REFERENCES


Knisel, W.G. (1980). CREAMS: A field scale model for chemicals, runoff and erosion from agricultural management systems. USDA.


Pagiola, S. (1999). The global environmental benefit of land degradation control on 
agricultural land. *World Bank Environmental Paper No. 16*, The World Bank, 
Washington D.C.

development in the Ethiopian highlands. *American Journal of Agricultural 
Economics*, 83:1231-1240.


Pindyck, R.S., and Rubinfeld, D.L. (1997). *Econometric models and economic forecasts*, 4th 


Sustainable Economic Development, Uppsala: SIAS.

tenure and land policy in Ethiopia after the Derg*, Norway: Reprocentralen AVH.

Ababa.


APPENDICES

Appendix I. Current Value Hamiltonian version of the soil nutrient mining control problem

The current value Hamiltonian of the nutrient mining problem is given by:

\[ H_C(F, L, L_s, K, \eta) = Pf(L_t, K, N) - (W_F F + W_L L_T + W_S L_S + W_K K_T) + \eta[G(F) - D(Y) + M(Z, L_S, Y)] \]  

(1.1)

Where: \( \eta = e^{\delta \mu} \)  

(1.2)

The FOC for this system:

\[ \frac{\partial H_C}{\partial F} = -W_F + \eta \frac{\partial G}{\partial F} = 0 \Rightarrow W_F = \eta G_F \]  

(1.3)

\[ \frac{\partial H_C}{\partial L_T} = P \frac{\partial f}{\partial L_T} - W_L - \eta \frac{\partial D}{\partial L_T} + \eta \frac{\partial M}{\partial L_T} = 0 \]  

(1.4)

\[ \frac{\partial H_C}{\partial K} = P \frac{\partial f}{\partial K} - W_K - \eta \frac{\partial D}{\partial K} + \eta \frac{\partial M}{\partial K} = 0 \]  

(1.5)

\[ \frac{\partial H_C}{\partial L_S} = -W_S + \eta \frac{\partial M}{\partial L_S} = 0 \]  

(1.6)

\[ \dot{\eta} = \delta \eta - \frac{\partial H_C}{\partial N} = \delta \eta - P \frac{\partial f}{\partial N} + \eta \left( \frac{\partial D}{\partial N} - \frac{\partial M}{\partial N} \right) \]  

(1.7)

\[ \frac{\partial H_C}{\partial \eta} = \dot{N} = G(F) - D(Y) + M(Z, L_S, Y) \]  

(1.8)

Assuming a steady state where \( \dot{\eta} = \ddot{N} = 0 \), the first order conditions shown in equations (1.1 through 1.8) could be restated as follows:
\[ \eta = \frac{W_F}{G_F} \quad (1.3b) \]

\[ \eta = \frac{Pf_{L_y} - W_L}{D_{L_y} - M_{L_y}} \quad (1.4b) \]

\[ \eta = \frac{Pf_{K_y} - W_K}{D_{K_y} - M_{K_y}} \quad (1.5b) \]

\[ \eta = \frac{W_S}{M_{L_y}} \quad (1.6b) \]

\[ \eta = \frac{Pf_N}{[(D_N - M_N) + \delta]} \quad (1.7b) \]

\[ G(F) = D(Y) - M(Z, L_S, Y) \quad (1.8b) \]

Equation (1.8b) above states that at steady state, the net nutrient depletion through crop harvest, erosion and natural processes are matched by external nutrients added to the soil. Further combining equations 1.3b with each of 1.4b through 1.7b, the following equations are derived:

\[ \frac{W_F}{G_F} = \frac{Pf_N}{[(D_N - M_N) + \delta]} \quad (1.9) \]

\[ \frac{Pf_{L_y} - W_L}{D_{L_y} - M_{L_y}} = \frac{Pf_N}{[(D_N - M_N) + \delta]} \quad (1.10) \]
\[ \frac{P_{f_{K_x}} - W_{K_y}}{D_{K_y} - M_{K_x}} = \frac{P_{f_N}}{[(D_N - M_N) + \delta]} \]  

(1.11)

\[ \frac{W_s}{M_{L_y}} = \frac{P_{f_N}}{[(D_N - M_N) + \delta]} \]  

(1.12)

Eliminating, common terms the following fundamental equation is derived:

\[ \eta = \frac{W_F}{G_F} = \frac{P_{f_N}}{[ (D_N - M_N) + \delta ]} = \frac{P_{f_{L_y}} - W_{L_y}}{D_{L_y} - M_{L_y}} = \frac{P_{f_{K_y}} - W_{K_y}}{D_{K_y} - M_{K_y}} = \frac{W_{L_y}}{M_{L_y}} \]  

(1.13)
Appendix II. Summary of specified functions and functional relationships used in the empirical soil degradation control model

The reduced form solutions of the control model for both scenarios (soil nutrient mining only) as well as (soil nutrient mining and physical soil degradation) are based on the following empirical specifications:

1. The production function:

\[ Y = AL^b_y K^c_y SD^d N^e \]  \hspace{1cm} (2.1)

2. The aggregate soil regeneration and decay function (H)

2.1. Relationship between soil conservation effort and erosion damage

\[ E_t = \alpha e^{ad_t} \]  \hspace{1cm} (2.2)

2.2. Contribution of canopy to soil decay

\[ J = \phi (1 - e^{-\nu Y}) \]  \hspace{1cm} (2.3)

Accordingly combining equations (2.2 and 2.3), we have

\[ h = E_t - J = \alpha e^{ad_t} - \phi (1 - e^{-\nu Y}) \]  \hspace{1cm} (2.4)

The natural rate of soil regeneration, Z, is constant. The aggregate soil regeneration and decay function, H (Z, L_S, Y) is given by

\[ H = Z - h = Z - E + J \]

\[ H = Z - \alpha e^{ad_t} + \phi (1 - e^{-\nu Y}) \]  \hspace{1cm} (2.5)
3. The nutrient regeneration and damage function
   
   3.1. The nutrient augmentation function, \( G(F) \)

   \[ G(F) = \beta_1 F \]  \hspace{1cm} (2.6)

   3.2. Nutrient depletion due to biomass removal

   \[ D(Y) = \beta_2 Y \]  \hspace{1cm} (2.7)

   3.3. Nutrient regeneration and damage due to erosion and natural processes

   \[ M = \beta_3[Z - \alpha e^{-\alpha L} + \phi(1 - e^{-\gamma Y})] \]  \hspace{1cm} (2.8)

   Accordingly the aggregate nutrient regeneration and damage function is given by

   \[ N = G(F) - D(Y) + M(Z, L, Y) \]  \hspace{1cm} (2.9)

   \[ N = \beta_1 F - \beta_2 Y + \beta_3[Z - \alpha e^{-\alpha L} + \phi(1 - e^{-\gamma Y})] \]  \hspace{1cm} (2.10)

   Reduced form solutions for the optimality conditions derived in appendices III, IV and V are based on the following functional relationships.

   The production function is specified as:

   \[ f(L, K, SD, N) = Y = AL^b Y^c SD^d N^g \]  \hspace{1cm} (2.11)

   Accordingly, the respective marginal products (partial derivatives with respect to its arguments) are given by:
The respective partial derivatives of the soil regeneration and damage function are given by:

\[
\frac{\partial f}{\partial L_Y} = f_{L_Y} = bL_Y^{b-1} K_Y^c SD^{d-1} N^\gamma = \frac{bY}{L_Y}
\]

(2.12)

\[
\frac{\partial f}{\partial K_Y} = f_{K_Y} = cL_Y^b K_Y^{-1} SD^{d} N^\gamma = \frac{cY}{K_Y}
\]

(2.13)

\[
\frac{\partial f}{\partial SD} = f_{SD} = dL_Y^b K_Y^c SD^{d-1} N^\gamma = \frac{dY}{SD}
\]

(2.14)

\[
\frac{\partial f}{\partial N} = f_{N} = gL_Y^b K_Y^c SD^{d} N^{\gamma-1} = \frac{gY}{N}
\]

(2.15)

Where: \( \phi e^{-\nu Y} = \varphi \)

\[
\frac{\partial H}{\partial L_Y} = H_{L_Y} = \phi e^{-\nu Y} \frac{\partial f}{\partial L_Y} = \varphi \frac{bY}{L_Y}
\]

(2.16)

\[
\frac{\partial H}{\partial K_Y} = H_{K_Y} = \phi e^{-\nu Y} \frac{\partial f}{\partial K_Y} = \varphi \frac{cY}{K_Y}
\]

(2.17)

\[
\frac{\partial H}{\partial SD} = H_{SD} = \phi e^{-\nu Y} \frac{\partial f}{\partial SD} = \varphi \frac{dY}{SD}
\]

(2.18)

\[
\frac{\partial H}{\partial N} = H_N = \phi e^{-\nu Y} \frac{\partial f}{\partial N} = \varphi \frac{gY}{N}
\]

