Chapter 6

Orthotropic flat shell elements

In this chapter, the constitutive relationship is extended to incorporate orthotropy.

Layered orthotropic materials are particularly demanding in terms of the kinematic requirements of finite elements, since the transverse shear flexibility could be significant. Therefore, a shear flexible through-thickness formulation is called for. The Mindlin theory includes shear deformations, and C^0 continuity of the shape functions only is required.

Hence, the 5β/SA, 8β/SA and 9β/SA elements developed in previous chapters are suitable candidates for orthotropic problems.

6.1 Constitutive relationship

The linear elastic three-dimensional stress-strain relation defined by

$$\sigma_{ij} = E_{ijkl}\epsilon_{kl} \tag{6.1}$$

is used as the basic building brick for laminated orthotropic materials. It is assumed in the plate theory of laminated orthotropic materials that the normal stress in each laminate vanishes, i.e. it is assumed that (6.1) reduces to

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} & 0 & 0 \\ & Q_{22} & Q_{26} & 0 & 0 \\ & & Q_{66} & 0 & 0 \\ & & & Q_{44} & Q_{45} \\ Symm & & & & Q_{55} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \\ \epsilon_{xz} \\ \epsilon_{yz} \end{Bmatrix} \tag{6.2}$$

where the stress-strain relations are written with respect to the reference coordinate system. The laminate staking convention is depicted in Figure 6.1.

Since the orthotropic layers are generally rotated with respect to the reference coordinate axis (see Figure 6.2), Q_{ij} relates the principal directions of the material orthotropy to the reference coordinate system. Q_{ij} is defined by

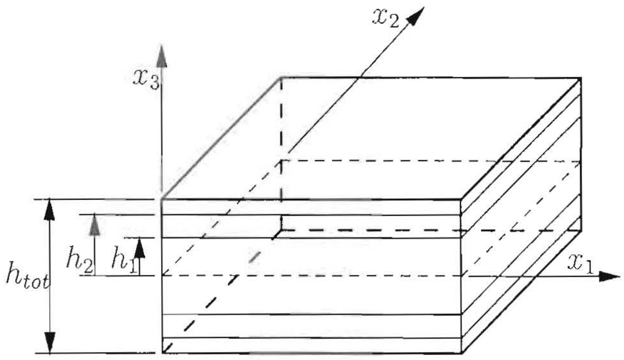

Figure 6.1: Laminate staking convention

$$
\begin{aligned}
Q_{11} &= c^4 E_{1111} + 2c^2 s^2 E_{1122} + s^4 E_{2222} + 4s^2 c^2 E_{1212} \\
Q_{12} &= c^2 s^2 E_{1111} + s^4 E_{1122} + c^4 E_{1122} + s^2 c^2 E_{2222} - 4s^2 c^2 E_{1212} \\
Q_{16} &= sc^3 E_{1111} + s^3 c E_{1122} - sc^3 E_{1122} - s^3 c E_{2222} + 2sc(s^2 - c^2) E_{1212} \\
Q_{22} &= s^4 E_{1111} + 2s^2 c^2 E_{1122} + c^4 E_{2222} + 4s^2 c^2 E_{1212} \\
Q_{26} &= cs^3 E_{1111} + c^3 s E_{1122} - cs^3 E_{1122} - c^3 s E_{2222} + 2sc(c^2 - s^2) E_{1212} \\
Q_{66} &= s^2 c^2 E_{1111} - 2s^2 c^2 E_{1122} + s^2 c^2 E_{2222} + (c^4 - 2s^2 c^2 + s^4) E_{1212} \\
Q_{44} &= c^2 E_{1313} + s^2 E_{2323} \\
Q_{45} &= cs E_{1313} - cs E_{2323} \\
Q_{55} &= s^2 E_{1313} + c^2 E_{2323}
\end{aligned}
\tag{6.3}
$$

where c and s respectively indicate $\cos\theta$ and $\sin\theta$, while θ indicates the fiber ply angle in respect to the positive x_1-axis (See Figure 6.2).

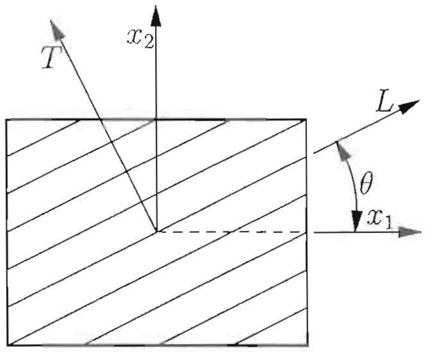

Figure 6.2: Local coordinate system for laminated structures

For orthotropic layered laminates E_{ijkl} are obtainable as

$$E_{1111} = \frac{E_L}{(1 - \nu_{LT}\nu_{TL})}$$

$$
\begin{aligned}
E_{2222} &= \frac{E_T}{(1-\nu_{LT}\nu_{TL})} \\
E_{1122} &= \frac{\nu_{LT}E_T}{(1-\nu_{LT}\nu_{TL})} \\
E_{1212} &= G_{LT} \\
E_{1313} &= G_{LO} \\
E_{2323} &= G_{TO}
\end{aligned} \tag{6.4}
$$

where the subscripts L and T indicate the in-plane longitudinal and transverse directions of the fibers, and O indicates the out-of-plane transverse direction. E_i denote the Young's moduli, G_{ij} denote the shear moduli and ν_{ij} denote the Poisson's ratio's.

