-l}'$ and take the expected value,

$$\begin{aligned}\boldsymbol{\Gamma}(l) &= E\left(\boldsymbol{y}_t \boldsymbol{y}_{t-l}'\right) \\ &= \boldsymbol{\Phi}_1 E\left(\boldsymbol{y}_{t-1}\boldsymbol{y}_{t-l}'\right) + \ldots + \boldsymbol{\Phi}_p E\left(\boldsymbol{y}_{t-p}\boldsymbol{y}_{t-l}'\right) + E\left(\boldsymbol{a}_t \boldsymbol{y}_{t-l}'\right) + \boldsymbol{\Theta}_1 E\left(\boldsymbol{a}_{t-1}\boldsymbol{y}_{t-l}'\right) + \ldots + \boldsymbol{\Theta}_q E\left(\boldsymbol{a}_{t-q}\boldsymbol{y}_{t-l}'\right)\end{aligned}$$

But, using similar reasoning as in section 2.3.1,

$$E\left(\boldsymbol{a}_t \boldsymbol{y}_{t-l}'\right) = \boldsymbol{0} \text{ for } l > 0$$

$$\ldots$$

$$E\left(\boldsymbol{a}_{t-q} \boldsymbol{y}_{t-l}'\right) = \boldsymbol{0} \text{ for } l > q,$$

therefore,

$$\boldsymbol{\Gamma}(l) = \boldsymbol{\Phi}_1 \boldsymbol{\Gamma}(l-1) + \ldots + \boldsymbol{\Phi}_p \boldsymbol{\Gamma}(l-p) \text{ if } l > q \tag{2.41}$$

Relation (2.41) can be used to calculate $\boldsymbol{\Gamma}(l)$ recursively if $l > q$ and $l \geq p$, in other words if $p > q$ and $\boldsymbol{\Gamma}(0), \boldsymbol{\Gamma}(1), \ldots, \boldsymbol{\Gamma}(p-1)$ are available, the autocovariance matrix $\boldsymbol{\Gamma}(l)$ can be computed for $l = p, p+1, \ldots$. If the VAR order, p, is less than the VMA order, q, we can overcome this by including lags of $\boldsymbol{y}_t$ with zero coefficient matrices until p is greater than q.

The autocovariance matrices $\boldsymbol{\Gamma}(0), \boldsymbol{\Gamma}(1), \ldots, \boldsymbol{\Gamma}(p-1)$ can be determined by first rewriting the VARMA(p,q) process as a VAR(1) process and by making use of the result derived in (2.18). The following system of equations

$$\begin{aligned}
&\boldsymbol{y}_t = \boldsymbol{\Phi}_1 \boldsymbol{y}_{t-1} + \ldots + \boldsymbol{\Phi}_p \boldsymbol{y}_{t-p} + \boldsymbol{a}_t + \boldsymbol{\Theta}_1 \boldsymbol{a}_{t-1} + \ldots + \boldsymbol{\Theta}_q \boldsymbol{a}_{t-q} \\
&\boldsymbol{y}_{t-1} = \boldsymbol{y}_{t-1} \\
&\vdots \\
&\boldsymbol{y}_{t-p+1} = \boldsymbol{y}_{t-p+1} \\
&\boldsymbol{a}_t = \boldsymbol{a}_t \\
&\vdots \\
&\boldsymbol{a}_{t-q+1} = \boldsymbol{a}_{t-q+1}
\end{aligned}$$

can be written in matrix form

$$\begin{pmatrix} \boldsymbol{y}_t \\ \boldsymbol{y}_{t-1} \\ \vdots \\ \boldsymbol{y}_{t-p+1} \\ \boldsymbol{a}_t \\ \boldsymbol{a}_{t-1} \\ \vdots \\ \boldsymbol{a}_{t-q+1} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\Phi}_1 & \cdots & \boldsymbol{\Phi}_{p-1} & \boldsymbol{\Phi}_p & \boldsymbol{\Theta}_1 & \cdots & \boldsymbol{\Theta}_{q-1} & \boldsymbol{\Theta}_q \\ \boldsymbol{I}_k & \cdots & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \boldsymbol{0} & \cdots & \boldsymbol{I}_k & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I}_k & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{I}_k & \boldsymbol{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{y}_{t-1} \\ \boldsymbol{y}_{t-2} \\ \vdots \\ \boldsymbol{y}_{t-p} \\ \boldsymbol{a}_{t-1} \\ \boldsymbol{a}_{t-2} \\ \vdots \\ \boldsymbol{a}_{t-q} \end{pmatrix} + \begin{pmatrix} \boldsymbol{a}_t \\ \boldsymbol{0} \\ \vdots \\ \boldsymbol{0} \\ \boldsymbol{a}_t \\ \boldsymbol{0} \\ \vdots \\ \boldsymbol{0} \end{pmatrix}$$

or

$$\boldsymbol{Y}_t = \boldsymbol{\Phi} \boldsymbol{Y}_{t-1} + \boldsymbol{A}_t \tag{2.42}$$

where

$$\boldsymbol{Y}_t : k(p+q)\times 1 = \begin{pmatrix} \boldsymbol{y}_t \\ \boldsymbol{y}_{t-1} \\ \vdots \\ \boldsymbol{y}_{t-p+1} \\ \boldsymbol{a}_t \\ \boldsymbol{a}_{t-1} \\ \vdots \\ \boldsymbol{a}_{t-q+1} \end{pmatrix}, \quad \boldsymbol{A}_t : k(p+q)\times 1 = \begin{pmatrix} \boldsymbol{a}_t \\ \boldsymbol{0} \\ \vdots \\ \boldsymbol{0} \\ \boldsymbol{a}_t \\ \boldsymbol{0} \\ \vdots \\ \boldsymbol{0} \end{pmatrix}$$

$$\boldsymbol{\Phi} : k(p+q)\times k(p+q) = \begin{pmatrix} \boldsymbol{\Phi}_{11} & \boldsymbol{\Phi}_{12} \\ \boldsymbol{\Phi}_{21} & \boldsymbol{\Phi}_{22} \end{pmatrix} \text{ with}$$

$$\boldsymbol{\Phi}_{11} : kp \times kp = \begin{pmatrix} \boldsymbol{\Phi}_1 & \cdots & \boldsymbol{\Phi}_{p-1} & \boldsymbol{\Phi}_p \\ \boldsymbol{I}_k & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \vdots & \ddots & \vdots & \vdots \\ \boldsymbol{0} & \cdots & \boldsymbol{I}_k & \boldsymbol{0} \end{pmatrix} \qquad \boldsymbol{\Phi}_{12} : kp \times kq = \begin{pmatrix} \boldsymbol{\Theta}_1 & \cdots & \boldsymbol{\Theta}_{q-1} & \boldsymbol{\Theta}_q \\ \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \vdots & \vdots & \vdots & \vdots \\ \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} \end{pmatrix}$$

$$\boldsymbol{\Phi}_{21} : kq \times kp = \boldsymbol{0} \qquad \boldsymbol{\Phi}_{22} : kq \times kq = \begin{pmatrix} \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{I}_k & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \vdots & \ddots & \vdots & \vdots \\ \boldsymbol{0} & \cdots & \boldsymbol{I}_k & \boldsymbol{0} \end{pmatrix}$$

and $\boldsymbol{\Sigma}_A : k(p+q) \times k(p+q) = E(\boldsymbol{A}_t \boldsymbol{A}_t') = \begin{pmatrix} \boldsymbol{\Sigma}_a & \boldsymbol{0} & \cdots & \boldsymbol{\Sigma}_a & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \boldsymbol{\Sigma}_a & \boldsymbol{0} & \cdots & \boldsymbol{\Sigma}_a & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} & \cdots & \boldsymbol{0} \end{pmatrix}$

From the VAR(1) representation in (2.42), it follows by applying (2.17) that

$$\boldsymbol{\Gamma}^*(0) = \boldsymbol{\Phi}\boldsymbol{\Gamma}^*(0)\boldsymbol{\Phi}' + \boldsymbol{\Sigma}_A \tag{2.43}$$

where

$$\boldsymbol{\Gamma}^*(0) = E(\boldsymbol{Y}_t \boldsymbol{Y}_t') = E\left[\begin{pmatrix} \boldsymbol{y}_t \\ \boldsymbol{y}_{t-1} \\ \vdots \\ \boldsymbol{y}_{t-p+1} \\ \boldsymbol{a}_t \\ \boldsymbol{a}_{t-1} \\ \vdots \\ \boldsymbol{a}_{t-q+1} \end{pmatrix} \begin{pmatrix} \boldsymbol{y}_t' & \boldsymbol{y}_{t-1}' & \cdots & \boldsymbol{y}_{t-p+1}' & \boldsymbol{a}_t' & \boldsymbol{a}_{t-1}' & \cdots & \boldsymbol{a}_{t-q+1}' \end{pmatrix}\right]$$

$$= E\begin{pmatrix} \boldsymbol{y}_t \boldsymbol{y}_t' & \cdots & \boldsymbol{y}_t \boldsymbol{y}_{t-p+1}' & \boldsymbol{y}_t \boldsymbol{a}_t' & \cdots & \boldsymbol{y}_t \boldsymbol{a}_{t-q+1}' \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \boldsymbol{y}_{t-p+1} \boldsymbol{y}_t' & \cdots & \boldsymbol{y}_{t-p+1} \boldsymbol{y}_{t-p+1}' & \boldsymbol{y}_{t-p+1} \boldsymbol{a}_t' & \cdots & \boldsymbol{y}_{t-p+1} \boldsymbol{a}_{t-q+1}' \\ \boldsymbol{a}_t \boldsymbol{y}_t' & \cdots & \boldsymbol{a}_t \boldsymbol{y}_{t-p+1}' & \boldsymbol{a}_t \boldsymbol{a}_t' & \cdots & \boldsymbol{a}_t \boldsymbol{a}_{t-q+1}' \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \boldsymbol{a}_{t-q+1} \boldsymbol{y}_t' & \cdots & \boldsymbol{a}_{t-q+1} \boldsymbol{y}_{t-p+1}' & \boldsymbol{a}_{t-q+1} \boldsymbol{a}_t' & \cdots & \boldsymbol{a}_{t-q+1} \boldsymbol{a}_{t-q+1}' \end{pmatrix}$$

$$\boldsymbol{\Gamma}^{*}(0)=\begin{pmatrix} \boldsymbol{\Gamma}(0) & \cdots & \boldsymbol{\Gamma}(p-1) & E(\boldsymbol{y}_t\boldsymbol{a}_t') & \cdots & E(\boldsymbol{y}_t\boldsymbol{a}_{t-q+1}') \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \boldsymbol{\Gamma}(-p+1) & \cdots & \boldsymbol{\Gamma}(0) & \boldsymbol{0} & \cdots & E(\boldsymbol{y}_{t-p+1}\boldsymbol{a}_{t-q+1}') \\ E(\boldsymbol{a}_t\boldsymbol{y}_t') & \cdots & \boldsymbol{0} & \boldsymbol{\Sigma}_a & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ E(\boldsymbol{a}_{t-q+1}\boldsymbol{y}_t') & \cdots & E(\boldsymbol{a}_{t-q+1}\boldsymbol{y}_{t-p+1}') & \boldsymbol{0} & \cdots & \boldsymbol{\Sigma}_a \end{pmatrix}$$
$$=\begin{pmatrix} \boldsymbol{\Gamma}_{11}^{*}(0) & \boldsymbol{\Gamma}_{12}^{*}(0) \\ \boldsymbol{\Gamma}_{12}^{*}(0)' & \boldsymbol{\Gamma}_{22}^{*}(0) \end{pmatrix}$$

$$\boldsymbol{\Gamma}_{11}^{*}(0): kp\times kp=\begin{pmatrix} \boldsymbol{\Gamma}(0) & \boldsymbol{\Gamma}(1) & \cdots & \boldsymbol{\Gamma}(p-1) \\ \boldsymbol{\Gamma}(-1) & \boldsymbol{\Gamma}(0) & \cdots & \boldsymbol{\Gamma}(p-2) \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{\Gamma}(-p+1) & \boldsymbol{\Gamma}(-p+2) & \cdots & \boldsymbol{\Gamma}(0) \end{pmatrix}$$

$$\boldsymbol{\Gamma}_{12}^{*}(0): kp\times kq=\begin{pmatrix} E(\boldsymbol{y}_t\boldsymbol{a}_t') & E(\boldsymbol{y}_t\boldsymbol{a}_{t-1}') & \cdots & E(\boldsymbol{y}_t\boldsymbol{a}_{t-q+1}') \\ \boldsymbol{0} & E(\boldsymbol{y}_{t-1}\boldsymbol{a}_{t-1}') & \cdots & E(\boldsymbol{y}_{t-1}\boldsymbol{a}_{t-q+1}') \\ \vdots & \vdots & \vdots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & E(\boldsymbol{y}_{t-p+1}\boldsymbol{a}_{t-q+1}') \end{pmatrix}$$

$$\boldsymbol{\Gamma}_{22}^{*}(0): kq\times kq=\begin{pmatrix} \boldsymbol{\Sigma}_a & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Sigma}_a & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{\Sigma}_a \end{pmatrix}$$

We can solve for $\boldsymbol{\Gamma}^{*}(0)$ by applying the *vec* operator, using (2.18)

$$vec\boldsymbol{\Gamma}^{*}(0)=\left(\boldsymbol{I}_{k^2(p+q)^2}-\boldsymbol{\Phi}\otimes\boldsymbol{\Phi}\right)^{-1}vec\boldsymbol{\Sigma}_A \tag{2.44}$$

This VAR(1) representation is stationary if the modulus of the roots of $\det\left(\boldsymbol{I}_{k(p+q)}-\boldsymbol{\Phi}z\right)=0$ are all greater than one. From the properties of the determinant, together with the partitioning of $\boldsymbol{\Phi}$, it can be shown that

$$\begin{aligned}\det\left(\boldsymbol{I}_{k(p+q)}-\boldsymbol{\Phi}z\right)&=\det\begin{pmatrix} \boldsymbol{I}_{kp}-\boldsymbol{\Phi}_{11}z & -\boldsymbol{\Phi}_{12} \\ -\boldsymbol{\Phi}_{21} & \boldsymbol{I}_{kq}-\boldsymbol{\Phi}_{22}z \end{pmatrix}\\ &=\det\begin{pmatrix} \boldsymbol{I}_{kp}-\boldsymbol{\Phi}_{11}z & -\boldsymbol{\Phi}_{12} \\ \boldsymbol{0} & \boldsymbol{I}_{kq}-\boldsymbol{\Phi}_{22}z \end{pmatrix}\\ &=\det\left(\boldsymbol{I}_{kp}-\boldsymbol{\Phi}_{11}z\right)\det\left(\boldsymbol{I}_{kq}-\boldsymbol{\Phi}_{22}z\right)\end{aligned}$$

The matrix $\left(\boldsymbol{I}_{kq}-\boldsymbol{\Phi}_{22}z\right)$ is a lower triangular matrix with ones on the main diagonal, therefore

$$\det\left(\boldsymbol{I}_{kp}-\boldsymbol{\Phi}_{11}z\right)\det\left(\boldsymbol{I}_{kq}-\boldsymbol{\Phi}_{22}z\right)=\det\left(\boldsymbol{I}_{kp}-\boldsymbol{\Phi}_{11}z\right)$$

It can be shown that $\det\left(\boldsymbol{I}_{kp}-\boldsymbol{\Phi}_{11}z\right)=\det\left(\boldsymbol{I}_{k}-\boldsymbol{\Phi}_{1}z-\boldsymbol{\Phi}_{2}z^{2}-\ldots-\boldsymbol{\Phi}_{p}z^{p}\right)$. The modulus of the roots of $\det\left(\boldsymbol{I}_{k}-\boldsymbol{\Phi}_{1}z-\boldsymbol{\Phi}_{2}z^{2}-\ldots-\boldsymbol{\Phi}_{p}z^{p}\right)=0$ are greater than one if the VARMA(p,q) process, $\{\boldsymbol{y}_t\}$, is stationary. If this is the case, the VAR(1) representation is also stationary. Since the VAR(1) process is stationary the existence of the inverse of $\left(\boldsymbol{I}_{k^2(p+q)^2}-\boldsymbol{\Phi}\otimes\boldsymbol{\Phi}\right)$ used in (2.44) follows from similar reasoning as in section 2.3.1.

Once $\boldsymbol{\Gamma}(l)$ has been determined it is easy to obtain the autocorrelation matrices of the VARMA(p,q) model by applying relation (2.4).

The following example considers a VARMA(2,1) model. The tests for stationarity and invertibility are illustrated. The model is expressed in the form of a VAR(1) model in order to calculate the matrices of autocovariances at lag 0 and 1. For lags greater than one, the calculated $\boldsymbol{\Gamma}(0)$ and $\boldsymbol{\Gamma}(1)$ are used together with the Yule-Walker equations.

Example 2.6*

Consider the bivariate VARMA(2,1) model,

$$\boldsymbol{y}_t=\begin{pmatrix}-0.2 & 0.1\\ 0.5 & 0.1\end{pmatrix}\boldsymbol{y}_{t-1}+\begin{pmatrix}0.8 & 0.5\\ -0.4 & 0.5\end{pmatrix}\boldsymbol{y}_{t-2}+\boldsymbol{a}_t+\begin{pmatrix}0.2 & 0.1\\ 0.1 & 0.4\end{pmatrix}\boldsymbol{a}_{t-1}\text{ with }\boldsymbol{\Sigma}_a=\begin{pmatrix}1.0 & 0.5\\ 0.5 & 0.9\end{pmatrix}.$$

The model is stationary if the modulus of the roots of $\det\left(\boldsymbol{I}_2-\boldsymbol{\Phi}_1z-\boldsymbol{\Phi}_2z^2\right)=0$ are greater than one. This is satisfied since the roots are $-1.013\pm0.351i$, 1.160 and 1.250 with modulus equal to 1.072, 1.072, 1.160 and 1.250 , respectively. The invertibility follows from the fact that the absolute value of the roots of $\det\left(\boldsymbol{I}_2+\boldsymbol{\Theta}_1z\right)=0$ are 6.306 and 2.265, which are both greater than one. Another way to establish the stationarity and invertibility of a model, is by determining the roots of $\det\left(\boldsymbol{I}_2\lambda^2-\boldsymbol{\Phi}_1\lambda-\boldsymbol{\Phi}_2\right)=0$ and $\det\left(\boldsymbol{I}_2\lambda-\boldsymbol{\Theta}_1\right)=0$, respectively. The modulus of these roots should be less than one.

* The SAS program is provided in Appendix B page 126 and the *Mathematica*® calculations in Appendix C page 170.

The VAR(1) representation of this model is needed to determine the autocovariance matrices at different lags. According to (2.42)

$$\boldsymbol{Y}_t = \boldsymbol{\Phi Y}_{t-1} + \boldsymbol{A}_t \text{ where } \boldsymbol{Y}_t = \begin{pmatrix} \boldsymbol{y}_t \\ \boldsymbol{y}_{t-1} \\ \boldsymbol{a}_t \end{pmatrix}, \boldsymbol{\Phi} = \begin{pmatrix} \boldsymbol{\Phi}_{11} & \boldsymbol{\Phi}_{12} \\ \boldsymbol{\Phi}_{21} & \boldsymbol{\Phi}_{22} \end{pmatrix} = \left(\begin{array}{cc|c} \boldsymbol{\Phi}_1 & \boldsymbol{\Phi}_2 & \boldsymbol{\Theta}_1 \\ \boldsymbol{I}_2 & \boldsymbol{0} & \boldsymbol{0} \\ \hline \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{array}\right) \boldsymbol{A}_t = \begin{pmatrix} \boldsymbol{a}_t \\ \boldsymbol{0} \\ \boldsymbol{a}_t \end{pmatrix} \text{ and}$$

$$\boldsymbol{\Sigma}_A = \begin{pmatrix} \boldsymbol{\Sigma}_a & \boldsymbol{0} & \boldsymbol{\Sigma}_a \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{\Sigma}_a & \boldsymbol{0} & \boldsymbol{\Sigma}_a \end{pmatrix}$$

The autocovariance matrices, $\boldsymbol{\Gamma}(0)$ and $\boldsymbol{\Gamma}(1)$, are calculated using (2.44)

$$vec\boldsymbol{\Gamma}^*(0) = (\boldsymbol{I}_{36} - \boldsymbol{\Phi} \otimes \boldsymbol{\Phi})^{-1} vec\boldsymbol{\Sigma}_A$$

$$\therefore \boldsymbol{\Gamma}^*(0) = \begin{pmatrix} \boldsymbol{\Gamma}^*_{11}(0) & \boldsymbol{\Gamma}^*_{12}(0) \\ \boldsymbol{\Gamma}^*_{21}(0) & \boldsymbol{\Gamma}^*_{22}(0) \end{pmatrix} = \left(\begin{array}{cccc|cc} 8.286 & 2.583 & 5.144 & 4.355 & 1 & 0.5 \\ 2.583 & 5.260 & 4.821 & 0.598 & 0.5 & 0.9 \\ 5.144 & 4.821 & 8.286 & 2.583 & 0 & 0 \\ 4.355 & 0.598 & 2.583 & 5.260 & 0 & 0 \\ \hline 1 & 0.5 & 0 & 0 & 1 & 0.5 \\ 0.5 & 0.9 & 0 & 0 & 0.5 & 0.9 \end{array}\right)$$

$$\therefore \boldsymbol{\Gamma}(0) = \begin{pmatrix} 8.286 & 2.583 \\ 2.583 & 5.260 \end{pmatrix} \text{ and } \boldsymbol{\Gamma}(1) = \begin{pmatrix} 5.144 & 4.355 \\ 4.821 & 0.598 \end{pmatrix} \text{ using (2.43)}$$

From (2.41), for $l > 1$, for example

$$\boldsymbol{\Gamma}(2) = \boldsymbol{\Phi}_1 \boldsymbol{\Gamma}(1) + \boldsymbol{\Phi}_2 \boldsymbol{\Gamma}(0) = \begin{pmatrix} 7.373 & 3.885 \\ 1.031 & 3.835 \end{pmatrix}$$

Refer to examples 2.1 and 2.5 for information regarding the built in SAS functions.

2.6 CONCLUSION

This chapter presented an overview of vector autoregressive moving average time series models. Conditions for stationarity and invertibility were given. The population moments for each of these models were derived under the restriction of stationarity. The formulae obtained were illustrated by means of numerical examples that were programmed using the IML module of SAS.

The properties of the population moments will later on be used to identify a possible model for an observed time series vector. The next two chapters will focus on the estimation of the parameters of these multivariate time series models.

CHAPTER 3

ESTIMATION OF VECTOR AUTOREGRESSIVE PROCESSES

3.1 INTRODUCTION

Vector autoregressive models are often used in practice due to the simplicity of the estimation thereof. The VAR(p) model can be written in the form of a multivariate linear model. The results of such a model can then be used to obtain least squares estimators. When the assumption of a Gaussian error distribution is added, it is possible to obtain the likelihood function and subsequently the maximum likelihood estimators of the unknown parameters. These procedures are described by both Reinsel (1997) and Lütkepohl (2005) while Draper & Smith (1998) provides a detailed discussion of generalised least squares estimation. Estimation of VAR models was also considered by Hannan (1970) in the spectral domain, who also derived the asymptotic distribution of the estimators.

Estimation is presented in two chapters. This chapter is used to describe the autoregressive case. Closed form expressions are available. If a moving average component is added to the model, estimation becomes much more complex, since the normal equations are nonlinear. That will be the topic of the next chapter.

This chapter describes two methods used for estimating the parameters of a VAR(p) model, namely least squares estimation and the method of maximum likelihood. The asymptotic properties of these estimators are also briefly discussed. Both methods are illustrated with an example; the SAS programs for these examples are available in Appendix B. In the derivations of the estimators properties of the Kronecker product and *vec* operator are used, as well as rules of vector and matrix differentiation. These properties and rules are given in Appendix A.

Suppose we have k time series processes that were generated by a stationary VAR(p) process as defined in (2.19). For each time series a sample of size T is observed. Assume that p presample values for each of the k variables are available, namely $\boldsymbol{y}_{-p+1}, \boldsymbol{y}_{-p+2} \ldots, \boldsymbol{y}_{-1}, \boldsymbol{y}_0$.

In what follows it is assumed that the vector of constant terms and the autoregressive coefficient matrices are unknown, hence the aim is to estimate them.

3.2 MULTIVARIATE LEAST SQUARES ESTIMATION

In this section some basic notation is introduced, the least squares estimator is derived and its asymptotic properties given. An example is provided to illustrate this method of estimation.

3.2.1 Notation

In this paragraph, the notation that will be used in the derivation of the least squares estimator is defined.

$$\left.\begin{aligned}
&\boldsymbol{Y} : k \times T = \begin{pmatrix} \boldsymbol{y}_1 & \boldsymbol{y}_2 & \cdots & \boldsymbol{y}_T \end{pmatrix} = \begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1T} \\ y_{21} & y_{22} & \cdots & y_{2T} \\ \vdots & \vdots & \vdots & \vdots \\ y_{k1} & y_{k2} & \cdots & y_{kT} \end{pmatrix} \\
&\boldsymbol{B} : k \times (kp+1) = \begin{pmatrix} \boldsymbol{c} & \boldsymbol{\Phi}_1 & \boldsymbol{\Phi}_2 & \cdots & \boldsymbol{\Phi}_p \end{pmatrix} \\
&\boldsymbol{Z}_t : (kp+1) \times 1 = \begin{pmatrix} 1 \\ \boldsymbol{y}_t \\ \vdots \\ \boldsymbol{y}_{t-p+1} \end{pmatrix} \\
&\boldsymbol{Z} : (kp+1) \times T = \begin{pmatrix} \boldsymbol{Z}_0 & \boldsymbol{Z}_1 & \cdots & \boldsymbol{Z}_{T-1} \end{pmatrix} \\
&\boldsymbol{A} : k \times T = \begin{pmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_T \end{pmatrix}
\end{aligned}\right\} \tag{3.1}$$

Furthermore, the dimensions of these matrices after applying the *vec* operator become

$$vec(\boldsymbol{Y}) : kT \times 1$$

$$vec(\boldsymbol{B}) : (k^2 p + k) \times 1$$

$$vec(\boldsymbol{A}) : kT \times 1$$

Using notation (3.1), the VAR(p) model in (2.19) can be written as

$$\begin{aligned}
\boldsymbol{y}_t &= \boldsymbol{c} + \boldsymbol{\Phi}_1 \boldsymbol{y}_{t-1} + \boldsymbol{\Phi}_2 \boldsymbol{y}_{t-2} + \ldots + \boldsymbol{\Phi}_p \boldsymbol{y}_{t-p} + \boldsymbol{a}_t \\
&= \boldsymbol{B}\boldsymbol{Z}_{t-1} + \boldsymbol{a}_t
\end{aligned} \tag{3.2}$$

Equation (3.2) can be expanded to model $\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_T$ simultaneously,

$$\begin{aligned}(\boldsymbol{y}_1 \quad \boldsymbol{y}_2 \quad \cdots \quad \boldsymbol{y}_T) &= \boldsymbol{B}(\boldsymbol{Z}_0 \quad \boldsymbol{Z}_1 \quad \cdots \quad \boldsymbol{Z}_{T-1}) + \boldsymbol{A} \\ \boldsymbol{Y} &= \boldsymbol{BZ} + \boldsymbol{A} \end{aligned} \tag{3.3}$$

Applying the *vec* operator and its properties, (3.3) becomes

$$\begin{aligned} vec(\boldsymbol{Y}) &= vec(\boldsymbol{BZ}) + vec(\boldsymbol{A}) \quad \text{using (A1.1)} \\ &= (\boldsymbol{Z}' \otimes \boldsymbol{I}_k)vec(\boldsymbol{B}) + vec(\boldsymbol{A}) \quad \text{using (A1.2)} \end{aligned} \tag{3.4}$$

The covariance matrix of $vec(\boldsymbol{A})$ is

$$\begin{aligned} E[vec(\boldsymbol{A})vec(\boldsymbol{A})'] &= E\left[\begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \vdots \\ \boldsymbol{a}_T \end{pmatrix} (\boldsymbol{a}_1' \quad \boldsymbol{a}_2' \quad \cdots \quad \boldsymbol{a}_T')\right] \\ &= E\begin{bmatrix} \boldsymbol{a}_1\boldsymbol{a}_1' & \boldsymbol{a}_1\boldsymbol{a}_2' & \cdots & \boldsymbol{a}_1\boldsymbol{a}_T' \\ \boldsymbol{a}_2\boldsymbol{a}_1' & \boldsymbol{a}_2\boldsymbol{a}_2' & \cdots & \boldsymbol{a}_2\boldsymbol{a}_T' \\ \vdots & \vdots & \vdots & \vdots \\ \boldsymbol{a}_T\boldsymbol{a}_1' & \boldsymbol{a}_T\boldsymbol{a}_2' & \cdots & \boldsymbol{a}_T\boldsymbol{a}_T' \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{\Sigma}_a & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Sigma}_a & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{\Sigma}_a \end{bmatrix} \\ &= \boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a \end{aligned} \tag{3.5}$$

where

$$\boldsymbol{\Sigma}_a = E(\boldsymbol{a}_t\boldsymbol{a}_t') \quad \text{(from (2.7))}$$

and $E(\boldsymbol{a}_t\boldsymbol{a}_s') = \boldsymbol{0}$ for $t \neq s$.

3.2.2 Least squares estimation

In order to estimate $vec(\boldsymbol{B})$ by means of multivariate least squares estimation (generalised least squares estimation), we need to select the estimator that minimises the sum of squares of the difference between the observed values ($\boldsymbol{Y}$) and the estimated values ($\boldsymbol{BZ}$), namely $vec(\boldsymbol{Y}) - vec(\boldsymbol{BZ}) = vec(\boldsymbol{A})$. (Draper & Smith, 1998) Let the sum of squares be denoted by $S(\ )$. Therefore, minimise

$$S(vec(\boldsymbol{B})) = vec(\boldsymbol{A})'(\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a)^{-1} vec(\boldsymbol{A})$$

$$
\begin{aligned}
&= vec(\boldsymbol{A})'\left(\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a^{-1}\right)vec(\boldsymbol{A}) \quad \text{using (A2.1)} \\
&= vec(\boldsymbol{Y}-\boldsymbol{BZ})'\left(\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a^{-1}\right)vec(\boldsymbol{Y}-\boldsymbol{BZ}) \\
&= \left[vec(\boldsymbol{Y}) - vec(\boldsymbol{BZ})\right]'\left(\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a^{-1}\right)\left[vec(\boldsymbol{Y}) - vec(\boldsymbol{BZ})\right] \quad \text{using (A1.1)} \\
&= \left[vec(\boldsymbol{Y}) - (\boldsymbol{Z}'\otimes \boldsymbol{I}_k)vec(\boldsymbol{B})\right]'\left(\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a^{-1}\right)\left[vec(\boldsymbol{Y}) - (\boldsymbol{Z}'\otimes \boldsymbol{I}_k)vec(\boldsymbol{B})\right] \quad \text{using (A1.2)} \qquad (3.6)
\end{aligned}
$$

Take note that by multiplying (3.6),

$$
\begin{aligned}
S\left(vec(\boldsymbol{B})\right) = {}& vec(\boldsymbol{Y})'\left(\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a^{-1}\right)vec(\boldsymbol{Y}) \\
&+ vec(\boldsymbol{B})'(\boldsymbol{Z}'\otimes \boldsymbol{I}_k)'\left(\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a^{-1}\right)(\boldsymbol{Z}'\otimes \boldsymbol{I}_k)vec(\boldsymbol{B}) \\
&- 2vec(\boldsymbol{B})'(\boldsymbol{Z}'\otimes \boldsymbol{I}_k)'\left(\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a^{-1}\right)vec(\boldsymbol{Y})
\end{aligned}
\qquad (3.7)
$$

Applying the properties of the Kronecker product and the *vec* operator, (3.7) simplifies to

$$
\begin{aligned}
S\left(vec(\boldsymbol{B})\right) &= vec(\boldsymbol{Y})'\left(\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a^{-1}\right)vec(\boldsymbol{Y}) + vec(\boldsymbol{B})'(\boldsymbol{Z}\otimes \boldsymbol{I}_k)\left(\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a^{-1}\right)(\boldsymbol{Z}'\otimes \boldsymbol{I}_k)vec(\boldsymbol{B}) \\
&\quad - 2vec(\boldsymbol{B})'(\boldsymbol{Z}\otimes \boldsymbol{I}_k)\left(\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a^{-1}\right)vec(\boldsymbol{Y}) \quad \text{using (A2.2)} \\
&= vec(\boldsymbol{Y})'\left(\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a^{-1}\right)vec(\boldsymbol{Y}) + vec(\boldsymbol{B})'\left(\boldsymbol{Z}\otimes \boldsymbol{\Sigma}_a^{-1}\right)(\boldsymbol{Z}'\otimes \boldsymbol{I}_k)vec(\boldsymbol{B}) \\
&\quad - 2vec(\boldsymbol{B})'\left(\boldsymbol{Z}\otimes \boldsymbol{\Sigma}_a^{-1}\right)vec(\boldsymbol{Y}) \quad \text{using (A2.3)} \\
&= vec(\boldsymbol{Y})'\left(\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a^{-1}\right)vec(\boldsymbol{Y}) + vec(\boldsymbol{B})'\left(\boldsymbol{ZZ}'\otimes \boldsymbol{\Sigma}_a^{-1}\right)vec(\boldsymbol{B}) \\
&\quad - 2vec(\boldsymbol{B})'\left(\boldsymbol{Z}\otimes \boldsymbol{\Sigma}_a^{-1}\right)vec(\boldsymbol{Y}) \quad \text{using (A2.3)}
\end{aligned}
\qquad (3.8)
$$

Differentiating $S\left(vec(\boldsymbol{B})\right)$ in (3.8) with respect to $vec(\boldsymbol{B})$,

$$
\begin{aligned}
\frac{\partial S\left(vec(\boldsymbol{B})\right)}{\partial vec(\boldsymbol{B})} &= \left[\left(\boldsymbol{ZZ}'\otimes \boldsymbol{\Sigma}_a^{-1}\right) + \left(\boldsymbol{ZZ}'\otimes \boldsymbol{\Sigma}_a^{-1}\right)'\right]vec(\boldsymbol{B}) - 2\left(\boldsymbol{Z}\otimes \boldsymbol{\Sigma}_a^{-1}\right)vec(\boldsymbol{Y}) \\
&\quad \text{using (A3.2), (A3.1)} \\
&= \left[\left(\boldsymbol{ZZ}'\otimes \boldsymbol{\Sigma}_a^{-1}\right) + \left(\boldsymbol{ZZ}'\otimes \left(\boldsymbol{\Sigma}_a'\right)^{-1}\right)\right]vec(\boldsymbol{B}) - 2\left(\boldsymbol{Z}\otimes \boldsymbol{\Sigma}_a^{-1}\right)vec(\boldsymbol{Y}) \\
&\quad \text{using (A2.2)} \\
&= \left[\left(\boldsymbol{ZZ}'\otimes \boldsymbol{\Sigma}_a^{-1}\right) + \left(\boldsymbol{ZZ}'\otimes \boldsymbol{\Sigma}_a^{-1}\right)\right]vec(\boldsymbol{B}) - 2\left(\boldsymbol{Z}\otimes \boldsymbol{\Sigma}_a^{-1}\right)vec(\boldsymbol{Y}) \\
&= 2\left(\boldsymbol{ZZ}'\otimes \boldsymbol{\Sigma}_a^{-1}\right)vec(\boldsymbol{B}) - 2\left(\boldsymbol{Z}\otimes \boldsymbol{\Sigma}_a^{-1}\right)vec(\boldsymbol{Y}) \qquad (3.9)
\end{aligned}
$$

Setting the partial derivatives in (3.9) equal to zero, the normal equations are

$$
\left(\boldsymbol{ZZ}'\otimes \boldsymbol{\Sigma}_a^{-1}\right)vec(\hat{\boldsymbol{B}}) = \left(\boldsymbol{Z}\otimes \boldsymbol{\Sigma}_a^{-1}\right)vec(\boldsymbol{Y}) \qquad (3.10)
$$

From the normal equations in (3.10) the least squares estimator, $vec(\hat{\boldsymbol{B}})$, is

$$
\begin{aligned}
vec(\hat{\boldsymbol{B}}) &= \left(\boldsymbol{ZZ}' \otimes \boldsymbol{\Sigma}_a^{-1}\right)^{-1}\left(\boldsymbol{Z} \otimes \boldsymbol{\Sigma}_a^{-1}\right) vec(\boldsymbol{Y}) \\
&= \left((\boldsymbol{ZZ}')^{-1} \otimes \boldsymbol{\Sigma}_a\right)\left(\boldsymbol{Z} \otimes \boldsymbol{\Sigma}_a^{-1}\right) vec(\boldsymbol{Y}) \quad \text{using (A2.1)} \\
&= \left((\boldsymbol{ZZ}')^{-1}\boldsymbol{Z} \otimes \boldsymbol{I}_k\right) vec(\boldsymbol{Y}) \qquad \text{using (A2.3)}
\end{aligned} \tag{3.11}
$$

Take note that the existence of the inverse of $\boldsymbol{ZZ}'$ follows from the fact that we assume $\boldsymbol{ZZ}'$ is positive definite, which implies that it is nonsingular.

The least squares estimator, $vec(\hat{\boldsymbol{B}})$, minimises $S(vec(\boldsymbol{B}))$ since the Hessian of $S(vec(\boldsymbol{B}))$, which is the partial derivative of (3.9) with respect to $vec(\boldsymbol{B})'$,

$$
\frac{\partial^2 S(vec(\boldsymbol{B}))}{\partial vec(\boldsymbol{B})\partial(vec(\boldsymbol{B})')} = 2\left(\boldsymbol{ZZ}' \otimes \boldsymbol{\Sigma}_a^{-1}\right) \quad \text{using (A3.4)} \tag{3.12}
$$

is positive definite.

Note that the multivariate least squares estimator $vec(\hat{\boldsymbol{B}})$ is identical to the ordinary least squares estimator obtained by minimising $\bar{S}(vec(\boldsymbol{B}))$,

$$
\begin{aligned}
\bar{S}(vec(\boldsymbol{B})) &= vec(\boldsymbol{A})'vec(\boldsymbol{A}) \\
&= [vec(\boldsymbol{Y}) - vec(\boldsymbol{BZ})]'[vec(\boldsymbol{Y}) - vec(\boldsymbol{BZ})] \\
&= [vec(\boldsymbol{Y}) - (\boldsymbol{Z}' \otimes \boldsymbol{I}_k)vec(\boldsymbol{B})]'[vec(\boldsymbol{Y}) - (\boldsymbol{Z}' \otimes \boldsymbol{I}_k)vec(\boldsymbol{B})] \quad \text{using (A1.2)} \\
&= vec(\boldsymbol{Y})'vec(\boldsymbol{Y}) + vec(\boldsymbol{B})'(\boldsymbol{Z}' \otimes \boldsymbol{I}_k)'(\boldsymbol{Z}' \otimes \boldsymbol{I}_k)vec(\boldsymbol{B}) \\
&\quad - 2vec(\boldsymbol{B})'(\boldsymbol{Z}' \otimes \boldsymbol{I}_k)'vec(\boldsymbol{Y}) \\
&= vec(\boldsymbol{Y})'vec(\boldsymbol{Y}) + vec(\boldsymbol{B})'(\boldsymbol{Z} \otimes \boldsymbol{I}_k)(\boldsymbol{Z}' \otimes \boldsymbol{I}_k)vec(\boldsymbol{B}) \\
&\quad - 2vec(\boldsymbol{B})'(\boldsymbol{Z} \otimes \boldsymbol{I}_k)vec(\boldsymbol{Y}) \quad \text{using (A2.2)} \\
&= vec(\boldsymbol{Y})'vec(\boldsymbol{Y}) + vec(\boldsymbol{B})'(\boldsymbol{ZZ}' \otimes \boldsymbol{I}_k)vec(\boldsymbol{B}) \\
&\quad - 2vec(\boldsymbol{B})'(\boldsymbol{Z} \otimes \boldsymbol{I}_k)vec(\boldsymbol{Y}) \quad \text{using (A2.3)}
\end{aligned} \tag{3.13}
$$

The derivative of $\bar{S}(vec(\boldsymbol{B}))$ in (3.13) with respect to $vec(\boldsymbol{B})$ is

$$
\begin{aligned}
\frac{\partial \bar{S}(vec(\boldsymbol{B}))}{\partial vec(\boldsymbol{B})} &= \left[(\boldsymbol{ZZ}' \otimes \boldsymbol{I}_k) + (\boldsymbol{ZZ}' \otimes \boldsymbol{I}_k)'\right]vec(\boldsymbol{B}) - 2(\boldsymbol{Z} \otimes \boldsymbol{I}_k)vec(\boldsymbol{Y}) \quad \text{using (A3.2), (A3.1)} \\
&= [(\boldsymbol{ZZ}' \otimes \boldsymbol{I}_k) + (\boldsymbol{ZZ}' \otimes \boldsymbol{I}_k)]vec(\boldsymbol{B}) - 2(\boldsymbol{Z} \otimes \boldsymbol{I}_k)vec(\boldsymbol{Y}) \quad \text{using (A2.2)} \\
&= 2(\boldsymbol{ZZ}' \otimes \boldsymbol{I}_k)vec(\boldsymbol{B}) - 2(\boldsymbol{Z} \otimes \boldsymbol{I}_k)vec(\boldsymbol{Y})
\end{aligned} \tag{3.14}
$$

Setting (3.14) equal to zero, we obtain $(\boldsymbol{ZZ}' \otimes \boldsymbol{I}_k)vec(\hat{\boldsymbol{B}}) = (\boldsymbol{Z} \otimes \boldsymbol{I}_k)vec(\boldsymbol{Y})$.

Then the ordinary least squares estimator, $vec(\hat{\boldsymbol{B}})$, is

$$\begin{aligned} vec(\hat{\boldsymbol{B}}) &= \left(\boldsymbol{ZZ}' \otimes \boldsymbol{I}_k\right)^{-1}\left(\boldsymbol{Z} \otimes \boldsymbol{I}_k\right)vec(\boldsymbol{Y}) \\ &= \left((\boldsymbol{ZZ}')^{-1} \otimes \boldsymbol{I}_k\right)\left(\boldsymbol{Z} \otimes \boldsymbol{I}_k\right)vec(\boldsymbol{Y}) \quad \text{using (A2.1)} \\ &= \left((\boldsymbol{ZZ}')^{-1}\boldsymbol{Z} \otimes \boldsymbol{I}_k\right)vec(\boldsymbol{Y}) \quad \text{using (A2.3)} \end{aligned}$$

which is the same as the multivariate least squares estimator obtained in (3.11).

The Hessian, $\dfrac{\partial^2 \bar{S}\left(vec(\boldsymbol{B})\right)}{\partial vec(\boldsymbol{B})\partial\left(vec(\boldsymbol{B})'\right)} = 2\left(\boldsymbol{ZZ}' \otimes \boldsymbol{I}_k\right)$ (using (A3.4)) is positive definite, therefore $vec(\hat{\boldsymbol{B}})$ minimises $\bar{S}\left(vec(\boldsymbol{B})\right)$.

The least squares estimator $vec(\hat{\boldsymbol{B}})$ in (3.11) can also be written in an alternative form,

$$\begin{aligned} vec(\hat{\boldsymbol{B}}) &= \left((\boldsymbol{ZZ}')^{-1}\boldsymbol{Z} \otimes \boldsymbol{I}_k\right)vec(\boldsymbol{Y}) \\ &= vec\left(\boldsymbol{YZ}'(\boldsymbol{ZZ}')^{-1}\right) \quad \text{using (A1.2)} \end{aligned} \tag{3.15}$$

implying that

$$\hat{\boldsymbol{B}} = \boldsymbol{YZ}'(\boldsymbol{ZZ}')^{-1} \tag{3.16}$$

3.2.3 Asymptotic properties of the least squares estimator

Now that the least squares estimator is determined, a way is needed to establish the significance of the individual estimates. Usually the estimate is divided with its standard error to obtain a t-ratio that can be compared with a critical value. In order to do this, the distribution of the estimator is needed.

Proposition 3.1 of Lütkepohl (2005) addresses the consistency and the asymptotic normality of the least squares estimator, namely

"Let $\{\boldsymbol{y}_t\}$ be a stable, k-dimensional VAR(p) process with standard white noise residuals, $\hat{\boldsymbol{B}} = \boldsymbol{YZ}'(\boldsymbol{ZZ}')^{-1}$ is the LS estimator of the VAR coefficients $\boldsymbol{B}$. Then

$$\text{plim}\hat{\boldsymbol{B}} = \boldsymbol{B}$$

and

$$\sqrt{T}vec\left(\hat{\boldsymbol{B}} - \boldsymbol{B}\right) \xrightarrow{d} N\left(\boldsymbol{0}, \boldsymbol{\Gamma}^{-1} \otimes \boldsymbol{\Sigma}_a\right) \tag{3.17}$$

where $\boldsymbol{\Gamma} = \text{plim}\dfrac{\boldsymbol{ZZ}'}{T}$."

