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Interlacing zeros of linear combinations of classical orthogonal polynomials

by

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Declaration

I, the undersigned, hereby declare that the thesis submitted herewith for the degree Philosophiae Doctor to the University of Pretoria contains my own independent work and has not been submitted for any degree at any other university.

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Abstract

Our objective in this thesis is to investigate the interlacing of zeros of real polynomials of the same degree and of adjacent degree in three different contexts.

Our first results concern the interlacing of zeros of p_n and q_m, $m = n$ or $n-1$, where $\{p_n\}_{n=1}^{\infty}$ and $\{q_m\}_{m=1}^{\infty}$ are sequences of Jacobi polynomials corresponding to different values of the parameters α and β. We prove an extension of a conjecture of Askey [cf [9], p.28] by showing that the zeros of $P_n^{(\alpha,\beta)}$ and $P_n^{(\alpha',\beta')}$ interlace when $\alpha < \alpha' \leq \alpha + 2$ and $\beta < \beta' \leq \beta + 2$, allowing a continuous shift of both α and β.

Next, we investigate the interlacing of zeros of linear combinations $p_n + aq_m$, $a \in R$, $a \neq 0$, with the zeros of the component polynomials p_n and q_m, where $\{p_n\}_{n=1}^{\infty}$ and $\{q_n\}_{n=1}^{\infty}$ are different sequences of Jacobi polynomials. Numerical examples are given to illustrate situations where the zeros do not interlace. We also prove that the zeros of the linear combination $p_n + aq_m$ interlace with the zeros of certain other Jacobi polynomials that are not components of the linear combination.

Finally, we investigate the interlacing of zeros of polynomials of consecutive degree in the sequences $\{r_n\}_{n=1}^{\infty}$ and $\{s_n\}_{n=1}^{\infty}$ where

$$r_n = p_n + a_n q_n \quad \text{and} \quad s_n = p_n + b_n q_{n-1}, \quad a_n, b_n \neq 0, a_n, b_n \in R$$

and $\{p_n\}_{n=1}^{\infty}$ and $\{q_n\}_{n=1}^{\infty}$ are different sequences of either Laguerre or Jacobi polynomials.

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Contents

Chapter 1

Introduction

1.1 Orthogonal polynomials and their properties

Orthogonal polynomials were used by R. Murphy (cf. [40]) in 1835 although he referred to them as "reciprocal functions".
P.L. Chebyshev (cf. [16]) recognized the importance of orthogonal polynomials during the course of his work on Fourier series, continued fractions and approximation theory.

The classical orthogonal polynomials are often thought to be the Jacobi, Laguerre and Hermite polynomials, which are orthogonal on the real line, with respect to the beta, gamma and normal distributions respectively. It is well known that the Jacobi polynomial $P_n^{(\alpha,\beta)}$ has a ${}_2F_1$ representation, while Laguerre and Hermite polynomials have ${}_1F_1$ and ${}_2F_0$ representations respectively; so there are natural connections between these orthogonal polynomials and hypergeometric functions. Gegenbauer, Chebyshev and Legendre polynomials are special cases of the Jacobi polynomial $P_n^{(\alpha,\beta)}$. Sequences of Jacobi, Laguerre and Hermite polynomials have several useful and important

properties in common.
These include the following:

- They all satisfy a second order linear differential equation of the Sturm-Liouville type

$$g_2(x)\frac{d^2y}{dx^2} + g_1(x)\frac{dy}{dx} + a_n y = 0$$

  where $g_2(x)$ is a polynomial of degree ≤ 2 and $g_1(x)$ is a linear polynomial, both independent of n, and a_n depends only on n.

- Their derivatives form sequences of orthogonal polynomials (cf. [31] and [50]).

- They all satisfy a Rodrigues' formula

$$p_n = \frac{1}{e_n w(x)}\frac{d^n}{dx^n}\{w(x)[g(x)]^n\} \qquad n = 0, 1, 2, \ldots$$

  where $w(x)$ is a positive function on certain interval, $g(x)$ is a polynomial in x independent of n, and e_n is independent of x.
  Note that the Rodrigues' formula provides transparent and immediate information about the interval of orthogonality, the weight function and the range of parameters for which orthogonality holds.

- They are all orthogonal with respect to a weight function that satisfies a Pearson differential equation, namely,

$$\frac{w(x)'}{w(x)} = \frac{N(x)}{g_2(x)}, \qquad (g_2(x)w(x))' = g_1(x)w(x), \quad N(x) = g_1(x) - g_2(x)'.$$

The notion of which properties define "classical orthogonal polynomials" has been extensively discussed during the last few decades. One of the most recent views is that classical orthogonal polynomials are those with hypergeometric representations (cf. [8] and [11]). In this case, they satisfy difference differential equations on a linear lattice, a quadratic lattice, a q-linear lattice, or a q-quadratic lattice, and they can all be obtained as limits of the q-Racah polynomials or the Askey-Wilson polynomials. It has been shown by Atakishiyev et al. in [11] that this broader definition of classical orthogonal polynomials can also be reformulated in terms of difference equations, Rodrigues formula and moments. Furthermore, details of the solutions to these characterization problems are worked out, such as the explicit orthogonality, boundary conditions, moments, and integral representations. A classification of continuous and discrete classical orthogonal polynomials, based on the lattice type, is also presented.

Agarwal and Manocha introduced, in [3], a sequence of polynomials defined by a Rodrigues type formula. They obtained linear and trilinear generating functions and operational formulas. Their results generalize those of Srivastava and Singhal (cf. [54]) and also the results for the classical orthogonal polynomials, including the Bessel polynomials via the extended Jacobi polynomials as discussed by Patil and Thakare in [41].

A significant contemporary contribution to orthogonal polynomials was made by Askey and Wilson in [10], in which they introduce a q-analogue of the Wigner $6-j$ symbols. This q-analogue defines a sequence of orthogonal polynomials $\{p_n(x)\}_{n=0}^{\infty}$, where $p_n(x)$ is a constant multiple of

$$ {}_{\mathbf{4}}\varphi_{\mathbf{3}}\left(\begin{array}{c} q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{array} ; q, q\right), \quad x = \cos\theta. $$

These polynomials are called the Askey-Wilson polynomials. Askey and Wilson evaluated the integral

$$ I := \frac{(q;q)_\infty}{2\pi} \int_0^\pi \left| \frac{(e^{2i\theta};q)_\infty}{\prod_{j=1}^{4}(a_j e^{i\theta};q)_\infty} \right|^2 d\theta $$

using an elliptic function argument. The integral turns out to be

$$I = \frac{(a_1 a_2 a_3 a_4; q)_\infty}{\displaystyle\prod_{1 \leq j < i \leq 4} (a_i a_j; q)_\infty}.$$

This is then used to establish the orthogonality of $\{p_n(\cos\theta)\}$
when $\{a, b, c, d\} = \{a_1, a_2, a_3, a_4\}$. The choices $a + c = b + d = 0$ make the weight function an even function of $\cos\theta$, and $p_{2n}(x)$ becomes a polynomial of degree n in x^2 so that $p_{2n}(\sqrt{x})$ are orthogonal and turn out to be another set of Askey-Wilson polynomials. This leads to a quadratic transformation connecting balanced ${}_4\varphi_3$ polynomials with four free parameters. Several known special cases of the Askey-Wilson polynomials are mentioned and they include continuous q-ultraspherical and q-Jacobi polynomials and the Al-Salam-Chihara polynomials. Rodrigues type formulas are derived using the finite difference operator $(\delta_q f)(e^{i\theta}) = f(e^{i\theta\sqrt{q}}) - f(ei\theta/\sqrt{q})$.
Askey and Wilson also solve the connection coefficient problem for their polynomials and make some remarks on the zeros of the polynomials.

In [4], Al-Salam gives a survey of various characterization theorems for orthogonal polynomials on the real line. In addition to the standard characterization theorems, Al-Salam also describes the discrete cases by replacing the derivation operator with a finite difference operator and a q-difference operator. Several other orthogonal polynomial sets are thus characterized, including the Charlier, Meixner and Hahn polynomials and some of their q-analogues. In particular, it is shown that the Askey-Wilson polynomials have a q-difference operator such that there is a second-order difference equation. Their differences are again orthogonal polynomials and have an analogue of Rodrigues' formula. Other characterization results given in this paper are results based on the generating function of polynomials (for Sheffer polynomials, Brenke polynomials, Fejér's generalized Legendre polynomials) and a classification of polynomial sets such that the convolution is again an orthogonal polynomial set.

Abdelkarim and Maroni, in [1], used the operator $D_h f(x) = \dfrac{f(x+h)-f(x)}{h}$ as opposed to the q-difference operator used by Hahn in [33]. The most general set of orthogonal polynomials they got was found by Chebyshev. To find all of the orthogonal polynomials, h must be allowed to be complex, for both real h and purely imaginary h lead to polynomials orthogonal with respect to a positive measure.
[1] contains examples where there is orthogonality, but not with respect to a positive measure. One interesting case is the Charlier polynomials, where an analogue of the Meixner-Pollaczek polynomials is treated.

Linear combinations of orthogonal polynomials are also discussed extensively in the literature. Franz Peherstorfer, in [43], approaches the question of orthogonality and quasi-orthogonality (which we will define later) by investigating when a certain linear combination $\sum_{j=0}^{k} \mu_j p_{n-j}$, $k \le n$, $k, n \in \mathbb{N}$, generates a positive quadrature formula. He establishes sufficient conditions on the real $\{\mu_j\}_{j=0}^{k}$ such that

$$Q_n(x) = p_n(x) + \mu_1 p_{n-1}(x) + \mu_2 p_{n-2}(x) + \cdots + \mu_k p_{n-k}(x)$$

has n simple zeros in $(-1,1)$, when $\{p_n\}_{n=0}^{\infty}$ is a sequence of monic orthogonal polynomials on $[-1,1]$ with respect to the positive measure μ and $supp(\mu) = (-1,1)$.
Marcellán et al., in [5], derive necessary and sufficient conditions for the orthogonality of $\{Q_n\}_{n=0}^{\infty}$ when $\mu_k \neq 0$. Their work extends the results of Peherstorfer in [43].

Brezinski et al., in [15], using the Christoffel-Darboux identity, established results on the location of zeros of $R_n(x) = p_n(x) + a_n p_{n-1}(x)$ with $a_n \neq 0$.

They also gave sufficient conditions on the interlacing of zeros (which we also define later) of $R_n(x)$, $p_n(n)$ and $p_{n-1}(x)$. In addition, they dealt with the location of zeros of $p_n(x) + a_n p_{n-1}(x) + b_n p_{n-2}(x)$ when a_n and b_n are both different from zero.

Stieltjes, in [52], established that for any sequence of orthogonal polynomials $\{p_n\}$, if $m < n$, then there are m distinct intervals of the form (x_k, x_{k+1}) each containing one zero of p_m, where x_k, x_{k+1} are consecutive zeros of p_n. Beardon and Driver, in [12], extend Stieltjes' result to some linear combinations $a_k p_k + \cdots + a_m p_m$, with $a_k a_m \neq 0, 1 \leq k \leq m \leq n$. They also discuss the interlacing property of zeros of $ap_n + bp_{n+1}$ and those of $cp_n + dp_{n+1}$, when $ad - bc \neq 0$, using the Wronskian operator.

Recently, Joulak, in [36], gives characterizations of the quasi-orthogonality of order r by using linear algebra techniques. He extends the results in [15] and gives new results on the location of zeros of quasi-orthogonal polynomials.

Let us recall the definition of orthogonality of a sequence of polynomials.

Definition 1.1.1 *Let μ be a positive Borel measure supported on an infinite subset of the real line. Assume that for all $n = 0, 1, 2. \ldots,$ $\int x^n d\mu(x)$ exists. A system of polynomials $\{p_n(x) : n \in I\}$, with deg$(p_k) = k$ and $I = \{0, 1, 2, \ldots\}$ or $I = \{0, 1, 2, \ldots, N\}$, $N \in \mathbb{N}$, is orthogonal with respect to μ, if*

$$\int p_n(x)p_m(x)d\mu(x) = h_n\delta_n^m, \quad n, m \in I \tag{1.1}$$

where the constants h_n are strictly positive and δ_n^m is the Kronecker delta.

If the measure μ is absolutely continuous, it has a Radon-Nikodym derivative and we can write $d\mu(x) = w(x)dx$. Then (1.1) becomes

$$\int p_n(x)p_m(x)w(x)dx = h_n\delta_n^m, \quad n, m \in I. \tag{1.2}$$

The density function $w(x)$ is also called the weight function and one speaks of orthogonality of a sequence of polynomials with respect to the weight function $w(x)$.
If the measure μ is a discrete measure with the weights (often called masses) ρ_i at the points x_i, then (1.1) becomes

$$\sum_{i=0}^{N} p_n(x_i)p_m(x_i)\rho_i = h_n\delta_n^m \quad n, m \in I;$$

where N may be finite or infinite.

Given a positive Borel measure μ, an orthogonal sequence can always be generated using the Gram-Schmidt orthogonalization process (cf. [34], [57], [56]). It is important to note that, for a given Borel measure μ, the orthogonal sequence $\{p_n\}_{n=0}^{\infty}$ is uniquely determined up to normalization.

It is useful to replace the orthogonality condition (1.2) by an equivalent formulation, namely (cf. [45, Theorem 54, p.148])

$$\int x^k p_n(x)w(x)dx = 0 \text{ for } k = 0, 1, 2, \ldots, n-1 \text{ where } n \in \mathbb{N}. \tag{1.3}$$

A remarkable property of any infinite sequence $\{p_n\}_{n=0}^{\infty}$ of orthogonal polynomials is that it satisfies a three term recurrence relation given by

$$p_{n+1}(x) = (A_n x + B_n)p_n(x) - C_n p_{n-1}(x) \tag{1.4}$$

where $A_n A_{n-1} C_n > 0$ and we take $p_0(x) \equiv 1$, $p_{-1}(x) \equiv 0$.
If we impose the normalization condition that p_n is monic, the three term recurrence relation simplifies into the form

$$p_{n+1}(x) = (x - c_n)p_n(x) - \lambda_n p_{n-1}(x) \tag{1.5}$$

where c_n and λ_n are sequences of real numbers with $\lambda_n > 0$ for each $n \in \mathbb{N}$. As a partial converse, in [29], Favard proved that if a sequence of polynomials $\{p_n\}_{n=0}^{\infty}$ satisfies a three term recurrence relation of type (1.4) with $C_n > 0$, then there exists a positive Borel measure μ such that $\{p_n\}_{n=0}^{\infty}$ is orthogonal with respect to μ.

Another important property of an orthogonal sequence that we shall find useful is the Christoffel-Darboux formula given by

$$\sum_{j=0}^{n} \frac{p_j(x)p_j(y)}{h_j} = \frac{k_n}{k_{n+1}h_n} \cdot \frac{p_{n+1}(y)p_n(x) - p_{n+1}(x)p_n(y)}{x-y}$$

where k_n is the leading coefficient of p_n and $h_j = \displaystyle\int w(x)p_j^2(x)dx$.
If the polynomials $p_n(x)$ are monic, the Christoffel-Darboux formula simplifies to

$$\sum_{j=0}^{n} \frac{p_j(x)p_j(y)}{h_j} = \frac{p_{n+1}(y)p_n(x) - p_{n+1}(x)p_n(y)}{h_n(y-x)}.$$

If $\{p_n\}_{n=0}^{\infty}$ is orthogonal with respect to the weight function $w(x)$, then there exists a polynomial $g(x)$ independent of n and a constant e_n which depends only on n such that

$$p_n = \frac{1}{e_n w(x)} \frac{d^n}{dx^n}\{w(x)[g(x)]^n\}.$$

This type of formula is known as Rodrigues' formula.

A key focus for this thesis is the following fact.

Theorem 1.1.2 ([51]) *If $\{p_n\}_{n=0}^{\infty}$ is a sequence of orthogonal polynomials, with respect to the positive Borel measure μ, then the zeros of p_n are real and simple and lie in the interior of the convex hull of the support of the measure*

μ . Moreover, if $x_1 < x_2 < \cdots < x_n$ and $y_1 < y_2 < \cdots < y_{n+1}$ are the zeros of p_n and p_{n+1} respectively, then

$$y_1 < x_1 < y_2 < x_2 < \cdots < y_n < x_n < y_{n+1}, \tag{1.6}$$

a property referred to as the interlacing of zeros.

Proof. For the convenience of the reader since this is central to this thesis, we give a brief outline of a proof of the interlacing property.

We assume that $\{p_n(x)\}_{n=0}^{\infty}$ is a sequence of monic orthogonal polynomials and let

$$K_n(x,y) = \frac{p_{n+1}(x)p_n(y) - p_{n+1}(y)p_n(x)}{h_n(x-y)}$$

be the Christoffel-Darboux formula. Then

$$\begin{aligned} h_n K_n(x,x) &= h_n \lim_{y\to x} K_n(x,y) \\ &= \lim_{y\to x} \frac{p_{n+1}(x)p_n(y) - p_{n+1}(y)p_n(x)}{x-y} \\ &= p'_{n+1}(x)p_n(x) - p'_n(x)p_{n+1}(x). \end{aligned}$$

So

$$h_n K_n(x,x) = \sum_{j=0}^{n} p_j^2(x) = p'_{n+1}(x)p_n(x) - p'_n(x)p_{n+1}(x).$$

Let x_i, x_{i+1}, $i = 1, 2, \ldots, n$ be any two consecutive zeros of $p_{n+1}(x)$. Then

$$\sum_{j=0}^{n} p_j^2(x_i) = p'_{n+1}(x_i)p_n(x_i) > 0$$

$$\text{and } \sum_{j=0}^{n} p_j^2(x_{i+1}) = p'_{n+1}(x_{i+1})p_n(x_{i+1}) > 0.$$

Since $p_{n+1}(x)$ is continuous, it follows from Rolle's theorem
that $p'_{n+1}(x_i)p'_{n+1}(x_{i+1}) < 0$, and hence $p_n(x_i)p_n(x_{i+1}) < 0$. So there is at least one zero of odd multiplicity of $p_n(x)$ between any two consecutive zeros of $p_{n+1}(x)$. □

Remarks

(1) From the proof above, it is clear that if polynomials $P(x)$ and $Q(x)$ have interlacing zeros, then $P(y_i)P(y_{i+1}) < 0$ for any two consecutive zeros y_i and y_{i+1} of $Q(x)$, an idea central to many of our later proofs.