(2.19)

\[
\frac{\partial H}{\partial L_S} = H_{L_S} = \tau e^{-\alpha Y}
\]

(2.20)
Partial derivatives of the nutrient regeneration and damage function with respect to its arguments:

\[
\frac{\partial M}{\partial L_y} = M_{L_y} = \beta_2 \phi e^{-\gamma} \frac{\partial f}{\partial L_y} = \beta_2 \phi \frac{b_y}{L_y}
\]
\[
(2.21)
\]

\[
\frac{\partial M}{\partial K_y} = M_{K_y} = \beta_2 \phi e^{-\gamma} \frac{\partial f}{\partial K_y} = \beta_2 \phi \frac{c_y}{K_y}
\]
\[
(2.22)
\]

\[
\frac{\partial M}{\partial SD} = M_{SD} = \beta_2 \phi e^{-\gamma} \frac{\partial f}{\partial SD} = \beta_2 \phi \frac{dY}{SD}
\]
\[
(2.23)
\]

\[
\frac{\partial M}{\partial N} = M_N = \beta_2 \phi e^{-\gamma} \frac{\partial f}{\partial N} = \beta_2 \phi \frac{g_y}{N}
\]
\[
(2.24)
\]

\[
\frac{\partial M}{\partial L_s} = M_{L_s} = \beta_2 \phi e^{-\gamma} \frac{\partial f}{\partial L_s} = \beta_2 \phi \frac{e^{-\gamma}}{L_s}
\]
\[
(2.25)
\]

Partial derivatives of the nutrient depletion function due to biomass removal, D(Y), with respect to its arguments:

\[
\frac{\partial D}{\partial L_y} = D_{L_y} = \beta_2 \frac{\partial f}{\partial L_y} = \beta_2 \frac{b_y}{L_y}
\]
\[
(2.26)
\]

\[
\frac{\partial D}{\partial K_y} = D_{K_y} = \beta_2 \frac{\partial f}{\partial K_y} = \beta_2 \frac{c_y}{K_y}
\]
\[
(2.27)
\]

\[
\frac{\partial D}{\partial SD} = D_{SD} = \beta_2 \frac{\partial f}{\partial SD} = \beta_2 \frac{dY}{SD}
\]
\[
(2.28)
\]

\[
\frac{\partial D}{\partial N} = D_N = \beta_2 \frac{\partial f}{\partial N} = \beta_2 \frac{g_y}{N}
\]
\[
(2.29)
\]

Partial derivative of the nutrient augmentation function:

\[
\frac{\partial G}{\partial F} = \beta_1
\]
\[
(2.30)
\]
Accordingly,

\[ H_{\lambda \gamma} = \frac{g L_{\lambda}}{b N} \]  \hspace{1cm} (2.31) 

\[ H_{\kappa \gamma} = \frac{g K_{\gamma}}{c N} \]  \hspace{1cm} (2.32) 

\[ H_{\lambda N} = \frac{\varphi g Y}{\tau \alpha e^{-a t_{d}} N} \]  \hspace{1cm} (2.33) 

\[ D_{\lambda \gamma} - M_{\lambda \gamma} = \frac{b Y}{L_{\gamma}} \left( \beta_{2} - \beta_{s} \phi \alpha e^{-\alpha \gamma} \right) = \frac{b Y}{L_{\gamma}} \xi \]  \hspace{1cm} (2.34) 

Where: \( \beta_{2} - \beta_{s} \phi \alpha e^{-\alpha \gamma} = \xi \)

\[ D_{\kappa \gamma} - M_{\kappa \gamma} = \frac{c Y}{K_{\gamma}} \left( \beta_{2} - \beta_{s} \phi \alpha e^{-\alpha \gamma} \right) = \frac{c Y}{K_{\gamma}} \xi \]  \hspace{1cm} (2.35) 

\[ D_{SD} - M_{SD} = \frac{d Y}{SD} \left( \beta_{2} - \beta_{s} \phi \alpha e^{-\alpha \gamma} \right) = \frac{d Y}{SD} \xi \]  \hspace{1cm} (2.36) 

\[ D_{N} - M_{N} = \frac{g Y}{N} \left( \beta_{2} - \beta_{s} \phi \alpha e^{-\alpha \gamma} \right) = \frac{g Y}{N} \xi \]  \hspace{1cm} (2.37) 

Given the above formulations, the optimal solutions derived from the first order conditions for the soil-mining scenario are specified below:

\[ \frac{W_{F}}{G_{F}} = \frac{W_{F}}{\beta_{1}} \]  \hspace{1cm} (2.38) 

\[ \frac{P_{f} L_{\gamma} - W_{L}}{D_{\lambda \gamma} - M_{\lambda \gamma}} = \frac{P b Y - W_{L} L_{\gamma}}{b Y \xi} \]  \hspace{1cm} (2.39)
\[
\frac{Pf_{k_y} - W_K}{D_{k_y} - M_{k_y}} = \frac{PcY - W_K K_Y}{cY^2_S} \quad (2.40)
\]

\[
W_{L_S} = \frac{W_S}{M_{L_S}} \beta_3 \tau \alpha e^{-\alpha L_S} \quad (2.41)
\]

\[
\frac{Pf_N}{(D_N - M_N) + \delta} = \frac{PgY}{gY^2 + \delta N} \quad (2.42)
\]

\[
G(F) = D(Y) - M(Z, L_S, Y) \Rightarrow \beta_1 F = \beta_2 Y - \beta_3 [Z - \tau e^{-\alpha L_S} + \phi(1 - e^{-\gamma Y})] \quad (2.43)
\]
Appendix III. Derivation of the reduced form solutions for the choice variables (L_Y, L_S, K_Y and F) and the optimal nutrient stock (N) for the soil-mining scenario

Equating equation (2.39) with equation (2.40)

\[
\frac{PbY - W_L L_Y}{bY \xi} = \frac{PcY - W_K K_Y}{cY \xi}
\]

\(c(PbY - W_L L_Y) = b(PcY - W_K K_Y)\)

\(cW_L L_Y = bW_K K_Y\)

\[L_Y = \frac{bW_K K_Y}{cW_L}\] \hspace{1cm} (3.1a)

\[K_Y = \frac{cW_L L_Y}{bW_K}\] \hspace{1cm} (3.2a)

Equating equation (2.38) with equation (2.39) to solve for N and assuming \(\beta_i = 1:\)

\[W_F = \frac{PbY - W_L L_Y}{bY \xi}\]

\[W_F bY \xi = PbY - W_L L_Y\]

\[W_L L_Y = PbY - W_F bY \xi\]

\[W_L L_Y = bY(P - W_F \xi)\]

\[\frac{bY}{W_L L_Y} = \frac{b(\beta_i K^c N^\xi)}{W_L} = \frac{1}{(P - W_F \xi)}\] \hspace{1cm} (3.3)
Substituting equation (3.2a) into equation (3.3):

\[
\left( \frac{b}{W_L} \right) A L_{x}^{b-1} \left( \frac{c W_L}{W_K b} \right)^c L_{y} N^c = \frac{1}{(P - W_f \xi)} \]

\[
AL_{x}^{(b+c-1)} \left( \frac{b}{W_L} \right)^{1-c} \left( \frac{c}{W_K} \right)^c N^c = \frac{1}{(P - W_f \xi)}
\]

\[
L_y = \left( \frac{b}{W_L} \right)^{c-1} \left( \frac{c}{W_K} \right)^{-c} \left( \frac{1}{AN^c (P - W_f \xi)} \right)^{\frac{1}{b+c-1}} \]

(3.1b)

Substituting equation (3.1b) back to equation (3.2a)

\[
K_y = \left( \frac{c}{W_K} \right)^{-b} \left( \frac{b}{W_L} \right) \left[ \left( \frac{b}{W_L} \right)^{c-1} \left( \frac{c}{W_K} \right)^{-c} \left( \frac{1}{AN^c (P - W_f \xi)} \right)^{\frac{1}{b+c-1}} \right]
\]

\[
K_y = \left( \frac{b}{W_L} \right)^{b-1} \left( \frac{c}{W_K} \right)^{b-1} \left( \frac{1}{AN^c (P - W_f \xi)} \right)^{\frac{1}{b+c-1}} \]

(3.2b)

Equating equation (2.38) with equation (2.42) to solve for N:

\[
W_f = \frac{P g Y}{g Y \xi + \delta N}
\]

\[
W_f g Y \xi + W_f \delta N = P g Y
\]

\[
W_f \delta N = P g Y - W_f g Y \xi
\]

\[
g Y (P - W_f \xi) = W_f \delta N
\]
\[ \frac{Y}{N} = A L^b K^c N^{e-1} = \frac{W_F \delta}{g(P - W_F \xi)} \]  

(3.4a)