A standard white noise process is a white noise process as described in section 2.3, with the additional property that all the fourth moments must exist and be bounded.

Consistent estimators of the unknown parameters $\boldsymbol{\Gamma}$ and $\boldsymbol{\Sigma}_a$ in (3.17) are given by Lütkepohl (2005),

$$\hat{\boldsymbol{\Gamma}} = \frac{\boldsymbol{ZZ}'}{T} \tag{3.18}$$

$$\tilde{\boldsymbol{\Sigma}}_a = \frac{1}{T}\sum_{t=1}^{T}\hat{\boldsymbol{a}}_t\hat{\boldsymbol{a}}_t' \tag{3.19}$$

where $\hat{\boldsymbol{a}}_t$ is the vector of estimated residuals. The estimate of $\boldsymbol{\Sigma}_a$ in (3.19) can be written in terms of the notation defined in (3.1),

$$\begin{aligned}
\tilde{\boldsymbol{\Sigma}}_a &= \frac{1}{T}\hat{\boldsymbol{A}}\hat{\boldsymbol{A}}' \\
&= \frac{1}{T}\left(\boldsymbol{Y} - \hat{\boldsymbol{B}}\boldsymbol{Z}\right)\left(\boldsymbol{Y} - \hat{\boldsymbol{B}}\boldsymbol{Z}\right)' \quad \text{(from (3.3))} \\
&= \frac{1}{T}\left(\boldsymbol{Y} - \boldsymbol{YZ}'(\boldsymbol{ZZ}')^{-1}\boldsymbol{Z}\right)\left(\boldsymbol{Y} - \boldsymbol{YZ}'(\boldsymbol{ZZ}')^{-1}\boldsymbol{Z}\right)' \\
&= \frac{1}{T}\left(\boldsymbol{Y} - \boldsymbol{YZ}'(\boldsymbol{ZZ}')^{-1}\boldsymbol{Z}\right)\left(\boldsymbol{Y}' - \boldsymbol{Z}'(\boldsymbol{ZZ}')^{-1}\boldsymbol{ZY}'\right) \\
&= \frac{1}{T}\boldsymbol{Y}\left(\boldsymbol{I}_T - \boldsymbol{Z}'(\boldsymbol{ZZ}')^{-1}\boldsymbol{Z}\right)\left(\boldsymbol{I}_T - \boldsymbol{Z}'(\boldsymbol{ZZ}')^{-1}\boldsymbol{Z}\right)\boldsymbol{Y}' \\
&= \frac{1}{T}\boldsymbol{Y}\left(\boldsymbol{I}_T - \boldsymbol{Z}'(\boldsymbol{ZZ}')^{-1}\boldsymbol{Z}\right)\boldsymbol{Y}'
\end{aligned} \tag{3.20}$$

$\tilde{\boldsymbol{\Sigma}}_a$ is a biased estimator which can be adjusted to obtain an unbiased estimator $\hat{\boldsymbol{\Sigma}}_a$,

$$\hat{\boldsymbol{\Sigma}}_a = \frac{T}{T-kp-1}\tilde{\boldsymbol{\Sigma}}_a = \frac{1}{T-kp-1}\boldsymbol{Y}\left(\boldsymbol{I}_T - \boldsymbol{Z}'(\boldsymbol{ZZ}')^{-1}\boldsymbol{Z}\right)\boldsymbol{Y}' \tag{3.21}$$

Lütkepohl (2005) showed that (3.18), (3.19) and (3.20) are consistent under certain constraints.

Substituting $\hat{\boldsymbol{\Gamma}}$ and $\hat{\boldsymbol{\Sigma}}_a$ into (3.17), it follows that

$$vec\left(\hat{\boldsymbol{B}} - \boldsymbol{B}\right) \xrightarrow{d} N\left(\boldsymbol{0}, \frac{1}{T}\left(\frac{\boldsymbol{ZZ}'}{T}\right)^{-1} \otimes \hat{\boldsymbol{\Sigma}}_a\right)$$

$$vec\left(\hat{\boldsymbol{B}} - \boldsymbol{B}\right) \xrightarrow{d} N\left(\boldsymbol{0}, (\boldsymbol{ZZ}')^{-1} \otimes \hat{\boldsymbol{\Sigma}}_a\right) \tag{3.22}$$

The square root of the diagonal elements of $(\boldsymbol{ZZ}')^{-1} \otimes \hat{\boldsymbol{\Sigma}}_a$, denoted by $\hat{s}_i$, is the estimated standard deviation of the corresponding $\hat{\beta}_i - \beta_i$, the i-th element of $vec\left(\hat{\boldsymbol{B}} - \boldsymbol{B}\right)$. Equation (3.22) implies that $\dfrac{\hat{\beta}_i - \beta_i}{\hat{s}_i}$ has an approximate t-distribution which is asymptotically standard normal. This can be used for hypothesis testing regarding the significance of the least squares estimator.

The following example illustrates the calculation, using the expressions derived in this section, of the least squares estimates of a generated VAR(1) model. The asymptotic results are used to obtain t-ratios that can be used in testing for the significance of the parameter values. The results are compared to the output of the VARMAX procedure in the SAS/ETS module.

Example 3.1*

Consider the bivariate VAR(1) model $\boldsymbol{y}_t = \begin{pmatrix} 0.5 & 0.6 \\ 0.1 & 0.4 \end{pmatrix} \boldsymbol{y}_{t-1} + \boldsymbol{a}_t$ with $\boldsymbol{\Sigma}_a = \begin{pmatrix} 1.0 & 0.5 \\ 0.5 & 0.9 \end{pmatrix}$.

A sample of size 500 is generated. The method used to generate data from a multivariate normal distribution is discussed after the example. The least squares estimate of $vec(\boldsymbol{B})$ in (3.11), is

$$vec(\hat{\boldsymbol{B}}) = \begin{pmatrix} -0.055 \\ 0.028 \\ 0.516 \\ 0.115 \\ 0.503 \\ 0.320 \end{pmatrix}$$

$$\therefore \hat{\boldsymbol{B}} = \left(\hat{\boldsymbol{c}} \quad \hat{\boldsymbol{\Phi}}_1\right) = \begin{pmatrix} -0.055 & 0.516 & 0.503 \\ 0.028 & 0.115 & 0.320 \end{pmatrix}$$

The estimates for $\boldsymbol{\Gamma}$ and $\boldsymbol{\Sigma}_a$ according to (3.18) and (3.21) are

$$\hat{\boldsymbol{\Gamma}} = \begin{pmatrix} 1.000 & -0.070 & 0.033 \\ -0.070 & 2.598 & 1.131 \\ 0.033 & 1.131 & 1.211 \end{pmatrix}$$

* The SAS program is provided in Appendix B page 127.

$$\hat{\boldsymbol{\Sigma}}_a = \begin{pmatrix} 1.005 & 0.532 \\ 0.532 & 0.974 \end{pmatrix}$$

and

$$(\boldsymbol{ZZ}')^{-1} \otimes \hat{\boldsymbol{\Sigma}}_a = \begin{pmatrix} 0.0020256 & 0.0010729 & 0.0001326 & 0.0000703 & -0.000179 & -0.000095 \\ 0.0010729 & 0.0019633 & 0.0000703 & 0.0001286 & -0.000095 & -0.000173 \\ 0.0001326 & 0.0000703 & 0.001312 & 0.000695 & -0.001228 & -0.000651 \\ 0.0000703 & 0.0001286 & 0.000695 & 0.0012717 & -0.000651 & -0.001191 \\ -0.000179 & -0.000095 & -0.001228 & -0.000651 & 0.0028115 & 0.0014892 \\ -0.000095 & -0.000173 & -0.000651 & -0.001191 & 0.0014892 & 0.0027251 \end{pmatrix}$$

Using these estimates together with (3.22) makes it possible to determine the standard errors and t-ratios of the least squares estimate. The results are summarised in the table below.

	$vec(\hat{\boldsymbol{B}})$	**Standard error** $(\hat{s}_i)$	**t-ratio**
$\hat{c}_1$	-0.055	0.045	-1.229
$\hat{c}_2$	0.028	0.044	0.628
$\hat{\phi}_{11}$	0.516	0.036	14.244
$\hat{\phi}_{21}$	0.115	0.036	3.222
$\hat{\phi}_{12}$	0.503	0.053	9.491
$\hat{\phi}_{22}$	0.320	0.052	6.137

This is comparable to the SAS output that is provided below. The slight differences are due to the assumption that the presample values are known when calculating $vec(\hat{\boldsymbol{B}})$ in (3.11), and in this example $\boldsymbol{y}_0$ was generated the same way as the process was generated.

```
                         The VARMAX Procedure

                      Model Parameter Estimates

                                             Standard
Equation Parameter        Estimate              Error t Value Pr > |t|
Variable
y1       CONST1           -0.05856          0.04500    -1.30   0.1938 1
         AR1_1_1           0.51289          0.03623    14.15   0.0001 y1(t-1)
         AR1_1_2           0.50467          0.05297     9.53   0.0001 y2(t-1)
y2       CONST2            0.02543          0.04435     0.57   0.5666 1
         AR1_2_1           0.11264          0.03571     3.15   0.0017 y1(t-1)
         AR1_2_2           0.32141          0.05220     6.16   0.0001 y2(t-1)
```

Covariances of Innovations

Variable	y1	y2
y1	1.00270	0.53016
y2	0.53016	0.97378

All the parameter values are significant except the constant terms. This is expected since the data was generated with the constant vector equal to zero.

Generating data from a multivariate normal distribution, $\boldsymbol{X} \sim N(\boldsymbol{\mu},\boldsymbol{\Sigma})$

The VARMASIM CALL in SAS IML was used to generate data from a multivariate normal distribution. Alternatively, data can be generated using the method described below.

Let $\boldsymbol{D} \sim N(\boldsymbol{0},\boldsymbol{I})$. This implies that the components of $\boldsymbol{D}$ are independent $N(0,1)$ variables, which can easily be generated separately using, for example, the RANNOR function is SAS IML. The positive definite covariance matrix $\boldsymbol{\Sigma}$ can be factored, using the Choleski decomposition, as

$$\boldsymbol{\Sigma} = \boldsymbol{P}\boldsymbol{P}'$$

where $\boldsymbol{P}$ is a lower triangular matrix with positive elements on the main diagonal. $\boldsymbol{P}'$ can be obtained with the HALF function in SAS IML.

Let $\boldsymbol{X} = \boldsymbol{P}\boldsymbol{D} + \boldsymbol{\mu}$. $\boldsymbol{X}$ has a multivariate normal distribution, since it is a linear function of a multivariate normal random vector. The parameters are

$$E(\boldsymbol{X}) = \boldsymbol{P}E(\boldsymbol{D}) + \boldsymbol{\mu} = \boldsymbol{\mu}$$

$$\begin{aligned} \operatorname{cov}(\boldsymbol{X},\boldsymbol{X}') &= \operatorname{cov}\left(\boldsymbol{P}\boldsymbol{D} + \boldsymbol{\mu}, (\boldsymbol{P}\boldsymbol{D} + \boldsymbol{\mu})'\right) \\ &= \operatorname{cov}(\boldsymbol{P}\boldsymbol{D}, \boldsymbol{D}'\boldsymbol{P}') \\ &= \boldsymbol{P}\operatorname{cov}(\boldsymbol{D},\boldsymbol{D}')\boldsymbol{P}' \\ &= \boldsymbol{P}\boldsymbol{P}' \\ &= \boldsymbol{\Sigma} \end{aligned}$$

$$\therefore \boldsymbol{X} \sim N(\boldsymbol{\mu},\boldsymbol{\Sigma})$$

The method described above can be employed to generate the multivariate white noise series $\{\boldsymbol{a}_t\}$ with mean zero and covariance matrix $\boldsymbol{\Sigma}_a$. The white noise series can then be used to generate observations from any specified model. As an illustration the data from the bivariate VAR(1) model, stated in Example 3.1, was also generated using this method. The SAS program is given in Appendix B.

3.3 MAXIMUM LIKELIHOOD ESTIMATION

In this section the maximum likelihood estimator of the mean, the coefficient matrices and the white noise covariance matrix are derived by obtaining the likelihood functions and maximising them with respect to each of the unknown parameters. The asymptotic properties of the maximum likelihood estimators are provided. The section is concluded with a numerical example using the matrix expressions derived.

3.3.1 The likelihood function

When the distribution of a process is known, the maximum likelihood estimator can be determined. Assume that we have a Gaussian VAR(p) process, this means that the white noise process $\{\boldsymbol{a}_t\}$ is normally distributed with mean zero and covariance matrix $\boldsymbol{\Sigma}_a$. This, together with (3.5) implies that $vec(\boldsymbol{A})$ has a $N(\boldsymbol{0}, \boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a)$ distribution, with probability density function given by

$$f(vec(\boldsymbol{A})) = \frac{1}{(2\pi)^{kT/2}} \left|\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a\right|^{-1/2} \exp\left[-\frac{1}{2} vec(\boldsymbol{A})'(\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a)^{-1} vec(\boldsymbol{A})\right] \tag{3.23}$$

The aim is to utilise (3.23) to determine the probability density function of $vec(\boldsymbol{Y})$ using the transformation theorem (A5.1). Rewriting the deviation from the mean form in (2.22) yields

$$\boldsymbol{a}_t = (\boldsymbol{y}_t - \boldsymbol{\mu}) - \boldsymbol{\Phi}_1(\boldsymbol{y}_{t-1} - \boldsymbol{\mu}) - \boldsymbol{\Phi}_2(\boldsymbol{y}_{t-2} - \boldsymbol{\mu}) - \ldots - \boldsymbol{\Phi}_p(\boldsymbol{y}_{t-p} - \boldsymbol{\mu})$$

then

$$\begin{aligned}
\boldsymbol{a}_1 &= (\boldsymbol{y}_1 - \boldsymbol{\mu}) - \boldsymbol{\Phi}_1(\boldsymbol{y}_0 - \boldsymbol{\mu}) - \boldsymbol{\Phi}_2(\boldsymbol{y}_{-1} - \boldsymbol{\mu}) - \ldots - \boldsymbol{\Phi}_p(\boldsymbol{y}_{-p+1} - \boldsymbol{\mu}) \\
\boldsymbol{a}_2 &= (\boldsymbol{y}_2 - \boldsymbol{\mu}) - \boldsymbol{\Phi}_1(\boldsymbol{y}_1 - \boldsymbol{\mu}) - \boldsymbol{\Phi}_2(\boldsymbol{y}_0 - \boldsymbol{\mu}) - \ldots - \boldsymbol{\Phi}_p(\boldsymbol{y}_{-p+2} - \boldsymbol{\mu}) \\
&\vdots \\
\boldsymbol{a}_T &= (\boldsymbol{y}_T - \boldsymbol{\mu}) - \boldsymbol{\Phi}_1(\boldsymbol{y}_{T-1} - \boldsymbol{\mu}) - \boldsymbol{\Phi}_2(\boldsymbol{y}_{T-2} - \boldsymbol{\mu}) - \ldots - \boldsymbol{\Phi}_p(\boldsymbol{y}_{-p+T} - \boldsymbol{\mu})
\end{aligned} \tag{3.24}$$

Let $\boldsymbol{a}_t = g(\boldsymbol{Y}_t)$ where $\boldsymbol{Y}_t = \begin{pmatrix} \boldsymbol{y}_t \\ \boldsymbol{y}_{t-1} \\ \vdots \\ \boldsymbol{y}_{t-p} \end{pmatrix}$.

Using matrix notation, (3.24) becomes

$$\begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \vdots \\ \boldsymbol{a}_T \end{pmatrix} = \begin{pmatrix} \boldsymbol{I}_k & \boldsymbol{0} & \dots & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ -\boldsymbol{\Phi}_1 & \boldsymbol{I}_k & \cdots & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -\boldsymbol{\Phi}_p & -\boldsymbol{\Phi}_{p-1} & \cdots & \boldsymbol{I}_k & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & -\boldsymbol{\Phi}_p & \cdots & \boldsymbol{I}_k \end{pmatrix} \begin{pmatrix} \boldsymbol{y}_1 - \boldsymbol{\mu} \\ \boldsymbol{y}_2 - \boldsymbol{\mu} \\ \vdots \\ \boldsymbol{y}_T - \boldsymbol{\mu} \end{pmatrix} + \begin{pmatrix} -\boldsymbol{\Phi}_1 & -\boldsymbol{\Phi}_2 & \cdots & -\boldsymbol{\Phi}_{p-1} & -\boldsymbol{\Phi}_p \\ -\boldsymbol{\Phi}_2 & -\boldsymbol{\Phi}_3 & \cdots & -\boldsymbol{\Phi}_p & \boldsymbol{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -\boldsymbol{\Phi}_p & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{y}_0 - \boldsymbol{\mu} \\ \boldsymbol{y}_{-1} - \boldsymbol{\mu} \\ \vdots \\ \boldsymbol{y}_{-p+1} - \boldsymbol{\mu} \end{pmatrix}$$

The partial derivative of $vec(\boldsymbol{A})$ with respect to $vec(\boldsymbol{Y})'$ is

$$\frac{\partial vec(\boldsymbol{A})}{\partial vec(\boldsymbol{Y})'} = \begin{pmatrix} \boldsymbol{I}_k & \boldsymbol{0} & \dots & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ -\boldsymbol{\Phi}_1 & \boldsymbol{I}_k & \cdots & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -\boldsymbol{\Phi}_p & -\boldsymbol{\Phi}_{p-1} & \cdots & \boldsymbol{I}_k & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & -\boldsymbol{\Phi}_p & \cdots & \boldsymbol{I}_k \end{pmatrix} \tag{3.25}$$

therefore the Jacobian of the transformation from $vec(\boldsymbol{A})$ to $vec(\boldsymbol{Y})$ is

$$\left| \frac{\partial vec(\boldsymbol{A})}{\partial vec(\boldsymbol{Y})'} \right| = 1 \tag{3.26}$$

since the derivative in (3.25) is a lower triangular matrix with ones on the main diagonal.

The next step is to rewrite $vec(\boldsymbol{A})$ as a function of $vec(\boldsymbol{Y})$. From the deviation of the mean form (3.24),

$$\begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \vdots \\ \boldsymbol{a}_T \end{pmatrix} = \begin{pmatrix} \boldsymbol{y}_1 - \boldsymbol{\mu} \\ \boldsymbol{y}_2 - \boldsymbol{\mu} \\ \vdots \\ \boldsymbol{y}_T - \boldsymbol{\mu} \end{pmatrix} - vec\left[\begin{pmatrix} \boldsymbol{\Phi}_1 & \boldsymbol{\Phi}_2 & \cdots & \boldsymbol{\Phi}_p \end{pmatrix} \begin{pmatrix} \boldsymbol{y}_0 - \boldsymbol{\mu} & \boldsymbol{y}_1 - \boldsymbol{\mu} & \cdots & \boldsymbol{y}_{T-1} - \boldsymbol{\mu} \\ \boldsymbol{y}_{-1} - \boldsymbol{\mu} & \boldsymbol{y}_0 - \boldsymbol{\mu} & \cdots & \boldsymbol{y}_{T-2} - \boldsymbol{\mu} \\ \vdots & \vdots & \vdots & \vdots \\ \boldsymbol{y}_{-p+1} - \boldsymbol{\mu} & \boldsymbol{y}_{-p+2} - \boldsymbol{\mu} & \cdots & \boldsymbol{y}_{-p+T} - \boldsymbol{\mu} \end{pmatrix} \right]$$

or

$$\begin{aligned} vec(\boldsymbol{A}) &= vec(\boldsymbol{Y}) - \boldsymbol{\mu}^* - vec(\boldsymbol{B}^* \boldsymbol{X}) \\ &= vec(\boldsymbol{Y}) - \boldsymbol{\mu}^* - (\boldsymbol{X}' \otimes \boldsymbol{I}_k) vec(\boldsymbol{B}^*) \quad \text{using (A1.2)} \end{aligned} \tag{3.27}$$

where

$$\boldsymbol{B}^* : k \times kp = \begin{pmatrix} \boldsymbol{\Phi}_1 & \boldsymbol{\Phi}_2 & \cdots & \boldsymbol{\Phi}_p \end{pmatrix} \tag{3.28}$$

$$\boldsymbol{\mu}^* : kT \times 1 = \begin{pmatrix} \boldsymbol{\mu}' & \boldsymbol{\mu}' & \cdots & \boldsymbol{\mu}' \end{pmatrix}'$$

$$\boldsymbol{X} : kp \times T = \begin{pmatrix} \boldsymbol{y}_0 - \boldsymbol{\mu} & \boldsymbol{y}_1 - \boldsymbol{\mu} & \cdots & \boldsymbol{y}_{T-1} - \boldsymbol{\mu} \\ \boldsymbol{y}_{-1} - \boldsymbol{\mu} & \boldsymbol{y}_0 - \boldsymbol{\mu} & \cdots & \boldsymbol{y}_{T-2} - \boldsymbol{\mu} \\ \vdots & \vdots & \vdots & \vdots \\ \boldsymbol{y}_{-p+1} - \boldsymbol{\mu} & \boldsymbol{y}_{-p+2} - \boldsymbol{\mu} & \cdots & \boldsymbol{y}_{-p+T} - \boldsymbol{\mu} \end{pmatrix} \tag{3.29}$$

According to the transformation theorem (A5.1) together with (3.23), (3.26) and (3.27) the probability density function of $vec(\boldsymbol{Y})$ is

$$h\left(vec(\boldsymbol{y}_1 \quad \boldsymbol{y}_2 \quad \ldots \quad \boldsymbol{y}_T)\right) = f\left(vec(g(\boldsymbol{Y}_1) \quad g(\boldsymbol{Y}_2) \quad \ldots \quad g(\boldsymbol{Y}_T))\right)\left|\frac{\partial vec(\boldsymbol{A})}{\partial vec(\boldsymbol{Y})'}\right|$$

$$h\left(vec(\boldsymbol{Y})\right) = f\left(vec(\boldsymbol{a}_1 \quad \boldsymbol{a}_2 \quad \ldots \quad \boldsymbol{a}_T)\right)\left|\frac{\partial vec(\boldsymbol{A})}{\partial vec(\boldsymbol{Y})'}\right|$$

$$= f\left(vec(\boldsymbol{A})\right)\left|\frac{\partial vec(\boldsymbol{A})}{\partial vec(\boldsymbol{Y})'}\right|$$

$$\begin{aligned} = \frac{1}{(2\pi)^{kT/2}} \left|\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a\right|^{-1/2} \exp\Big\{ -\frac{1}{2}\left[vec(\boldsymbol{Y}) - \boldsymbol{\mu}^* - (\boldsymbol{X}' \otimes \boldsymbol{I}_k)vec(\boldsymbol{B}^*)\right]' \times \\ (\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a)^{-1}\left[vec(\boldsymbol{Y}) - \boldsymbol{\mu}^* - (\boldsymbol{X}' \otimes \boldsymbol{I}_k)vec(\boldsymbol{B}^*)\right]\Big\} \end{aligned} \tag{3.30}$$

The log-likelihood function is obtained by taking the natural logarithm of (3.30),

$$\begin{aligned} &\ln L\left(\boldsymbol{\mu}, \boldsymbol{B}^*, \boldsymbol{\Sigma}_a\right) \\ &= -\frac{kT}{2}\ln(2\pi) - \frac{1}{2}\ln\left|\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a\right| - \frac{1}{2}\left[vec(\boldsymbol{Y}) - \boldsymbol{\mu}^* - (\boldsymbol{X}' \otimes \boldsymbol{I}_k)vec(\boldsymbol{B}^*)\right]' \times \\ &\quad (\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a)^{-1}\left[vec(\boldsymbol{Y}) - \boldsymbol{\mu}^* - (\boldsymbol{X}' \otimes \boldsymbol{I}_k)vec(\boldsymbol{B}^*)\right] \end{aligned} \tag{3.31}$$

$$\begin{aligned} &= -\frac{kT}{2}\ln(2\pi) - \frac{1}{2}\ln\left|\boldsymbol{I}_T\right|^k\left|\boldsymbol{\Sigma}_a\right|^T - \frac{1}{2}\left[vec(\boldsymbol{Y}) - \boldsymbol{\mu}^* - (\boldsymbol{X}' \otimes \boldsymbol{I}_k)vec(\boldsymbol{B}^*)\right]' \times \\ &\quad (\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a)^{-1}\left[vec(\boldsymbol{Y}) - \boldsymbol{\mu}^* - (\boldsymbol{X}' \otimes \boldsymbol{I}_k)vec(\boldsymbol{B}^*)\right] \text{ using (A2.4)} \end{aligned}$$

$$\begin{aligned} &= -\frac{kT}{2}\ln(2\pi) - \frac{T}{2}\ln\left|\boldsymbol{\Sigma}_a\right| - \frac{1}{2}\left[vec(\boldsymbol{Y}) - \boldsymbol{\mu}^* - (\boldsymbol{X}' \otimes \boldsymbol{I}_k)vec(\boldsymbol{B}^*)\right]' \times \\ &\quad (\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a)^{-1}\left[vec(\boldsymbol{Y}) - \boldsymbol{\mu}^* - (\boldsymbol{X}' \otimes \boldsymbol{I}_k)vec(\boldsymbol{B}^*)\right] \end{aligned}$$

$$
\begin{aligned}
&= -\frac{kT}{2}\ln(2\pi) - \frac{T}{2}\ln\left|\boldsymbol{\Sigma}_a\right| - \frac{1}{2}\sum_{t=1}^{T}\left[(\boldsymbol{y}_t - \boldsymbol{\mu}) - \sum_{i=1}^{p}\boldsymbol{\Phi}_i(\boldsymbol{y}_{t-i} - \boldsymbol{\mu})\right]' \boldsymbol{\Sigma}_a^{-1} \times \\
&\quad \left[(\boldsymbol{y}_t - \boldsymbol{\mu}) - \sum_{i=1}^{p}\boldsymbol{\Phi}_i(\boldsymbol{y}_{t-i} - \boldsymbol{\mu})\right]
\end{aligned} \tag{3.32}
$$

$$
\begin{aligned}
&= -\frac{kT}{2}\ln(2\pi) - \frac{T}{2}\ln\left|\boldsymbol{\Sigma}_a\right| - \frac{1}{2}\sum_{t=1}^{T}\left(\boldsymbol{y}_t - \sum_{i=1}^{p}\boldsymbol{\Phi}_i\boldsymbol{y}_{t-i}\right)' \boldsymbol{\Sigma}_a^{-1}\left(\boldsymbol{y}_t - \sum_{i=1}^{p}\boldsymbol{\Phi}_i\boldsymbol{y}_{t-i}\right) \\
&\quad - \sum_{t=1}^{T}\left(-\boldsymbol{\mu} + \sum_{i=1}^{p}\boldsymbol{\Phi}_i\boldsymbol{\mu}\right)' \boldsymbol{\Sigma}_a^{-1}\left(\boldsymbol{y}_t - \sum_{i=1}^{p}\boldsymbol{\Phi}_i\boldsymbol{y}_{t-i}\right) \\
&\quad - \frac{1}{2}\sum_{t=1}^{T}\left(-\boldsymbol{\mu} + \sum_{i=1}^{p}\boldsymbol{\Phi}_i\boldsymbol{\mu}\right)' \boldsymbol{\Sigma}_a^{-1}\left(-\boldsymbol{\mu} + \sum_{i=1}^{p}\boldsymbol{\Phi}_i\boldsymbol{\mu}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{kT}{2}\ln(2\pi) - \frac{T}{2}\ln\left|\boldsymbol{\Sigma}_a\right| - \frac{1}{2}\sum_{t=1}^{T}\left(\boldsymbol{y}_t - \sum_{i=1}^{p}\boldsymbol{\Phi}_i\boldsymbol{y}_{t-i}\right)' \boldsymbol{\Sigma}_a^{-1}\left(\boldsymbol{y}_t - \sum_{i=1}^{p}\boldsymbol{\Phi}_i\boldsymbol{y}_{t-i}\right) \\
&\quad + \boldsymbol{\mu}'\left(\boldsymbol{I}_k - \sum_{i=1}^{p}\boldsymbol{\Phi}_i\right)' \boldsymbol{\Sigma}_a^{-1}\sum_{t=1}^{T}\left(\boldsymbol{y}_t - \sum_{i=1}^{p}\boldsymbol{\Phi}_i\boldsymbol{y}_{t-i}\right) \\
&\quad - \frac{T}{2}\boldsymbol{\mu}'\left(\boldsymbol{I}_k - \sum_{i=1}^{p}\boldsymbol{\Phi}_i\right)' \boldsymbol{\Sigma}_a^{-1}\left(\boldsymbol{I}_k - \sum_{i=1}^{p}\boldsymbol{\Phi}_i\right)\boldsymbol{\mu}
\end{aligned} \tag{3.33}
$$

A different expression for the log-likelihood function, in terms of the deviation from the mean, follows from (3.32)

$$
\begin{aligned}
&\ln L(\boldsymbol{\mu}, \boldsymbol{B}^*, \boldsymbol{\Sigma}_a) \\
&= -\frac{kT}{2}\ln(2\pi) - \frac{T}{2}\ln\left|\boldsymbol{\Sigma}_a\right| - \frac{1}{2}tr\left[(\boldsymbol{Y}^0 - \boldsymbol{B}^*\boldsymbol{X})' \boldsymbol{\Sigma}_a^{-1}(\boldsymbol{Y}^0 - \boldsymbol{B}^*\boldsymbol{X})\right]
\end{aligned} \tag{3.34}
$$

where

$\boldsymbol{Y}^0 : k \times T = (\boldsymbol{y}_1 - \boldsymbol{\mu} \quad \boldsymbol{y}_2 - \boldsymbol{\mu} \quad \ldots \quad \boldsymbol{y}_T - \boldsymbol{\mu})$

$\boldsymbol{B}^*$ and $\boldsymbol{X}$ are defined as in (3.28) and (3.29), respectively.

3.3.2 The maximum likelihood estimators

To find the maximum likelihood estimators of $\boldsymbol{\mu}$, $vec(\boldsymbol{B}^*)$ and $\boldsymbol{\Sigma}_a$ we need to determine the partial derivative of the log-likelihood function with respect to each of these unknown parameters.

From (3.33) it follows that

$$\frac{\partial \ln L}{\partial \boldsymbol{\mu}} = \left(\boldsymbol{I}_k - \sum_{i=1}^{p} \boldsymbol{\Phi}_i\right)' \boldsymbol{\Sigma}_a^{-1} \sum_{t=1}^{T} \left(\boldsymbol{y}_t - \sum_{i=1}^{p} \boldsymbol{\Phi}_i \boldsymbol{y}_{t-i}\right) - T\left(\boldsymbol{I}_k - \sum_{i=1}^{p} \boldsymbol{\Phi}_i\right)' \boldsymbol{\Sigma}_a^{-1} \left(\boldsymbol{I}_k - \sum_{i=1}^{p} \boldsymbol{\Phi}_i\right)\boldsymbol{\mu}$$

using (A3.1), (A3.2)

$$= \left(\boldsymbol{I}_k - \sum_{i=1}^{p} \boldsymbol{\Phi}_i\right)' \boldsymbol{\Sigma}_a^{-1} \left[\sum_{t=1}^{T} \left(\boldsymbol{y}_t - \sum_{i=1}^{p} \boldsymbol{\Phi}_i \boldsymbol{y}_{t-i}\right) - T\left(\boldsymbol{I}_k - \sum_{i=1}^{p} \boldsymbol{\Phi}_i\right)\boldsymbol{\mu}\right] \quad (3.35)$$

Setting $\frac{\partial \ln L}{\partial \boldsymbol{\mu}}$ in (3.35) equal to zero, the maximum likelihood estimator of $\boldsymbol{\mu}$, namely $\tilde{\boldsymbol{\mu}}$, is:

$$\sum_{t=1}^{T} \left(\boldsymbol{y}_t - \sum_{i=1}^{p} \tilde{\boldsymbol{\Phi}}_i \boldsymbol{y}_{t-i}\right) = T\left(\boldsymbol{I}_k - \sum_{i=1}^{p} \tilde{\boldsymbol{\Phi}}_i\right)\tilde{\boldsymbol{\mu}}$$

$$\therefore \tilde{\boldsymbol{\mu}} = \frac{1}{T}\left(\boldsymbol{I}_k - \sum_{i=1}^{p} \tilde{\boldsymbol{\Phi}}_i\right)^{-1} \sum_{t=1}^{T} \left(\boldsymbol{y}_t - \sum_{i=1}^{p} \tilde{\boldsymbol{\Phi}}_i \boldsymbol{y}_{t-i}\right) \quad (3.36)$$

where $\tilde{\boldsymbol{\Phi}}_i$ is the maximum likelihood estimator of $\boldsymbol{\Phi}_i$.

From (3.31) the terms involving $vec(\boldsymbol{B}^*)$ are

$$\begin{aligned}
&\frac{1}{2} vec(\boldsymbol{B}^*)' (\boldsymbol{X}' \otimes \boldsymbol{I}_k)' (\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a)^{-1} (vec(\boldsymbol{Y}) - \boldsymbol{\mu}^*) \\
&- \frac{1}{2} vec(\boldsymbol{B}^*)' (\boldsymbol{X}' \otimes \boldsymbol{I}_k)' (\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a)^{-1} (\boldsymbol{X}' \otimes \boldsymbol{I}_k) vec(\boldsymbol{B}^*) \\
&+ \frac{1}{2} (vec(\boldsymbol{Y}) - \boldsymbol{\mu}^*)' (\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a)^{-1} (\boldsymbol{X}' \otimes \boldsymbol{I}_k) vec(\boldsymbol{B}^*) \\
&= vec(\boldsymbol{B}^*)' (\boldsymbol{X} \otimes \boldsymbol{I}_k)(\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a)^{-1} (vec(\boldsymbol{Y}) - \boldsymbol{\mu}^*) \\
&\quad - \frac{1}{2} vec(\boldsymbol{B}^*)' (\boldsymbol{X} \otimes \boldsymbol{I}_k)(\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a)^{-1} (\boldsymbol{X}' \otimes \boldsymbol{I}_k) vec(\boldsymbol{B}^*) \quad \text{using (A2.2)}
\end{aligned} \quad (3.37)$$

Therefore, from (3.37) it follows that

$$\begin{aligned}
&\frac{\partial \ln L}{\partial vec(\boldsymbol{B}^*)} \\
&= (\boldsymbol{X} \otimes \boldsymbol{I}_k)(\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a)^{-1} (vec(\boldsymbol{Y}) - \boldsymbol{\mu}^*) \\
&- \frac{1}{2}\left[(\boldsymbol{X} \otimes \boldsymbol{I}_k)(\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a)^{-1} (\boldsymbol{X}' \otimes \boldsymbol{I}_k) + \left((\boldsymbol{X} \otimes \boldsymbol{I}_k)(\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a)^{-1} (\boldsymbol{X}' \otimes \boldsymbol{I}_k)\right)'\right] vec(\boldsymbol{B}^*)
\end{aligned}$$

using (A3.1), (A3.2)

$$= (\boldsymbol{X} \otimes \boldsymbol{I}_k)(\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a)^{-1}\left(vec(\boldsymbol{Y}) - \boldsymbol{\mu}^*\right) - (\boldsymbol{X} \otimes \boldsymbol{I}_k)(\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a)^{-1}(\boldsymbol{X}' \otimes \boldsymbol{I}_k)vec\left(\boldsymbol{B}^*\right)$$
$$= (\boldsymbol{X} \otimes \boldsymbol{I}_k)\left(\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a^{-1}\right)\left(vec(\boldsymbol{Y}) - \boldsymbol{\mu}^*\right) - (\boldsymbol{X} \otimes \boldsymbol{I}_k)\left(\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a^{-1}\right)(\boldsymbol{X}' \otimes \boldsymbol{I}_k)vec\left(\boldsymbol{B}^*\right)$$

using (A2.1)

$$= \left(\boldsymbol{X} \otimes \boldsymbol{\Sigma}_a^{-1}\right)\left(vec(\boldsymbol{Y}) - \boldsymbol{\mu}^*\right) - \left(\boldsymbol{X}\boldsymbol{X}' \otimes \boldsymbol{\Sigma}_a^{-1}\right)vec\left(\boldsymbol{B}^*\right) \text{ using (A2.3)} \tag{3.38}$$

Setting $\frac{\partial \ln L}{\partial vec(\boldsymbol{B}^*)}$ in (3.38) equal to zero, the maximum likelihood estimator of $vec\left(\boldsymbol{B}^*\right)$, namely $vec\left(\tilde{\boldsymbol{B}}^*\right)$, is:

$$\left(\tilde{\boldsymbol{X}} \otimes \tilde{\boldsymbol{\Sigma}}_a^{-1}\right)\left(vec(\boldsymbol{Y}) - \tilde{\boldsymbol{\mu}}^*\right) = \left(\tilde{\boldsymbol{X}}\tilde{\boldsymbol{X}}' \otimes \tilde{\boldsymbol{\Sigma}}_a^{-1}\right)vec\left(\tilde{\boldsymbol{B}}^*\right)$$
$$\therefore vec\left(\tilde{\boldsymbol{B}}^*\right) = \left(\tilde{\boldsymbol{X}}\tilde{\boldsymbol{X}}' \otimes \tilde{\boldsymbol{\Sigma}}_a^{-1}\right)^{-1}\left(\tilde{\boldsymbol{X}} \otimes \tilde{\boldsymbol{\Sigma}}_a^{-1}\right)\left(vec(\boldsymbol{Y}) - \tilde{\boldsymbol{\mu}}^*\right)$$
$$= \left(\left(\tilde{\boldsymbol{X}}\tilde{\boldsymbol{X}}'\right)^{-1} \otimes \tilde{\boldsymbol{\Sigma}}_a\right)\left(\tilde{\boldsymbol{X}} \otimes \tilde{\boldsymbol{\Sigma}}_a^{-1}\right)\left(vec(\boldsymbol{Y}) - \tilde{\boldsymbol{\mu}}^*\right) \text{ using (A2.1)}$$
$$= \left(\left(\tilde{\boldsymbol{X}}\tilde{\boldsymbol{X}}'\right)^{-1}\tilde{\boldsymbol{X}} \otimes \boldsymbol{I}_k\right)\left(vec(\boldsymbol{Y}) - \tilde{\boldsymbol{\mu}}^*\right) \text{ using (A2.3)} \tag{3.39}$$

From (3.34) it follows that

$$\frac{\partial \ln L}{\partial \boldsymbol{\Sigma}_a} = -\frac{T}{2}\boldsymbol{\Sigma}_a^{-1} - \frac{1}{2}\left[-\boldsymbol{\Sigma}_a^{-1}\left(\boldsymbol{Y}^0 - \boldsymbol{B}^*\boldsymbol{X}\right)\left(\boldsymbol{Y}^0 - \boldsymbol{B}^*\boldsymbol{X}\right)'\boldsymbol{\Sigma}_a^{-1}\right] \text{ using (A3.5), (A3.6)}$$
$$= -\frac{T}{2}\boldsymbol{\Sigma}_a^{-1} + \frac{1}{2}\boldsymbol{\Sigma}_a^{-1}\left(\boldsymbol{Y}^0 - \boldsymbol{B}^*\boldsymbol{X}\right)\left(\boldsymbol{Y}^0 - \boldsymbol{B}^*\boldsymbol{X}\right)'\boldsymbol{\Sigma}_a^{-1} \tag{3.40}$$

Setting $\frac{\partial \ln L}{\partial \boldsymbol{\Sigma}_a}$ in (3.40) equal to zero, the maximum likelihood estimator of $\boldsymbol{\Sigma}_a$, namely $\tilde{\boldsymbol{\Sigma}}_a$, is:

$$\frac{T}{2}\tilde{\boldsymbol{\Sigma}}_a^{-1} = \frac{1}{2}\tilde{\boldsymbol{\Sigma}}_a^{-1}\left(\tilde{\boldsymbol{Y}}^0 - \tilde{\boldsymbol{B}}^*\tilde{\boldsymbol{X}}\right)\left(\tilde{\boldsymbol{Y}}^0 - \tilde{\boldsymbol{B}}^*\tilde{\boldsymbol{X}}\right)'\tilde{\boldsymbol{\Sigma}}_a^{-1}$$
$$\therefore \tilde{\boldsymbol{\Sigma}}_a = \frac{1}{T}\left(\tilde{\boldsymbol{Y}}^0 - \tilde{\boldsymbol{B}}^*\tilde{\boldsymbol{X}}\right)\left(\tilde{\boldsymbol{Y}}^0 - \tilde{\boldsymbol{B}}^*\tilde{\boldsymbol{X}}\right)' \tag{3.41}$$

Take note that $\tilde{\boldsymbol{X}}$ and $\tilde{\boldsymbol{Y}}^0$ are obtained by replacing $\boldsymbol{\mu}$ with the estimated value, $\tilde{\boldsymbol{\mu}}$.

3.3.3 Asymptotic properties of the maximum likelihood estimator

As explained in section 3.2.3 it is useful to know the asymptotic distribution of the estimator. Proposition 3.4 of Lütkepohl (2005) states:

"Let $\{\boldsymbol{y}_t\}$ be a stationary, stable Gaussian VAR(p) process. Then the ML estimator of $vec(\tilde{\boldsymbol{B}}^*) = \left((\tilde{\boldsymbol{X}}\tilde{\boldsymbol{X}}')^{-1}\tilde{\boldsymbol{X}} \otimes \boldsymbol{I}_k\right)(vec(\boldsymbol{Y}) - \tilde{\boldsymbol{\mu}}^*)$ is consistent and

$$\sqrt{T}vec(\tilde{\boldsymbol{B}}^* - \boldsymbol{B}^*) \xrightarrow{d} N\left(\boldsymbol{0}, \boldsymbol{\Gamma}_Y(0)^{-1} \otimes \boldsymbol{\Sigma}_a\right) \tag{3.42}$$

where $\boldsymbol{\Gamma}_Y(0) = E\left(\frac{\boldsymbol{X}\boldsymbol{X}'}{T}\right)$."

Rewriting (3.42) and substituting $\boldsymbol{\Sigma}_a$ with the maximum likelihood estimator obtained in (3.41) and estimating $\boldsymbol{\Gamma}_Y(0)$ with $\hat{\boldsymbol{\Gamma}}_Y(0) = \frac{\tilde{\boldsymbol{X}}\tilde{\boldsymbol{X}}'}{T}$,

$$vec(\tilde{\boldsymbol{B}}^* - \boldsymbol{B}^*) \xrightarrow{d} N\left(\boldsymbol{0}, \frac{1}{T}\left(\frac{\tilde{\boldsymbol{X}}\tilde{\boldsymbol{X}}'}{T}\right)^{-1} \otimes \tilde{\boldsymbol{\Sigma}}_a\right)$$

$$vec(\tilde{\boldsymbol{B}}^* - \boldsymbol{B}^*) \xrightarrow{d} N\left(\boldsymbol{0}, (\tilde{\boldsymbol{X}}\tilde{\boldsymbol{X}}')^{-1} \otimes \tilde{\boldsymbol{\Sigma}}_a\right) \tag{3.43}$$

Dividing the individual elements of $vec(\tilde{\boldsymbol{B}}^* - \boldsymbol{B}^*)$ with the square root of the diagonal elements of $(\tilde{\boldsymbol{X}}\tilde{\boldsymbol{X}}')^{-1} \otimes \tilde{\boldsymbol{\Sigma}}_a$, yields an approximate asymptotic standard normal distribution. This can be used for hypothesis testing regarding the significance of the maximum likelihood estimators.

Due to the complex nature of the iterative process of maximisation, Example 3.2 only considers a simple case where it is assumed that it is known that the mean of the process is equal to zero.

In the following example the maximum likelihood estimates of a VAR(1) model are calculated using matrix operations. Approximate standard errors of the coefficient matrices are determined. All the results are compared to the output produced by the VARMAX procedure on the same sample.

Example 3.2*

Consider the bivariate VAR(1) model $\boldsymbol{y}_t = \begin{pmatrix} 0.5 & 0.6 \\ 0.1 & 0.4 \end{pmatrix} \boldsymbol{y}_{t-1} + \boldsymbol{a}_t$ with $\boldsymbol{\Sigma}_a = \begin{pmatrix} 1.0 & 0.5 \\ 0.5 & 0.9 \end{pmatrix}$.