Integration of the shell stresses yields the shell resultants as

$$
\begin{aligned}
N_{ij} &= \int_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_{ij}\,\mathrm{d}z && i,j=1,2 \\
M_{ij} &= \int_{\frac{-h}{2}}^{\frac{h}{2}} z\sigma_{ij}\,\mathrm{d}z && i,j=1,2 \\
V_{ij} &= \int_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_{ij}\,\mathrm{d}z && i=1,2;j=3
\end{aligned} \tag{6.5}
$$

For isotropic materials the relevant constitutive relationships are given in (2.13), (4.18) and (4.24) respectively. For orthotropy the constitutive relationships are given by [59]

$$
\begin{aligned}
C_{ij}^m &= \textstyle\sum_{k=1}^{n}(Q_{ij})_k(h_{k+1}-h_k) && i,j=1,2,6 \\
C_{ij}^b &= \tfrac{1}{3}\textstyle\sum_{k=1}^{n}(Q_{ij})_k(h_{k+1}^3-h_k^3) && i,j=1,2,6 \\
C_{ij}^s &= \textstyle\sum_{k=1}^{n}(Q_{ij})_k(h_{k+1}-h_k) && i,j=4,5
\end{aligned} \tag{6.6}
$$

6.2 Compliance matrix

Complementary to the general relationship between stress and strain, (6.1), one can define the inverse relationships as:

$$\epsilon_{ij} = S_{ijkl}\sigma_{kl} \tag{6.7}$$

where S_{ijkl} is the 'compliance tensor'. (6.1) and (6.7) can be rewritten in matrix form

$$\boldsymbol{\sigma} = \boldsymbol{E}\boldsymbol{\epsilon} \tag{6.8}$$

$$\boldsymbol{\epsilon} = \boldsymbol{S}\boldsymbol{\sigma} \tag{6.9}$$

This means that $\boldsymbol{S}$ is the inverse of $\boldsymbol{E}$. Therefore, S_{ijkl} has the same symmetries as E_{ijkl}. (6.7) now reduces to

$$\left\{ \begin{array}{c} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \\ \epsilon_{xz} \\ \epsilon_{yz} \end{array} \right\} = \left[\begin{array}{ccccc} S_{11} & S_{12} & S_{16} & 0 & 0 \\ & S_{22} & S_{26} & 0 & 0 \\ & & S_{66} & 0 & 0 \\ & & & S_{44} & S_{45} \\ Symm & & & & S_{55} \end{array} \right] \left\{ \begin{array}{c} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{array} \right\} \tag{6.10}$$

S_{ij} relates the principal directions of the material orthotropy to the reference coordinate system and are defined by

$$\begin{aligned}
S_{11} &= c^4 S_{1111} + 2c^2 s^2 S_{1122} + s^4 S_{2222} + s^2 c^2 S_{1212} \\
S_{12} &= c^2 s^2 S_{1111} + s^4 S_{1122} + c^4 S_{1122} + s^2 c^2 S_{2222} - s^2 c^2 S_{1212} \\
S_{16} &= 2sc^3 S_{1111} + 2s^3 c S_{1122} - 2sc^3 S_{1122} - 2s^3 c S_{2222} - sc(c^2 - s^2) S_{1212} \\
S_{22} &= s^4 S_{1111} + 2s^2 c^2 S_{1122} + c^4 S_{2222} + s^2 c^2 S_{1212} \\
S_{26} &= 2cs^3 S_{1111} + 2c^3 s S_{1122} - 2cs^3 S_{1122} - 2c^3 s S_{2222} + sc(c^2 - s^2) S_{1212} \\
S_{66} &= 4s^2 c^2 S_{1111} - 8s^2 c^2 S_{1122} + 4s^2 c^2 S_{2222} + (c^4 - 2s^2 c^2 + s^4) S_{1212} \\
S_{44} &= c^2 S_{1313} + s^2 S_{2323} \\
S_{45} &= cs S_{1313} - cs S_{2323} \\
S_{55} &= s^2 S_{1313} + c^2 S_{2323}
\end{aligned} \tag{6.11}$$

where c and s respectively indicate $\cos\theta$ and $\sin\theta$, while θ indicates the fiber ply angle in respect to the positive x_1-axis (See Figure 6.2).

For orthotropic layered laminates S_{ijkl} are obtainable as

$$\begin{aligned}
S_{1111} &= \frac{1}{E_L} \\
S_{2222} &= \frac{1}{E_T} \\
S_{1122} &= \frac{-\nu_{LT}}{E_T} \\
S_{1212} &= \frac{1}{G_{LT}} \\
S_{1313} &= \frac{1}{G_{LO}} \\
S_{2323} &= \frac{1}{G_{TO}}
\end{aligned} \tag{6.12}$$

where the subscripts L and T indicate the in-plane longitudinal and transverse directions of the fibers, and O indicates the out-of-plane transverse direction. E_i denote the Young's moduli, G_{ij} denote the shear moduli and ν_{ij} denote the Poisson's ratio's.