CHAPTER 2

Frequency Domain Analysis

2. FREQUENCY DOMAIN ANALYSIS

The force identification techniques will be performed in the frequency domain. This section motivates the use of the frequency domain and highlights the related theory used in Fourier Transforms, as well the frequency response function's formulation, measurement and modal parameter extraction.

2.1 ADVANTAGE OF USING FREQUENCY DOMAIN

The frequency response function is one of the functions used to describe the input-output relation in a linear system, in the frequency domain. Equation (1.3) is repeated as an example:

$$\{X(\omega)\}=[H(\omega)]\{F(\omega)\} \tag{2.1}$$

where

$\{X(\omega)\}$ is the $(n\times 1)$ response vector,

$[H(\omega)]$ is the $(n\times n)$ frequency response function matrix,

$\{F(\omega)\}$ is the $(n\times 1)$ force vector.

One important benefit, which is evident from the equation above, is that the Fourier Transform, transforms a convolution in the time domain, into a multiplication in the frequency domain (Randall, 1977). The equivalent convolution in the time domain is evidently a much more complicated procedure. This is one of the reasons for the great success of the Fourier Transform technique in signal processing.

Dealing with stationary random excitations also benign the use of the frequency domain. As examples we can mention flow-induced vibration in a piping system and the fluctuating pressure gusts on the wing of an airplane in flight. These systems can only be formulated in terms of their statistical properties and can be completely defined by the spectral density functions.

It may also be justifiable to mention some of the disadvantages associated with the frequency domain. Windowing functions need to be enforced on the time signals to suppress the affect of 'leakage'. Furthermore, the Auto Spectral Density (ASD) functions contain no phase information and are unable to capture transient phenomena of systems.

2.2 DISCRETE FOURIER TRANSFORM

The Discrete Fourier Transform (DFT) technique will be employed to transform any time function, $x(t)$ into the frequency domain by use of the following equation (Broch, 1990):

$$X(n) = \frac{1}{N}\sum_{k=0}^{N-1} x(k)\, e^{-i2\pi\frac{jk}{N}} \qquad (2.2)$$

where

$x(k)$ is a discrete series of a sampled time function $x(t)$,

N is the number of sampled points,

$X(n)$ is the Fourier series coefficients,

for $j = 0,1,2,\ldots,N-1$; $k = 0,1,2,\ldots,N-1$.

Conversely, a discrete time series may be calculated from knowledge of the Fourier series coefficients:

$$x(k) = \sum_{j=0}^{N-1} X(n)\, e^{i2\pi\frac{jk}{N}} \qquad (2.3)$$

which is the Inverse of the Discrete Fourier Transform (IDFT). The DFT and IDFT equations are implemented in *Matlab*®, which uses efficient Fast Fourier Transform (FFT) algorithms.

Thus, the use of DFT permits any time response to be transformed into the frequency domain. The force identification technique will yield the estimated forces. These forces may be transformed back into discrete time series, using the IDFT. The flowchart for this procedure is as follows:

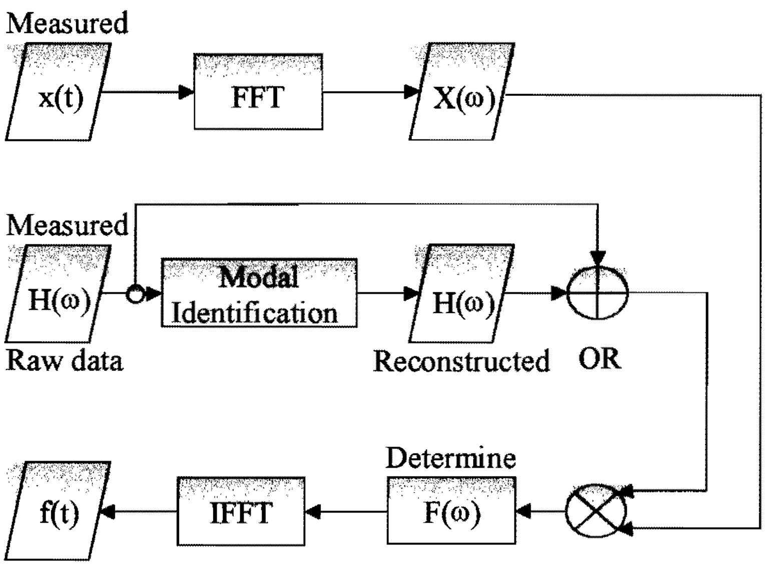

Figure 2.1 - Flowchart of a typical force identification procedure

For more detailed information on the discrete Fourier transform, the FFT algorithms or implementation of Fourier analysis in the *Matlab*® environment the interested reader is referred to the original references (Randall, 1977; Broch, 1990; McConnell, 1992 and *Matlab*®, *Version 5.3*).