(2) We noted that in any sequence $\{p_n\}_{n=0}^{\infty}$ of orthogonal polynomials, the polynomials p_n and p_{n+1} have interlacing zeros if $n \geq 1$. As a partial converse, Wendroff (cf. [58]) proved that, given any n real distinct points
$x_1 < x_2 < \cdots < x_n$ and $n+1$ real distinct points $y_1 < y_2 < \cdots < y_{n+1}$ such that

$$y_1 < x_1 < y_2 < x_2 < \cdots < y_n < x_n < y_{n+1} \text{ holds,}$$

then the polynomials

$$p_n(x) = \prod_{i=1}^{n}(x - x_i) \text{ and } p_{n+1}(x) = \prod_{i=1}^{n+1}(x - y_i)$$

can be embedded in a sequence of monic orthogonal polynomials.

However, in [22], Driver has shown that if the zeros of polynomials of successive degree in an infinite sequence satisfy the interlacing property, this by no means ensures the orthogonality of the sequence $\{p_n\}_{n=0}^{\infty}$ with respect to some positive Borel measure. Indeed, in [22], it is proved that if $\{p_n\}_{n=0}^{\infty}$ is a sequence of real monic polynomials with $deg(p_n) = n$, such that the zeros of p_n are real and simple and p_n and p_{n+1} have no common zero for any n. In addition, we assume that

$$\left(\frac{p_{n+1}}{p_{n-1}}\right)(x_{i,n}) = \left(\frac{p_{n+1}}{p_{n-1}}\right)(x_{j,n}) \text{ for each } n \tag{1.7}$$

and each $i, j = 1, 2, 3, \ldots, n$ where $\{x_{k,n}\}_{k=1}^{n}$ denote the zeros of p_n. Then the following statements are equivalent (cf. [22], Theorem):

(a) the sequence $\{p_n\}_{n=0}^{\infty}$ is orthogonal with respect to some Borel measure;

(b) λ_n is positive for each n, where $-\lambda_n$ is the common value of the ratios in (1.7);

(c) the zeros of p_n and p_{n+1} interlace for each $n \geq 1$.

1.2 Interlacing property

We have seen in Section 1.1, that if $\{p_n\}_{n=0}^{\infty}$ is a sequence of orthogonal polynomials, then for $n > 0$, if $x_{1,n} < x_{2,n} < x_{3,n} < \cdots < x_{n,n}$ are the zeros of $p_n(x)$
and $x_{1,n+1} < x_{2,n+1} < \cdots < x_{n+1,n+1}$ are zeros of p_{n+1}, the polynomials p_n and p_{n+1} have interlacing zeros; namely,

$$x_{1,n+1} < x_{1,n} < x_{2,n+1} < x_{2,n} < \cdots < x_{n,n+1} < x_{n,n} < x_{n+1,n+1}.$$

It is interesting to note that the interlacing of zeros is meaningful outside the context of orthogonality and, in the classical situation, deals with polynomials of successive degree in a sequence.

Obreschkoff proved (cf. [14]) that p_n and q_m have interlacing zeros if and only if any polynomial of the form $\alpha p_n + \beta q_m$, with $\alpha \in \mathrm{R}$, $\beta \in \mathrm{R}$, $\alpha \neq 0$ or $\beta \neq 0$, has all roots real and simple.

The Hermite-Biehler Theorem and Hermite-Kakeya Theorem (cf. [44], pp.197-198) also give necessary and sufficient conditions for two non-constant polynomials with real coefficients to have interlacing zeros.

Let $p_n(x)$ and $q_m(x)$ be two real polynomials. Consider the two-variable symmetric polynomial

$$\begin{aligned} f(x,y) &= \frac{p_n(x)q_m(y) - p_n(y)q_m(x)}{x-y} \\ &= \sum_{k,l=1}^{n} a_{kl} x^{k-1} y^{l-1}. \end{aligned}$$

The n-order symmetric matrix $B[a_{kl}]$ is called the Bezoutian matrix of the polynomials $p_n(x)$ and $q_m(x)$.

Alvarez and Sansigre established in [6, Theorem 2] that the monic polynomials $p_n(x)$ and $q_{n+1}(x)$ have interlacing zeros if and only if the corresponding Bezoutian matrix is positive definite.

In this thesis, we will investigate the interlacing property of zeros not only for polynomials of successive degrees from different orthogonal sequences, but also polynomials of the same degree from different orthogonal sequences. We shall say that two real and non-constant polynomials $p_n(x)$ and $q_m(x)$, with $m = n$ or $m = n+1$, have interlacing zeros if the zeros of $p_n(x)$ and $q_m(x)$ are all real and simple and between any two consecutive zeros of $q_m(x)$, there is exactly one zero of $p_n(x)$.

1.3 Importance of interlacing of zeros

The interlacing of zeros of polynomials is important in a wide variety of applications.

The interlacing of zeros plays a critical role ensuring the positivity of quadrature formulae (and hence their convergence) (cf. [49]), the approximation of zeros by fixed point iterations techniques (cf. [47]), the completeness of the set of eigenfunctions to a Sturm-Liouville eigenvalue problem (cf. [13]) and the uniform convergence of derivatives arising in the extended Lagrange interpolation (cf. [18]).

In [39], Mastroianni and Occorsio propose a new method to approximate the Hilbert transform using interlacing of zeros of associated orthogonal polynomials.

1.4 Monotonicity result

One of the techniques used in our discussion of the interlacing of zeros is the Markoff Theorem. Since we shall make extensive use of this monotonicity result, we state the theorem and corollaries.

Theorem 1.4.1 ([35], Theorem 7.1.1, p.204) *Let $\{p_n(x,\tau)\}_n$ be a sequence of polynomials orthogonal on the interval $I = (a,b)$ with respect to $d\alpha(x;\tau)$, with $d\alpha(x;\tau) = \rho(x;\tau)d\alpha(x)$, and we assume that $\rho(x;\tau)$ is positive and has a continuous first derivative with respect to τ for $x \in I,\ \tau \in T = (\tau_1, \tau_2)$. Furthermore, we assume that*

$$\int_a^b x^j \rho_\tau(x;\tau)d\alpha(x), j = 0,1,2,\ldots,2n-1,$$

*converge uniformly for τ in every closed subinterval of T. Then the zeros of $p_n(x:\tau)$ are increasing (decreasing) functions of $\tau, \tau \in T$,
if $\partial\{\ln\rho(x;\tau)\}/\partial\tau$ is an increasing (decreasing) function of x, $x \in I$.*

Corollary 1.4.2 ([53], Theorem 6.12.2, p.116) *Let $w(x)$ and $W(x)$ be two weight functions on $[a,b]$, both positive and continuous
for $a < x < b$. Let $W(x)/w(x)$ be increasing. Then if x_k and y_k denote the zeros the corresponding orthogonal polynomials of degree n in decreasing order, one has*

$$x_k < y_k,\ \ k = 1,2,\ldots,n.$$

Corollary 1.4.3 ([35], Theorem 7.1.2, p. 205) *The zeros of a Jacobi polynomial $P_n^{(\alpha,\beta)}$ increase with β and decrease with α.*

1.5 Interlacing of zeros of Laguerre polynomials and Gegenbauer polynomials

In this section, our aim is to review recent results on the interlacing property of zeros of polynomials of the same or adjacent degree within single parameter families of classical orthogonal polynomials (cf. [24], [15], [48]).

Among the classical orthogonal polynomials that depend on one parameter, we have the Laguerre and Gegenbauer polynomials.

Laguerre polynomials $L_n^{\alpha}(x)$ are defined by

$$L_n^{\alpha}(x) = \frac{(1+\alpha)_n}{n!} {}_1F_1(-n; 1+\alpha; x).$$

and are orthogonal on $[0, \infty)$ with respect to the weight function $w(x) = e^{-x}x^{\alpha}$ for $\alpha > -1$. The three-term-recurrence relation for Laguerre polynomials is given by

$$(n+1)L_{n+1}^{\alpha}(x) = (2n+1+\alpha-x)L_n^{\alpha}(x) - (n+\alpha)L_n^{\alpha}(x)$$

and useful mixed recurrence relations are (cf. [2], 22.729 and 22.7.30)

$$\begin{aligned} xL_n^{\alpha+1}(x) &= (x-n)L_n^{\alpha}(x) + (\alpha+n)L_{n-1}^{\alpha}(x) \\ L_n^{\alpha}(x) &= L_n^{\alpha+1}(x) - L_{n-1}^{\alpha+1}(x). \end{aligned}$$

Let $\alpha > -1$ and let

$$\begin{aligned} 0 < x_1 < x_2 < \cdots < x_n \quad &\text{be the zeros of} \quad L_n^{\alpha}(x) \\ 0 < z_1 < z_2 < \cdots < z_n \quad &\text{be the zeros of} \quad L_n^{\alpha+2}(x) \quad \text{and} \\ 0 < t_1 < t_2 < \cdots < t_n \quad &\text{be the zeros of} \quad L_n^{\alpha+t}(x) \end{aligned}$$

where $0 < t < 2$.

The ratio of the weight functions corresponding to the orthogonal sequences $\{L_n^{\alpha}\}_{n=1}^{\infty}$ and $\{L_n^{\alpha+t}\}_{n=1}^{\infty}$ where $\alpha > -1$ and $0 < t \leq 2$ is

$$\frac{e^{-x}x^{\alpha}}{e^{-x}x^{\alpha+t}} = x^{-t}$$

and using the Markoff monotonicity result (Theorem 1.4.1), Driver and Jordaan (cf. [24, Theorem 2,3]) proved that

$$0 < x_1 < t_1 < z_1 < x_2 < \cdots < x_{n-1} < t_{n-1} < z_{n-1} < x_n.$$

For Laguerre polynomials of adjacent degree, they proved that the zeros of L_n^α and those of $L_{n-1}^{\alpha+1}$ interlace. This result was extended in ([25]) to show that the interlacing of zeros holds not only for an integer shift of the parameter, but also when the parameter α is shifted continuously to 2. Indeed letting $\alpha > -1$ and letting

$$\begin{aligned} 0 < x_1 < x_2 < \ldots x_n \quad & \text{be the zeros of} \quad L_n^\alpha(x), \\ 0 < y_1 < y_2 < \ldots y_{n-1} \quad & \text{be the zeros of} \quad L_{n-1}^\alpha(x), \\ 0 < t_1 < t_2 < \ldots t_{n-1} \quad & \text{be the zeros of} \quad L_{n-1}^{\alpha+t}(x) \quad \text{and} \\ 0 < z_1 < z_2 < \cdots < z_n \quad & \text{be the zeros of} \quad L_{n-1}^{\alpha+2}(x) \end{aligned}$$

where $0 < t < 2$, they established in [25, Theorem 3.1] that
$0 < x_1 < y_1 < t_1 < z_1 < x_2 < \cdots < x_{n-1} < y_{n-1} < t_{n-1} < z_{n-1} < x_n < z_n$.

Moving to the question of the interlacing property for zeros of Gegenbauer polynomials, let us recall that the Gegenbauer polynomial C_n^α can be defined by (cf. [45], p.279)

$$C_n^\lambda(x) = \frac{2^{2n}(\lambda)_n}{n!}(\frac{x+1}{2})^n {}_2F_1\left(\begin{matrix} -n, \frac{1}{2}-\lambda-n \\ 1-2\lambda-2n \end{matrix}; \frac{2}{1-x}\right);$$

and for $\lambda > -\frac{1}{2}$, the sequence $\{C_n^\lambda(x)\}_{n=0}^\infty$ is orthogonal over the interval $[-1,1]$ with respect to the weight function $w(x) = (1-x^2)^{\lambda-\frac{1}{2}}$. The three term recurrence relation is given by (cf. [7], p.303)

$$nC_n^\lambda(x) = 2(n+\lambda-1)C_{n-1}^\lambda(x) - (n+2\lambda-2)C_{n-2}^\lambda(x)$$

In addition to the three term recurrence relation, there are mixed recurrence relations (cf. [2], formulae 22.7.21 and 22.7.22)

$$\begin{aligned} 2\lambda(1-x^2)C_{n-1}^{\lambda+1}(x) &= (2\lambda+n-1)C_{n-1}^{\lambda}(x) - nxC_n^{\lambda}(x) \\ 2\lambda(1-x^2)C_{n-1}^{\lambda+1}(x) &= (2\lambda+n)xC_n^{\lambda}(x) - (n+1)C_{n+1}^{\lambda}(x) \\ (n+\lambda)C_{n+1}^{\lambda-1}(x) &= (\lambda-1)C_{n+1}^{\lambda}(x) - C_{n-1}^{\lambda}(x) \end{aligned}$$

Another useful mixed relation (see [24, Lemma 3.2]) is

$$\begin{aligned} &(2\lambda+2)(1-x^2)C_n^{\lambda+2}(x) \\ &= \frac{n+2\lambda}{2\lambda(1-x^2)}[(2\lambda+n+2)-(n+1)x^2]C_n^{\lambda}(x) \\ &\quad -\frac{(n+1)x}{2\lambda(1-x^2)}[(2n+4\lambda+3)-(2n+2\lambda+2)x^2]C_{n+1}^{\lambda}(x) \end{aligned}$$

From these mixed recurrence relations and the Markoff result Theorem 1.4.1, Driver and Jordaan established in [24] that for $\lambda > -\frac{1}{2}$, the zeros of $C_n^{\lambda+2}$ and C_{n+1}^{λ} interlace and if

$$\begin{aligned} 0 < y_1 < y_2 < \cdots < y_{[\frac{n}{2}]} < 1 \quad &\text{are the positive zeros of} \quad C_n^{\lambda+2}(x), \\ 0 < x_1 < x_2 < \cdots < x_{[\frac{n}{2}]} < 1 \quad &\text{are the positive zeros of} \quad C_n^{\lambda}(x) \quad \text{and} \\ 0 < t_1 < t_2 < \cdots < t_{[\frac{n}{2}]} < 1 \quad &\text{are the positive zeros of} \quad C_n^{\lambda+t}(x) \end{aligned}$$

with $0 < t < 2$, then

$$0 < y_1 < t_1 < x_1 < y_2 < t_2 < x_2 < \cdots < y_{[\frac{n}{2}]} < t_{[\frac{n}{2}]} < x_{[\frac{n}{2}]} < 1.$$

Segura, in ([48]), studied the interlacing property of zeros of contiguous hypergeometric functions, using first order difference-differential equations. From the continuity of coefficients of the related difference-differential equations, he deduces the interlacing properties of zeros. The Laguerre function $L_\nu^{(\alpha)}(x)$ is defined by

$$L_\nu^{(\alpha)}(x) = \binom{\nu+\alpha}{\nu} {}_1F_1(-\nu; 1+\alpha; x)$$

while the Gegenbauer function $C_\nu^\lambda(x)$ is defined by

$$C_\nu^{(\lambda)}(x) = \begin{pmatrix} \nu+\lambda \\ \nu \end{pmatrix} {}_2F_1(-\nu, \nu+2\lambda+1; 1+\lambda; \frac{1-x}{2}).$$

Since Laguerre functions and Gegenbauer functions are linked to hypergeometric functions, he proves that (cf. [48], Theorem 6) the zeros of the Laguerre functions $L_\nu^{(\alpha)}(x)$ and $L_{\nu'}^{(\alpha')}(x)$ interlace in $(0, \infty)$ when the differences

$$\delta\nu = \nu - \nu' \in Z \quad \text{and } \delta\alpha = \alpha - \alpha' \in Z$$

(not all of them equal to zero) satisfy:

1. $|\delta\nu| \leq 1$;

2. $|\delta\nu + \delta\alpha| \leq 1$

We note that the results proved in [24], as well as those which will be proved in the next chapter, are valid for continuous shifts of the parameters in the classical range of orthogonality for Jacobi, Laguerre and Gegenbauer polynomials. In [48], similar results are given only for integer shifts of parameters but for the parameter ranges beyond the classical range where the orthogonality holds.

1.6 Brief overview

The overarching theme of this thesis is an investigation of the interlacing (or not) of zeros of polynomials of the same or consecutive degree from different sequences. The polynomials in the sequences we consider are either classical orthogonal polynomials, namely Laguerre and Jacobi, or the linear combinations of these classical orthogonal polynomials.

In Chapter 2, we study interlacing properties for the zeros of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ and $P_m^{(\alpha',\beta')}(x)$, where $m = n$ or $m = n-1$ and $(\alpha', \beta') =$

$(\alpha \pm t, \beta \pm k)$, $0 < t \leq 2$ and $0 < k \leq 2$. We use Markoff's monotonicity result and mixed recurrence relations satisfied by Jacobi polynomials. These results prove a significant extension of the Askey conjecture (cf. [9]).

