Consider the signals from D sources impinging on the elements of an array. The received signal of source \( d \) at the array elements is:

\[
X_d(t) = S_d(t)U_d
\]

where \( S_d(t) \) is the data transmitted by source \( d \) and \( U_d \) is the array vector of source \( d \). The total received signal at the array is the sum of all the signals plus noise received at the array, or:

\[
X(t) = \sum_{d=1}^{D} S_d(t)U_d + n(t)
\]

where \( n \) is zero mean Gaussian noise at the antenna elements. The co-variance matrix of the received signals at the array elements is:

\[
R = E\{X(t)X^H(t)\}
\]

Inserting now (A2) in (A3) and assuming that the signals and noise are uncorrelated, the co-variance matrix becomes:

\[
R = E\left\{ \sum_{d=1}^{D} S_d(t)U_d \left( \sum_{d=1}^{D} S_d(t)U_d^H \right)^H \right\} + \sigma^2 I
\]

where \( I \) is a unity matrix and \( \sigma^2 \) is the noise power. Expanding (A4) the following is obtained:

\[
R = E\left\{ S_1(t)S_1^*(t)U_1U_1^H + S_1(t)S_2^*(t)U_1U_2^H + ... + S_1(t)S_D^*(t)U_1U_D^H + S_2(t)S_1^*(t)U_2U_1^H + S_2(t)S_2^*(t)U_2U_2^H + ... + S_2(t)S_D^*(t)U_2U_D^H + ... + S_D(t)S_1^*(t)U_DU_1^H + S_D(t)S_2^*(t)U_DU_2^H + ... + S_D(t)S_D^*(t)U_DU_D^H \right\} + \sigma^2 I
\]

Now, if the sources are uncorrelated, then:
Appendix A

\[ E\{S_1(t)S_2^*(t)\} = E\{S_1(t)S_2^*(t)\} = \ldots = E\{S_1(t)S_D^*(t)\} = \ldots = E\{S_D(t)S_1^*(t)\} = \ldots = 0 \quad (A6) \]

Since the power \( P_d \) of each signal \( d \in \{1, 2, \ldots, D\} \) is:

\[ P_d = E\{S_d(t)S_d^*(t)\} \quad (A7) \]

and using (A6) and (A7), equation (A5) becomes:

\[ R = P_1 U_1 U_1^H + P_2 U_2 U_2^H + \ldots + P_D U_D U_D^H + \sigma^2 I \quad (A8) \]

which can be written as:

\[ R = \sum_{d=1}^{D} P_d U_d U_d^H + \sigma^2 I \quad (A9) \]
APPENDIX B

PROOF THAT THE OPTIMUM COMBINED SINR OF TWO ARRAYS WITH INDIVIDUAL BEAMFORMING IS EQUAL TO THE SUM OF THE SINR OF EACH INDEPENDENT BEAMFORMING ARRAY

It was shown in section 4.2.1 that the signal to interference plus noise ratio after optimum combining of two individual arrays, each with signals combined with independent optimum beamforming, is the sum of the individual array signal to interference ratios. However, it was done for the special case where $\psi_{12} = \psi_{22}$. The derivation will be extended to the general case ($\psi_{12} \neq \psi_{22}$) in this appendix.

The signal to interference plus noise ratio of optimum combining of the individual array signals is given in (116) as

$$\text{SINR}_C = U_d^H R_{\text{nnC}}^{-1} U_{\text{dc}}$$

with $R_{\text{nnC}}$ the covariance matrix of the signals from the two arrays, given by:

$$R_{\text{nnC}} = \begin{bmatrix} R_{\text{nnC},11} & R_{\text{nnC},12} \\ R_{\text{nnC},21} & R_{\text{nnC},22} \end{bmatrix}$$

and $U_{\text{dc}}$ is the array steering vector in the direction of the desired signal, given by:

$$U_{\text{dc}} = [1 \ 1]^T$$

The inverse of a two by two matrix is [66]:

$$R_{\text{nnC}}^{-1} = \left(R_{\text{nnC},11} R_{\text{nnC},22} - R_{\text{nnC},12} R_{\text{nnC},21}\right)^{-1} \begin{bmatrix} R_{\text{nnC},22} & -R_{\text{nnC},12} \\ -R_{\text{nnC},21} & R_{\text{nnC},11} \end{bmatrix}$$

