Appendix A

CORRELATED MULTIVARIATE GAMMA DISTRIBUTION

In mobile communication systems the most frequently used statistical models to describe the amplitude fading process are Rayleigh, Rician and Nakagami distributions. When the power of the fading amplitude is of interest, these statistical fading models are all related to the gamma distribution. In diversity based systems the correlated multivariate gamma distribution is of interest. With reference to Figure 3.1, this appendix presents a very general result when arbitrary Nakagami fading, arbitrary correlation and arbitrary signal powers are present on each MRC receive diversity branch.

A.1 CORRELATED MULTIVARIATE GAMMA DISTRIBUTION

With reference to Figure 3.1, the following assumptions are made in deriving the model

- arbitrary signal power, $\Omega_l$, on branch $l$,
- arbitrary correlation, $\rho_{k,l}$, between branches $k$ and $l$, and
- arbitrary Nakagami fading, $m_i$, on each branch.

Two well-known correlation models are the constant correlation model, where

$$\rho_{i,j} = \rho \quad \forall \quad i, j = 1, 2, \cdots, M_D, \tag{A.1}$$

and the exponential correlation model where

$$\rho_{i,j} = \rho^{i-j} \quad \forall \quad i, j = 1, 2, \cdots, M_D. \tag{A.2}$$

The constant correlation model is applicable to closely spaced diversity antennas. When the diversity signals are taken from a configuration which, in some physical sense, is equi-spaced (either in space, time, frequency, etc), the correlation model can be exponential. The validity of this model stems from the assumption that, given the stationary nature of the overall diversity process (assuming statistical equivalence of the signals),
the correlation between a pair of signals decreases as the separation between them increases (see Chapter 2 for a more detailed discussion on this issue).

For a fixed set of received fading amplitudes \( \{ \beta(k) \} \), the random variables \( X_{ck} \) and \( X_{sk} \) are normally assumed Gaussian, with received power given by

\[
S = \sum_{k=1}^{M_D} \beta_k^2 = \sum_{k=1}^{M_D} \{X_{ck}^2 + X_{sk}^2\} = \sum_{k=1}^{M_D} S_k, \tag{A.3}
\]

with \( S_k \) the instantaneous power of the \( k \)th channel. It is noted that

\[
E\{\beta_k^2\} = E\{S_k\} = \Omega_k. \tag{A.4}
\]

The received instantaneous SNR per bit can be written as

\[
\gamma_k = \frac{E_b}{N_0} \sum_{k=1}^{M_D} \beta_k^2 = \sum_{k=1}^{M_D} \gamma_k, \tag{A.5}
\]

and

\[
\bar{\gamma}_k = \frac{E_b}{N_0} E\{\beta_k^2\} = \frac{E_b}{N_0} \Omega_k, \tag{A.6}
\]

the average SNR of the \( k \)th diversity branch.

The general characteristic function for an \( M_D \) branch MRC diversity system can be derived as [244]

\[
\Phi_S(t) = \Pi_{k=1}^{M_D} |I_{M_{D_k} \times M_{D_k}} - i t D_{M_{D_k}} (\bar{m}^{-1}) D_{M_{D_k}} (\Omega_k) J_{M_{D_k} \times M_{D_k}} |^{-\left(m_k - m_{k+1}\right)}, \tag{A.7}
\]

where \( m_{M_D} + 1 = 0, i = \sqrt{-1}, \) and

\[
D_{M_{D_k}} (\Omega) = \text{diag}\{\Omega_1, \Omega_2, \ldots, \Omega_{M_{D_k}}\}, \tag{A.8}
\]

\[
D_{M_{D_k}} (\bar{m}^{-1}) = \text{diag}\{m_1^{-1}, m_2^{-1}, \ldots, m_{M_{D_k}}^{-1}\},
\]

and \( J_{M_{D_k} \times M_{D_k}} \) is an \( M_{D_k} \times M_{D_k} \) correlation matrix, given by

\[
J_{k,l} = \rho_{k,l} \sqrt{\frac{m_k}{m_l}}, \quad k \geq l,
\]

\[
k = 1, 2, \ldots, M_D. \tag{A.9}
\]

The restriction

\[
m_1 \geq m_2 \geq \ldots \geq m_{M_D} \tag{A.10}
\]
The multivariate gamma distribution is finally obtained by taking the inverse Fourier transform of \( \Phi_S(t) \) w.r.t. \( t \),

\[
\phi_S(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_S(t) e^{-its} dt.
\]  

(A.12)

It is emphasized again that the characteristic function of (A.7) is very general and valid within the constraints of (A.10) and (A.11).