Considering the LHS and substituting equations (3.1b and 3.2b)

\[ A L^b = A \left( \frac{b}{W_L} \right)^{bc-b} \left( \frac{c}{W_K} \right)^{-bc} \left( \frac{1}{AN^e (P - W_F \xi)} \right)^{b+c-1} \]

\[ K^c N^{e-1} = \left( \frac{b}{W_L} \right)^{bc-c} \left( \frac{c}{W_K} \right)^{c} \left( \frac{1}{AN^e (P - W_F \xi)} \right)^{b+c-1} N^{e-1} \]

Pulling the components of the LHS together and equating with the RHS

\[ \left[ A^{-1} \left( \frac{b}{W_L} \right)^{-b} \left( \frac{c}{W_K} \right)^{-c} \left( \frac{1}{AN^e (P - W_F \xi)} \right)^{b+c-1} \right]^{e-1} = \frac{W_F \delta}{g(P - W_F \xi)} \]

\[ N^{\frac{-e(b+c)}{b+c-1}} (N^{e-1}) = \left[ A \left( \frac{c}{W_K} \right)^{c} \left( \frac{b}{W_L} \right)^{b} (P - W_F \xi)^{b+c} \right]^{1/b+c-1} \]

\[ N^{\frac{1-g-c-b}{b+c-1}} = \left[ A \left( \frac{b}{W_K} \right)^{b} \left( \frac{b}{W_L} \right)^{b+c-1} \left( \frac{W_F \delta}{g} (P - W_F \xi) \right)^{1/b+c-1} \right]^{1} \]

Let: \( 1 - g - c - b = \sigma \),

Accordingly, substituting the above expression and solving for N:

\[ \sigma \]

\[ N^{\frac{1}{b+c-1}} = A^{\frac{1}{b+c-1}} \left( \frac{c}{W_K} \right)^{\frac{c}{b+c-1}} \left( \frac{b}{W_L} \right)^{\frac{b}{b+c-1}} \left( \frac{W_F \delta}{g} (P - W_F \xi) \right)^{\frac{1}{b+c-1}} \]
\[ N^* = A^{\frac{1}{\alpha}} \left( \frac{c}{W_K} \right)^{\frac{c}{\alpha}} \left( \frac{b}{W_L} \right)^{\frac{b}{\alpha}} \left( \frac{W_F \delta}{g} \right)^{\frac{b+c-1}{\alpha}} \left( P - W_F \xi \right)^{\frac{1}{\alpha}} \] \hspace{1cm} \text{(3.4b)}

Substituting the values of \( N \) from equation (3.4b) into equation (3.1b) to solve for the optimal value of \( L_Y \):

From equation (3.1b), we have

\[ L_Y = \left( \frac{b}{W_L} \right)^{\frac{b-1}{b+c-1}} \left( \frac{c}{W_K} \right)^{\frac{c-1}{b+c-1}} \left( \frac{1}{AN^* (P - W_F \xi)} \right)^{\frac{1}{b+c-1}} \]

Substituting equation (3.4b) to the above and solving for \( L_Y \):

\[ L_Y = \left( \frac{c}{W_K} \right)^{\frac{-c}{b+c-1}} \left( \frac{b}{W_L} \right)^{\frac{c-1}{b+c-1}} A^{\frac{1}{(b+c-1)}} \left( P - W_F \xi \right)^{\frac{-1}{b+c-1}} \left[ A^{\frac{1}{\alpha}} \left( \frac{c}{W_K} \right)^{\frac{c}{\alpha}} \left( \frac{b}{W_L} \right)^{\frac{b}{\alpha}} \left( \frac{W_F \delta}{g} \right)^{\frac{b+c-1}{\alpha}} \left( P - W_F \xi \right)^{\frac{1}{\alpha}} \right]^{\frac{-g}{(b+c-1)}} \]

\[ L_Y = A^{\frac{1}{\alpha}} \left( \frac{c}{W_K} \right)^{\frac{c}{\alpha}} \left( \frac{b}{W_L} \right)^{\frac{b-1}{b+c-1}} \left( \frac{W_F \delta}{g} \right)^{\frac{-g}{b+c-1}} \left( P - W_F \xi \right)^{\frac{1}{b+c-1}} \] \hspace{1cm} \text{(3.1c)}

Substituting the values of \( N \) from equation (3.4b) to equation (3.2b) to solve for the optimal value of \( K_Y \):

From equation (3.2b), we have

\[ K_Y = \left( \frac{b}{W_L} \right)^{\frac{-b}{b+c-1}} \left( \frac{c}{W_K} \right)^{\frac{b-1}{b+c-1}} \left( \frac{1}{AN^* (P - W_F \xi)} \right)^{\frac{1}{b+c-1}} \]
Substituting equation (3.4b) to the above and solving for $K_Y$:

$$
K_Y = \left( \frac{c}{W_k} \right)^{\frac{b}{b+1}} \left( \frac{b}{W_L} \right)^{\frac{-b}{b+1}} A^{\frac{-1}{b+1}} \left( P - W_L \xi \right)^{\frac{-1}{\sigma}} \left[ \frac{1}{A} \left( \frac{c}{W_k} \right)^{\frac{b}{b+1}} \left( \frac{b}{W_L} \right)^{\frac{-b}{b+1}} \left( P - W_L \xi \right)^{\frac{1}{\sigma}} \right]^{-\frac{1}{b+1}}
$$

$$
K^* = A^{\frac{b(1-\alpha)-c}{\sigma(b+c-1)}} \left( \frac{b}{W_L} \right)^{\frac{b}{b+c-1}} \left( \frac{b}{g} \right)^{\frac{-1}{\sigma}} \left( P - W_L \xi \right)^{\frac{1}{\sigma}}
$$

Equating equation (2.38) with equation (2.41) to solve for $L_S$:

$$
W_F = \frac{W_s}{\beta \gamma \alpha e^{-\alpha s}}
$$

$$
W_F \beta \gamma \alpha e^{-\alpha s} = W_s
$$

$$
e^{-\alpha s} = W_s
$$

$$
e^{\alpha s} = \frac{W_F \beta \gamma \alpha}{W_s}
$$

$$
\alpha L_S = \ln \left( \frac{W_F \beta \gamma \alpha}{W_s} \right)
$$

$$
L^*_S = \left( \frac{1}{\alpha} \right) \ln \left( \frac{W_F \beta \gamma \alpha}{W_s} \right)
$$

Given the optimal values for $L_Y$, $K_Y$ and $N$ above, the optimal output at a desirable steady state is given by:

$$
Y^* = A L^*_Y K^*_Y N^*
$$

Where : $L^*_Y$, $K^*_Y$, $N^*$ are given by equations (3.1c, 3.2c and 3.4b), respectively.
Solving equation (2.43) provides the steady state optimal fertilizer use as follows

\[ \beta_i F = \beta_2 Y - \beta_3 \left[ Z - e^{-\alpha s} + \phi(1 - e^{-\gamma Y}) \right] \]

\[ F^* = \left\{ \beta_2 Y^* - \beta_3 \left[ Z - e^{-\alpha s} + \phi(1 - e^{-\gamma Y}) \right] \right\} / \beta_1 \]  

(3.7)

Where: \( Y^* \) and \( L^*_s \) are given by equations (3.6 and 3.5), respectively.
Appendix IV. Current Value Hamiltonian Version and Derivation of Reduced form Solutions for the problem of physical soil degradation and nutrient mining (scenario II)

The current value Hamiltonian of this scenario is given by:

\[
\Pi_c(F, L_y, L_s, K_y, SD, N, \psi, \eta) = Pf(L_y, K_y, SD, N) - \left(W_f F + W_L L_y + W_s L_s + W_k K_y \right) \\
+ \psi[H(Z, L_s, Y)] + \eta[G(F) - D(Y) + M(Z, L_s, Y)]
\]  

(4.1)

Where: \( \psi = e^{\delta\lambda} \) and \( \eta = e^{\delta\mu} \)  

(4.2)

The FOC for this system:

\[
\frac{\partial \Pi_c}{\partial F} = -W_f + \eta \frac{\partial G}{\partial F} = 0
\]  

(4.3)

\[
\frac{\partial \Pi_c}{\partial L_y} = P \frac{\partial f}{\partial L_y} - W_L + \psi \frac{\partial H}{\partial L_y} - \eta \frac{\partial D}{\partial L_y} + \eta \frac{\partial M}{\partial L_y} = 0
\]  

(4.4)

\[
\frac{\partial \Pi_c}{\partial K_y} = P \frac{\partial f}{\partial K_y} - W_k + \psi \frac{\partial H}{\partial K_y} - \eta \frac{\partial D}{\partial K_y} + \eta \frac{\partial M}{\partial K_y} = 0
\]  

(4.5)

\[
\frac{\partial \Pi_c}{\partial L_s} = -W_L + \psi \frac{\partial H}{\partial L_s} + \eta \frac{\partial M}{\partial L_s} = 0
\]  

(4.6)

\[
\psi = \delta \psi - \frac{\partial \Pi_c}{\partial SD} = \delta \psi - P \frac{\partial f}{\partial SD} - \psi \frac{\partial H}{\partial SD} + \eta(\frac{\partial D}{\partial SD} - \frac{\partial M}{\partial SD})
\]  

(4.7)

\[
\eta = \delta \eta - \frac{\partial \Pi_c}{\partial N} = \delta \eta - P \frac{\partial f}{\partial N} - \psi \frac{\partial H}{\partial N} + \eta(\frac{\partial D}{\partial N} - \frac{\partial M}{\partial N})
\]  

(4.8)

\[
\frac{\partial \Pi_c}{\partial \psi} = \frac{\partial \Pi_c}{\partial SD} = Z - E + J
\]  

(4.9)

\[
\frac{\partial \Pi_c}{\partial \eta} = \frac{\partial \Pi_c}{\partial N} = G(F) - D(Y) + M(Z, L_s, Y)
\]  

(4.10)
In a steady state the rate of change of the resource stock and hence the implicit prices are zero. That is $\dot{\psi} = \dot{\eta} = \dot{N} = \dot{SD} = 0$.