For simplicity it is assumed that it is know that $\boldsymbol{\mu} = \boldsymbol{0}$. A sample of size 500 is generated. The maximum likelihood estimates of $\boldsymbol{B}^*$ in (3.39) and $\boldsymbol{\Sigma}_a$ in (3.40) are

$$\tilde{\boldsymbol{B}}^* = \left(\hat{\boldsymbol{\Phi}}_1\right) = \begin{pmatrix} 0.520 & 0.498 \\ 0.113 & 0.323 \end{pmatrix}$$

$$\tilde{\boldsymbol{\Sigma}}_a = \begin{pmatrix} 1.002 & 0.528 \\ 0.528 & 0.969 \end{pmatrix}$$

The standard errors and t-ratios of the maximum likelihood estimates of the autoregressive coefficients can be obtained using (3.43). The results are summarised in the table below.

	$vec(\tilde{\boldsymbol{B}}^*)$	**Standard error**	**t-ratio**
$\hat{\phi}_{11}$	0.520	0.036	14.413
$\hat{\phi}_{21}$	0.113	0.035	3.190
$\hat{\phi}_{12}$	0.498	0.053	9.440
$\hat{\phi}_{22}$	0.323	0.052	6.218

This compares well with the SAS output that is provided below. As mentioned in Example 3.1, the slight differences are due to the presample values. All the parameter values are significant.

```
                        The VARMAX Procedure

                      Model Parameter Estimates

                                    Standard
Equation Parameter       Estimate       Error t Value Pr > |t|
Variable
y1       AR1_1_1          0.51822     0.03620   14.31   0.0001 y1(t-1)
         AR1_1_2          0.50127     0.05285    9.48   0.0001 y2(t-1)
y2       AR1_2_1          0.11255     0.03557    3.16   0.0017 y1(t-1)
         AR1_2_2          0.32529     0.05200    6.26   0.0001 y2(t-1)
```

* The SAS program is provided in Appendix B page 129.

Covariances of Innovations

Variable	y1	y2
y1	1.00342	0.52871
y2	0.52871	0.97038

3.4 CONCLUSION

The least squares estimator and the maximum likelihood estimator of the parameters of a vector autoregressive model were derived for the general case of order p. Chapter 5 will consider some methods to determine a tentative value for p. The distributions of the estimators were also discussed. This gave rise to a hypothesis test to establish the significance of the individual estimates. Examples were given in which the estimates were calculated from theoretical results and compared to the corresponding results provided by the VARMAX procedure in the SAS/ETS module on computer generated multivariate time series. Close correspondence was achieved throughout. In Chapter 4 the estimation procedure will be expanded to also include moving average parameters.

CHAPTER 4

ESTIMATION OF VARMA PROCESSES

4.1 INTRODUCTION

The simplicity of the estimation of VAR models makes them very attractive in practice. The opposite is however true for VARMA models, because for VARMA models it is complicated to obtain a unique representation. Hannan (1969) derived conditions for a VARMA model to be uniquely identified, while Lütkepohl and Poskitt (1996) proposed the echelon form that leads to a parsimonious and unique structure.

Hannan (1970) considered the estimation of a VMA model in the spectral domain, Osborn (1977) derived an exact likelihood function for a VMA model and Phadke & Kedem (1978) were concerned about the computation and maximisation of the exact likelihood function of a VMA model. The problem of estimating the parameters of VARMA models has been considered by Wilson (1973), Nicholls & Hall (1979), Hillmer & Tiao (1979) and more recently by Mauricio (1995) and Ma (1997). De Frutos & Serrano (2002) proposed a generalised least squares procedure for estimating VARMA models. This chapter will however only focus on maximum likelihood estimation because it is the most common procedure the moment moving average parameters are included. The primary source used for this chapter is Lütkepohl (2005).

In sections 4.2, 4.3 and 4.4 we will only derive the likelihood function for the VMA(1), VMA(q) and VARMA(1,1) processes, respectively. The VARMA(p,q) process will not be presented since the VARMA representation is not unique. In order to overcome this identification problem, the VARMA representation must be in final equations or echelon form. This problem is briefly discussed in section 4.5.

The maximum likelihood estimates can be obtained by setting the normal equations equal to zero and solving for the parameters. Since this is nonlinear in the parameters, numerical optimisation methods are employed to obtain maximum likelihood estimates.

4.2 THE LIKELIHOOD FUNCTION OF A VMA(1) PROCESS

Suppose we have k time series processes each comprising of T equally spaced observations that were generated by a Gaussian, invertible, zero mean with covariance matrix $\boldsymbol{\Sigma}_a$, VMA(1) process, $\boldsymbol{y}_t = \boldsymbol{a}_t + \boldsymbol{\Theta}_1 \boldsymbol{a}_{t-1}$. The constant term is set equal to zero for convenience. It can be shown in a similar way as (3.5) that

$$\begin{pmatrix} \boldsymbol{a}_0 \\ \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_T \end{pmatrix} \sim N\left(\boldsymbol{0}, \boldsymbol{I}_{T+1} \otimes \boldsymbol{\Sigma}_a\right) \tag{4.1}$$

The matrix of time series observations is denoted by $\boldsymbol{Y} : k \times T$ as in (2.1), where each column represents the k observations at a specific point in time, while each row represents all the observations of one of the k time series processes. $vec(\boldsymbol{Y})$ is a linear function of the white noise vectors (4.1), therefore the multivariate normal distribution can be used to determine the likelihood function.

$$\begin{aligned} vec(\boldsymbol{Y}) : kT \times 1 = \begin{pmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \\ \vdots \\ \boldsymbol{y}_T \end{pmatrix} &= \begin{pmatrix} \boldsymbol{a}_1 + \boldsymbol{\Theta}_1 \boldsymbol{a}_0 \\ \boldsymbol{a}_2 + \boldsymbol{\Theta}_1 \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_T + \boldsymbol{\Theta}_1 \boldsymbol{a}_{T-1} \end{pmatrix} \\ &= \begin{pmatrix} \boldsymbol{\Theta}_1 & \boldsymbol{I}_k & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Theta}_1 & \boldsymbol{I}_k & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{\Theta}_1 & \boldsymbol{I}_k \end{pmatrix} \begin{pmatrix} \boldsymbol{a}_0 \\ \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_T \end{pmatrix} = \overline{\boldsymbol{\Theta}}_1 \begin{pmatrix} \boldsymbol{a}_0 \\ \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_T \end{pmatrix} \end{aligned} \tag{4.2}$$

where $\overline{\boldsymbol{\Theta}}_1 : kT \times k(T+1)$.

By applying result (A5.2) to (4.2) and taking into account the distribution in (4.1) it follows that $vec(\boldsymbol{Y}) \sim N\left(\boldsymbol{0}, \overline{\boldsymbol{\Theta}}_1 (\boldsymbol{I}_{T+1} \otimes \boldsymbol{\Sigma}_a) \overline{\boldsymbol{\Theta}}_1'\right)$. Therefore, the likelihood function is proportional to

$$L(\boldsymbol{\Theta}_1, \boldsymbol{\Sigma}_a) \propto \left| \overline{\boldsymbol{\Theta}}_1 (\boldsymbol{I}_{T+1} \otimes \boldsymbol{\Sigma}_a) \overline{\boldsymbol{\Theta}}_1' \right|^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2} vec(\boldsymbol{Y})' \left[\overline{\boldsymbol{\Theta}}_1 (\boldsymbol{I}_{T+1} \otimes \boldsymbol{\Sigma}_a) \overline{\boldsymbol{\Theta}}_1' \right]^{-1} vec(\boldsymbol{Y}) \right\} \tag{4.3}$$

To simplify (4.3) it can be assumed that the starting residuals are equal to zero ($\boldsymbol{a}_0 = \boldsymbol{0}$), then (4.2) becomes

$$vec(\boldsymbol{Y}) = \begin{pmatrix} \boldsymbol{I}_k & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{\Theta}_1 & \boldsymbol{I}_k & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{\Theta}_1 & \boldsymbol{I}_k \end{pmatrix} \begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \vdots \\ \boldsymbol{a}_T \end{pmatrix} = \tilde{\boldsymbol{\Theta}}_1 vec(\boldsymbol{A}) \tag{4.4}$$

where $\tilde{\boldsymbol{\Theta}}_1 : kT \times kT$.

The covariance matrix of $vec(\boldsymbol{A})$, as derived in (3.5), is $(\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a)$. By applying result (A5.2) to (4.4) we have that $vec(\boldsymbol{Y}) \sim N\left(\boldsymbol{0}, \tilde{\boldsymbol{\Theta}}_1 (\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a) \tilde{\boldsymbol{\Theta}}_1'\right)$ and therefore the conditional likelihood function is proportional to

$$\hat{L}(\boldsymbol{\Theta}_1, \boldsymbol{\Sigma}_a) \propto \left|\tilde{\boldsymbol{\Theta}}_1 (\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a) \tilde{\boldsymbol{\Theta}}_1'\right|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2} vec(\boldsymbol{Y})' \left[\tilde{\boldsymbol{\Theta}}_1 (\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a) \tilde{\boldsymbol{\Theta}}_1'\right]^{-1} vec(\boldsymbol{Y})\right\} \tag{4.5}$$

Take note that according to the properties of the determinant $\left|\tilde{\boldsymbol{\Theta}}_1 (\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a) \tilde{\boldsymbol{\Theta}}_1'\right| = \left|\tilde{\boldsymbol{\Theta}}_1\right| \left|\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a\right| \left|\tilde{\boldsymbol{\Theta}}_1'\right| = \left|\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a\right|$ since $\tilde{\boldsymbol{\Theta}}_1$ is a lower triangular matrix with ones on the main diagonal and therefore $\left|\tilde{\boldsymbol{\Theta}}_1\right| = 1$. Furthermore, from property (A2.4) of the Kronecker product, $\left|\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a\right| = \left|\boldsymbol{I}_T\right|^k \left|\boldsymbol{\Sigma}_a\right|^T = \left|\boldsymbol{\Sigma}_a\right|^T$. The conditional likelihood function in (4.5) simplifies to

$$\begin{aligned} \hat{L}(\boldsymbol{\Theta}_1, \boldsymbol{\Sigma}_a) &\propto \left|\boldsymbol{\Sigma}_a\right|^{-\frac{T}{2}} \exp\left\{-\frac{1}{2} vec(\boldsymbol{Y})' \left(\tilde{\boldsymbol{\Theta}}_1'\right)^{-1} (\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a)^{-1} \tilde{\boldsymbol{\Theta}}_1^{-1} vec(\boldsymbol{Y})\right\} \\ &= \left|\boldsymbol{\Sigma}_a\right|^{-\frac{T}{2}} \exp\left\{-\frac{1}{2} \left(\tilde{\boldsymbol{\Theta}}_1^{-1} vec(\boldsymbol{Y})\right)' \left(\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a^{-1}\right) \left(\tilde{\boldsymbol{\Theta}}_1^{-1} vec(\boldsymbol{Y})\right)\right\} \text{ using (A2.1)} \end{aligned} \tag{4.6}$$

Rewriting (4.4) in terms of $vec(\boldsymbol{A})$, we have that

$$\tilde{\boldsymbol{\Theta}}_1^{-1} vec(\boldsymbol{Y}) = vec(\boldsymbol{A}) \tag{4.7}$$

Take note that the existence of the inverse of $\tilde{\boldsymbol{\Theta}}_1$ follows from the fact that the determinant of $\tilde{\boldsymbol{\Theta}}_1$ is unequal to zero.

Substituting (4.7) into (4.6) a simplified form of the conditional likelihood is obtained,

$$\begin{aligned}\hat{L}(\boldsymbol{\Theta}_1, \boldsymbol{\Sigma}_a) &\propto |\boldsymbol{\Sigma}_a|^{-\frac{T}{2}} \exp\left\{-\frac{1}{2} vec(\boldsymbol{A})' \left(\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a^{-1}\right) vec(\boldsymbol{A})\right\} \\ &= |\boldsymbol{\Sigma}_a|^{-\frac{T}{2}} \exp\left\{-\frac{1}{2}\sum_{t=1}^{T} \boldsymbol{a}_t' \boldsymbol{\Sigma}_a^{-1} \boldsymbol{a}_t\right\} \qquad (4.8)\end{aligned}$$

where $\boldsymbol{a}_t$ can be determined by rewriting the VMA(1) process as a VAR process and setting $\boldsymbol{a}_0 = \boldsymbol{0}$.

The following example employs dual quasi-Newton optimisation techniques to determine the parameter estimates that maximise the log-likelihood.

Example 4.1*

Consider the bivariate VMA(1) model $\boldsymbol{y}_t = \boldsymbol{a}_t - \begin{pmatrix} 0.2 & 0.1 \\ 0.1 & 0.4 \end{pmatrix} \boldsymbol{a}_{t-1}$ with $\boldsymbol{\Sigma}_a = \begin{pmatrix} 1.0 & 0.5 \\ 0.5 & 0.9 \end{pmatrix}$. A sample of size 500 is generated.

The maximum likelihood estimates of $\boldsymbol{\Theta}_1$ and $\boldsymbol{\Sigma}_a$ are those values that maximise the likelihood function in (4.8) or alternatively the log-likelihood function,

$$\ln \hat{L}(\boldsymbol{\Theta}_1, \boldsymbol{\Sigma}_a) \propto -\frac{T}{2} \ln|\boldsymbol{\Sigma}_a| - \frac{1}{2}\sum_{t=1}^{T} \boldsymbol{a}_t' \boldsymbol{\Sigma}_a^{-1} \boldsymbol{a}_t \quad \text{where } \boldsymbol{a}_t = \boldsymbol{y}_t - \boldsymbol{\Theta}_1 \boldsymbol{a}_{t-1}$$

Using the dual quasi-Newton optimisation method in PROC IML, the maximum likelihood estimates are

$$\hat{\boldsymbol{\Theta}}_1 = \begin{pmatrix} -0.221 & -0.181 \\ -0.090 & -0.462 \end{pmatrix}, \ \hat{\boldsymbol{\Sigma}}_a = \begin{pmatrix} 0.996 & 0.525 \\ 0.525 & 0.976 \end{pmatrix}$$

Therefore,

$$\begin{aligned}\hat{\boldsymbol{y}}_t &= \boldsymbol{a}_t + \hat{\boldsymbol{\Theta}}_1 \boldsymbol{a}_{t-1} \\ &= \boldsymbol{a}_t + \begin{pmatrix} -0.221 & -0.181 \\ -0.090 & -0.462 \end{pmatrix} \boldsymbol{a}_{t-1} \\ &= \boldsymbol{a}_t - \begin{pmatrix} 0.221 & 0.181 \\ 0.090 & 0.462 \end{pmatrix} \boldsymbol{a}_{t-1}\end{aligned}$$

* The SAS program is provided in Appendix B page 130.

This can be compared to the maximum likelihood estimates obtained using the VARMAX procedure. The estimated model is,

$\hat{\boldsymbol{y}}_t = \boldsymbol{a}_t - \begin{pmatrix} 0.223 & 0.183 \\ 0.096 & 0.459 \end{pmatrix} \boldsymbol{a}_{t-1}$ with $\hat{\boldsymbol{\Sigma}}_a = \begin{pmatrix} 0.996 & 0.524 \\ 0.524 & 0.969 \end{pmatrix}$

Take note of the SAS program in Appendix B that illustrates the NLPQN CALL in SAS IML that was used to solve the optimisation problem.

4.3 THE LIKELIHOOD FUNCTION OF A VMA(q) PROCESS

Osborn derived the exact likelihood function for vector moving average processes in 1977. Suppose that $\{\boldsymbol{y}_t\}$ is generated by a Gaussian, invertible, zero mean VMA(q) process, $\boldsymbol{y}_t = \boldsymbol{a}_t + \boldsymbol{\Theta}_1 \boldsymbol{a}_{t-1} + \ldots + \boldsymbol{\Theta}_q \boldsymbol{a}_{t-q}$. Then

$$vec(\boldsymbol{Y}) : kT \times 1 = \begin{pmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \\ \vdots \\ \boldsymbol{y}_T \end{pmatrix} = \begin{pmatrix} \boldsymbol{a}_1 + \boldsymbol{\Theta}_1 \boldsymbol{a}_0 + \ldots + \boldsymbol{\Theta}_q \boldsymbol{a}_{1-q} \\ \boldsymbol{a}_2 + \boldsymbol{\Theta}_1 \boldsymbol{a}_1 + \ldots + \boldsymbol{\Theta}_q \boldsymbol{a}_{2-q} \\ \vdots \\ \boldsymbol{a}_T + \boldsymbol{\Theta}_1 \boldsymbol{a}_{T-1} + \ldots + \boldsymbol{\Theta}_q \boldsymbol{a}_{T-q} \end{pmatrix}$$

$$= \begin{pmatrix} \boldsymbol{\Theta}_q & \boldsymbol{\Theta}_{q-1} & \cdots & \boldsymbol{\Theta}_1 & \boldsymbol{I}_k & \boldsymbol{0} & \cdots & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Theta}_q & \ddots & \boldsymbol{\Theta}_2 & \boldsymbol{\Theta}_1 & \boldsymbol{I}_k & \cdots & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \cdots & \boldsymbol{\Theta}_q & \cdots & \cdots & \boldsymbol{\Theta}_1 & \boldsymbol{I}_k \end{pmatrix} \begin{pmatrix} \boldsymbol{a}_{-q+1} \\ \vdots \\ \boldsymbol{a}_0 \\ \vdots \\ \boldsymbol{a}_T \end{pmatrix} = \overline{\boldsymbol{\Theta}}_q \begin{pmatrix} \boldsymbol{a}_{-q+1} \\ \vdots \\ \boldsymbol{a}_0 \\ \vdots \\ \boldsymbol{a}_T \end{pmatrix} \qquad (4.9)$$

where $\overline{\boldsymbol{\Theta}}_q : kT \times k(T+q)$.

In a similar way as in (3.5) it can be shown that

$$\begin{pmatrix} \boldsymbol{a}_{-q+1} \\ \vdots \\ \boldsymbol{a}_0 \\ \vdots \\ \boldsymbol{a}_T \end{pmatrix} \sim N\left(\boldsymbol{0}, \boldsymbol{I}_{T+q} \otimes \boldsymbol{\Sigma}_a\right) \qquad (4.10)$$

$vec(\boldsymbol{Y})$ in (4.9) is $N\left(\boldsymbol{0}, \overline{\boldsymbol{\Theta}}_q\left(\boldsymbol{I}_{T+q} \otimes \boldsymbol{\Sigma}_a\right)\overline{\boldsymbol{\Theta}}_q{}'\right)$ distributed, this follows from the distribution in (4.10) together with result (A5.2). The likelihood function is therefore proportional to

$$L\left(\boldsymbol{\Theta}_1, \ldots, \boldsymbol{\Theta}_q, \boldsymbol{\Sigma}_a\right) \propto \left|\overline{\boldsymbol{\Theta}}_q\left(\boldsymbol{I}_{T+q} \otimes \boldsymbol{\Sigma}_a\right)\overline{\boldsymbol{\Theta}}_q{}'\right|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2} vec(\boldsymbol{Y})'\left[\overline{\boldsymbol{\Theta}}_q\left(\boldsymbol{I}_{T+q} \otimes \boldsymbol{\Sigma}_a\right)\overline{\boldsymbol{\Theta}}_q{}'\right]^{-1} vec(\boldsymbol{Y})\right\} \tag{4.11}$$

An approximation to the likelihood function in (4.11) is obtained by setting the starting residuals $\boldsymbol{a}_0 = \boldsymbol{a}_{-1} = \ldots \boldsymbol{a}_{-q+1} = \boldsymbol{0}$, then (4.9) simplifies to

$$vec(\boldsymbol{Y}) = \begin{pmatrix} \boldsymbol{I}_k & \boldsymbol{0} & \cdots & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{\Theta}_1 & \boldsymbol{I}_k & \cdots & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \vdots & \ddots & \ddots & & & \vdots & \vdots \\ \boldsymbol{\Theta}_q & \boldsymbol{\Theta}_{q-1} & \ddots & \ddots & & \vdots & \vdots \\ \boldsymbol{0} & \boldsymbol{\Theta}_q & & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & & \ddots & \ddots & \\ \boldsymbol{0} & \boldsymbol{0} & & \boldsymbol{\Theta}_q & & \boldsymbol{\Theta}_1 & \boldsymbol{I}_k \end{pmatrix} \begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \vdots \\ \boldsymbol{a}_T \end{pmatrix} = \tilde{\boldsymbol{\Theta}}_q vec(\boldsymbol{A}) \tag{4.12}$$

where $\tilde{\boldsymbol{\Theta}}_q : kT \times kT$.

By applying result (A5.2) to (4.12) and taking into account the covariance matrix of $vec(\boldsymbol{A})$ in (3.5), it follows that $vec(\boldsymbol{Y}) \sim N\left(\boldsymbol{0}, \tilde{\boldsymbol{\Theta}}_q\left(\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a\right)\tilde{\boldsymbol{\Theta}}_q{}'\right)$. The conditional likelihood function is therefore proportional to

$$\begin{aligned} \hat{L}\left(\boldsymbol{\Theta}_1, \ldots, \boldsymbol{\Theta}_q, \boldsymbol{\Sigma}_a\right) &\propto \left|\tilde{\boldsymbol{\Theta}}_q\left(\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a\right)\tilde{\boldsymbol{\Theta}}_q{}'\right|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2} vec(\boldsymbol{Y})'\left[\tilde{\boldsymbol{\Theta}}_q\left(\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a\right)\tilde{\boldsymbol{\Theta}}_q{}'\right]^{-1} vec(\boldsymbol{Y})\right\} \\ &= \left|\boldsymbol{\Sigma}_a\right|^{-\frac{T}{2}} \exp\left\{-\frac{1}{2} vec(\boldsymbol{Y})'\left(\tilde{\boldsymbol{\Theta}}_q{}'\right)^{-1}\left(\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a\right)^{-1}\tilde{\boldsymbol{\Theta}}_q^{-1} vec(\boldsymbol{Y})\right\} \\ &= \left|\boldsymbol{\Sigma}_a\right|^{-\frac{T}{2}} \exp\left\{-\frac{1}{2} vec(\boldsymbol{A})'\left(\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a^{-1}\right) vec(\boldsymbol{A})\right\} \quad \text{(from (4.12))} \\ &= \left|\boldsymbol{\Sigma}_a\right|^{-\frac{T}{2}} \exp\left\{-\frac{1}{2}\sum_{t=1}^{T} \boldsymbol{a}_t' \boldsymbol{\Sigma}_a^{-1} \boldsymbol{a}_t\right\} \end{aligned} \tag{4.13}$$

where $\boldsymbol{a}_t$ can be determined by rewriting the VMA(q) process as a VAR process and setting $\boldsymbol{a}_0 = \boldsymbol{a}_{-1} = \ldots \boldsymbol{a}_{-q+1} = \boldsymbol{0}$.

Note that $\left|\tilde{\boldsymbol{\Theta}}_q (\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a)\tilde{\boldsymbol{\Theta}}_q'\right| = \left|\tilde{\boldsymbol{\Theta}}_q\right|\left|\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a\right|\left|\tilde{\boldsymbol{\Theta}}_q'\right| = \left|\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a\right|$ since $\left|\tilde{\boldsymbol{\Theta}}_q\right| = 1$ and furthermore $\left|\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a\right| = \left|\boldsymbol{I}_T\right|^k \left|\boldsymbol{\Sigma}_a\right|^T = \left|\boldsymbol{\Sigma}_a\right|^T$ using (A2.4). The existence of the inverse of $\tilde{\boldsymbol{\Theta}}_q$, used in the derivation of (4.13), follows from the fact that the determinant of $\tilde{\boldsymbol{\Theta}}_q$ is unequal to zero.

The maximum likelihood estimators of the unknown parameters can be obtained by maximising the conditional likelihood function (4.13) using numerical optimisation methods.

4.4 THE LIKELIHOOD FUNCTION OF A VARMA(1,1) PROCESS

Suppose that $\{\boldsymbol{y}_t\}$ is a zero mean, Gaussian, stationary and invertible VARMA(1,1) process, $\boldsymbol{y}_t = \boldsymbol{\Phi}_1 \boldsymbol{y}_{t-1} + \boldsymbol{a}_t + \boldsymbol{\Theta}_1 \boldsymbol{a}_{t-1}$. Then

$$
\begin{aligned}
&\boldsymbol{y}_1 - \boldsymbol{\Phi}_1 \boldsymbol{y}_0 = \boldsymbol{a}_1 + \boldsymbol{\Theta}_1 \boldsymbol{a}_0 \\
&\boldsymbol{y}_2 - \boldsymbol{\Phi}_1 \boldsymbol{y}_1 = \boldsymbol{a}_2 + \boldsymbol{\Theta}_1 \boldsymbol{a}_1 \\
&\vdots \\
&\boldsymbol{y}_T - \boldsymbol{\Phi}_1 \boldsymbol{y}_{T-1} = \boldsymbol{a}_T + \boldsymbol{\Theta}_1 \boldsymbol{a}_{T-1}
\end{aligned}
$$

or, in matrix notation

$$
\begin{pmatrix}
\boldsymbol{I}_k & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} \\
-\boldsymbol{\Phi}_1 & \boldsymbol{I}_k & \cdots & \boldsymbol{0} & \boldsymbol{0} \\
\vdots & \ddots & \ddots & & \vdots \\
\vdots & & \ddots & \ddots & \vdots \\
\boldsymbol{0} & \boldsymbol{0} & \cdots & -\boldsymbol{\Phi}_1 & \boldsymbol{I}_k
\end{pmatrix}
\begin{pmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \\ \vdots \\ \boldsymbol{y}_T \end{pmatrix}
+
\begin{pmatrix} -\boldsymbol{\Phi}_1 \boldsymbol{y}_0 \\ \boldsymbol{0} \\ \vdots \\ \boldsymbol{0} \end{pmatrix}
=
\begin{pmatrix}
\boldsymbol{\Theta}_1 & \boldsymbol{I}_k & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} \\
\boldsymbol{0} & \boldsymbol{\Theta}_1 & \boldsymbol{I}_k & \cdots & \boldsymbol{0} & \boldsymbol{0} \\
\vdots & \vdots & \ddots & \ddots & & \vdots \\
\vdots & \vdots & & \ddots & \ddots & \vdots \\
\boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{\Theta}_1 & \boldsymbol{I}_k
\end{pmatrix}
\begin{pmatrix} \boldsymbol{a}_0 \\ \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_T \end{pmatrix}
$$

$$
\therefore \boldsymbol{U}_1 vec(\boldsymbol{Y}) + \begin{pmatrix} -\boldsymbol{\Phi}_1 \boldsymbol{y}_0 \\ \boldsymbol{0} \\ \vdots \\ \boldsymbol{0} \end{pmatrix} = \overline{\boldsymbol{\Theta}}_1 \begin{pmatrix} \boldsymbol{a}_0 \\ \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_T \end{pmatrix} \tag{4.14}
$$

Solving for $vec(\boldsymbol{Y})$ in (4.14),

$$
vec(\boldsymbol{Y}) = \boldsymbol{U}_1^{-1} \overline{\boldsymbol{\Theta}}_1 \begin{pmatrix} \boldsymbol{a}_0 \\ \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_T \end{pmatrix} + \boldsymbol{U}_1^{-1} \begin{pmatrix} \boldsymbol{\Phi}_1 \boldsymbol{y}_0 \\ \boldsymbol{0} \\ \vdots \\ \boldsymbol{0} \end{pmatrix} \tag{4.15}
$$

By utilising (4.1) and result (A5.2), assuming fixed presample values $\boldsymbol{y}_0$, the distribution of $vec(\boldsymbol{Y})$ is,

$$vec(\boldsymbol{Y}) \sim N\left(\boldsymbol{U}_1^{-1}\begin{pmatrix}\boldsymbol{\Phi}_1\boldsymbol{y}_0\\ \boldsymbol{0}\\ \vdots\\ \boldsymbol{0}\end{pmatrix}, \boldsymbol{U}_1^{-1}\overline{\boldsymbol{\Theta}}_1(\boldsymbol{I}_{T+1}\otimes\boldsymbol{\Sigma}_a)\overline{\boldsymbol{\Theta}}_1'\boldsymbol{U}_1^{-1'}\right)$$

The likelihood function is proportional to

$$L(\boldsymbol{\Phi}_1,\boldsymbol{\Theta}_1,\boldsymbol{\Sigma}_a) \propto \left|\boldsymbol{U}_1^{-1}\overline{\boldsymbol{\Theta}}_1(\boldsymbol{I}_{T+1}\otimes\boldsymbol{\Sigma}_a)\overline{\boldsymbol{\Theta}}_1'\boldsymbol{U}_1^{-1'}\right|^{-\frac{1}{2}}\times$$
$$\exp\left\{-\frac{1}{2}\left(vec(\boldsymbol{Y})-\boldsymbol{U}_1^{-1}\begin{pmatrix}\boldsymbol{\Phi}_1\boldsymbol{y}_0\\ \boldsymbol{0}\\ \vdots\\ \boldsymbol{0}\end{pmatrix}\right)'\boldsymbol{U}_1'\left[\overline{\boldsymbol{\Theta}}_1(\boldsymbol{I}_{T+1}\otimes\boldsymbol{\Sigma}_a)\overline{\boldsymbol{\Theta}}_1'\right]^{-1}\boldsymbol{U}_1\left(vec(\boldsymbol{Y})-\boldsymbol{U}_1^{-1}\begin{pmatrix}\boldsymbol{\Phi}_1\boldsymbol{y}_0\\ \boldsymbol{0}\\ \vdots\\ \boldsymbol{0}\end{pmatrix}\right)\right\}$$
$$=\left|\overline{\boldsymbol{\Theta}}_1(\boldsymbol{I}_{T+1}\otimes\boldsymbol{\Sigma}_a)\overline{\boldsymbol{\Theta}}_1'\right|^{-\frac{1}{2}}\times$$
$$\exp\left\{-\frac{1}{2}\left(\boldsymbol{U}_1 vec(\boldsymbol{Y})-\begin{pmatrix}\boldsymbol{\Phi}_1\boldsymbol{y}_0\\ \boldsymbol{0}\\ \vdots\\ \boldsymbol{0}\end{pmatrix}\right)'\left[\overline{\boldsymbol{\Theta}}_1(\boldsymbol{I}_{T+1}\otimes\boldsymbol{\Sigma}_a)\overline{\boldsymbol{\Theta}}_1'\right]^{-1}\left(\boldsymbol{U}_1 vec(\boldsymbol{Y})-\begin{pmatrix}\boldsymbol{\Phi}_1\boldsymbol{y}_0\\ \boldsymbol{0}\\ \vdots\\ \boldsymbol{0}\end{pmatrix}\right)\right\} \quad (4.16)$$

Take note that the determinant, $|\boldsymbol{U}_1| = 1$ since it is a lower diagonal matrix with ones on the main diagonal.

An approximation of the likelihood function in (4.16) can be obtained by assuming that $\boldsymbol{y}_0 = \boldsymbol{a}_0 = \boldsymbol{0}$, then (4.14) simplifies to

$$\boldsymbol{U}_1 vec(\boldsymbol{Y}) = \begin{pmatrix}\boldsymbol{I}_k & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{\Theta}_1 & \boldsymbol{I}_k & \cdots & \boldsymbol{0} & \boldsymbol{0}\\ \vdots & \ddots & \ddots & & \vdots\\ \vdots & & \ddots & \ddots & \vdots\\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{\Theta}_1 & \boldsymbol{I}_k\end{pmatrix} vec(\boldsymbol{A}) = \tilde{\boldsymbol{\Theta}}_1 vec(\boldsymbol{A})$$

and solving for $vec(\boldsymbol{Y})$,

$$vec(\boldsymbol{Y}) = \boldsymbol{U}_1^{-1}\tilde{\boldsymbol{\Theta}}_1 vec(\boldsymbol{A}) \quad (4.17)$$

By applying result (A5.2) and (3.5) to (4.17), $vec(\boldsymbol{Y})$ is $N\left(\boldsymbol{0}, \boldsymbol{U}_1^{-1}\tilde{\boldsymbol{\Theta}}_1(\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a)\tilde{\boldsymbol{\Theta}}_1' \boldsymbol{U}_1^{-1'}\right)$ distributed. Thus, the conditional likelihood function is proportional to

$$\hat{L}(\boldsymbol{\Phi}_1, \boldsymbol{\Theta}_1, \boldsymbol{\Sigma}_a) \propto \left|\boldsymbol{U}_1^{-1}\tilde{\boldsymbol{\Theta}}_1(\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a)\tilde{\boldsymbol{\Theta}}_1' \boldsymbol{U}_1^{-1'}\right|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2} vec(\boldsymbol{Y})' \left[\boldsymbol{U}_1^{-1}\tilde{\boldsymbol{\Theta}}_1(\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a)\tilde{\boldsymbol{\Theta}}_1' \boldsymbol{U}_1^{-1'}\right]^{-1} vec(\boldsymbol{Y})\right\} \tag{4.18}$$

Utilising the properties of the determinant and Kronecker product, $\left|\boldsymbol{U}_1^{-1}\tilde{\boldsymbol{\Theta}}_1(\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a)\tilde{\boldsymbol{\Theta}}_1' \boldsymbol{U}_1^{-1'}\right| = \left|\boldsymbol{U}_1^{-1}\right|\left|\tilde{\boldsymbol{\Theta}}_1\right|\left|\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a\right|\left|\tilde{\boldsymbol{\Theta}}_1'\right|\left|\boldsymbol{U}_1^{-1'}\right| = \left|\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a\right|$ since $\boldsymbol{U}_1$ and $\tilde{\boldsymbol{\Theta}}_1$ are lower triangular matrices with ones on the main diagonal, therefore their determinants are equal to one; and $\left|\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a\right| = \left|\boldsymbol{I}_T\right|^k \left|\boldsymbol{\Sigma}_a\right|^T = \left|\boldsymbol{\Sigma}_a\right|^T$ using (A2.4). Taking this into account, the conditional likelihood function in (4.18) simplifies to

$$\begin{aligned}
\hat{L}(\boldsymbol{\Phi}_1, \boldsymbol{\Theta}_1, \boldsymbol{\Sigma}_a) &\propto \left|\boldsymbol{\Sigma}_a\right|^{-\frac{T}{2}} \exp\left\{-\frac{1}{2} vec(\boldsymbol{Y})' \boldsymbol{U}_1' \left(\tilde{\boldsymbol{\Theta}}_1'\right)^{-1} (\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a)^{-1} \tilde{\boldsymbol{\Theta}}_1^{-1} \boldsymbol{U}_1 vec(\boldsymbol{Y})\right\} \\
&= \left|\boldsymbol{\Sigma}_a\right|^{-\frac{T}{2}} \exp\left\{-\frac{1}{2} \left(\tilde{\boldsymbol{\Theta}}_1^{-1} \boldsymbol{U}_1 vec(\boldsymbol{Y})\right)' \left(\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a^{-1}\right) \tilde{\boldsymbol{\Theta}}_1^{-1} \boldsymbol{U}_1 vec(\boldsymbol{Y})\right\} \\
&= \left|\boldsymbol{\Sigma}_a\right|^{-\frac{T}{2}} \exp\left\{-\frac{1}{2} vec(\boldsymbol{A})' \left(\boldsymbol{I}_T \otimes \boldsymbol{\Sigma}_a^{-1}\right) vec(\boldsymbol{A})\right\} \quad \text{(from (4.17))} \\
&= \left|\boldsymbol{\Sigma}_a\right|^{-\frac{T}{2}} \exp\left\{-\frac{1}{2} \sum_{t=1}^{T} \boldsymbol{a}_t' \boldsymbol{\Sigma}_a^{-1} \boldsymbol{a}_t\right\}
\end{aligned} \tag{4.19}$$

where $\boldsymbol{a}_t$ can be determined by rewriting the VARMA(1,1) process as a VAR process and setting $\boldsymbol{y}_0 = \boldsymbol{a}_0 = \boldsymbol{0}$.

The maximum likelihood estimators of the unknown parameters can be obtained by maximising the likelihood function (4.19) using numerical optimisation techniques. The dual quasi-Newton optimisation technique is used to illustrate maximum likelihood estimation of a VARMA(1,1) model in the following example.

Example 4.2*

Consider the bivariate VARMA(1,1) model $\boldsymbol{y}_t = \begin{pmatrix} -0.2 & 0.1 \\ 0.5 & 0.1 \end{pmatrix} \boldsymbol{y}_{t-1} + \boldsymbol{a}_t - \begin{pmatrix} 0.2 & 0.1 \\ 0.1 & 0.4 \end{pmatrix} \boldsymbol{a}_{t-1}$ with $\boldsymbol{\Sigma}_a = \begin{pmatrix} 1.0 & 0.5 \\ 0.5 & 0.9 \end{pmatrix}$. A sample of size 500 is generated.

The `NLPQN CALL` in `SAS IML` was used to maximise the log-likelihood function,

$$\ln \hat{L}(\boldsymbol{\Phi}_1, \boldsymbol{\Theta}_1, \boldsymbol{\Sigma}_a) = -\frac{T}{2} \ln|\boldsymbol{\Sigma}_a| - \frac{1}{2} \sum_{t=1}^{T} \boldsymbol{a}_t' \boldsymbol{\Sigma}_a^{-1} \boldsymbol{a}_t \text{ where } \boldsymbol{a}_t = \boldsymbol{y}_t - \boldsymbol{\Phi}_1 \boldsymbol{y}_{t-1} - \boldsymbol{\Theta}_1 \boldsymbol{a}_{t-1}$$

The maximum likelihood estimates are

$$\hat{\boldsymbol{\Phi}}_1 = \begin{pmatrix} -0.215 & 0.150 \\ 0.467 & 0.053 \end{pmatrix}, \hat{\boldsymbol{\Theta}}_1 = \begin{pmatrix} -0.206 & -0.235 \\ -0.050 & -0.428 \end{pmatrix}, \hat{\boldsymbol{\Sigma}}_a = \begin{pmatrix} 0.995 & 0.525 \\ 0.525 & 0.975 \end{pmatrix}$$

Therefore, the estimated model is

$$\begin{aligned} \hat{\boldsymbol{y}}_t &= \hat{\boldsymbol{\Phi}}_1 \boldsymbol{y}_{t-1} + \boldsymbol{a}_t + \hat{\boldsymbol{\Theta}}_1 \boldsymbol{a}_{t-1} \\ &= \begin{pmatrix} -0.215 & 0.150 \\ 0.467 & 0.053 \end{pmatrix} \boldsymbol{y}_{t-1} + \boldsymbol{a}_t + \begin{pmatrix} -0.206 & -0.235 \\ -0.050 & -0.428 \end{pmatrix} \boldsymbol{a}_{t-1} \\ &= \begin{pmatrix} -0.215 & 0.150 \\ 0.467 & 0.053 \end{pmatrix} \boldsymbol{y}_{t-1} + \boldsymbol{a}_t - \begin{pmatrix} 0.206 & 0.235 \\ 0.050 & 0.428 \end{pmatrix} \boldsymbol{a}_{t-1} \end{aligned}$$

These estimates are very similar to the maximum likelihood estimates obtained using the `VARMAX` procedure, the estimated model using this procedure is

$$\hat{\boldsymbol{y}}_t = \begin{pmatrix} -0.209 & 0.157 \\ 0.470 & 0.065 \end{pmatrix} \boldsymbol{y}_{t-1} + \boldsymbol{a}_t - \begin{pmatrix} 0.213 & 0.238 \\ 0.062 & 0.434 \end{pmatrix} \boldsymbol{a}_{t-1} \text{ with } \hat{\boldsymbol{\Sigma}}_a = \begin{pmatrix} 1.003 & 0.530 \\ 0.530 & 0.969 \end{pmatrix}$$

4.5 THE IDENTIFICATION PROBLEM

Let $\{\boldsymbol{y}_t\}$ be a stationary, invertible VARMA(p,q) process, as defined in (2.39), with zero mean. In terms of the lag operator this process can be represented as

* The `SAS` program is provided in Appendix B page 131.

$$\boldsymbol{\Phi}(L)\boldsymbol{y}_t = \boldsymbol{\Theta}(L)\boldsymbol{a}_t \tag{4.20}$$

where the operators $\boldsymbol{\Phi}(L)$ and $\boldsymbol{\Theta}(L)$ are defined in (2.40).

It is possible that two VARMA(p,q) representations are observationally equivalent, that is, two VARMA(p,q) models with different coefficient matrices will have the same VMA(∞) representation. This will be the case when the two sets of operators, say $\boldsymbol{\Phi}^*(L)$ and $\boldsymbol{\Theta}^*(L)$ are related to $\boldsymbol{\Phi}(L)$ and $\boldsymbol{\Theta}(L)$ by premultiplying with a non-singular matrix $\boldsymbol{U}(L)$, for example $\boldsymbol{\Phi}^*(L) = \boldsymbol{U}(L)\boldsymbol{\Phi}(L)$ and $\boldsymbol{\Theta}^*(L) = \boldsymbol{U}(L)\boldsymbol{\Theta}(L)$. (Reinsel, 1997)

In order to specify a unique set of parameters we need to put certain restrictions on the VAR and VMA operators. The representation must be such that there are no common factors in the $\boldsymbol{\Phi}(L)$ and $\boldsymbol{\Theta}(L)$ operators, except for unimodular operators. A unimodular operator is an operator with its determinant equal to a non zero constant, which implies that the determinant is not a function of L, the lag operator. If this is the case, the operators $\boldsymbol{\Phi}(L)$ and $\boldsymbol{\Theta}(L)$ are called left-coprime. The only unimodular operator that will ensure uniqueness of the left-coprime operators is the one equal to the identity matrix. (Lütkepohl, 2005)

The final equations form and the echelon form result in a unique representation of the VARMA(p,q) process. Before defining these forms we need to consider a more general representation of the standard VARMA representation in (2.39) by including coefficient matrices for $\boldsymbol{y}_t$ and $\boldsymbol{a}_t$, namely

$$\boldsymbol{\Phi}_0\boldsymbol{y}_t = \boldsymbol{c} + \boldsymbol{\Phi}_1\boldsymbol{y}_{t-1} + \boldsymbol{\Phi}_2\boldsymbol{y}_{t-2} + \ldots + \boldsymbol{\Phi}_p\boldsymbol{y}_{t-p} + \boldsymbol{\Theta}_0\boldsymbol{a}_t + \boldsymbol{\Theta}_1\boldsymbol{a}_{t-1} + \boldsymbol{\Theta}_2\boldsymbol{a}_{t-2} + \ldots + \boldsymbol{\Theta}_q\boldsymbol{a}_{t-q} \tag{4.21}$$

or in terms of the lag operator

$$\begin{aligned} \left(\boldsymbol{\Phi}_0 - \boldsymbol{\Phi}_1 L - \boldsymbol{\Phi}_2 L^2 - \ldots - \boldsymbol{\Phi}_p L^p\right)\boldsymbol{y}_t &= \boldsymbol{c} + \left(\boldsymbol{\Theta}_0 + \boldsymbol{\Theta}_1 L + \boldsymbol{\Theta}_2 L^2 + \ldots + \boldsymbol{\Theta}_q L^q\right)\boldsymbol{a}_t \\ \boldsymbol{\Phi}(L)\boldsymbol{y}_t &= \boldsymbol{c} + \boldsymbol{\Theta}(L)\boldsymbol{a}_t \end{aligned} \tag{4.22}$$

where

$$\begin{aligned} \boldsymbol{\Phi}(L) &= \boldsymbol{\Phi}_0 - \boldsymbol{\Phi}_1 L - \boldsymbol{\Phi}_2 L^2 - \ldots - \boldsymbol{\Phi}_p L^p \\ \boldsymbol{\Theta}(L) &= \boldsymbol{\Theta}_0 + \boldsymbol{\Theta}_1 L + \boldsymbol{\Theta}_2 L^2 + \ldots + \boldsymbol{\Theta}_q L^q \end{aligned}$$

with $\boldsymbol{\Phi}(L)$ and $\boldsymbol{\Theta}(L)$ left-coprime.

