CHAPTER 3

The Frequency Response Function Method

3.1 THEORY

Preamble

The intent of this section is twofold: Firstly, we will formulate the relevant theory relating to the frequency response function method as applied to the force identification process. We will start with the more familiar inverse of a square matrix and progress to the pseudo-inverse of a rectangular matrix. The second objective is concerned with the calculation of the pseudo-inverse itself. There are currently a number of different matrix decomposition methods that are used in the calculation of the pseudo-inverse. It is not the intention to present a detailed mathematical explanation of the derivation of the pseudo-inverse, but rather to highlight some of the important issues. This is explained on the basis of the frequency response function method, but is also relevant to other force identification procedures, among others the modal coordinate transformation method which also features in this work.

3.1.1 Direct Inverse

By assuming that the number of forces to be identified and the number of responses are equal $(m = n)$, the frequency response matrix becomes a square matrix and thus an ordinary inversion routine can be applied, as follows:

$$\{F(\omega)\} = [H(\omega)]^{-1}\{X(\omega)\} \tag{3.1}$$

The above equation suppresses many of the responses for computational purposes, since the number of forces is usually only a few even if the structure is very complex or many responses are available.

3.1.2 Moore-Penrose Pseudo-Inverse

Accordingly, it is proposed to use a method of least-squares regression analysis, which allows the use of more equations than unknowns, whence the name over-determined. The advantage of being able to use redundant information minimises the consequence of errors in measured signals due to noise, which are always present. Adopting the least-squares method the following set of inconsistent linear equations are formulated:

$$\{X(\omega)\} = [H(\omega)]\{F(\omega)\} \tag{3.2}$$

where

$\{X(\omega)\}$ is the $(n \times 1)$ response vector,

$[H(\omega)]$ is the $(n \times m)$ frequency response function matrix,

$\{F(\omega)\}$ is the $(m \times 1)$ force vector.

The difference between the above equation and equation (3.1) is that here m unknowns (forces) are to be estimated from n equations (responses), with $n \geq m$. The least-squares solution of equation (3.2) is given by:

$$\{\hat{F}(\omega)\} = [H(\omega)]^{+}\{X(\omega)\} \tag{3.3.a}$$

where

$$[H(\omega)]^{+} = \left([H(\omega)]^{*}[H(\omega)]\right)^{-1}[H(\omega)]^{*} \tag{3.3.b}$$

which is known as the Moore-Penrose pseudo-inverse of the rectangular matrix $[H(\omega)]$. Since the force and response vectors are always functions of the frequency, the functional notation (ω) will be dropped in further equations.

The least-squares solution $\{\hat{F}\}$ is thus given by:

$$\{\hat{F}\} = \left([H]^{*}[H]\right)^{-1}[H]^{*}\{X\} \tag{3.4}$$

where

$[\cdot]^{*}$ is the complex conjugate transpose of the indicated matrix and $(\cdot)^{-1}$ is the inverse of a square matrix.

Now, we would like to investigate the conditions under which the pseudo-inverse, as stated in equation (3.4), are valid. For this reason, we first need to consider what is meant by the rank of a matrix.

a) The Rank of a Matrix

The rank of a matrix can be defined as the number of linearly independent rows or columns of the matrix. A square matrix is of full rank if all the rows are linearly independent and rank deficient if one or more rows of the matrix are a linear combination of the other rows. Rank deficiency implies that the matrix is singular, i.e. its determinant equals zero and its inverse cannot be calculated. An $n \times m$ rectangular matrix with $n \geq m$ is said to be 'full rank' if its rank equals m, but rank deficient if its rank is less than m. (Maia, 1991)

b) Limitation Regarding the Moore-Penrose Pseudo-Inverse

It should be noted that equation (3.4) is only unique when $[H]$ *is of full column rank* ($rank([H]) = m$; m = number of forces), i.e. the equations in (3.2) are linearly independent. Or in other words, the inverse of $([H(\omega)]^{*}[H(\omega)])^{-1}$ in equation (3.4) is only feasible if all the columns and at least m rows of the $(n \times m)$ rectangular matrix $[H]$ are linearly independent.

If $[H]$ is rank deficient ($rank([H]) < m$), the matrix to be inverted will be singular and the pseudo-inverse cannot be computed. This, however, does not mean that the pseudo-inverse does not exist, but merely that another method needs to be employed for its determination.

Based on the above-mentioned requirement Brandon (1988) refers to the Moore-Penrose pseudo-inverse as the 'restricted' pseudo-inverse. He investigated the use of the restricted pseudo-inverse method in modal analysis and only his conclusions will be represented here.

- "The most common representation of pseudo-inverse, in modal analysis is the full rank restricted form. This will fail if the data is rank deficient, due to the singularity of the product matrices. In cases where the data is full rank, but is poorly conditioned (common in identification problems), the common formulation of the restricted pseudo-inverse will worsen the condition *unnecessarily*.

- In applications where the rank of the data matrix is uncertain, the singular value decomposition gives a reliable numerical procedure, which includes an explicit measure of the rank."

It is to be hoped that the reader will be convinced in view of the above that certain restrictions exist regarding the use of the Moore-Penrose pseudo-inverse. The Singular Value Decomposition will prove to be an alternative for calculating the pseudo-inverse of a matrix.

c) *Further Limitations Regarding the Least-Square Solution*

Up to now, it may seem possible to apply the least-squares solution to the force identification problem, simply by ensuring that the columns of $[H]$ are all linearly independent. But this in itself introduces further complications. The number of significantly participating modes, as introduced by Fabunmi (1986), plays an important role in the linear dependency of the columns of the frequency response function matrix.

The components of the forces acting on a structure are usually independent. Conversely, the different responses caused by each one of the forces may have quite similar spatial distributions. As a result, the columns of the frequency response function matrix are "almost" linearly dependent, resulting in a rank deficient matrix. This can be circumvented by taking more measurements, or by moving the measurement positions along the structure. However, situations exist where the above action will have little effect.

As is generally known, the response at a particular frequency will be dominated only by a number of significantly participating modes, p. This is particularly true at or near resonance. In such a situation only a limited number of columns of the frequency response function matrix are linearly independent, while some can be written as linear combinations of the dominated modes. The linear dependency may be disguised by measurement errors. This leads to ill-conditioning of the matrix, which can be prone to significant errors when inverted.

In order to successfully implement the least-squares technique, Fabunmi (1986) suggests that the number of forces one attempts to predict should be less or equal to the significantly participating modes at some frequency ($m \leq p$). This will ensure that all the columns and at least m rows will be linearly independent.

To conclude:

- The number of response coordinates must be at least as many as the number of forces. In the least-squares estimation, the response coordinates should considerably outnumber the estimated forces ($n \geq m$).

- Furthermore, the selection of the response coordinates must be such as to ensure that at least m rows of the frequency response function matrix are linearly independent. If there are fewer than m independent rows, the estimated forces will be in error, irrespective of how many rows there are altogether ($p \geq m$).

