Chapter 13: Appendix F - Calculations

The general equation for a first order transfer function linking a specific output (y) with a specific input (x) can be written as follows (in the Laplace domain):

$$y = \frac{a}{1+\tau s}x$$

This Laplace-transform equation can be rewritten in the discrete incremental form for the time domain as follows (where $\bar{y}$ and $\bar{x}$ are the outputs and inputs in deviation variable form):

$$\bar{y} + \tau\frac{d\bar{y}}{dt} = a\bar{x}$$

$$\frac{d\bar{y}}{dt} = \frac{a\bar{x}-\bar{y}}{\tau}$$

$$\int_{\bar{y}_k}^{\bar{y}_{k+1}} \frac{d\bar{y}}{a\bar{x}-\bar{y}} = \int_t^{t+\Delta t}\frac{dt}{\tau}$$

$$-\ln\left(\frac{a\bar{x}-\bar{y}_{k+1}}{a\bar{x}-\bar{y}_k}\right) = \frac{\Delta t}{\tau}$$

$$a\bar{x}-\bar{y}_{k+1} = (a\bar{x}-\bar{y}_k)e^{-\frac{\Delta t}{\tau}}$$

$$\bar{y}_{k+1} = \bar{y}_k e^{-\frac{\Delta t}{\tau}} + a\bar{x}\left(1-e^{-\frac{\Delta t}{\tau}}\right)$$

The following equation also incorporates an immediate initial response:

$$\bar{y} = \frac{a+b\tau s}{1+\tau s}\bar{x}$$

If a step change of size $\Delta x = x_1 - x_0$, are applied at $t = 0$, the following equation results:

$$\bar{y} = \left(\frac{a + b\tau s}{1 + \tau s}\right)\left(\frac{\Delta x}{s}\right)$$
$$= \frac{a\Delta x}{s} - \frac{a\Delta x\tau}{1 + \tau s} + \frac{b\Delta x\tau}{1 + \tau s}$$

The effect on the output after one time step ($t = \Delta t$) written in time domain is now as follows:

$$\begin{aligned}\bar{y}_1 &= a\Delta x + (b - a)(\Delta x)e^{-\frac{\Delta t}{\tau}} \\ &= a(\bar{x}_1 - \bar{x}_0) + (b - a)(\bar{x}_1 - \bar{x}_0)e^{-\frac{\Delta t}{\tau}}\end{aligned} \quad \text{.. (13.1)}$$

The effect of the first step change after two time steps ($t = 2\Delta t$) will be as follows:

$$\bar{y}_2 = a(\bar{x}_1 - \bar{x}_0) + (b - a)(\bar{x}_1 - \bar{x}_0)e^{-\frac{2\Delta t}{\tau}}$$

If another step change ($x_2 - x_1$) is applied after one time step (at $t = \Delta t$), the combined effect of the two step changes after two time steps ($t = 2\Delta t$) will be:

$$\bar{y}_2 = a(\bar{x}_1 - \bar{x}_0) + (b - a)(\bar{x}_1 - \bar{x}_0)e^{-\frac{2\Delta t}{\tau}} + a(\bar{x}_2 - \bar{x}_1) + (b - a)(\bar{x}_2 - \bar{x}_1)e^{-\frac{\Delta t}{\tau}} \quad \text{... (13.2)}$$

Substituting Equation 3.12 into Equation 3.7 and simplifying results in:

$$\begin{aligned}\bar{y}_2 &= a(\bar{x}_2 - \bar{x}_0) + [\bar{y}_1 - a(\bar{x}_1 - \bar{x}_0)]e^{-\frac{\Delta t}{\tau}} + (b - a)\left[(\bar{x}_2 - \bar{x}_1)e^{-\frac{\Delta t}{\tau}}\right] \\ &= a \cdot \bar{x}_2 - a \cdot \bar{x}_0 + \bar{y}_1 e^{-\frac{\Delta t}{\tau}} - a \cdot \bar{x}_1 e^{-\frac{\Delta t}{\tau}} + a \cdot \bar{x}_0 e^{-\frac{\Delta t}{\tau}} + b(\bar{x}_2 - \bar{x}_1)e^{-\frac{\Delta t}{\tau}} - a \cdot \bar{x}_2 e^{-\frac{\Delta t}{\tau}} + a \cdot \bar{x}_1 e^{-\frac{\Delta t}{\tau}} \\ &= \bar{y}_1 e^{-\frac{\Delta t}{\tau}} + a(\bar{x}_2)\left(1 - e^{-\frac{\Delta t}{\tau}}\right) - a(\bar{x}_0)\left(1 - e^{-\frac{\Delta t}{\tau}}\right) + b(\bar{x}_2 - \bar{x}_1)e^{-\frac{\Delta t}{\tau}} \\ &= \bar{y}_1 e^{-\frac{\Delta t}{\tau}} + a(\bar{x}_2 - \bar{x}_0)\left(1 - e^{-\frac{\Delta t}{\tau}}\right) + b(\bar{x}_2 - \bar{x}_1)e^{-\frac{\Delta t}{\tau}}\end{aligned}$$

Through mathematical induction this equation can be written as a general recursive equation as follows:

$$\overline{y}_{k+1} = \overline{y}_k e^{-\frac{\Delta t}{\tau}} + a(\overline{x}_k - \overline{x}_0)\left(1 - e^{-\frac{\Delta t}{\tau}}\right) + b(\overline{x}_k - \overline{x}_{k-1})e^{-\frac{\Delta t}{\tau}}$$

$$y_{k+1} - y_{ss} = (y_k - y_{ss})e^{-\frac{\Delta t}{\tau}} + a(x_k - x_0)\left(1 - e^{-\frac{\Delta t}{\tau}}\right) + b(x_k - x_{k-1})e^{-\frac{\Delta t}{\tau}}$$

$$y_{k+1} = y_k e^{-\frac{\Delta t}{\tau}} + [y_{ss} + a(x_k - x_0)]\left(1 - e^{-\frac{\Delta t}{\tau}}\right) + b(x_k - x_{k-1})e^{-\frac{\Delta t}{\tau}}$$

The above equation was used to determine the outputs to changes in the inputs after each time step in the computer algorithm for a first order transfer function with a time constant of τ, an initial response of b and a steady state gain of a.