Since $R_{\text{nnC},12} = R_{\text{nnC},21}^*$, the inverse in (B4) becomes:

$$R_{\text{nnC}}^{-1} = \left(R_{\text{nnC},11} R_{\text{nnC},22} - R_{\text{nnC},12} R_{\text{nnC},21}^*\right)^{-1} \begin{bmatrix} R_{\text{nnC},22} & -R_{\text{nnC},12} \\ -R_{\text{nnC},21}^* & R_{\text{nnC},11} \end{bmatrix}$$

Inserting (B5) in (B1), the SINR is:
If it can now be shown that $R_{nnC,11} R_{nnC,22} \gg R_{nnC,12} R_{nnC,12}$, then the SINR in (B7) becomes:

$$\text{SINR}_C = \left( \frac{1}{R_{nnC,11}} + \frac{1}{R_{nnC,22}} \right)$$  \hspace{1cm} (B8)

Insert the components of (93) in (B8), the SINR becomes:

$$\text{SINR}_C = \left( \frac{1}{W_1^H R_{mn11} W_1} + \frac{1}{W_2^H R_{mn22} W_2} \right)$$  \hspace{1cm} (B9)

And using (88) and (89) the following is obtained:

$$\text{SINR}_C = \left( U_d^H R_{mn11}^{-1} U_d + U_d^H R_{mn22}^{-1} U_d \right)$$  \hspace{1cm} (B10)

After cancellation of the components in (B10), the following simplified equation is obtained:

$$\text{SINR}_C = \left( U_d^H R_{mn11}^{-1} U_d + U_d^H R_{mn22}^{-1} U_d \right)$$  \hspace{1cm} (B11)

Comparing this to (108) and (109) it can be seen that:

$$\text{SINR}_C = (\text{SINR}_1 + \text{SINR}_2)$$  \hspace{1cm} (B12)

Using (93), (102) and (105), the product $R_{nnC,11} R_{nnC,22}$ in (B6) is:

$$R_{nnC,11} R_{nnC,22} = \frac{(2 + \sigma^2) \sigma^4}{4((\cos(\pi \sin \psi_{12}) - 1 - \sigma^2)(\cos(\pi \sin \psi_{22}) - 1 - \sigma^2))}$$  \hspace{1cm} (B13)

The relation between the angles $\psi_{12}$ and $\psi_{22}$ is:

$$\psi_{22} = \arctan \left( \frac{\sin \psi_{12}}{\xi - \cos \psi_{12}} \right)$$  \hspace{1cm} (B14)

Where $\xi$ is the proportion of the range from array 1 to the mobile relative to the distance between the two arrays. Inserting this angle relationship in (B13) into (B14), the following results:
\[ R_{nnC,11}R_{nnC,22} = \frac{\sigma^4}{\cos\left(\frac{\omega}{\Gamma}\right)\cos(\omega) - \cos(\omega) - \cos(\frac{\omega}{\Gamma}) + 1} \]  

(B15)

where

\[ \omega = \pi \sin \psi_{12} \]  

(B16)

and

\[ \Gamma = \sqrt{\xi^2 - 2\xi \cos \psi_{12} + 1} \]  

(B17)

Let \( \alpha = \frac{\omega}{\Gamma} \), then (B17) can be written as:

\[ R_{nnC,11}R_{nnC,22} = \frac{\sigma^4}{\cos(\alpha)\cos(\omega) - \cos(\omega) - \cos(\alpha) + 1} \]  

(B18)

Using (93), (103), (104) and (B14) the product \( R_{nnC,12}^* R_{nnC,12}^* \) in (B6) becomes:

\[
\frac{1}{4} \sigma^4 \left( \cos\left(-\frac{1}{2} \omega + \frac{1}{2} \alpha\right) - \cos(\omega) \cos\left(-\frac{1}{2} \omega + \frac{1}{2} \alpha\right) + \sin(\omega) \sin\left(-\frac{1}{2} \omega + \frac{1}{2} \alpha\right) \right) + \sigma^2 \cos\left(-\frac{1}{2} \omega + \frac{1}{2} \alpha\right) + \frac{1}{2} \omega + \frac{1}{2} \alpha \right) + \cos\left(\frac{1}{2} \omega + \frac{1}{2} \alpha\right) \sigma^2 \right) \\
- \cos\left(\frac{1}{2} \omega + \frac{1}{2} \alpha\right) \cos(\omega) - \sin\left(\frac{1}{2} \omega + \frac{1}{2} \alpha\right) \sin(\omega) \right)^2 / (1 + 8 \cos(\alpha) \sigma^2 \cos(\omega) \\
- 2 \cos(\alpha) \cos(\omega)^2 \sigma^2 - 2 \cos(\alpha) \sigma^2 + 4 \cos(\alpha) \sigma^4 \cos(\omega) - 2 \cos(\alpha) + \cos(\alpha)^2 + 4 \sigma^2 - 2 \cos(\omega) + \cos(\omega)^2 + 6 \sigma^4 + \sigma^8 + 4 \sigma^6 + 4 \cos(\alpha) \cos(\omega) \\
+ \cos(\alpha)^2 \cos(\omega)^2 - 2 \cos(\alpha) \cos(\omega)^2 - \cos(\alpha)^2 \cos(\omega)^2 - 6 \cos(\alpha) \sigma^4 \\
+ \cos(\alpha)^2 \sigma^4 - 6 \cos(\alpha) \sigma^2 - 2 \cos(\alpha) \sigma^4 + 2 \cos(\alpha)^2 \sigma^2 - 6 \sigma^4 \cos(\omega) \\
+ \sigma^4 \cos(\omega)^2 + 2 \sigma^2 \cos(\omega)^2 - 6 \sigma^2 \cos(\omega) - 2 \sigma^6 \cos(\omega) \right) \]  

(B19)

Using (B18) and (B19), the ratio of \( R_{nnC,11} R_{nnC,22} \) to \( R_{nnC,12}^* R_{nnC,12}^* \) can be calculated.

Once (B18) is divided in (B19), the fact that \( \sigma^2 << 1 \) is used to simplify the result, given by:

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where

\[ A = \cos(\alpha)^2 \cos^2(\omega) - 2 \cos(\alpha)^2 \cos(\omega) - 2 \cos(\alpha) \cos(\omega)^2 + 4 \cos(\alpha) \cos(\omega) \]
\[ + \cos(\alpha)^2 - 2 \cos(\alpha) + \cos(\omega)^2 - 2 \cos(\omega) + 1 \]  \hspace{1cm} (B21)

\[ B = \left( \cos\left(\frac{1}{2} \omega - \frac{1}{2} \alpha \right) - \cos(\omega) \cos\left(\frac{1}{2} \omega - \frac{1}{2} \alpha \right) \sin(\omega) \sin\left(\frac{1}{2} \omega - \frac{1}{2} \alpha \right) \right. \]
\[ + \cos\left(\frac{1}{2} \omega + \frac{1}{2} \alpha \right) - \cos\left(\frac{1}{2} \omega + \frac{1}{2} \alpha \right) \cos(\omega) \sin\left(\frac{1}{2} \omega + \frac{1}{2} \alpha \right) \sin(\omega) \right)^2 \]  \hspace{1cm} (B22)

\[ C = \cos(\alpha) \cos(\omega) - \cos(\alpha) - \cos(\omega) + 1 \]  \hspace{1cm} (B23)

Applying the following trigonometry properties to (B23):

\[ \cos(a-b) = \cos(a) \cos(b) + \sin(a) \sin(b) \]  \hspace{1cm} (B24)

and

\[ \cos(a+b) = \cos(a) \cos(b) - \sin(a) \sin(b) \]  \hspace{1cm} (B25)

the equation becomes zero (i.e. \( B = 0 \)), which results in a large value for the ratio

\[ \frac{R_{nn,11} R_{nn,22}}{R_{nn,12} R_{nn,12}^*} \] and thereby proving that the SINR of the arrays combined with optimum combining is equal to the individual array SINRs.
BER OF ARRAY WITH MULTIPLE NON-UNIQUE EIGENVALUES
(MULTIPLICITY > 1)

In the case of one eigenvalue with multiplicity equal to one and M-1 eigenvalues with multiplicity equal to M-1, the characteristic function of the probability density function can be written as:

\[
\Psi(z) = \frac{\langle \lambda_1 \rangle^{M-1} \langle \lambda_M \rangle}{(z + \langle \lambda_1 \rangle)^{M-1} (z + \langle \lambda_M \rangle)}
\]  
(C1)