3.1.3 Singular Value Decomposition (SVD)

In the force identification the number of modes that contribute to the data is not always precisely known. As a result the order of the data matrix may not match the number of modes represented in the data. Another method must then be employed to calculate the pseudo-inverse, for instance Singular Value Decomposition (SVD).

It is not the intent to present a detailed mathematical explanation of the derivation of the SVD technique, but rather to highlight some of the important issues. The reader is referred to the original references for specific details (Menke, 1984; Maia, 1991; Brandon, 1988).

The SVD of an $n \times m$ matrix $[H]$ is defined by:

$$[H]=[U]\,[\Sigma][V]^T \tag{3.5}$$

where

$[U]$ is the $(n \times n)$ matrix, the columns comprise the normalised eigenvectors of $[H][H]^T$,

$[V]$ is the $(m \times m)$ matrix and the columns are composed of the eigenvectors of $[H]^T[H]$, and

$[\Sigma]$ is the $(n \times m)$ matrix with the singular values of $[H]$ on its leading diagonal (off-diagonal elements are all zero).

The following mathematical properties follow from the SVD:

a) The Rank of a Matrix

The singular values in the matrix $[\Sigma]$ are arranged in decreasing order ($\sigma_1 > \sigma_2 > ... > \sigma_m$). Thus,

$$[\Sigma]=\underbrace{\left[\begin{array}{cccc} \sigma_1 & & & 0 \\ & \sigma_2 & & \\ & & \ddots & \\ 0 & & & \sigma_m \\ \hdashline & & 0 & \end{array}\right]}_{m} \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right\} m \\ \\ \end{array} \left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\end{array}}\right\} n \tag{3.6}$$

Some of these singular values may be zero. The number of non-zero singular values defines the rank of a matrix $[\Sigma]$. However, some singular values may not be zero because of experimental measurements, but instead are very small compared to the other singular values. The significance of a particular singular value can be determined by expressing it as the ratio of the largest singular value to that particular singular value. This gives rise to the condition number.

b) Condition Number of Matrix

After decomposition, the condition number, $\kappa_2([H])$, of a matrix can be expressed as the ratio of the largest to the smallest singular value.

$$\kappa_2([H])=\|[H]\|_2\,\|[H]^+\|_2=\sigma_{\max}/\sigma_{\min} \tag{3.7}$$

If this ratio is so large that the smaller one might as well be considered zero, the matrix $[H]$ is 'almost singular' and has a large condition number. This reflects an ill-conditioned matrix. We can establish a criterion whereby any singular value smaller than a tolerance value will be set to zero. This will avoid numerical problems, as the inverse of a small number is very large and would, falsely, dominate the pseudo-inverse if not excluded. By setting the singular value equal to zero, the rank of the matrix $[H]$ will in turn be reduced, and in effect the number of force predictions allowed, as referred to in Section 3.1.2.c.

c) *Pseudo-Inverse*

Since the matrices $[U]$ and $[V]$ are orthogonal matrices, i.e.,

$$[U][U]^T = [U]^T[U] = [V][V]^T = [V]^T[V] = [I] \qquad (3.8a)$$

and

$$[U]^T = [U]^{-1} \qquad \text{and} \qquad [V]^T = [V]^{-1} \qquad (3.8b)$$

the pseudo-inverse is related to the least-squares problem, as the value of $\{\hat{F}\}$ that minimises $\left\| [H]\{\hat{F}\} - \{X\} \right\|_2$ and can be expressed as:

$$[H]^+ = ([V]^T)^+ [\Sigma]^+ [U]^+ \qquad (3.9)$$

Therefore,

$$[H]^+ = [V][\Sigma]^+ [U]^T \qquad (3.10)$$

where

$[H]^+$ is an $(m \times n)$ pseudo-inverse of the frequency response matrix,

$[V]$ is an $(m \times m)$ matrix containing the eigenvectors of $[H][H]^T$,

$[U]^T$ is an $(n \times n)$ unitary matrix comprising the eigenvectors of $[H]^T[H]$,

$[\Sigma]^+$ is an $(m \times n)$ real diagonal matrix, constituted by the inverse values of the non-zero singular values.

The force estimates can then be obtained as follows:

$$\{\hat{F}\} = [V][\Sigma]^+[U]^T\{X\} \qquad (3.11)$$

3.1.4 QR Decomposition

The QR Decomposition (Dongarra *et al.*, 1979) provides another means of determining the pseudo-inverse of a matrix. This method is used when the matrix is ill-conditioned, but not singular.

The QR Decomposition of the $(n \times m)$ matrix $[H]$ is given as:

$$[H]=[Q][R] \tag{3.12}$$

where

$[Q]$ is the $(n \times m)$ orthogonal matrix, and

$[R]$ is the $(m \times m)$ upper triangular matrix with the diagonal elements in descending order.

The least-squares solution follows from the fact that

$$[H]^T[H]=[R]^T[Q]^T[Q][R] \tag{3.13}$$

and taking the inverse of the triangular and well-conditioned matrix $[R]$ it follows that

$$\{\hat{F}\}=[R]^{-1}[Q]^T\{X\} \tag{3.14}$$

3.1.5 Tikhonov Regularisation

As described earlier, the SVD deals with the inversion of an ill-conditioned matrix by setting the very small singular values to zero and thus, averting their contribution to the pseudo-inverse. In some instances the removal of the small singular values will still result in undesirable solutions. The Tikhonov Regularisation (Sarkar *et al.*, 1981; Hashemi and Hammond, 1996) differs from the previously mentioned procedures in the sense that it is not a matrix decomposition method, but rather a stable approximate solution to an ill-conditioned problem, and whence the name regularisation methods. The basic idea behind regularisation methods is to replace the unconstrained least-squares solution by a constrained optimisation problem which would force the inversion problem to have a unique solution.

The optimisation problem can be stated as the minimising of $\left\|[H]\{\hat{F}\}-\{X\}\right\|^2$ subjected to the constraint $\left\|[\hat{L}]\{\hat{F}\}\right\|^2$, where $[\hat{L}]$ is a suitably chosen linear operator.

It has been shown that this problem is equivalent to the following one

$$\min_{\hat{F}}\left\{\left\|[H]\{\hat{F}\}-\{X\}\right\|^2+\mu^2\left\|[\hat{L}]\{\hat{F}\}\right\|^2\right\} \tag{3.15}$$

where

μ plays the role of the Lagrange multiplier.