In Chapter 3, we consider the linear combinations of Jacobi polynomials of the form $p_n + \nu q_m$, where $p_n = P_n^{(\alpha,\beta)}$ and $q_m = P_m^{(\alpha',\beta')}$ with $(\alpha', \beta') \neq (\alpha, \beta)$. We focus on the interlacing of zeros of these linear combinations with those of the component polynomials p_n and q_m where $m = n$ and $m = n - 1$. We also study the interlacing of zeros of $P_n^{(\alpha,\beta)} + \nu P_m^{(\alpha',\beta')}$ with those of other selected Jacobi polynomials that are different from $P_n^{(\alpha,\beta)}$ and $P_m^{(\alpha',\beta')}$.

In Chapter 4, we consider $R_n = L_n^{\alpha} + b_n L_n^{\alpha+1}$ and study the interlacing of zeros of R_n and those of R_{n+1}. Similarly we define $E_n = P_n^{(\alpha,\beta)}(x) + r_n P_n^{(\alpha,\beta+1)}(x)$ and study the interlacing of zeros of E_n and those of E_{n+1}. For this chapter, our main tool is Joulak's result (cf. [36]).

Chapter 2

Interlacing of zeros of Jacobi polynomials from sequences corresponding to different parameters

2.1 Introduction

Let $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ be the sequence of Jacobi polynomials orthogonal on the interval $[-1,1]$ with respect to the weight function
$w(x) = (1-x)^{\alpha}(1+x)^{\beta}$ for $\alpha > -1$ and $\beta > -1$. Each fixed value of the parameters $\alpha > -1$ and $\beta > -1$ generates a distinct infinite orthogonal sequence $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ and within each of these distinct infinite sequences, we know from the classical result that the zeros of the polynomials $P_n^{(\alpha,\beta)}(x)$ and $P_{n+1}^{(\alpha,\beta)}(x)$ interlace. An interesting question is whether interlacing occurs for the zeros of polynomials of adjacent degree corresponding to different parameters α and β. More specifically, for what values of α' and β', with $\alpha' \neq \alpha$, $\beta' \neq \beta$ such that $\alpha', \alpha > -1$ and $\beta', \beta > -1$, do the zeros of

$P_n^{(\alpha,\beta)}(x)$ and $P_{n-1}^{(\alpha',\beta')}(x)$ interlace? A related question is whether we can establish interlacing of the zeros of Jacobi polynomials of the same degree corresponding to different choices of the parameters α and β.

In addressing this question, we recall a result proved by R. Askey on the interlacing zeros of Jacobi polynomials. In [9], Askey proved that $P_n^{(\alpha,\beta)}(x)$ and $P_{n+1}^{(\alpha+1,\beta)}(x)$ have interlacing zeros. Indeed if one denotes by $x_{k,n}^{(\alpha,\beta)}$ the zeros of $P_n^{(\alpha,\beta)}(x)$, then

$$x_{1,n+1}^{(\alpha,\beta)} < x_{1,n}^{(\alpha+1,\beta)} < x_{1,n}^{(\alpha,\beta)} < x_{2,n+1}^{(\alpha,\beta)} < x_{2,n}^{(\alpha+1,\beta)} < x_{2,n}^{(\alpha,\beta)} <$$
$$\cdots < x_{n,n+1}^{(\alpha,\beta)} < x_{n,n}^{(\alpha+1,\beta)} < x_{n,n}^{(\alpha,\beta)} < x_{n+1,n+1}^{(\alpha,\beta)}.$$

He conjectured that the zeros of Jacobi polynomials $P_n^{(\alpha,\beta)}$ and $P_n^{(\alpha+2,\beta)}$ interlace; namely

$$x_{1,n+1}^{(\alpha,\beta)} < x_{1,n}^{(\alpha+2,\beta)} < x_{1,n}^{(\alpha,\beta)} < x_{2,n+1}^{(\alpha,\beta)} < x_{2,n}^{(\alpha+2,\beta)} < x_{2,n}^{(\alpha,\beta)} <$$
$$\cdots < x_{n,n+1}^{(\alpha,\beta)} < x_{n,n}^{(\alpha+2,\beta)} < x_{n,n}^{(\alpha,\beta)} < x_{n+1,n+1}^{(\alpha,\beta)}.$$

He also posed the question whether the zeros of $P_n^{(\alpha,\beta)}$ and $P_n^{(\alpha',\beta)}$ interlace when $\alpha < \alpha' \leq \alpha + 2$.

In this chapter we investigate interlacing properties for the zeros of Jacobi polynomials $P_n^{(\alpha,\beta)}$ and $P_m^{(\alpha',\beta')}$, where $m = n$ or $m = n-1$ and $(\alpha',\beta') = (\alpha \pm t, \beta \pm k)$, $0 < t \leq 2$ and $0 < k \leq 2$.
This chapter is organized as follows. In Section 2, we study the interlacing of zeros of Jacobi polynomials of consecutive degree from different sequences while in Section 3, we study the interlacing of zeros of Jacobi polynomials of the same degree from different sequences.

2.2 Interlacing of zeros of Jacobi polynomials of consecutive degree from different sequences

2.2.1 Some recurrence relations of Jacobi polynomials

The Jacobi polynomials $\{P_n^{(\alpha,\beta)}(x)\}_{n=1}^{\infty}$, satisfy the well known three term recurrence relation given by

$$\begin{aligned}
&2n(\alpha+\beta+n)(\alpha+\beta+2n-2)P_n^{(\alpha,\beta)}(x)\\
&\quad= (\alpha+\beta+2n-1)[\alpha^2-\beta^2+x(\alpha+\beta+2n)(\alpha+\beta+2n-2)]P_{n-1}^{(\alpha,\beta)}(x)\\
&\qquad -2(\alpha+n-1)(\beta+n-1)(\alpha+\beta+2n)P_{n-2}^{(\alpha,\beta)}(x),\ \alpha>-1,\ \beta>-1
\end{aligned}$$

Using contiguous relations of ${}_2F_1$ (cf. [45] for a discussion on contiguous hypergeometric functions) together with connections between ${}_2F_1$ polynomials and Jacobi polynomials, one can generate mixed recurrence relations for Jacobi polynomials. We shall state a number of identities of this type that will be useful tools in our proofs.

Lemma 2.2.1 *(a) For $\alpha>-1$ and $\beta>-1$,*

$$\begin{aligned}
&2n[2+\alpha+3\beta+2n+(\alpha+\beta+2n)x]P_n^{(\alpha,\beta)}(x)\\
&\quad= (1+\alpha+\beta+n)(\alpha+\beta+2n)(1+x)^2P_{n-1}^{(\alpha,\beta+2)}(x)\\
&\qquad -4(\beta+n)(\beta+1)P_{n-1}^{(\alpha,\beta)}(x).
\end{aligned}$$

(b) For $\alpha>-1$ and $\beta>-1$,

$$\begin{aligned}
&2n[2+3\alpha+\beta+2n-(\alpha+\beta+2n)x]P_n^{(\alpha,\beta)}(x)\\
&\quad -4(1+\alpha)(\alpha+n)P_{n-1}^{(\alpha,\beta)}(x)\\
&\quad +(1+\alpha+\beta+n)(\alpha+\beta+2n)(1-x)^2P_{n-1}^{(\alpha+2,\beta)}(x)=0
\end{aligned}$$

(c) For $\alpha > -1$ and $\beta > -1$, one has

$$\begin{aligned}
&(1+\alpha+\beta+n)_2(1+x)P_n^{(\alpha,\beta+2)}(x) \\
&= \{(1+\alpha+\beta+2n)_2(1+x) - 2n(\alpha+n)\}P_n^{(\alpha,\beta)}(x) \\
&\quad -\{4+\alpha+3\beta+2n+(2+\alpha+\beta+2n)x\}(\alpha+n)P_{n-1}^{(\alpha,\beta)}(x).
\end{aligned}$$

(d) For $\alpha > 1$ and $\beta > -1$, one obtains

$$\begin{aligned}
&-2(\alpha+n-1)P_n^{(\alpha-2,\beta)}(x) \\
&= (\alpha+\beta+n)(1-x)P_n^{(\alpha,\beta)}(x) \\
&+ [3\alpha-2+\beta+2n-(\alpha+\beta+2n)x]P_n^{(\alpha-1,\beta)}(x)
\end{aligned}$$

(e) For $\alpha > 1$ and $\beta > 1$,

$$\begin{aligned}
&(\beta+n+1)[\beta+2\alpha+n-(\beta+n+2)x]P_n^{(\alpha,\beta)}(x) \\
&+ (1+\alpha+\beta+n)[\alpha+\beta-(2-\alpha+\beta)x]P_n^{(\alpha,\beta+1)}(x) \\
&- (1+x)(\alpha+n)(\alpha+n-1)P_n^{(\alpha-2,\beta-2)}(x) = 0 \qquad (2.1)
\end{aligned}$$

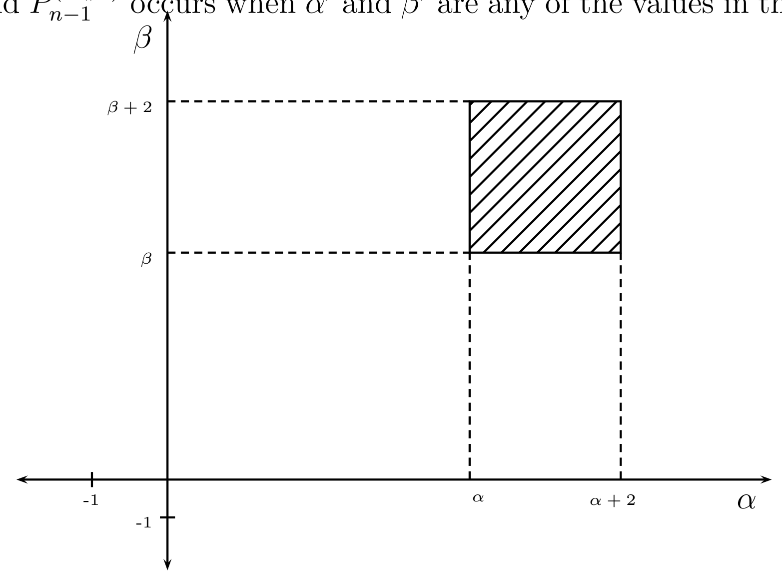
Remark The proofs of (a) to (e) in Lemma 2.2.1 are given in the Appendix
.

2.2.2 Interlacing of zeros of Jacobi polynomials of consecutive degree from different sequences

We turn our attention to Jacobi polynomials of the same degree with different parameters, where one or both parameters α and β increase or decrease by t, $0 < t \leq 2$.

In this section and the following, as in ([24]), we will make extensive use of the monotonicity result Theorem 1.4.1 of Markoff on the variation of the zeros of a polynomial with the parameter, as applied to Jacobi polynomials.

The results we will prove may be represented by shaded regions in the $\alpha\beta$-plane. For a fixed $\alpha > -1$ and $\beta > -1$, interlacing of the zeros of $P_n^{(\alpha,\beta)}$ and $P_{n-1}^{(\alpha',\beta')}$ occurs when α' and β' are any of the values in the shaded region.



Our first two results show that the zeros of Jacobi polynomials of consecutive degree interlace when one of the parameters α or β is increased by t where $0 < t \leq 2$, while the other parameter remains fixed.

Theorem 2.2.2 *Let $\alpha > -1$ and $\beta > -1$ and $t \in (0,2)$. Let*

$$\begin{aligned}
-1 < x_1 < x_2 < \cdots < x_n < 1 \quad &\textit{be the zeros of} \quad P_n^{(\alpha,\beta)},\\
-1 < y_1 < y_2 < \cdots < y_{n-1} < 1 \quad &\textit{be the zeros of} \quad P_{n-1}^{(\alpha,\beta)},\\
-1 < t_1 < t_2 < \cdots < t_{n-1} < 1 \quad &\textit{be the zeros of} \quad P_{n-1}^{(\alpha,\beta+t)} \textit{ and}\\
-1 < z_1 < z_2 < \cdots < z_{n-1} < 1 \quad &\textit{be those of} \quad P_{n-1}^{(\alpha,\beta+2)}.
\end{aligned}$$

Then

$$-1 < x_1 < y_1 < t_1 < z_1 < x_2 < \cdots < x_{n-1} < y_{n-1} < t_{n-1} < z_{n-1} < x_n < 1.$$

Proof. We know from the classical theory that $x_i < y_i < x_{i+1}$, for $i = 1, 2, \ldots n-1$. Also, it follows from Corollary 1.4.3 that $y_i < t_i < z_i$ for

$i = 1, 2, \ldots, n-1$. Thus

$$x_i < y_i < t_i < z_i \text{ for } i = 1, 2, \ldots, n-1. \tag{2.2}$$

From Lemma 2.2.1(a), we have

$$\begin{aligned}
&(1+\alpha+\beta+n)(\alpha+\beta+2n)(1+x)^2 P_{n-1}^{(\alpha,\beta+2)}(x)\\
&= 2n[2+\alpha+3\beta+2n+(\alpha+\beta+2n)x]P_n^{(\alpha,\beta)}(x)\\
&\quad +4(\beta+1)(\beta+n)P_{n-1}^{(\alpha,\beta)}(x).
\end{aligned}$$

Evaluating this equation at successive zeros x_i and x_{i+1} of $P_n^{(\alpha,\beta)}(x)$ we obtain

$$\begin{aligned}
&[(1+x_i)(1+x_{i+1})]^2 P_{n-1}^{(\alpha,\beta+2)}(x_i) P_{n-1}^{(\alpha,\beta+2)}(x_{i+1})\\
&= \left[\frac{4(\beta+1)(\beta+n)}{(1+\alpha+\beta+n)(\alpha+\beta+2n)}\right]^2 P_{n-1}^{(\alpha,\beta)}(x_i) P_{n-1}^{(\alpha,\beta)}(x_{i+1}).
\end{aligned}$$

Since $P_{n-1}^{(\alpha,\beta)}$ has a different sign at successive zeros of $P_n^{(\alpha,\beta)}$, we deduce that $P_{n-1}^{(\alpha,\beta+2)}(x_i)P_{n-1}^{(\alpha,\beta+2)}(x_{i+1}) < 0$. Therefore at least one zero of $P_{n-1}^{(\alpha,\beta+2)}$ lies in each interval (x_i, x_{i+1}), $i = 1, 2, \ldots, n-1$ and, together with (2.2), this yields the required result. □

Notice that the following theorem is a dual of Theorem 2.2.2 according to a very well known result about Jacobi polynomials.

Theorem 2.2.3 *Let $\alpha,\ \beta > -1$ and $t \in (0, 2)$. Let*

$$\begin{aligned}
-1 < x_1 < x_2 < \cdots < x_{n-1} < x_n < 1, &\quad \textit{be the zeros of} \quad P_n^{(\alpha,\beta)},\\
-1 < y_1 < y_2 < \cdots < y_{n-1} < 1, &\quad \textit{be the zeros of} \quad P_{n-1}^{(\alpha,\beta)},\\
-1 < t_1 < t_2 < \cdots < t_{n-1} < 1, &\quad \textit{be the zeros of} \quad P_{n-1}^{(\alpha+t,\beta)}\\
\textit{and } -1 < z_1 < z_2 < \cdots < z_{n-1} < 1, &\quad \textit{be the zeros of} \quad P_{n-1}^{(\alpha+2,\beta)}.
\end{aligned}$$

Then

$$-1 < x_1 < z_1 < t_1 < y_1 < x_2 < \cdots < z_{n-1} < t_{n-1} < y_{n-1} < x_n < 1$$

Proof. The zeros of polynomials of adjacent degree in a sequence of orthogonal polynomials interlace. Thus

$$x_i < y_i < x_{i+1} \quad \text{for} \quad i = 1, 2, \dots n-1 \tag{2.3}$$

and, by Corollary 1.4.3,

$$z_i < t_i < y_i \quad \text{for} \quad i = 1, 2, \dots n-1. \tag{2.4}$$

From Lemma 2.2.1(b) we know that

$$\begin{aligned}
&(1+\alpha+\beta+n)(\alpha+\beta+2n)\frac{(1-x)^2}{4}P_{n-1}^{(\alpha+2,\beta)}(x) \\
&\quad = \frac{[-2-3\alpha-\beta-2n+(2n+\alpha+\beta)x]}{2}P_n^{(\alpha,\beta)}(x) + (1+\alpha)(n+\alpha)P_{n-1}^{(\alpha,\beta)}(x).
\end{aligned}$$

Evaluating this equation at the consecutive zeros x_i and x_{i+1} of $P_n^{(\alpha,\beta)}(x)$, we have

$$\begin{aligned}
&P_{n-1}^{(\alpha+2,\beta)}(x_i)P_{n-1}^{(\alpha+2,\beta)}(x_{i+1}) \\
&\quad = \left[\frac{4(1+\alpha)(n+\alpha)}{[(1+\alpha+\beta+n)(\alpha+\beta+2n)(1-x_i)(1-x_{i+1})]}\right]^2 P_{n-1}^{(\alpha,\beta)}(x_i)P_{n-1}^{(\alpha,\beta)}(x_{i+1})
\end{aligned}$$

which is negative since the zeros of $P_n^{(\alpha,\beta)}$ and $P_{n-1}^{(\alpha,\beta)}$ interlace. Hence

$$x_i < z_i \quad \text{for} \quad i = 1, 2, \dots, n-1$$

and together with (2.3) and (2.4) we are done. □

The next result shows that interlacing of the zeros also occurs for Jacobi polynomials of adjacent degree when both the parameters α and β are increased by t and k respectively for any $t, k \in (0, 2]$.