A partial fraction expansion of (C1) is:

\[
\Psi(z) = \frac{-C}{(z + \lambda_1)^{M-1} (\lambda_1 - \lambda_M)} + \frac{-C}{(z + \lambda_1)^{M-2} \left( \lambda_1 - \lambda_M \right)^2} + \ldots \ldots.
\]  
(C2)

\[
\frac{+ C}{(z + \lambda_5) (\lambda_1 - \lambda_M)^{M-1}}
\]

where

\[
C = \langle \lambda_1 \rangle^{M-1} \langle \lambda_M \rangle
\]  
(C3)

Equation (C2) can also be written as:

\[
\Psi(z) = C \left\{ \frac{\Omega_1}{(z + \lambda_1)^{M-1}} + \frac{\Omega_2}{(z + \lambda_1)^{M-2}} + \ldots + \frac{\Omega_M}{(z + \lambda_5)} \right\}
\]  
(C4)

where

\[
\Omega_1 = \frac{-1}{(\lambda_1 - \lambda_M)}
\]  
(C5)

\[
\Omega_2 = \frac{-1}{(\lambda_1 - \lambda_M)^2}
\]  
(C6)

and
\[
\Omega_M = \frac{1}{(\lambda_1 - \lambda_M)^{M-1}}
\]  

(C7)

The inverse Laplace transform of (C4) is given by:

\[
p(\eta) = L^{-1}\{\Psi(z)\} = C \{\Omega_{M-1} e^{-\lambda_1 \eta} + \eta \Omega_{M-2} e^{-\lambda_2 \eta} + \ldots + \eta^{M-2} \Omega_1 e^{-\lambda_1 \eta} + \Omega_M e^{-\lambda_M \eta}\}
\]

(C8)

The average bit error rate (BER) of phased shift keyed signals is given by [11]:

\[
BER = \frac{1}{2} \int_{-\infty}^{\infty} p(\eta) \text{erfc} (\sqrt{\eta}) \, d\eta
\]

(C9)

Inserting (C8) in (C9) the following is obtained:

\[
BER = \frac{C}{2} \int_{-\infty}^{\infty} \{\Omega_{M-1} e^{-\lambda_1 \eta} + \eta \Omega_{M-2} e^{-\lambda_2 \eta} + \ldots + \eta^{M-2} \Omega_1 e^{-\lambda_1 \eta} + \Omega_M e^{-\lambda_M \eta}\} \text{erfc} (\sqrt{\eta}) \, d\eta
\]

(C10)

The following general integral formula [67] is used to solve (C10)

\[
\frac{1}{2(K-1)!} \int_0^\infty x^{K-1} e^{-(ax+b)} \, \text{erfc} (\sqrt{bx}) \, dx = \left( \frac{1+\frac{a}{b}+1}{2a\sqrt{1+\frac{a}{b}}} \right)^K \sum_{k=0}^{K-1} \left( \frac{K-1+k}{k} \right) \left( \frac{1+\frac{a}{b}}{2\sqrt{1+\frac{a}{b}}} \right)^k
\]

(C11)

where

\[
\left( \frac{K-1+k}{k} \right) = \frac{(K-1+k)!}{k!(K-1)!}
\]

(C12)

Using (C11) in (C10) the bit error rate is

\[
BER = -C \sum_{m=2}^{M-1} \left\{ \Omega^{M-1} \zeta_1^m \right\} \sum_{i=0}^{m-1} \left\{ \frac{(m-1+i)! \mu^i}{i!(m-1)!} \right\} + C \Omega^{M-1} \zeta_M
\]

(C13)

where

\[
\zeta_1 = \frac{\sqrt{1+\lambda_1} - 1}{2\lambda_1 \sqrt{1+\lambda_1}}
\]

(C14)
Appendix C

\[ \zeta_M = \frac{\sqrt{1 + \lambda_M} - 1}{2 \lambda_5 \sqrt{1 + \lambda_5}} \]  \hfill (C15)

and

\[ \mu = \frac{\sqrt{1 + \lambda_1} + 1}{2 \sqrt{1 + \lambda_1}} \]  \hfill (C16)
APPENDIX D