The following matrix equation is equivalent to equation (3.14):

$$\left([H]^*[H]+\mu^2[\hat{L}]^*[\hat{L}]\right)\{\hat{F}\}=[H]^*\{X\} \tag{3.16}$$

leading to

$$\{\hat{F}\}=\left([H]^*[H]+\mu^2[\hat{L}]^*[\hat{L}]\right)^{-1}[H]^*\{X\} \tag{3.17}$$

3.2 TWO DEGREE-OF-FREEDOM SYSTEM

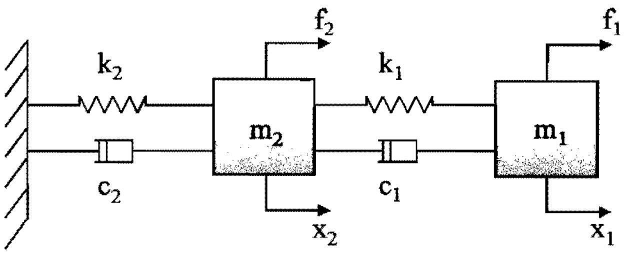
Noise, as encountered in experimental measurements, consists of correlated and uncorrelated noise. The former includes errors due to signal conditioning, transduction, signal processing and the interaction of the measurement system with the structure. The latter comprises errors arising from thermal noise in electronic circuits, as well as external disturbances. (Ziaei-Rad and Imregun, 1995)

The added effect of noise, as encountered in experimental measurements, may further degrade the inversion process. This is especially true for a system with a high condition number, indicating an ill-conditioned matrix. As was stated previously, the inverse of a small number is very large and would, falsely, dominate the pseudo-inverse.

There are primarily two sources of error in the force identification process. The first is the noise encountered in the structure's response measurements. Another source of errors arises from the measured frequency response functions and the modal parameter extraction. Bartlett and Flannelly (1979) indicated that noise contaminating the frequency response measurements could create instabilities in the inversion process. Hillary (1983) concluded that noise in both the structure's response measurement and the frequency response function can affect the force estimates. Elliott *et al.* (1988) showed that the measurement noise increased the rank of the strain response matrix, which circumvented the force predictions.

The measured frequency response functions can be applied directly to the force identification process. As an alternative, the frequency response functions can also be reconstructed from the modal parameters, but this approach requires that an experimental modal analysis be done in advance. The latter has the advantage that it leads to a considerable reduction in the amount of data to be stored. The curve-fitted frequency response functions can be seen as a way of filtering some of the unwanted noise from the frequency response functions. However, the reconstruction of the frequency response functions from the identified eigensolutions might give rise to difficulties in taking into account the effect of out-of-band modes in the reconstruction.

To illustrate the ill-conditioning of a system near and at the resonant frequencies, consider the following lumped-mass system:



$m_1 = 0.5$ kg	$k_1 = 1.4212 \times 10^5$ N/m	$\zeta_1 = 0.001$
$m_2 = 1.0$ kg	$k_2 = 2k_1$ N/m	$\zeta_2 = 0.001$

Figure 3.1 – 2 DOF lumped mass system

A harmonic forcing function is used to excite each of the masses.

$$f_1(t) = 150\cos(60\pi t) \qquad f_2(t) = 100\cos(60\pi t) \tag{3.18}$$

It can be shown that the natural frequencies and mass-normalised mode-shapes for the undamped system are:

$$\begin{aligned}\omega_1 &= 60\ Hz \\ \omega_2 &= 120\ Hz\end{aligned} \qquad [\Phi] = \begin{bmatrix} 1.1547 & 0.8165 \\ 0.5773 & -0.8165 \end{bmatrix} \tag{3.19}$$

The forward problem was solved to obtain the response for each degree-of-freedom from

$$\{\ddot{X}(\omega)\}_{EXACT} = [A(\omega)]_{EXACT}\{F(\omega)\}_{APPLIED} \tag{3.20}$$

where

$\{\ddot{X}(\omega)\}$ is the (2×1) acceleration vector,

$[A(\omega)]$ is the (2×2) inertance matrix,

$\{F(\omega)\}$ is the (2×1) force vector.

The frequency response function was recalculated for the perturbated modal parameters. The inverse problem was solved subsequently to obtain the force estimates

$$\{\hat{F}(\omega)\} = [\tilde{A}(\omega)]^{+}\{\tilde{\tilde{X}}(\omega)\} \tag{3.25}$$

where

$(\tilde{\cdot})$ denotes the contaminated values.

The relative error is given by the Force Error Norm (FEN), $\varepsilon_f(\omega)$, of the forces, evaluated at each frequency line, and is defined as

$$\varepsilon_f(\omega) = \frac{\left\| \{F(\omega)\} - \{\hat{F}(\omega)\} \right\|_2}{\left\| \{F(\omega)\} \right\|_2} \times 100\,\% \tag{3.26}$$

where

$\{F(\omega)\}$ is the actual/applied force vector,

$\{\hat{F}(\omega)\}$ is the estimated force vector,

$\| \cdot \|_2$ is the vector 2-norm.

Results and Discussion

Figures 3.2 and 3.3 show the ill-conditioning of equation (3.25) in the vicinity of the modes, where the force estimates are in error. The FEN at the first mode (Figure 3.4) is considerably higher than at the second mode. Since the applied forces are not in the vicinity of the system's resonances, they are not affected by this ill-conditioned behaviour and are correctly determined.

The modes of this system are well-separated, and near and at the resonant frequencies the response of the system is dominated by a single mode. As Fabunmi (1986) concluded, the response of which content is primarily that of one mode only cannot be used to determine more than one force.

In another numerical simulation of the same system, the influence of the perturbation of the different modal parameter on the force identification was considered. The force estimates were calculated for the case where a single modal parameter was polluted to the prescribed error level. It was concluded that the perturbation of the mode-shapes had the most significant effect in producing large errors in the force estimates. This result confirmed findings of Hillary (1983) and Okubo *et al.* (1985).

Although the frequency response function matrix is a square matrix, it is still singular at the resonant frequencies. This implies that the pseudo-inverse of the frequency response function can only be obtained by the SVD.

Since two forces were determined from two response measurements the least-square solution is of no use. In practice one would include as many response measurements as possible to utilise the least-squares solution for the over-determined case.

The maximum error levels were considered as realistic of what one could expect during vibration testing. These values were gathered from similar perturbation analyses performed by Hillary (1983), Genaro and Rade (1998), and Han and Wicks (1990). No explanations or references were provided for the error levels adopted. A literature survey conducted by the author concerning this issue also failed to produce satisfactory information. These error values proved to produce very large errors in the identified forces, beyond the point where the estimated forces could be meaningful.

To conclude, the aim of this section was to prove that small errors can have adverse effects on the force identification at the resonant frequencies of a system.