Theorem 2.2.4 *Let* $\alpha,\ \beta > -1$ *and let* $0 \le t \le 2$ *and* $0 \le k \le 2$. *Let*

$$\begin{aligned}
-1 < x_1 < x_2 < \cdots < x_n < 1 &\quad \textit{be the zeros of} \quad P_n^{(\alpha,\beta)} \quad \textit{and} \\
-1 < t_1 < t_2 < \cdots < t_{n-1} < 1 &\quad \textit{be the zeros of} \quad P_{n-1}^{(\alpha+t,\beta+k)}.
\end{aligned}$$

Then

$$-1 < x_1 < t_1 < x_2 < \cdots < x_{n-1} < t_{n-1} < x_n < 1.$$

Proof. Let $t, k \in [0, 2]$, fixed. We denote the zeros of $P_{n-1}^{(\alpha+t,\beta)}$ by $-1 < y_1 < y_2 < \cdots < y_{n-1} < 1$ and those of $P_{n-1}^{(\alpha,\beta+k)}$ by $-1 < z_1 < z_2 < \cdots < z_{n-1} < 1$. Then

$$x_i < y_i \quad \text{for} \quad i = 1, 2, \ldots, n-1$$

by Theorem 2.2.3. In addition, $y_i < t_i$ and $t_i < z_i$ for $i = 1, 2, \ldots, n-1$ Corollary 1.4.3. Lastly, it follows from Theorem 2.2.2 that

$$z_i < x_{i+1} \quad \text{for} \quad i = 1, \ldots, n$$

which proves the result. $\square$

Remark Some restrictions on the ranges of t and k are required in the theorems since the interlacing property is not retained, in general, when one or both of the parameters α, β are increased by more than 2. This can be seen by considering, for example, the zeros of $P_n^{(\alpha,\beta)}$ and $P_{n-1}^{(\alpha+2.1,\ \beta)}$ when $n = 4$, $\alpha = -0.866$ and $\beta = 1.85$. The zeros of $P_4^{(-0.866,\ 1.85)}(x)$ are

$$\{-0.54137,\ 0.0888398,\ 0.673625,\ 0.988166\} \tag{2.5}$$

while those of $P_3^{(1.234,\ 1.85)}(x)$ are

$$\{-0.546253,\ 0.0805805,\ 0.669107\}$$

and interlacing clearly fails.

Furthermore, the interlacing property is also not retained in general for the zeros of $P_n^{(\alpha,\beta)}$ and those of $P_{n-1}^{(\alpha-t,\beta)}$ or $P_{n-1}^{(\alpha,\beta-k)}$ or $P_{n-1}^{(\alpha-t,\beta-k)}$ where $t, k > 0$. For example when n, α and β are chosen as in the example above, the zeros of $P_n^{(\alpha,\beta)}(x)$ are given by (2.5) and those of $P_{n-1}^{(\alpha-0.1,\ \beta)}(x)$ are

$$\{-0.292451,\ 0.524348,\ 0.995296\}.$$

Also, the zeros of $P_4^{(40.2,\ 0.05)}(x)$ are

$$\{-0.984779, \ -0.921655, \ -0.80448, \ -0.617579\}$$

while the zeros of $P_3^{(40.2,\ 0.02)}(x)$ are

$$\{-0.986307, \ -0.908504, \ -0.746103\}$$

and we see that $P_4^{(40.2,0.05)}$ and $P_3^{(40.2,0.02)}$ do not have interlacing zeros.

2.3 Interlacing of zeros of Jacobi polynomials of the same degree with different parameters

We now consider whether the zeros of Jacobi polynomials of the same degree with different parameters are interlacing if we allow one or both of the parameters α and β to increase or decrease by t, $0 < t \leq 2$.

Theorem 2.3.1 *Let $n \in \mathbb{N}$, $\alpha > -1$, $\beta > -1$ and let*

$$\begin{aligned} -1 < x_1 < x_2 < \cdots < x_n < 1 &\quad \textit{be the zeros of} \quad P_n^{(\alpha,\beta)}, \\ -1 < t_1 < t_2 < \cdots < t_n < 1 &\quad \textit{be the zeros of} \quad P_n^{(\alpha,\beta+t)} \textit{ and} \\ -1 < y_1 < y_2 < \ldots < y_n < 1 &\quad \textit{be the zeros of} \quad P_n^{(\alpha,\beta+2)} \end{aligned}$$

where $0 < t < 2$. Then

$$-1 < x_1 < t_1 < y_1 < x_2 < t_2 < y_2 < \cdots < x_n < t_n < y_n < 1.$$

Proof. It follows from Corollary 1.4.3 that

$$x_i < t_i < y_i, \quad \text{for} \quad i = 1, 2, \ldots, n. \tag{2.6}$$

Lemma 2.2.1(c) gives

$$\begin{aligned}&(1+\alpha+\beta+n)_2(1+x)P_n^{(\alpha,\beta+2)}(x)\\&\quad=\{(1+\alpha+\beta+2n)_2(1+x)-2n(\alpha+n)\}P_n^{(\alpha,\beta)}(x)\\&\quad-\{4+\alpha+3\beta+2n+(2+\alpha+\beta+2n)x\}(\alpha+n)P_{n-1}^{(\alpha,\beta+1)}(x).\end{aligned}\quad(2.7)$$

Note that the expression $4+\alpha+3\beta+2n+(2+\alpha+\beta+2n)x$ does not change sign for x belonging to $(-1,1)$ since

$$-\frac{4+\alpha+3\beta+2n}{2+\alpha+\beta+2n}=-1-2\frac{1+\beta}{2+\alpha+\beta+2n}<-1$$

for $\alpha>-1$ and $\beta>-1$. Evaluating (2.7) at consecutive zeros x_i and x_{i+1} of $P_n^{(\alpha,\beta)}(x)$, we obtain

$$\begin{aligned}&(1+\alpha+\beta+n)_2^2(1+x_i)(1+x_{i+1})P_n^{(\alpha,\beta+2)}(x_i)P_n^{(\alpha,\beta+2)}(x_{i+1})\\&\quad=\{4+\alpha+3\beta+2n+(2+\alpha+\beta+2n)x_i\}\{4+\alpha+3\beta+2n+\\&\quad(2+\alpha+\beta+2n)x_{i+1}\}(\alpha+n)^2P_{n-1}^{(\alpha,\beta+1)}(x_i)P_{n-1}^{(\alpha,\beta+1)}(x_{i+1})<0\end{aligned}$$

since the zeros of $P_n^{(\alpha,\beta)}$ and $P_{n-1}^{(\alpha,\beta+1)}$ interlace by Theorem 2.2.2 and hence $y_i<x_{i+1}$ for $i=1,2,\ldots,n-1$. Together with (2.6), this yields the required result. $\square$

Theorem 2.3.2 *Let $\beta>-1$, $\alpha>1$ and $t\in(0,2)$. Let*

$$\begin{aligned}&-1<x_1<x_2<\cdots<x_n<1 \quad \textit{be the zeros of} \quad P_n^{(\alpha,\beta)},\\&-1<t_1<t_2<\cdots<t_n<1 \quad \textit{be the zeros of} \quad P_n^{(\alpha-t,\beta)} \ \textit{and}\\&-1<y_1<y_2<\cdots<y_n<1 \quad \textit{be those of} \quad P_n^{(\alpha-2,\beta)}.\end{aligned}$$

Then

$$-1<x_1<t_1<y_1<x_2<t_2<y_2<\cdots<x_n<t_n<y_n<1.$$

Proof. Note that we need $\alpha > 1$ to ensure that $\{P_n^{(\alpha-2,\beta)}\}_{n=0}^{\infty}$ is an orthogonal sequence. From Corollary 1.4.3, $x_i < t_i < y_i$ for $i = 1, 2, \ldots, n$. It remains to prove that $y_i < x_{i+1}$ for $i = 1, 2, \ldots, n-1$.

By virtue of the recurrence relation in Lemma 2.2.1(d), we have

$$\begin{aligned} -2(\alpha+n-1)P_n^{(\alpha-2,\beta)}(x) &= (\alpha+\beta+n)(1-x)P_n^{(\alpha,\beta)}(x) \\ &+[3\alpha-2+\beta+2n-(\alpha+\beta+2n)x]P_n^{(\alpha-1,\beta)}(x) \end{aligned}$$

Evaluating this equation at the consecutive zeros x_1 and x_{i+1} of $P_n^{(\alpha,\beta)}(x)$,

$$\begin{aligned} &[2(\alpha+n-1)]^2 P_n^{(\alpha-2,\beta)}(x_i)P_n^{(\alpha-2,\beta)}(x_{i+1}) \\ &= [3\alpha-2+\beta+2n-(\alpha+\beta+2n)x_i] \\ &\quad \times[3\alpha-2+\beta+2n-(\alpha+\beta+2n)x_{i+1}]P_n^{(\alpha-1,\beta)}(x_i)P_n^{(\alpha-1,\beta)}(x_{i+1}). \end{aligned}$$

Note that the expression $3\alpha-2+\beta+2n-(\alpha+\beta+2n)x$ does not change sign for x belonging to $(-1,1)$. In fact $x = 1 + \dfrac{2\alpha-2}{\alpha+\beta+2n} > 1$ because $\alpha > 1$. Since the zeros of $P_n^{(\alpha,\beta)}$ and $P_n^{(\alpha-1,\beta)}$ interlace, we deduce that $y_i < x_{i+1}$ for each $i = 1, 2, \ldots, n-1$. $\square$

Theorem 2.3.3 *Let $\alpha > 1$, $\beta > -1$ and $t \in (0,2)$, $k \in (0,2)$.*

$$\begin{aligned} \text{Let} -1 < x_1 < x_2 < \cdots < x_n < 1, &\quad \text{be the zeros of} \quad P_n^{(\alpha,\beta)}, \\ -1 < t_1 < t_2 < \cdots < t_n < 1, &\quad \text{be the zeros of} \quad P_n^{(\alpha-k,\beta+t)}, \\ \text{and} \quad -1 < y_1 < y_2 < \cdots < y_n < 1, &\quad \text{be those of} \quad P_n^{(\alpha-2,\beta+2)}. \end{aligned}$$

Then

$$-1 < x_1 < t_1 < y_1 < x_2 < t_2 < y_2 < \cdots < x_n < t_n < y_n < 1.$$

Proof. According to Corollary 1.4.3,

$$x_i < t_i < y_i \quad \text{for} \quad i = 1, 2, \ldots n.$$

We need to prove that

$$y_i < x_{i+1} \quad \text{for} \quad i = 1, 2, \dots n-1.$$

Let us consider the expression

$$\begin{aligned}
&(\beta+n+1)[\beta+2\alpha+n-(\beta+n+2)x]P_n^{(\alpha,\beta)}(x) \\
&+ \quad (1+\alpha+\beta+n)[\alpha+\beta-(2-\alpha+\beta)x]P_n^{(\alpha,\beta+1)}(x) \\
&- \quad (1+x)(\alpha+n)(\alpha+n-1)P_n^{(\alpha-2,\beta-2)}(x) = 0 \qquad (2.8)
\end{aligned}$$

given in Lemma 2.2.1(e). We observe that the term $\alpha+\beta-(2-\alpha+\beta)x$ does not change sign for x belonging to $(-1,1)$. Indeed, $\alpha+\beta-(2-\alpha+\beta)x = 0$ if and only if

$$x = \frac{\alpha+\beta}{2-\alpha+\beta} = 1 + \frac{2\alpha-2}{2-\alpha+\beta}.$$

It is straightforward to show that for $\alpha > 1$ and $\beta > -1$, $\alpha+\beta-(2-\alpha+\beta)x = 0$ only when $|x| \geq 1$. Also, $\beta+2\alpha+n-(\beta+n+2)x$ does not change sign on $(-1,1)$. Evaluating expression (2.8) in x_i and x_{i+1}, consecutive zeros of $P_n^{(\alpha,\beta)}$ we have that

$$P_n^{(\alpha-2,\beta+2)}(x_i)P_n^{(\alpha-2,\beta+2)}(x_{i+1}) < 0$$

since the zeros of $P_n^{(\alpha,\beta)}$ and $P_{n-1}^{(\alpha,\beta+1)}$ interlace by Theorem 2.2.2. Thus

$$y_i < x_{i+1} \quad \text{for} \quad i = 1, 2, \dots, n-1. \qquad \square$$

$\square$

We note that analogous interlacing results will follow for the zeros of $P_n^{(\alpha,\beta)}$ and those of $P_n^{(\alpha+t,\beta)}$, $P_n^{(\alpha,\beta-t)}$ and $P_n^{(\alpha+k,\beta-t)}$ respectively where $t,\ k \in (0,2]$, by replacing α with $\alpha+t$ and β with $\beta-t$ in Theorems 2.4, 2.5 and 2.6.

Remark The zeros of $P_n^{(\alpha,\beta)}$ and $P_n^{(\alpha',\beta')}$ do not interlace in general when both the parameters are increased simultaneously. For example, taking $n = 4$, $\alpha = 1.266$ and $\beta = 1.85$, the zeros of $P_n^{(\alpha,\beta)}$ are

$$\{-0.67979, \ -0.201233, \ 0.326414, \ 0.764756\}$$

and those of $P_n^{(\alpha+0.2,\beta+0.2)}$ are

$$\{-0.667543, \ -0.197421, \ 0.317377, \ 0.750436\}.$$

Remark In [20], conditions for monotonicity of the zeros of Jacobi polynomials $P_n^{(\alpha,\beta)}$ and $P_n^{(\alpha',\beta')}$ are characterised. It is interesting to compare the differences between the conditions on the parameters that guarantee monotonicity and interlacing. Clearly, when the zeros are interlacing, they are also monotone and therefore both properties hold for the zeros of $P_n^{(\alpha,\beta)}$ and $P_n^{(\alpha-k,\beta+t)}$, or $P_n^{(\alpha+k,\beta-t)}$, when $k, t \in (0, 2]$. However, the monotonicity property holds more generally for any $k, t > 0$ (cf. [20], Theorem 1.2). We note that, when $n = 4$, $\alpha = -0.8$, $\beta = 2$, $k = 3.2$ and $t = 1.8$, the zeros of $P_n^{(\alpha,\beta)}$

$$\{-0.5273, \ 0.09428, \ 0.66789, \ 0.9825\}$$

while those of $P_n^{(\alpha+k,\beta-t)}$ are

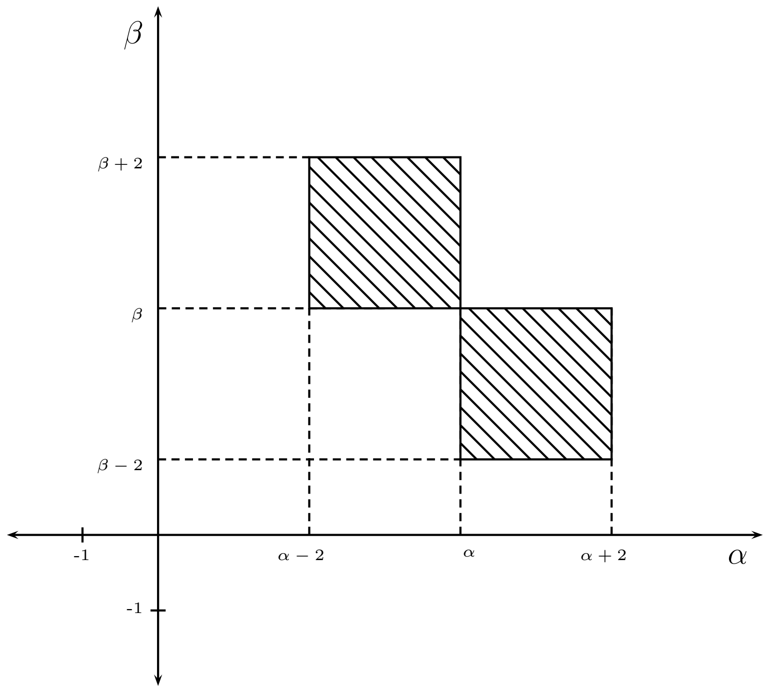
$$\{-0.9333, \ -0.61144, \ -0.100326 \ , 0.46048\}.$$

and we see that the monotonicity result holds but the zeros are not interlacing.

2.4 Conclusion

As remarked in the introduction to this chapter, Richard Askey conjectured in 1989 (cf. [9], p.29) that the zeros of $P_n^{(\alpha+2,\beta)}(x)$ and those of $P_{n+1}^{(\alpha,\beta)}(x)$ are interlacing. The results of this chapter prove this conjecture. We moreover prove more than the integer increment cases, we show that the interlacing property of zeros is retained also for continuous variation of both the parameters α and β within a specified range.

We can represent the situation when the interlacing property for the zeros of $P_n^{(\alpha,\beta)}$ and $P_n^{(\alpha',\beta')}$, for a fixed α and β with $(\alpha,\beta) \neq (\alpha',\beta')$; α, α', β and $\beta' > -1$ holds, by the shaded area in the $\alpha\beta$ plane.



Chapter 3

Interlacing of zeros of linear combinations of Jacobi polynomials from different sequences

3.1 Introduction

In this chapter, we focus on the interlacing of the zeros of linear combinations of Jacobi polynomials of the form $p_n + \mu q_m$, where $p_n = P_n^{(\alpha,\beta)}$ and $q_m = P_m^{(\alpha',\beta')}$, $(\alpha, \beta) \neq (\alpha', \beta')$, with the zeros of the component polynomials p_n and q_m when $m = n$ and $m = n-1$. We will also examine when interlacing takes place between the zeros of the linear combination and the zeros of certain Jacobi polynomials that are different from the component polynomials p_n and q_m.