### A.1.1 Example

In this example we show how to calculate the pdf for the sum of \( M_D = 4 \) MRC receive diversity signals. We begin by calculating the characteristic function given in (A.7), and then taking the inverse Fourier transform as shown in (A.12).

In our calculations it is important to adhere to the constraints given by (A.10) and (A.11). For our example the fading parameters on the \( k \) diversity branches is arbitrarily chosen as \( \{m_k\} = 4 \) with average received power on each branch \( \{\Omega_k\} = 1 \). The correlation matrix is given by

\[
J = \begin{pmatrix}
1 & \rho_{21} \sqrt{m_2/m_1} & \rho_{31} \sqrt{m_3/m_1} & \rho_{41} \sqrt{m_4/m_1} \\
\rho_{21} \sqrt{m_2/m_1} & 1 & \rho_{32} \sqrt{m_3/m_2} & \rho_{42} \sqrt{m_4/m_2} \\
\rho_{31} \sqrt{m_3/m_1} & \rho_{32} \sqrt{m_3/m_2} & 1 & \rho_{43} \sqrt{m_4/m_3} \\
\rho_{41} \sqrt{m_4/m_1} & \rho_{42} \sqrt{m_4/m_2} & \rho_{43} \sqrt{m_4/m_3} & 1
\end{pmatrix}
\]  

(A.13)

In general \( \rho_{ij} = \rho_{ji} \) due to symmetry (i.e. \( \rho_{12} = \rho_{21} \) etc.). In our example, let \( \rho_{21} = \rho_{31} = \rho_{41} = \rho_{32} = \rho_{31} = \rho_{42} = \rho_{43} = 0.65 \). The correlation matrix \( J \) therefore reduces to

\[
J = \begin{pmatrix}
1 & 0.65 & 0.65 & 0.65 \\
0.65 & 1 & 0.65 & 0.65 \\
0.65 & 0.65 & 1 & 0.65 \\
0.65 & 0.65 & 0.65 & 1
\end{pmatrix}
\]  

(A.14)

The calculation of \( D_{M_D} (\Omega) \) and \( D_{M_D} (\Omega^{-1}) \) follows trivially from (A.8). Using the identity

\[
|B| = [1 - a(1 - b)]^{(k - 1)} \cdot [1 - a(1 - b + bk)],
\]  

(A.15)

with

\[
B = \begin{pmatrix}
1 - a & -ab & \cdots & -ab \\
-ab & 1 - a & \cdots & -ab \\
& -ab & \cdots & 1 - a
\end{pmatrix}_{k \times k}
\]  

(A.16)
the characteristic function is obtained with the pdf shown in Figure A.1. Figure A.1 also displays the pdf for different values of $\rho$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figureA1.png}
\caption{Probability density function for $M = 4$ and $m = 4$.}
\end{figure}

A.1.2 Special Case

For the special case where the received signal strength is constant, i.e. $\Omega_k = \Omega$, and the fading on the branches is constant, i.e. $m_k = m = 1$, and we make use of the constant correlation model given by (A.1), the characteristic function of (A.7) reduces to

$$\Phi_S(t) = [1 - i \Omega^2(1 - \rho)]^{-(M-1)} \cdot [1 - i \Omega^2(1 - \rho + \rho M)]^{-1}. \tag{A.17}$$

Taking the inverse Fourier transform of (A.17), gives a closed form expression for the pdf of interest as

$$p_S(s) = \frac{1}{\Omega^2 \Gamma(M)} \left( \frac{s}{\Omega^2} \right)^{M-1} \times \exp\left(\frac{-s}{(1-\rho)\Omega^2}\right) \cdot \frac{\rho^M}{(1-\rho)(M-1)(1 - \rho + \rho M)} \cdot {}_{1}F_{1}\left(1, M, \frac{\rho^M}{(1-\rho)(1-\rho+\rho M)\Omega^2}\right), \tag{A.18}$$

where $\,_{1}F_{1}(\cdot)$ is the confluent hyper geometric function [147].
Appendix B

CONVOLUTIONAL AND TURBO CODE TRANSFER FUNCTIONS

B.1 CONVOLUTIONAL CODE PERFORMANCE

B.1.1 Convolutional Codes

In this appendix, the transfer functions, $T(L, I, D)$ for the rate-1/2 and 1/3, constraint length $L_{cc} = 9$ codes are considered as required in Chapter 4. The exponents of $l, i$, and $d$ of the monomial $L^l I^i D^d$ indicate, respectively, the path length, input word weight and output word weight.