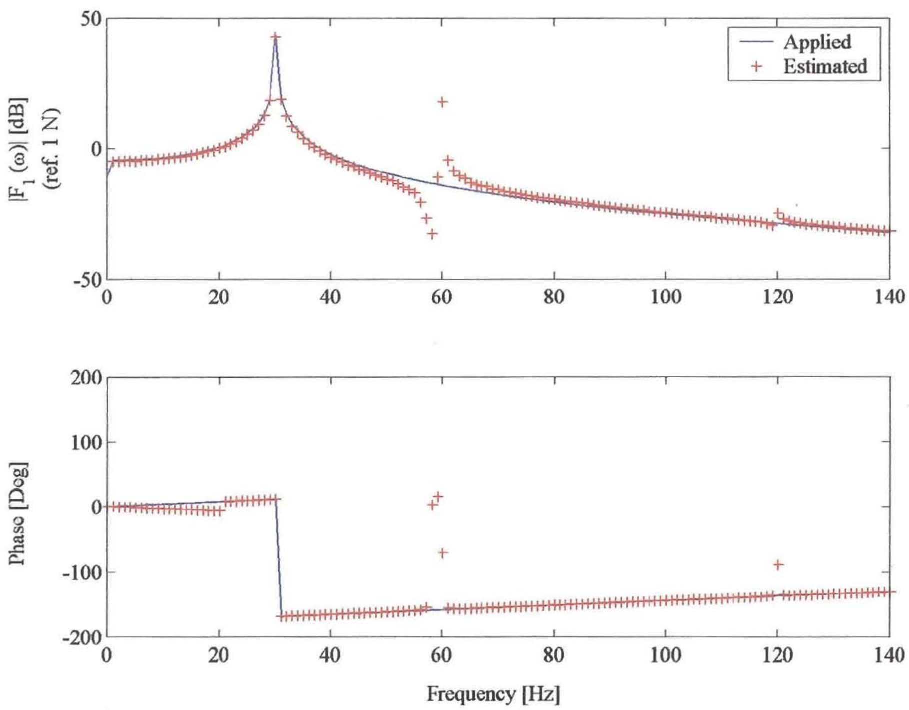


Figure 3.2 – Applied and estimated force no.1 for the 2 DOF lumped-mass system in the frequency domain

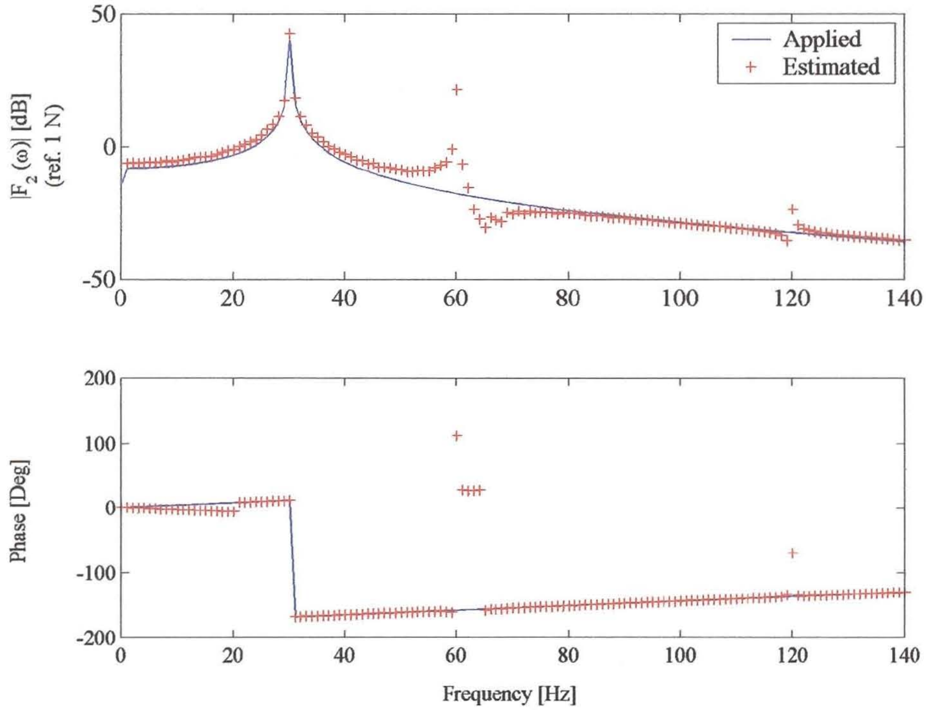


Figure 3.3 – Applied and estimated force no.2 for the 2 DOF lumped-mass system in the frequency domain

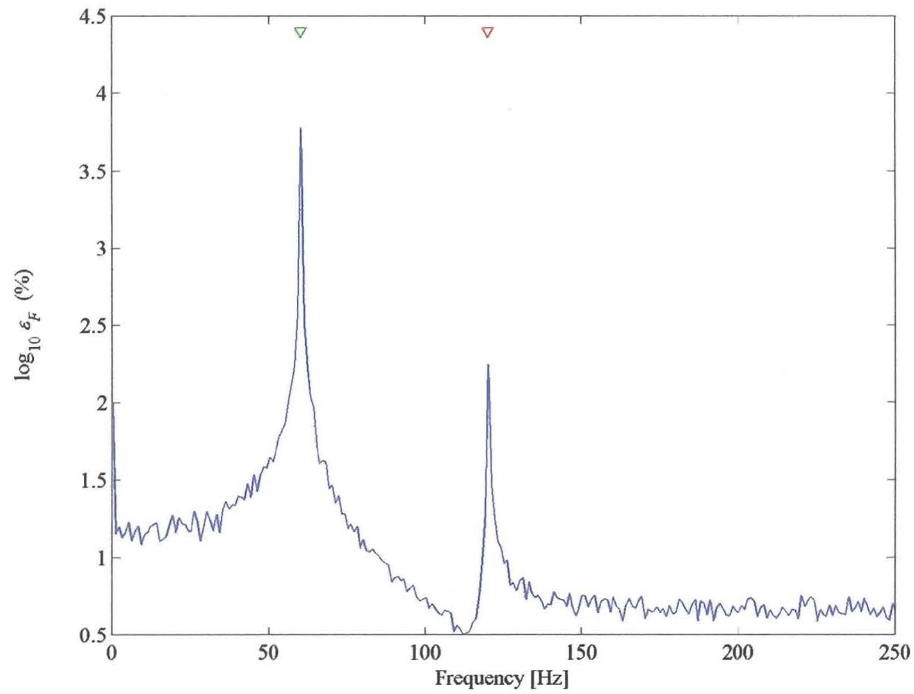


Figure 3.4 – Force Error Norm (FEN) of the estimated forces, ∇ indicates the 2 DOF systems' resonant frequencies

3.3 SIGNIFICANCE OF THE CONDITION NUMBER

In the previous section the effect of noise in the force identification was introduced. Consequently, errors will always be present in the measurements. In this section it is suggested that the condition number of the frequency response function matrix serves as a measure of the sensitivity of the pseudo-inverse.

Again consider the 2 DOF lumped-mass system. This time the system was subjected to randomly generated forces with uniform distribution on the interval $[-1,1]$. The responses were obtained utilising the forward problem. The contaminated frequency response function matrix was generated through the perturbation of the modal parameters, as previously described. Finally, the inverse problem was solved to obtain the force estimates. The above procedure was repeated 100 times. In each run new random variables were generated. Figure 3.5 represents the average FEN, $\varepsilon_f(\omega)$ for these 100 runs.