The first order conditions could, therefore, be restated as follows:

$$\eta = \frac{W_f}{G_f} \quad (4.3b)$$

$$\eta = \frac{(P_f L_r - W_L) + \psi H_{L_r}}{D_{L_r} - M_{L_r}} \quad (4.4b)$$

$$\eta = \frac{(P_f K_r - W_K) + \psi H_{K_r}}{D_{K_r} - M_{K_r}} \quad (4.5b)$$

$$\eta = \frac{W_S - \psi H_{L_s}}{M_{L_s}} \quad (4.6b)$$

$$\eta = \frac{P_f SD + \psi (H_{SD} - \delta)}{D_{SD} - M_{SD}} \quad (4.7b)$$

$$\eta = \frac{P_f N + \psi H_{K_N}}{D_N - M_N + \delta} \quad (4.8b)$$

$$Z = E - J \quad (4.9b)$$

$$G(F) = D(Y) - M(Z, L_z, Y) \quad (4.10b)$$

Equation (4.9b) above states that at the steady state the net rate of natural soil regeneration ($Z$) is exactly matched by the net soil loss due to erosion and cultivation ($E - J$). Analogously, equation (4.10b) describes that the sum of nutrients lost through crop harvest, damage function $D(Y)$ and net nutrient gains/losses though $M$ are matched by the nutrients added to the soil through the nutrient augmentation function $G(F)$.
Combining equations 4.3b with each of 4.4b through 4.8b and eliminating \( \eta \), the following equations are derived:

\[
\psi = \frac{(W_F / G_F) (D_{L_T} - M_{L_T}) - (Pf_{L_T} - W_L)}{H_{L_T}} \tag{4.11}
\]

\[
\psi = \frac{-[(W_F / G_F) M_{L_S} - W_S]}{H_{L_S}} \tag{4.12}
\]

\[
\psi = \frac{(W_F / G_F) (D_{K_T} - M_{K_T}) - (Pf_{K_T} - W_K)}{H_{K_T}} \tag{4.13}
\]

\[
\psi = \frac{(W_F / G_F) (D_{SD} - M_{SD}) - Pf_{SD}}{H_{SD} - \delta} \tag{4.14}
\]

\[
\psi = \frac{(W_F / G_F) (D_N - M_N + \delta) - Pf_N}{H_N} \tag{4.15}
\]

Further combining equations (4.11 through 4.13) first with Equations (4.14) and then with equation (4.15), the following equations are derived.

\[
Pf_N + \frac{H_N}{H_{L_T}} \left[ \left( \frac{W_F}{G_F} \right) (D_{L_T} - M_{L_T}) - (Pf_{L_T} - W_L) \right] = \frac{W_F}{G_F} (\delta + D_N - M_N) \tag{4.16}
\]

\[
Pf_N + \frac{H_N}{H_{K_T}} \left[ \left( \frac{W_F}{G_F} \right) (D_{K_T} - M_{K_T}) - (Pf_{K_T} - W_K) \right] = \frac{W_F}{G_F} (\delta + D_N - M_N) \tag{4.17}
\]

\[
Pf_N + \frac{H_N}{H_{L_S}} \left[ (W_S - M_{L_S}) \left( \frac{W_F}{G_F} \right) \right] = \frac{W_F}{G_F} (\delta + D_N - M_N) \tag{4.18}
\]
The above equations along with equations (4.9b) and (4.10b) form a system of eight equations with eight unknowns and could be solved simultaneously for optimal steady state values of the four choice variables (F, L\textsubscript{Y}, K\textsubscript{Y}, L\textsubscript{S}), the optimal resource stocks (SD and N) and the respective dynamic prices (Ψ and η).

Given the above formulations and the functional relationships given in Appendix II equations (4.16 – 4.21) and (4.9b and 4.10b) are specified as follows:

\[ P_{fSD} + \left( \frac{H_{SD} - \delta}{H_{K_Y}} \right) \left( \frac{W_{F}}{G_{F}} \right) (D_{K} - M_{K}) - (P_{fK} - W_{K}) = \frac{W_{F}}{G_{F}} (D_{SD} - M_{SD}) \]  
(4.20)

\[ P_{fSD} + \left( \frac{H_{SD} - \delta}{H_{L_Y}} \right) \left( \frac{W_{F}}{G_{F}} \right) (D_{L} - M_{L}) - (P_{fL} - W_{L}) = \frac{W_{F}}{G_{F}} (D_{SD} - M_{SD}) \]  
(4.21)

The above equations along with equations (4.9b) and (4.10b) form a system of eight equations with eight unknowns and could be solved simultaneously for optimal steady state values of the four choice variables (F, L\textsubscript{Y}, K\textsubscript{Y}, L\textsubscript{S}), the optimal resource stocks (SD and N) and the respective dynamic prices (Ψ and η).

Given the above formulations and the functional relationships given in Appendix II equations (4.16 – 4.21) and (4.9b and 4.10b) are specified as follows:

\[ P_{gY} + \frac{G}{b} \left[ W_{F} b Y \xi - P b Y + W_{L} L \xi \right] = W_{F} \left( \delta N + g Y \xi \right) \]  
(4.22)

\[ P_{gY} + \frac{G}{c} \left[ W_{F} c Y \xi - P c Y + W_{K} K \xi \right] = W_{F} \left( \delta N + g Y \xi \right) \]  
(4.23)

\[ P_{gY} + \frac{\phi_{Y}}{\alpha_{Y}e^{-a_{Y}}} \left[ W_{S} - \beta_{Y} \alpha_{Y}e^{-a_{Y}} \right] = W_{F} \left( \delta N + g Y \xi \right) \]  
(4.24)

\[ P_{dY} + \left( \frac{\phi_{Y} - \delta_{SD}}{q_{Y}} \right) \left[ W_{F} b Y \xi - P b Y + W_{L} L \xi \right] = W_{F} d Y \xi \]  
(4.25)

\[ P_{dY} + \left( \frac{\phi_{Y} - \delta_{SD}}{q_{Y}} \right) \left[ W_{F} c Y \xi - P c Y + W_{K} K \xi \right] = W_{F} d Y \xi \]  
(4.26)

\[ P_{dY} + \left( \frac{\phi_{Y} - \delta_{SD}}{\alpha_{Y}e^{-a_{Y}}} \right) \left[ W_{S} - \beta_{Y} \alpha_{Y}e^{-a_{Y}} \right] = W_{F} d Y \xi \]  
(4.27)
\[ Z = \alpha e^{-ax} - \phi(1 - e^{-vy}) \quad (4.28) \]

\[ \beta_1 F = \beta_2 Y - \beta_3 [Z - \alpha e^{-ax} + \phi(1 - e^{vy})] \quad (4.29) \]

Using the above formulations, the reduced form solutions for the choice variables (L_Y, L_S, K_Y and F) and the optimal nutrient stocks (N and SD) are solved as follows:

Using equation (4.22) to solve for L_Y:

\[ P g Y + \left( \frac{g}{b} \right) (W_r b Y \xi - P b Y + W_L L_Y) = W_r \delta N + W_r g Y \epsilon \]

\[ P g Y + g W_r Y \xi - g P Y + \frac{g}{b} W_L L_Y = W_r \delta N + W_r g Y \epsilon \]

\[ \frac{g}{b} W_L L_Y = W_r \delta N \]

\[ L_Y = \frac{W_r \delta b}{g W_L} N = \left( \frac{W_r \delta}{g} \right) \left( \frac{b}{W_L} \right) N \quad (4.30a) \]