Our proofs make extensive use of the interlacing property of the zeros of $P_n^{(\alpha,\beta)}$ and $P_m^{(\alpha',\beta')}$ for $m = n$ and $m = n-1$ and $(\alpha', \beta') = (\alpha \pm t, \beta \mp k)$, $0 < t \leq 2$ and $0 < k \leq 2$ studied in Chapter 2.

The structure of the chapter is as follows. In Section 2, we consider the linear combination $P_n^{(\alpha,\beta)}(x) + \nu P_n^{(\alpha-k,\beta+t)}(x)$ for $t, k \in (0, 2]$ and we discuss the interlacing property between the zeros of this linear combination and the zeros of the component polynomials. In Section 3, we study the interlacing of the zeros of $P_n^{(\alpha,\beta)} + \nu P_n^{(\alpha',\beta')}$ and the zeros of $P_{n+1}^{(\tilde{\alpha},\tilde{\beta})}$. The results of Section 3 are of independent interest and will also have specific application in Chapter 4 where we consider interlacing properties of the zeros of the linear combination $p_n + \nu q_n$ with those of $p_{n+1} + \nu q_{n+1}$.

3.2 Interlacing of the zeros of linear combinations of different Jacobi polynomials with the component polynomials

Interlacing properties of linear combinations of orthogonal polynomials can often be derived from the following simple result that has been proved in several contexts, for example, in dealing with polynomials associated with sequences of power moment functions ([37], p.117) and when considering quasi-orthogonality ([15], Theorem 3)

Lemma 3.2.1 *Let $\{p_n\}$ and $\{q_n\}$ be two sequences of polynomials that are orthogonal with respect to positive Borel measures μ_1 and μ_2, $\mu_1 \neq \mu_2$.*

(a) Assume that the zeros of p_n interlace with the zeros of q_n.

(i) The zeros of $E_n = p_n + \nu q_n$, $\nu \neq 0$ are all real and simple and interlace with the zeros of p_n and q_n.

(ii) If $\nu_1 \neq \nu_2$ are any two real numbers, then the zeros of $p_n(x) + \nu_1 q_n(x)$ and those of $p_n(x) + \nu_2 q_n(x)$ interlace.

(b) Assume that the zeros of p_n and q_{n-1} interlace.

(i) *The zeros of* $F_n = p_n + \kappa q_{n-1}$, $\kappa \neq 0$ *are all real, simple and interlace with the zeros of* p_n *and* q_{n-1}.

(ii) *Let* κ_1 *and* κ_2 *be two real numbers such that* $\kappa_1 \neq \kappa_2$. *Then the zeros of* $p_n(x)+\kappa_1 q_{n-1}(x)$ *and those of* $p_n(x)+\kappa_2 q_{n-1}(x)$ *interlace.*

Note that Lemma 3.2.1 also holds if the constants ν and κ in the linear combinations depend on n.

Corollary 3.2.2 *Let* $\alpha > 1$, $\beta > -1$ *and* $\nu \neq 0$. *Let*

$$E_n^{(\alpha,\beta,k,t)}(x) = P_n^{(\alpha,\beta)}(x) + \nu P_n^{(\alpha-k,\beta+t)}(x) \text{ for } t, k \in (0,2].$$

The zeros of $E_n^{(\alpha,\beta,k,t)}(x)$ *are real, simple and interlace with the zeros of* $P_n^{(\alpha,\beta)}(x)$ *as well as those of* $P_n^{(\alpha-k,\beta+t)}(x)$.

Proof. It was shown in Theorem 2.3.3, that the zeros of $P_n^{(\alpha,\beta)}$ interlace with the zeros of $P_n^{(\alpha-k,\beta+t)}$ for $t, k \in (0,2]$. The result then follows from Lemma 3.2.1(a). $\square$

Remark The condition $\alpha > 1$ is necessary to ensure the orthogonality of $P_n^{(\alpha-2,\beta)}$ when $\beta > -1$. We note that analogous interlacing results will follow for the zeros of the linear combination

$$P_n^{(\alpha,\beta)}(x) + \nu P_n^{(\alpha+k,\beta-t)}(x), \ \nu \neq 0$$

and those of $P_n^{(\alpha,\beta)}(x)$ and $P_n^{(\alpha+k,\beta-t)}(x)$ respectively, where $t, k \in (0,2]$, by replacing α with $\alpha + k$ and β with $\beta - t$ in Corollary 3.2.2.

It is interesting to note that in the case of linear combinations of Jacobi polynomials of degree n, the zeros of $E_n^{(\alpha,\beta,k,t)}(x)$ do not necessarily interlace with the zeros of either $P_{n-1}^{(\alpha,\beta)}(x)$ or $P_{n-1}^{(\alpha-k,\beta+t)}(x)$. Indeed, even in the simplest case when $t = k = 1$ and $n = 6$, $\alpha = 2.3$, $\beta = 3.2$, $\nu = 3$, the zeros of $E_6^{(2.3,3.2,1,1)}(x)$ are given by

$x_1 = -0.666,\ x_2 = -0.347,\ x_3 = 0.0014,\ x_4 = 0.341,\ x_5 = 0.6359,\ x_6 = 0.8571$

while those of $P_5^{(2.3,3.2)}$ are

$x = -0.684915,\ x = -0.328066,\ x = 0.0711339,\ x = 0.457021$ and $x = 0.775148$

and those of $P_5^{(2.3-1,3.2+1)}$ are

$x = -0.571753,\ x = -0.177335,\ x = 0.22934,\ x = 0.592974,\ x = 0.862258.$

Figures 1 and 2 show the zeros of these polynomials.

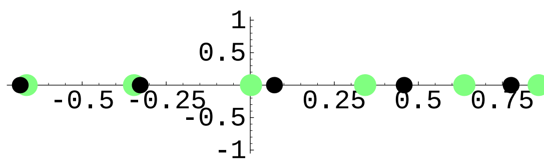


Figure 3.1: The zeros of $E_6^{(2.3,3.2,1,1)}$ are given by the larger grey dots, while those of $P_5^{(2.3,3.2)}$ are smaller and black

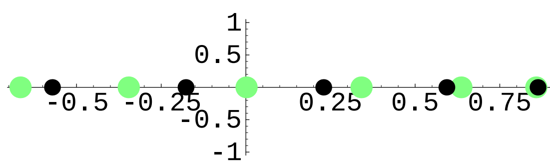


Figure 3.2: The larger grey dots represent the zeros of $E_6^{(2.3,3.2,1,1)}$ while the black dots are the zeros of $P_5^{(2.3-1,3.2+1)}$

Lemma 3.2.1(a) requires that the zeros of p_n and q_n are interlacing. We showed (in the remark after Theorem 2.3.3) that the zeros of Jacobi polynomials of the same degree do not interlace when both the parameters α and

β are increased simultaneously. Using this, it is not difficult to construct examples with $p_n = P_n^{(\alpha,\beta)}$ and $q_n = P_n^{(\alpha+k,\beta+t)}$ where the zeros of $p_n + \nu q_n$ and p_n or q_n do not interlace. For example, Figure 3 shows the zeros of the linear combination $P_n^{(\alpha,\beta)} + \nu P_n^{(\alpha+k,\beta+t)}$ and the component polynomial $P_n^{(\alpha,\beta)}$ for $n = 4$, $\alpha = 1.266$, $\beta = 1.85$, $\nu = 4.76$, $k = t = 0.5$.

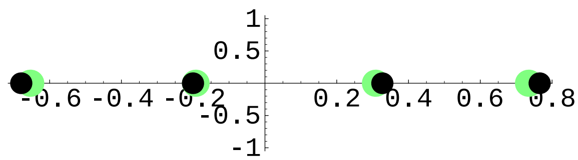


Figure 3.3: The zeros of $P_4^{(1.266,1.85)} + 4.76 P_4^{(1.266+0.5,1.85+0.5)}$ are represented by the larger dots in grey and those of $P_4^{(1.266,1.85)}$ are the smaller black dots.

The assumption made in Lemma 3.2.1(b) that the zeros of p_n and q_{n-1} interlace, is satisfied when $p_n = P_n^{(\alpha,\beta)}$ and $q_{n-1} = P_{n-1}^{(\alpha+t,\beta+k)}$ with $0 \le t \le 2$ and $0 \le k \le 2$ (cf. Theorem 2.2.4).

Corollary 3.2.3 *Let* $\alpha > -1$*,* $\beta > -1$*,* $t, k \in [0, 2]$ *and* $F_n^{(\alpha,\beta,t,k)}(x) = P_n^{(\alpha,\beta)}(x) + \mu P_{n-1}^{(\alpha+t,\beta+k)}(x)$*. Then the zeros of* $P_n^{(\alpha,\beta)}(x)$ *and the zeros of* $P_{n-1}^{(\alpha+t,\beta+k)}(x)$ *interlace with the zeros of* $F_n^{(\alpha,\beta,t,k)}(x)$*.*

Proof. It was shown in Theorem 2.2.4, that the zeros of $P_n^{(\alpha,\beta)}$ interlace with the zeros of $P_{n-1}^{(\alpha+t,\beta+k)}$ and the result follows as an immediate consequence of Lemma 3.2.1(b). $\square$

In general, the zeros of $F_n^{(\alpha,\beta,t,k)}(x)$ do not interlace with the zeros of $P_{n-1}^{(\alpha,\beta)}(x)$. Indeed, if $n = 7$, $\alpha = 2.3$, $\beta = 3.2$ and $\nu = 1.2$, $t = 1.7$, $k = 1$ then the zeros of $F_n^{(\alpha,\beta,t,k)}(x)$ are

$$x_1 = -1.56243,\ x_2 = -0.690889, \qquad x_3 = -0.403688,\ x_4 = -0.0921743,$$
$$x_5 = 0.21912,\ x_6 = 0.507874,\ x_7 = 0.756135$$

while those of $P_{n-1}^{(\alpha,\beta)}(x)$ are

$$\begin{aligned} y_1 = -0.748195,\ y_2 &= -0.454948,\ y_3 = -0.112964, \\ y_4 = 0.239816,\ y_5 &= 0.563443,\ y_6 = 0.821418. \end{aligned}$$

Remark More recently [cf. [25], Theorem 2.1, Theorem 3.1], K. Driver and K. Jordaan studied the zeros of linear combinations of Laguerre polynomials from different sequences. They established that if $\alpha > -1$, $a \neq 0$ and $0 < t < 2$, the zeros of $L_n^{\alpha}(x) + aL_n^{\alpha+t}(x)$ interlace with the zeros of $L_n^{\alpha}(x)$ and $L_n^{\alpha+t}(x)$. And if

$$\begin{aligned} 0 < x_1 < x_2 < \cdots < x_n &\text{ be the zeros of } L_n^{\alpha}, \\ 0 < y_1 < y_2 < \cdots < y_{n-1} &\text{ are the zeros of } L_{n-1}^{\alpha}, \\ 0 < t_1 < t_2 < \cdots < t_{n-1} &\text{ are the zeros of } L_{n-1}^{\alpha+t}, \text{ and} \\ 0 < z_1 < z_2 < \cdots < z_{n-1} &\text{ are the zeros of } L_{n-1}^{\alpha+2}, \end{aligned}$$

then we have $0 < x_1 < y_1 < t_1 < z_1 < x_2 < y_2 < t_2 < z_2 < \cdots < x_{n-1} < y_{n-1} < t_{n-}1 < z_{n-1} < x_n$

We note that the one parameter family of classical orthogonal polynomials, the Gegenbauer polynomials, are a special case of the Jacobi polynomials with $\alpha = \beta$.

3.3 Interlacing of the zeros of linear combinations of different Jacobi polynomials polynomials with other Jacobi polynomials

Our method of proof makes extensive use of the relationship between ${}_2F_1$ and Jacobi polynomials (cf. [45, p.254]), as well as the contiguous function relations of the hypergeometric polynomials. We state the Lemma we will use to establish our main results and note that its proof can be found in the Appendix.

Lemma 3.3.1 *(a) For $\alpha > 0$ and $\beta > -1$,*

$$\begin{aligned}
&(1+\alpha+\beta+2n)P_n^{(\alpha-1,\beta+1)}(x) \\
&= (1+\alpha+\beta+n)P_n^{(\alpha.\beta+1)}(x) - (\beta+n+1)P_{n-1}^{(\alpha,\beta+1)}(x).
\end{aligned}$$

(b) For $\alpha > -1$ and $\beta > 0$,

$$\begin{aligned}
&2(1+\alpha+\beta+2n)(n+1)P_{n+1}^{(\alpha,\beta-1)}(x) \\
&= (1+\alpha+\beta+n)[1+\alpha-\beta+2n+(1+\alpha+\beta+2n)x]P_n^{(\alpha,\beta+1)}(x) \\
&\quad -2\beta(\alpha+n)P_{n-1}^{(\alpha,\beta+1)}(x).
\end{aligned}$$

(c) For $\alpha > 0$ and $\beta > -1$,

$$\begin{aligned}
&2(1+\alpha+\beta+n)P_{n+1}^{(\alpha,\beta)}(x) \\
&= [2+\alpha+\beta+2n+(2+\alpha+\beta+2n)x](\alpha+n)P_n^{(\alpha-1,\beta+1)}(x) \\
&\quad +(\beta+n+1)[-\alpha-\beta+(2+\alpha+\beta+2n)x]P_n^{(\alpha,\beta)}(x).
\end{aligned}$$

(d) For $\alpha > 0$ and $\beta > -1$,

$$\begin{aligned} & 2(n+1)P_{n+1}^{(\alpha-1,\beta+1)}(x) \\ = \; & (\alpha+\beta+(2+\alpha+\beta+2n)x)P_n^{(\alpha,\beta+1)}(x) - 2(\beta+n+1)P_n^{(\alpha,\beta)}(x). \end{aligned}$$

Theorem 3.3.2 *Let $\beta > 0$, $0 < r < \dfrac{\alpha+n}{1+\beta+n}$ and let*

$$E_n^{(\alpha,\beta,1,1)} = P_n^{(\alpha,\beta)} + rP_n^{(\alpha-1,\beta+1)}.$$

a) If $\alpha > -1$ then $E_n^{(\alpha,\beta,1,1)}$ and $P_{n+1}^{(\alpha,\beta-1)}$ have interlacing zeros.

b) If $\alpha > 0$ then $E_n^{(\alpha,\beta,1,1)}$ and $P_{n+1}^{(\alpha,\beta)}$ have interlacing zeros.

Proof. a) The connection between Jacobi and hypergeometric polynomials, together with the contiguous relation (cf. [45, p.71, eqn.1]), yields

$$(1+\alpha+\beta+2n)P_n^{(\alpha,\beta)}(x) = (1+\alpha+\beta+n)P_n^{(\alpha,\beta+1)}(x) + (\alpha+n)P_{n-1}^{(\alpha,\beta+1)}(x),$$

while, according to Lemma 3.3.1 (a),

$$\begin{aligned} & (1+\alpha+\beta+2n)P_n^{(\alpha-1,\beta+1)}(x) \\ = \; & (1+\alpha+\beta+n)P_n^{(\alpha,\beta+1)}(x) - (\beta+n+1)P_{n-1}^{(\alpha,\beta+1)}(x). \end{aligned}$$

Since $E_n^{(\alpha,\beta,1,1)} = P_n^{(\alpha,\beta)} + rP_n^{(\alpha-1,\beta+1)}$,

$$\begin{aligned} & (1+\alpha+\beta+2n)E_n^{(\alpha,\beta,1,1)}(x) \\ = \; & (1+\alpha+\beta+n)(1+r)P_n^{(\alpha,\beta+1)}(x) + [\alpha+n-r(\beta+n+1)]P_{n-1}^{(\alpha,\beta+1)}(x). \end{aligned} \tag{3.1}$$

On the other hand, from Lemma 3.3.1 (b),

$$\begin{aligned} & 2(1+\alpha+\beta+2n)(n+1)P_{n+1}^{(\alpha,\beta-1)}(x) \\ = \; & (1+\alpha+\beta+n)[1+\alpha-\beta+2n+(1+\alpha+\beta+2n)x]P_n^{(\alpha,\beta+1)}(x) \\ & -2\beta(\alpha+n)P_{n-1}^{(\alpha,\beta+1)}(x). \end{aligned} \tag{3.2}$$

Thus

$$\begin{aligned}
&E_n^{(\alpha,\beta,1,1)}(x)\\
&= -\frac{2(1+\alpha+\beta+2n)(n+1)[\alpha+n-r(\beta+n+1)]}{2\beta(\alpha+n)(1+\alpha+\beta+2n)}P_{n+1}^{(\alpha,\beta-1)}(x)\\
&+ \frac{(1+\alpha+\beta+n)}{2\beta(\alpha+n)(1+\alpha+\beta+2n)}A_n^{(\alpha,\beta,r)}P_n^{(\alpha,\beta+1)}(x) \qquad (3.3)
\end{aligned}$$

where

$$A_n^{(\alpha,\beta,r)} = 2\beta(\alpha+n)(1+r)+[\alpha+n-r(\beta+n+1)][1+\alpha+\beta+2n+(1+\alpha+\beta+2n)x]$$

changes sign only if $x = -1 - \dfrac{2\beta r}{\alpha+n-r(\beta+n+1)}$. It is clear that if $\alpha > -1$, $\beta > 0$ and $0 < r < \dfrac{\alpha+n}{1+\beta+n}$, the coefficient of $P_n^{(\alpha,\beta+1)}$ in (3.3) does not change sign on $(-1,1)$. Evaluating (3.3) at consecutive zeros x_i and x_{i+1}, $i = 1,\ldots,n$, of $P_{n+1}^{(\alpha,\beta-1)}(x)$, one obtains $E_n^{(\alpha,\beta,1,1)}(x_i)E_n^{(\alpha,\beta,1,1)}(x_{i+1}) < 0$ since $P_{n+1}^{(\alpha,\beta-1)}$ and $P_n^{(\alpha,\beta+1)}$ have interlacing zeros (cf. Theorem 2.2.2).