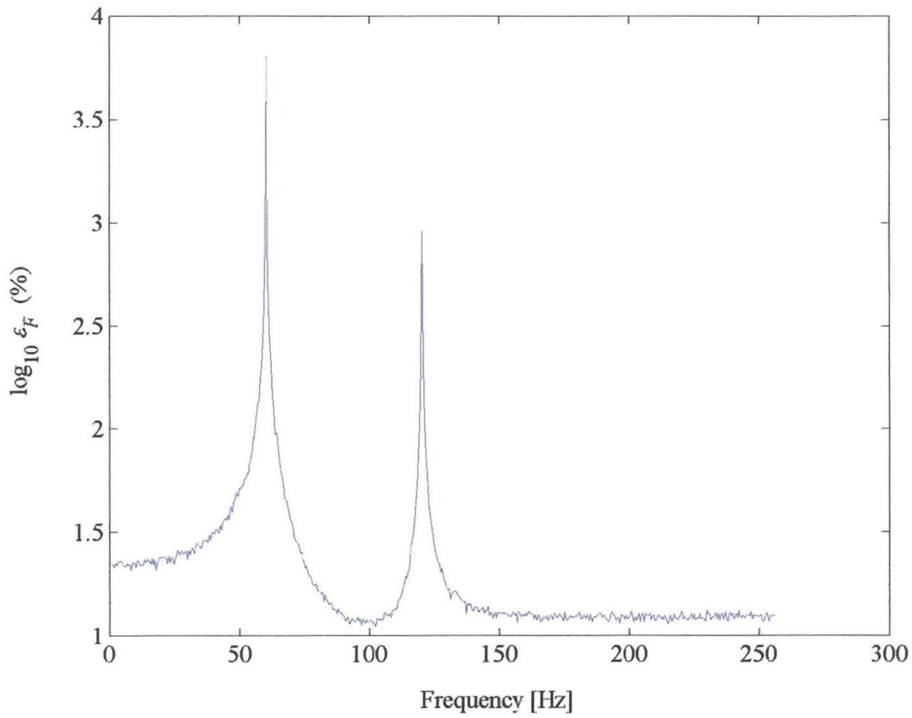


Figure 3.5 – Average Force Error Norm of estimated forces for the 2 DOF lumped-mass system

Golub and Van Loan (1989) describe the error propagation using the condition number, of the matrix to be inverted, as an error boundary for perturbation of linear systems of equations. Referring to the above case where only the frequency response function matrix was perturbated, errors in the calculation of $\{F(\omega)\}$ will be restricted by

$$\frac{\| \{F(\omega)\} - \{\hat{F}(\omega)\} \|_2}{\| \{F(\omega)\} \|_2} \le \kappa_2([H])^2 \frac{\| [\delta H(\omega)] \|_2}{\| [H(\omega)] \|_2} \tag{3.27}$$

where

$\{F(\omega)\}$ is the actual force vector,

$\{\hat{F}(\omega)\}$ is the estimated force vector,

$\kappa_2([H])$ is the condition number of $[H]$,

$[\delta H(\omega)]$ is the difference between the actual and perturbated $[H]$,

$\| \cdot \|_2$ is the vector 2-norm.

Similar expressions could be obtained for perturbation of the response, $\{X(\omega)\}$. Unfortunately, these expressions are of little practical use, since the actual force and response are unknown.

It was suggested that the condition number of the matrix to be inverted, should be used as a measure of the sensitivity of the pseudo-inverse (Starkey and Merrill, 1989). Although the exact magnitude of the error bound of the system at a particular frequency remains unknown, the condition number enables one to comment on the reliability of the force estimates within a given frequency range. A high condition number indicates that the columns of the frequency response function are linearly or "almost" linearly dependent, i.e. rank deficient. This can result in large errors in the identified forces. Conversely, a condition number close to unity indicates that the columns of the frequency response function are almost mutually perpendicular. One should not expect any large amplification of the measurement noise when inverting the frequency response function matrix. Figure 3.6 shows the condition number of the frequency response function matrix for the 2 DOF lumped-mass system. From this it is evident that the condition number follows the same trend as the relative error in the force estimates given in Figure 3.5.

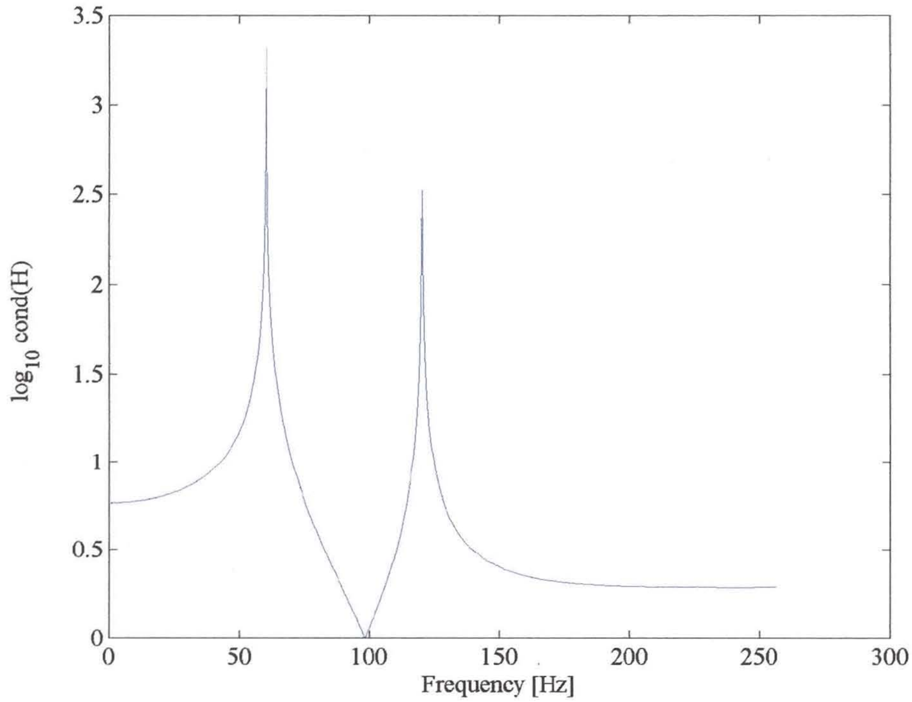


Figure 3.6 – Condition number of frequency response function matrix

In the next half of this section we would like to investigate the factors that have an influence on the value of the condition number. A more complex structure will be considered for this purpose.

A Finite Element Model (FEM) of a freely supported beam was constructed. Ten equally spaced beam elements were used to simulate the 2 metres long beam. The model was restricted to two dimensions, since only the transverse bending modes were of interest, for which the natural frequencies and normal modes were obtained solving the eigenvalue problem. The first eight bending modes were used in the reconstruction of the frequency response function matrix of which, the first three modes were the rigid body modes of the beam. A uniform damping factor of 0.001 was chosen. Each node point was considered as a possible sensor location.

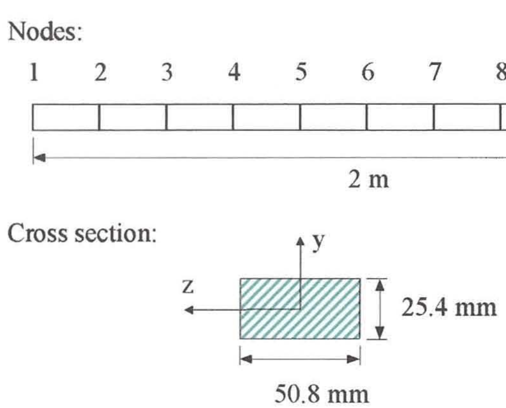


Figure 3.7 – FEM of free-free beam and response locations

3.3.1 Effect of the Number of Forces

Eleven response 'measurements' were taken and four sets of excitation forces were applied to the beam. These sets consisted of 1, 2, 3 and 4 forces, respectively and are shown in Figure 3.8.