Using equation (4.23) to solve for K_Y:

\[ P g Y + \left( \frac{g}{c} \right) (W_r c Y \xi - P c Y + W_k K_Y) = W_r \delta N + W_r g Y \epsilon \]

\[ P g Y + g W_r Y \xi - g P Y + \frac{g}{c} W_k K_Y = W_r \delta N + W_r g Y \epsilon \]

\[ \frac{g}{c} W_k K_Y = W_r \delta N \]

\[ K_Y = \frac{W_r \delta c}{g W_k} N = \left( \frac{W_r \delta}{g} \right) \left( \frac{c}{W_k} \right) N \quad (4.31a) \]
Using equation (4.25) to solve for SD:

\[ PdY + \left( \frac{\varphi dY - \delta SD}{\varphi b Y} \right) (W_r b Y \xi - P b Y + W_L L_y) = W_r d Y \varepsilon \]

\[ \varphi b Y P d Y + \varphi d Y W_r b Y \xi - \varphi d Y P b Y + \varphi d Y W_L L_y - \delta S D W_r b Y \xi + \delta S D P b Y - \delta S D W_L L_y = W_r d Y \xi \varphi b Y \]

\[ \varphi d Y W_L L_y - \delta S D W_r b Y \xi + \delta S D P b Y - \delta S D W_L L_y = 0 \]

\[ \varphi d Y W_L L_y - \delta S D W_r b Y \xi + \delta S D P b Y = \delta S D W_L L_y \]

\[ Y(\varphi d W_L L_y - \delta S D W_r b Y \xi + \delta S D P b) = \delta S D W_L L_y \]

\[ Y = \frac{\delta S D W_L L_y}{(\varphi d W_L L_y - \delta S D W_r b Y \xi + \delta S D P b)} \]

\[ Y = \frac{\delta S D W_L}{b \delta S D (P - W_r \xi) + \varphi d W_L L_y} \] (4.32)

Equating equations (4.25 and 4.27) and solving

\[ PdY + \left( \frac{\varphi d Y - \delta SD}{\varphi b Y} \right) (W_r b Y \xi - P b Y + W_L L_y) = PdY + \left( \frac{\varphi d Y - \delta SD}{\gamma \alpha e^{-\alpha t_s}} \right) (W_L - \beta_\gamma \alpha e^{-\alpha t_s}) \]

\[ W_r b Y \xi - P b Y + W_L L_y = \frac{\varphi b Y}{\gamma \alpha e^{-\alpha t_s}} (W_L - \beta_\gamma \alpha e^{-\alpha t_s}) \]

\[ W_r b Y \xi - P b Y + W_L L_y = \frac{\varphi b Y W_L e^{\alpha t_s}}{\gamma \alpha} - \beta_\gamma \varphi b Y \]

\[ \frac{\varphi b Y W_L e^{\alpha t_s}}{\gamma \alpha} - \beta_\gamma \varphi b Y - W_r b Y \xi + P b Y = W_L L_y \]
\[
b Y \left( \frac{\phi W_l e^{\alpha s}}{\gamma \alpha} - \beta_j \phi - W_f \xi + P \right) = W_L L_y
\]

\[
Y \frac{L_y}{L_y} = \frac{W_L}{b \left[ \phi \left( \frac{W_l e^{\alpha s}}{\gamma \alpha} - \beta_j \right) + (P - W_f \xi) \right]}
\]

Let: \( \phi \left( \frac{W_l e^{\alpha s}}{\gamma \alpha} - \beta_j \right) = \xi \), hence

\[
Y \frac{L_y}{L_y} = \frac{W_L}{b (\xi + P - W_f \xi)}
\]

(4.33)

Further equating equation (4.32 and 4.33)

\[
\frac{\delta SD W_L}{b \delta SD (P - W_f \xi) + \phi d L_y} = \frac{W_L}{b (\xi + P - W_f \xi)}
\]

\[
\delta SD (\xi + P - W_f \xi) = b \delta SD (P - W_f \xi) + \phi d L_y
\]

\[
\delta SD (\xi + P - W_f \xi) - b \delta SD (P - W_f \xi) = \phi d L_y
\]

\[
\delta SD (\xi + P - W_f \xi - P + W_f \xi) = \phi d L_y
\]

\[
\delta SD \xi = \phi d L_y
\]

\[
SD = \left( \frac{\phi d}{\delta \xi} \right) \left( \frac{W_L}{b} \right) L_y
\]

(4.34a)

Substituting equations (4.30a) into the above equation (4.34a)
Solving for N using equations (4.30a, 4.31a, 4.33 and 4.34b)

From equation (4.33), we have

\[
\frac{Y}{L_Y} = \frac{W_L}{b(\xi + P - W_f \xi)}
\]

Y has been specified as: \(Y = AL_Y^b K_Y^c N^d SD^e\)

Hence, \(\frac{Y}{L_Y} = AL_Y^{b-1} K_Y^c N^d SD^e = \left(\frac{W_L}{b}\right) \left(\frac{1}{\xi + P - W_f \xi}\right)\)

Substituting equations (4.30a, 4.31a, and 4.34b) for \(L_Y, K_Y\) and SD, respectively, into the above equation and solving for N:

\[
A \left[\left(\frac{W_f \delta}{g} \right) \left(\frac{b}{W_L}\right) \left(\frac{c}{W_k}\right) \left(\frac{\varphi l}{\delta \xi}\right) \left(\frac{W_r \delta}{g}\right) \right]^{N-1} \left[\left(\frac{W_f \delta}{g} \right) \left(\frac{b}{W_L}\right) \left(\frac{c}{W_k}\right) \left(\frac{\varphi l}{\delta \xi}\right) \left(\frac{W_r \delta}{g}\right) \right]^{d} (N)^{d} = \left(\frac{W_L}{b}\right) \left(\frac{1}{\xi + P - W_f \xi}\right)
\]

\[
A \left(\frac{W_f \delta}{g}\right)^{d+c+b-1} \left(\frac{b}{W_L}\right)^{b-1} \left(\frac{c}{W_k}\right)^{d} \left(\frac{\varphi l}{\delta \xi}\right)^{d} N^{d+c+b-1} = \left(\frac{W_L}{b}\right) \left(\frac{1}{\xi + P - W_f \xi}\right)
\]

Let \(g + d + c + b - 1 = \theta\), Substituting and solving for N:
\[ N^\theta = A^{-1} \left( \frac{W_F \delta}{g} \right)^{-(d+c+b-1)} \left( \frac{b}{W_L} \right)^b \left( \frac{c}{W_K} \right)^c \left( \frac{\partial \phi}{\partial \zeta} \right)^-d \left( \frac{1}{\zeta + P - W_F \xi} \right) \]

\[ N^\ast = A^{-1/\theta} \left( \frac{W_F \delta}{g} \right)^{-(d+c+b-1)} \left( \frac{b}{W_L} \right)^b \left( \frac{c}{W_K} \right)^c \left( \frac{\partial \phi}{\partial \zeta} \right)^-d \left( \frac{1}{\zeta + P - W_F \xi} \right)^{1/\theta} \quad (4.35) \]

Substituting the value of \( N \) from equation (4.35) into equation (4.30a) to solve for \( L_Y \):

From equation (4.30a), we have,

\[ L_Y = \left( \frac{W_F \delta}{g} \right) \left( \frac{b}{W_L} \right) N \]

Substituting equation (4.35) into the above equation and solving for \( L_Y \):

\[ L_Y = \left( \frac{W_F \delta}{g} \right) \left( \frac{b}{W_L} \right) \left[ A^{-1/\theta} \left( \frac{W_F \delta}{g} \right)^{-(d+c+b-1)} \left( \frac{b}{W_L} \right)^b \left( \frac{c}{W_K} \right)^c \left( \frac{\partial \phi}{\partial \zeta} \right)^-d \left( \frac{1}{\zeta + P - W_F \xi} \right)^{1/\theta} \right] \]

\[ L_Y = A^{-1/\theta} \left( \frac{W_F \delta}{g} \right)^{d+c+b-1} \left( \frac{b}{W_L} \right)^b \left( \frac{c}{W_K} \right)^c \left( \frac{\partial \phi}{\partial \zeta} \right)^-d \left( \frac{1}{\zeta + P - W_F \xi} \right)^{1/\theta} \quad (4.30b) \]

Likewise substituting the value of \( N \) from equation (4.35) into equation (4.31a) to solve for \( K_Y \):

From equation (4.31a), we have,

\[ K_Y = \left( \frac{W_F \delta}{g} \right) \left( \frac{c}{W_K} \right) N \]

Substituting equation (4.34) into the above equation
Substituting the value of $N$ from equation (4.35) into equation (4.34b) to solve for the optimal value of $SD$:

From equation (4.34b), we have,

$$SD = \left(\frac{\varphi d}{\delta}\right) \left(\frac{1}{\zeta}\right) \left(\frac{W_r \delta}{g}\right) N$$