2.3 FREQUENCY RESPONSE FUNCTION MODELLING

The *Structural Dynamics Toolbox*® (Balmès, 1997) provides a framework for the modelling of the input/output response of a linear system. Many engineering structures may be considered as lightly-damped structures. That is structures for which the damping is small so that the low frequency response is characterised mostly by the mass and stiffness contributions. Consequently, a normal mode model is used.

The eigenvalue problem of the normal modes may be defined as follows:

$$-[M]\{\phi\}_r\omega_r^2+[K]\{\phi\}_r=\{0\} \tag{2.4}$$

where

$[M]$ and $[K]$ are the $(N\times N)$ mass and stiffness matrices, respectively.

$\{\phi\}_r$ denotes $(N\times 1)$ independent eigenvectors (normal modes),

ω_r^2 is N independent eigenvalues (eigenfrequencies squared), and $r=1,\ldots,N$ where N is the number of degrees-of-freedom in the system.

The solutions of the eigenvalue problem of equation (2.4) yield the following mass-normalised modal matrix orthogonality properties:

$$[\Phi]^T[M][\Phi]=[I] \qquad \text{and} \qquad [\Phi]^T[K][\Phi]=[\Lambda] \tag{2.5}$$

The state space formulation of the normal mode model for the *damped system*, as expressed in terms of the principal coordinates, is as follows:

$$[[I]s^2+[\beta]s+[\Lambda]]\{p\}=[[\Phi]^T[b]]\{u\} \tag{2.6a}$$

$$\{y\}=[[c][\Phi]]\{p\} \tag{2.6b}$$

where

$s=i\omega$ is the Laplace variable,

$[\Lambda]$ is the diagonal modal stiffness matrix (eigenfrequency squared),

$[\beta]$ is the modal damping matrix, and $[\beta]=[\Phi]^T[C][\Phi]$,

$[[\Phi]^T[b]]$ is the modal input matrix,

$[b]$ is the input shape matrix, which is time invariant and characterises the spatial properties of the applied forces,

$[[c][\Phi]]$ is the modal output matrix,

$[c]$ is the output shape matrix, which is time invariant and characterises the spatial properties of the sensors,

$\{u\}$ is the input vector to the system,

$\{y\}$ is the output vector of the system,

$\{p\}$ is the principal/modal coordinates.

Assuming a unity modal mass matrix, the first-order state space model takes the form

$$\begin{Bmatrix} \{\dot{p}\} \\ \{\ddot{p}\} \end{Bmatrix} = \begin{bmatrix} [0] & [I] \\ -[\Lambda] & -[\beta] \end{bmatrix} \begin{Bmatrix} \{p\} \\ \{\dot{p}\} \end{Bmatrix} + \begin{bmatrix} [0] \\ [\Phi]^T[b] \end{bmatrix} \{u\} \tag{2.7a}$$

$$\{y\} = [[c][\Phi] \;\; [0]] \begin{Bmatrix} \{p\} \\ \{\dot{p}\} \end{Bmatrix} \tag{2.7b}$$

In the case of proportional damping, the diagonal modal damping matrix, $[\beta]$, may be expressed in terms of the damping ratios $\beta_r = 2\zeta_r\omega_r$.

The frequency response function is, by definition, the Fourier Transform of the system's response divided by the Fourier Transform of the applied force. The frequency response function for the linear system, which corresponds to the partial fraction expansion, can be written as:

$$[\alpha(s)] = \sum_{r=1}^{N} \frac{\{[c]\{\phi\}_r\}\{[b]^T\{\phi\}_r\}^T}{s^2 + 2\zeta_r\omega_r s + \omega_r^2} = \sum_{r=1}^{N} \frac{[T]_r}{s^2 + 2\zeta_r\omega_r s + \omega_r^2} \tag{2.8}$$

where

ω_r is the natural circular/normal mode frequencies for each mode,

ζ_r is the modal damping factor for each mode, and

$[T]_r$ is the residue matrix, which is equal to the product of the normal mode $[[c]\{\phi\}_r]$ and $[\{\phi\}_r^T[b]]$.