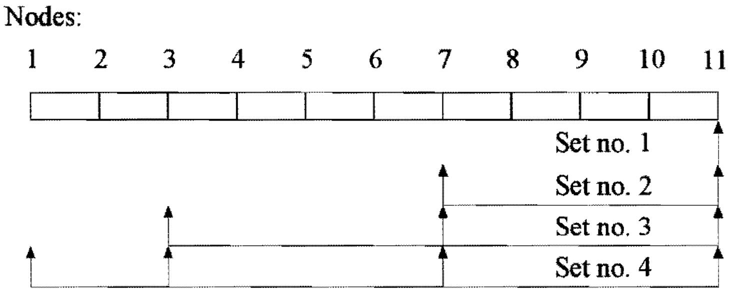


Figure 3.8 - FEM of free-free beam and force locations

Figure 3.9 shows the condition number of the frequency response function matrix for each of these sets. It is evident from the results that the condition number increases drastically as the number of force predictions increase. The condition number for a single force prediction is equal to unity over the entire frequency range. If an additional force is added, it implies that the column corresponding the force location be included in the frequency response function matrix, which in turn increases the condition number of the matrix. The number of force predictions is, however, limited by the number of modes included in the analysis (Fabunmi, 1986).

3.3.2 Effect of the Damping

The effect of the damping on the condition number was evaluated next. Only three forces were applied, while still measuring eleven responses. From Figure 3.10 one notices that the condition number varies for different modal damping factors. In fact, as the damping factors increase, the condition number decreases, especially at the beam's resonances. This can be attributed to higher modal overlap due to the higher damping. It was mentioned earlier that at a resonant frequency the system's response is dominated by that particular mode. In a system with higher damping the neighbouring modes have a larger contribution to the response of the system at that frequency.

3.3.3 Effect of the Number of Response Measurements

Figure 3.11 shows the condition number as a function of the number of response measurements across the frequency range of interest. The beam is once again subjected to three forces while considering 3 (locations 3, 7 and 11), 6 (locations 2, 3, 5, 7, 9 and 11) and 9 (locations 2, 3, 4, 5, 7, 8, 9, 10 and 11) response measurements, respectively.

The results show that there is a significant improvement in the condition number by increasing the number of response measurements. Since there is a direct relation between the condition number and force error, over-determination will improve the force estimates as well. Adding a response measurement implies that an additional row, and thus a new equation, is added to the frequency response function matrix. Mas *et al.* (1994) showed that the ratio of the number of response measurements to the number of force predictions should preferably be greater than or equal to 3 (i.e. $n/m \geq 3$).

3.3.4 Effect of the Response Type

Both accelerometers and strain gauges have been employed by Hillary and Ewins (1984) to determine sinusoidal forces on a uniform cantilever beam. The strain responses gave more accurate force estimates, since the strain responses are more influenced by the higher modes at low frequencies. Han and Wicks (1990) also studied the application of displacement, slope and strain measurements. From both these studies it is evident that proper selection of the measurement type can improve the condition of the frequency response function matrix and hence obtain better force predictions.

3.3.5 Conclusion

It is suggested that the condition number of the frequency response function matrix serves as a measure of the sensitivity of the pseudo-inverse. The frequency response function matrix needs to be inverted at each discrete frequency, and as a result the condition number varies with frequency. Large condition numbers exist near and at the system's resonances.

The condition number of the pseudo-inverse is a function of the number of response points included. The number of force predictions, system's damping, as well as the selection of the response type also influence the condition number.