b) From Lemma 3.3.1 (c), one has

$$\begin{aligned}
&2(1+\alpha+\beta+n)(n+1)P_{n+1}^{(\alpha,\beta)}(x)\\
&= [2+\alpha+\beta+2n+(2+\alpha+\beta+2n)x](\alpha+n)P_n^{(\alpha-1,\beta+1)}(x)\\
&\quad +(\beta+n+1)[-\alpha-\beta+(2+\alpha+\beta+2n)x]P_n^{(\alpha,\beta)}(x).
\end{aligned}$$

Replacing $P_n^{(\alpha-1,\beta+1)}$ by $\frac{1}{r}[E_n^{(\alpha,\beta,1,1)} - P_n^{(\alpha,\beta)}]$, we have

$$\begin{aligned}
&2r(1+\alpha+\beta+n)(n+1)P_{n+1}^{(\alpha,\beta)}(x)\\
&= (2+\alpha+\beta+2n)(1+x)(\alpha+n)E_n^{(\alpha,\beta,1,1)}(x)\\
&\quad +\{r(\beta+n+1)[-\alpha-\beta+(2+\alpha+\beta+2n)x]\\
&\quad -(2+\alpha+\beta+2n)(1+x)(\alpha+n)\}P_n^{(\alpha,\beta)}(x). \qquad (3.4)
\end{aligned}$$

For $\alpha > 0$, $\beta > 0$ and $0 < r < \dfrac{\alpha+n}{1+\beta+n}$, the coefficient of $P_n^{(\alpha,\beta)}$ in this equation does not change sign on $(-1,1)$ and since $P_{n+1}^{(\alpha,\beta)}$ and $P_n^{(\alpha,\beta)}$ have interlacing zeros, we deduce the result by evaluating (3.4) at consecutive zeros of $P_{n+1}^{(\alpha,\beta)}(x)$. $\square$

Theorem 3.3.3 *Let* $E_n^{\alpha,\beta,0,1} = P_n^{(\alpha,\beta)} + rP_n^{(\alpha,\beta+1)}$*. If* $\alpha > 0$, $\beta > -1$ *and* $r > \dfrac{n+1}{\beta+n+1}$*, the zeros of* $E_n^{(\alpha,\beta,0,1)}$ *and those of* $P_{n+1}^{(\alpha-1,\beta+1)}$ *interlace.*

Proof. We know (cf. Lemma 3.3.1 (d)) that

$$\begin{aligned} &2(n+1)P_{n+1}^{(\alpha-1,\beta+1)}(x) \\ &= (\alpha+\beta+(2+\alpha+\beta+2n)x)P_n^{(\alpha,\beta+1)}(x) - 2(\beta+n+1)P_n^{(\alpha,\beta)}(x). \end{aligned}$$

$$\text{Since} \quad P_n^{(\alpha,\beta)}(x) = E_n^{(\alpha,\beta,0,1)}(x) - rP_n^{(\alpha,\beta+1)}(x),$$

$$\begin{aligned} &2(n+1)P_{n+1}^{(\alpha-1,\beta+1)}(x) \\ &= [\alpha+\beta+2r(\beta+n+1)+(2+\alpha+\beta+2n)x]P_n^{(\alpha,\beta+1)}(x) \\ &\quad -2(\beta+n+1)E_n^{(\alpha,\beta,0,1)}(x). \end{aligned} \tag{3.5}$$

The coefficient of $P_n^{(\alpha,\beta+1)}$ is zero only if $x = -1 - \dfrac{2r(\beta+n+1)-2(n+1)}{\alpha+\beta+2+2n}$ and therefore the coefficient does not change sign on $(-1,1)$ when

$$r > \frac{n+1}{\beta+n+1}.$$

Evaluating (3.5) at consecutive zeros x_i and x_{i+1}, $i = 1, 2, \ldots, n$, of $P_{n+1}^{(\alpha-1,\beta+1)}$ we obtain $E_n^{(\alpha,\beta,0,1)}(x_i)E_n^{(\alpha,\beta,0,1)}(x_{i+1}) < 0$ since the zeros of $P_{n+1}^{(\alpha-1,\beta+1)}$ and $P_n^{(\alpha,\beta+1)}$ interlace (cf. Theorem 2.2.3). We deduce that there is at least one zero of $E_n^{(\alpha,\beta,0,1)}$ between any two consecutive zeros of $P_{n+1}^{(\alpha-1,\beta+1)}$ and the result follows. $\square$

3.4 Conclusion

A linear combination of two Jacobi polynomials from different sequences has real and simple zeros, as long as the components have interlacing zeros. In this case, the zeros interlace with those of component polynomials.
In the next chapter, we shall investigate the interlacing property for the zeros of consecutive polynomials in some sequences of linear combinations of Laguerre and Jacobi polynomials.

Chapter 4

Interlacing of zeros of linear combinations of classical orthogonal polynomials from different sequences

4.1 Introduction

Let us consider two sequences of polynomials $\{p_n\}_{n=0}^{\infty}$ and $\{q_n\}_{n=0}^{\infty}$ that are orthogonal respectively with respect to positive Borel measures μ_1 and μ_2, $\mu_1 \neq \mu_2$ on the same interval of orthogonality I. Let

$$r_n = p_n + a_n q_n \text{ and } s_n = p_n + b_n q_n$$

with a_n and b_n real and independent of x. The following questions arise in natural way: When do the zeros of r_n and r_{n+1} interlace and when do the zeros of s_n and s_{n+1} interlace?

The questions are challenging to answer in general and it is natural to first consider the simplest, or the best understood, or the most useful, orthogonal sequences $\{p_n\}_{n=0}^{\infty}$ and $\{q_n\}_{n=0}^{\infty}$. Two choices that serve as a reasonable starting point are

$$p_n = L_n^{\alpha},\ q_n = L_n^{\alpha+1}$$

where L_n^{α} are the Laguerre polynomials, orthogonal with respect to the weight function $e^{-x}x^{\alpha}$ on the interval $(0,\infty)$ for $\alpha > -1$; and

$$p_n = P_n^{(\alpha,\beta)},\ q_n = P_n^{(\alpha,\beta+1)}$$

where $P_n^{(\alpha,\beta)}$ are the Jacobi polynomials, orthogonal with respect to the weight function $(1-x)^{\alpha}(1+x)^{\beta}$ on the interval $(-1,1)$ for $\alpha > -1, \beta > -1$.

We define

$$\begin{aligned} R_n &= L_n^{\alpha} + b_n L_n^{\alpha+1} \\ S_n &= L_n^{\alpha} + d_n L_{n-1}^{\alpha+1} \\ E_n &= P_n^{(\alpha,\beta)} + r_n P_n^{(\alpha,\beta+1)}. \end{aligned}$$

We shall prove that the zeros of polynomials of consecutive degree in the sequences $\{R_n\}_{n=0}^{\infty}$, $\{S_n\}_{n=0}^{\infty}$ and $\{E_n\}_{n=0}^{\infty}$ are interlacing.

The chapter is organized as follows. In Section 4.2, we give the definition of quasi-orthogonality and preliminary results which are well known in the literature. Also we prove a lemma that will help us to establish our result in Section 4.3.

In Section 4.3, we prove the interlacing properties zeros of linear combinations of Laguerre polynomials. and in Section 4.4, we discuss the interlacing of zeros of linear combinations of Jacobi polynomials.

4.2 Quasi-orthogonality and preliminary results

The concept of quasi-orthogonality of order 1 was introduced by M. Riesz (see [46]). Since then the notion of quasi-orthogonality has been discussed by many authors. In [17], T. S. Chihara discussed quasi-orthogonality using the three term recurrence relations; and he established that every sequence of quasi-orthogonal polynomials satisfies a three term recurrence relation whose coefficients are polynomials of appropriate degrees. The work of Draux (cf. [21]) and Dickinson (cf. [19]) complete and improve Chihara's result. Brezinski, Driver and Redivo-Zaglia in [15] deal with the location of zeros of quasi-orthogonal polynomials of order 1 and of order 2 using classical tools such as the Christoffel-Darboux identity.

Recently, in [36], Joulak studied the quasi-orthogonality of polynomials and their associated polynomials using tools from linear algebra. He generalized the results in [15] and obtained new results on interlacing zeros of quasi-orthogonal polynomials of order 1, 2, and 3.

We recall the definition of quasi-orthogonality.

Definition 4.2.1 *A sequence $\{r_n\}_{n=0}^{\infty}$ of real polynomials with $deg(r_n) = n \geq k$ is quasi-orthogonal of order k, where k is a fixed non-negative integer, with respect to a positive weight function $w(x)$ on the interval I if*

$$\int_I x^m r_n(x) w(x) = \begin{cases} 0 \text{ for } m = 0, 1, \ldots, n-1-k \\ \neq 0 \text{ for } m = n-k. \end{cases}$$

When $k = 0$, the sequence is orthogonal. For the general case of the quasi-orthogonality, see [17] and [49].

In this chapter, we will use the following results by Brezinski-Driver-Redivo-Zaglia , as well as by Joulak, to obtain our main results.

Lemma 4.2.2 *[15, Theorem 3] Let* $\{p_n\}_{n=1}^{\infty}$ *be a sequence of monic orthogonal polynomials and consider*

$$R_n = p_n + a_n p_{n-1} \qquad \textit{where} \quad a_n \in \mathbb{R}.$$

Let

$$\begin{aligned} y_1 < y_2 < \cdots < y_n &\quad \textit{be the zeros of} \quad R_n, \\ x_1 < x_2 < \cdots < x_n &\quad \textit{be the zeros of} \quad p_n, \\ t_1 < t_2 < \cdots < t_{n-1} &\quad \textit{be the zeros of} \quad p_{n-1}. \end{aligned}$$

Then

(a) $a_n < 0 \quad \Leftrightarrow \quad x_1 < y_1 < t_1 < x_2 < y_2 < t_2 < \cdots < x_{n-1} < y_{n-1} < t_{n-1} < x_n < y_n,$

(b) $a_n > 0 \quad \Leftrightarrow \quad y_1 < x_1 < t_1 < y_2 < x_2 < t_2 < \cdots < y_{n-1} < x_{n-1} < t_{n-1} < y_n < x_n.$

Lemma 4.2.3 *[36, Theorem 6] Let* $\{p_n\}_{n=1}^{\infty}$ *be a sequence of monic orthogonal polynomials and let* $y_1 < y_2 < \cdots < y_n$ *be the zeros of* $R_n = p_n + a_n p_{n-1}$ *while* $z_1 < z_2 < \cdots < z_{n+1}$ *are the zeros of* $R_{n+1} = p_{n+1} + a_{n+1} p_n$. *If*

$$f_{n+1} = \frac{p_{n+1}}{p_n}$$

then

$$z_1 < y_1 < z_2 < y_2 < \cdots < z_n < y_n < z_{n+1}$$

if and only if

$$\left\{ \begin{array}{ll} f_{n+1}(y_n) + a_{n+1} < 0 & \textit{if } a_n < 0 \\ f_{n+1}(y_1) + a_{n+1} > 0 & \textit{if } a_n > 0 \end{array} \right\}$$

In addition to the three term recurrence relation for Laguerre polynomials, one can also derive the following mixed recurrence relations (cf. [2], 22.7.29, 22.7.30 and [45], p. 203)

$$\begin{aligned} xL_n^{\alpha+1}(x) &= (x-n)L_n^{\alpha}(x) + (\alpha+n)L_{n-1}^{\alpha}(x), && (4.1) \\ L_{n-1}^{\alpha+1}(x) &= L_n^{\alpha+1}(x) - L_n^{\alpha}(x), && (4.2) \\ xL_n^{\alpha+1}(x) &= (\alpha+n+1)L_n^{\alpha}(x) - (n+1)L_{n+1}^{\alpha}(x). && (4.3) \end{aligned}$$

Replacing in (4.1) n by $n+1$ and α by $\alpha-1$ while in (4.2) n by $n+1$, one deduces

$$xL_{n+1}^{\alpha+1}(x) = (x+\alpha)L_{n+1}^{\alpha}(x) - (\alpha+n+1)L_{n+1}^{\alpha-1}(x). \tag{4.4}$$

We will use the following notation for the zeros of the Laguerre polynomials. Let $\{x_{i,k}^{\gamma}\}_{i=1}^{k}$ denote the set of zeros of $L_k^{\gamma}(x)$ arranged in ascending order, $\gamma > -1, k \in \mathbb{N}$.

In [32], it is established that

$$\frac{\gamma+1}{k} < x_{1,k}^{\gamma} \text{ while } x_{k,k}^{\gamma} < 4k + 2\gamma + 1. \tag{4.5}$$

The following lemma will be useful for the proof of our result on interlacing zeros of linear combinations of Laguerre polynomials.

Lemma 4.2.4 *Let $n \in \mathbb{N}$, $\alpha > -1$, $b_n \in \mathbb{R}$, $b_n \neq -1, 0$ and*

$$R_n = L_n^{\alpha} + b_n L_n^{\alpha+1}.$$

(a)

$$\textit{If } b_n < -\frac{3n+2\alpha+4}{n+\alpha+1} \textit{ or } b_n > -\frac{n^2+2n-\alpha}{(n+1)(\alpha+n+1)},$$

the zeros of R_n and those of $L_{n+1}^{\alpha+1}$ interlace.

(b)

$$\textit{If } b_n < -\frac{4n+2\alpha+5}{\alpha+n+1} \textit{ or } b_n > -\frac{\alpha+1}{(n+1)(\alpha+n+1)},$$

the zeros of R_n and those of L_{n+1}^{α} interlace.

Proof. (a) We have

$$\begin{aligned} R_n &= L_n^{\alpha} + b_n L_n^{\alpha+1} \\ &= [L_{n+1}^{\alpha} - L_{n+1}^{\alpha-1}] + b_n[L_{n+1}^{\alpha+1} - L_{n+1}^{\alpha}], \text{ using (4.2)} \\ &= (1-b_n)L_{n+1}^{\alpha} + b_n L_{n+1}^{\alpha+1} - L_{n+1}^{\alpha-1}. \end{aligned}$$

Thus, from (4.4),

$$\begin{aligned} R_n(x) &= (1-b_n)L^{\alpha}_{n+1}(x) + b_n L^{\alpha+1}_{n+1}(x) \\ &\quad -[\frac{x+\alpha}{\alpha+n+1}L^{\alpha}_{n+1}(x) - \frac{x}{\alpha+n+1}L^{\alpha+1}_{n+1}] \\ &= [1-b_n-\frac{x+\alpha}{\alpha+n+1}]L^{\alpha}_{n+1}(x) + (b_n+\frac{x}{\alpha+n+1})L^{\alpha+1}_{n+1}(x). \end{aligned}$$

From (4.5), we know that the zeros of Laguerre polynomial L^{α}_{n+1} are bounded below by $\dfrac{\alpha+1}{n+1}$ and bounded above by $4n+2\alpha+5$. This, together with the conditions on b_n, ensures that the coefficient of $L^{\alpha}_{n+1}(x)$ does not change sign on the interval $(x^{\alpha+1}_{1,n+1}, x^{\alpha+1}_{n+1,n+1})$ which has endpoints at the smallest and largest zero of $L^{\alpha+1}_{n+1}$.

Evaluating $R_n(x)$ at consecutive zeros of $L^{\alpha+1}_{n+1}$ we see that

$$\begin{aligned} & R_n(x^{\alpha+1}_{i,n+1})R_n(x^{\alpha+1}_{i+1,n+1}) \\ = \; & \frac{[(b_n-1)(\alpha+n+1)+x^{\alpha+1}_{i+1,n+1}+\alpha][(b_n-1)(\alpha+n+1)+x^{\alpha+1}_{i,n+1}+\alpha]}{(\alpha+n+1)^2} \\ & \times L^{\alpha}_{n+1}(x_{i,n+1})(L^{\alpha}_{n+1}(x^{\alpha+1}_{i+1,n+1}). \end{aligned} \tag{4.6}$$

We know from [24, Theorem 2.3] that the zeros of L^{α}_{n+1} interlace with the zeros of $L^{\alpha+1}_{n+1}$ which implies that L^{α}_{n+1} has a different sign at successive zeros of $L^{\alpha+1}_{n+1}$. So the expression (4.6) is negative. Thus R_n and $L^{\alpha+1}_{n+1}$ have interlacing zeros.

(b) Substituting (4.3) into the left hand side of $L^{\alpha+1}_{n} = \dfrac{1}{b_n}[R_n - L^{\alpha}_{n}]$ we obtain

$$\begin{aligned} \frac{x}{b_n}[R_n - L^{\alpha}_{n}] &= (\alpha+n+1)L^{\alpha}_{n} - (n+1)L^{\alpha}_{n+1} \\ xR_n &= [x+b_n(\alpha+n+1)]L^{\alpha}_{n} - b_n(n+1)L^{\alpha}_{n+1}. \end{aligned} \tag{4.7}$$

Using the same upper and lower bounds for the zeros of L^{α}_{n+1} as above, together with the conditions on b_n, we deduce that the expression $x+b_n(\alpha+n+1)$ does not change sign on the interval $(x^{\alpha}_{1,n+1}, x^{\alpha}_{n+1,n+1})$. $\square$

Remark

1. In the assumptions, we assume that $b_n \neq -1$ to ensure that $R_n(x)$ is of exact degree n since it can be seen from (4.2) that $R_n(x) = (1 + b_n)L_n^{\alpha}(x) + b_n L_{n-1}^{\alpha+1}(x)$.