OPTIMUM COMBINING WEIGHT VECTOR

The average output signal to interference plus noise power ratio (SINR) is given in (44) as:

\[ \text{SINR} = \Gamma = \frac{W^H U_{\text{des}} U_{\text{des}}^H W}{W^H R_{\text{nn}} W} \]  \hspace{1cm} (D1)

where \( R_{\text{nn}} \) is the interference plus noise co-variance matrix, equal to [54]:

\[ R_{\text{nn}} = U_{\text{int}} U_{\text{int}}^H + \sigma^2 \]  \hspace{1cm} (D2)

The optimum weight vector, \( W_{\text{opt}} \), which maximizes the SINR is now required. The derivation given here is from [58]. Since the interference plus noise co-variance matrix \( R_{\text{nn}} \) is positive definite, i.e.

\[ W^H R_{\text{nn}} W > 0 \]  \hspace{1cm} (D3)

(positive definiteness is ensured since it has an uncorrelated noise component included) it can be factored into the product of two Hermitian (\( G = G^H \)) matrices:

\[ R_{\text{nn}} = G G \]  \hspace{1cm} (D4)

and

\[ R_{\text{nn}}^{-1} = G^{-1} G^{-1} \]  \hspace{1cm} (D5)

Matrix \( G \) can now be used to transform the weight vector \( W \) into a new vector \( V \) and visa versa:

\[ V = G W \]  \hspace{1cm} (D6)

and

\[ W = G^{-1} V = (G^{-1})^H V \]  \hspace{1cm} (D7)

Substituting equations (D6) and (D7) in (D1) yields:
The maximization of the SINR has now been reduced to a generalized eigenvalue problem, that of maximization of (D9), whose solution is:

\[ \mathbf{V} = \mathbf{E}_{\text{max}} \]  

(D11)

and

\[ \mathbf{B} \mathbf{E}_{\text{max}} = \lambda_{\text{max}} \mathbf{E}_{\text{max}} \]  

(D12)

where \( \lambda_{\text{max}} \) is the largest eigenvalue of \( \mathbf{B} \) and \( \mathbf{E}_{\text{max}} \) is the associated eigenvector. Inserting (D11) and (D12) in (D9), the following is obtained:

\[ \Gamma_{\text{max}} = \frac{\mathbf{E}_{\text{max}}^H \lambda_{\text{max}} \mathbf{E}_{\text{max}}}{\mathbf{E}_{\text{max}}^H \mathbf{E}_{\text{max}}} = \frac{\lambda_{\text{max}} \mathbf{E}_{\text{max}}^H \mathbf{E}_{\text{max}}}{\mathbf{E}_{\text{max}}^H \mathbf{E}_{\text{max}}} = \lambda_{\text{max}} \]  

(D13)

Inserting (D11) in (D7), the equivalent weighting vector is:

\[ \mathbf{W}_{\text{opt}} = \mathbf{G}^{-1} \mathbf{E}_{\text{max}} \]  

(D14)

Let now:

\[ \mathbf{C} = \mathbf{G}^{-1} \mathbf{U}_{\text{des}} \]  

(D15)

then equation (D10) becomes:

\[ \mathbf{B} = \mathbf{C} \mathbf{C}^H \]  

(D16)

inserting (D16) in (D12), the following is obtained:

\[ \mathbf{B} \mathbf{E}_{\text{max}} = \mathbf{C} \mathbf{C}^H \mathbf{E}_{\text{max}} = \lambda_{\text{max}} \mathbf{E}_{\text{max}} \]  

(D17)

and therefore the maximum eigenvalue is:
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\[ \lambda_{\text{max}} = CC^H \]  \hfill (D18)

\( \mathbf{B} \) is a rank one matrix and \( \mathbf{E}_{\text{max}} \) is the only eigenvector with non-zero eigenvalue. Substitution of (B17) and (D18) in (D14) and also using (D15) and (D16), the weight vector that will optimize the signal to interference ratio is:

\[ \mathbf{W}_{\text{opt}} = \mathbf{G}^{-1} \mathbf{G}^{-1} \mathbf{U}_{\text{des}} = \mathbf{R}^{-1}_{nn} \mathbf{U}_{\text{des}} \]  \hfill (D19)
APPENDIX E