Equation (2.8) is generally referred to as the *receptance*, since it gives the relation between the displacement and force. Usually, an alternative formulation known as *inertance* is used, which is the ratio of the acceleration to the force. This formulation is desired, since piezoelectric accelerometers are used for the measurement of the frequency response functions and responses. The inertance can be obtained simply by multiplying the receptance by $-\omega^2$, as follows:

$$[A(s)] = -\omega^2[\alpha(s)] \tag{2.9}$$

In practice, one can measure only a limited number of modes, N, within the frequency range of interest. However, the contribution of the modes outside this frequency range is evident in the measured frequency response function, and needs to be accounted for when one desires to reconstruct measured frequency response functions. Equation (2.8) is rewritten to include the high- and low-frequency corrections or generally referred to as residuals.

$$\left[\alpha(s)\right]=\sum_{j=1}^{N}\left(\frac{\left[T\right]_j}{s^2+2\zeta_j\omega_j s+\omega_j^2}\right)+\left[E\right]+\frac{\left[F\right]}{s^2} \tag{2.10}$$

where

$[E]$ denotes the high-frequency residual, and

$[F]$ is the low-frequency residual.

Figure 2.2 shows a typical reconstructed frequency response function with and without the residual terms included.

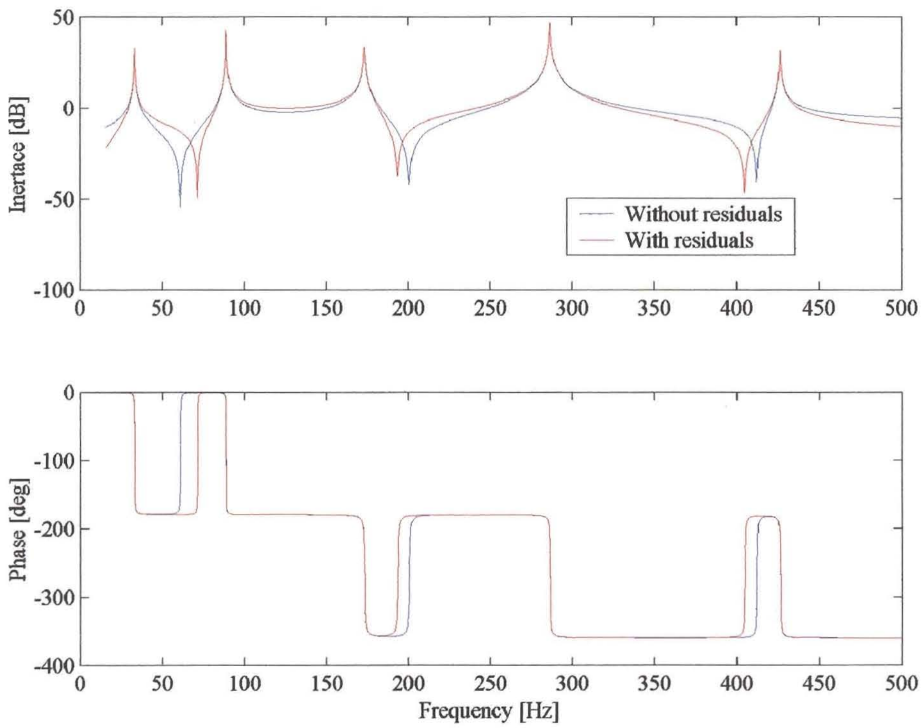

Figure 2.2 – Contribution of the residual terms for a typical system

2.4 MULTIPLE INPUT MULTIPLE OUTPUT EXCITATION

In the case of a single input excitation, the structure is excited sequentially at each of the desired locations, while measuring the responses, from which columns of the frequency response function are obtained successively.

Alternatively, Multiple Input Multiple Output (MIMO) excitation is used in an attempt to obtain the information from several rows or columns of the frequency response function matrix simultaneously during a single excitation run. This does not only reduce the test time required, but also contributes to better estimates of the natural frequencies, modal damping factors and modal vectors in the case of closely spaced modes. A more detailed mathematical treatment of the MIMO excitation can be found in Maia and Silva (1997) and Zaveri (1984).