For the rate-1/2 code, characterised by the generator polynomials $g_1 = (561)_8$ and $g_2 = (753)_8$, the coefficients of the transfer function have been found through computer search and they are tabulated in the literature [245]:

$$T(D) = 11D^{12} + 50D^{14} + 286D^{16} + 1630D^{18} + 9630D^{20} + \ldots,$$

where $D$ and $I$ are the distance and information error weight operators, respectively. Differentiating the transfer function with respect to $I$ and setting $I = 1$, the following expression is obtained

$$\frac{\delta T(I, D)}{\delta I} \bigg|_{I=1} = 33D^{12} + 281D^{14} + 2179D^{16} + 15035D^{18} + \ldots$$

Similarly, for the rate-1/3 code, characterised by the generator polynomials $g_1(D) = (557)_8$, $g_2 = (663)_8$ and $g_3 = (711)_8$, the coefficients of the transfer function are given by [245]:

$$T(D) = 5D^{18} + 7D^{20} + 36D^{22} + 85D^{24} + 204D^{26} + \ldots$$

with transfer function derivative written as

$$\frac{\delta T(I, D)}{\delta I} \bigg|_{I=1} = 11D^{18} + 32D^{20} + 195D^{22} + 564D^{24} + \ldots$$
B.1.2 Orthogonal and Super-orthogonal Convolutional Codes

Now consider the transfer functions of proposed low-rate codes for combined coding and spreading are considered. These codes are the orthogonal, bi-orthogonal and super-orthogonal convolutional codes [13]. Here, $L_{oc} = L$ is used to denote the constraint length of orthogonal encoders considered.

An expression for the transfer function of orthogonal convolutional codes is given by [16]

$$T(I,D) = \frac{ID^{L_2L-1}(1 - D^{2L-1})}{1 - D^{2L-1}[1 + I(1 - D^{(2L-1)(L-1)})]}.$$

(B.1)

The corresponding expression for the transfer function derivative is [16]

$$\frac{dT(I,D)}{dI} \bigg|_{I=1} = \frac{D^{L_2L-1}(1 - D^{2L-1})^2}{(1 - 2D^{2L-1} + D^{2L-1})^2}.$$

(B.2)

Performing a series expansion of (B.2), the number of bit errors corresponding to an error event of length $d$ is obtained as the multiplicity term of $D^d$. The results from the series expansion can then be applied to the BEP bounds discussed in Section 4.2.

When the second order terms are discarded, (B.2) can be approximated by

$$\frac{dT(I,D)}{dI} \bigg|_{I=1} \approx D^{L_2L-1}.$$

(B.3)

The distance spectrum of the super-orthogonal encoder can be obtained in the same way as for the orthogonal scheme, namely through a series expansion of the differentiated transfer function. This produces [13]

$$\frac{dT(I,D)}{dI} \bigg|_{I=1} = \frac{D^{L_2L-1}(1 - D^{2L-1})^2}{(1 - 2D^{2L-1} + D^{2L-1})^2} \approx D^{(L+2)2L-3}.$$

(B.4)

(B.5)

The upper bounds for the BEP for the orthogonal and super-orthogonal convolutional codes can then be written, respectively, as

$$P_e \leq \frac{1}{2} \exp\left(-\sigma_{oe} L 2^{L-1}\right),$$

(B.6)

$$P_e \leq \frac{1}{2} \exp\left(-\sigma_{oe} (L + 2) 2^{L-3}\right),$$

(B.7)

where $\sigma_{oe}$ is the effective received SNR.

B.2 TURBO CODE PERFORMANCE

B.2.1 Input-Output Weight Enumerator Recursion

The notation and terminology introduced by Divsalar et al. in [176] are used. The turbo code is the parallel concatenation of the constituent rate $R_e = 1$ components, which is referred to as constituent code fragments.
The constituent non-trivial 4-state \((2^m)\) code fragment is completely characterized by its state transition matrix, \(A(L, I, D)\), where

\[
A_{2m/4} = \begin{pmatrix}
L & LID & 0 & 0 \\
0 & 0 & LD & LI \\
LID & L & 0 & 0 \\
0 & 0 & LI & LD
\end{pmatrix}.
\]  

(B.8)

For a given constituent code the number of paths of length \(l\), input weight \(i\), and output weight \(d\), starting in the all-zero state, denoted by \(t(l, i, d)\). Then the corresponding transfer function, or complete path enumerator, is defined by [246]

\[
T(L, I, D) = \sum_{l \geq 0} \sum_{i \geq 0} \sum_{d \geq 0} L^l I^i D^d \cdot t(l, i, d)
\]