2. Applying (4.2), we also see that $R_n(x) = (b_n + 1)L_n^{\alpha+1}(x) - L_{n-1}^{\alpha+1}(x)$, a linear combination of orthogonal polynomials from the same sequence. Hence the zeros of R_n interlace with those of $L_n^{\alpha+1}(x)$ and $L_{n-1}^{\alpha+1}(x)$ respectively when $b_n \neq -1, 0$ (cf [25]); and the restrictions on b_n for $\{R_n(x)\}_{n=0}^{\infty}$ to be an orthogonal sequence can be deduced from [22, Theorem 3(v)].

4.3 Linear combinations of Laguerre polynomials

We state and prove our main interlacing result for the zeros of linear combinations of Laguerre polynomials.

Theorem 4.3.1 *Let* $R_n(x) = L_n^{\alpha}(x) + b_n L_n^{\alpha+1}(x)$, $n \in \mathbb{N}, \alpha > -1, b_n \in \mathbb{R}, b_n \neq -1, 0$. *If*

$$\begin{aligned} (a)\ b_n &> \frac{n^2 + 2n - \alpha}{(n+1)(\alpha + n + 1)} \ or \\ (b)\ b_n &< -\frac{4n + 2\alpha + 5}{\alpha + n + 1} \ or \ -\frac{\alpha + 1}{n + 1} < b_n < -1, \end{aligned}$$

then $R_n(x)$ *and* $R_{n+1}(x)$ *have interlacing zeros.*

Proof. We know that the leading coefficient of L_n^{α} is $\frac{(-1)^n}{n!}$; so we can renormalize $R_n(x)$. Let

$$\begin{aligned} K_n(x) &= \frac{(-1)^n n!}{b_n+1} R_n \\ &= \frac{(-1)^n n!}{b_n+1}[(b_n+1)L_n^{\alpha+1}(x) - L_{n-1}^{\alpha+1}(x)] \text{ from (4.2)} \\ &= \tilde{L}_n^{\alpha+1}(x) + a_n \tilde{L}_{n-1}^{\alpha+1}(x) \end{aligned}$$

where $\tilde{L}_n^{\alpha+1}(x) = (-1)^n n! L_n^{\alpha+1}(x)$ is the Laguerre polynomial of degree n renormalized to be monic and $a_n = \dfrac{n}{b_n+1}$. Let $y_1 < y_2 < y_3 < \cdots < y_n$ denote the zeros of $R_n(x)$.

(a) Obviously we have $a_n > 0$ if and only if $b_n > -1$. On the other hand,

$$\begin{aligned} b_n &> \frac{n^2+2n-\alpha}{(n+1)(\alpha+n+1)} \\ &= 1 - \frac{\alpha(n+2)+1}{(n+1)(n+\alpha+1)}, \end{aligned}$$

together with the condition on α, this is trivially greater than -1. Then it follows from Lemma 4.2.4(a), that

$$x_{i,n+1}^{\alpha+1} < y_i < x_{i+1,n+1}^{\alpha+1} \quad i = 1, 2, \ldots, n \tag{4.8}$$

For n even, since $\tilde{L}_{n+1}^{\alpha+1}$ is monic, $\lim\limits_{x\to-\infty} \tilde{L}_{n+1}^{\alpha+1}(x) = -\infty$; so according to (4.8) we have $\tilde{L}_{n+1}^{\alpha+1}(y_1) > 0$.

Similarly, since $b_n > -1$, we have

$$y_1 < x_{1,n}^{\alpha+1} \text{ from Lemma 4.2.4(b).}$$

Thus for n even, $\tilde{L}_n^{\alpha+1}(y_1) > 0$ because $\lim\limits_{x\to-\infty} \tilde{L}_n^{\alpha+1}(x) = +\infty$.

For n odd, we obtain $\tilde{L}_{n+1}^{\alpha+1}(y_1) < 0$ and $\tilde{L}_n^{\alpha+1}(y_1) < 0$. Therefore

$$\frac{\tilde{L}_{n+1}^{\alpha+1}(y_1)}{\tilde{L}_n^{\alpha+1}(y_1)} + \frac{n+1}{b_n+1} > 0 \text{ for any } n \in \mathbb{N} \text{ and } b_n > -1.$$

It now follows from Lemma 4.2.3 that R_n and R_{n+1} have interlacing zeros.

(b) Let $b_n < -\dfrac{4n+2\alpha+5}{\alpha+n+1}$ or $-\dfrac{\alpha+1}{n+1} < b_n < -1$. Then $a_n = \dfrac{n}{b_n+1} < 0$. From Lemma 4.2.2, Lemma 4.2.4(b) and [24, Theorem 2.3], we have

$$x_{i,n+1}^{\alpha+1} < x_{i,n}^{\alpha} < y_i < x_{i+1,n+1}^{\alpha} < x_{i+1,n+1}^{\alpha+1}, i = 1, 2, \ldots, n.$$

$$\text{Furthermore } \lim_{x\to\infty} \tilde{L}_{n+1}^{\alpha+1}(x) = \lim_{x\to\infty} \tilde{L}_{n}^{\alpha+1}(x) = \infty$$

and it follows that $\tilde{L}_{n+1}^{\alpha+1}(y_n) < 0$ and $\tilde{L}_{n}^{\alpha+1}(y_n) > 0$.
Therefore $\dfrac{\tilde{L}_{n+1}^{\alpha+1}(y_n)}{\tilde{L}_{n}^{\alpha+1}(y_n)} + \dfrac{n+1}{b_n+1} < 0$ and the result follows from Lemma 4.2.3.

□

Remark In [22], a necessary and sufficient condition is given for the orthogonality of a sequence of monic polynomials that have the property that two polynomials of consecutive degree in the sequence have interlacing zeros. In the context of context of Theorem 4.3.1, it is interesting to consider the implications of this necessary and sufficient condition for linear combinations of Laguerre polynomials. It is easy to find values of n, b and $\alpha > -1$ with $n \in \mathbb{N}$ such that the monic polynomials $K_n = \dfrac{(-1)^n n!}{b+1} R_n$ where $R_n = L_n^{\alpha} + bL_n^{\alpha+1}$ do not satisfy the condition for orthogonality given in [22]. For example, when $n = 3$, $\alpha = 0.8$ and $b = 3.2$, the conditions of Theorem 4.3.1(a) are satisfied and the zeros of $K_3 = \{1.31747,\ 3.89356,\ 8.47469\}$ interlace with the zeros of $K_4 = \{1.07131,\ 3.09399,\ 6.38603,\ 11.6963\}$. However, evaluating K_4/K_2 at the consecutive zeros of K_3 we obtain $\frac{K_4}{K_2}(1.31747) = -13.8835$, $\frac{K_4}{K_2}(3.89356) = -13.4189$ and $\frac{K_4}{K_2}(8.47469) = -12.9148$ respectively. Since these ratios are not equal, the sequence $\{K_n\}_{n=0}^{\infty}$ is not orthogonal with respect to any positive Borel measure.

Corollary 4.3.2 *Let $S_n = L_n^{\alpha} + d_n L_{n-1}^{\alpha+1}$, $d_n \neq -1$ and $\alpha > -1$. If* *(a)* $\dfrac{n^2+2n-\alpha}{2n^2+4n+n\alpha+1} < d_n < 1$ *or*

(b) *(i)* $1 < d_n < \dfrac{4n+2\alpha+5}{3n+\alpha+4}$ *or (ii)* $d_n > \dfrac{\alpha+1}{\alpha-n}$ *and* $\alpha > n$

then S_n *and* S_{n+1} *have interlacing zeros.*

Proof. We have

$$\begin{aligned} S_n &= L_n^{\alpha} + d_n L_{n-1}^{\alpha+1} \\ &= (1-d_n)L_n^{\alpha+1} \text{ from (4.2)} \end{aligned}$$

and consider

$$T_n = \frac{S_n}{1-d_n} = L_n^{\alpha} + \left(\frac{d_n}{1-d_n}\right)L_n^{\alpha+1}.$$

(a) The result follows immediately from Theorem 4.3.1(a) noting that

$$b_n = \frac{d_n}{1-d_n} > \frac{n^2+2n-\alpha}{(n+1)(n+\alpha+1)}$$

is equivalent to

$$d_n > \frac{n^2+2n-\alpha}{2n^2+4n+n\alpha+1} = 1 - \frac{n^2+2n+n\alpha+\alpha+1}{2n^2+4n+n\alpha+1}$$

when $d_n < 1$.

(b) (i) Note that $b_n < -\dfrac{4n+2\alpha+5}{n+\alpha+1}$ is equivalent to $d_n < 1 + \dfrac{n+\alpha+1}{3n+\alpha+1}$ if $d_n < 1$.

(ii) The inequality $-\dfrac{\alpha+1}{n+1} < b_n < -1$ is equivalent to $d_n(n-\alpha) < -(\alpha+1)$ when $d_n > 1$ and hence $d_n > 1 + \dfrac{n+1}{\alpha-n}$ when $d_n > 1$ and $\alpha > n$.

Remark Note that the conditions on the coefficient d_n in Corollary 4.3.2 are fairly restrictive. For example, the condition in the first part of the corollary that $\dfrac{n^2+2n-\alpha}{2n^2+4n+n\alpha+1} < d_n < 1$ reduces to $\dfrac{1}{2} < d_n < 1$ as $n \to \infty$ while the second condition becomes $1 < d_n < \dfrac{4}{3}$ asymptotically. $\square$

4.4 Linear combinations of Jacobi polynomials

We now consider the interlacing of zeros of linear combinations of Jacobi polynomials.

Theorem 4.4.1 *Let*

$$E_n = P_n^{(\alpha,\beta)} + r_n P_n^{(\alpha,\beta+1)}.$$

If $\alpha > 0$, $\beta > -1$ *and* $r_n > \dfrac{n+1}{\beta+n+1}$, *then* E_n *and* E_{n+1} *have interlacing zeros.*

Proof. From [45, p.71, eqn.1] with $a = -n$, we have

$${}_2F_1(-n,b;c;z) = \frac{n}{n+b}\ {}_2F_1(-n+1,b;c;z) + \frac{b}{n+b}\ {}_2F_1(-n,b+1;c;z),$$

and using the connection between hypergeometric and Jacobi polynomials given by

$${}_2F_1(-n,1+\alpha+\beta+n;1+\alpha;\frac{1-x}{2}) = \frac{n!}{(1+\alpha)_n} P_n^{(\alpha,\beta)}(x)$$

we obtain the relation

$$P_n^{(\alpha,\beta)}(x) = \frac{1+\alpha+\beta+n}{1+\alpha+\beta+2n} P_n^{(\alpha,\beta+1)}(x) + \frac{\alpha+n}{1+\alpha+\beta+2n} P_{n-1}^{(\alpha,\beta+1)}(x).$$

Thus

$$E_n = \frac{1+\alpha+\beta+n+r_n(1+\alpha+\beta+2n)}{1+\alpha+\beta+2n} P_n^{(\alpha,\beta+1)}(x) + \frac{\alpha+n}{1+\alpha+\beta+2n} P_{n-1}^{(\alpha,\beta+1)}(x).$$

Let

$$\begin{aligned} F_n &= \frac{(1+\alpha+\beta+2n)n!2^n}{(1+\alpha+\beta+n+r_n(1+\alpha+\beta+2n))(2+\alpha+\beta+n)_n} E_n \\ &= \tilde{P}_n^{(\alpha,\beta+1)} + a_n \tilde{P}_{n-1}^{(\alpha,\beta+1)} \end{aligned}$$

$$\text{where} \quad \tilde{P}_n^{(\alpha,\beta+1)} = \frac{n!2^n}{(2+\alpha+\beta+n)_n} P_n^{(\alpha,\beta+1)}$$

$$\text{and} \quad \tilde{P}_{n-1}^{(\alpha,\beta+1)} = \frac{(n-1)!2^{n-1}}{(1+\alpha+\beta+n)_{n-1}} P_{n-1}^{(\alpha,\beta+1)}$$

are the Jacobi polynomials re-normalized to be monic and

$$\begin{aligned} a_n &= \frac{\alpha+n}{1+\alpha+\beta+n+r_n(1+\alpha+\beta+2n)} \frac{n!2^n}{(2+\alpha+\beta+n)_n} . \frac{(1+\alpha+\beta+n)_{n-1}}{2^{n-1}(n-1)!} \\ &> 0 \quad \text{for} \quad \alpha > 0, \beta > -1 \quad \text{and} \quad r_n > \frac{n+1}{\beta+n+1}. \end{aligned}$$

If we denote the zeros of $P_n^{(\alpha,\beta+1)}$; F_n; $P_{n+1}^{(\alpha-1,\beta+1)}$ and $P_{n+1}^{(\alpha,\beta+1)}$ by $x_1 < x_2 < \cdots < x_n$, $y_1 < y_2 < \cdots < y_n$; $t_1 < t_2 < \cdots < t_{n+1}$ and $z_1 < z_2 < \cdots < z_{n+1}$ respectively then it follows from Lemma 4.2.2(b) that $y_1 < x_1$. Also, from Theorem 3.3.3, we have

$$t_1 < y_1 < t_2 < y_2 < \cdots < t_n < y_n < t_{n+1},$$

while from Theorem 2.3.2,

$$z_1 < t_1 < z_2$$

and

$$z_1 < x_1 < z_2$$

since $P_n^{(\alpha,\beta+1)}$ and $P_{n+1}^{(\alpha,\beta+1)}$ have interlacing zeros.
Hence $z_1 < t_1 < y_1 < x_1 < z_2$ and for n even and n odd we obtain

$$\frac{\tilde{P}_{n+1}^{(\alpha,\beta+1)}}{\tilde{P}_n^{(\alpha,\beta+1)}}(y_1) > 0.$$

Since $a_{n+1} > 0$, the result follows from Lemma 4.2.3. □

4.5 Conclusion and Future Research

In the second chapter, we have investigated the interlacing property for $P_n^{(\alpha,\beta)}$ and $P_m^{(\alpha',\beta')}$ where $\alpha' = \alpha \pm t$, $\beta = \beta \pm k$ and $m = n$ or $m = n-1$.
On the other hand, we have noticed from [10] that Askey-Wilson polynomials generalize the q-ultraspherical and q-Jacobi polynomials. The next step could

be the investigation of the interlacing behavior for the zeros of Askey-Wilson polynomials.

The main purpose of this thesis was the investigation of the interlacing of zeros of linear combinations of Jacobi polynomials and Laguerre polynomials from different sequences. One has fixed some ranges of parameters on which the interlacing of zeros occurs. Since the interlacing of zeros does not mean the orthogonality, the next step could be the investigation of the ranges of parameters for which one has the orthogonality of the linear combinations.

Chapter 5

Appendix

Hypergeometric function

Let a, b and c be complex numbers such that $c \notin \{0, -1, -2, -3, \ldots\}$. The series

$$\sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k \tag{5.1}$$

is called the Gauss hypergeometric series, where $(\alpha)_k$ is the Pochhammer symbol or shifted factorial and is defined by:

$$(\alpha)_k = \begin{cases} \alpha(\alpha+1)\ldots(\alpha+k-1) & \text{if } k = 1, 2, 3, \ldots \\ 1 \text{ if } k = 0 \text{ and } \alpha \neq 0 \end{cases} .$$

The series (5.1) converges for $|z| < 1$ and the hypergeometric function in (5.1) is usually denoted by ${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right)$ or ${}_2F_1(a, b; c; z)$.

For $n \in \mathbf{N}$, we have

$$(-n)_k = \begin{cases} (-1)^k \frac{n!}{(n-k)!} & \text{for } 0 \leq k \leq n \\ 0 & \text{for } k \geq n+1 \end{cases}$$

Hence, if either a or b, or both, is a negative integer, the series (5.1) terminates, and we see that

$$\begin{aligned} {}_2F_1\left(\begin{matrix} -n, b \\ c \end{matrix} ; z\right) &= 1 + \sum_{k=1}^{\infty} \frac{(-n)_k (b)_k}{(c)_k k!} z^k \\ &= \sum_{k=0}^{\infty} \frac{(-n)_k (b)_k}{(c)_k k!} z^k. \end{aligned}$$

is a polynomial of degree n in z, called the hypergeometric polynomial.

The contiguous function relations

In [30], Gauss defined each of the six functions obtained by increasing or decreasing one of the parameters a, b or c by unity as being contiguous to $F = {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} ; z\right)$.

We denote ${}_2F_1\left(\begin{matrix} a+1, b \\ c \end{matrix} ; z\right)$ by $F(a+)$, ${}_2F_1\left(\begin{matrix} a-1, b \\ c \end{matrix} ; z\right)$ by $F(a-)$ and so on.

Gauss proved that between F and any two of its contiguous functions, there exits a linear relation with coefficients at most linear in z.