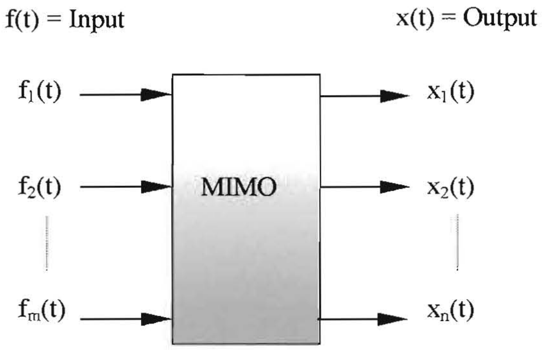

Figure 2.3 – Multiple-input multiple-output model

Consider the above MIMO system, which is excited at m input locations and whose response is measured at n points. The frequency response function matrix for this system can be written as:

$$[H(\omega)]=\begin{bmatrix} H_{11}(\omega) & H_{12}(\omega) & \cdots & H_{1m}(\omega) \\ H_{21}(\omega) & H_{22}(\omega) & \cdots & H_{2m}(\omega) \\ \vdots & \vdots & & \vdots \\ H_{n1}(\omega) & H_{n2}(\omega) & \cdots & H_{nm}(\omega) \end{bmatrix} \tag{2.11}$$

where $H_{ij}(\omega)$ is the frequency response function for excitation at point j and the response measured at point i.

The structure may be excited with two or more exciters simultaneously. For the sake of simplicity we will consider a dual-input, single-output system as shown in Figure 2.4.

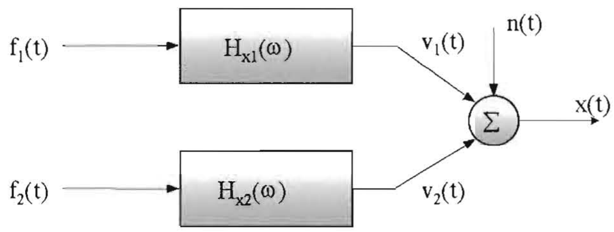

Figure 2.4 – Dual-input, single-output system

The Fourier Transform of the response $X(\omega)$ is given by

$$X(\omega) = H_{x1}(\omega)F_1(\omega) + H_{x2}(\omega)F_2(\omega) + N(\omega) \tag{2.12}$$

where

$F_1(\omega)$ and $F_2(\omega)$ are the Fourier Transform of the inputs at points 1 and 2, respectively,

$N(\omega)$ is the noise contaminating the response.

Assuming that averaging is employed it can be shown that

$$XF_1^* = H_{x1}F_1F_1^* + H_{x2}F_2F_1^* \tag{2.13a}$$

$$XF_2^* = H_{x1}F_1F_2^* + H_{x2}F_2F_2^* \tag{2.13b}$$

where

$*$ denotes the complex conjugate.

These equations can be expressed in terms of the auto- and cross-spectrums as:

$$G_{1x} = H_{x1}G_{11} + H_{x2}G_{12} \tag{2.14a}$$

$$G_{2x} = H_{x1}G_{21} + H_{x2}G_{22} \tag{2.14b}$$

Equation (2.14) can be solved for H_{x1} and H_{x2}, i.e.

$$H_{x1} = \frac{G_{1x}G_{22} - G_{2x}G_{12}}{G_{11}G_{22} - |G_{12}|^2} = \left(\frac{G_{1x}}{G_{11}}\right)\left\{\frac{1 - \dfrac{G_{2x}G_{12}}{G_{1x}G_{22}}}{1-\gamma_{12}^2}\right\} \tag{2.15a}$$

$$H_{x2} = \frac{G_{11}G_{2x} - G_{21}G_{1x}}{G_{11}G_{22} - |G_{12}|^2} = \left(\frac{G_{2x}}{G_{22}}\right)\left\{\frac{1 - \dfrac{G_{2x}G_{12}}{G_{2x}G_{22}}}{1-\gamma_{12}^2}\right\} \qquad (2.15b)$$

providing that the coherence function, γ_{12}^2 between the inputs $F_1(\omega)$ and $F_2(\omega)$ is not equal to unity, i.e.

$$\gamma_{12}^2 = \frac{|G_{12}|^2}{G_{11}G_{22}} \neq 1$$

When $G_{12} = G_{21}^* = 0$, $\gamma_{12}^2 = 0$, equation (2.15) reduces to the single-input expressions

$$H_{x1} = \frac{G_{1x}}{G_{11}} \qquad (2.16a)$$

$$H_{x2} = \frac{G_{2x}}{G_{22}} \qquad (2.16b)$$

As long as the inputs are uncorrelated, equation (2.15) can be used to obtain the frequency response functions when two inputs are acting simultaneously. The above analysis can be extended to apply to any arbitrary number of inputs and outputs.