Substituting equation (4.34) into the above equation

$$SD = \left(\frac{\varphi d}{\delta}\right) \left(\frac{W_r \delta}{g}\right) \left[ A^{-\frac{1}{\sigma}} \left(\frac{W_r \delta}{g}\right)^{\frac{d+\sigma+1}{\sigma}} \left(\frac{b}{W_L}\right)^{-\frac{b}{\sigma}} \left(\frac{c}{W_K}\right)^{-\frac{c}{\sigma}} \left(\frac{\varphi d}{\delta \zeta}\right)^{-\frac{d}{\sigma}} \left(\frac{1}{\zeta + P - W_r \xi}\right)^{\frac{1}{\sigma}} \right]$$

$$SD^* = A^{-\frac{1}{\sigma}} \left(\frac{W_r \delta}{g}\right)^{\frac{\sigma}{\sigma}} \left(\frac{b}{W_L}\right)^{-\frac{b}{\sigma}} \left(\frac{c}{W_K}\right)^{-\frac{c}{\sigma}} \left(\frac{\varphi d}{\delta \zeta}\right)^{-\frac{d}{\sigma}} \left(\frac{1}{\zeta + P - W_r \xi}\right)^{\frac{1}{\sigma}}$$

(4.34c)

Solving for the optimal value of $L_S$ using equations (4.24 and 4.27)

From equation (4.27)

$$PdY + \left(\frac{\varphi d Y - \delta SD}{\gamma c e^{-\alpha t_s}}\right) (W_S - \beta_S \gamma c e^{-\alpha t_s}) = W_r dY e$$
\[ PdY + (\alpha dY - \delta SD) \left( \frac{W_s e^{ad_s}}{\gamma \alpha} - \beta_3 \right) = W_f dY \epsilon \]

\[ PdY + \frac{\alpha dYW_s e^{ad_s}}{\gamma \alpha} - \alpha dY \beta_3 - \frac{\delta SD W_s e^{ad_s}}{\gamma \alpha} + \delta SD \beta_3 = W_f dY \epsilon \]

\[ PdY + \frac{\alpha dYW_s e^{ad_s}}{\gamma \alpha} - \alpha dY \beta_3 - W_f dY \epsilon = \frac{\delta SD W_s e^{ad_s}}{\gamma \alpha} - \delta SD \beta_3 \]

\[ dY \left( P + \frac{\varphi W_s e^{ad_s}}{\gamma \alpha} - \varphi \beta_3 - W_f \epsilon \right) = \delta SD \left( \frac{W_s e^{ad_s}}{\gamma \alpha} - \beta_3 \right) \]

\[ Y = \frac{\delta SD \left( \frac{W_s e^{ad_s}}{\gamma \alpha} - \beta_3 \right)}{d \left( P + \frac{\varphi W_s e^{ad_s}}{\gamma \alpha} - \varphi \beta_3 - W_f \epsilon \right)} = \frac{\delta SD \left( \frac{W_s e^{ad_s}}{\gamma \alpha} - \beta_3 \right)}{\varphi \left( \frac{W_s e^{ad_s}}{\gamma \alpha} - \beta_3 \right) + (P - W_f \epsilon)} \]

Since : \( \varphi \left( \frac{W_s e^{ad_s}}{\gamma \alpha} - \beta_3 \right) = \zeta \), the above equation could be rewritten as

\[ Y = \left( \frac{\delta}{d} \right) \left( \frac{W_s e^{ad_s}}{\gamma \alpha} - \beta_3 \right) SD \]

(4.36)

Similarly using equation (4.24)

\[ PgY + \left( \frac{\varphi g Y}{\gamma \alpha e^{-ad_s}} \right) (W_s - \beta_3 \gamma \alpha e^{-ad_s}) = W_f (\delta N + gY \epsilon) \]
Equating equations (4.36 and 4.37) to solve for $L_S$:

\[
\delta \left( \frac{W_s e^{ad_s}}{\gamma \alpha} - \beta_3 \right) \frac{d}{d} \left( N^* \right) = \left( \frac{W_f \delta}{g} \right) \left( \frac{1}{\zeta + P - W_f e} \right) SD^* \\

W_s e^{ad_s} \frac{1}{\gamma \alpha} - \beta_3 = \left( \frac{W_f \delta}{g} \right) \left( \frac{d}{d} \right) \left( \frac{N^*}{SD^*} \right)
\]

Substituting equations (4.35 and 4.34c) for $N^*$ and $SD^*$ and solving

\[
W_s e^{ad_s} \frac{1}{\gamma \alpha} - \beta_3 = \frac{\zeta}{\phi}
\]

\[
e^{ad_s} = \frac{\gamma \alpha}{W_s} \left( \frac{\zeta}{\phi} + \beta_3 \right)
\]
\[ aL_s = \ln \left( \frac{\gamma x}{W_s} \left( \frac{\zeta}{\varphi} + \beta_3 \right) \right) \]

\[ L_s = \left( \frac{1}{\alpha} \right) \ln \left( \frac{\gamma x}{W_s} \left( \frac{\zeta}{\varphi} + \beta_3 \right) \right) \]  

(4.38)

Given the optimal values for \( L_Y, K_Y, N \) and \( SD \) above, the optimal output at a desirable steady state is given by:

\[ Y^* = AL^*_Y K_Y^* N^* SD^* \]  

(4.39)

Where: \( L_Y^*, K_Y^*, N^* \) and \( SD^* \) are given by equations (4.30b, 4.31b, 4.35 and 4.34c), respectively.

Solving equation (4.29) provides the steady state optimal fertilizer use:

\[ \beta_1 F = \beta_2 Y - \beta_3 \left[ Z - \alpha e^{-\alpha z} + \phi (1 - e^{-y}) \right] \]

\[ F^* = \left\{ \beta_2 Y^* - \beta_3 \left[ Z - \alpha e^{-\alpha z} + \phi (1 - e^{-y}) \right] \right\} / \beta_1 \]  

(4.40)

Where: \( Y^* \) and \( L_s^* \) are given by equations (4.39 and 4.38), respectively.
Appendix V. Derivation of static optimal solutions

\[ \text{Max} \Pi(F, L_y, L_s, K_y) = Pf(L_y, K_y, F) - W_L F - W_L L_y - W_k K_y \]  \hspace{1cm} (5.1)

The FOC for this system:

\[ \frac{\partial \Pi}{\partial F} = P \frac{\partial f}{\partial F} - W_F = 0 \quad \Rightarrow \quad P \frac{Y}{F} = W_F \]  \hspace{1cm} (5.2)

\[ \frac{\partial \Pi}{\partial L_y} = P \frac{\partial f}{\partial L_y} - W_L = 0 \quad \Rightarrow \quad P \frac{b}{L_y} = W_L \]  \hspace{1cm} (5.3)

\[ \frac{\partial \Pi}{\partial K_y} = P \frac{\partial f}{\partial K_y} - W_k = 0 \quad \Rightarrow \quad P \frac{c}{K_y} = W_k \]  \hspace{1cm} (5.4)

Equations (5.2 to 5.4) could be solved simultaneously for the optimal values of $L_y$, $K_y$ and $F$ as follows.

Combining equations (5.2 and 5.3),

\[ \frac{P_y}{F} \frac{L_y}{P_b} = \frac{W_F}{W_L} \]

\[ L_y = \frac{W_F}{b} \frac{g}{W_L} F \]  \hspace{1cm} (5.5)

Combining equations 5.2 and 5.4

\[ \frac{P_y}{F} \frac{K_y}{P_c} = \frac{W_F}{W_K} \]

\[ K_y = \frac{W_F}{g} \frac{c}{W_K} F \]  \hspace{1cm} (5.6)
Combining equations 5.3 and 5.4

\[
\frac{PbY}{L_y} * \frac{K_y}{PcY} = \frac{W_L}{W_K}
\]

\[
K_y = \frac{W_L}{b} \frac{c}{W_K} L_y
\]  

(5.7)

Solving for \(L_y\) using equation 5.2:

\[
\frac{PgY}{F_y} = W_f
\]

Since, \(Y = AL_y^b K_y^c F_s^\gamma\), the above can be written as:

\[
AL_y^b K_y^c F_s^\gamma = \frac{W_L}{gP}
\]

Substituting equations (5.5 and 5.6) into the above expression

\[
A \left(\frac{W_f}{g} \frac{b}{W_L} F\right)^b \left(\frac{W_L}{g} \frac{c}{W_K} F\right)^c F_s^\gamma = \frac{W_L}{gP}
\]

\[
F^{c+b-1} \left(\frac{W_f}{g}\right)^{c+b} \left(\frac{b}{W_L}\right)^b \left(\frac{c}{W_K}\right)^c = \frac{W_f}{PAg}
\]

Let \(1-g-c-b = \sigma\), then \(g+c+b-1 = -\sigma\)

Substituting and solving for \(F\):

\[
F^{-\sigma} = \left(\frac{W_f}{g}\right)^{1-c-b} \left(\frac{b}{W_L}\right)^b \left(\frac{c}{W_K}\right)^c \left(\frac{1}{PA}\right)
\]

\[
F^\gamma = \left(\frac{1}{PA}\right)^{\frac{1}{\sigma}} \left(\frac{b}{W_L}\right)^{\frac{b}{\sigma}} \left(\frac{c}{W_K}\right)^{\frac{c}{\sigma}} \left(\frac{W_f}{g}\right)^{\frac{c+b-1}{\sigma}}
\]  