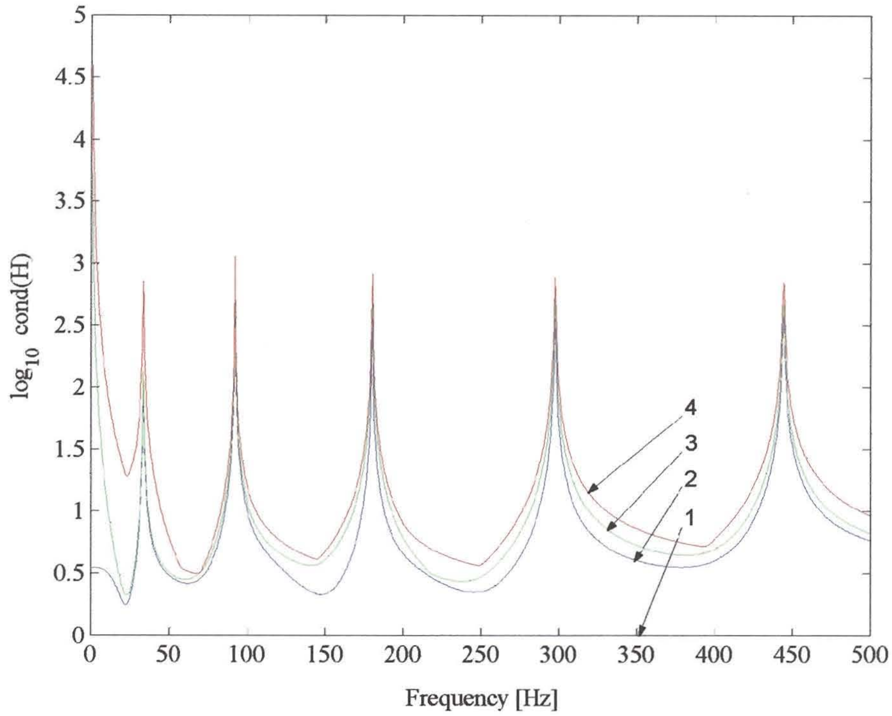


Figure 3.9 – Effect of the number of forces on the condition number

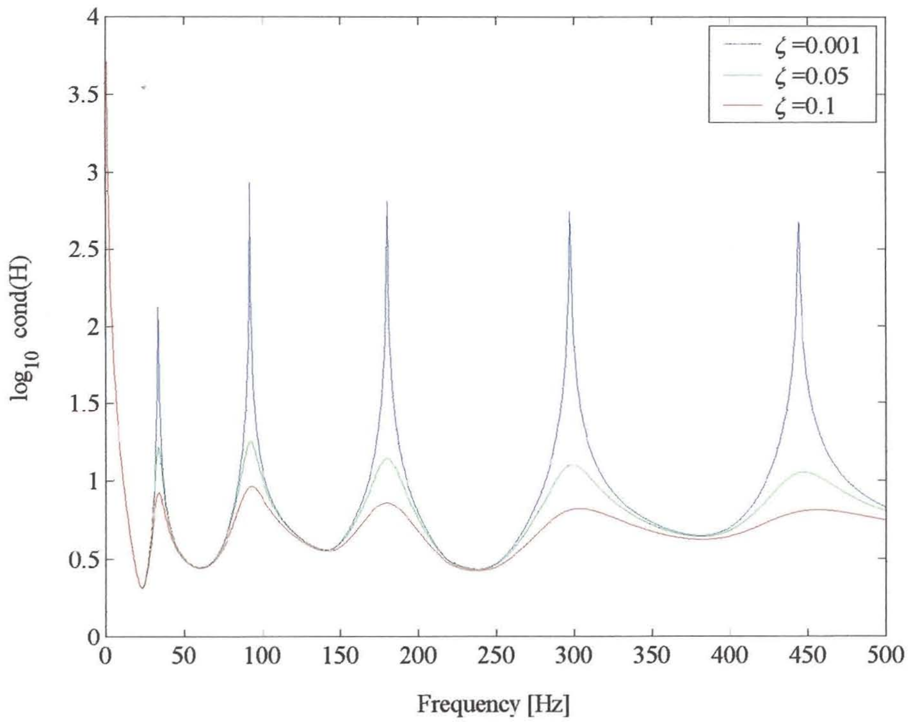


Figure 3.10 – Effect of the damping on the condition number

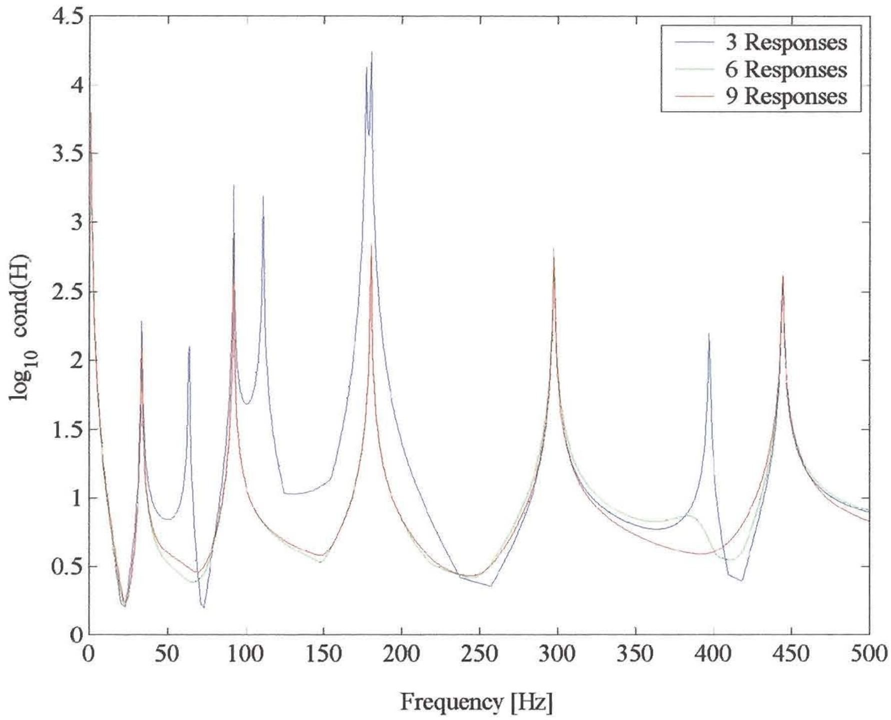


Figure 3.11 – Effect of the number of response measurements on the condition number

3.4 NUMERICAL STUDY OF A FREE-FREE BEAM

This section examines the different matrix decomposition and regularisation methods employed for the calculation of the pseudo-inverse. For this purpose it was decided to continue using the FEM of the freely supported aluminium beam, as introduced in Section 3.3.

The eigenvalue problem was solved to obtain the natural frequencies and mode shapes. In addition to the RBM, five bending modes were in the chosen frequency range of 0 to 500 Hz. The natural frequencies and mode shapes were taken as the 'exact' values. A proportional damping model was assumed with values obtained from the experimental modal analysis performed on a similar beam.

The beam was then subjected to two simultaneous harmonic forces applied at positions 5 and 11 with forcing frequencies of 220 Hz and 140 Hz, respectively. The frequency content of the force signals was not determined with an FFT algorithm, thus presenting zero amplitude values in the frequency range except at the discrete forcing frequencies.

The 'exact' response at each of the eleven sensor locations was calculated from:

$$\{\ddot{X}(\omega)\}_{EXACT} = [A(\omega)]_{EXACT}\{F(\omega)\}_{APPLIED} \tag{3.28}$$

where

$\{\ddot{X}(\omega)\}$ is the (11×1) acceleration vector,

$[A(\omega)]$ is the (11×2) inertance matrix,

$\{F(\omega)\}$ is the (2×1) force vector.

The $[A(\omega)]$ matrix was constructed from the RBM and five bending modes, while omitting the residual terms. The response and modal parameters were perturbated, as described in Section 3.2, to resemble experimental data. Successively, the force identification problem was solved while including only six response locations (positions 1, 3, 5, 6, 9 and 11) in the analysis.

$$\{\hat{F}(\omega)\} = [\tilde{A}(\omega)]^{+}\{\tilde{\ddot{X}}(\omega)\} \tag{3.29}$$

where

$(\tilde{\ })$ denotes the contaminated values.

Figure 3.12 shows the effect of the perturbation analysis on the reconstructed inertance matrix $[\tilde{A}(\omega)]$.

Each of the previously explained pseudo-inverse methods was employed to evaluate their ability to correctly determine the two harmonic forces.

A major difficulty associated with the Tikhonov regularisation is the choice of the regularisation parameter μ (Da Silva and Rade, 1999). The value of this parameter was obtained from using the L-curve (Hansen and O'Leary, 1991). The L-curve is a plot of the semi-norm $\left\|[\hat{L}]\{\hat{F}\}\right\|$, as a function of the residual norm $\left\|[H]\{\hat{F}\}-\{X\}\right\|$ for various values of μ. The corner of the L-curve is identified and the corresponding regularisation parameter μ is returned. This procedure has to be performed at each discrete frequency line and a result is computationally very expensive.

The results of the analysis are presented in Table 3.2. Only, the Tikhonov regularisation failed to predict the two forces correctly. This constraint optimisation algorithm identified the force applied at position 11 correctly, but calculated a significant force amplitude with the same frequency content as the force applied at node 11 at the other force location, position 5. Furthermore, it also under-estimated the force at node 5. Increasing the number of response locations had no improvement on the result.

Table 3.2 - Force results of the different matrix decomposition and regularisation methods

	$\lvert F_1 \rvert$ [N]	$\lvert F_2 \rvert$ [N]
Applied	10.000	0
Force amplitudes	0	23.000
Singular Value Decomposition	9.338	21.432
QR Decomposition	9.338	21.432
Moore-Penrose	9.338	21.432
Tikhonov Regularisation	5.2139	19.512

Next, the author evaluated the force identification process for the entire frequency range. In this case, the frequency content of the harmonic force time signals was determined with an FFT-algorithm, thus presenting non-zero values in the frequency range considered. Figure 3.13 illustrates the ill-conditioning of the force estimates at the resonant frequencies of the beam. The ill-conditioning in this particular case is a result of the perturbation analysis, the FEM approximations and the FFT-algorithms. Changing the excitation points on the beam produced the same trends. Once again the Tikhonov regularisation produced poorer results than the other methods. The Singular Value Decomposition, QR Decomposition and Moore-Penrose pseudo-inverse produced exactly the same results.