MIMO Applied To Experimental Setup:

Despite the advantages referred to earlier, associated with the MIMO excitation the use of this type of excitation was motivated by the exciter-structure interaction. The exciter-structure interaction inherently creates difficulties, since the dynamic characteristics of the exciter becomes combined with those of the structure (the exciter adds some of its own mass, stiffness and damping to that of the structure). From experiments conducted on a beam-like structure Han (1998) confirmed that the exciter-structure interaction distorted the natural frequencies and damping values of the structure. In the experimental studies that follow in Chapter 5 two exciters were attached to the beam, in some instances. The distortion of the natural frequencies of the beam due to the two exciters were so severe that the frequency response functions obtained from single-input excitation could not be reconciled with the frequency response functions measured with both exciters attached.

2.5 EXPERIMENTAL MODAL ANALYSIS

The aim of Experimental Modal Analysis is to construct a mathematical model of the structure, which will resemble the characteristics of the experimentally measured data. In the case of measured frequency response functions, one needs to curve-fit an expression to the measured data and thereby finding the appropriate modal parameters (i.e. natural frequencies, damping ratios and mode shapes).

Reducing the frequency response function to terms involving only the modal parameters as in equation (2.10) leads to considerable reduction in the amount of data to be handled. The frequency response function may now be reconstructed for any frequency, simply from use of the modal parameters. Another advantage is that the regenerated frequency response curve is smoother than the experimentally measured data, which always contains noise.

The author made extensive use of the *Structural Dynamic Toolbox*® (Balmès, 1997 (1)) to identify the modal parameters and reconstruct the frequency response functions.

It is not the intention of this work to include a detailed discussion regarding the experimental modal analysis technique. However, the author has spent a considerable amount of time and effort mastering this Toolbox and gaining insight into the technique proposed by Balmès. Only the methodology that has been followed in the analysis will be discussed briefly.

Experimental Modal Analysis Methodology:

Step 1: The measured frequency response function data is imported into the Toolbox, in the desired format.

Step 2: At this stage, the user needs to specify the appropriate type of model that will be used in the identification. The type of model may be either a complex mode model or a normal mode model. An experimentally identified model will have complex eigenvectors. The normal mode model can then be obtained through the use of a transformation procedure, which allows the identification of the normal mode model (i.e. real modes) from the complex mode identification result (Balmès, 1997 (2)).

Step 3: Next, one iteratively computes an approximation of the measured response. This is done in three separate procedures:

a) First, finding initial complex pole estimates. The Toolbox obtains estimates of the poles by searching for the minima of the Multivariable Mode Indicator Function (MMIF) within a frequency region specified by the user. Additional poles may be added or removed to obtain the best fit to the data.

b) Once the user is satisfied with the set of complex poles, the Toolbox continues to estimate the residues and residual terms for the given set of poles.

c) These complex poles and residues are then optimised using a broad or narrow band update algorithm.

Step 4: The poles, damping ratios and complex mode shapes at the sensor locations are extracted from the mathematical model. If the user requires the normal mode model the above-mentioned transformation will be performed and will produce the modal parameters corresponding to the normal mode model.

Step 5: Lastly the frequency response functions may be reconstructed from use of the modal parameters.

The above procedure is depicted in Figure 2.5.

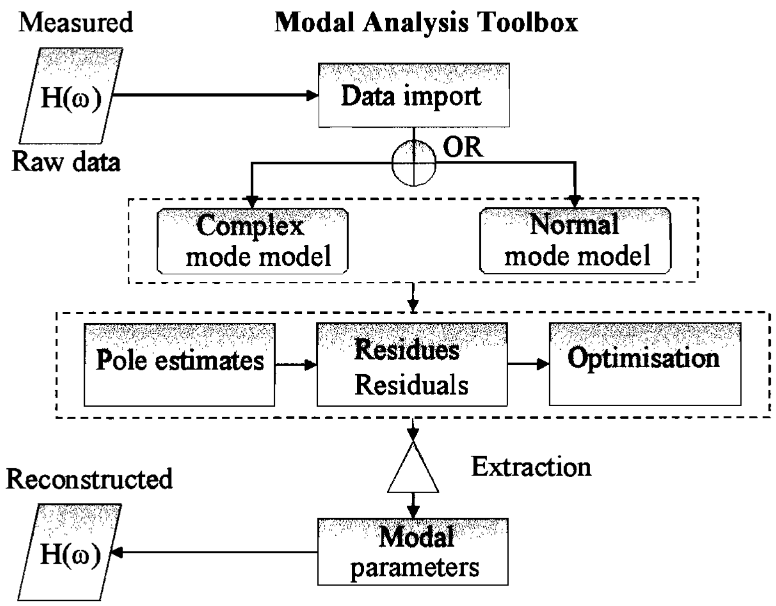

Figure 2.5 - Flowchart of a typical modal analysis and frequency response function reconstruction procedure