(B.9)

The first element of the matrix given in (B.9) produces the transfer function of the constituent code

\[
T(L, I, D) = T^{(1,1)}(L, I, D) = \frac{T_N}{T_D}.
\]

(B.10)

where

\[
T_N = 1 - LD - L^2 I - L^3 (D^2 - I^2),
\]

(B.11)

and

\[
T_D = 1 - L(1 + I) - L^3(D^2 - I - I^2 - I^3 D^2) - L^4(D^2 - I^2 - I^2 D^2 + I^4 D^2).
\]

(B.12)

For a constituent code, the path input/output weight index is used to determine the output weight probability distribution function. For the constituent encoder, the path input/output weight index, is

\[
t(l, i, d) = t(l - 1, i - 1, d) + t(l - 1, i, d) + t(l - 3, i - 3, d - 2) - t(l - 3, i - 2, d)
- t(l - 3, i - 1, d) + t(l - 3, i, d - 2) - t(l - 4, i - 4, d - 2) + t(l - 4, i - 2, d - 4)
+ t(l - 4, i, d - 2) - t(l - 4, i, d - 2) + \delta(l, i, d) - \delta(l - 1, i - 1, d)
- \delta(l - 2, i - 1, d) - \delta(l - 3, i, d - 2) + \delta(l - 3, i - 2, d)
\]

(B.13)

with initial conditions such that \(t(l, i, d) = 0\) for any negative index, where \(\delta(l, i, d) = 1\) if \(l = i = d = 0\) and \(\delta(l, i, d) = 0\) otherwise.

B.2.2 Input-Output Conditional Probability Density Function

In this section the conditional probability of producing a codeword fragment of weight \(d\) given a randomly selected input sequence of weight \(i\) is evaluated. This is given by

\[
p(d \mid i) = \frac{t(N_{tc}, i, d)}{\sum_d t(N_{tc}, i, d')} = \frac{t(N_{tc}, i, d)}{\binom{N_{tc}}{i}}.
\]

(B.14)
The conditional pdf (cpdf) of the constituent recursive convolutional encoder is shown in Figure B.1, for an interleaver size of $N_{tc} = 100$. Since the code fragment only produces even output weights, the uneven weight probabilities are not shown. Similar distributions are obtained when the interleaver size is increased. In this thesis the latter is referred to as the Divsalar cpdf. It was shown in [176] that, given a sufficiently large input codeword weight $i$, the cpdf approaches a binomial distribution. Given a balanced source with $N_{tc}$ sufficiently large (typically $> 100$), this condition will always be met. Thus, in addition to the calculated weight distributions the cpdf, described as a binomial probability distribution with probability $1/2$ (taken over $N_{tc} = 100$ trails) is also shown for reference purposes.

![Figure B.1](image.png)

**Figure B.1.** Conditional probability density function (cpdf) as a function of the output weight $d$ of the constituent recursive convolutional encoder, given input weight $i$, with interleaver size $N_{tc} = 100$.

If the interleavers of the encoder are selected randomly and independently, the cpdf $p(d | i)$ that any input sequence $u$ of weight $i$ will be mapped into code fragments of weights $d_s, d_p$ is

$$R_c = 1/2 : p(d | i) = p^s(d_s | i) * p^p(d_p | i),$$

$$R_c = 1/3 : p(d | i) = p^s(d_s | i) * p^p(d_p | i) * p^p(d_p | i),$$

(B.15)

where $*$ denotes convolution.

In (B.15), $p^s(d_s | i)$ corresponds to the systematic output weight probability, and $p^p(d_p | i)$ corresponds to the non-punctured parity output weight probability. Therefore, the total codeword output weight relative to the all-zero codeword is $d = d_s + d_p$ and $d = d_s + 2d_p$, for the rate-$1/2$ and rate-$1/3$ coders, respectively. The extension to the low-rate code, e.g. $R_c = 1/6$ follows in a similar manner.

In the calculation of the BER for turbo codes of different code rate we have to integrate over the total code input-output weight probability distribution, denoted by $p(d_0, d_1, \ldots, d_Z | i)$. This is achieved by the introduction of the conditional expectation $E_{d_i \{i\}}$ which should be taken over the probability distribution $p(d_0, d_1, \ldots, d_Z | i)$. Here, $Z$ constitutes the number of constituent code fragments.


REFERENCES


REFERENCES


REFERENCES


Department of Electrical and Electronic Engineering

University of Pretoria