In view of the above-mentioned the author decided to use the SVD from here onwards for the calculation of the pseudo-inverse matrix. The motivation being the ease of the implementation of this algorithm in the *Matlab®* environment. Another advantage of the SVD is the ability to ascertain the rank of a matrix and to truncate the singular values accordingly.

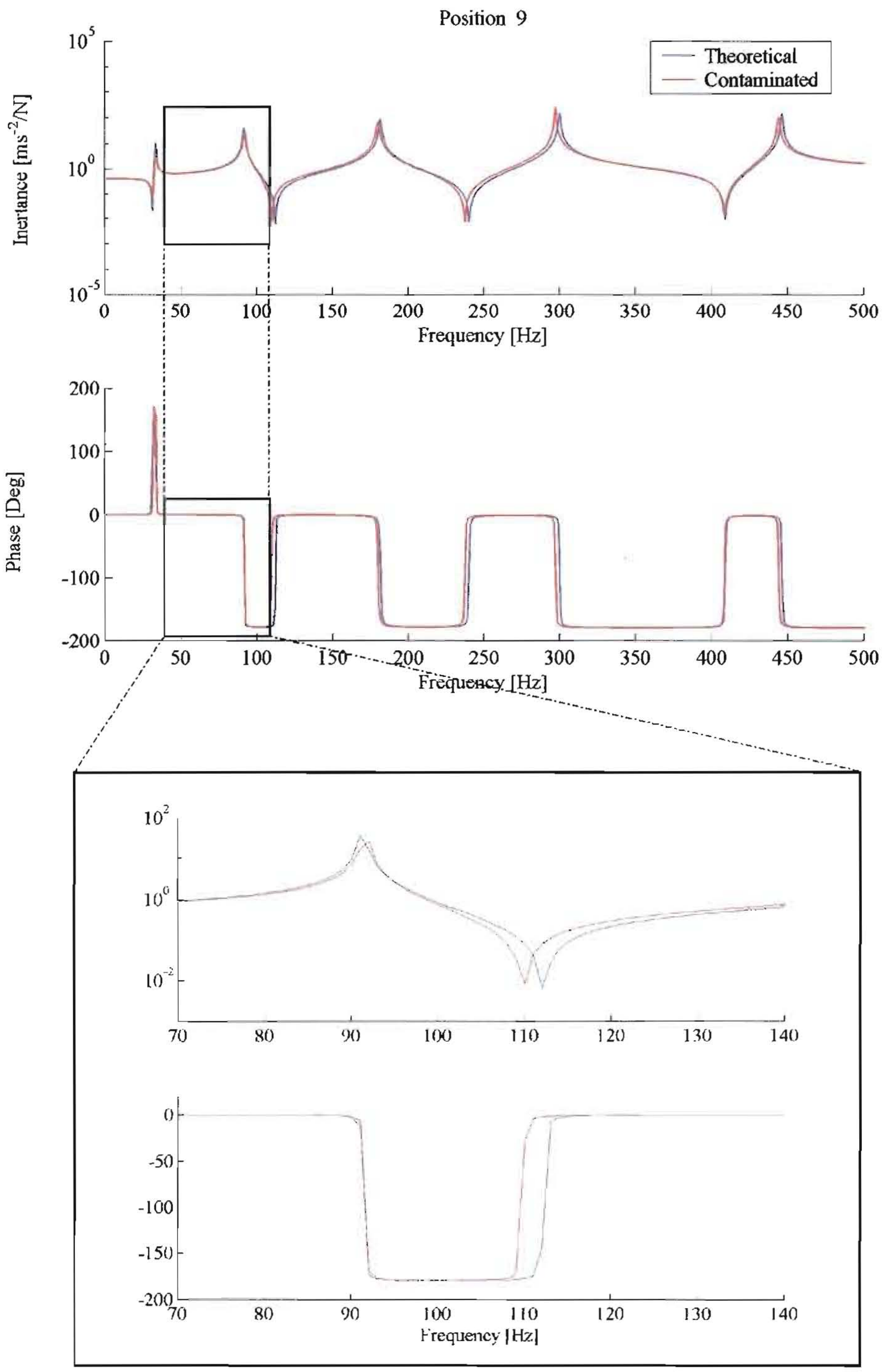


Figure 3.12 - Comparison of the 'exact' and perturbated inertance frequency response function

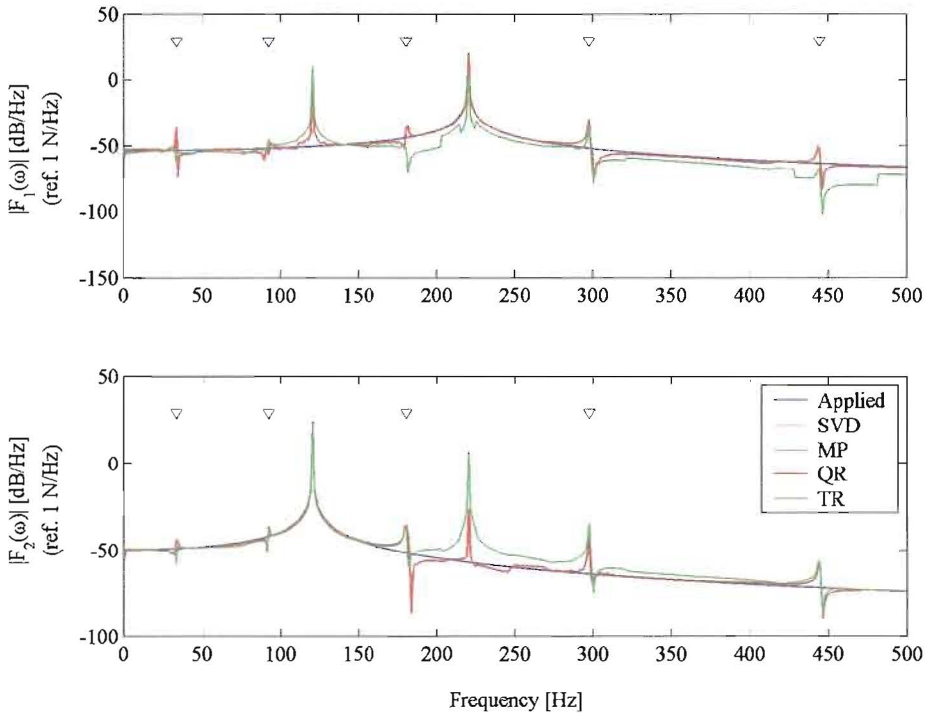


Figure 3.13 – Comparison of the different decomposition and regularisation methods. ∇ indicates the resonant frequencies