WEIGHT VECTOR WITH INTERFERENCE CO-VARIANCE MATRIX

In this appendix a derivation will be shown for re-writing the optimum weight vector containing the full co-variance matrix in terms of only the interference plus noise co-variance matrix. It will also be shown how the constant in the optimum weight vector cancels out when estimating the SINR. The optimum weight vector using the received co-variance matrix is given in [54] as:

\[ W_{opt} = \mu R^{-1} U_{des} \]  

(E1)

where \( \mu \) is a constant given as:

\[ \mu = \frac{1}{U_{des}^H R^{-1} U_{des}} \]  

(E2)

and

\[ R = U_{des} U_{des}^H + R_{nn} \]  

(E3)

is the full received signal co-variance matrix. Using the matrix inversion lemma, the inverse of the full received signal co-variance matrix can be written as:

\[ R^{-1} = R_{nn}^{-1} - \frac{R_{nn}^{-1} U_{des} U_{des}^H R_{nn}^{-1}}{1 + U_{des}^H R_{nn}^{-1} U_{des}} = \frac{\Omega R_{nn}^{-1} - R_{nn}^{-1} U_{des} U_{des}^H R_{nn}^{-1}}{\Omega} \]  

(E4)

where \( \Omega \) is a scalar given by:

\[ \Omega = 1 + U_{des}^H R_{nn}^{-1} U_{des} \]  

(E5)

Inserting now (E4) in (E1) the following is obtained:

\[ W_{opt} = \frac{\left( \frac{\Omega R_{nn}^{-1} - R_{nn}^{-1} U_{des} U_{des}^H R_{nn}^{-1}}{\Omega} \right) U_{des}}{U_{des}^H \left( \frac{\Omega R_{nn}^{-1} - R_{nn}^{-1} U_{des} U_{des}^H R_{nn}^{-1}}{\Omega} \right) U_{des}} = \frac{\left( \frac{\Omega R_{nn}^{-1} - R_{nn}^{-1} U_{des} U_{des}^H R_{nn}^{-1}}{\Omega} \right) U_{des}}{U_{des}^H \left( \frac{\Omega R_{nn}^{-1} - R_{nn}^{-1} U_{des} U_{des}^H R_{nn}^{-1}}{\Omega} \right) U_{des}} \]

\[ = \frac{R_{nn}^{-1} \left( \Omega - U_{des} U_{des}^H R_{nn}^{-1} \right) U_{des}}{U_{des}^H R_{nn}^{-1} \left( \Omega - U_{des} U_{des}^H R_{nn}^{-1} \right) U_{des}} = \frac{R_{nn}^{-1} U_{des}}{U_{des}^H R_{nn}^{-1} U_{des}} \]
Equation (E6) gives the weight vector in terms of a constant and co-variance matrix of the interference signals alone (as well as desired signal array vector). Using now (E6) in the average SINR as given in equation (41), the average SINR becomes:

\[
\text{SINR} = \frac{P_{\text{des}} (\mu_{nn} R_{nn}^{-1} U_{\text{des}}) H U_{\text{des}} H (\mu_{nn} R_{nn}^{-1} U_{\text{des}})}{(\mu_{nn} R_{nn}^{-1} U_{\text{des}}) H P_{\text{int}} U_{\text{int}} H + \sigma_N^2 (\mu_{nn} R_{nn}^{-1} U_{\text{des}})}
\]

\[
= \frac{(\mu_{nn})^2 P_{\text{des}} (R_{nn}^{-1} U_{\text{des}}) H U_{\text{des}} H (R_{nn}^{-1} U_{\text{des}})}{(\mu_{nn})^2 (R_{nn}^{-1} U_{\text{des}}) H P_{\text{int}} U_{\text{int}} H + \sigma_N^2 (R_{nn}^{-1} U_{\text{des}})}
\]

\[
= \frac{P_{\text{des}} (R_{nn}^{-1} U_{\text{des}}) H U_{\text{des}} H (R_{nn}^{-1} U_{\text{des}})}{(R_{nn}^{-1} U_{\text{des}}) H P_{\text{int}} U_{\text{int}} H + \sigma_N^2 (R_{nn}^{-1} U_{\text{des}})}
\]

It can be seen in (E7) that the constant \( \mu_{nn} \) cancels out of the SINR.
REFERENCES


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