We list a selection of contiguous relations (cf [45], p.71-72)

$$(a-b)F = aF(a+) - bF(b+) \tag{5.2}$$

$$(a-c+1)F = aF(a+) - (c-1)F(c-) \tag{5.3}$$

$$\begin{aligned} c[a+(b-c)z]F &= ca(1-z)F(a+) \\ &\quad - (c-a)(c-b)zF(c+) \end{aligned} \tag{5.4}$$

$$c(1-z)F = cF(a-) - (c-b)zF(c+) \tag{5.5}$$

$$c(1-z)F = cF(b-) - (c-a)zF(c+) \tag{5.6}$$

$$(c-a-b)F = (c-a)F(a-) - b(1-z)F(b+) \tag{5.7}$$

$$(b-a)(1-z)F = (c-a)F(a-) - (c-b)F(b-) \tag{5.8}$$

$$[2b-c+(a-b)z]F = (1-z)bF(b+) - (c-b)F(b-) \tag{5.9}$$

$$c[b+(a-c)z]F = bc(F(b+) - (c-a)(c-b)zF(c+) \tag{5.10}$$

$$(b-c+1)F = bF(b+) - (c-1)F(c-) \tag{5.11}$$

$$[1-b+(c-a-1)z]F = (c-b)F(b-) - (c-1)(1-z)F(c-) \tag{5.12}$$

$$\begin{aligned} [c-1+(a+b+1-2c)z]cF &= c(c-1)(1-z)F(c-) \\ &\quad - (c-a)(c-b)zF(c+) \end{aligned} \tag{5.13}$$

From (5.5) and (5.6), one has

$$cF(a-) - cF(b-) + (b-a)zF(c+) = 0 \tag{5.14}$$

Further, in (5.8), shifting b to $b+1$ gives

$$(c-b-1)F = (c-a)F(a-,b+) + (a-b-1)(1-z)F(b+). \tag{5.15}$$

Jacobi polynomials

The Jacobi polynomial $P_n^{\alpha,\beta}(x)$ can be defined by the Rodrigues' formula

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{(1-x)^\alpha(1+x)^\beta}\frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}[(1-x)^{n+\alpha}(1+x)^{n+\beta}].$$

The following connections between $_2F_1$ polynomials and Jacobi polynomials are well known (cf [45], p.254-256)

$$\begin{aligned} P_n^{\alpha,\beta}(x) &= \frac{(1+\alpha)_n}{n!} {}_2F_1\left(\begin{matrix} -n, 1+\alpha+\beta+n \\ 1+\alpha \end{matrix}; \frac{1-x}{2}\right) \qquad (5.16) \\ &= \frac{(1+\alpha)_n}{n!} (\frac{x+1}{2})^n {}_2F_1\left(\begin{matrix} -n, -\beta-n \\ 1+\alpha \end{matrix}; \frac{x-1}{x+2}\right) \\ &= \frac{(-1)^n(1+\beta)_n}{n!} {}_2F_1\left(\begin{matrix} -n, 1+\alpha+\beta+n \\ 1+\beta \end{matrix}; \frac{1+x}{2}\right) \\ &= \frac{(1+\alpha+\beta)_{2n}}{n!(1+\alpha+\beta)_n} (\frac{x-1}{2})^n {}_2F_1\left(\begin{matrix} -n, -\alpha-n \\ -\alpha-\beta-2n \end{matrix}; \frac{2}{1-x}\right) \\ &= \frac{(1+\beta)_n}{n!} (\frac{x-1}{2})^n {}_2F_1\left(\begin{matrix} -n, -\alpha-n \\ 1+\beta \end{matrix}; \frac{x+1}{x-1}\right) \\ &= \frac{(1+\alpha+\beta)_{2n}}{n!(1+\alpha+\beta)_n} (\frac{x+1}{2})^n {}_2F_1\left(\begin{matrix} -n, -\beta-n \\ -\alpha-\beta-2n \end{matrix}; \frac{2}{x+1}\right). \end{aligned}$$

where $\alpha, \beta \notin \{..., -3, -2, -1\}$.

The infinite sequence of Jacobi polynomials $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$, with $\alpha > -1$ and $\beta > -1$, satisfies a three-term recurrence relation which is given by

$$\begin{aligned} & 2n(\alpha+\beta+n)(\alpha+\beta+2n-2)P_n^{(\alpha,\beta)}(x) \\ &= (\alpha+\beta+2n-1)[\alpha^2-\beta^2+x(\alpha+\beta+2n)(\alpha+\beta+2n-2)]P_{n-1}^{(\alpha,\beta)}(x) \\ &\quad -2(\alpha+n-1)(\beta+n-1)(\alpha+\beta+2n)P_{n-2}^{(\alpha,\beta)}(x). \end{aligned}$$

Using contiguous relations for $_2F_1's$, together with connections between $_2F_1$ polynomials and Jacobi polynomials, one can generate mixed recurrence relations for Jacobi polynomials.

Mixed recurrence relations

We derive 9 mixed recurrence relations that are used in our proofs in the main text

1. For $\alpha > -1$ and $\beta > -1$, we have

$$\begin{aligned} &2n[2+\alpha+3\beta+2n+(\alpha+\beta+2n)x]P_n^{(\alpha,\beta)}(x) \\ &\quad = \quad (1+\alpha+\beta+n)(\alpha+\beta+2n)(1+x)^2 P_{n-1}^{(\alpha,\beta+2)}(x) \\ &\qquad\quad -4(\beta+n)(\beta+1)P_{n-1}^{(\alpha,\beta)}(x). \end{aligned}$$

2. For $\alpha > -1$ and $\beta > -1$,

$$\begin{aligned} &2n[2+3\alpha+\beta+2n-(\alpha+\beta+2n)x]P_n^{(\alpha,\beta)}(x) \\ &\quad -4(1+\alpha)(\alpha+n)P_{n-1}^{(\alpha,\beta)}(x) \\ &\quad +(1+\alpha+\beta+n)(\alpha+\beta+2n)(1-x)^2 P_{n-1}^{(\alpha+2,\beta)}(x) = 0 \end{aligned}$$

3. Let $\alpha > -1$ and $\beta > -1$. Then

$$\begin{aligned} &(1+\alpha+\beta+n)_2(1+x)P_n^{(\alpha,\beta+2)}(x) \\ &\quad = \quad \{(1+\alpha+\beta+2n)_2(1+x)-2n(\alpha+n)\}P_n^{(\alpha,\beta)}(x) \\ &\qquad\quad -\{4+\alpha+3\beta+2n+ \\ &\qquad\quad (2+\alpha+\beta+2n)x\}(\alpha+n)P_{n-1}^{(\alpha,\beta)}(x) \end{aligned}$$

4. For $\alpha > 1$ and $\beta > -1$,

$$\begin{aligned} &-2(\alpha+n-1)P_n^{(\alpha-2,\beta)}(x) \\ &\quad = \quad (\alpha+\beta+n)(1-x)P_n^{(\alpha,\beta)}(x) \\ &\qquad\quad +[3\alpha-2+\beta+2n-(\alpha+\beta+2n)x]P_n^{(\alpha-1,\beta)}(x) \end{aligned}$$

5. For $\alpha > 1$ and $\beta > -1$, we have

$$\begin{aligned}
&(\beta+n+1)[\beta+2\alpha+n-(\beta+n+2)x]P_n^{(\alpha,\beta)}(x) \\
&+ \quad (1+\alpha+\beta+n)[\alpha+\beta-(2-\alpha+\beta)x]P_n^{(\alpha,\beta+1)}(x) \\
&- \quad (1+x)(\alpha+n)(\alpha+n-1)P_n^{(\alpha-2,\beta+2)}(x)=0.
\end{aligned}$$

6. For $\alpha > 0$ and $\beta > -1$,

$$\begin{aligned}
&(1+\alpha+\beta+2n)P_n^{(\alpha-1,\beta+1)}(x) \\
&= \quad (1+\alpha+\beta+n)P_n^{(\alpha.\beta+1)}(x)-(\beta+n+1)P_{n-1}^{(\alpha,\beta+1)}(x).
\end{aligned}$$

7. For $\alpha > -1$ and $\beta > 0$,

$$\begin{aligned}
&2(1+\alpha+\beta+2n)(n+1)P_{n+1}^{(\alpha,\beta-1)}(x) \\
&= \quad (1+\alpha+\beta+n)[1+\alpha-\beta+2n+(1+\alpha+\beta+2n)x]P_n^{(\alpha,\beta+1)}(x) \\
&\quad -2\beta(\alpha+n)P_{n-1}^{(\alpha,\beta+1)}(x).
\end{aligned}$$

8. For $\alpha > 0$ and $\beta > -1$,

$$\begin{aligned}
&2(1+\alpha+\beta+n)P_{n+1}^{(\alpha,\beta)}(x) \\
&= \quad [2+\alpha+\beta+2n+(2+\alpha+\beta+2n)x](\alpha+n)P_n^{(\alpha-1,\beta+1)}(x) \\
&\quad +(\beta+n+1)[-\alpha-\beta+(2+\alpha+\beta+2n)x]P_n^{(\alpha,\beta)}(x).
\end{aligned}$$

9. For $\alpha > 0$ and $\beta > -1$,

$$\begin{aligned}
&2(n+1)P_{n+1}^{(\alpha-1,\beta+1)}(x) \\
&= \quad (\alpha+\beta+(2+\alpha+\beta+2n)x)P_n^{(\alpha,\beta+1)}(x)-2(\beta+n+1)P_n^{(\alpha,\beta)}(x).
\end{aligned}$$

Proof. 1. In (5.9) and (5.7), replacing a by $a+1$, we have

$$\begin{aligned}
&[2b-c+(a-b+1)z]F(a+) \\
&= \quad b(1-z)F(a+,b+)-(c-b)F(a+,b-) \qquad (5.17)
\end{aligned}$$

and

$$\begin{aligned}
&(c-a-b-1)F(a+) \\
&= \quad (c-a-1)F-b(1-z)F(a+,b+) \qquad (5.18)
\end{aligned}$$

respectively. From (5.17) and (5.18), one obtains

$$\begin{aligned}
&[2b-c+(a-b+1)z](c-a-1)F \\
&\quad = \ b(b-a-1)(1-z)^2F(a+,b+)-(c-b)(c-a-b-1)F(a+,b-)
\end{aligned}$$

and with $a=-n$, $b=1+\alpha+\beta+n$, $c=1+\alpha$, $z=\dfrac{1-x}{2}$ and (5.16), the result follows.

2. In (5.14), with a and c replaced by $a+1$ and $c+1$ respectively, one has

$$(c+1)F(a+,c+)=(c+1)F(b+,c+)+(b-a)zF(a+,b+,c+2)$$

and in (5.5), the replacing of a by $a+1$ gives

$$c(1-z)F(a+)=cF-(c-b)zF(a+,c+);$$

$$\begin{aligned}
\text{then} \quad & c(c+1)F-c(c+1)(1-z)F(a+) \\
= \ & (c+1)(c-b)zF(b+,c+) \\
& +(c-b)(b-a)z^2F(a+,b+,c+2). \qquad (5.19)
\end{aligned}$$

Also replacing c by $c+1$ in (5.11) and using (5.4), one obtains

$$\begin{aligned}
&[(-b+a)z-a]cF \\
&\quad = \ ca(z-1)F(a+)+b(a-c)zF(b+,c+). \qquad (5.20)
\end{aligned}$$

From (5.15), by symmetry, one has

$$(c-1-a)F=(c-b)F(a+,b-)+(b-1-a)(1-z)F(a+). \quad (5.21)$$

Then (5.19), (5.20) and (5.21) together give

$$\begin{aligned}
&-c(c+1)[(-b+1+a)z-c]F-c^2(c+1)F(a+,b-) \\
&+ \ b(-b+1+a)z^2(a-c)F(a+,b+,c+2)=0;
\end{aligned}$$

with $a=-n$, $b=1+\alpha+\beta+n$, $c=1+\alpha$ and $z=\frac{1-x}{2}$ and using (5.16), we obtain the result.

3. In (5.9), replacing b by $b+1$, this relation becomes

$$\begin{aligned} & [2(b+1)-c+(a-b-1)z]F(b+) \\ = \; & (b+1)(1-z)F(b+2)-(c-b-1)F. \end{aligned} \tag{5.22}$$

Combining (5.22) and (5.2) implies using

$$\begin{aligned} & \{(a-b)[2(b+1)-c+(a-b-1)z]-b(c-b-1)\}F \\ = \; & a[2(b+1)-c+(a-b-1)z]F(a+)+b(b+1)(1-z)F(b+2) \end{aligned}$$

Letting $a=-n$, $b=1+\alpha+\beta+n$, $c=1+\alpha$, $z=\dfrac{1-x}{2}$ and considering (5.16), we obtain

$$\begin{aligned} & (1+\alpha+\beta+n)(2+\alpha+\beta+n)(1+x)P_n^{(\alpha,\beta+2)}(x) \\ = \; & \{(1+\alpha+\beta+2n)(2+\alpha+\beta+n)(1+x)\}P_n^{(\alpha,\beta)}(x) \\ - \; & \{4+\alpha+3\beta+2n+(2+\alpha+\beta+2n)x\}(\alpha+n)P_{n-1}^{(\alpha,\beta+1)}(x). \end{aligned}$$

4. Consider the equation (5.6)

$$c(1-z)F = cF(b-)-(c-a)zF(c+)$$

in which replacing b by $b-1$ and c by $c-1$, one gets

$$(c-1)(1-z)F(b-,c-) = (c-1)F(b-2,c-)-(c-a-1)zF(b-).$$

Now let us consider (5.11) in which we change b to $b-1$, this becomes

$$(b-c)F(b-) = (b-1)F-(c-1)F(b-,c-).$$

Then we have

$$\begin{aligned} & (b-c)(c-1)F(b-2,c-) \\ = \; & (c-a-1)(b-1)zF \\ & +(c-1)[b-c-(b-a-1)z]F(b-,c-). \end{aligned}$$

In this last expression, we change c to $c-1$; then

$$\begin{aligned}
&(b-c+1)(c-2)F(b-2,c-2)\\
&= (c-a-2)(b-1)zF(c-)\\
&\quad +(c-2)[b-c+1-(b-a-1)z]F(b-,c-2). \qquad (5.23)
\end{aligned}$$

Now, in (5.13), we shift c to $c-1$ and b to $b-1$, so we obtain

$$\begin{aligned}
&[c-2+(a+b-2c+2)z](c-1)F(b-,c-)\\
&= (c-1)(c-2)(1-z)F(b-,c-2)\\
&- (c-a-1)(c-b)zF(b-). \qquad (5.24)
\end{aligned}$$

Combining (5.23), (5.24) and (5.12), we have

$$(c-2)_2F(b-2,c-2) = (b-1)(c-a-1)zF + [c-2+(b-a-1)z](c-1)F(b-,c-).$$

Finally, putting $a=-n$, $b=1+\alpha+\beta+n$, $c=1+\alpha$, $z=\dfrac{1-x}{2}$ and using (5.16), one has

$$\begin{aligned}
&-2(\alpha+n-1)P_n^{(\alpha-2,\beta)}(x)\\
&= (\alpha+\beta+n)(1-x)P_n^{(\alpha,\beta)}(x)\\
&\quad +[3\alpha-2+\beta+2n-(\alpha+\beta+2n)x]P_n^{(\alpha-1,\beta)}(x),
\end{aligned}$$

which proves the result.

5. Changing c to $c-1$ in (5.10) and (5.11) yields

$$\begin{aligned}
&(c-1)[b+(a-c+1)z]F(c-)\\
&= b(c-1)(1-z)F(b+,c-)\\
&\quad -(c-a-1)(c-b-1)zF \qquad (5.25)
\end{aligned}$$

and $$(b-c+2)F(c-) = bF(b+,c-)-(c-2)F(c-2) \qquad (5.26)$$

From (5.10), (5.25) and (5.26) we obtain

$$\begin{aligned}&(-c+b+1)[(-c+2+b)z+c-2]F\\&\quad=b[c-2+(-2c+3+a+b)z]F(b+)+(z-1)(c-2)_2F(c-2).\end{aligned}$$

Putting $a=-n$, $b=1+\alpha+\beta+n$, $c=1+\alpha$ and $z=\dfrac{1-x}{2}$, we get the relation.

6. From equations (5.2) and (5.3), one obtains

$$[c-b-1]aF(a+)+[a-c+1]bF(b+)+(c-1)(b-a)F(c-)=0.$$

Thus with $a=-n$, $b=1+\alpha+\beta+n$, $c=1+\alpha$, $z=\dfrac{1-x}{2}$ and using (5.16), the relation holds.

7. Equations (5.2) and (5.7) together give

$$a(c-a-b)F(a+)+[2a-c+(b-a)z]bF(b+)+(a-b)(a-c)F(a-)=0.$$

Letting $a=-n$, $b=1+\alpha+\beta+n$, $c=1+\alpha$, $z=\dfrac{1-x}{2}$ and using the relation (5.16), one has the relation.

8. Let us consider the equation (5.8) in which replacing b by $b+1$ in , we obtain

$$(b-a+1)(1-z)F(b+)=(c-a)F(a-,b+)-(c-b-1)F. \quad (5.27)$$

Equations (5.11) and (5.27) together give

$$\begin{aligned}&(b-c+1)[a-1+(b-a+1)z]F\\&\quad=\quad(a-c)bF(a-,b+)+(b-a+1)(1-c)(z-1)F(c-).\end{aligned}$$

Letting $a=-n$, $b=1+\alpha+\beta+n$, $c=1+\alpha$, $z=\dfrac{1-x}{2}$ and considering (5.16), we obtain the result.

9. Replacing b by $b+1$ and c by $c-1$ in 5.5, we obtain

$$\begin{aligned}
&(c-1)(1-z)F(b+,c-) \\
&= \quad (c-1)F(a-,b+,c-)-(c-b-2)zF(b+)
\end{aligned} \tag{5.28}$$

while replacing b by $b+1$ in (5.12), yields

$$\begin{aligned}
&[-b+(c-a-1)z]F(b+) \\
&= \quad (c-b-1)F-(c-1)(1-z)F(b+,c-).
\end{aligned} \tag{5.29}$$

From (5.28) and (5.29) we obtain

$$(1-c)F(a-,b+,c-)+[b+(a-b-1)z]F(b+)+(c-b-1)F=0.$$

Letting $a=-n$, $b=1+\alpha+\beta+n$, $c=1+\alpha$, $z=\dfrac{1-x}{2}$ and using (5.16), we obtain the result.

□

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