Definitions 12.1 and 12.2 of Lütkepohl (2005) define the final equations form and the echelon form respectively, namely:

"The VARMA representation (4.22) is said to be in final equations form if $\boldsymbol{\Theta}_0 = \boldsymbol{I}_k$ and $\boldsymbol{\Phi}(L) = \phi(L)\boldsymbol{I}_k$, where $\phi(L) = 1 - \phi_1 L - \ldots \phi_p L^p$ is a scalar operator with $\phi_p \neq 0$. "

"The VARMA representation (4.22) is said to be in echelon form or $\mathrm{ARMA_E}$ form if the VAR and VMA operators $\boldsymbol{\Phi}(L) = [\phi_{mi}(L)]_{m,i=1,\ldots,k}$ and $\boldsymbol{\Theta}(L) = [\theta_{mi}(L)]$ are left-coprime and satisfy the following conditions: the operators $\phi_{mi}(L)$ $(i = 1,\ldots,k)$ and $\theta_{mj}(L)$ $(j = 1,\ldots,k)$ in the m-th row of $\boldsymbol{\Phi}(L)$ and $\boldsymbol{\Theta}(L)$ have degree p_m and they have the form

$$\phi_{mm}(L) = 1 - \sum_{j=1}^{p_m} \phi_{mm,j} L^j \text{, for } m = 1,\ldots,k$$

$$\phi_{mi}(L) = -\sum_{j=p_m-p_{mi}+1}^{p_m} \phi_{mi,j} L^j \text{, for } m \neq i$$

and

$$\theta_{mi}(L) = \sum_{j=0}^{p_m} \theta_{mi,j} L^j \text{, for } m,i = 1,\ldots,k \text{ with } \boldsymbol{\Phi}_0 = \boldsymbol{\Theta}_0$$

In the VAR operators $\phi_{mi}(L)$

$$p_{mi} = \begin{cases} \min(p_m + 1, p_i) \text{ for } m \geq i & m,i = 1,\ldots,k \\ \min(p_m, p_i) \text{ for } m < i & \end{cases}$$

That is, p_{mi} specifies the number of free coefficients in the operator $\phi_{mi}(L)$ for $i \neq m$. The row degrees $(p_1,\ldots,p_k)$ are called the Kronecker indices and their sum $\sum_{i=1}^{k} p_i$ is the McMillan degree"

For more detail and examples we refer to chapter 12 of Lütkepohl (2005).

4.6 CONCLUSION

This chapter focused on maximum likelihood estimation of VARMA processes. Due to the nonlinear nature of the normal equations with respect to the parameters, only the likelihood functions were derived. The examples employed numerical optimisation techniques to maximise the likelihood function in order to determine the parameter estimates. An overview was given of the identification problem regarding the uniqueness of the VARMA representation. Before a model can be estimated, one has to determine the values of p and q. Chapter 5 will discuss some guidelines to select the appropriate order.

CHAPTER 5

ORDER SELECTION

5.1 INTRODUCTION

The order of the model is not known in most applications; therefore order selection forms part of the model building process. We are looking for a parsimonious model, a model with as little as possible parameters that explains most of the variation in the data.

Before the vector autoregressive coefficients, $\boldsymbol{\Phi}_i$, $i = 1,2,\ldots p$ and the vector moving average coefficients $\boldsymbol{\Theta}_i$, $i = 1,2,\ldots q$ can be estimated, the order of the VARMA process need to be determined. Thus we are searching for unique numbers p and q such that $\boldsymbol{\Phi}_p \neq 0$ and $\boldsymbol{\Phi}_i = 0$ for $i > p$ while $\boldsymbol{\Theta}_q \neq 0$ and $\boldsymbol{\Theta}_j = 0$ for $j > q$.

The problem of finding appropriate values for p and q was tackled by, amongst others, Tiao & Box (1981). They considered methods based on the sample autocorrelations and the sample partial autoregression matrices to select the order of pure VMA and VAR models, respectively. They introduced a way of visualising the sample autocorrelation and sample partial autoregression matrices by replacing the values with symbols. The challenge of determining the order for mixed models was addressed by, for example, Quinn (1980) who extended the Hannan-Quinn information criterion to the multivariate environment. This method entails fitting different models and then selecting the model that minimises the information criterion. This can be a time consuming exercise. Spliid (1983) was one of the people who proposed an algorithm for the MINIC (minimum information criterion) method, which is another way of tentatively identifying the order.

In section 5.2 the use of the sample autocovariance and autocorrelation matrices, to identify the order of a pure VMA process, is considered. Section 5.3 focuses on identifying the order of a pure VAR process by determining the partial autoregression matrices. Finally in section 5.4 a method to determine the order of a VARMA process, based on the information criteria, is discussed.

The following bivariate models will be used in the examples to illustrate the different techniques of determining the order of a VARMA process:

VAR(1) model: $\boldsymbol{y}_t = \begin{pmatrix} 0.5 & 0.6 \\ 0.1 & 0.4 \end{pmatrix} \boldsymbol{y}_{t-1} + \boldsymbol{a}_t$

VMA(2) model: $\boldsymbol{y}_t = \boldsymbol{a}_t - \begin{pmatrix} 0.2 & 0.1 \\ 0.1 & 0.4 \end{pmatrix} \boldsymbol{a}_{t-1} - \begin{pmatrix} 0.4 & 0 \\ 0.6 & 0.1 \end{pmatrix} \boldsymbol{a}_{t-2}$

VARMA(2,1) model: $\boldsymbol{y}_t = \begin{pmatrix} -0.2 & 0.1 \\ 0.5 & 0.1 \end{pmatrix} \boldsymbol{y}_{t-1} + \begin{pmatrix} 0.8 & 0.5 \\ -0.4 & 0.5 \end{pmatrix} \boldsymbol{y}_{t-2} + \boldsymbol{a}_t - \begin{pmatrix} 0.2 & 0.1 \\ 0.1 & 0.4 \end{pmatrix} \boldsymbol{a}_{t-1}$

with $\boldsymbol{\Sigma}_a = \begin{pmatrix} 1.0 & 0.5 \\ 0.5 & 0.9 \end{pmatrix}$ for all the models.

5.2 SAMPLE AUTOCOVARIANCE AND AUTOCORRELATION MATRICES

In this section, expression for sample autocovariance and sample autocorrelation matrices are given, a large sample test for the significance of the elements of the autocorrelation matrix is provided and illustrated by means of a numerical example.

Suppose we have k time series processes each comprising of T equally spaced observations denoted by $\boldsymbol{Y} : k \times T$.

The sample estimate of the process mean is

$$\hat{\boldsymbol{\mu}} = \bar{\boldsymbol{y}} = \begin{pmatrix} \bar{y}_1 & \bar{y}_2 & \cdots & \bar{y}_k \end{pmatrix}' = \frac{1}{T} \sum_{t=1}^{T} \boldsymbol{y}_t \qquad (5.1)$$

This estimate, $\bar{\boldsymbol{y}}$, is an unbiased estimator for the process mean since,

$$E(\bar{\boldsymbol{y}}) = E\left(\frac{1}{T} \sum_{t=1}^{T} \boldsymbol{y}_t \right) = \frac{1}{T} \sum_{t=1}^{T} E(\boldsymbol{y}_t) = \frac{1}{T} \sum_{t=1}^{T} \boldsymbol{\mu} = \boldsymbol{\mu}$$

The autocovariance matrix at lag l, $\boldsymbol{\Gamma}(l) = E\left[(\boldsymbol{y}_t - \boldsymbol{\mu})(\boldsymbol{y}_{t-l} - \boldsymbol{\mu})' \right]$ can be estimated from the sample values to determine the sample autocovariance matrix,

$$\hat{\boldsymbol{\Gamma}}(l) = \frac{1}{T} \sum_{t=l+1}^{T} (\boldsymbol{y}_t - \bar{\boldsymbol{y}})(\boldsymbol{y}_{t-l} - \bar{\boldsymbol{y}})' \qquad \text{for } l = 0,1,\ldots \qquad (5.2)$$

The (i,j)-th element of $\hat{\boldsymbol{\Gamma}}(l)$ is given by $\hat{\gamma}_{ij}(l) = \frac{1}{T}\sum_{t=l+1}^{T}(y_{it} - \bar{y}_i)(y_{j,t-l} - \bar{y}_j)$.

The formula for the sample autocovariance matrix in (5.2) only adds T-l observations and then divides this by T, not by T-l. This means that as l increases the estimate decreases and eventually will be zero. This is in line with the population autocovariance matrix because for a stationary process $\boldsymbol{\Gamma}(l) \to \boldsymbol{0}$ as $l \to \infty$. (Hamilton, 1994)

From the sample autocovariance matrices in (5.2) the sample autocorrelations can be calculated by

$$\hat{\rho}_{ij}(l) = \frac{\hat{\gamma}_{ij}(l)}{\sqrt{\hat{\gamma}_{ii}(0)\hat{\gamma}_{jj}(0)}} \tag{5.3}$$

or in matrix form at lag l,

$$\hat{\boldsymbol{\rho}}(l) = \hat{\boldsymbol{V}}^{-\frac{1}{2}}\hat{\boldsymbol{\Gamma}}(l)\hat{\boldsymbol{V}}^{-\frac{1}{2}} \tag{5.4}$$

where $\hat{\boldsymbol{V}}^{\frac{1}{2}}$ is the $k \times k$ diagonal matrix with the sample standard deviations.

In Chapter 2 the autocovariance matrices, $\boldsymbol{\Gamma}(l)$, for a VMA(q) process were derived. It was shown in (2.35) that $\boldsymbol{\Gamma}(l) = \boldsymbol{0}$ for $l > q$. Since the autocorrelation matrices $\boldsymbol{\rho}(l)$ are a function of the autocovariance matrices, it can be shown that $\boldsymbol{\rho}(l) = \boldsymbol{0}$ for $l > q$. This property can be used to determine the order of a pure VMA process. We will calculate the sample autocorrelation matrices at different lags and determine whether they differ significantly from zero. If they do not differ significantly from zero at lag $j+1$, it can be concluded that the data was generated by a VMA(j) model.

This 'significance test' is more of an informal guideline developed by Tiao & Box (1981). It has to be determined whether the autocorrelation matrices differ significantly from zero. In other words one can test whether the autocorrelation matrix at different lags corresponds to that of a white noise process. It is known that for large T, the individual elements of a sample autocorrelation matrix of a white noise process are normally distributed with zero mean and variance equal to $\frac{1}{T}$. This will be considered in more detail in Section 6.2.1. Based on this distribution, Tiao & Box constructed a confidence interval with the following symbols:

" – " : less than - 2 estimated standard errors $\left(< -\frac{2}{\sqrt{T}} \right)$

" . " : within two estimated standard errors $\left(-\frac{2}{\sqrt{T}}, \frac{2}{\sqrt{T}} \right)$

" + " : greater than 2 estimated standard errors $\left(> \frac{2}{\sqrt{T}} \right)$

In the following example PROC IML was used to determine the sample autocovariance and sample autocorrelation matrices up to lag 3 using formulae (5.1), (5.2) and (5.4). The individual elements of the sample autocorrelation matrices are tested for significance using the guideline developed by Tiao & Box. The results were compared with the results produced by the VARMAX procedure.

Example 5.1*

The sample autocovariances and sample autocorrelations, for the three generated time series processes with $T = 500$ and a Gaussian error distribution, were calculated using (5.2) and (5.4), respectively and are tabulated below.

	Process 1	**Process 2**	**Process 3**
Generated by	**VAR(1)**	**VMA(2)**	**VARMA(2,1)**
	Autocovariances		
$\hat{\Gamma}(0)$	$\begin{pmatrix} 2.584 & 1.131 \\ 1.131 & 1.210 \end{pmatrix}$	$\begin{pmatrix} 1.251 & 0.925 \\ 0.925 & 1.640 \end{pmatrix}$	$\begin{pmatrix} 6.771 & -3.173 \\ -3.173 & 8.543 \end{pmatrix}$
$\hat{\Gamma}(1)$	$\begin{pmatrix} 1.896 & 1.190 \\ 0.654 & 0.516 \end{pmatrix}$	$\begin{pmatrix} -0.202 & -0.171 \\ -0.121 & -0.279 \end{pmatrix}$	$\begin{pmatrix} -4.355 & 2.247 \\ 5.668 & -7.005 \end{pmatrix}$
$\hat{\Gamma}(2)$	$\begin{pmatrix} 1.358 & 0.882 \\ 0.425 & 0.297 \end{pmatrix}$	$\begin{pmatrix} -0.342 & -0.185 \\ -0.613 & -0.392 \end{pmatrix}$	$\begin{pmatrix} 5.262 & 0.567 \\ -5.979 & 5.915 \end{pmatrix}$
$\hat{\Gamma}(3)$	$\begin{pmatrix} 0.916 & 0.616 \\ 0.239 & 0.207 \end{pmatrix}$	$\begin{pmatrix} 0.013 & 0.040 \\ 0.015 & 0.082 \end{pmatrix}$	$\begin{pmatrix} -2.303 & -1.189 \\ 6.615 & -3.413 \end{pmatrix}$

* The SAS program is provided in Appendix B page 132.

	Process 1	**Process 2**	**Process 3**
Generated by	**VAR(1)**	**VMA(2)**	**VARMA(2,1)**
		Autocorrelations	
$\hat{\rho}(0)$	$\begin{pmatrix} 1 & 0.639 \\ 0.639 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0.646 \\ 0.646 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -0.417 \\ -0.417 & 1 \end{pmatrix}$
$\hat{\rho}(1)$	$\begin{pmatrix} 0.734 & 0.673 \\ 0.370 & 0.426 \end{pmatrix}$	$\begin{pmatrix} -0.162 & -0.119 \\ -0.084 & -0.170 \end{pmatrix}$	$\begin{pmatrix} -0.643 & 0.295 \\ 0.745 & -0.820 \end{pmatrix}$
$\hat{\rho}(2)$	$\begin{pmatrix} 0.526 & 0.499 \\ 0.240 & 0.245 \end{pmatrix}$	$\begin{pmatrix} -0.274 & -0.129 \\ -0.428 & -0.239 \end{pmatrix}$	$\begin{pmatrix} 0.777 & 0.075 \\ -0.786 & 0.692 \end{pmatrix}$
$\hat{\rho}(3)$	$\begin{pmatrix} 0.354 & 0.348 \\ 0.135 & 0.171 \end{pmatrix}$	$\begin{pmatrix} 0.010 & 0.028 \\ 0.010 & 0.050 \end{pmatrix}$	$\begin{pmatrix} -0.340 & -0.156 \\ 0.870 & -0.400 \end{pmatrix}$

$\hat{\Gamma}(3)$ and $\hat{\rho}(3)$ for process 2 are very close to zero, this is in line with what is expected for a VMA(2) process. In terms of the guideline developed by Tiao & Box, an element will be "significant" if the absolute value thereof is greater than $\dfrac{2}{\sqrt{500}} = 0.089$. This confirms that $\hat{\rho}(3)$ for process 2 does not differ significantly from zero.

For comparison purposes, the corresponding SAS output is provided below. Take note that the values calculated by SAS are the transpose of those in the table above, this is due to the definition of the autocovariance matrix at lag *l*, as explained in Example 2.1.

Process 1 (VAR(1))	**Process 2 (VMA(2))**

<pre>Cross Covariances of Dependent Series
 	
Lag   Variable   y1   y2
 	
  0   y1   2.58426   1.13083
      y2   1.13083   1.21028
  1   y1   1.89590   0.65443
      y2   1.19016   0.51605
  2   y1   1.35804   0.42457
      y2   0.88176   0.29683
  3   y1   0.91573   0.23894
      y2   0.61588   0.20748</pre>
	<pre>Cross Covariances of Dependent Series
	
	Lag Variable y1 y2
	
	  0 y1 1.25146 0.92534
	      y2 0.92534 1.64031
	  1 y1 -0.20234 -0.12100
	      y2 -0.17081 -0.27860
	  2 y1 -0.34233 -0.61275
	      y2 -0.18537 -0.39239
	  3 y1 0.01307 0.01501
	      y2 0.04020 0.08177</pre>

Process 1 (VAR(1))

Cross Correlations of Dependent Series

Lag	Variable	y1	y2
0	y1	1.00000	0.63942
	y2	0.63942	1.00000
1	y1	0.73363	0.37004
	y2	0.67297	0.42639
2	y1	0.52550	0.24007
	y2	0.49859	0.24526
3	y1	0.35435	0.13511
	y2	0.34824	0.17143

Schematic Representation of Cross Correlations

Variable/ Lag	0	1	2	3
y1	++	++	++	++
y2	++	++	++	++

+ is > 2*std error, - is < -2*std error, . is between

Process 2 (VMA(2))

Cross Correlations of Dependent Series

Lag	Variable	y1	y2
0	y1	1.00000	0.64585
	y2	0.64585	1.00000
1	y1	-0.16168	-0.08445
	y2	-0.11922	-0.16984
2	y1	-0.27354	-0.42767
	y2	-0.12938	-0.23921
3	y1	0.01044	0.01048
	y2	0.02806	0.04985

Schematic Representation of Cross Correlations

Variable/ Lag	0	1	2	3
y1	++	-.	--	..
y2	++	--	--	..

+ is > 2*std error, - is < -2*std error, . is between

Process 3 (VARMA(2,1))

Cross Covariances of Dependent Series

Lag	Variable	y1	y2
0	y1	6.77120	-3.17312
	y2	-3.17312	8.54336
1	y1	-4.35510	5.66751
	y2	2.24744	-7.00455
2	y1	5.26153	-5.97919
	y2	0.56684	5.91469
3	y1	-2.30307	6.61509
	y2	-1.18913	-3.41342

Cross Correlations of Dependent Series

Lag	Variable	y1	y2
0	y1	1.00000	-0.41719
	y2	-0.41719	1.00000
1	y1	-0.64318	0.74515
	y2	0.29549	-0.81988
2	y1	0.77705	-0.78613
	y2	0.07453	0.69231
3	y1	-0.34013	0.86974
	y2	-0.15634	-0.39954

Schematic Representation of Cross Correlations

Variable/ Lag	0	1	2	3
y1	+-	-+	+-	-+
y2	-+	+-	.+	--

+ is > 2*std error, - is < -2*std error, . is between

The schematic representation of the autocorrelations summarises the significance of the individual elements. Each lag is represented by four symbols corresponding to the elements of the autocorrelation matrix. A "+" or "-" indicates significance, while a "." means that the hypothesis $H_0 : \rho_{mn,i} = 0$ cannot be rejected. In this example no autocorrelation from lag 3 onwards for process 2 is significant, implying that this process is generated by a VMA(2) model. The other two processes both have significant autocorrelations at lag 3, implying either higher order VMA or mixed models. The partial autoregression matrices may shed more light on the autoregressive order.

5.3 PARTIAL AUTOREGRESSION MATRICES

In this section the Yule-Walker equations of a VAR model are utilised to derive formulae for the partial autoregression matrices up to lag 2. Another method of obtaining partial autoregression matrices and a test for the significance of individual elements is given. The section is concluded with a numerical example.

The partial autoregression matrix is a measure of the autocovariance between the observed values at two time points after the effect of the terms in between the two time points has been removed. These matrices can be used to identify the order of a VAR process, since the partial autoregression matrices of a VAR(p) process are equal to zero from lag $p+1$ onwards. The Yule-Walker equations in (2.25) that calculate the autocovariance matrix recursively, can be used to determine the partial autoregression matrices. Consider as an example the VAR(1) and VAR(2) processes.

For a VAR(1) process the Yule-Walker equation for the autocovariance matrix is,

$$\boldsymbol{\Gamma}(l) = \boldsymbol{\Phi}_1 \boldsymbol{\Gamma}(l-1), \text{ and therefore}$$

$$\boldsymbol{\Gamma}(1) = \boldsymbol{\Phi}_{11} \boldsymbol{\Gamma}(0)$$

where $\boldsymbol{\Phi}_{11}$ is called the partial autoregression matrix of lag 1. Solving for $\boldsymbol{\Phi}_{11}$ we have that

$$\boldsymbol{\Phi}_{11} = \boldsymbol{\Gamma}(1)\boldsymbol{\Gamma}(0)^{-1} \tag{5.5}$$

In case of a VAR(1) process $\boldsymbol{\Phi}_{11} = \boldsymbol{\Phi}_1$ and $\boldsymbol{\Phi}_{22} = \boldsymbol{\Phi}_{33} = \ldots = \boldsymbol{0}$.

Consider a VAR(2) process. The autocovariance matrix at lag *l*,

$$\boldsymbol{\Gamma}(l) = \boldsymbol{\Phi}_1 \boldsymbol{\Gamma}(l-1) + \boldsymbol{\Phi}_2 \boldsymbol{\Gamma}(l-2), \text{ and therefore}$$

$$\boldsymbol{\Gamma}(1) = \boldsymbol{\Phi}_{12} \boldsymbol{\Gamma}(0) + \boldsymbol{\Phi}_{22} \boldsymbol{\Gamma}(-1) = \boldsymbol{\Phi}_{12} \boldsymbol{\Gamma}(0) + \boldsymbol{\Phi}_{22} \boldsymbol{\Gamma}(1)' \quad (5.6)$$

$$\boldsymbol{\Gamma}(2) = \boldsymbol{\Phi}_{12} \boldsymbol{\Gamma}(1) + \boldsymbol{\Phi}_{22} \boldsymbol{\Gamma}(0) \quad (5.7)$$

By solving these two equations simultaneously, the partial autoregression matrix of lag 2, $\boldsymbol{\Phi}_{22}$, can be determined,

$$\boldsymbol{\Phi}_{12} \boldsymbol{\Gamma}(0) = \boldsymbol{\Gamma}(1) - \boldsymbol{\Phi}_{22} \boldsymbol{\Gamma}(1)' \qquad \text{(from (5.6))}$$

$$\therefore \boldsymbol{\Phi}_{12} = \boldsymbol{\Gamma}(1)\boldsymbol{\Gamma}(0)^{-1} - \boldsymbol{\Phi}_{22} \boldsymbol{\Gamma}(1)' \boldsymbol{\Gamma}(0)^{-1} \quad (5.8)$$

Substituting (5.8) into (5.7),

$$\boldsymbol{\Gamma}(2) = \boldsymbol{\Gamma}(1)\boldsymbol{\Gamma}(0)^{-1}\boldsymbol{\Gamma}(1) - \boldsymbol{\Phi}_{22}\boldsymbol{\Gamma}(1)'\boldsymbol{\Gamma}(0)^{-1}\boldsymbol{\Gamma}(1) + \boldsymbol{\Phi}_{22}\boldsymbol{\Gamma}(0)$$

$$\boldsymbol{\Gamma}(2) - \boldsymbol{\Gamma}(1)\boldsymbol{\Gamma}(0)^{-1}\boldsymbol{\Gamma}(1) = \boldsymbol{\Phi}_{22}\left[\boldsymbol{\Gamma}(0) - \boldsymbol{\Gamma}(1)'\boldsymbol{\Gamma}(0)^{-1}\boldsymbol{\Gamma}(1)\right]$$

$$\therefore \boldsymbol{\Phi}_{22} = \left[\boldsymbol{\Gamma}(2) - \boldsymbol{\Gamma}(1)\boldsymbol{\Gamma}(0)^{-1}\boldsymbol{\Gamma}(1)\right]\left[\boldsymbol{\Gamma}(0) - \boldsymbol{\Gamma}(1)'\boldsymbol{\Gamma}(0)^{-1}\boldsymbol{\Gamma}(1)\right]^{-1} \quad (5.9)$$

For a VAR(2) process $\boldsymbol{\Phi}_{22} = \boldsymbol{\Phi}_2$ and $\boldsymbol{\Phi}_{33} = \boldsymbol{\Phi}_{44} = \ldots = \boldsymbol{0}$.

In general, for a VAR(*p*) process, $p = 1, 2, \ldots$, the partial autoregression matrices of lag *p*, $\boldsymbol{\Phi}_{pp}$, can be determined by solving the *p* Yule-Walker equations,

$$\boldsymbol{\Gamma}(l) = \sum_{i=1}^{p} \boldsymbol{\Phi}_{ip} \boldsymbol{\Gamma}(l-i) \quad \text{where } l = 1, 2, \ldots, p \quad (5.10)$$

The partial autoregression matrix of order *p*, $\boldsymbol{\Phi}_{pp}$ is equal to $\boldsymbol{\Phi}_p$ and $\boldsymbol{\Phi}_{mm} = \boldsymbol{0}$ for $m > p$. (Reinsel, 1997)

Note that the Yule-Walker equation system is used to derive expressions for the partial autoregression matrices in terms of autocovariance matrices. The expressions are general, they hold for all VARMA models.

This characteristic can be used to determine the order of a pure VAR process by determining whether the matrix of partial autoregressions at lag $j+1$, $\boldsymbol{\Phi}_{j+1,j+1}$, differs significantly from zero. If $\boldsymbol{\Phi}_{j+1,j+1}$ does not differ significantly from zero it can be concluded that the data was generated by a VAR(j) model.

The partial autoregression matrices are estimated by replacing the autocovariance matrices in (5.5) and (5.9) with their sample estimates. The sample estimates of $\boldsymbol{\Phi}_{11}$ and $\boldsymbol{\Phi}_{22}$ are given by,

$$\hat{\boldsymbol{\Phi}}_{11} = \hat{\boldsymbol{\Gamma}}(1)\hat{\boldsymbol{\Gamma}}(0)^{-1} \tag{5.11}$$

$$\hat{\boldsymbol{\Phi}}_{22} = \left[\hat{\boldsymbol{\Gamma}}(2) - \hat{\boldsymbol{\Gamma}}(1)\hat{\boldsymbol{\Gamma}}(0)^{-1}\hat{\boldsymbol{\Gamma}}(1)\right]\left[\hat{\boldsymbol{\Gamma}}(0) - \hat{\boldsymbol{\Gamma}}(1)'\hat{\boldsymbol{\Gamma}}(0)^{-1}\hat{\boldsymbol{\Gamma}}(1)\right]^{-1} \tag{5.12}$$

Another way of obtaining estimates for the partial autoregression matrices and their standard errors are by fitting VAR models of increasing order. Tiao & Box (1981) also suggested a guideline to tentatively determine the order of the VAR model by constructing a confidence interval of $\pm 2\,\text{estimated standard errors}$. Each element of the partial autoregression matrix is classified as a " – ", " . " or " + " depending on whether it is below the confidence limit, between the confidence limits or above the confidence limit.

In Example 5.2 PROC IML was used to calculate the estimated partial autoregression matrices using the formulae derived in (5.5) and (5.9), respectively. It is shown that the estimates obtained are the same as the results of the VARMAX procedure.

Example 5.2*

The sample partial autoregression matrices, $\hat{\boldsymbol{\Phi}}_{11}$ and $\hat{\boldsymbol{\Phi}}_{22}$, can be calculated using (5.11) and (5.12), respectively. The sample partial autoregression matrices for the three generated time series are tabulated below.

	Process 1	**Process 2**	**Process 3**
Generated by	**VAR(1)**	**VMA(2)**	**VARMA(2,1)**
Partial Autoregression Matrices			
$\hat{\boldsymbol{\Phi}}_{11}$	$\begin{pmatrix} 0.513 & 0.504 \\ 0.113 & 0.321 \end{pmatrix}$	$\begin{pmatrix} -0.145 & -0.022 \\ 0.050 & -0.198 \end{pmatrix}$	$\begin{pmatrix} -0.629 & 0.029 \\ 0.548 & -0.616 \end{pmatrix}$
$\hat{\boldsymbol{\Phi}}_{22}$	$\begin{pmatrix} 0.049 & -0.002 \\ 0.002 & -0.005 \end{pmatrix}$	$\begin{pmatrix} -0.356 & 0.066 \\ -0.535 & 0.027 \end{pmatrix}$	$\begin{pmatrix} 0.890 & 0.361 \\ -0.187 & 0.221 \end{pmatrix}$

* The SAS program is provided in Appendix B page 132.

$\hat{\boldsymbol{\Phi}}_{22}$ for process 1 is very close to zero, this is in line with what is expected for a VAR(1) process. The partial autoregression matrices for the other two processes are not close to zero implying they are either VAR models of a higher order, VMA models or mixed model.

The SAS output of the partial autoregression matrices for the processes is provided below. The schematic representations can be interpreted as in Example 5.1.

Process 1 (VAR(1))

```
         Partial Autoregression

Lag    Variable        y1             y2

  1    y1          0.51311        0.50395
       y2          0.11276        0.32103
  2    y1          0.04868       -0.00223
       y2          0.00179       -0.00529
  3    y1         -0.03094       -0.00512
       y2         -0.05409        0.03963

        Schematic Representation
         of Partial Autoregression
       Variable/
       Lag           1     2     3

       y1            ++    ..    ..
       y2            ++    ..    ..

       + is > 2*std error,  - is <
       -2*std error,  . is between
```

Process 2 (VMA(2))

```
         Partial Autoregression

Lag    Variable        y1             y2

  1    y1         -0.14529       -0.02217
       y2          0.04958       -0.19781
  2    y1         -0.35634        0.06602
       y2         -0.53476        0.02730
  3    y1         -0.16908        0.05576
       y2         -0.28812        0.08834

        Schematic Representation
         of Partial Autoregression
       Variable/
       Lag           1     2     3

       y1            -.    -.    -.
       y2            .-    -.    -.

       + is > 2*std error,  - is <
       -2*std error,  . is between
```

Process 3 (VARMA(2,1))

```
         Partial Autoregression

Lag    Variable        y1             y2

  1    y1         -0.62946        0.02927
       y2          0.54820       -0.61627
  2    y1          0.88979        0.36075
       y2         -0.18681        0.22123
  3    y1          0.17631        0.12897
       y2         -0.03528        0.16250

        Schematic Representation
         of Partial Autoregression
       Variable/
       Lag           1     2     3

       y1            -.    ++    ++
       y2            +-    -+    .+

       + is > 2*std error,  - is <
       -2*std error,  . is between
```

The partial autoregression matrices for process 1 do not differ significantly from zero from lag 2 onwards, implying that this process is generated by a VAR(1) model. It is clear from the schematic representation that processes 2 and 3 are not pure VAR models since the elements of the partial autoregression matrices differ significantly from zero.

5.4 THE MINIMUM INFORMATION CRITERION METHOD

Up to now methods for determining the order of a VMA process, as well as a VAR process, were considered. In this section a method for establishing the tentative order of a VARMA is discussed.

One of the objectives of time series analysis is to determine a suitable model in order to predict future values. The minimum information criterion method utilises the forecasting accuracy to determine the order of a VARMA(p,q) model. In particular the one step forecast MSE is minimised, which is a function of the white noise covariance matrix, $\boldsymbol{\Sigma}_a$.

In order to choose an appropriate model, the value of an information criterion for several values of p and q will be determined. The pair (p,q) for which the information criterion attains a minimum will be the order of the VARMA(p,q) model. Any one of the information criteria listed in Table 5.1 can be used for this method. The determinant of the estimated white noise covariance matrix plays a key role in all the criteria. The criteria also depend on the sample size, the number of parameters estimated and the dimension of the time series.

Table 5.1. Information criteria (Source: SAS/ETS 9.1 User's Guide)

Criterion	**Abbreviation**	**Formula**
Akaike Information Criterion	AIC	$\ln\lvert\tilde{\boldsymbol{\Sigma}}_a\rvert + \frac{2r}{T}$
Corrected Akaike Information Criterion	AAIC	$\ln\lvert\tilde{\boldsymbol{\Sigma}}_a\rvert + \frac{2r}{T - \frac{r}{k}}$

Table 5.1. Information criteria (Source: SAS/ETS 9.1 User's Guide)

Criterion	Abbreviation	Formula
Final Prediction Error	FPE	$\left(\dfrac{T+\frac{r}{k}}{T-\frac{r}{k}}\right)^{k}\left\lvert\tilde{\boldsymbol{\Sigma}}_a\right\rvert$
Hannan-Quinn Criterion	HQC / HQ	$\ln\left\lvert\tilde{\boldsymbol{\Sigma}}_a\right\rvert+\dfrac{2r\ln(\ln(T))}{T}$
Schwarz Bayesian Criterion	SBC / SC	$\ln\left\lvert\tilde{\boldsymbol{\Sigma}}_a\right\rvert+\dfrac{r\ln(T)}{T}$

where

$\tilde{\boldsymbol{\Sigma}}_a$: maximum likelihood estimate of $\boldsymbol{\Sigma}_a$

r : number of parameters estimated

T : sample size

k : dimension of the time series

Instead of fitting several models and comparing the information criteria, one can also make use of the MINIC (minimum information criterion) method, which tentatively identifies the order of a VARMA(p,q) process. (Spliid, 1983; Reinsel, 1997) This method estimates the innovation series by fitting a high order VAR process to the original time series. Using the original observations and these residuals, it fits several models with different values of p and q. It finally selects the model with the minimum value for a selected information criterion. Any one of the information criteria mentioned in Table 5.1 may be used, the default is AICC. This method is often used when a value for p and/or q is not known.

In the following example the use of the information criteria, to select a model, is demonstrated using the generated VARMA(2,1) process. All the values were calculated from first principles and subsequently compared with the values provided by the VARMAX procedure. The MINIC method is also illustrated.

Example 5.3[*]

Consider the process generated by the VARMA(2,1) model. If the underlying data generating process is unknown, one can fit several models with different values for p and q and then select the model with the minimum value for the information criterion. A VAR(3) model, a VARMA(1,1) model and a VARMA(2,1) model were fitted to the generated data. The information criteria, of the fitted models, according to the formulae in Table 5.1 as well as the corresponding VARMAX output are tabulated below.

Information criteria	**VAR(3)**	**VARMA(1,1)**	**VARMA(2,1)**
AICC	-0.317	0.763	-0.332
HQC	-0.278	0.789	-0.293
AIC	-0.318	0.763	-0.333
SBC	-0.217	0.830	-0.232
FPE	0.728	2.144	0.717
	with $r = 12$, $T = 500$ and $k = 2$ $\tilde{\boldsymbol{\Sigma}}_a = \begin{pmatrix} 0.998 & 0.534 \\ 0.534 & 0.980 \end{pmatrix}$	with $r = 8$, $T = 500$ and $k = 2$ $\tilde{\boldsymbol{\Sigma}}_a = \begin{pmatrix} 2.486 & 0.987 \\ 0.987 & 1.227 \end{pmatrix}$	with $r = 12$, $T = 500$ and $k = 2$ $\tilde{\boldsymbol{\Sigma}}_a = \begin{pmatrix} 0.990 & 0.523 \\ 0.523 & 0.966 \end{pmatrix}$
VARMAX output	Information Criteria AICC -0.31734 HQC -0.27805 AIC -0.31793 SBC -0.21632 FPEC 0.727652	Information Criteria AICC 0.763021 HQC 0.789265 AIC 0.762762 SBC 0.830298 FPEC 2.144191	Information Criteria AICC -0.3322 HQC -0.29297 AIC -0.33279 SBC -0.23133 FPEC 0.716922

Irrespective of which information criterion is used, the minimum is attained when a VARMA(2,1) model is fitted.

[*] The SAS program is provided in Appendix B page 132.

Instead of fitting several models with different values of p and q, the MINIC method can be used. The SAS output regarding the MINIC method as well as the information criteria for the fitted VAR(3) model is given below.

Minimum Information Criterion

Lag	MA 0	MA 1	MA 2	MA 3	MA 4
AR 0	3.8514402	3.8350536	3.5061428	3.4004058	3.0793454
AR 1	1.4133135	1.3859316	1.0870527	1.052723	0.8185272
AR 2	-0.277616	$\boxed{-0.322909}$	-0.310653	-0.300108	-0.29529
AR 3	$\boxed{-0.324909}$	-0.312945	-0.301504	-0.291147	-0.279252
AR 4	-0.318086	-0.301794	-0.293259	-0.278826	-0.263467

Covariances of Innovations

Variable	y1	y2
y1	1.01016	0.54044
y2	0.54044	0.99239

Information Criteria

AICC	-0.31734
HQC	-0.27805
AIC	-0.31793
SBC	-0.21632
FPEC	0.727652

According to the MINIC method, the smallest value of the criterion (-0.325) implies that a VAR(3) model was selected. Take note that this minimum value is very close to the criterion value for the VARMA(2,1) model (-0.323).

Since a VAR(3) model was estimated using the method of least squares, the matrix of autocovariances for the innovations must be mulitplied with $\frac{T-kp}{T}$ to obtain the estimate $\tilde{\Sigma}_a$ used in the formulae in Table 5.1. This is due to the fact that SAS adjusts the estimate of the white noise covariance matrix to be unbiased. Since there is not an intercept included in the model, this adjustment differs slightly from (3.21).

According to the information criteria for the two models, the VARMA(2,1) model performs better than the VAR(3) model. One must keep in mind that the MINIC method only tentatively selects the order, Chapter 6 will still look at the model diagnostics in order to determine whether a selected model is an adequate representation of the underlying data generating process.

5.5 CONCLUSION

This chapter was concerned with tentatively determining the order of a VARMA model. It is relatively easy to determine the order of a pure VMA and a pure VAR model simply by examining the sample autocorrelation matrices and the sample partial autoregression matrices, respectively. However, the moment there is a combination of these models (VARMA models), the above mentioned methods do not contribute in finding the order. For the more complex models, the MINIC method was introduced. The MINIC method proved to be successful, also for mixed (VARMA) models.

CHAPTER 6

MODEL DIAGNOSTICS

6.1 INTRODUCTION

In this chapter the goodness of fit of a selected model is assessed. The significance of the estimated parameters (as determined in Sections 3.2.3 and 3.3.3) is a good starting point since it is not desirable to have extra parameters that do not contribute to the model. On the other side it may also be misleading, because the parameter estimates of a poor model may also be significant. Thus, we can not solely rely on the significance of the parameters to assess the model. As in most modeling situations the fit is assessed through the behaviour of the residuals. If a model is an adequate representation of the process that generated the time series, the residuals should have no significant trend or pattern. One way to establish this is to look at the individual elements of the autocorrelation matrices of the residual vectors, this is done in Section 6.2.1. In Section 6.2.2 the Portmanteau test statistic will be discussed, which determines the overall significance of the residual autocorrelations.

Testing the adequacy of a fitted model based on the multivariate residual autocorrelation matrices became popular since Chitturi (1974) derived the asymptotic distribution of residual autocorrelations and proposed a Chi-squared statistic to test the fit of pure autoregressive models. This was generalised to VARMA models by Hosking (1980) and Li & McLeod (1981) who proposed the multivariate Portmanteau test statistic.

The estimated multivariate time series model can also be decomposed into univariate time series models. These univariate models can be evaluated separately by means of a R^2 value, the Durbin-Watson test for serial correlation and the Jarque-Bera test for normality of the residuals, to name only a few. These tests will be discussed in more detail in Section 6.3.

The multivariate and univariate diagnostic checks described in this chapter will be used in an example in Section 6.4.1 to distinguish between a good and a poor model. The rest of Section 6.4 is devoted to examples of the whole model building process, based on two multivariate time series datasets, namely temperatures and electricity demand.

6.2 MULTIVARIATE DIAGNOSTIC CHECKS

In this section the residual autocorrelation matrices of the fitted model are analysed, using two methods. The first method tests the individual elements of the residual autocorrelation matrix at different lags for significance, while the second method considers the autocorrelation matrices up to a certain lag as a whole and tests that for significance.

6.2.1 Residual autocorrelation matrices

This section starts off by determining the distribution of the autocorrelation matrices of a white noise process. The reason being that the residuals of a fitted model should behave the same as a white noise process if the model fits well.

Let $\{\boldsymbol{a}_t\}$ be a k-dimensional white noise process with covariance matrix $\boldsymbol{\Sigma}_a$ and corresponding correlation matrix $\boldsymbol{R}_a$. The sample autocovariance matrix and the sample autocorrelation matrix of $\{\boldsymbol{a}_t\}$ at lag i are given by

$$\boldsymbol{C}_i = \frac{1}{T}\sum_{t=i+1}^{T}\boldsymbol{a}_t\boldsymbol{a}'_{t-i} \qquad i = 0,1,\ldots,h < T \tag{6.1}$$

$$\boldsymbol{R}_i = \boldsymbol{V}_a^{-\frac{1}{2}}\boldsymbol{C}_i\boldsymbol{V}_a^{-\frac{1}{2}} \qquad i = 0,1,\ldots,h < T \tag{6.2}$$

where T is the length of the time series and $\boldsymbol{V}_a^{\frac{1}{2}}$ is a $k \times k$ diagonal matrix with the square root of the diagonal elements of $\boldsymbol{C}_0$ on the main diagonal. Let $\boldsymbol{R}_h^* = (\boldsymbol{R}_1 \quad \ldots \quad \boldsymbol{R}_h)$.

Proposition 4.4 of Lütkepohl (2005) states:
"Let $\{\boldsymbol{a}_t\}$ be a k-dimensional identically distributed standard white noise process, that is, $\boldsymbol{a}_t$ and $\boldsymbol{a}_s$ have the same multivariate distribution with nonsingular covariance matrix $\boldsymbol{\Sigma}_a$ and corresponding correlation matrix $\boldsymbol{R}_a$. Then, for $h \geq 1$,

$$\sqrt{T}vec(\boldsymbol{R}_h^*) \xrightarrow{d} N(\boldsymbol{0}, \boldsymbol{I}_h \otimes \boldsymbol{R}_a \otimes \boldsymbol{R}_a)\text{"} \tag{6.3}$$

From (6.3) it follows that $\sqrt{T}vec(\boldsymbol{R}_i)$ and $\sqrt{T}vec(\boldsymbol{R}_j) \xrightarrow{d} N(\boldsymbol{0}, \boldsymbol{R}_a \otimes \boldsymbol{R}_a)$ and that if $i \neq j$ they are asymptotically independent. (Lütkepohl, 2005)

The elements on the main diagonal of the correlation matrix, $\boldsymbol{R}_a$, are equal to one. This is then also true for the elements on the main diagonal of $\boldsymbol{R}_a \otimes \boldsymbol{R}_a$. Consequently, the asymptotic distributions of the elements of $\sqrt{T}vec\left(\boldsymbol{R}_h^*\right)$ are approximate standard normal distributions. This follows from the property of the multivariate normal distribution that all subsets also have a (multivariate) normal distribution. (Johnson & Wichern, 2002) Consider as an example a bivariate white noise process with the sample autocorrelation matrix at lag *i*, $\boldsymbol{R}_i = \begin{pmatrix} r_{11,i} & r_{12,i} \\ r_{21,i} & r_{22,i} \end{pmatrix}$ and $\boldsymbol{R}_a = \begin{pmatrix} 1 & * \\ * & 1 \end{pmatrix}$, then

$$\sqrt{T}vec\left(\boldsymbol{R}_i\right) = \sqrt{T}\begin{pmatrix} r_{11,i} \\ r_{21,i} \\ r_{12,i} \\ r_{22,i} \end{pmatrix} \xrightarrow{d} N\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & * & * & * \\ * & 1 & * & * \\ * & * & 1 & * \\ * & * & * & 1 \end{pmatrix}\right) \text{ where * is an arbitrary number,}$$

therefore $\sqrt{T}r_{mn,i} \xrightarrow{d} N(0,1)$.

This property can be used to test whether the elements of the sample autocorrelation matrices at different lags of a white noise process differ significantly from zero. Let $\rho_{mn,i}$ be the true correlation in row *m*, column *n* at lag *i*. The hypothesis tested is:

$$H_0 : \rho_{mn,i} = 0 \text{ against } H_a : \rho_{mn,i} \neq 0 \tag{6.4}$$

The null hypothesis will be rejected on an approximate 5% level of significance if

$$\left|\sqrt{T}r_{mn,i}\right| > 2 \text{ or } \left|r_{mn,i}\right| > \frac{2}{\sqrt{T}} \tag{6.5}$$

This hypothesis test can be used as a guideline to determine whether the residuals of a fitted model are correlated. If the null hypothesis cannot be rejected, it can be concluded that the residuals behave like a white noise process and therefore the model fitted is adequate. This test is performed on the non-duplicated elements of the autocorrelation matrices individually.

6.2.2 The Portmanteau statistic

The Box & Pierce (1970) goodness-of-fit test, the Portmanteau test, was extended to multivariate VARMA models by Hosking (1980) and Li & McLeod (1981). This test determines whether the residual autocorrelations, up to a specific lag, are zero.

Let $\hat{\boldsymbol{C}}_i = \frac{1}{T}\sum_{t=i+1}^{T}\hat{\boldsymbol{a}}_t\hat{\boldsymbol{a}}'_{t-i}$ be the i-th residual autocovariance matrix, where $\hat{\boldsymbol{a}}_t$ contains the residuals of the estimated model at time t, and let $\hat{\boldsymbol{R}}_i$ be the corresponding residual autocorrelation matrix. The hypothesis tested is:

$$H_0 : \boldsymbol{R}_h^* = (\boldsymbol{R}_1 \quad \dots \quad \boldsymbol{R}_h) = \boldsymbol{0} \text{ against } H_a : \boldsymbol{R}_h^* = (\boldsymbol{R}_1 \quad \dots \quad \boldsymbol{R}_h) \neq \boldsymbol{0} \tag{6.6}$$

An inability to reject the null hypothesis will indicate that the residuals behave like a white noise process, and hence adequacy of the fitted model.