(5.8)
Solving for $L_Y$ using equation (5.5):

$$L_Y = \frac{W_F}{g} \frac{b}{W_L}$$

Substituting equation (5.8) into the above expression and solving for $L_Y$:

$$L_Y = \frac{W_F}{g} \frac{b}{W_L} \left( \frac{W_F}{g} \right)^{\frac{c-b-1}{\sigma}} \left( \frac{b}{W_L} \right)^{\frac{1}{\sigma}} \left( \frac{c}{W_K} \right)^{\frac{1}{\sigma}} \left( \frac{1}{PA} \right)^{\frac{1}{\sigma}}$$

$$L_Y^{*} = \left( \frac{1}{PA} \right)^{\frac{1}{\sigma}} \left( \frac{b}{W_L} \right)^{\frac{1-g-c}{\sigma}} \left( \frac{c}{W_K} \right)^{\frac{1}{\sigma}} \left( \frac{W_F}{g} \right)^{\frac{g}{\sigma}} \left( \frac{b}{W_L} \right)^{\frac{b}{\sigma}} \left( \frac{c}{W_K} \right)^{\frac{1-g-b}{\sigma}} \left( \frac{W_F}{g} \right)^{\frac{g}{\sigma}}$$

Solving for $L_K$ using equation (5.7),

$$K_Y = \frac{W_L}{b} \frac{c}{W_K} L_Y$$

Substituting equation (5.9) into the above expression

$$K_Y = \frac{W_L}{b} \frac{c}{W_K} \left( \frac{1}{PA} \right)^{\frac{1}{\sigma}} \left( \frac{W_F}{g} \right)^{\frac{g}{\sigma}} \left( \frac{b}{W_L} \right)^{\frac{b}{\sigma}} \left( \frac{c}{W_K} \right)^{\frac{1-g-b}{\sigma}} \left( \frac{W_F}{g} \right)^{\frac{g}{\sigma}}$$

$$K_Y^{*} = \left( \frac{1}{PA} \right)^{\frac{1}{\sigma}} \left( \frac{b}{W_L} \right)^{\frac{b}{\sigma}} \left( \frac{c}{W_K} \right)^{\frac{1-g-b}{\sigma}} \left( \frac{W_F}{g} \right)^{\frac{g}{\sigma}}$$

(5.10)
Appendix VI. Soil loss for two plot categories estimated using the USLE modified for Ethiopia

<table>
<thead>
<tr>
<th>Conservation structure</th>
<th>Crop type</th>
<th>Plot category</th>
<th>Rainfall erosivity</th>
<th>Soil erodibility</th>
<th>Slope length</th>
<th>Slope gradient</th>
<th>Land cover</th>
<th>Management factor</th>
<th>Soil loss (Ton/ha)</th>
</tr>
</thead>
<tbody>
<tr>
<td>No</td>
<td>Tef</td>
<td>Uplands</td>
<td>430.2</td>
<td>0.25</td>
<td>2.1</td>
<td>1.78</td>
<td>0.25</td>
<td>0.75</td>
<td>75.38</td>
</tr>
<tr>
<td>No</td>
<td>Tef</td>
<td>Bottomlands</td>
<td>430.2</td>
<td>0.15</td>
<td>3.5</td>
<td>0.4</td>
<td>0.25</td>
<td>0.75</td>
<td>16.94</td>
</tr>
<tr>
<td>Yes</td>
<td>Tef</td>
<td>Uplands</td>
<td>430.2</td>
<td>0.25</td>
<td>0.6</td>
<td>1.78</td>
<td>0.25</td>
<td>0.9</td>
<td>25.84</td>
</tr>
<tr>
<td>Yes</td>
<td>Tef</td>
<td>Bottomlands</td>
<td>430.2</td>
<td>0.15</td>
<td>1.2</td>
<td>0.4</td>
<td>0.25</td>
<td>0.9</td>
<td>6.97</td>
</tr>
<tr>
<td>No</td>
<td>Other cereals</td>
<td>Uplands</td>
<td>430.2</td>
<td>0.25</td>
<td>2.1</td>
<td>1.78</td>
<td>0.18</td>
<td>0.75</td>
<td>54.27</td>
</tr>
<tr>
<td>No</td>
<td>Other cereals</td>
<td>Bottomlands</td>
<td>430.2</td>
<td>0.15</td>
<td>3.5</td>
<td>0.4</td>
<td>0.18</td>
<td>0.75</td>
<td>12.20</td>
</tr>
<tr>
<td>Yes</td>
<td>Other cereals</td>
<td>Uplands</td>
<td>430.2</td>
<td>0.25</td>
<td>0.6</td>
<td>1.78</td>
<td>0.18</td>
<td>0.9</td>
<td>18.61</td>
</tr>
<tr>
<td>Yes</td>
<td>Other cereals</td>
<td>Bottomlands</td>
<td>430.2</td>
<td>0.15</td>
<td>1.2</td>
<td>0.4</td>
<td>0.18</td>
<td>0.9</td>
<td>5.02</td>
</tr>
<tr>
<td>No</td>
<td>Pulses</td>
<td>Uplands</td>
<td>430.2</td>
<td>0.25</td>
<td>2.1</td>
<td>1.78</td>
<td>0.15</td>
<td>0.75</td>
<td>45.23</td>
</tr>
<tr>
<td>No</td>
<td>Pulses</td>
<td>Bottomlands</td>
<td>430.2</td>
<td>0.15</td>
<td>3.5</td>
<td>0.4</td>
<td>0.15</td>
<td>0.75</td>
<td>10.16</td>
</tr>
<tr>
<td>Yes</td>
<td>Pulses</td>
<td>Uplands</td>
<td>430.2</td>
<td>0.25</td>
<td>0.6</td>
<td>1.78</td>
<td>0.15</td>
<td>0.9</td>
<td>15.51</td>
</tr>
<tr>
<td>Yes</td>
<td>Pulses</td>
<td>Bottomlands</td>
<td>430.2</td>
<td>0.15</td>
<td>1.2</td>
<td>0.4</td>
<td>0.15</td>
<td>0.9</td>
<td>4.18</td>
</tr>
</tbody>
</table>

Source: Shiferaw and Holden (1999)
Appendix VII. Parameter estimates of the multinomial logit soil fertility adoption model, Central highlands of Ethiopia, 2003

<table>
<thead>
<tr>
<th>Explanatory Variables</th>
<th>Seasonal fallowing (SF) or Crop rotations (LG)</th>
<th>Animal manure (AM) alone</th>
<th>Animal manure associated with either SF or LR</th>
<th>Inorganic fertilizers (IF) alone</th>
<th>Inorganic fertilizer associated with either SF, LR or MR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant</td>
<td>-3.6566**</td>
<td>0.000</td>
<td>-1.4414***</td>
<td>0.003</td>
<td>-3.3788***</td>
</tr>
<tr>
<td>Education</td>
<td>0.0327</td>
<td>0.378</td>
<td>0.1124***</td>
<td>0.042</td>
<td>0.0897</td>
</tr>
<tr>
<td>Off-farm income</td>
<td>0.1462</td>
<td>0.476</td>
<td>-0.1372</td>
<td>0.609</td>
<td>-0.1457</td>
</tr>
<tr>
<td>Livestock</td>
<td>-0.0137</td>
<td>0.582</td>
<td>0.0295</td>
<td>0.305</td>
<td>0.0136</td>
</tr>
<tr>
<td>Plot size</td>
<td>2.3760***</td>
<td>0.000</td>
<td>0.1526</td>
<td>0.797</td>
<td>1.5744</td>
</tr>
<tr>
<td>No. of plots</td>
<td>0.0365</td>
<td>0.349</td>
<td>-0.1194***</td>
<td>0.002</td>
<td>-0.0951</td>
</tr>
<tr>
<td>Plot distant</td>
<td>-0.0016</td>
<td>0.724</td>
<td>-0.1075***</td>
<td>0.000</td>
<td>-0.1086</td>
</tr>
<tr>
<td>Light</td>
<td>-0.1013</td>
<td>0.596</td>
<td>0.6588***</td>
<td>0.007</td>
<td>1.4831***</td>
</tr>
<tr>
<td>Medium</td>
<td>-0.1109</td>
<td>0.627</td>
<td>0.1745</td>
<td>0.517</td>
<td>0.9849</td>
</tr>
<tr>
<td>Sever</td>
<td>0.3095</td>
<td>0.262</td>
<td>-0.0326</td>
<td>0.932</td>
<td>0.8488</td>
</tr>
<tr>
<td>Tenure</td>
<td>0.0666</td>
<td>0.705</td>
<td>1.2011***</td>
<td>0.001</td>
<td>0.7503</td>
</tr>
<tr>
<td>Credit</td>
<td>0.5557**</td>
<td>0.029</td>
<td>0.4815**</td>
<td>0.075</td>
<td>-0.1725</td>
</tr>
<tr>
<td>Extension</td>
<td>0.0094</td>
<td>0.981</td>
<td>0.2558</td>
<td>0.624</td>
<td>0.9767</td>
</tr>
<tr>
<td>Agro-ecology</td>
<td>3.1615***</td>
<td>0.000</td>
<td>0.9560***</td>
<td>0.001</td>
<td>1.6719***</td>
</tr>
<tr>
<td>Kossi</td>
<td>-1.2191***</td>
<td>0.049</td>
<td>1.3675***</td>
<td>0.001</td>
<td>-0.0151</td>
</tr>
</tbody>
</table>