The multivariate Portmanteau test proposed by Hosking (1980) is

$$P = T\sum_{i=1}^{h} tr\left(\hat{\boldsymbol{C}}'_i\hat{\boldsymbol{C}}_0^{-1}\hat{\boldsymbol{C}}_i\hat{\boldsymbol{C}}_0^{-1}\right) \tag{6.7}$$

and it has an approximate Chi-squared distribution with $k^2(h-p-q)$ degrees of freedom under the null hypothesis, where p and q are the orders of the estimated VARMA(p,q) model and h is the number of lags included in the test for overall significance. Ljung & Box (1978) proposed a modification that leads to better small sample properties in the univariate case. Hosking considered a similar modification for the multivariate case. The modified Portmanteau test statistic is given by

$$P' = T^2\sum_{i=1}^{h}(T-i)^{-1} tr\left(\hat{\boldsymbol{C}}'_i\hat{\boldsymbol{C}}_0^{-1}\hat{\boldsymbol{C}}_i\hat{\boldsymbol{C}}_0^{-1}\right) \tag{6.8}$$

Hosking (1980) used a simulation study to illustrate the effectiveness of this modification for a sample of size 200. We expanded this simulation by also including other sample sizes. Samples of size 1000, 200, 100 and 30 from a bivariate normal VAR(1) process, $\boldsymbol{y}_t = \boldsymbol{\Phi}\boldsymbol{y}_{t-1} + \boldsymbol{a}_t$, with $\boldsymbol{\Phi} = \begin{pmatrix} 0.9 & 0.1 \\ -0.6 & 0.4 \end{pmatrix}$ and $\boldsymbol{\Sigma}_a = \begin{pmatrix} 1 & 0.4 \\ 0.4 & 1 \end{pmatrix}$ were generated. The residuals of the estimated VAR(1) model were used to calculate P and P' with $h = 20$. The results from 1000 simulations as well as the approximate theoretical values are summarised in Table 6.1. The SAS IML program used for the simulation is provided in Appendix B.

Table 6.1 Simulation study for P and P'

		Mean	Variance	Significance level (%)			
	χ^2_{76}	76	152	20.0	10.0	5.0	1.0
$T=1000$	P	75.14	153.11	17.0	9.7	4.2	1.0
	P'	75.97	156.44	19.0	10.9	5.9	1.3
$T=200$	P	70.54	138.62	10.2	4.3	2	0.4
	P'	74.58	154.62	16.8	8.5	4.3	1.0
$T=100$	P	65.19	115.22	3.9	1.1	0.5	0.2
	P'	73.14	145.07	13.4	6.5	3.3	0.6
$T=30$	P	46.46	57.10	0.1	0.1	0	0
	P'	73.43	121.71	12.1	4.6	2.5	0.7

For a large sample ($T=1000$) the distributions of P and P' are similar, and very close to the asymptotic distribution. As the sample size decreases, the distribution of P' is closer to the asymptotic χ^2_{76} distribution. P performs poorly for samples of size 100 and smaller. These conclusions should only be considered as guidelines, since it is based on a simulation study. In practice P' is generally used for both small and large samples, since for large samples P and P' are very similar. For example, SAS includes only P' by default when a model is estimated.

6.3 UNIVARIATE DIAGNOSTIC CHECKS

The fitted k-dimensional VARMA(p,q) model can also be written as k univariate regression equations. In Section 6.3.1 we will assess the fit of the individual models by interpreting the R^2-value and also discuss the overall F - test for the significance of the models. This section will focus on the residual analysis of the individual univariate models. The residuals of one of these equations will be denoted by $\hat{\varepsilon}_t$ where $t=1,\ldots,T$. The residuals of an adequate model should be independent normally distributed random variables with zero mean. Test procedures to establish these properties will be formulated in Sections 6.3.2 and 6.3.3, respectively. Section 6.3.4 deals with a test for heteroscedasticity of the residuals. A test for higher order autocorrelation in the residuals is the subject of Section 6.3.5.

6.3.1 The multiple coefficient of determination and the F - test for overall significance

In a regression context the multiple regression model is given by

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 \cdots + \beta_p x_p + \varepsilon \tag{6.9}$$

where

y: dependent variable

$\beta' s$: parameters

$x' s$: explanatory variables

ε: error term

For our purpose, the explanatory variables may include lagged values of the dependent variable.

The multiple coefficient of determination, R^2, is a measure of the portion of the variability in the dependent variable (a single time series) that can be explained by the estimated regression equation (lagged observations of the single time series, together with observations from the (k-1) other time series processes). The calculation formula for R^2 is

$$R^2 = \frac{SSR}{SST} \tag{6.10}$$

where

SSR: sum of squared differences of the estimated value and the mean

SST: sum of squared differences of the observed value and the mean

The F - test is used to establish whether a significant relationship exists between the dependent and explanatory variables. The hypothesis,

$$\begin{aligned} &H_0 : \beta_1 = \beta_2 = \ldots = \beta_p = 0 \\ &H_a : \text{Not all the parameters are equal to zero} \end{aligned} \tag{6.11}$$

can be tested using an F - statistic,

$$F = \frac{\frac{SSR}{p}}{\frac{SSE}{T-p-1}} \tag{6.12}$$

where

SSE : sum of squared differences of the observed and estimated values

p : number of explanatory variables

T : sample size

Under the null hypothesis, the F - statistic follows an F distribution with p and $T-p-1$ degrees of freedom. The null hypothesis will be rejected when the F - statistic exceeds an appropriate critical value. (Williams, Sweeney, Anderson, 2006)

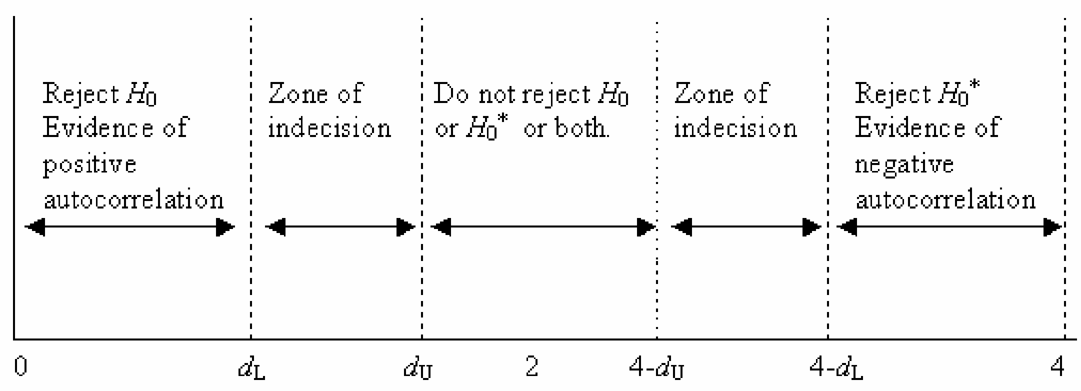
6.3.2 Durbin-Watson test

The Durbin-Watson *d* statistic for detecting serial correlation of the error term originates from regression analysis. Some of the assumptions underlying this statistic, summarised by Gujarati (1995), include that the regression model has an intercept term and that the regression model should not include lagged values of the dependent variable as explanatory variables. The nature of time series analysis violates the last mentioned assumption. Durbin (1970) proposed the *h* statistic for testing serial correlation in regression when some of the regressors are lagged dependent variables. Nonetheless statistical packages still calculate the Durbin-Watson *d* statistic.

The *d* statistic is derived in a paper by Durbin & Watson (1950), while the critical values of this statistic are tabulated in a paper by the same authors (1951). Using the notation specified in section 6.3, the *d* statistic is

$$d = \frac{\sum_{t=2}^{T} \left(\hat{\varepsilon}_t - \hat{\varepsilon}_{t-1}\right)^2}{\sum_{t=2}^{T} \hat{\varepsilon}_t^2} \tag{6.13}$$

The Durbin-Watson *d* statistic tests the null hypothesis of independence of the error terms against an alternative that the error terms are generated by an AR(1) process. This is an indication that some of the variation is not captured by the model, but included in the error term. The decision rule for this test is graphically represented in Figure 12.9 of Gujarati (1995) and is as follows:



where

H_0: No positive autocorrelation

H_0^*: No negative autocorrelation

As a rule of thumb, a d statistic equal to 2 is an indication of no first order autocorrelation.

6.3.3 Jarque-Bera normality test

Jarque & Bera (1987) established a test statistic to test for the normality of observations. This statistic is based on the skewness and kurtosis of the residuals, which are calculated using the sample moments. The sample skewness and kurtosis coefficients can be calculated by

$$S = \frac{\hat{\mu}_3}{\hat{\mu}_2^{\frac{3}{2}}} \qquad \text{(skewness)} \qquad (6.14)$$

$$K = \frac{\hat{\mu}_4}{\hat{\mu}_2^2} \qquad \text{(kurtosis)} \qquad (6.15)$$

where $\hat{\mu}_j$ is the j-th order central sample moment, $\hat{\mu}_j = \frac{1}{T}\sum\left(\hat{\varepsilon}_t - \bar{\varepsilon}\right)^j$ with $\bar{\varepsilon} = \frac{1}{T}\sum\hat{\varepsilon}_t$.

When there is an intercept in the model the Jarque-Bera test statistic for the null hypothesis, that the observations (residuals) are normally distributed, is

$$JB = T\left[\frac{S^2}{6} + \frac{(K-3)^2}{24}\right] \tag{6.16}$$

The Jaque-Bera test has a Chi-squared distribution with 2 degrees of freedom asymptotically, and the null hypothesis is rejected if the computed value exceeds a Chi-squared critical value.

6.3.4 Autoregressive conditional heteroscedasticity (ARCH) model

Consider the univariate AR(p) model

$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \ldots + \phi_p y_{t-p} + a_t \tag{6.17}$$

where a_t is a white noise process with zero mean and $E(a_t a_\tau) = \begin{cases} \sigma^2 & \text{if} \quad t = \tau \\ 0 & \text{if} \quad t = \tau \end{cases}$

Engle (1982) proposed a class of models with nonconstant variances conditional on the past, called ARCH models. The idea behind the ARCH model is that the conditional variance of a_t changes over time. For example, a_t^2 may also follow an AR(m) process,

$$a_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \alpha_2 a_{t-2}^2 + \ldots + \alpha_m a_{t-m}^2 + w_t \tag{6.18}$$

where w_t is a white noise process. The conditional variance of a_t is then given by

$$E\left(a_t^2 \middle| a_{t-1}^2, \ldots, a_{t-m}^2\right) = \alpha_0 + \alpha_1 a_{t-1}^2 + \alpha_2 a_{t-2}^2 + \ldots + \alpha_m a_{t-m}^2 \tag{6.19}$$

If this is the case then a_t can be described by an ARCH(m) model. Based on this, the null hypothesis to test for ARCH disturbances is

$$H_0 : \alpha_1 = \alpha_2 = \ldots = \alpha_m = 0 \tag{6.20}$$

In practice we are usually interested in ARCH(1) disturbances. The hypothesis in (6.20) can be tested by means of the F - test of overall significance of the regression

$$\hat{\varepsilon}_t^2 = \hat{\alpha}_0 + \hat{\alpha}_1 \hat{\varepsilon}_{t-1}^2 \tag{6.21}$$

where $\hat{\varepsilon}_t$ denotes the residuals of the estimated model. (Hamilton, 1994; Gujarati, 1995) Statistical packages usually report this F - statistic.

An alternative test procedure derived by Engle (1982) is to compare TR^2 (R^2 is the coefficient of determination for the regression in (6.21)) to a Chi-squared critical value with one degree of freedom.

6.3.5 F - test for AR disturbances

The Durbin-Watson d statistic tests for independence of the error terms against an alternative that they are generated by an AR(1) process. Another approach to test for autocorrelation in the residuals is to fit an AR(1) model to the residuals,

$$\varepsilon_t = \hat{c} + \hat{\phi}_1^{res} \varepsilon_{t-1} \tag{6.22}$$

and test the hypothesis

$$H_0 : \hat{\phi}_1^{res} = 0 \text{ against the alternative } H_a : \hat{\phi}_1^{res} \neq 0 \tag{6.23}$$

This is called the F - test for AR(1) disturbances.

The significance of higher order models can also be tested, for example the F - test for AR(4) disturbances. This is done by fitting an AR(4) model to the residuals,

$$\varepsilon_t = \hat{c} + \hat{\phi}_1^{res} \varepsilon_{t-1} + \hat{\phi}_2^{res} \varepsilon_{t-2} + \hat{\phi}_3^{res} \varepsilon_{t-3} + \hat{\phi}_4^{res} \varepsilon_{t-4} \tag{6.24}$$

and testing for overall significance of the model by means of the F - test for the hypothesis

$$\begin{aligned} &H_0 : \hat{\phi}_1^{res} = \hat{\phi}_2^{res} = \hat{\phi}_3^{res} = \hat{\phi}_4^{res} = 0 \\ &H_a : \text{not all the coefficients are equal to zero} \end{aligned} \tag{6.25}$$

(Williams, Sweeney, Anderson, 2006)

6.4 EXAMPLES

This section consists out of three examples. The first example is based on a generated VAR(2) process. The purpose of this example is twofold, firstly the diagnostic tests described in this chapter are calculated using the formulae provided to show that it is comparable to the results obtained using the VARMAX procedure; and secondly to establish whether the diagnostic checks can be used to distinguish between a poor and a good fitted model. The other two examples illustrate the model building process, including a test for stationarity, order selection, estimation and diagnostic checks, using observed multivariate time series datasets.

6.4.1 Simulated data*

In this example VAR(1) and VAR(2) models are fitted to a computer generated VAR(2) process. Diagnostic checks are compared for the two cases. Take note that all the test statistics for the residual diagnostics for the fitted VAR(2) model were also calculated by programming the formulae (as given in this chapter) in SAS IML. The program is given in Appendix B and the results are summarised in Table 6.2.

To illustrate the use of the diagnostic checks, 500 observations were generated from a stationary bivariate VAR(2) model,

$$\begin{pmatrix} y_t \\ x_t \end{pmatrix} = \begin{pmatrix} 5.5 \\ 42 \end{pmatrix} + \begin{pmatrix} 0.6 & -0.8 \\ 0.2 & 0.3 \end{pmatrix}\begin{pmatrix} y_{t-1} \\ x_{t-1} \end{pmatrix} + \begin{pmatrix} 0.3 & 0.7 \\ -0.6 & -0.5 \end{pmatrix}\begin{pmatrix} y_{t-2} \\ x_{t-2} \end{pmatrix} + \boldsymbol{a}_t \text{ with } \boldsymbol{\Sigma}_a = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 0.9 \end{pmatrix}$$

The method of least squares was used to fit a VAR(1) and a VAR(2) model to the generated data. Selected SAS output of the model estimation and diagnostics is provided below.

VAR(1) model

Model Parameter Estimates

Equation	Parameter	Estimate	Standard Error	t Value	Pr > |t|	Variable
y	CONST1	33.83842	1.57822	21.44	0.0001	1
	AR1_1_1	0.51157	0.02748	18.62	0.0001	y(t-1)
	AR1_1_2	-0.76455	0.04340	-17.62	0.0001	x(t-1)
x	CONST2	20.98801	1.71523	12.24	0.0001	1
	AR1_2_1	-0.00837	0.02986	-0.28	0.7794	y(t-1)
	AR1_2_2	0.16704	0.04716	3.54	0.0004	x(t-1)

Cross Correlations of Residuals

Lag	Variable	y	x
0	y	1.00000	-0.40091
	x	-0.40091	1.00000
1	y	0.09378	0.12757
	x	-0.07765	0.06109
2	y	0.07038	-0.13764
	x	0.54508	-0.48413
3	y	0.36908	-0.43414
	x	-0.15818	0.36027

* The SAS program is provided in Appendix B page 136.

VAR(1) model

Schematic Representation of Cross Correlations of Residuals

Variable/Lag	0	1	2	3
y	+-	++	.-	+-
x	-+	..	+-	-+

+ is > 2*std error, - is < -2*std error, . is between

Portmanteau Test for Cross Correlations of Residuals

Up To Lag	DF	Chi-Square	Pr > ChiSq
2	4	302.12	<.0001
3	8	443.19	<.0001

Univariate Model ANOVA Diagnostics

Variable	R-Square	Standard Deviation	F Value	Pr > F
y	0.6699	1.58160	503.26	<.0001
x	0.0296	1.71890	7.57	0.0006

Univariate Model White Noise Diagnostics

Variable	Durbin Watson	Normality Chi-Square	Normality Pr > ChiSq	ARCH F Value	ARCH Pr > F
y	1.80738	1.70	0.4283	0.53	0.4661
x	1.87744	6.17	0.0458	0.07	0.7907

Univariate Model AR Diagnostics

Variable	AR1 F Value	AR1 Pr > F	AR2 F Value	AR2 Pr > F	AR3 F Value	AR3 Pr > F	AR4 F Value	AR4 Pr > F
y	4.42	0.0360	3.13	0.0447	27.23	<.0001	21.07	<.0001
x	1.86	0.1734	79.45	<.0001	160.03	<.0001	120.45	<.0001

VAR(2) model

Model Parameter Estimates

Equation	Parameter	Estimate	Standard Error	t Value	Pr > |t|	Variable
y	CONST1	5.95057	1.77370	3.35	0.0009	1
	AR1_1_1	0.58546	0.03197	18.31	0.0001	y(t-1)
	AR1_1_2	-0.80929	0.02942	-27.51	0.0001	x(t-1)
	AR2_1_1	0.30620	0.02421	12.65	0.0001	y(t-2)
	AR2_1_2	0.70258	0.03643	19.28	0.0001	x(t-2)
x	CONST2	42.00651	1.64503	25.54	0.0001	1
	AR1_2_1	0.18944	0.02965	6.39	0.0001	y(t-1)
	AR1_2_2	0.30175	0.02729	11.06	0.0001	x(t-1)
	AR2_2_1	-0.57041	0.02245	-25.41	0.0001	y(t-2)
	AR2_2_2	-0.52556	0.03379	-15.55	0.0001	x(t-2)

VAR(2) model

Cross Correlations of Residuals

Lag	Variable	y	x
0	y	1.00000	0.52148
	x	0.52148	1.00000
1	y	0.00368	0.00779
	x	-0.00691	-0.00681
2	y	0.02469	-0.00763
	x	-0.01631	-0.01248
3	y	0.00126	0.00668
	x	-0.00666	0.00230

Schematic Representation of Cross Correlations of Residuals

Variable/ Lag	0	1	2	3
y	++	..	..	..
x	++	..	..	..

+ is > 2*std error, - is < -2*std error, . is between

Portmanteau Test for Cross Correlations of Residuals

Up To Lag	DF	Chi-Square	Pr > ChiSq
3	4	1.41	0.8423

Univariate Model ANOVA Diagnostics

Variable	R-Square	Standard Deviation	F Value	Pr > F
y	0.8596	1.03151	754.32	<.0001
x	0.7012	0.95668	289.27	<.0001

Univariate Model White Noise Diagnostics

Variable	Durbin Watson	Normality Chi-Square	Normality Pr > ChiSq	ARCH F Value	ARCH Pr > F
y	1.99066	0.84	0.6580	0.02	0.8753
x	2.01359	1.21	0.5459	0.20	0.6512

Univariate Model AR Diagnostics

Variable	AR1 F Value	AR1 Pr > F	AR2 F Value	AR2 Pr > F	AR3 F Value	AR3 Pr > F	AR4 F Value	AR4 Pr > F
y	0.01	0.9347	0.15	0.8574	0.10	0.9577	0.10	0.9814
x	0.02	0.8795	0.05	0.9515	0.03	0.9916	0.08	0.9875

Table 6.2 Summary of the diagnostic checks of the fitted VAR(2) model using explicit formulae

Diagnostic Check	Formula	Result
Residual autocorrelation matrices	(6.2)	$\hat{\boldsymbol{R}}_0 = \begin{pmatrix} 1 & 0.521 \\ 0.521 & 1 \end{pmatrix}$ $\hat{\boldsymbol{R}}_1 = \begin{pmatrix} 0.004 & -0.007 \\ 0.008 & -0.007 \end{pmatrix}$ $\hat{\boldsymbol{R}}_2 = \begin{pmatrix} 0.025 & -0.016 \\ -0.008 & -0.012 \end{pmatrix}$ $\hat{\boldsymbol{R}}_3 = \begin{pmatrix} 0.001 & -0.007 \\ 0.007 & 0.002 \end{pmatrix}$
Portmanteau statistic	(6.7) (6.8)	$P = 1.405$ $P' = 1.411$ and $p-value = 0.842$
Multiple coefficient of determination	(6.10)	$y:\ R^2 = 0.8596$ $x:\ R^2 = 0.7012$
F **- statistic**	(6.12)	$y:\ F = 754.354$ and $p-value = 0$ $x:\ F = 289.266$ and $p-value = 0$
Durbin-Watson test	(6.13)	$y:\ d = 1.991$ $x:\ d = 2.014$
Jarque-Bera normality test	(6.16)	$y:\ JB = 0.837$ and $p-value = 0.658$ $x:\ JB = 1.211$ and $p-value = 0.546$
ARCH model	(6.21)	$y:\ F = 0.02$ and $p-value = 0.875$ $x:\ F = 0.20$ and $p-value = 0.651$
AR disturbances: AR(1)	(6.22)	$y:\ F = 0.01$ and $p-value = 0.935$ $x:\ F = 0.02$ and $p-value = 0.880$
AR(2)		$y:\ F = 0.15$ and $p-value = 0.857$ $x:\ F = 0.05$ and $p-value = 0.952$
AR(3)		$y:\ F = 0.10$ and $p-value = 0.958$ $x:\ F = 0.03$ and $p-value = 0.992$
AR(4)	(6.24)	$y:\ F = 0.10$ and $p-value = 0.981$ $x:\ F = 0.08$ and $p-value = 0.988$

The estimated models are

$$\text{VAR(1)}: \quad \begin{pmatrix} \hat{y}_t \\ \hat{x}_t \end{pmatrix} = \begin{pmatrix} 33.838 \\ 20.988 \end{pmatrix} + \begin{pmatrix} 0.512 & -0.765 \\ -0.008 & 0.167 \end{pmatrix} \begin{pmatrix} y_{t-1} \\ x_{t-1} \end{pmatrix} + \boldsymbol{a}_t$$

$$\text{VAR(2):} \quad \begin{pmatrix} \hat{y}_t \\ \hat{x}_t \end{pmatrix} = \begin{pmatrix} 5.951 \\ 42.007 \end{pmatrix} + \begin{pmatrix} 0.585 & -0.809 \\ 0.189 & 0.302 \end{pmatrix} \begin{pmatrix} y_{t-1} \\ x_{t-1} \end{pmatrix} + \begin{pmatrix} 0.306 & 0.703 \\ -0.570 & -0.526 \end{pmatrix} \begin{pmatrix} y_{t-2} \\ x_{t-2} \end{pmatrix} + \boldsymbol{a}_t$$

In what follows, the goodness of fit of these two models will be evaluated with regards to the diagnostic checks discussed in this chapter.

The parameter estimates for both models are significant, except $\hat{\phi}_{21,1}$ (p-value = 0.7794) for the VAR(1) model.

The residual autocorrelation matrices from lag 1 onwards must be close to zero for the model to be adequate. The hypothesis test in (6.4) considers the individual elements of the residual autocorrelation matrices at different lags and test whether they differ significantly from zero. The null hypothesis will be rejected if the absolute value of any of the individual elements of the residual autocorrelation matrices exceed $\left(\frac{2}{\sqrt{500}}\right) = 0.0894$. SAS summarises this with a schematical representation where a "+" and "-" indicates significance, while a "." means the null hypothesis cannot be rejected. Based on the residual autocorrelation matrices, only the VAR(2) model is adequate.

Instead of considering the individual elements of the residual autocorrelation matrices, the Portmanteau statistic rather looks at the matrices as a whole up to a specific lag. The null hypothesis in (6.6) with $h = 3$ will be rejected for the VAR(1) model (p-value < 0.0001), while the residuals of the VAR(2) model behave like a white noise process (p-value = 0.8423).

These models can be assessed individually by writing them in terms of univariate equations,

$$\text{VAR(1):} \quad \begin{aligned} \hat{y}_t &= 33.838 + 0.512 y_{t-1} - 0.765 x_{t-1} + a_{1t} \\ \hat{x}_t &= 20.988 - 0.008 y_{t-1} + 0.167 x_{t-1} + a_{2t} \end{aligned}$$

$$\text{VAR(2):} \quad \begin{aligned} \hat{y}_t &= 5.951 + 0.585 y_{t-1} - 0.809 x_{t-1} + 0.306 y_{t-2} + 0.703 x_{t-2} + a_{1t} \\ \hat{x}_t &= 42.007 + 0.189 y_{t-1} + 0.302 x_{t-1} - 0.570 y_{t-2} - 0.526 x_{t-2} + a_{2t} \end{aligned}$$

For the VAR(1) model about 67% of the total variation in y at time t can be explained by y and x at time $t-1$, while only approximately 3% of the total variation of x at time t can be explained by these variables. The R^2 values increase drastically for the VAR(2) model, for example 70% of the total variation in x at time t can be explained by x and y at time $t-1$ and time $t-2$. According to the F - test in (6.12) all four equations explain a significant proportion of the total variability in y and x.

The residuals of the univariate equations of the VAR(1) model are not independent. This is evident since the Durbin Watson d - statistic is not close to two, as well as the AR(1) to AR(4) models fitted to the residuals are significant, with the exception of an AR(1) model for x (p-value = 0.1734). The residuals for x are not normally distributed (p-value = 0.0458). The F - test for ARCH(1) disturbances shows that the variance of the residuals do not change over time. The VAR(2) model fits the data better since the residuals of the univariate equations are uncorrelated, normally distributed and the variance does not change over time.

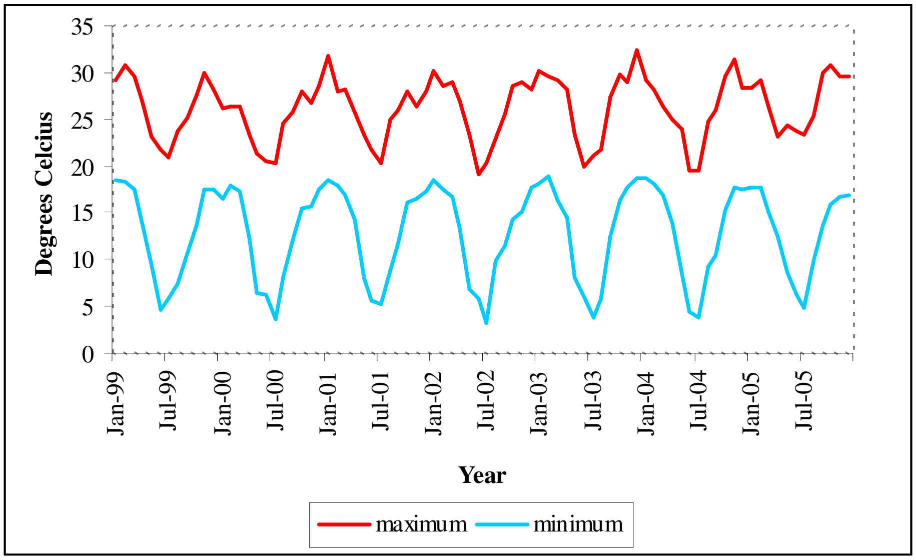
The conclusion is that the diagnostic tests were able to distinguish between a good and a bad fit and that only the VAR(2) model gives an adequate representation of the generated data.

6.4.2 Temperature data[1*]

Consider the average monthly maximum and minimum temperatures from January 1999 to December 2005. Figure 6.1 shows a clear pattern with higher temperatures during summer and lower temperatures during the winter months.

[1] Source: South African Weather Service
[*] The SAS program is provided in Appendix B page 140.

Figure 6.1 The average monthly maximum and minimum temperature from January 1999 to December 2005



This seasonal pattern can be isolated by means of the seasonal indices. The seasonal indices were calculated using the multiplicative model *LSCI* where *L* represents the long term movement, *S* the seasonal fluctuation, *C* the cyclical movement and *I* the irregular variation. Dividing *LSCI* by the 12 month moving averages yields the seasonal irregular values. Determining the averages of these seasonal irregular values and adjusting them, results in the seasonal indices summarised in Table 6.3.

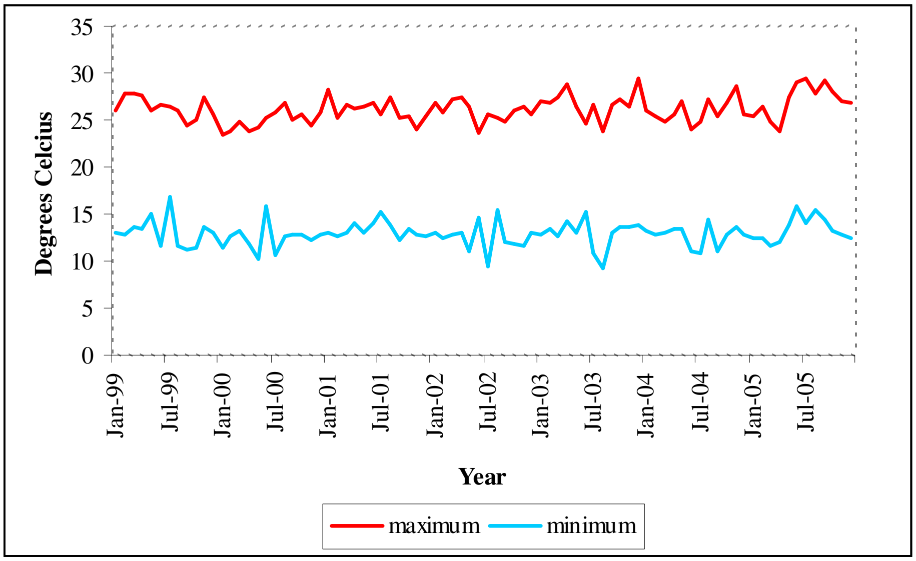
Table 6.3 Seasonal Indices

	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Max	112.1	110.7	106.2	97.9	88.5	81.5	79.3	91.4	102.5	109.7	109.6	110.6
Min	143.1	142.7	130.2	102.6	62.4	39.7	35.0	63.8	95.0	119.5	129.7	136.3

According to the seasonal indices, the maximum temperature for January is 12.1% above the monthly average, while July is 20.7% below the monthly average. The seasonal indices for the minimum temperature are similar, but more extreme. For example, the minimum temperature for January is 43.1% above the average monthly minimum temperature.

Dividing the observations by the corresponding seasonal index eliminates this seasonal pattern. The seasonally adjusted data is plotted in Figure 6.2. (Steyn *et al*, 1998)

Figure 6.2 The seasonally adjusted average monthly maximum and minimum temperature from January 1999 to December 2005



The rest of this example will be concerned with the seasonally adjusted data. The following notation will be used for the seasonally adjusted data

y_t : maximum temperature at time t

x_t : minimum temperature at time t

Figure 6.2 suggests that the two time series are stationary. This can be established by performing the Dickey Fuller Unit Root test. (Dickey & Fuller, 1979; Said & Dickey, 1984) The null hypothesis that the series is non-stationary can be rejected for both y_t (p-value = 0.0002) and x_t (p-value < 0.0001).

Dickey-Fuller Unit Root Tests

Variable	Type	Rho	Pr < Rho	Tau	Pr < Tau
yt	Zero Mean	-0.10	0.6583	-0.25	0.5940
	Single Mean	-48.03	0.0008	-4.84	0.0002
	Trend	-55.29	0.0003	-5.25	0.0002
xt	Zero Mean	-0.28	0.6159	-0.39	0.5392
	Single Mean	-75.16	0.0008	-6.01	<.0001
	Trend	-76.07	0.0003	-6.00	<.0001

The correlation at lag 0 of 0.4052 indicates the existence of a very weak linear relationship between y_t and x_t. The linear relationship between x_t and x_{t-1} has a negative coefficient and is not significant. This is an indication that past values of the minimum temperature cannot be used to explain / predict future values. On the other hand, there does exist a very weak relationship between x_t and y_{t-1} and between y_t and x_{t-1}. The autocorrelation matrices from lag 2 onwards do not differ significantly from zero, implying that the underlying data generating process could be a VMA(1) model.

Cross Correlations of Dependent Series

Lag	Variable	yt	xt
0	yt	1.00000	0.40520
	xt	0.40520	1.00000
1	yt	0.38973	0.26589
	xt	0.26655	-0.12035
2	yt	0.16721	0.08074
	xt	0.00946	0.11299
3	yt	0.16865	0.19466
	xt	0.05809	-0.01309
4	yt	-0.06268	-0.05705
	xt	-0.05808	0.02797
5	yt	-0.07644	-0.04729
	xt	-0.11339	-0.14811
6	yt	-0.10172	-0.09873
	xt	0.00434	-0.01098

Schematic Representation of Cross Correlations

Variable/ Lag	0	1	2	3	4	5	6
yt	++	++	..	..	..	..	..
xt	++	+.	..	..	..	..	..

+ is > 2*std error, - is < -2*std error, . is between

The minimum information criterion as well as the partial autoregression matrices suggest that a VAR(1) model might be appropriate.

Minimum Information Criterion

Lag	MA 0	MA 1	MA 2	MA 3	MA 4
AR 0	1.0795714	1.026612	0.99467	0.9895747	1.0708531
AR 1	0.8968038	1.1023662	1.0502803	1.0964425	1.1727411
AR 2	0.9083807	1.0464227	1.1152725	1.2130536	1.3059295
AR 3	1.0060682	1.1375668	1.1960255	1.3256859	1.4010507
AR 4	1.0668047	1.2196028	1.3064426	1.3848342	1.5053257

Partial Autoregression

Lag	Variable	yt	xt
1	yt	0.33707	0.13449
	xt	0.36381	-0.27290
2	yt	0.03673	-0.09182
	xt	0.01952	-0.03103
3	yt	0.14303	-0.02943
	xt	0.20747	-0.07518

```
   4    yt               -0.20763        -0.04550
        xt               -0.13935        -0.00861
   5    yt                0.07752        -0.08563
        xt                0.04123        -0.10498
   6    yt               -0.14251         0.09700
        xt               -0.10796        -0.00754


             Schematic Representation
             of Partial Autoregression
  Variable/
  Lag           1     2     3     4     5     6

  yt            +.    ..    ..    ..    ..    ..
  xt            +-    ..    ..    ..    ..    ..

            + is > 2*std error,  - is <
            -2*std error,  . is between
```

A VAR(1) model was fitted using the method of least squares,

$$\begin{pmatrix} \hat{y}_t \\ \hat{x}_t \end{pmatrix} = \begin{pmatrix} 15.582 \\ 6.879 \end{pmatrix} + \begin{pmatrix} 0.338 & 0.134 \\ 0.366 & -0.275 \end{pmatrix} \begin{pmatrix} y_{t-1} \\ x_{t-1} \end{pmatrix} + \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix}$$

```
                         Model Parameter Estimates

                                            Standard
  Equation Parameter       Estimate           Error t Value Pr > |t|
Variable

  yt       CONST1          15.58188       2.69091    5.79   0.0001 1
           AR1_1_1          0.33798       0.11198    3.02   0.0034 yt(t-1)
           AR1_1_2          0.13358       0.11585    1.15   0.2523 xt(t-1)
  xt       CONST2           6.87850       2.65044    2.60   0.0112 1
           AR1_2_1          0.36631       0.11029    3.32   0.0014 yt(t-1)
           AR1_2_2         -0.27540       0.11411   -2.41   0.0181 xt(t-1)
```

All the coefficients are significant except for $\hat{\phi}_{12}$ (p-value = 0.2523). The coefficient $\hat{\phi}_{22}$ is negative; this is in line with the negative relationship mentioned when discussing the autocorrelations.

The individual elements of the residual autocorrelation matrices do not differ significantly from zero for lags greater than zero. This is an indication that the residuals behave like a white noise process, implying that the model is adequate. This conclusion is confirmed by the Portmanteau test, which considers the autocorrelation matrices as a whole up to a specific lag.

Cross Correlations of Residuals

Lag	Variable	yt	xt
0	yt	1.00000	0.38944
	xt	0.38944	1.00000
1	yt	0.01169	0.00001
	xt	-0.02227	-0.01040
2	yt	-0.04435	-0.05570
	xt	-0.10986	-0.08921
3	yt	0.19436	0.20926
	xt	0.08690	0.04573
4	yt	-0.13768	-0.03699
	xt	-0.09781	-0.01346
5	yt	-0.03784	-0.01155
	xt	-0.06174	-0.09844
6	yt	-0.13645	-0.11150
	xt	0.07417	-0.03165

Schematic Representation of Cross Correlations of Residuals

Variable/ Lag	0	1	2	3	4	5	6
yt	++	..	..	..	..	..	..
xt	++	..	..	..	..	..	..

+ is > 2*std error, - is < -2*std error, . is between

Portmanteau Test for Cross Correlations of Residuals

Up To Lag	DF	Chi-Square	Pr > ChiSq
2	4	1.38	0.8483
3	8	6.65	0.5753
4	12	8.57	0.7393
5	16	9.63	0.8853
6	20	13.54	0.8530

The VAR(1) model can be regarded in terms of two univariate equations,

$$\hat{y}_t = 15.582 + 0.338 y_{t-1} + 0.134 x_{t-1} + a_{1t}$$
$$\hat{x}_t = 6.879 + 0.366 y_{t-1} - 0.275 x_{t-1} + a_{2t}$$

The portion of the variability explained by each of these univariate models only amounts to 16.62% and 13.39%, respectively. Even though this does not seem to be a lot, it is a vast improvement from the result obtained when analysing these time series on their own. For comparison purposes an AR(1) model was also fitted to both y_t and x_t. 15.23% of the variation in y_t can be explained by y_{t-1}, while only 1% of the variation in x_t can be explained by x_{t-1}. It turned out that when looking at x_t alone, it is a white noise process.

The residuals of the univariate equations of the VAR(1) model are normally distributed and there is no sign of serial correlation or ARCH disturbances.

Univariate Model ANOVA Diagnostics

Variable	R-Square	Standard Deviation	F Value	Pr > F
yt	0.1662	1.31835	7.97	0.0007
xt	0.1339	1.29853	6.19	0.0032

Univariate Model White Noise Diagnostics

Variable	Durbin Watson	Normality Chi-Square	Normality Pr > ChiSq	ARCH F Value	ARCH Pr > F
yt	1.95389	1.75	0.4163	0.06	0.8107
xt	2.01497	3.94	0.1392	0.03	0.8630

Univariate Model AR Diagnostics

Variable	AR1 F Value	AR1 Pr > F	AR2 F Value	AR2 Pr > F	AR3 F Value	AR3 Pr > F	AR4 F Value	AR4 Pr > F
yt	0.01	0.9164	0.08	0.9229	1.13	0.3421	1.36	0.2552
xt	0.01	0.9259	0.32	0.7260	0.26	0.8556	0.20	0.9377

Based on the residual analysis it is apparent that the VAR(1) model is an adequate representation of the relationship between the maximum and minimum monthly temperature. This model can definitely be improved by taking into account more related variables, for example the rainfall pattern and the humidity index, to mention only a few. Another advantage of multivariate time series analysis is that it can be used to determine the cause and effect relation between variables. Examining the results, we realised that the maximum temperature of the previous month has a greater impact on the minimum of the current month than the minimum of previous month has on the maximum of the current month. For a novice in climatology this seems realistic since the minimum temperature will depend on how much it cooled down during the night, implying that it depends on the maximum temperature.

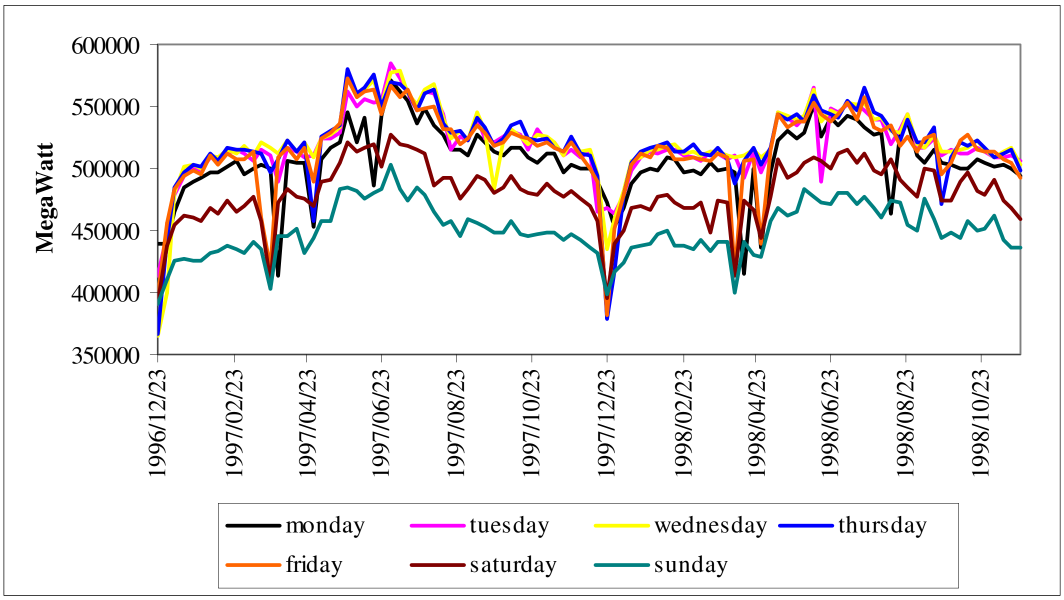
6.4.3 Electricity data*

The possibilities with multivariate time series analysis are endless. In this example the daily electricity consumption will be analysed, but instead of considering it as a single variable the seven weekdays can be regarded as a 7-dimensional vector, corresponding to each day of the

* The SAS program is provided in Appendix B page 143.

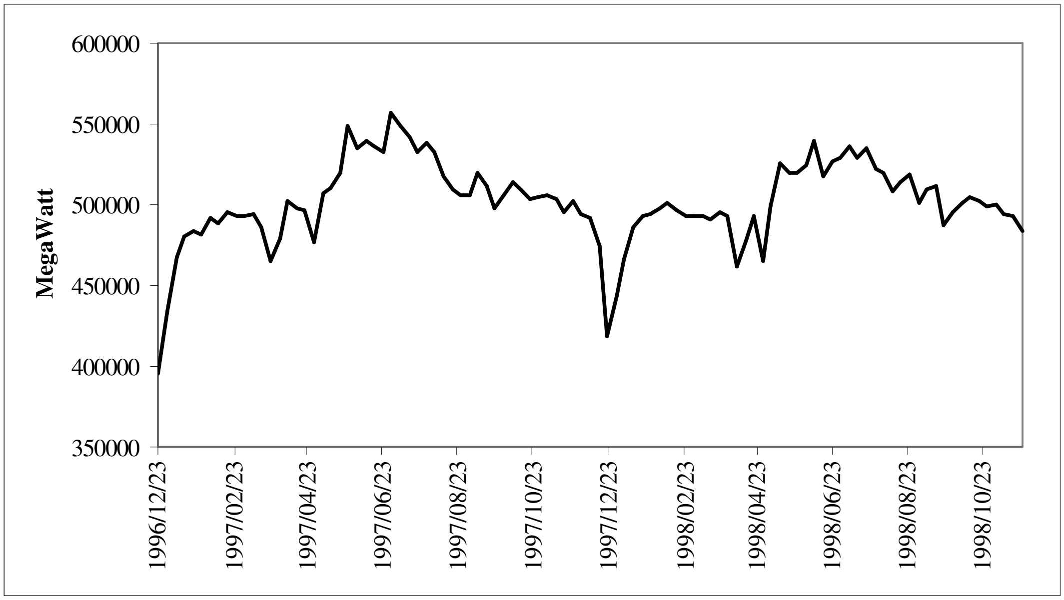
week. An application can be to use this as part of a one-week ahead planning process to estimate the electricity demand for the week. Figure 6.3 shows the graph of the electricity consumption from 23 December 1996 to 29 November 1998. Every line represents a different day of the week (variable). The electricity consumption for Sundays is the lowest, followed by that for Saturdays. The aim is to observe the relationship of variables over time and utilise it to build a model for the electricity consumption.

Figure 6.3 The electricity consumption from 23 December 1996 to 29 November 1998



The average electricity consumption, for every week, is graphed in Figure 6.4. The minimum values were observed for the weeks that included a public holiday, more particularly Christmas day, Easter weekend and the time between Freedom day and Workers' day. The maximum values correspond to the winter months where everyone uses more electricity to keep warm.

Figure 6.4 The average weekly electricity consumption from 23 December 1996 to 29 November 1998



Due to the high dimension of this multivariate time series problem, the SAS output used in the discussion below is provided in Appendix B.

The purpose of this exercise is to use the correlation structure between the different weekdays to build a model for short-term electricity load predictions. Although seasonality due to annual weather patterns is not explicitly addressed in this example, it is unlikely that the effect would be non-stationary, or be revealed as such through a seasonal unit root test when considering the duration of time considered.

The correlations between the variables during the same week are very high. They range from 0.67515 between a Monday and a Saturday, to 0.96147 for a Friday and a Saturday. For the model building purpose we are more interested in the lagged correlations, since only lagged values of the variables can be included in the model. Table 6.4 contains the lag 1 autocorrelations. The highest correlation of 0.83647 is between Monday and the Sunday of the previous week, which is also the previous day. Based on this, it seems likely that the fitted model will be able to explain the variation on a Monday the best. All the variables are more correlated with the Sunday of the previous week (most recent observation) and this pattern decreases towards Monday, with the exception of Tuesday and Wednesday. A possible explanation for this is that Sundays would serve as a minimum for electricity consumption, since most businesses are closed and the consumption is more for private use.

Tuesdays and Wednesdays, on the other hand, are days where most people are at work and therefore can be considered as an upper bound for the electricity consumption.

Table 6.4 Lag 1 autocorrelations

Lag 1	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
Monday	0.52633	0.56355	0.54639	0.50333	0.47835	0.52091	0.55937
Tuesday	0.65060	0.71417	0.72278	0.68567	0.64404	0.67381	0.71087
Wednesday	0.63268	0.70101	0.70808	0.63985	0.58013	0.60570	0.66288
Thursday	0.64872	0.69729	0.68762	0.64983	0.57696	0.59659	0.64629
Friday	0.75922	0.70250	0.64937	0.63297	0.59030	0.61755	0.64606
Saturday	0.79386	0.73928	0.68412	0.65566	0.61282	0.65766	0.70378
Sunday	0.83647	0.79010	0.71914	0.68720	0.65802	0.71209	0.77082

The autocorrelations at lag 2 are all in the order of 0.5 and they decrease rapidly as the lag increases. Based on the autocorrelations, the possibility of a pure VMA model is excluded. The partial autoregressions do not differ significantly from zero for lags greater than one, implying that a VAR(1) model might be appropriate. This is confirmed using the MINIC method. A VAR(1) model was fitted using the method of least squares,

$$\begin{pmatrix} mon_t \\ tue_t \\ wed_t \\ thu_t \\ fri_t \\ sat_t \\ sun_t \end{pmatrix} = \begin{pmatrix} 38813 \\ 84221 \\ 35625 \\ 20087 \\ 33067 \\ 72891 \\ 75048 \end{pmatrix} + \begin{pmatrix} -0.11 & 0.08 & -0.06 & -0.16 & 0.44 & -0.51 & 1.36 \\ -0.11 & 0.21 & 0.001 & 0.12 & 0.09 & -0.51 & 1.15 \\ -0.19 & 0.50 & 0.04 & 0.17 & -0.07 & -0.28 & 0.84 \\ -0.32 & 0.90 & -0.40 & 0.30 & 0.11 & -0.49 & 0.94 \\ -0.30 & 0.98 & -0.40 & 0.06 & 0.31 & -0.73 & 1.10 \\ -0.16 & 0.67 & -0.28 & 0.01 & 0.10 & -0.37 & 0.91 \\ -0.10 & 0.38 & -0.13 & 0.08 & -0.09 & -0.24 & 0.92 \end{pmatrix} \begin{pmatrix} mon_{t-1} \\ tue_{t-1} \\ wed_{t-1} \\ thu_{t-1} \\ fri_{t-1} \\ sat_{t-1} \\ sun_{t-1} \end{pmatrix} + \boldsymbol{a}_t$$

Most of the elements of the coefficient matrix corresponding to a lagged Monday, Tuesday and Sunday are significant. The highest coefficients are those of sun_{t-1}. This is in line with what is expected of the high correlation mentioned earlier.

Some of the individual elements of the residual autocorrelation matrices, at higher lags, differ significantly from zero. These are few and far between. The Portmanteau test, with null hypothesis of no autocorrelaion in the residuals, cannot be rejected. This implies that the residuals behave like a white noise process.

The multivariate VAR(1) model can also be considered as univariate equations. According to the F - test all the univariate equations explain a significant portion of the total variability. 80% of the variability of the electricity consumption on a Wednesday can be explained by the consumption of the previous week, while only 64% of the variation for a Saturday can be explained by the same variables.

Based on the Durbin-Watson test and the AR(1) to AR(4) disturbances, the residuals of the univariate models seem to be independent. According to the ARCH disturbances, the variance of the residuals is also constant. The major concern is the normality. The null hypothesis of normally distributed residuals is rejected for all the univariate equations. This could be due to the extreme values for the holiday periods and possibly a seasonal pattern in the data that was not accounted for in the model.

6.5 CONCLUSION

This chapter discussed procedures to determine whether the fitted model was an adequate representation of the underlying data generating process. These procedures were grouped into multivariate and univariate diagnostic checks. The multivariate checks were based on the residual autocorrelation matrices. The aim was to show that the residuals behave like a multivariate white noise process. This was achieved by testing whether the individual elements of the autocorrelation matrices at different lags, as well as the whole matrix up to a certain lag, differ significantly from zero. The univariate checks included several testing procedures to establish whether the residuals of the univariate equations are independent, normally distributed random variables with zero mean. The chapter was concluded with some examples to illustrate the diagnostic checks and the model building process.

CHAPTER 7

CONCLUSION

The ultimate aim of this study was to explore the field of multivariate time series analysis, and more particularly stationary processes. After defining the different multivariate time series models, an overview was given of all the techniques used in finding a suitable model for an observed multivariate time series. The model building process comprised of investigating the sample autocorrelations and sample partial autoregressions to tentatively select the order of a model; fitting a model using the method of least squares or the method of maximum likelihood; and assessing the adequacy of the fitted model through analysing the residuals.

Throughout the study examples were used to illustrate the different techniques. Most formulae were programmed using the IML procedure in SAS. The results obtained were compared to the output of the built-in SAS functions. *Mathematica*® was used to do some algebraic calculations and to show that it is possible to derive formulae for specific models in terms of their coefficient matrices and the white noise covariance matrix. Since the last mentioned formulae were computationally intense an Excel spreadsheet was developed where one can just enter some information and Excel will calculate the answer.

Finally, fitting a model to observed data provided a practical overview of the model building process. In the one example, the challenge was to estimate a model for the average monthly minimum and maximum temperature. Using related variables to improve the model was evident from this example. When the minimum temperature was analysed separately, it could not be modeled because it was just a white noise process. When the extra information of the maximum temperature was utilised, the model improved substantially. The other example was concerned with the daily electricity consumption. Instead of considering the consumption as a univariate time series, it was decomposed into a 7-dimensional multivariate time series, where each day of the week was considered individually. This way, the weekly pattern was taken into account. These two examples highlighted some of the advantages of multivariate time series analysis.

In the future, statistical software packages especially open source packages (for example R) can be explored to determine what other procedures are available and how to utilise them to improve the model building process. Using the SAS code developed for this dissertation as a basis it will be relatively simple to develop a module in open source for multivariate time series analysis that would be available to a much wider user group than those who have access to high-cost commercial software products. On a more theoretical note, forecasting of multivariate time series models as well as the field of nonstationary processes could be addressed.

APPENDIX A

CONTENTS

APPENDIX A1

PROPERTIES OF THE *VEC* OPERATOR

(Source: Lütkepohl, 2005)

Let $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ be matrices with appropriate dimensions.

1. $vec(\boldsymbol{A}+\boldsymbol{B}) = vec(\boldsymbol{A}) + vec(\boldsymbol{B})$ (A1.1)
2. $vec(\boldsymbol{AB}) = (\boldsymbol{I} \otimes \boldsymbol{A})vec(\boldsymbol{B}) = (\boldsymbol{B}' \otimes \boldsymbol{I})vec(\boldsymbol{A})$ (A1.2)
3. $vec(\boldsymbol{ABC}) = (\boldsymbol{C}' \otimes \boldsymbol{A})vec(\boldsymbol{B})$ (A1.3)

APPENDIX A2

PROPERTIES OF THE KRONECKER PRODUCT

(Source: Lütkepohl, 2005)

1. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are invertible, then $(\boldsymbol{A} \otimes \boldsymbol{B})^{-1} = \boldsymbol{A}^{-1} \otimes \boldsymbol{B}^{-1}$ (A2.1)
2. $(\boldsymbol{A} \otimes \boldsymbol{B})' = \boldsymbol{A}' \otimes \boldsymbol{B}'$ (A2.2)
3. $(\boldsymbol{A} \otimes \boldsymbol{B})(\boldsymbol{C} \otimes \boldsymbol{D}) = \boldsymbol{AC} \otimes \boldsymbol{BD}$ (A2.3)
4. If $\boldsymbol{A} : (m \times m)$ and $\boldsymbol{B} : (n \times n)$ then $|\boldsymbol{A} \otimes \boldsymbol{B}| = |\boldsymbol{A}|^n |\boldsymbol{B}|^m$ (A2.4)
5. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are square matrices with eigenvalues λ_A and λ_B respectively, then $\lambda_A \lambda_B$ is an eigenvalue of $(\boldsymbol{A} \otimes \boldsymbol{B})$ (A2.5)

APPENDIX A3

RULES FOR VECTOR AND MATRIX DIFFERENTIATION

(Source: Lütkepohl, 2005)

1. Let $\boldsymbol{A}:(m\times n)$ and $\boldsymbol{b}:(n\times 1)$. Then $\dfrac{\partial \boldsymbol{Ab}}{\partial \boldsymbol{b}'}=\boldsymbol{A}$ and $\dfrac{\partial \boldsymbol{b}'\boldsymbol{A}'}{\partial \boldsymbol{b}}=\boldsymbol{A}'$ (A3.1)

2. Let $\boldsymbol{A}:(m\times m)$ and $\boldsymbol{b}:(m\times 1)$. Then $\dfrac{\partial \boldsymbol{b}'\boldsymbol{Ab}}{\partial \boldsymbol{b}}=(\boldsymbol{A}+\boldsymbol{A}')\boldsymbol{b}$ and $\dfrac{\partial \boldsymbol{b}'\boldsymbol{Ab}}{\partial \boldsymbol{b}'}=\boldsymbol{b}'(\boldsymbol{A}'+\boldsymbol{A})$ (A3.2)

3. Let $\boldsymbol{A}:(m\times m)$ and $\boldsymbol{b}:(m\times 1)$. Then $\dfrac{\partial^2 \boldsymbol{b}'\boldsymbol{Ab}}{\partial \boldsymbol{b}\partial \boldsymbol{b}'}=(\boldsymbol{A}+\boldsymbol{A}')$ (A3.3)

4. If $\boldsymbol{A}:(m\times m)$ is symmetric and $\boldsymbol{b}:(m\times 1)$. Then $\dfrac{\partial^2 \boldsymbol{b}'\boldsymbol{Ab}}{\partial \boldsymbol{b}\partial \boldsymbol{b}'}=2\boldsymbol{A}$ (A3.4)

5. If $\boldsymbol{A}:(m\times m)$ is nonsingular with $|\boldsymbol{A}|>0$, then $\dfrac{\partial \ln|\boldsymbol{A}|}{\partial \boldsymbol{A}}=(\boldsymbol{A}')^{-1}$ (A3.5)

6. Let $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ be $(m\times m)$ matrices with $\boldsymbol{A}$ non-singular. Then

$$\frac{\partial tr(\boldsymbol{BA}^{-1}\boldsymbol{C})}{\partial \boldsymbol{A}}=-(\boldsymbol{A}^{-1}\boldsymbol{CBA}^{-1}) \quad \text{(A3.6)}$$

APPENDIX A4

DEFINITION OF MODULUS

(Source: Hamilton, 1994)

The modulus of a complex number $(a+bi)$ is

$|a+bi| = \sqrt{a^2 + b^2}$

The modulus of a real number ($b = 0$) is the absolute value of that number.

APPENDIX A5

MULTIVARIATE RESULTS

Transformation Theorem (A5.1)

(Source: Anderson, 1984)

Let the density function of $X_1,\ldots,X_p$ be $f(x_1,\ldots,x_p)$. Consider the p real-valued functions

$$y_i = y_i(x_1,\ldots,x_p) \quad i=1,\ldots,p$$

We assume that the transformation from the $x-$space to the $y-$space is one-to-one; the inverse transformation is

$$x_i = x_i(y_1,\ldots,y_p) \quad i=1,\ldots,p$$

Let the random variables $Y_1,\ldots,Y_p$ be defined by

$$Y_i = y_i(X_1,\ldots,X_p) \quad i=1,\ldots,p$$

Then the density function of $Y_1,\ldots,Y_p$ is

$$h(y_1,\ldots,y_p) = f[x_1(y_1,\ldots,y_p),\ldots,x_p(y_1,\ldots,y_p)]J(y_1,\ldots,y_p)$$

where $J(y_1,\ldots,y_p)$ is the Jacobian

$$J(y_1,\ldots,y_p) = \operatorname{mod}\begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_p} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \cdots & \frac{\partial x_2}{\partial y_p} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial x_p}{\partial y_1} & \frac{\partial x_p}{\partial y_2} & \cdots & \frac{\partial x_p}{\partial y_p} \end{vmatrix}$$

where "mod" means the absolute value of the expression following it.

Multivariate normal distribution result (A5.2)

(Source: Johnson & Wichern, 2002)

If $\boldsymbol{X}$ is distributed as $N_p(\boldsymbol{\mu},\boldsymbol{\Sigma})$, the q linear combinations $\boldsymbol{AX}:(q\times p)(p\times 1)$ are distributed as $N_q(\boldsymbol{A\mu},\boldsymbol{A\Sigma A}')$

APPENDIX B

CONTENTS

DESCRIPTION OF SOME OF THE FUNCTIONS AND PROCEDURES USED IN THE SAS PROGRMAS

(Quoted from: SAS/ETS 9.1 User's Guide, 2004)

PROC IML: Statements, Functions, and Subroutines

APPEND Statement: adds observations to the end of a SAS data set

APPEND FROM *from-name;*
from-name is the name of a matrix containing data to append

CREATE Statement: creates a new SAS data set

CREATE *SAS-data-set* **FROM** *matrix-name*[***COLNAME***=*column*-]***;***
SAS-data-set is the name of the new data set
matrix-name names a matrix containing the data
column-name is a character matrix containing descriptive names to associate with data set variables

EIGVAL Function: computes the eigenvalues of a square matrix

EIGVAL(*square-matrix***)**

DET Function: computes the determinant of a square matrix

DET(*square-matrix***)**

DIAG Function: creates a diagonal matrix from a square matrix or a vector

DIAG(*square-matrix / vector***)**

I Function: creates an identity matrix

I(*dimension***)**
dimension specifies the size of the identity matrix

INV Function: computes the inverse of a square nonsingular matrix

INV(*square-matrix***)**

J Function: creates a matrix of identical values

J(*nrow, ncol, value***)**
nrow is the number of rows
ncol is a the number of columns.
value is the value used to fill the rows and columns of the matrix

LOG Function: takes the natural logarithm

LOG(*matrix***)**
matrix is a numeric matrix or literal

NCOL Function: finds the number of columns of a matrix

NCOL(*matrix***)**

PROBCHI Function: returns the probability that an observation from a Chi-square distribution, with degrees of freedom *df* is less than or equal to *x*

PROBCHI(*x,df***)**

PROBF Function: returns the probability that an observation from an F distribution, with numerator degrees of freedom *ndf*, denominator degrees of freedom *ddf*, is less than or equal to *x*

PROBF(*x,ndf,ddf***)**

SHAPE Function: reshapes a matrix

SHAPE(*matrix, nrow, ncol***)**
nrow gives the number of rows of the new matrix
ncol gives the number of columns of the new matrix

SQRT Function: calculates the square root

SQRT(*matrix***)**
matrix is a numeric matrix or literal

TRACE Function: sums diagonal elements of a matrix

TRACE(*matrix***)**

VARMACOV Call: computes the theoretical cross-covariance matrices for a stationary VARMA(***p***,***q***) model

CALL VARMACOV(*cov***)** *phi= theta= sigma= lag=***;**
phi specifies the autoregressive coefficient matrices
theta specifies the moving average coefficient matrices
sigma specifies the covariance matrix of the innovation series
lag specifies the number of lags

The **VARMACOV** subroutine returns the following value:
cov is a matrix that contains the theoretical cross-covariance matrices

VARMASIM Call: generates a VARMA(***p***,***q***) time series

CALL VARMASIM(*series*) *phi*= *theta*= *mu*= *sigma*= *n*= *seed*=**;**
phi specifies the autoregressive coefficient matrices
theta specifies the moving average coefficient matrices
mu specifies the mean vector of the series
sigma specifies the covariance matrix of the innovation series
n specifies the length of the series
seed specifies the random number seed

The **VARMASIM** subroutine returns the following value:
series is a matrix containing the generated time series.

VECDIAG Function: creates a vector from the diagonal elements of a square matrix

VECDIAG(*square-matrix*)

VTSROOT Call: calculates the characteristic roots of the model from AR and MA characteristic functions

CALL VTSROOT(*root, phi, theta*)**;**
phi specifies the autoregressive coefficient matrices
theta specifies the moving average coefficient matrices

The **VTSROOT** subroutine returns the following value:
root is a matrix, where
the first column contains the real parts, *x*, of eigenvalues
the second column contains the imaginary parts, *y*, of the eigenvalues
the third column contains the modulus of the eigenvalues

PROC IML: Operators

Addition Operator	+	adds corresponding matrix elements
Concatenation Operator, Horizontal	||	concatenates matrices horizontally
Concatenation Operator, Vertical	//	concatenates matrices vertically
Kronecker Product Operator	@	takes the *Kronecker product* of two matrices
Division Operator	/	performs elementwise division
Multiplication Operator, Elementwise	#	performs elementwise multiplication
Multiplication Operator, Matrix	*	performs matrix multiplication
Power Operator, Elementwise	##	raises each element to a power
Power Operator, Matrix	**	raises a matrix to a power
Subscripts	[ ]	select submatrices matrix[rows,columns]
Subtraction Operator	-	subtracts corresponding matrix elements
Transpose Operator	`	transposes a matrix

The VARMAX Procedure

Syntax

PROC VARMAX options **;**
    MODEL dependent variables </ options > **;**
    OUTPUT < options > **;**

PROC VARMAX Statement

PROC VARMAX *options ;*

Options

DATA= ***SAS-data-set***
specifies the input SAS data set

MODEL Statement

MODEL *dependents </ options > ;*

The MODEL statement specifies dependent variables for the VARMAX model.

General Options

METHOD= ***value***
requests the type of estimates to be computed, the possible values are:
LS: specifies least-squares estimates
ML: specifies maximum likelihood estimates

NOINT
suppresses the intercept parameter

Printing Control Options

LAGMAX= ***number***
specifies the number of lags to display in the output

Printing Options

PRINT=(*options*)
The following options can be used in the PRINT=( ) option:

CORRY(*number*)
prints the cross-correlation matrices of dependent variables

COVY(*number*)
prints the cross-covariance matrices of dependent variables

PARCOEF(*number*)
prints the partial autoregression coefficient matrices

Lag Specification Options

P= ***number***
specifies the order of the vector autoregressive process

Q= ***number***
specifies the order of the moving-average error process

Tentative Order Selection Options

MINIC= (TYPE=*value* P=*number* Q=*number*)
prints the information criterion for the appropriate AR and MA tentative order selection

P= ***number***
specifies the order of the vector autoregressive process

Q= ***number***
specifies the order of the moving-average error process

TYPE= ***value***
specifies the criterion for the model order selection, valid criteria are as follows:
AIC: Akaike Information Criterion
AICC: Corrected Akaike Information Criterion (this is the default criterion)
FPE: Final Prediction Error criterion
HQC: Hanna-Quinn Criterion
SBC: Schwarz Bayesian Criterion

Cointegration Related Options

DFTEST
prints the Dickey-Fuller unit root tests

OUTPUT Statement

OUTPUT < *options* >***;***

The OUTPUT statement generates and prints forecasts based on the model estimated in the previous MODEL statement and, optionally, creates an output SAS data set that contains these forecasts.

Options

LEAD= ***number***
specifies the number of multistep-ahead forecast values to compute

OUT= ***SAS-data-set***
writes the forecast values to an output data set

The ARIMA Procedure

Syntax

PROC ARIMA options;
    IDENTIFY VAR=variable options;
    ESTIMATE options;
    FORECAST options;

PROC ARIMA Statement

PROC ARIMA *options;*

Options

DATA= ***SAS-data-set***
specifies the name of the SAS data set containing the time series

OUT= SAS-data-set
specifies a SAS data set to which the forecasts are output

IDENTIFY Statement

IDENTIFY *VAR=variable;*

The IDENTIFY statement specifies the time series to be modeled.

ESTIMATE Statement

ESTIMATE *options;*

The ESTIMATE statement specifies an ARMA model for the response variable specified in the previous IDENTIFY statement, and produces estimates of its parameters. The ESTIMATE statement also prints diagnostic information by which to check the model.

Options

P= order
specifies the autoregressive part of the model

FORECAST Statement

FORECAST *options;*

The FORECAST statement generates forecast values for a time series using the parameter estimates produced by the previous ESTIMATE statement

Options

LEAD= ***n***
specifies the number of multistep forecast values to compute

SAS PROGRAMS

Example 2.1

```
proc iml;
sig={1.0 0.5,0.5 0.9};
vecsig=sig[,1]//sig[,2];
phi= {0.5 0.6,0.1 0.4};
print sig vecsig phi;

e=eigval(phi);
print e;

call vtsroot(root,phi);
print root;

call varmacov(cov,phi) sigma=sig lag=2;
print cov;

k=phi@phi;
vec00=inv(I(4)-phi@phi)*vecsig;
gamma0=vec00[1:2,]||vec00[3:4,];
gamma1=phi*gamma0;
gamma2=phi*gamma1;

print k, vec00, gamma0, gamma1, gamma2;
run;
```

Example 2.3

```
proc iml;
siga={1.0 0.5,0.5 0.9};
sig=siga||J(2,2,0)//J(2,4,0);
vecsig=sig[,1]//sig[,2]//sig[,3]//sig[,4];
phi1={-0.2 0.1,0.5 0.1};
phi2={0.8 0.5,-0.4 0.5};
phi=phi1//phi2;
F=(phi1||phi2)//(I(2)||J(2,2,0));
print siga, sig, vecsig, phi1, phi2, phi, F;

e=eigval(F);
print e;

call vtsroot(root,phi);
print root;

call varmacov(cov,phi) sigma=siga lag=2;
print cov;

vec00=inv(I(16)-F@F)*vecsig;
gamma0=vec00[1:2,]||vec00[5:6,];
gamma1=vec00[9:10,]||vec00[13:14,];
gamma2=phi1*gamma1+phi2*gamma0;
print vec00, gamma0, gamma1, gamma2;
run;
```

Example 2.5

```
proc iml;
siga={1.0 0.5,0.5 0.9};
theta1={0.2 0.1,0.1 0.4};
theta2={0.4 0,0.6 0.1};
teta=theta1//theta2;
print siga theta1 theta2 teta;

call vtsroot(root) theta=teta;
print root;

call varmacov(cov) theta=-teta sigma=siga lag=3;
print cov;

gamma0=siga+theta1*siga*theta1`+theta2*siga*theta2`;
gamma1=theta1*siga+theta2*siga*theta1`;
gamma2=theta2*siga;
print gamma0, gamma1, gamma2;
run;
```

Example 2.6

```
proc iml;
siga={1.0 0.5,0.5 0.9};
sig=(siga||J(2,2,0)||siga)//J(2,6,0)//(siga||J(2,2,0)||siga);
vecsig=sig[,1]//sig[,2]//sig[,3]//sig[,4]//sig[,5]//sig[,6];
phi1={-0.2 0.1,0.5 0.1};
phi2={0.8 0.5,-0.4 0.5};
theta1={0.2 0.1,0.1 0.4};
phi12=phi1//phi2;
phi=(phi1||phi2||theta1)//(I(2)||J(2,4,0))//J(2,6,0);
print siga sig vecsig phi1 phi2 theta1 phi12 phi;

call vtsroot(root,phi12,theta1);
print root;

call varmacov(cov,phi12,-theta1) sigma=siga lag=2;
print cov;

vec00=inv(I(36)-phi@phi)*vecsig;
g0star=vec00[1:6,]||vec00[7:12,]||vec00[13:18,]||vec00[19:24,]||
       vec00[25:30,]||vec00[31:36,];
print vec00 g0star;

gamma0=g0star[1:2,1:2];
gamma1=g0star[1:2,3:4];
gamma2=phi1*gamma1+phi2*gamma0;
print gamma0, gamma1, gamma2;
run;
```

Example 3.1

```
proc iml;
T=500;
k=2;
p=1;

sig={1.0 0.5,0.5 0.9};
phi={0.5 0.6,0.1 0.4};
call varmasim(yy,phi) sigma=sig n=T seed=1;
cn={'y1' 'y2'};
create simul1 from yy[colname=cn];
append from yy;

y=yy`;
*print y;

vecy=y[,1];
do i= 2 to T;
      vecy=vecy//y[,i];
end;
*print vecy;

call varmasim(yyy,phi) sigma=sig n=1 seed=2;

z=J(1,T,1)//(yyy`||y[,1:T-1]);

vecb=((inv(z*z`)*z)@I(k))*vecy;
print vecb;

b=vecb[1:2,]||vecb[3:4,]||vecb[5:6,];
print b;

gamhat=(1/T)#z*z`;
print gamhat;

sighat=1/(T-k*p-1)*(y*(I(T)-z`*inv(z*z`)*z)*y`);
print sighat;

var=inv(z*z`)@sighat;
print var;

stderr=sqrt(vecdiag(inv(z*z`)@sighat));
print stderr;

t=vecb/stderr;
print t;

proc varmax data=simul1;
model y1 y2 / p=1 lagmax=3;
run;
```

Example 3.1 (Alternative way of generating data)

```
proc iml;
T=500;
k=2;
porder=1;
```

```
siga={1.0 0.5,0.5 0.9};
phi={0.5 0.6,0.1 0.4};
p=half(siga)`;
print p;

*generate random starting point y0;
yp=J(2,1,0);
do j=1 to 50;
      d=J(2,1,0);
      d[1,1]=rannor(0);
      d[2,1]=rannor(0);
      aa=p*d;
      yy=phi*yp+aa;
      yp=yy;
end;
*print yp;

a=J(2,T,0);
y=J(2,T,0);
a[,1]=p*J(2,1,rannor(0));
y[,1]=phi*yp+a[,1];
do i=2 to T;
      d=J(2,1,0);
      d[1,1]=rannor(0);
      d[2,1]=rannor(0);
      a[,i]=p*d;
      y[,i]=phi*y[,i-1]+a[,i];
end;
*print a y;

vecy=y[,1];
do m= 2 to T;
      vecy=vecy//y[,m];
end;

z=J(1,T,1)//(yp||y[,1:T-1]);

vecb=((inv(z*z`)*z)@I(k))*vecy;
print vecb;

b=vecb[1:2,]||vecb[3:4,]||vecb[5:6,];
print b;

gamhat=(1/T)#z*z`;
print gamhat;

sighat=1/(T-k*porder-1)*(y*(I(T)-z`*inv(z*z`)*z)*y`);
print sighat;

stderr=sqrt(vecdiag(inv(z*z`)@sighat));
print stderr;

tstat=vecb/stderr;
print tstat;

yy=y`;
cn={'y1' 'y2'};
create simul1 from yy[colname=cn];
append from yy;
quit;
```

```
proc varmax data=simul1;
model y1 y2 / p=1 lagmax=3;
run;
```

Example 3.2

```
*assume mean is zero;
proc iml;
T=500;
k=2;
p=1;

sig={1.0 0.5,0.5 0.9};
phi={0.5 0.6,0.1 0.4};
call varmasim(yy,phi) sigma=sig n=T seed=1;
cn={'y1' 'y2'};
create simul1 from yy[colname=cn];
append from yy;

y=yy`;
*print y;

mean=1/T#y[,+];
print mean;

mu=J(2,1,0);

vecy=y[,1];
do i= 2 to T;
      vecy=vecy//y[,i];
end;
*print vecy;

vecmu=mu;
do j=2 to T;
      vecmu=vecmu//mu;
end;

call varmasim(yyy,phi) sigma=sig n=1 seed=2;

x=(yyy`-mu)||(y[,1:T-1]-mu*j(1,T-1,1));

vecb=((inv(x*x`)*x)@I(k))*(vecy-vecmu);
print vecb;

b=vecb[1:2,]||vecb[3:4,];
print b;

ynul=(y[,1:T]-mu*j(1,T,1));

sighat=1/T#(ynul-b*x)*(ynul-b*x)`;
print sighat;

stderr=sqrt(vecdiag(inv(x*x`)@sighat));
print stderr;

t=vecb/stderr;
print t;
```

```
proc varmax data=simul1;
model y1 y2 / p=1 method=ml noint lagmax=3;
run;
```

Example 4.1

```
*IML Program to optimise likelihood function of VMA(1) by means of dual
quasi Newton optimisation algorithm;
proc iml;

*simulate VMA(1) time series;

sig={1.0 0.5, 0.5 0.9 };
T=500;
theta1={0.2 0.1, 0.1 0.4};

call varmasim(yy) theta=theta1 sigma=sig n=T seed=1;
cn={'y1' 'y2'};
create vma1 from yy[colname=cn];
append from yy;

*Calculate -2*logL for VMA(1);

start loglike(x) global(yy);
 theta=(x[1]||x[2])//(x[3]||x[4]);
 siga=(x[5]||x[6])//(x[6]||x[7]);

 *invertibility test;
 e=eigval(theta);
 norme=sqrt(e##2);
 testi=(norme>=1)[+,];

 *determinant sigma_a test;
 detsiga=det(siga);
 testd=(detsiga<=0);

 test=testi+testd;

 if test=0 then do;
   a=j(nrow(yy),2,0);
   do i=2 to nrow(a);
    a[i,]=yy[i,]-a[i-1,]*theta`;
   end;
   aa=a[11:nrow(a),];
   capt=nrow(aa);
 sum=0;
 isiga=inv(siga);
 do j=1 to capt;
  sum=sum+aa[j,]*isiga*aa[j,]`;
 end;
 logl=capt#log(det(siga))+sum;
 end;
 return(logl);
finish loglike;

x={-0.1 0.1 -0.1 0.1 1.01 0.01 1.01}; *Starting values for parameters;
optn = {0 2 . 2}; *Options for optimisation procedure;
call nlpqn(rc,xr,"loglike",x,optn);

run;
```

```
proc varmax data=vma1;
model y1 y2 / q=1 method=ml noint lagmax=3;
run;

quit;
```

Example 4.2

```
*IML Program to optimise likelihood function of VARMA(1,1) by means of dual
quasi Newton optimisation algorithm;
proc iml;

*simulate VARMA(1,1) time series;

sig={1.0 0.5, 0.5 0.9 };
T=500;
phi1={-0.2 0.1,0.5 0.1};
theta1={0.2 0.1,0.1 0.4};

call varmasim(yy) phi=phi1 theta=theta1 sigma=sig n=T seed=1;
cn={'y1' 'y2'};
create varma11 from yy[colname=cn];
append from yy;

*Calculate -2*logL for VMA(1);

start loglike(x) global(yy);
 phi=(x[1]||x[2])//(x[3]||x[4]);
 theta=(x[5]||x[6])//(x[7]||x[8]);
 siga=(x[9]||x[10])//(x[10]||x[11]);

 *stationarity test;
 ep=eigval(phi);
 normep=sqrt(ep##2);
 tests=(normep>=1)[+,];

 *invertibility test;
 et=eigval(theta);
 normet=sqrt(et##2);
 testi=(normet>=1)[+,];

 *determinant sigma_a test;
 detsiga=det(siga);
 testd=(detsiga<=0);

 test=tests+testi+testd;

 if test=0 then do;
   a=j(nrow(yy),2,0);
   do i=2 to nrow(a);
    a[i,]=yy[i,]-yy[i-1,]*phi`-a[i-1,]*theta`;
   end;
   aa=a[11:nrow(a),];
   capt=nrow(aa);
 sum=0;
 isiga=inv(siga);
 do j=1 to capt;
  sum=sum+aa[j,]*isiga*aa[j,]`;
 end;
 logl=capt#log(det(siga))+sum;
```

```
 end;
 return(logl);
finish loglike;

x={-0.1 0.1 -0.1 0.1 -0.1 0.1 -0.1 0.1 1.01 0.01 1.01}; *Starting values for
parameters;
optn = {0 2 . 2}; *Options for optimisation procedure;
call nlpqn(rc,xr,"loglike",x,optn);

run;

proc varmax data=varma11;
model y1 y2 /p=1 q=1 method=ml noint lagmax=3;
run;

quit;
```

Examples 5.1, 5.2, 5.3

```
proc iml;

start
autocovcor(T,y,gamma0,gamma1,gamma2,gamma3,rho0,rho1,rho2,rho3,phi11,phi22);
      y=y`;
      mean=(1/T)#y[,+]*J(1,500,1);
      gamma0=(1/T)#(y-mean)*(y-mean)`;
      gamma1=(1/T)#(y[,2:T]-mean[,2:T])*(y[,1:T-1]-mean[,1:T-1])`;
      gamma2=(1/T)#(y[,3:T]-mean[,3:T])*(y[,1:T-2]-mean[,1:T-2])`;
      gamma3=(1/T)#(y[,4:T]-mean[,4:T])*(y[,1:T-3]-mean[,1:T-3])`;
      print gamma0,gamma1,gamma2,gamma3;

      vhalf=sqrt(diag(gamma0));
      rho0=inv(vhalf)*gamma0*inv(vhalf);
      rho1=inv(vhalf)*gamma1*inv(vhalf);
      rho2=inv(vhalf)*gamma2*inv(vhalf);
      rho3=inv(vhalf)*gamma3*inv(vhalf);
      print rho0,rho1,rho2,rho3;

      phi11=gamma1*inv(gamma0);
      phi22=(gamma2-gamma1*inv(gamma0)*gamma1)*inv(gamma0-
      gamma1`*inv(gamma0)*gamma1);
      print phi11, phi22;
finish autocovcor;

sig={1.0 0.5,0.5 0.9};
T=500;

phi={0.5 0.6,0.1 0.4};

call varmasim(y,phi) sigma=sig n=T seed=1;
cn={'y1' 'y2'};
create var1 from y[colname=cn];
append from y;

call
autocovcor(T,y,gamma0,gamma1,gamma2,gamma3,rho0,rho1,rho2,rho3,phi11,phi22);
```

```
theta1={0.2 0.1,0.1 0.4};
theta2={0.4 0,0.6 0.1};
theta12=theta1//theta2;
call varmasim(yy) theta=theta12 sigma=sig n=T seed=1;
cn={'y1' 'y2'};
create vma2 from yy[colname=cn];
append from yy;

call
autocovcor(T,yy,gamma0,gamma1,gamma2,gamma3,rho0,rho1,rho2,rho3,phi11,phi22)
;

phi1={-0.2 0.1,0.5 0.1};
phi2={0.8 0.5,-0.4 0.5};
theta1={0.2 0.1,0.1 0.4};
phi12=phi1//phi2;
call varmasim(yyy) phi=phi12 theta=theta1 sigma=sig n=T seed=1;
cn={'y1' 'y2'};
create varma21 from yyy[colname=cn];
append from yyy;

call
autocovcor(T,yyy,gamma0,gamma1,gamma2,gamma3,rho0,rho1,rho2,rho3,phi11,phi22
);

proc varmax data=var1;
model y1 y2 / noint lagmax=3 print=(covy(3)) print=(corry(3))
print=(parcoef);
run;

proc varmax data=vma2;
model y1 y2 / noint lagmax=3 print=(covy(3)) print=(corry(3))
print=(parcoef);
run;

proc varmax data=varma21 outstat=out21;
model y1 y2 /p=2 q=1 noint lagmax=3 method=ml print=(covy(3))
print=(corry(3)) print=(parcoef);
run;

proc iml;
use out21;
read all into out21;

T=500;
r21=12;
k=2;

siga21=out21[,1:2];
print siga21;

aic21=log(det(siga21))+2*r21/T;
aaic21=log(det(siga21))+(2*r21)/(T-r21/k);
fpe21=(((T+r21/k)/(T-r21/k))**k)*det(siga21);
hqc21=log(det(siga21))+2*r21*log(log(T))/T;
sbc21=log(det(siga21))+r21*log(T)/T;
print aic21 aaic21 fpe21 hqc21 sbc21;
run;

proc varmax data=varma21 outstat=out3;
model y1 y2 /p=3 noint lagmax=3;
run;
```

```
proc iml;
use out3;
read all into out3;

T=500;
r3=12;
k=2;
p=3;

siga3=out3[,1:2];
siga3=((t-k#p)/T)#siga3;
print siga3;

aic3=log(det(siga3))+2*r3/T;
aaic3=log(det(siga3))+(2*r3)/(T-r3/k);
fpe3=(((T+r3/k)/(T-r3/k))**k)*det(siga3);
hqc3=log(det(siga3))+2*r3*log(log(T))/T;
sbc3=log(det(siga3))+r3*log(T)/T;
print aic3 aaic3 fpe3 hqc3 sbc3;
run;

proc varmax data=varma21 outstat=out11;
model y1 y2 /p=1 q=1 noint lagmax=3;
run;

proc iml;
use out11;
read all into out11;

T=500;
r11=8;
k=2;

siga11=out11[,1:2];
print siga11;

aic11=log(det(siga11))+2*r11/T;
aaic11=log(det(siga11))+(2*r11)/(T-r11/k);
fpe11=(((T+r11/k)/(T-r11/k))**k)*det(siga11);
hqc11=log(det(siga11))+2*r11*log(log(T))/T;
sbc11=log(det(siga11))+r11*log(T)/T;
print aic11 aaic11 fpe11 hqc11 sbc11;
run;

proc varmax data=varma21;
model y1 y2 /noint minic=(p=4 q=4) lagmax=3;
run;
```

Hosking simulation

```
proc iml;

pp=J(1000,2,0);
do i=1 to 1000;

*******************************************;
* Generating the VAR(1) process           *;
*******************************************;
```

```
k=2;
t=1000;
sig={1.0 0.4,0.4 1};
phi1={0.9 0.1,-0.6 0.4};
call varmasim(yy) phi=phi1 sigma=sig n=t seed=0;

*****************************************************;
* Fit a VAR(1) model using method of least squares *;
*****************************************************;

y=yy`;

call varmasim(yyy,phi1) sigma=sig n=1 seed=0;

z=J(1,T,1)//(yyy`||y[,1:T-1]);

b=y*z`*inv(z*z`);
*print bb;

bz=b*z;
*print bz;

resid=y-bz;
*print resid;

************************************************;
* Portmanteau Statistic                        *;
************************************************;

ncolr=ncol(resid);

c0=(1/ncolr)#resid*resid`;
c1=(1/ncolr)#resid[,2:ncolr]*(resid[,1:ncolr-1])`;
c2=(1/ncolr)#resid[,3:ncolr]*(resid[,1:ncolr-2])`;
c3=(1/ncolr)#resid[,4:ncolr]*(resid[,1:ncolr-3])`;
c4=(1/ncolr)#resid[,5:ncolr]*(resid[,1:ncolr-4])`;
c5=(1/ncolr)#resid[,6:ncolr]*(resid[,1:ncolr-5])`;
c6=(1/ncolr)#resid[,7:ncolr]*(resid[,1:ncolr-6])`;
c7=(1/ncolr)#resid[,8:ncolr]*(resid[,1:ncolr-7])`;
c8=(1/ncolr)#resid[,9:ncolr]*(resid[,1:ncolr-8])`;
c9=(1/ncolr)#resid[,10:ncolr]*(resid[,1:ncolr-9])`;
c10=(1/ncolr)#resid[,11:ncolr]*(resid[,1:ncolr-10])`;
c11=(1/ncolr)#resid[,12:ncolr]*(resid[,1:ncolr-11])`;
c12=(1/ncolr)#resid[,13:ncolr]*(resid[,1:ncolr-12])`;
c13=(1/ncolr)#resid[,14:ncolr]*(resid[,1:ncolr-13])`;
c14=(1/ncolr)#resid[,15:ncolr]*(resid[,1:ncolr-14])`;
c15=(1/ncolr)#resid[,16:ncolr]*(resid[,1:ncolr-15])`;
c16=(1/ncolr)#resid[,17:ncolr]*(resid[,1:ncolr-16])`;
c17=(1/ncolr)#resid[,18:ncolr]*(resid[,1:ncolr-17])`;
c18=(1/ncolr)#resid[,19:ncolr]*(resid[,1:ncolr-18])`;
c19=(1/ncolr)#resid[,20:ncolr]*(resid[,1:ncolr-19])`;
c20=(1/ncolr)#resid[,21:ncolr]*(resid[,1:ncolr-20])`;

port=ncolr#(trace(c1`*inv(c0)*c1*inv(c0))+
      trace(c2`*inv(c0)*c2*inv(c0))+
      trace(c3`*inv(c0)*c3*inv(c0))+
      trace(c4`*inv(c0)*c4*inv(c0))+
      trace(c5`*inv(c0)*c5*inv(c0))+
      trace(c6`*inv(c0)*c6*inv(c0))+
      trace(c7`*inv(c0)*c7*inv(c0))+
      trace(c8`*inv(c0)*c8*inv(c0))+
```

```
       trace(c9`*inv(c0)*c9*inv(c0))+
       trace(c10`*inv(c0)*c10*inv(c0))+
       trace(c11`*inv(c0)*c11*inv(c0))+
       trace(c12`*inv(c0)*c12*inv(c0))+
       trace(c13`*inv(c0)*c13*inv(c0))+
       trace(c14`*inv(c0)*c14*inv(c0))+
       trace(c15`*inv(c0)*c15*inv(c0))+
       trace(c16`*inv(c0)*c16*inv(c0))+
       trace(c17`*inv(c0)*c17*inv(c0))+
       trace(c18`*inv(c0)*c18*inv(c0))+
       trace(c19`*inv(c0)*c19*inv(c0))+
       trace(c20`*inv(c0)*c20*inv(c0)));

portprime=(ncolr##2)#(1/(ncolr-1)#trace(c1`*inv(c0)*c1*inv(c0))+
       1/(ncolr-2)#trace(c2`*inv(c0)*c2*inv(c0))+
       1/(ncolr-3)#trace(c3`*inv(c0)*c3*inv(c0))+
       1/(ncolr-4)#trace(c4`*inv(c0)*c4*inv(c0))+
       1/(ncolr-5)#trace(c5`*inv(c0)*c5*inv(c0))+
       1/(ncolr-6)#trace(c6`*inv(c0)*c6*inv(c0))+
       1/(ncolr-7)#trace(c7`*inv(c0)*c7*inv(c0))+
       1/(ncolr-8)#trace(c8`*inv(c0)*c8*inv(c0))+
       1/(ncolr-9)#trace(c9`*inv(c0)*c9*inv(c0))+
       1/(ncolr-10)#trace(c10`*inv(c0)*c10*inv(c0))+
       1/(ncolr-11)#trace(c11`*inv(c0)*c11*inv(c0))+
       1/(ncolr-12)#trace(c12`*inv(c0)*c12*inv(c0))+
       1/(ncolr-13)#trace(c13`*inv(c0)*c13*inv(c0))+
       1/(ncolr-14)#trace(c14`*inv(c0)*c14*inv(c0))+
       1/(ncolr-15)#trace(c15`*inv(c0)*c15*inv(c0))+
       1/(ncolr-16)#trace(c16`*inv(c0)*c16*inv(c0))+
       1/(ncolr-17)#trace(c17`*inv(c0)*c17*inv(c0))+
       1/(ncolr-18)#trace(c18`*inv(c0)*c18*inv(c0))+
       1/(ncolr-19)#trace(c19`*inv(c0)*c19*inv(c0))+
       1/(ncolr-20)#trace(c20`*inv(c0)*c20*inv(c0)));

*print port portprime;
pp[i,]=port||portprime;
end;
*print pp;

col={'p' 'pprime'};
create hosking from pp[colname=col];
append from pp;
run;

proc univariate data=hosking;
var p pprime;
run;
```