Diagnostics

- No. Observations: 1411
- Wald Chi-Square: 771.08***
- Log pseudo likelihood: -1810.0929
- Pseudo R-Square: 0.2314

***, **, *= Significant at 1%, 5% and 10% probability level, respectively
Appendix VIII. Coefficient estimates of the multinomial logit soil conservation adoption model, Central highlands of Ethiopia, 2003

<table>
<thead>
<tr>
<th>Variable</th>
<th>Cut-off drainage (golenta)</th>
<th>Stone and soil bunds</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Coefficient</td>
<td>P-level</td>
</tr>
<tr>
<td>Constant</td>
<td>-3.3845***</td>
<td>0.000</td>
</tr>
<tr>
<td>Education</td>
<td>0.0552</td>
<td>0.358</td>
</tr>
<tr>
<td>Plot area</td>
<td>-0.2998</td>
<td>0.302</td>
</tr>
<tr>
<td>No. of plots</td>
<td>-0.0196</td>
<td>0.816</td>
</tr>
<tr>
<td>Plot distance</td>
<td>0.0079</td>
<td>0.263</td>
</tr>
<tr>
<td>Tenure</td>
<td>0.4967</td>
<td>0.163</td>
</tr>
<tr>
<td>Livestock</td>
<td>0.0671</td>
<td>0.193</td>
</tr>
<tr>
<td>Off-farm income</td>
<td>-0.0814</td>
<td>0.854</td>
</tr>
<tr>
<td>Extension</td>
<td>-0.2043</td>
<td>0.826</td>
</tr>
<tr>
<td>Credit</td>
<td>-0.6764</td>
<td>0.169</td>
</tr>
<tr>
<td>Plot slope</td>
<td>0.2570</td>
<td>0.300</td>
</tr>
<tr>
<td>Soil degradation</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sever</td>
<td>1.4714***</td>
<td>0.002</td>
</tr>
<tr>
<td>Medium</td>
<td>1.9818***</td>
<td>0.000</td>
</tr>
<tr>
<td>Light</td>
<td>1.9555***</td>
<td>0.000</td>
</tr>
<tr>
<td>Assistance</td>
<td>1.9876</td>
<td>0.056</td>
</tr>
<tr>
<td>District</td>
<td>-1.2241**</td>
<td>0.043</td>
</tr>
<tr>
<td>Diagnostics</td>
<td></td>
<td></td>
</tr>
<tr>
<td>No. Observations</td>
<td>97</td>
<td></td>
</tr>
<tr>
<td>Wall Chi-Square</td>
<td>270.03***</td>
<td></td>
</tr>
<tr>
<td>Pseudo Chi-Square</td>
<td>0.4017</td>
<td></td>
</tr>
</tbody>
</table>

***, **, *= Significant at 1%, 5% and 10% probability levels, respectively
### Appendix IX. Parameter estimates of the Tobit adoption model for the intensity of inorganic fertilizer use (kg/ha), Central highlands of Ethiopia, 2003

<table>
<thead>
<tr>
<th>Variable</th>
<th>Coefficient</th>
<th>P-level</th>
<th>Adoption (index)</th>
<th>P-level</th>
<th>Expected use (kg/ha)</th>
<th>P-level</th>
<th>Marginal effects</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant</td>
<td>-116.3897</td>
<td>0.000</td>
<td>N.A.</td>
<td>N.A.</td>
<td>2.0126***</td>
<td>0.000</td>
<td></td>
</tr>
<tr>
<td>Education¹</td>
<td>9.2427</td>
<td>0.000</td>
<td>0.0210**</td>
<td>0.000</td>
<td>1.4822***</td>
<td>0.000</td>
<td></td>
</tr>
<tr>
<td>Off-farm income²</td>
<td>29.0357</td>
<td>0.025</td>
<td>0.0689**</td>
<td>0.030</td>
<td>6.5489**</td>
<td>0.029</td>
<td></td>
</tr>
<tr>
<td>Livestock³</td>
<td>6.8069</td>
<td>0.000</td>
<td>0.0154**</td>
<td>0.000</td>
<td>1.08850***</td>
<td>0.002</td>
<td></td>
</tr>
<tr>
<td>Plot size⁴</td>
<td>49.9898</td>
<td>0.002</td>
<td>0.1133**</td>
<td>0.002</td>
<td>10.8850***</td>
<td>0.002</td>
<td></td>
</tr>
<tr>
<td>No. of plots</td>
<td>-10.4497**</td>
<td>0.000</td>
<td>-0.0237**</td>
<td>0.000</td>
<td>-2.2754**</td>
<td>0.000</td>
<td></td>
</tr>
<tr>
<td>Plot distant⁵</td>
<td>0.4583</td>
<td>0.094</td>
<td>0.0010**</td>
<td>0.093</td>
<td>0.0998**</td>
<td>0.093</td>
<td></td>
</tr>
<tr>
<td>Severity of soil degradation⁶</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Light</td>
<td>21.8407</td>
<td>0.097</td>
<td>0.0509</td>
<td>0.106</td>
<td>4.8545</td>
<td>0.104</td>
<td></td>
</tr>
<tr>
<td>Medium</td>
<td>38.0750</td>
<td>0.004</td>
<td>0.0927**</td>
<td>0.007</td>
<td>8.7697**</td>
<td>0.007</td>
<td></td>
</tr>
<tr>
<td>Sever</td>
<td>39.4582</td>
<td>0.013</td>
<td>0.0983**</td>
<td>0.022</td>
<td>9.2578**</td>
<td>0.021</td>
<td></td>
</tr>
<tr>
<td>Tenure⁷</td>
<td>17.9643</td>
<td>0.165</td>
<td>0.0391</td>
<td>0.146</td>
<td>3.8033</td>
<td>0.153</td>
<td></td>
</tr>
<tr>
<td>Credit⁸</td>
<td>99.6655</td>
<td>0.000</td>
<td>0.2419**</td>
<td>0.000</td>
<td>23.2970**</td>
<td>0.000</td>
<td></td>
</tr>
<tr>
<td>Extension⁹</td>
<td>74.3334</td>
<td>0.000</td>
<td>0.2017**</td>
<td>0.000</td>
<td>19.0840**</td>
<td>0.000</td>
<td></td>
</tr>
<tr>
<td>Agro-ecology¹⁰</td>
<td>-51.9767**</td>
<td>0.000</td>
<td>-0.1281**</td>
<td>0.000</td>
<td>-12.1180**</td>
<td>0.000</td>
<td></td>
</tr>
<tr>
<td>SFM used previous year¹¹</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Legume rotations</td>
<td>-26.8184**</td>
<td>0.025</td>
<td>-0.0610</td>
<td>0.024</td>
<td>-5.8586**</td>
<td>0.025</td>
<td></td>
</tr>
<tr>
<td>Manure</td>
<td>-119.2528**</td>
<td>0.000</td>
<td>-0.2190</td>
<td>0.000</td>
<td>-23.1081**</td>
<td>0.000</td>
<td></td>
</tr>
<tr>
<td>Fallow</td>
<td>33.1012</td>
<td>0.034</td>
<td>0.0806</td>
<td>0.046</td>
<td>7.6161**</td>
<td>0.044</td>
<td></td>
</tr>
<tr>
<td>District*Bund¹²</td>
<td>-20.5157</td>
<td>0.196</td>
<td>-0.0448</td>
<td>0.177</td>
<td>-4.3600</td>
<td>0.184</td>
<td></td>
</tr>
</tbody>
</table>

**Diagnostics**

<p>| | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>No. Observations</td>
<td>1293</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>LR Chi-Square</td>
<td>492.44</td>
<td>0.000</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

***, **, *= Significant at 1%, 5% and 10% probability levels, respectively; N.A.=Not applicable;
¹Number of years; ²Dummy variable, 1 denoting participation in off-farm activities; ³Tropical Livestock Unit (TLU); ⁴hectares; ⁵Minutes walked from residence; ⁶comparison category is plots perceived not having shown any form of soil degradation; ⁷dummy variable, 1 denoting PA allotted plots, 0 otherwise; ⁸dummy variable, 1 denoting access to institutional credit; ⁹dummy variable, 1 representing access to government extension; ¹⁰dummy variable, 1 referring to upper highlands; ¹¹dummy variables with 1 indicating use of the respective practices. ¹²dummy variable with 1 indicating plots with stone_soil bunds in Debre Berihan district.