Example 6.4.1 (Simulated Data)

```
proc iml;

********************************************;
* Generating the VAR(2) process           *;
********************************************;

t=500;
sig={1.0 0.5,0.5 0.9};
mean={30,25};
```

```
phi1={0.6 -0.8,0.2 0.3};
phi2={0.3 0.7,-0.6 -0.5};
phi12=phi1//phi2;
call varmasim(y) phi=phi12 mu=mean sigma=sig n=t seed=12;
cn={'y' 'x'};
create var2 from y[colname=cn];
append from y;
call vtsroot(root,phi12);
print root;
run;

*****************************************;
* Fit a VAR(1) model using proc varmax   *;
*****************************************;

proc varmax data=var2;
model y x / p=1 method=ls lagmax=3;
run;

*****************************************;
* Fit a VAR(2) model using proc varmax   *;
*****************************************;

proc varmax data=var2;
model y x / p=2 method=ls lagmax=3;
output out=forecast lead=0;
run;

**********************************************;
*  Multivariate Model Diagnostics             *;
**********************************************;

*********************************************;
* Residual Autocorrelation Matrices          *;
*********************************************;

proc iml;
t=500;
p=2;                 *var order;
q=0;                 *vma order;

use forecast;
read all into forecast;

resid=(forecast[p+1:t,3]||forecast[p+1:t,9]);
*print resid;

rmean=(J(nrow(resid),1,1)*((1/nrow(resid))#resid[+,]))`;
resid=resid`;
ncolr=ncol(resid);
*print rmean;

gamma0=(resid-rmean)*(resid-rmean)`;
gamma1=(resid[,2:ncolr]-rmean[,2:ncolr])*(resid[,1:ncolr-1]-
        rmean[,1:ncolr-1])`;
gamma2=(resid[,3:ncolr]-rmean[,3:ncolr])*(resid[,1:ncolr-2]-
        rmean[,1:ncolr-2])`;
gamma3=(resid[,4:ncolr]-rmean[,4:ncolr])*(resid[,1:ncolr-3]-
        rmean[,1:ncolr-3])`;
```

```
vhalf=sqrt(diag(gamma0));
rho0=inv(vhalf)*gamma0*inv(vhalf);
rho1=inv(vhalf)*gamma1*inv(vhalf);
rho2=inv(vhalf)*gamma2*inv(vhalf);
rho3=inv(vhalf)*gamma3*inv(vhalf);
print rho0,rho1,rho2,rho3;

**********************************************;
* Portmanteau Statistic                      *;
**********************************************;

k=2;                *dimension;
h=3;                *number of lags;

c0=(1/ncolr)#resid*resid`;
c1=(1/ncolr)#resid[,2:ncolr]*(resid[,1:ncolr-1])`;
c2=(1/ncolr)#resid[,3:ncolr]*(resid[,1:ncolr-2])`;
c3=(1/ncolr)#resid[,4:ncolr]*(resid[,1:ncolr-3])`;
*print c0 c1 c2 c3;

port=ncolr#(trace(c1`*inv(c0)*c1*inv(c0))+trace(c2`*inv(c0)*c2*inv(c0))+
     trace(c3`*inv(c0)*c3*inv(c0)));
portprime=(ncolr##2)#(1/(ncolr-1)#trace(c1`*inv(c0)*c1*inv(c0))+
           1/(ncolr-2)#trace(c2`*inv(c0)*c2*inv(c0))+
           1/(ncolr-3)#trace(c3`*inv(c0)*c3*inv(c0)));
critpp=1-probchi(portprime,(k##2)#(h-p-q));
print port portprime critpp;
run;

**********************************************;
*  Univariate Model Diagnostics               ;
**********************************************;

**********************************************;
* R Square and F                             *;
**********************************************;

parm=2*p;           *parameters of individual equation;

average=(1/t#forecast[+,1]*J(t-p,1,1))||(1/t#forecast[+,7]*J(t-p,1,1));
*print average;

ssr1=((forecast[p+1:t,2]-average[,1])##2)||((forecast[p+1:t,8]-
      average[,2])##2);
ssr=ssr1[+,];
*print ssr;

sst1=((forecast[p+1:t,1]-average[,1])##2)||((forecast[p+1:t,7]-
      average[,2])##2);
sst=sst1[+,];
*print sst;

sse1=((forecast[p+1:t,3])##2)||((forecast[p+1:t,9])##2);
sse=sse1[+,];
*print sse;

rsquare=ssr/sst;
print rsquare;

f=(ssr/parm)/(sse/(t-p-parm-1));
critf=1-probf(f,parm,t-p-parm-1);
print f critf;
```

```
*********************************************;
* Durbin Watson                             *;
*********************************************;

residual=(forecast[,3]||forecast[,9]);

d1=(residual[2:t,]-residual[1:t-1,])##2;
dbo=d1[+,];
d2=residual[2:t,]##2;
dond=d2[+,];
d=dbo/dond;
print d;

*********************************************;
* Jarqu-Bera                                *;
*********************************************;

m22=(resid-rmean)##2;
m2=(1/ncolr)#m22[,+];

m33=(resid-rmean)##3;
m3=(1/ncolr)#m33[,+];

m44=(resid-rmean)##4;
m4=(1/ncolr)#m44[,+];

s=m3/(m2##(3/2));
k=m4/(m2##2)-3;
jb=ncolr#((s##2)/6+(k##2)/24);
critjb=1-probchi(jb,2);
*print s k;
print jb critjb;

*********************************************;
* ARCH                                      *;
*********************************************;

at=residual;
at1=J(1,2,.)//residual[1:t-1,];
atat=at##2;
at1at1=at1##2;
archreg=atat||at1at1;
*print atat at1at1 archreg;
col={'yat' 'xat' 'yat1' 'xat1'};
create archreg from archreg[colname=col];
append from archreg;

*********************************************;
* AR(1) - AR(4)                             *;
*********************************************;

a=residual;
a1=J(1,2,.)//a[1:t-1,];
a2=J(2,2,.)//a[1:t-2,];
a3=J(3,2,.)//a[1:t-3,];
a4=J(4,2,.)//a[1:t-4,];
ardist=a||a1||a2||a3||a4;
*print ar14;
col={'ya' 'xa' 'ya1' 'xa1' 'ya2' 'xa2' 'ya3' 'xa3' 'ya4' 'xa4'};
create ardist from ardist[colname=col];
append from ardist;
```

```
proc reg data=ardist;
 model ya=ya1;

proc reg data=ardist;
 model ya=ya1 ya2;

proc reg data=ardist;
 model ya=ya1 ya2 ya3;

proc reg data=ardist;
 model ya=ya1 ya2 ya3 ya4;
run;

proc reg data=ardist;
 model xa=xa1;

proc reg data=ardist;
 model xa=xa1 xa2;

proc reg data=ardist;
 model xa=xa1 xa2 xa3;

proc reg data=ardist;
 model xa=xa1 xa2 xa3 xa4;
run;

***********************************************;
* ARCH                                        *;
***********************************************;

proc reg data=archreg;
 model yat=yat1;
run;

proc reg data=archreg;
 model xat=xat1;
run;
```

Example 6.4.2 (Temperature Data)

```
data a; *jan1999-des2005;
input t yt xt;
cards;
1      25.94868712 12.92476356
2      27.81361957 12.82533716
3      27.8724688  13.51923671
4      27.59007817 13.45073375
5      26.08907616 15.05343873
6      26.62787737 11.57998624
7      26.37219889 16.87041318
8      26.04728595 11.60759142
9      24.49451792 11.26737696
10     25.06128845 11.37850999
11     27.36336071 13.57102939
12     25.50606834 12.91000835
13     23.36273548 11.45762824
14     23.74994139 12.61508573
15     24.76506518 13.21198133
16     23.80921561 11.89122839
```

17	24.16910086	10.24914977
18	25.27807714	15.85954637
19	25.74128504	10.57975064
20	26.8133826	12.54874748
21	25.08004424	12.7416132
22	25.51694824	12.88448925
23	24.44460224	12.10597508
24	25.77740949	12.83665603
25	28.26712652	12.99462715
26	25.2851087	12.54500192
27	26.55417635	12.98153979
28	26.261667	13.93807917
29	26.31495561	12.97158018
30	26.75058649	14.09737455
31	25.61510227	15.15477794
32	27.36059449	13.80362223
33	25.27521968	12.21510026
34	25.42581628	13.47014786
35	23.98854623	12.72284005
36	25.32517424	12.61659907
37	26.84039458	12.99462715
38	25.82693246	12.33475049
39	27.21332258	12.75109826
40	27.4878927	12.96338832
41	26.31495561	11.0498646
42	23.56014956	14.60085221
43	25.61510227	9.435993811
44	25.17174693	15.37221567
45	24.8848688	12.00449508
46	25.97260803	11.88050308
47	26.45124869	11.64332635
48	25.50606834	13.05671299
49	26.92956532	12.7151728
50	26.72997205	13.31592383
51	27.49581381	12.52065672
52	28.81630387	14.13301734
53	26.31495561	12.97158018
54	24.54182246	15.10432988
55	26.62456443	10.86568984
56	23.74899602	9.254701269
57	26.64144778	13.05752097
58	27.15732348	13.63747888
59	26.45124869	13.64813751
60	29.30484448	13.71688387
61	26.03785786	13.13435432
62	25.37541266	12.75525335
63	24.76506518	12.98153979
64	25.54636868	13.45073375
65	26.99259395	13.45200908
66	23.9282769	11.07650858
67	24.73182288	10.86568984
68	27.14170973	14.43105961
69	25.3728074	11.05677179
70	26.88392761	12.80082374
71	28.54910634	13.64813751
72	25.59651539	12.76330371
73	25.32449189	12.43571845
74	26.36875622	12.4048343
75	24.85922893	11.59889059
76	23.70703013	12.08616656
77	27.44435285	13.77229501

```
78    29.08205962 15.85954637
79    29.40058537 14.01102111
80    27.798364   15.37221567
81    29.17872852 14.42645462
82    28.06864306 13.21915131
83    26.9073047  12.79994817
84    26.77232705 12.32318979
;
/*
proc print data=a;
run;
*/

goptions reset=all i=join;
proc gplot data=a;
plot (yt xt)*t / overlay;
run;

***********************************;
* Multivariate time series model *;
***********************************;

proc varmax data=a;
model yt xt /lagmax=6 print=(covy(6)) print=(corry(6)) print=(parcoef(6))
             minic=(p=4 q=4) dftest;
run;

****************************************;
* Univariate time series model for yt *;
****************************************;

proc arima data=a out=b;
identify var=yt;
estimate p=1;
forecast lead=0;
quit;
run;

proc iml;
use b;
read all into forecasty;

averagey=(1/84#forecasty[+,1]*J(84,1,1));
*print averagey;

ssr1y=((forecasty[,2]-averagey)##2);
ssry=ssr1y[+,];
*print ssry;

sst1y=((forecasty[,1]-averagey)##2);
ssty=sst1y[+,];
*print ssty;

rsquarey=ssry/ssty;
print rsquarey;

****************************************;
* Univariate time series model for xt *;
****************************************;

proc arima data=a out=c;
```

```
identify var=xt;
estimate p=1;
forecast lead=0;
quit;
run;

proc iml;
use c;
read all into forecastx;

averagex=(1/84#forecastx[+,1]*J(84,1,1));
*print averagex;

ssr1x=((forecastx[,2]-averagex)##2);
ssrx=ssr1x[+,];
*print ssrx;

sst1x=((forecastx[,1]-averagex)##2);
sstx=sst1x[+,];
*print sstx;

rsquarex=ssrx/sstx;
print rsquarex;
run;
```

Example 6.4.3 (Electricity Data)

```
data a;
infile 'C:\electricity.txt';
input zt;
*proc print data=a;
run;


proc iml;
use a;
read all into zt;
*print zt;
b=shape(zt,3458,24);
*print b;
c=b[,+];
*print c;
d=shape(c,494,7);
*print d;
dd=d[51:151,];
e=dd[,+]/7;
*print e;
cn={'mon' 'tue' 'wed' 'thu' 'fri' 'sat' 'sun'};
create b from dd[colname=cn];
append from dd;
quit;
run;

proc varmax data=b;
model mon tue wed thu fri sat sun /p=1 print=(corry(10)) print=(parcoef(10))
                                   minic=(p=4 q=4) dftest;
run;
```

SAS output for the electricity data

The VARMAX Procedure

Number of Observations	101
Number of Pairwise Missing	0

Simple Summary Statistics

Variable	Type	N	Mean	Standard Deviation	Min	Max
mon	Dependent	101	506107.22772	27801.09331	413462.00000	570960.00000
tue	Dependent	101	519399.70297	25886.24454	414017.00000	584331.00000
wed	Dependent	101	521400.14851	30524.71220	365376.00000	578179.00000
thu	Dependent	101	519856.47525	33250.20021	367024.00000	580381.00000
fri	Dependent	101	514810.88119	32898.64547	382226.00000	572089.00000
sat	Dependent	101	480481.64356	25001.34177	391812.00000	526765.00000
sun	Dependent	101	450104.31683	21146.11114	390919.00000	502992.00000

Cross Correlations of Dependent Series

Lag	Variable	mon	tue	wed	thu	fri	sat	sun
0	mon	1.00000	0.82870	0.74060	0.73000	0.68002	0.67515	0.69176
	tue	0.82870	1.00000	0.93110	0.88044	0.79670	0.81337	0.82452
	wed	0.74060	0.93110	1.00000	0.93229	0.82275	0.82877	0.81091
	thu	0.73000	0.88044	0.93229	1.00000	0.89639	0.87123	0.81904
	fri	0.68002	0.79670	0.82275	0.89639	1.00000	0.96147	0.88871
	sat	0.67515	0.81337	0.82877	0.87123	0.96147	1.00000	0.95346
	sun	0.69176	0.82452	0.81091	0.81904	0.88871	0.95346	1.00000
1	mon	0.52633	0.56355	0.54639	0.50333	0.47835	0.52091	0.55937
	tue	0.65060	0.71417	0.72278	0.68567	0.64404	0.67381	0.71087
	wed	0.63268	0.70101	0.70808	0.63985	0.58013	0.60570	0.66288
	thu	0.64872	0.69729	0.68762	0.64983	0.57696	0.59659	0.64629
	fri	0.75922	0.70250	0.64937	0.63297	0.59030	0.61755	0.64606
	sat	0.79386	0.73928	0.68412	0.65566	0.61282	0.65766	0.70378
	sun	0.83647	0.79010	0.71914	0.68720	0.65802	0.71209	0.77082
2	mon	0.46317	0.44027	0.37018	0.37706	0.43444	0.45739	0.48630
	tue	0.56638	0.59658	0.50145	0.51281	0.50716	0.55549	0.61423
	wed	0.52220	0.56580	0.47000	0.46318	0.45616	0.51340	0.58336
	thu	0.51680	0.56705	0.47025	0.45370	0.43792	0.49250	0.56392
	fri	0.52829	0.53648	0.44774	0.43554	0.41612	0.46703	0.54207
	sat	0.54053	0.57060	0.48574	0.46468	0.45487	0.51712	0.59905
	sun	0.56804	0.59936	0.52721	0.49885	0.51982	0.58851	0.66618

Lag	Variable	mon	tue	wed	thu	fri	sat	sun
3	mon	0.33750	0.39269	0.32925	0.31466	0.31683	0.39422	0.43475
	tue	0.44138	0.50400	0.42488	0.40759	0.40360	0.48501	0.55736
	wed	0.40781	0.47192	0.40161	0.37732	0.37838	0.45513	0.52998
	thu	0.38890	0.45013	0.38319	0.36389	0.36414	0.42220	0.47726
	fri	0.41392	0.44265	0.37019	0.36328	0.40973	0.44196	0.48699
	sat	0.45234	0.48375	0.40631	0.40190	0.47685	0.52421	0.57818
	sun	0.50603	0.52626	0.44153	0.44409	0.51017	0.57194	0.63240
4	mon	0.32804	0.28768	0.24887	0.26212	0.22470	0.26065	0.30453
	tue	0.41627	0.42970	0.36703	0.34809	0.34917	0.41324	0.47993
	wed	0.38488	0.41571	0.34866	0.31095	0.32277	0.40171	0.47799
	thu	0.35664	0.36546	0.29396	0.25708	0.26469	0.34633	0.41377
	fri	0.35147	0.37340	0.30584	0.26364	0.26710	0.35373	0.41093
	sat	0.42682	0.43065	0.35687	0.32321	0.33749	0.43176	0.48971
	sun	0.45650	0.48041	0.40328	0.37626	0.39661	0.49545	0.55533
5	mon	0.22575	0.22133	0.19159	0.17882	0.17990	0.20246	0.23142
	tue	0.34934	0.35006	0.31188	0.27422	0.28184	0.34567	0.39569
	wed	0.34974	0.35053	0.30866	0.27811	0.28704	0.35938	0.41087
	thu	0.31344	0.30942	0.26704	0.24402	0.23974	0.30735	0.34426
	fri	0.32792	0.28174	0.24710	0.26803	0.23902	0.28417	0.31846
	sat	0.37568	0.31571	0.27873	0.29883	0.29398	0.34805	0.39339
	sun	0.42641	0.38983	0.33396	0.33628	0.35365	0.40782	0.46732
6	mon	0.14755	0.15583	0.11843	0.13519	0.13183	0.17243	0.20156
	tue	0.28322	0.28435	0.23169	0.22274	0.23322	0.29408	0.32652
	wed	0.28326	0.28585	0.23536	0.22267	0.23056	0.29436	0.33785
	thu	0.23932	0.24291	0.20391	0.19047	0.18971	0.24487	0.27802
	fri	0.22788	0.19806	0.16908	0.17839	0.16625	0.21177	0.24090
	sat	0.28840	0.25106	0.20255	0.19335	0.22806	0.27499	0.31340
	sun	0.35354	0.32187	0.26484	0.25204	0.28763	0.33659	0.37915
7	mon	0.16524	0.15712	0.06588	0.06420	0.09287	0.13187	0.10076
	tue	0.23713	0.22532	0.15569	0.16334	0.20233	0.24575	0.26460
	wed	0.24508	0.23808	0.17531	0.18045	0.20902	0.25442	0.28367
	thu	0.23772	0.21594	0.14583	0.14311	0.17133	0.21501	0.24442
	fri	0.19819	0.22699	0.13444	0.11334	0.13494	0.17481	0.20453
	sat	0.24268	0.24605	0.17012	0.14016	0.17758	0.22810	0.25935
	sun	0.29265	0.28284	0.20512	0.20207	0.25221	0.30425	0.33263

Lag	Variable	mon	tue	wed	thu	fri	sat	sun
8	mon	0.03161	0.00438	-0.01857	-0.00417	0.02708	0.06532	0.07115
	tue	0.21202	0.16768	0.09940	0.09206	0.13469	0.17758	0.20178
	wed	0.18669	0.19068	0.13809	0.12981	0.16812	0.22315	0.25704
	thu	0.14011	0.15145	0.11083	0.10583	0.13643	0.18727	0.20166
	fri	0.10875	0.12472	0.09102	0.09107	0.10937	0.13399	0.11843
	sat	0.16617	0.15974	0.10429	0.11121	0.15035	0.17937	0.16786
	sun	0.25718	0.21827	0.13762	0.15726	0.20680	0.24095	0.22806
9	mon	0.02718	0.06770	-0.00932	-0.00906	0.01568	0.03053	0.01827
	tue	0.13433	0.14336	0.09334	0.09376	0.14630	0.18621	0.17667
	wed	0.16657	0.16031	0.11586	0.11023	0.15349	0.20834	0.22525
	thu	0.09827	0.09030	0.06873	0.05614	0.10035	0.15385	0.15483
	fri	0.03720	0.01444	-0.00813	-0.01263	0.02121	0.05941	0.04865
	sat	0.08822	0.04509	0.01172	0.02554	0.07349	0.11404	0.10112
	sun	0.13765	0.10806	0.05990	0.08070	0.13613	0.17784	0.16673
10	mon	-0.02571	-0.05058	-0.08768	-0.05901	0.00019	0.02065	0.02182
	tue	0.09870	0.06680	0.01792	0.04384	0.10803	0.13976	0.14353
	wed	0.13996	0.12831	0.07429	0.08748	0.12991	0.15703	0.17946
	thu	0.08713	0.08131	0.01866	0.02277	0.07169	0.10012	0.12071
	fri	0.03523	0.05682	-0.03996	-0.04042	0.00389	0.02226	0.03124
	sat	0.07662	0.07964	-0.00702	0.00090	0.04978	0.06870	0.07062
	sun	0.12560	0.12094	0.03798	0.05573	0.10437	0.11754	0.10865

Schematic Representation of Cross Correlations

Variable/ Lag	0	1	2	3	4	5	6	7	8	9	10
mon	+++++++	+++++++	+++++++	+++++++	+++++++	++...++	......+	....	....	....	....
tue	+++++++	+++++++	+++++++	+++++++	+++++++	+++++++	+++++++	++..+++	+.....+	....	....
wed	+++++++	+++++++	+++++++	+++++++	+++++++	+++++++	+++++++	++..+++	.....++	.....++	....
thu	+++++++	+++++++	+++++++	+++++++	+++++++	+++++++	+++..++	++...++	......+	....	....
fri	+++++++	+++++++	+++++++	+++++++	+++++++	+++++++	+....++	.+....+	....	....	....
sat	+++++++	+++++++	+++++++	+++++++	+++++++	+++++++	+++.+++	++...++	....	....	....
sun	+++++++	+++++++	+++++++	+++++++	+++++++	+++++++	+++++++	+++++++	++..+++	....	....

+ is > 2*std error, - is < -2*std error, . is between

Minimum Information Criterion

Lag	MA 0	MA 1	MA 2	MA 3	MA 4
AR 0	129.79648	130.26498	130.92364	132.02659	133.24856
AR 1	128.1816	129.24413	130.16247	131.50464	133.17751
AR 2	128.79886	130.0952	131.28796	132.95227	134.95407
AR 3	129.83109	131.33926	132.87816	135.21806	137.65251
AR 4	131.10309	132.84838	134.71381	137.508	141.46216

Partial Autoregression

Lag	Variable	mon	tue	wed	thu	fri	sat	sun
1	mon	-0.10753	0.08080	-0.05934	-0.16000	0.45173	-0.54001	1.37564
	tue	-0.11389	0.21218	0.00155	0.11995	0.10195	-0.53210	1.15920
	wed	-0.19207	0.50110	0.04507	0.16910	-0.05629	-0.30361	0.85653
	thu	-0.31894	0.90499	-0.40180	0.30207	0.12298	-0.51533	0.94936
	fri	-0.29639	0.98006	-0.39578	0.05358	0.31608	-0.75560	1.11308
	sat	-0.16408	0.67432	-0.28248	0.00872	0.10760	-0.39019	0.92102
	sun	-0.09849	0.38089	-0.12700	0.07419	-0.08001	-0.26029	0.93311
2	mon	0.05479	0.08251	-0.02623	-0.11404	0.35050	-0.01985	-0.67117
	tue	-0.08260	0.21581	-0.13648	0.09565	-0.01104	0.23372	-0.50699
	wed	-0.05670	0.14571	-0.20820	0.06684	-0.04960	0.19526	-0.51856
	thu	-0.09037	0.48166	-0.18845	-0.10186	0.01738	0.32495	-0.79887
	fri	0.11873	0.21018	-0.12289	-0.08504	-0.04093	-0.06811	0.05445
	sat	0.03223	0.13313	-0.04586	0.00396	-0.10499	-0.05986	0.18888
	sun	-0.02007	0.08020	-0.01430	-0.00442	-0.04071	-0.06360	0.29144
3	mon	-0.04477	0.07579	-0.03870	-0.20730	0.16346	0.01153	-0.13257
	tue	0.05116	0.01217	0.08504	-0.18405	0.03232	0.03058	0.12769
	wed	0.09786	-0.07628	0.15653	-0.20074	0.02343	-0.04745	0.22477
	thu	0.09393	-0.09418	-0.00570	-0.19112	0.01330	-0.14839	0.51227
	fri	0.08571	-0.11893	-0.03639	-0.39235	-0.11844	0.64777	0.16452
	sat	0.10965	-0.08202	0.04925	-0.30923	-0.22624	0.55183	0.22602
	sun	0.07410	-0.04869	0.16159	-0.32722	-0.19005	0.47248	0.24553
4	mon	-0.04533	0.16164	-0.23133	0.36552	-0.18083	0.11486	-0.30280
	tue	-0.13758	0.02836	0.02727	-0.00949	0.20779	-0.42978	0.19521
	wed	-0.12483	0.06871	0.01210	-0.12426	0.19435	-0.30296	0.07383
	thu	0.00674	0.10279	-0.01796	-0.16578	0.02365	-0.11752	-0.12860
	fri	-0.14988	0.08864	0.11708	0.01981	-0.21326	-0.12762	-0.03367
	sat	-0.17459	-0.00329	0.10495	0.03653	-0.07727	-0.21189	0.20493
	sun	-0.18748	0.00768	0.12055	-0.01055	0.04020	-0.20461	0.12423

Lag	Variable	mon	tue	wed	thu	fri	sat	sun
5	mon	0.10721	-0.02117	-0.05178	-0.06550	0.11983	-0.12847	0.17597
	tue	-0.00293	-0.04184	-0.06846	0.06300	0.27266	-0.80512	0.58823
	wed	-0.01613	0.14084	-0.14557	0.12873	0.12642	-0.44738	0.15222
	thu	0.02965	0.00554	-0.13020	-0.14154	0.23077	-0.00570	-0.17489
	fri	0.00537	-0.04949	-0.19129	-0.09901	0.00308	0.09692	0.27832
	sat	-0.06038	-0.02404	-0.04140	-0.06304	-0.05189	0.15360	0.08876
	sun	-0.08118	-0.02514	0.05007	-0.12716	-0.00900	-0.01820	0.31045
6	mon	-0.01384	0.04260	-0.17987	0.14698	0.12905	-0.26112	0.11032
	tue	-0.01777	0.18962	-0.21482	0.19184	-0.06256	-0.01139	-0.18662
	wed	-0.02230	0.20776	-0.26567	0.11663	-0.01165	0.06466	-0.33172
	thu	0.06215	0.25412	-0.27552	0.24561	0.15206	-0.26576	-0.64238
	fri	-0.01420	0.26701	-0.24279	0.27360	-0.30776	0.52550	-0.77587
	sat	-0.05655	0.18603	-0.13783	0.12871	-0.08970	0.26138	-0.43535
	sun	-0.02874	0.01232	-0.05275	0.06785	-0.01856	0.14785	-0.21496
7	mon	0.03548	-0.02892	-0.18302	0.20178	0.22287	-0.34906	0.14331
	tue	0.09912	-0.05539	-0.06851	-0.26773	0.62207	-0.51171	0.20338
	wed	0.01896	-0.07679	-0.02456	-0.16488	0.09318	0.14910	-0.07187
	thu	-0.08513	0.06297	-0.10277	-0.17071	0.27247	-0.39171	0.29107
	fri	-0.10632	0.14604	-0.14334	-0.18573	0.40791	-0.78656	0.66632
	sat	-0.02742	0.04311	-0.11682	-0.04173	0.22054	-0.63129	0.59166
	sun	-0.17401	0.14229	-0.10883	0.03079	0.16739	-0.49094	0.38788
8	mon	-0.14748	0.42232	0.02503	-0.19122	-0.15441	0.01812	0.07140
	tue	0.00210	-0.04385	0.04079	0.18680	-0.38168	0.25976	-0.22087
	wed	-0.03166	-0.16601	0.10992	0.23095	-0.26537	0.13102	-0.39362
	thu	0.12225	-0.37397	0.05117	0.28252	-0.60444	0.47977	-0.14368
	fri	0.04778	-0.20065	-0.11406	0.40073	-0.94446	0.80901	-0.13886
	sat	0.03537	-0.27587	0.00735	0.39532	-0.82827	0.69545	-0.18314
	sun	0.05096	-0.22501	0.09367	0.32206	-0.58962	0.45178	-0.19822
9	mon	-0.05485	0.21714	-0.08481	-0.41050	0.38555	-0.24030	0.06835
	tue	0.00501	0.22327	-0.11244	-0.18069	0.18297	-0.26877	0.08115
	wed	-0.10969	0.44468	-0.26248	0.00967	0.15502	-0.34556	0.04348
	thu	-0.13920	0.51652	-0.16673	0.00007	-0.22492	0.02240	-0.07740
	fri	-0.11650	0.44993	-0.18906	0.00582	-0.25750	-0.12808	0.22466
	sat	-0.15749	0.26486	-0.11182	0.25444	-0.32662	-0.11506	0.18842
	sun	-0.08814	0.02199	0.00770	0.16319	-0.20552	-0.22333	0.24303

Lag	Variable	mon	tue	wed	thu	fri	sat	sun
10	mon	0.05337	-0.12988	-0.04327	-0.22895	-0.07349	0.72667	-0.30916
	tue	-0.21895	0.11910	-0.01910	-0.07634	0.41431	-0.22583	-0.02396
	wed	-0.29749	0.01379	0.10097	-0.03438	-0.02008	0.13427	0.09661
	thu	-0.42908	0.32909	0.00103	-0.15419	0.01986	0.17456	0.03504
	fri	-0.51918	0.78093	-0.31171	-0.07206	-0.39349	0.95529	-0.65809
	sat	-0.34670	0.67498	-0.27045	0.04169	-0.24690	0.58138	-0.52651
	sun	-0.21125	0.43646	-0.12754	0.12845	-0.14196	0.29934	-0.45520

Schematic Representation of Partial Autoregression

Variable/ Lag	1	2	3	4	5	6	7	8	9	10
mon	....+.+	....	....	....	....	....	....	....	....	....
tue	......+	....	....	....	....	....	....	....	....	....
wed	....	....	....	....	....	....	....	....	....	....
thu	-+.....	....	....	....	....	....	....	....	....	....
fri	.+....+	....	....	....	....	....	....	....	....	....
sat	.+....+	....	....	....	....	....	....	....	....	....
sun	.+....+	....	...-...	....	....	....	....	....	....	....

+ is > 2*std error, - is < -2*std error, . is between

Type of Model	VAR(1)
Estimation Method	Least Squares Estimation

Constant Estimates

Variable	Constant
mon	38813.35385
tue	84220.67472
wed	35624.61758
thu	20086.51410
fri	33066.53330
sat	72890.60087
sun	75048.24356

AR Coefficient Estimates

Lag	Variable	mon	tue	wed	thu	fri	sat	sun
1	mon	-0.10655	0.07896	-0.05943	-0.15808	0.44015	-0.51142	1.36202
	tue	-0.11306	0.21063	0.00147	0.12157	0.09216	-0.50793	1.14769
	wed	-0.19112	0.49932	0.04498	0.17095	-0.06748	-0.27598	0.84337
	thu	-0.31803	0.90328	-0.40189	0.30385	0.11221	-0.48875	0.93670
	fri	-0.29565	0.97867	-0.39585	0.05503	0.30732	-0.73397	1.10278
	sat	-0.16340	0.67306	-0.28254	0.01003	0.09965	-0.37056	0.91167
	sun	-0.09781	0.37961	-0.12707	0.07551	-0.08803	-0.24048	0.92367

Schematic Representation of Parameter Estimates

Variable/ Lag	C	AR1
mon	.	....+.+
tue	+	......+
wed	.	-+....+
thu	.	-+....+
fri	.	-+....+
sat	+	.+....+
sun	+	.+....+

+ is > 2*std error, - is < -2*std error, . is between, * is N/A

Model Parameter Estimates

Equation	Parameter	Estimate	Standard Error	t Value	Pr > |t|	Variable
mon	CONST1	38813.35385	33042.53116	1.17	0.2432	1
	AR1_1_1	-0.10655	0.08944	-1.19	0.2366	mon(t-1)
	AR1_1_2	0.07896	0.18101	0.44	0.6637	tue(t-1)
	AR1_1_3	-0.05943	0.16511	-0.36	0.7197	wed(t-1)
	AR1_1_4	-0.15808	0.14838	-1.07	0.2895	thu(t-1)
	AR1_1_5	0.44015	0.18202	2.42	0.0176	fri(t-1)
	AR1_1_6	-0.51142	0.32420	-1.58	0.1181	sat(t-1)
	AR1_1_7	1.36202	0.24345	5.59	0.0001	sun(t-1)

Equation	Parameter	Estimate	Standard Error	t Value	Pr > |t|	Variable
tue	CONST2	84220.67472	28057.38223	3.00	0.0035	1
	AR1_2_1	-0.11306	0.07595	-1.49	0.1400	mon(t-1)
	AR1_2_2	0.21063	0.15370	1.37	0.1739	tue(t-1)
	AR1_2_3	0.00147	0.14020	0.01	0.9917	wed(t-1)
	AR1_2_4	0.12157	0.12599	0.96	0.3371	thu(t-1)
	AR1_2_5	0.09216	0.15456	0.60	0.5525	fri(t-1)
	AR1_2_6	-0.50793	0.27529	-1.85	0.0682	sat(t-1)
	AR1_2_7	1.14769	0.20672	5.55	0.0001	sun(t-1)
wed	CONST3	35624.61758	29893.70433	1.19	0.2364	1
	AR1_3_1	-0.19112	0.08092	-2.36	0.0203	mon(t-1)
	AR1_3_2	0.49932	0.16376	3.05	0.0030	tue(t-1)
	AR1_3_3	0.04498	0.14937	0.30	0.7640	wed(t-1)
	AR1_3_4	0.17095	0.13424	1.27	0.2061	thu(t-1)
	AR1_3_5	-0.06748	0.16468	-0.41	0.6829	fri(t-1)
	AR1_3_6	-0.27598	0.29331	-0.94	0.3492	sat(t-1)
	AR1_3_7	0.84337	0.22025	3.83	0.0002	sun(t-1)
thu	CONST4	20086.51410	41434.19250	0.48	0.6290	1
	AR1_4_1	-0.31803	0.11216	-2.84	0.0056	mon(t-1)
	AR1_4_2	0.90328	0.22698	3.98	0.0001	tue(t-1)
	AR1_4_3	-0.40189	0.20704	-1.94	0.0553	wed(t-1)
	AR1_4_4	0.30385	0.18606	1.63	0.1059	thu(t-1)
	AR1_4_5	0.11221	0.22825	0.49	0.6242	fri(t-1)
	AR1_4_6	-0.48875	0.40654	-1.20	0.2324	sat(t-1)
	AR1_4_7	0.93670	0.30528	3.07	0.0028	sun(t-1)
fri	CONST5	33066.53330	50914.32299	0.65	0.5177	1
	AR1_5_1	-0.29565	0.13782	-2.15	0.0346	mon(t-1)
	AR1_5_2	0.97867	0.27892	3.51	0.0007	tue(t-1)
	AR1_5_3	-0.39585	0.25441	-1.56	0.1232	wed(t-1)
	AR1_5_4	0.05503	0.22863	0.24	0.8103	thu(t-1)
	AR1_5_5	0.30732	0.28048	1.10	0.2761	fri(t-1)
	AR1_5_6	-0.73397	0.49956	-1.47	0.1452	sat(t-1)
	AR1_5_7	1.10278	0.37513	2.94	0.0042	sun(t-1)
sat	CONST6	72890.60087	35688.45966	2.04	0.0440	1
	AR1_6_1	-0.16340	0.09661	-1.69	0.0941	mon(t-1)
	AR1_6_2	0.67306	0.19551	3.44	0.0009	tue(t-1)
	AR1_6_3	-0.28254	0.17833	-1.58	0.1165	wed(t-1)
	AR1_6_4	0.01003	0.16026	0.06	0.9502	thu(t-1)
	AR1_6_5	0.09965	0.19660	0.51	0.6135	fri(t-1)
	AR1_6_6	-0.37056	0.35016	-1.06	0.2927	sat(t-1)
	AR1_6_7	0.91167	0.26294	3.47	0.0008	sun(t-1)

Equation	Parameter	Estimate	Standard Error	t Value	Pr > |t|	Variable
sun	CONST7	75048.24356	28763.57025	2.61	0.0106	1
	AR1_7_1	-0.09781	0.07786	-1.26	0.2122	mon(t-1)
	AR1_7_2	0.37961	0.15757	2.41	0.0180	tue(t-1)
	AR1_7_3	-0.12707	0.14373	-0.88	0.3790	wed(t-1)
	AR1_7_4	0.07551	0.12916	0.58	0.5602	thu(t-1)
	AR1_7_5	-0.08803	0.15845	-0.56	0.5798	fri(t-1)
	AR1_7_6	-0.24048	0.28222	-0.85	0.3964	sat(t-1)
	AR1_7_7	0.92367	0.21192	4.36	0.0001	sun(t-1)

Covariances of Innovations

Variable	mon	tue	wed	thu	fri	sat	sun
mon	182181535.88	55221956.951	26971672.810	46441491.140	35668438.122	-7202663.566	-9428262.610
tue	55221956.951	131356642.69	96207471.107	101990962.07	87669360.115	54528491.684	43199213.615
wed	26971672.810	96207471.107	149113585.11	149423884.73	121207604.86	74979050.658	48029383.771
thu	46441491.140	101990962.07	149423884.73	286467594.67	249768934.94	146845991.89	87098640.222
fri	35668438.122	87669360.115	121207604.86	249768934.94	432551367.46	273068857.16	177315755.52
sat	-7202663.566	54528491.684	74979050.658	146845991.89	273068857.16	212526627.38	150162911.19
sun	-9428262.610	43199213.615	48029383.771	87098640.222	177315755.52	150162911.19	138052197.94

Information Criteria

AICC	128.7481
HQC	129.2411
AIC	128.6507
SBC	130.1096
FPEC	7.47E55

Cross Covariances of Residuals

Lag	Variable	mon	tue	wed	thu	fri	sat	sun
0	mon	167607013.01	50804200.395	24813938.986	42726171.849	32814963.072	-6626450.481	-8674001.602
	tue	50804200.395	120848111.27	88510873.419	93831685.104	80655811.306	50166212.349	39743276.526
	wed	24813938.986	88510873.419	137184498.31	137469973.96	111510996.47	68980726.606	44187033.069
	thu	42726171.849	93831685.104	137469973.96	263550187.10	229787420.14	135098312.54	80130749.004
	fri	32814963.072	80655811.306	111510996.47	229787420.14	397947258.06	251223348.59	163130495.07
	sat	-6626450.481	50166212.349	68980726.606	135098312.54	251223348.59	195524497.19	138149878.29
	sun	-8674001.602	39743276.526	44187033.069	80130749.004	163130495.07	138149878.29	127008022.10
1	mon	9436925.5743	9134215.9620	18724242.286	37016717.414	11034928.843	802131.60167	-12717027.81
	tue	7783047.4265	2509468.8893	6341562.2490	33458067.051	9230289.3927	-2367688.920	-10306155.18
	wed	1267642.2904	-6125378.363	-4935986.814	29567981.543	11761363.646	-2205878.749	-9004764.438
	thu	-2201291.587	-10922206.60	-9256797.168	22468513.777	10008893.331	-2447007.571	-9899830.120
	fri	1030638.5483	-5018453.614	-1916393.944	16774104.694	523552.67814	-6888232.858	-12563118.36
	sat	5790248.1391	-225211.9018	1633443.9293	16300534.817	-2334465.192	-7364594.957	-11369836.95
	sun	13947513.958	4399433.3753	10828733.622	20762003.979	-1842255.283	-8530492.158	-12471967.04
2	mon	24000994.776	-7950275.098	-13713729.45	-9036058.457	1940557.5810	-8211912.688	-17295641.39
	tue	20017514.419	14023874.627	2390024.2723	24142487.358	13350116.171	5626908.9608	2298886.8635
	wed	21854569.333	13667037.007	-4047373.920	11640330.034	4450281.4148	2236436.1502	4890654.7028
	thu	26409737.709	25814070.862	4853376.0870	2455544.2694	-9011532.422	-2878407.760	7013823.1298
	fri	38711648.293	15250002.063	-1164836.625	-3410679.109	-25576313.13	-23522636.83	-4823627.665
	sat	9556941.6650	5156405.9313	-767533.6123	-7519036.415	-25356577.15	-21170761.92	-7098895.312
	sun	-2530372.894	-6852761.997	-3207076.611	-13842239.72	-12261886.02	-12805056.31	-6369790.852
3	mon	2806094.0202	12007432.098	16939833.314	28439067.622	15616769.470	8527748.0167	2517864.0094
	tue	1964521.4727	5619017.3662	2816227.1273	5257672.1173	-4110466.613	-3872623.548	-391138.8551
	wed	-14508509.22	-755995.5515	4393575.0519	-2021030.480	-10425483.57	-7424509.890	-3034519.571
	thu	-22142442.67	-12137964.01	-11382499.38	-12122163.74	-21709892.43	-28423250.24	-30638853.72
	fri	21535980.719	-2885006.315	-11099826.20	4226030.7685	45852751.436	4798864.8857	-7257393.591
	sat	12433954.601	-1188406.852	-8660031.289	5367544.6121	60205975.642	29174424.479	18851774.220
	sun	9517582.1411	201221.97602	-5102075.859	15444209.820	51619289.429	27900046.665	19867214.561
4	mon	-2797741.168	-15750842.59	-14551889.49	-12663478.32	-48109626.36	-39863203.18	-39195245.30
	tue	-921632.9785	4191986.6465	-1786629.783	-510005.5662	-8309889.850	-5880599.027	-5668528.313
	wed	-2931070.061	13529297.392	-1095029.742	-6394316.721	-5881141.120	1747894.8709	6389036.6437
	thu	3005384.8805	922630.07791	-29748280.04	-41817001.21	-37289595.40	-10916744.56	-2094270.362
	fri	-27953979.03	5153432.4418	-10373157.77	-38055497.01	-44750514.52	-5110290.574	2615872.0484
	sat	-25177731.23	2975893.0769	-3130554.655	-19011646.35	-21642628.75	6254286.1792	7246079.5248
	sun	-25659319.91	7013023.7098	4407384.6283	-1562244.289	-4182890.545	15115031.418	12066196.775

Lag	Variable	mon	tue	wed	thu	fri	sat	sun
5	mon	13334082.372	10235625.315	11007816.748	7737478.7670	-638705.6351	-12341769.81	-13144461.33
	tue	1271732.8286	-4127361.943	2600220.1133	-12425314.56	-18133346.94	-11667518.05	-5567944.325
	wed	6381385.7106	-5017221.182	2415053.4773	-4447092.247	-15869465.84	-8685853.452	-739421.0125
	thu	16106088.325	3233088.9701	4480194.6514	1672770.6239	-19856419.56	-11853320.01	-6915419.456
	fri	28415307.972	-17273160.11	-19280403.22	21073887.259	-19782738.32	-27134218.94	-20955846.87
	sat	12443489.967	-25470538.40	-23431102.28	9036472.3366	-8642400.310	-13713190.13	-8091607.514
	sun	8352689.0260	-9876657.600	-13392047.43	4448648.7169	336426.33958	-6135825.660	1072545.5986
6	mon	-12700087.21	-20612402.32	-4928968.938	10067209.051	-26037540.84	-17035193.57	-13856419.25
	tue	574271.12233	3882002.6328	10662440.227	7616762.9780	-12071025.32	-6942890.708	-14439127.71
	wed	-621957.0416	9401131.3228	14438413.082	8744588.5750	-11095440.05	-6483420.739	-12648195.53
	thu	2037524.0612	8528907.0301	21022443.915	10564199.292	-19708100.27	-13713853.18	-24098339.60
	fri	11157286.712	-22235209.69	1293112.0184	5923547.7252	-30769860.05	-22879844.37	-31207619.40
	sat	7882420.3244	-13257351.17	-655783.3109	-8457664.814	-2376569.077	-6330771.496	-11067162.32
	sun	4201642.2103	-3186840.043	4341936.9054	-6997903.559	-7857477.848	-10608166.64	-10354073.73
7	mon	16363900.930	20785810.214	-1977464.818	-18475306.77	-15048868.97	1305238.6171	-9137599.605
	tue	-2423762.947	1798080.6881	-6453723.389	-4010752.395	2785532.7297	902444.21748	6674360.7835
	wed	-4200431.261	3435859.3318	-1476022.153	6094009.4666	3653178.1765	-1785484.262	3714409.6736
	thu	25292244.426	21207545.102	5997952.6639	10076228.927	14062803.067	13778896.061	23620378.129
	fri	6412960.5169	52247151.018	25973004.322	11870763.765	13591903.237	16503089.239	26691979.112
	sat	-14430572.09	20540737.240	21536764.235	-1041441.133	1325605.8875	8876295.0725	20278945.335
	sun	-12094884.66	7887153.9031	9254658.0172	5631471.8170	11357087.861	13108649.575	19669164.765
8	mon	8891256.1580	-2528970.648	-7737755.712	-10993216.77	-24808766.83	-15486739.74	-756993.5139
	tue	38174915.602	12049318.098	-7376752.880	-18938908.47	-23458185.80	-20463532.76	-7925327.359
	wed	9966544.4003	13448115.798	1403456.2491	-6366096.722	-12544048.21	-5111655.970	8803176.7905
	thu	-5406474.409	6414121.9535	-3637792.002	-7888563.079	-26374956.29	-15642205.49	-5866838.157
	fri	-17004321.68	-9769018.889	-23439013.39	-33315089.23	-64900641.42	-57291246.69	-54320122.17
	sat	-3710462.598	-6161150.959	-21806211.34	-19367564.96	-28651701.55	-29041690.23	-30713934.23
	sun	15948424.555	989409.63211	-18810694.20	-12110464.47	-13541508.81	-16614643.72	-21927023.16
9	mon	-15932026.81	4312643.4588	5155195.3868	-1098790.202	-14811436.27	-11546131.62	-13024297.61
	tue	4894338.1957	7092502.6837	11582380.783	5996442.1506	9342067.1975	11316258.210	5354250.6728
	wed	3366223.3912	-119718.6294	-1067311.271	-13423696.07	-16288385.77	-3416850.411	2592847.5549
	thu	-21377861.72	-28060046.39	-12706301.09	-35008545.86	-36594471.25	-16169529.86	-14718164.84
	fri	-19755278.18	-46469941.70	-45251285.70	-66273710.21	-73018425.64	-44421095.86	-38103599.30
	sat	-9656939.472	-35654721.72	-36384347.14	-40698004.52	-38985351.31	-22166303.93	-19310772.37
	sun	-7129507.163	-18291089.30	-24272178.12	-25587319.71	-18887117.70	-7522730.068	-5973355.473

Lag	Variable	mon	tue	wed	thu	fri	sat	sun
10	mon	-10313457.63	-7715786.680	-16457289.44	-21585772.47	-7287537.184	5736171.3053	14174609.246
	tue	-6868682.246	-4120170.606	-2703328.947	-1426922.721	6499645.5117	10366743.779	10745955.604
	wed	-3638425.052	9385114.6327	17228378.736	13083692.966	-8116416.750	-9048366.358	299673.98999
	thu	2870306.7169	20829857.496	12388036.503	8704516.5778	7078256.3763	-59158.65310	9592335.4634
	fri	24276766.782	63298718.280	17783672.800	4219182.3868	-462059.8200	-4406764.163	4570855.1646
	sat	14430442.609	43160823.414	24602812.435	17351181.440	16088121.721	8017142.1885	8397695.8721
	sun	9917489.3828	31853115.338	22381745.862	18238460.226	14126337.533	4728681.7017	1577043.3234
11	mon	-9287201.835	-5790971.889	5247384.0016	-4178392.962	-2622182.024	-4946900.777	-8127712.695
	tue	-1988220.797	5024571.9591	3871085.7232	-5807242.051	766085.91655	5810131.6409	3233175.6349
	wed	-13380826.60	-6938805.549	-11643231.16	-15674593.10	-8681026.438	-7074188.258	-2139671.540
	thu	-31304766.94	-5935071.048	-5413255.970	-28439466.14	-42147748.44	-8755785.335	-2541304.235
	fri	-26705767.63	-19392560.97	-20242186.24	-52413329.77	-74103251.92	-38861121.77	-21166077.02
	sat	-20688951.06	-13670665.42	-20467766.69	-37622517.67	-51322909.83	-28173605.38	-13324950.24
	sun	-13518992.93	-7631630.760	-13017571.56	-23348551.59	-39089701.89	-22484084.67	-8877285.829
12	mon	-30230098.32	-9356144.832	-11729144.66	-14096033.44	-5985043.199	-3495234.893	-12146541.25
	tue	-15823318.82	-172800.2600	-12485308.70	-16285650.27	-8274837.930	-4471182.129	-14864214.50
	wed	-22583963.35	13504571.749	-2022450.521	-2546320.909	18183554.031	28407769.913	10587056.648
	thu	-14954154.43	-2342517.542	-1928817.245	-195611.3100	-5851385.420	14436610.507	-3303865.668
	fri	10289145.418	-13667759.46	-18856197.30	-30832653.89	-39758523.38	-10476619.85	-13602776.96
	sat	14288420.544	-16451434.93	-25683445.72	-34311761.87	-39862109.78	-15845149.34	-14837300.06
	sun	5723184.3550	-18518759.83	-32784697.37	-33591093.06	-42492142.94	-24201412.98	-20425448.46

Cross Correlations of Residuals

Lag	Variable	mon	tue	wed	thu	fri	sat	sun
0	mon	1.00000	0.35697	0.16364	0.20329	0.12706	-0.03660	-0.05945
	tue	0.35697	1.00000	0.68742	0.52577	0.36779	0.32636	0.32080
	wed	0.16364	0.68742	1.00000	0.72298	0.47726	0.42119	0.33475
	thu	0.20329	0.52577	0.72298	1.00000	0.70955	0.59514	0.43798
	fri	0.12706	0.36779	0.47726	0.70955	1.00000	0.90063	0.72562
	sat	-0.03660	0.32636	0.42119	0.59514	0.90063	1.00000	0.87667
	sun	-0.05945	0.32080	0.33475	0.43798	0.72562	0.87667	1.00000

Lag	Variable	mon	tue	wed	thu	fri	sat	sun
1	mon	0.05630	0.06418	0.12348	0.17612	0.04273	0.00443	-0.08716
	tue	0.05469	0.02077	0.04925	0.18748	0.04209	-0.01540	-0.08319
	wed	0.00836	-0.04757	-0.03598	0.15550	0.05034	-0.01347	-0.06822
	thu	-0.01047	-0.06120	-0.04868	0.08525	0.03091	-0.01078	-0.05411
	fri	0.00399	-0.02288	-0.00820	0.05180	0.00132	-0.02469	-0.05588
	sat	0.03199	-0.00147	0.00997	0.07181	-0.00837	-0.03767	-0.07215
	sun	0.09559	0.03551	0.08204	0.11348	-0.00819	-0.05413	-0.09820
2	mon	0.14320	-0.05586	-0.09044	-0.04299	0.00751	-0.04536	-0.11854
	tue	0.14065	0.11605	0.01856	0.13528	0.06088	0.03661	0.01856
	wed	0.14413	0.10615	-0.02950	0.06122	0.01905	0.01366	0.03705
	thu	0.12566	0.14465	0.02552	0.00932	-0.02783	-0.01268	0.03834
	fri	0.14989	0.06954	-0.00499	-0.01053	-0.06427	-0.08433	-0.02146
	sat	0.05279	0.03354	-0.00469	-0.03312	-0.09090	-0.10828	-0.04505
	sun	-0.01734	-0.05531	-0.02430	-0.07566	-0.05454	-0.08126	-0.05015
3	mon	0.01674	0.08437	0.11171	0.13531	0.06047	0.04711	0.01726
	tue	0.01380	0.04650	0.02187	0.02946	-0.01874	-0.02519	-0.00316
	wed	-0.09568	-0.00587	0.03203	-0.01063	-0.04462	-0.04533	-0.02299
	thu	-0.10535	-0.06801	-0.05986	-0.04600	-0.06704	-0.12521	-0.16747
	fri	0.08339	-0.01316	-0.04751	0.01305	0.11522	0.01720	-0.03228
	sat	0.06869	-0.00773	-0.05288	0.02365	0.21584	0.14921	0.11963
	sun	0.06523	0.00162	-0.03865	0.08441	0.22961	0.17705	0.15642
4	mon	-0.01669	-0.11067	-0.09597	-0.06025	-0.18628	-0.22020	-0.26864
	tue	-0.00648	0.03469	-0.01388	-0.00286	-0.03789	-0.03826	-0.04575
	wed	-0.01933	0.10508	-0.00798	-0.03363	-0.02517	0.01067	0.04840
	thu	0.01430	0.00517	-0.15645	-0.15867	-0.11514	-0.04809	-0.01145
	fri	-0.10824	0.02350	-0.04440	-0.11751	-0.11245	-0.01832	0.01164
	sat	-0.13908	0.01936	-0.01911	-0.08375	-0.07759	0.03199	0.04598
	sun	-0.17587	0.05661	0.03339	-0.00854	-0.01861	0.09592	0.09500
5	mon	0.07956	0.07192	0.07259	0.03681	-0.00247	-0.06818	-0.09009
	tue	0.00894	-0.03415	0.02019	-0.06962	-0.08269	-0.07590	-0.04494
	wed	0.04208	-0.03897	0.01760	-0.02339	-0.06792	-0.05303	-0.00560
	thu	0.07663	0.01812	0.02356	0.00635	-0.06131	-0.05222	-0.03780
	fri	0.11003	-0.07877	-0.08252	0.06507	-0.04971	-0.09728	-0.09321
	sat	0.06874	-0.16570	-0.14307	0.03981	-0.03098	-0.07014	-0.05135
	sun	0.05725	-0.07972	-0.10146	0.02432	0.00150	-0.03894	0.00844

Lag	Variable	mon	tue	wed	thu	fri	sat	sun
6	mon	-0.07577	-0.14483	-0.03251	0.04790	-0.10082	-0.09410	-0.09497
	tue	0.00404	0.03212	0.08281	0.04268	-0.05504	-0.04517	-0.11655
	wed	-0.00410	0.07301	0.10525	0.04599	-0.04749	-0.03959	-0.09582
	thu	0.00969	0.04779	0.11056	0.04008	-0.06086	-0.06041	-0.13172
	fri	0.04320	-0.10139	0.00553	0.01829	-0.07732	-0.08202	-0.13881
	sat	0.04354	-0.08625	-0.00400	-0.03726	-0.00852	-0.03238	-0.07023
	sun	0.02880	-0.02572	0.03289	-0.03825	-0.03495	-0.06732	-0.08152
7	mon	0.09763	0.14605	-0.01304	-0.08791	-0.05827	0.00721	-0.06263
	tue	-0.01703	0.01488	-0.05012	-0.02247	0.01270	0.00587	0.05387
	wed	-0.02770	0.02668	-0.01076	0.03205	0.01564	-0.01090	0.02814
	thu	0.12034	0.11883	0.03154	0.03823	0.04342	0.06070	0.12910
	fri	0.02483	0.23825	0.11116	0.03666	0.03416	0.05916	0.11873
	sat	-0.07971	0.13363	0.13150	-0.00459	0.00475	0.04540	0.12869
	sun	-0.08290	0.06366	0.07011	0.03078	0.05052	0.08318	0.15487
8	mon	0.05305	-0.01777	-0.05103	-0.05231	-0.09606	-0.08555	-0.00519
	tue	0.26823	0.09971	-0.05729	-0.10612	-0.10697	-0.13313	-0.06397
	wed	0.06573	0.10445	0.01023	-0.03348	-0.05369	-0.03121	0.06669
	thu	-0.02572	0.03594	-0.01913	-0.02993	-0.08144	-0.06891	-0.03207
	fri	-0.06584	-0.04455	-0.10032	-0.10287	-0.16309	-0.20539	-0.24162
	sat	-0.02050	-0.04008	-0.13315	-0.08532	-0.10272	-0.14853	-0.19490
	sun	0.10931	0.00799	-0.14251	-0.06619	-0.06023	-0.10543	-0.17264
9	mon	-0.09506	0.03030	0.03400	-0.00523	-0.05735	-0.06378	-0.08927
	tue	0.03439	0.05869	0.08996	0.03360	0.04260	0.07362	0.04322
	wed	0.02220	-0.00093	-0.00778	-0.07060	-0.06971	-0.02086	0.01964
	thu	-0.10172	-0.15723	-0.06682	-0.13283	-0.11300	-0.07123	-0.08045
	fri	-0.07649	-0.21190	-0.19367	-0.20464	-0.18349	-0.15925	-0.16949
	sat	-0.05334	-0.23195	-0.22216	-0.17928	-0.13976	-0.11337	-0.12254
	sun	-0.04886	-0.14764	-0.18388	-0.13985	-0.08401	-0.04774	-0.04703
10	mon	-0.06153	-0.05421	-0.10853	-0.10270	-0.02822	0.03169	0.09715
	tue	-0.04826	-0.03409	-0.02100	-0.00800	0.02964	0.06744	0.08674
	wed	-0.02399	0.07289	0.12559	0.06881	-0.03474	-0.05525	0.00227
	thu	0.01366	0.11672	0.06515	0.03303	0.02186	-0.00026	0.05243
	fri	0.09400	0.28864	0.07611	0.01303	-0.00116	-0.01580	0.02033
	sat	0.07971	0.28078	0.15022	0.07644	0.05768	0.04100	0.05329
	sun	0.06797	0.25711	0.16956	0.09969	0.06283	0.03001	0.01242

Lag	Variable	mon	tue	wed	thu	fri	sat	sun
11	mon	-0.05541	-0.04069	0.03461	-0.01988	-0.01015	-0.02733	-0.05571
	tue	-0.01397	0.04158	0.03006	-0.03254	0.00349	0.03780	0.02610
	wed	-0.08824	-0.05389	-0.08487	-0.08244	-0.03715	-0.04319	-0.01621
	thu	-0.14895	-0.03326	-0.02847	-0.10791	-0.13015	-0.03857	-0.01389
	fri	-0.10341	-0.08843	-0.08663	-0.16184	-0.18621	-0.13932	-0.09415
	sat	-0.11429	-0.08893	-0.12497	-0.16574	-0.18399	-0.14409	-0.08456
	sun	-0.09266	-0.06160	-0.09862	-0.12762	-0.17387	-0.14268	-0.06990
12	mon	-0.18036	-0.06574	-0.07735	-0.06707	-0.02317	-0.01931	-0.08325
	tue	-0.11118	-0.00143	-0.09697	-0.09125	-0.03773	-0.02909	-0.11998
	wed	-0.14894	0.10488	-0.01474	-0.01339	0.07782	0.17345	0.08021
	thu	-0.07115	-0.01313	-0.01014	-0.00074	-0.01807	0.06360	-0.01806
	fri	0.03984	-0.06233	-0.08070	-0.09521	-0.09991	-0.03756	-0.06051
	sat	0.07893	-0.10702	-0.15682	-0.15115	-0.14290	-0.08104	-0.09415
	sun	0.03923	-0.14948	-0.24837	-0.18360	-0.18901	-0.15358	-0.16082

```
                     Schematic Representation of Cross Correlations of Residuals
Variable/
Lag        0       1       2       3       4       5       6       7       8       9       10      11      12

mon        ++.+... ....... ....... ....... .....-- ....... ....... ....... ....... ....... ....... ....... .......
tue        +++++++ ....... ....... ....... ....... ....... ....... ....... +...... ....... ....... ....... .......
wed        .++++++ ....... ....... ....... ....... ....... ....... ....... ....... ....... ....... ....... .......
thu        +++++++ ....... ....... ....... ....... ....... ....... ....... ....... ....... ....... ....... .......
fri        .++++++ ....... ....... ....... ....... ....... ....... .+..... .....-- .-.-... .+..... ....... .......
sat        .++++++ ....... ....... ....+.. ....... ....... ....... ....... ....... .--.... .+..... ....... .......
sun        .++++++ ....... ....... ....+.. ....... ....... ....... ....... ....... ....... .+..... ....... ..-....

                       + is > 2*std error,  - is < -2*std error,  . is between
```

Portmanteau Test for Cross
Correlations of Residuals

Up To Lag	DF	Chi-Square	Pr > ChiSq
2	49	64.14	0.0720
3	98	113.86	0.1305
4	147	151.14	0.3904
5	196	190.96	0.5883
6	245	236.74	0.6357
7	294	308.44	0.2698
8	343	362.99	0.2194
9	392	396.35	0.4292
10	441	445.49	0.4313
11	490	482.60	0.5856
12	539	545.24	0.4170

Univariate Model ANOVA Diagnostics

Variable	R-Square	Standard Deviation	F Value	Pr > F
mon	0.7697	13497.46405	43.92	<.0001
tue	0.7834	11461.09256	47.54	<.0001
wed	0.8000	12211.20736	52.57	<.0001
thu	0.6970	16925.35361	30.23	<.0001
fri	0.5769	20797.86930	17.92	<.0001
sat	0.6417	14578.29302	23.54	<.0001
sun	0.6916	11749.56161	29.47	<.0001

Univariate Model White Noise Diagnostics

Variable	Durbin Watson	Normality		ARCH	
		Chi-Square	Pr > ChiSq	F Value	Pr > F
mon	1.87785	156.78	<.0001	0.01	0.9153
tue	1.95476	509.92	<.0001	1.43	0.2351
wed	1.99356	99.59	<.0001	0.06	0.8068
thu	1.82587	566.28	<.0001	1.46	0.2306
fri	1.99452	341.07	<.0001	0.43	0.5121
sat	2.06884	160.30	<.0001	0.37	0.5427
sun	2.19030	29.44	<.0001	0.00	0.9892

Univariate Model AR Diagnostics

Variable	AR1 F Value	AR1 Pr > F	AR2 F Value	AR2 Pr > F	AR3 F Value	AR3 Pr > F	AR4 F Value	AR4 Pr > F
mon	0.31	0.5779	1.11	0.3346	0.75	0.5259	0.61	0.6583
tue	0.04	0.8381	0.67	0.5122	0.50	0.6844	0.37	0.8267
wed	0.14	0.7111	0.05	0.9542	0.04	0.9897	0.07	0.9909
thu	0.71	0.4005	0.34	0.7107	0.29	0.8355	0.76	0.5565
fri	0.00	0.9896	0.20	0.8210	0.56	0.6401	0.75	0.5625
sat	0.14	0.7106	0.66	0.5195	1.11	0.3504	0.83	0.5082
sun	0.95	0.3321	0.70	0.4984	1.17	0.3247	1.29	0.2811

APPENDIX C

MATHEMATICA CALCULATIONS

CONTENTS

Explicit expression for $\Gamma(0)$ for a bivariate VAR(1) model

Determine the roots of $\det(\boldsymbol{I}_2 - \boldsymbol{\Phi}_1 z) = 0$:

```
z /. Solve[Det[(1 0; 0 1) - (φ11 φ12; φ21 φ22) z] == 0, z]
```

$$\left\{\frac{\phi_{11}+\phi_{22}-\sqrt{\phi_{11}^2+4\phi_{12}\phi_{21}-2\phi_{11}\phi_{22}+\phi_{22}^2}}{2(-\phi_{12}\phi_{21}+\phi_{11}\phi_{22})},\ \frac{\phi_{11}+\phi_{22}+\sqrt{\phi_{11}^2+4\phi_{12}\phi_{21}-2\phi_{11}\phi_{22}+\phi_{22}^2}}{2(-\phi_{12}\phi_{21}+\phi_{11}\phi_{22})}\right\}$$

Determine $vec\boldsymbol{\Gamma}(0)$ using (2.18)

```
(mm = {{φ11, φ12}, {φ21, φ22}}) // MatrixForm
```

$$\begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}$$

```
(siga = {{σ11, σ12}, {σ12, σ22}}) // MatrixForm
```

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}$$

```
(vecsiga = {{σ11}, {σ12}, {σ12}, {σ22}}) // MatrixForm
```

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{12} \\ \sigma_{22} \end{pmatrix}$$

```
<< LinearAlgebra`MatrixManipulation`
```

```
KroneckerProduct[a_?SquareMatrixQ, b_?SquareMatrixQ] :=
 BlockMatrix[Outer[Times, a, b]]
```

```
KroneckerProduct[mm, mm] // MatrixForm
```

$$\begin{pmatrix} \phi_{11}^2 & \phi_{11}\phi_{12} & \phi_{11}\phi_{12} & \phi_{12}^2 \\ \phi_{11}\phi_{21} & \phi_{11}\phi_{22} & \phi_{12}\phi_{21} & \phi_{12}\phi_{22} \\ \phi_{11}\phi_{21} & \phi_{12}\phi_{21} & \phi_{11}\phi_{22} & \phi_{12}\phi_{22} \\ \phi_{21}^2 & \phi_{21}\phi_{22} & \phi_{21}\phi_{22} & \phi_{22}^2 \end{pmatrix}$$

$$\mathbf{MatrixForm}\left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} \phi_{11}^2 & \phi_{11}\phi_{12} & \phi_{11}\phi_{12} & \phi_{12}^2 \\ \phi_{11}\phi_{21} & \phi_{11}\phi_{22} & \phi_{12}\phi_{21} & \phi_{12}\phi_{22} \\ \phi_{11}\phi_{21} & \phi_{12}\phi_{21} & \phi_{11}\phi_{22} & \phi_{12}\phi_{22} \\ \phi_{21}^2 & \phi_{21}\phi_{22} & \phi_{21}\phi_{22} & \phi_{22}^2 \end{pmatrix}\right]$$

$$\begin{pmatrix} 1-\phi_{11}^2 & -\phi_{11}\phi_{12} & -\phi_{11}\phi_{12} & -\phi_{12}^2 \\ -\phi_{11}\phi_{21} & 1-\phi_{11}\phi_{22} & -\phi_{12}\phi_{21} & -\phi_{12}\phi_{22} \\ -\phi_{11}\phi_{21} & -\phi_{12}\phi_{21} & 1-\phi_{11}\phi_{22} & -\phi_{12}\phi_{22} \\ -\phi_{21}^2 & -\phi_{21}\phi_{22} & -\phi_{21}\phi_{22} & 1-\phi_{22}^2 \end{pmatrix}$$

$$\mathbf{FullSimplify}\left[\mathbf{Inverse}\left[\begin{pmatrix} 1-\phi_{11}^2 & -\phi_{11}\phi_{12} & -\phi_{11}\phi_{12} & -\phi_{12}^2 \\ -\phi_{11}\phi_{21} & 1-\phi_{11}\phi_{22} & -\phi_{12}\phi_{21} & -\phi_{12}\phi_{22} \\ -\phi_{11}\phi_{21} & -\phi_{12}\phi_{21} & 1-\phi_{11}\phi_{22} & -\phi_{12}\phi_{22} \\ -\phi_{21}^2 & -\phi_{21}\phi_{22} & -\phi_{21}\phi_{22} & 1-\phi_{22}^2 \end{pmatrix}\right] \mathbin{//} \mathbf{MatrixForm}\right] \mathbin{//}$$

MatrixForm

$(4\times 4 \text{ matrix})$

```
FullSimplify[MatrixForm[%.vecsiga]] // MatrixForm
```

$$\begin{pmatrix} -\dfrac{-\sigma_{11}(-(-1+\phi_{22})(1+\phi_{22})(-1+\phi_{11}\phi_{22})+\phi_{12}\phi_{21}(1+\phi_{22}^2))+\phi_{12}(\sigma_{22}\phi_{12}(1-\phi_{12}\phi_{21}+\phi_{11}\phi_{22})+2\sigma_{12}(\phi_{12}\phi_{21}\phi_{22}-\phi_{11}(-1+\phi_{22}^2)))}{(-1+\phi_{12}\phi_{21}-\phi_{11}(-1+\phi_{22})+\phi_{22})(1+\phi_{12}\phi_{21}-\phi_{11}\phi_{22})(1-\phi_{12}\phi_{21}+\phi_{22}+\phi_{11}(1+\phi_{22}))} \\ -\dfrac{\sigma_{22}\phi_{12}(\phi_{11}\phi_{12}\phi_{21}-(-1+\phi_{11}^2)\phi_{22})+\sigma_{11}\phi_{21}(\phi_{12}\phi_{21}\phi_{22}-\phi_{11}(-1+\phi_{22}^2))+\sigma_{12}(1-\phi_{12}^2\phi_{21}^2-\phi_{22}^2+\phi_{11}^2(-1+\phi_{22}^2))}{(-1+\phi_{12}\phi_{21}-\phi_{11}(-1+\phi_{22})+\phi_{22})(1+\phi_{12}\phi_{21}-\phi_{11}\phi_{22})(1-\phi_{12}\phi_{21}+\phi_{22}+\phi_{11}(1+\phi_{22}))} \\ -\dfrac{\sigma_{22}\phi_{12}(\phi_{11}\phi_{12}\phi_{21}-(-1+\phi_{11}^2)\phi_{22})+\sigma_{11}\phi_{21}(\phi_{12}\phi_{21}\phi_{22}-\phi_{11}(-1+\phi_{22}^2))+\sigma_{12}(1-\phi_{12}^2\phi_{21}^2-\phi_{22}^2+\phi_{11}^2(-1+\phi_{22}^2))}{(-1+\phi_{12}\phi_{21}-\phi_{11}(-1+\phi_{22})+\phi_{22})(1+\phi_{12}\phi_{21}-\phi_{11}\phi_{22})(1-\phi_{12}\phi_{21}+\phi_{22}+\phi_{11}(1+\phi_{22}))} \\ -\dfrac{\phi_{21}(\sigma_{11}\phi_{21}(1-\phi_{12}\phi_{21}+\phi_{11}\phi_{22})+2\sigma_{12}(\phi_{11}\phi_{12}\phi_{21}-(-1+\phi_{11}^2)\phi_{22}))+\sigma_{22}(1-\phi_{12}\phi_{21}+\phi_{11}(-\phi_{11}(1+\phi_{12}\phi_{21})+(-1+\phi_{11}^2)\phi_{22}))}{(-1+\phi_{12}\phi_{21}-\phi_{11}(-1+\phi_{22})+\phi_{22})(1+\phi_{12}\phi_{21}-\phi_{11}\phi_{22})(1-\phi_{12}\phi_{21}+\phi_{22}+\phi_{11}(1+\phi_{22}))} \end{pmatrix}$$

Example 2.1

Determine the roots of $\det(\boldsymbol{I}_2 - \boldsymbol{\Phi}_1 z) = 0$:

$$\mathbf{z\ /.\ Solve}\left[\mathbf{Det}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.5 & 0.6 \\ 0.1 & 0.4 \end{pmatrix} \mathbf{z}\right] == 0,\ \mathbf{z}\right]$$

```
{1.42857, 5.}
```

Determine the roots of $\det(\boldsymbol{I}_2\lambda - \boldsymbol{\Phi}_1) = 0$:

$$\boldsymbol{\lambda}\ \mathbf{/.\ Solve}\left[\mathbf{Det}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \boldsymbol{\lambda} - \begin{pmatrix} 0.5 & 0.6 \\ 0.1 & 0.4 \end{pmatrix}\right] == 0,\ \boldsymbol{\lambda}\right]$$

```
{0.2, 0.7}
```

Determine $vec\boldsymbol{\Gamma}(0)$ using (2.18)

```
<< LinearAlgebra`MatrixManipulation`
```

```
KroneckerProduct[a_?SquareMatrixQ, b_?SquareMatrixQ] :=
 BlockMatrix[Outer[Times, a, b]]
```

```
(phi1 = {{0.5, 0.6}, {0.1, 0.4}}) // MatrixForm
```

$$\begin{pmatrix} 0.5 & 0.6 \\ 0.1 & 0.4 \end{pmatrix}$$

```
(vecsiga = {{1}, {0.5}, {0.5}, {0.9}}) // MatrixForm
```

$$\begin{pmatrix} 1 \\ 0.5 \\ 0.5 \\ 0.9 \end{pmatrix}$$

$$\text{MatrixForm}\left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \text{KroneckerProduct[phi1, phi1] // MatrixForm}\right]$$

$$\begin{pmatrix} 0.75 & -0.3 & -0.3 & -0.36 \\ -0.05 & 0.8 & -0.06 & -0.24 \\ -0.05 & -0.06 & 0.8 & -0.24 \\ -0.01 & -0.04 & -0.04 & 0.84 \end{pmatrix}$$

```
Inverse[%] // MatrixForm
```

$$\begin{pmatrix} 1.43069 & 0.632695 & 0.632695 & 0.974692 \\ 0.105449 & 1.32524 & 0.162449 & 0.470246 \\ 0.105449 & 0.162449 & 1.32524 & 0.470246 \\ 0.0270748 & 0.0783744 & 0.0783744 & 1.24687 \end{pmatrix}$$

```
MatrixForm[%.vecsiga]
```

$$\begin{pmatrix} 2.94061 \\ 1.27251 \\ 1.27251 \\ 1.22763 \end{pmatrix}$$

Example 2.3

Determine the roots of $\det\left(\boldsymbol{I}_2 - \boldsymbol{\Phi}_1 z - \boldsymbol{\Phi}_2 z^2\right) = 0$:

$$z\ /.\ \text{Solve}\left[\text{Det}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} -0.2 & 0.1 \\ 0.5 & 0.1 \end{pmatrix} z - \begin{pmatrix} 0.8 & 0.5 \\ -0.4 & 0.5 \end{pmatrix} z^2\right] == 0,\ z\right]$$

```
{-1.01315 - 0.351147 i, -1.01315 + 0.351147 i, 1.15964, 1.25}
```

Determine the modulus of the roots of $\det\left(\boldsymbol{I}_2 - \boldsymbol{\Phi}_1 z - \boldsymbol{\Phi}_2 z^2\right) = 0$:

```
Abs[{-1.0131528370199567` - 0.3511473042995436` i,
  -1.0131528370199567` + 0.3511473042995436` i, 1.1596390073732474`,
  1.2499999999999996`}]
```

```
{1.07228, 1.07228, 1.15964, 1.25}
```

Determine the roots of $\det\left(\boldsymbol{I}_2 \lambda^2 - \boldsymbol{\Phi}_1 \lambda - \boldsymbol{\Phi}_2\right) = 0$:

$$\lambda\ /.\ \text{Solve}\left[\text{Det}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \lambda^2 - \begin{pmatrix} -0.2 & 0.1 \\ 0.5 & 0.1 \end{pmatrix} \lambda - \begin{pmatrix} 0.8 & 0.5 \\ -0.4 & 0.5 \end{pmatrix}\right] == 0,\ \lambda\right]$$

```
{-0.881169 - 0.305403 i, -0.881169 + 0.305403 i, 0.8, 0.862337}
```

Determine the modulus of the roots of $\det\left(\boldsymbol{I}_2 \lambda^2 - \boldsymbol{\Phi}_1 \lambda - \boldsymbol{\Phi}_2\right) = 0$:

```
Abs[{-0.8811686626794089` - 0.30540308354978585` i,
  -0.8811686626794089` + 0.30540308354978585` i, 0.8`, 0.8623373253588177`}]
```

```
{0.932593, 0.932593, 0.8, 0.862337}
```

Determine $vec\boldsymbol{\Gamma}(0)^*$ using (2.27)

```
(phi1 = {{-0.2, 0.1}, {0.5, 0.1}}) // MatrixForm
```

$$\begin{pmatrix} -0.2 & 0.1 \\ 0.5 & 0.1 \end{pmatrix}$$

```
(phi2 = {{0.8, 0.5}, {-0.4, 0.5}}) // MatrixForm
```

$$\begin{pmatrix} 0.8 & 0.5 \\ -0.4 & 0.5 \end{pmatrix}$$

```
(siga = {{1.0, 0.5}, {0.5, 0.9}}) // MatrixForm
```

$$\begin{pmatrix} 1. & 0.5 \\ 0.5 & 0.9 \end{pmatrix}$$

```
(I2 = IdentityMatrix[2]) // MatrixForm
```

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

```
<< LinearAlgebra`MatrixManipulation`
```

```
(nul = ZeroMatrix[2] ) // MatrixForm
```

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

```
BlockMatrix[(f = {{phi1, phi2}, {I2, nul}})] // MatrixForm
```

$$\begin{pmatrix} -0.2 & 0.1 & 0.8 & 0.5 \\ 0.5 & 0.1 & -0.4 & 0.5 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\left(\mathbf{f} = \begin{pmatrix} -0.2` & 0.1` & 0.8` & 0.5` \\ 0.5` & 0.1` & -0.4` & 0.5` \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\right) \text{ // MatrixForm}$$

$$\begin{pmatrix} -0.2 & 0.1 & 0.8 & 0.5 \\ 0.5 & 0.1 & -0.4 & 0.5 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

```
BlockMatrix[(sigmaa = {{siga, nul}, {nul, nul}})] // MatrixForm
```

$$\begin{pmatrix} 1. & 0.5 & 0 & 0 \\ 0.5 & 0.9 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

```
(vecsigmaa = {{1}, {0.5}, {0}, {0}, {0.5}, {0.9}, {0}, {0},
    {0}, {0}, {0}, {0}, {0}, {0}, {0}, {0}}) // MatrixForm
```

$$\begin{pmatrix} 1 \\ 0.5 \\ 0 \\ 0 \\ 0.5 \\ 0.9 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

```
KroneckerProduct[a_?SquareMatrixQ, b_?SquareMatrixQ] :=
 BlockMatrix[Outer[Times, a, b]]
```

```
MatrixForm[IdentityMatrix[16] - KroneckerProduct[f, f] // MatrixForm]
```

$$\begin{pmatrix}
0.96 & 0.02 & 0.16 & 0.1 & 0.02 & -0.01 & -0.08 & -0.05 & 0.16 & -0.08 & -0.64 & -0.4 & 0.1 & -0.05 & -0.4 & -0.25 \\
0.1 & 1.02 & -0.08 & 0.1 & -0.05 & -0.01 & 0.04 & -0.05 & -0.4 & -0.08 & 0.32 & -0.4 & -0.25 & -0.05 & 0.2 & -0.25 \\
0.2 & 0 & 1 & 0 & -0.1 & 0 & 0 & 0 & -0.8 & 0 & 0 & 0 & -0.5 & 0 & 0 & 0 \\
0 & 0.2 & 0 & 1 & 0 & -0.1 & 0 & 0 & 0 & -0.8 & 0 & 0 & 0 & -0.5 & 0 & 0 \\
0.1 & -0.05 & -0.4 & -0.25 & 1.02 & -0.01 & -0.08 & -0.05 & -0.08 & 0.04 & 0.32 & 0.2 & 0.1 & -0.05 & -0.4 & -0.25 \\
-0.25 & -0.05 & 0.2 & -0.25 & -0.05 & 0.99 & 0.04 & -0.05 & 0.2 & 0.04 & -0.16 & 0.2 & -0.25 & -0.05 & 0.2 & -0.25 \\
-0.5 & 0 & 0 & 0 & -0.1 & 0 & 1 & 0 & 0.4 & 0 & 0 & 0 & -0.5 & 0 & 0 & 0 \\
0 & -0.5 & 0 & 0 & 0 & -0.1 & 0 & 1 & 0 & 0.4 & 0 & 0 & 0 & -0.5 & 0 & 0 \\
0.2 & -0.1 & -0.8 & -0.5 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.5 & -0.1 & 0.4 & -0.5 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.2 & -0.1 & -0.8 & -0.5 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -0.5 & -0.1 & 0.4 & -0.5 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

```
Inverse[%] // MatrixForm
```

(16×16 matrix)

```
MatrixForm[%.vecsigmaa]
```

$$\begin{pmatrix} 6.39789 \\ -0.130435 \\ 0.594203 \\ 2.82292 \\ -0.130435 \\ 5.59091 \\ 4.35968 \\ -2.5 \\ 0.594203 \\ 4.35968 \\ 6.39789 \\ -0.130435 \\ 2.82292 \\ -2.5 \\ -0.130435 \\ 5.59091 \end{pmatrix}$$

Explicit expression for $\Gamma(l)$ for a bivariate VMA(1) model

Determine $\boldsymbol{\Gamma}(0)$ and $\boldsymbol{\Gamma}(1)$ for a VMA(1) model using (2.34) and (2.35):

```
(theta1 = {{θ11, θ12}, {θ21, θ22}}) // MatrixForm
```

$$\begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix}$$

```
(siga = {{σ11, σ12}, {σ12, σ22}}) // MatrixForm
```

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}$$

```
z /. Solve[Det[IdentityMatrix[2] + theta1 * z] == 0, z]
```

$$\left\{\frac{-\theta_{11}-\theta_{22}-\sqrt{\theta_{11}^2+4\theta_{12}\theta_{21}-2\theta_{11}\theta_{22}+\theta_{22}^2}}{2(-\theta_{12}\theta_{21}+\theta_{11}\theta_{22})}, \frac{-\theta_{11}-\theta_{22}+\sqrt{\theta_{11}^2+4\theta_{12}\theta_{21}-2\theta_{11}\theta_{22}+\theta_{22}^2}}{2(-\theta_{12}\theta_{21}+\theta_{11}\theta_{22})}\right\}$$

```
(gamma0 = siga + theta1.siga.Transpose[theta1]) // MatrixForm
```

$$\begin{pmatrix} \sigma_{11}+\theta_{11}(\theta_{11}\sigma_{11}+\theta_{12}\sigma_{12})+\theta_{12}(\theta_{11}\sigma_{12}+\theta_{12}\sigma_{22}) & \sigma_{12}+\theta_{21}(\theta_{11}\sigma_{11}+\theta_{12}\sigma_{12})+\theta_{22}(\theta_{11}\sigma_{12}+\theta_{12}\sigma_{22}) \\ \sigma_{12}+\theta_{11}(\theta_{21}\sigma_{11}+\theta_{22}\sigma_{12})+\theta_{12}(\theta_{21}\sigma_{12}+\theta_{22}\sigma_{22}) & \theta_{21}(\theta_{21}\sigma_{11}+\theta_{22}\sigma_{12})+\sigma_{22}+\theta_{22}(\theta_{21}\sigma_{12}+\theta_{22}\sigma_{22}) \end{pmatrix}$$

```
Simplify[%] // MatrixForm
```

$$\begin{pmatrix} (1+\theta_{11}^2)\,\sigma_{11}+\theta_{12}\,(2\,\theta_{11}\,\sigma_{12}+\theta_{12}\,\sigma_{22}) & \sigma_{12}+\theta_{21}\,(\theta_{11}\,\sigma_{11}+\theta_{12}\,\sigma_{12})+\theta_{22}\,(\theta_{11}\,\sigma_{12}+\theta_{12}\,\sigma_{22}) \\ \sigma_{12}+\theta_{11}\,(\theta_{21}\,\sigma_{11}+\theta_{22}\,\sigma_{12})+\theta_{12}\,(\theta_{21}\,\sigma_{12}+\theta_{22}\,\sigma_{22}) & \theta_{21}^2\,\sigma_{11}+2\,\theta_{21}\,\theta_{22}\,\sigma_{12}+(1+\theta_{22}^2)\,\sigma_{22} \end{pmatrix}$$

```
(gamma1 = theta1.siga) // MatrixForm
```

$$\begin{pmatrix} \theta_{11}\,\sigma_{11}+\theta_{12}\,\sigma_{12} & \theta_{11}\,\sigma_{12}+\theta_{12}\,\sigma_{22} \\ \theta_{21}\,\sigma_{11}+\theta_{22}\,\sigma_{12} & \theta_{21}\,\sigma_{12}+\theta_{22}\,\sigma_{22} \end{pmatrix}$$

Example 2.5

Determine the roots of $\det\left(\boldsymbol{I}_2+\boldsymbol{\Theta}_1 z+\boldsymbol{\Theta}_2 z^2\right)=0$:

$$z\ /.\ \text{Solve}\left[\text{Det}\left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+\begin{pmatrix}0.2 & 0.1\\ 0.1 & 0.4\end{pmatrix}z+\begin{pmatrix}0.4 & 0\\ 0.6 & 0.1\end{pmatrix}z^2\right]==0,\ z\right]$$

```
{-0.987318 - 2.94179 i, -0.987318 + 2.94179 i,
 -0.512682 - 1.52758 i, -0.512682 + 1.52758 i}
```

Determine the modulus of the roots of $\det\left(\boldsymbol{I}_2+\boldsymbol{\Theta}_1 z+\boldsymbol{\Theta}_2 z^2\right)=0$:

```
Abs[{-0.9873175412331976` - 2.9417923553506107` i,
  -0.9873175412331976` + 2.9417923553506107` i,
  -0.5126824587668029` - 1.527578792977516` i,
  -0.5126824587668029` + 1.527578792977516` i}]
```

```
{3.10305, 3.10305, 1.61132, 1.61132}
```

Determine the roots of $\det\left(\boldsymbol{I}_2\lambda^2-\boldsymbol{\Theta}_1\lambda-\boldsymbol{\Theta}_2\right)=0$:

$$\lambda\ /.\ \text{Solve}\left[\text{Det}\left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\lambda^2-\begin{pmatrix}0.2 & 0.1\\ 0.1 & 0.4\end{pmatrix}\lambda-\begin{pmatrix}0.4 & 0\\ 0.6 & 0.1\end{pmatrix}\right]==0,\ \lambda\right]$$

```
{-0.47102, -0.228575, 0.42461, 0.874985}
```

Determine the absolute values of the roots of $\det\left(\boldsymbol{I}_2\lambda^2-\boldsymbol{\Theta}_1\lambda-\boldsymbol{\Theta}_2\right)=0$:

```
Abs[{-0.4710203057916489`, -0.22857524167865026`,
  0.42461014427787336`, 0.8749854031924259`}]
```

```
{0.47102, 0.228575, 0.42461, 0.874985}
```

Example 2.6

Determine the roots of $\det\left(\boldsymbol{I}_2 - \boldsymbol{\Phi}_1 z - \boldsymbol{\Phi}_2 z^2\right) = 0$:

$$z \text{ /. Solve}\left[\text{Det}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} -0.2 & 0.1 \\ 0.5 & 0.1 \end{pmatrix} z - \begin{pmatrix} 0.8 & 0.5 \\ -0.4 & 0.5 \end{pmatrix} z^2\right] == 0, z\right]$$

```
{-1.01315 - 0.351147 i, -1.01315 + 0.351147 i, 1.15964, 1.25}
```

Determine the modulus of the roots of $\det\left(\boldsymbol{I}_2 - \boldsymbol{\Phi}_1 z - \boldsymbol{\Phi}_2 z^2\right) = 0$:

```
Abs[{-1.0131528370199567` - 0.3511473042995436` i,
  -1.0131528370199567` + 0.3511473042995436` i,
  1.1596390073732474`, 1.2499999999999996`}]
```

```
{1.07228, 1.07228, 1.15964, 1.25}
```

Determine the roots of $\det\left(\boldsymbol{I}_2 \lambda^2 - \boldsymbol{\Phi}_1 \lambda - \boldsymbol{\Phi}_2\right) = 0$:

$$\lambda \text{ /. Solve}\left[\text{Det}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \lambda^2 - \begin{pmatrix} -0.2 & 0.1 \\ 0.5 & 0.1 \end{pmatrix} \lambda - \begin{pmatrix} 0.8 & 0.5 \\ -0.4 & 0.5 \end{pmatrix}\right] == 0, \lambda\right]$$

```
{-0.881169 - 0.305403 i, -0.881169 + 0.305403 i, 0.8, 0.862337}
```

Determine the modulus of the roots of $\det\left(\boldsymbol{I}_2 \lambda^2 - \boldsymbol{\Phi}_1 \lambda - \boldsymbol{\Phi}_2\right) = 0$:

```
Abs[{-0.8811686626794089` - 0.30540308354978585` i,
  -0.8811686626794089` + 0.30540308354978585` i, 0.8`,
  0.8623373253588177`}]
```

```
{0.932593, 0.932593, 0.8, 0.862337}
```

Determine the roots of $\det\left(\boldsymbol{I}_2 + \boldsymbol{\Theta}_1 z\right) = 0$:

$$z \text{ /. Solve}\left[\text{Det}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0.2 & 0.1 \\ 0.1 & 0.4 \end{pmatrix} z\right] == 0, z\right]$$

```
{-6.30602, -2.26541}
```

Determine the absolute values of the roots of $\det(\boldsymbol{I}_2 + \boldsymbol{\Theta}_1 z) = 0$:

```
Abs[{-6.30601937481871`, -2.265409196609863`}]

{6.30602, 2.26541}
```

Determine the roots of $\det(\boldsymbol{I}_2 \lambda - \boldsymbol{\Theta}_1) = 0$:

$$\lambda \,/.\, \mathrm{Solve}\left[\mathrm{Det}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\lambda - \begin{pmatrix} 0.2 & 0.1 \\ 0.1 & 0.4 \end{pmatrix}\right] == 0,\ \lambda\right]$$

```
{0.158579, 0.441421}
```

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SUMMARY

STATIONARY MULTIVARIATE TIME SERIES ANALYSIS

by

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Department: **Statistics**

Degree: **MSc (Course Work) Mathematical Statistics**

Multivariate time series analysis became popular in the early 1950s when the need to analyse time series simultaneously arose in the field of economics. This study provides an overview of some of the aspects of multivariate time series analysis in the case of stationarity.

The VARMA (vector autoregressive moving average) class of multivariate time series models, including pure vector autoregressive (VAR) and vector moving average (VMA) models is considered. Methods based on moments and information criteria for the determination of the appropriate order of a model suitable for an observed multivariate time series are discussed. Feasible methods of estimation based on the least squares and/or maximum likelihood are provided for the different types of VARMA models. In some cases, the estimation is more complicated due to the identification problem and the nonlinearity of the normal equations. It is shown that the significance of individual estimates can be established by using hypothesis tests based on the asymptotic properties of the estimators. Diagnostic tests for the adequacy of the fitted model are discussed and illustrated. These include methods based on both univariate and multivariate procedures. The complete model building process is illustrated by means of case studies on multivariate electricity demand and temperature time series.

Throughout the study numerical examples are used to illustrate concepts. Computer program code (using basic built-in multivariate functions) is given for all the examples. The results are benchmarked against those produced by a dedicated procedure for multivariate time series. It is envisaged that the program code (given in SAS/IML) could be made available to a much wider user community, without much difficulty, by translation into open source platforms.