The property of k-colourable graphs is uniquely decomposable

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Abstract

An additive hereditary graph property is a class of simple graphs which is closed under unions, subgraphs and isomorphisms. If $\mathcal{P}_1, \ldots, \mathcal{P}_n$ are graph properties, then a $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-decomposition of a graph G is a partition $E_1, \ldots, E_n$ of $E(G)$ such that $G[E_i]$, the subgraph of G induced by E_i, is in $\mathcal{P}_i$, for $i = 1, \ldots, n$. The sum of the properties $\mathcal{P}_1, \ldots, \mathcal{P}_n$ is the property $\mathcal{P}_1 \oplus \cdots \oplus \mathcal{P}_n = \{G \in \mathcal{I} : G \text{ has a } (\mathcal{P}_1, \ldots, \mathcal{P}_n)\text{-decomposition}\}$. A property $\mathcal{P}$ is said to be decomposable if there exist non-trivial additive hereditary properties $\mathcal{P}_1$ and $\mathcal{P}_2$ such that $\mathcal{P} = \mathcal{P}_1 \oplus \mathcal{P}_2$. A property is uniquely decomposable if, apart from the order of the factors, it can be written as a sum of indecomposable properties in only one way. We show that not all properties are uniquely decomposable; however, the property of k-colourable graphs $\mathcal{O}^k$ is a uniquely decomposable property.

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1. Introduction

For any undefined basic graph theoretical concepts the reader is referred to [3]. The class of all finite simple graphs is denoted by $\mathcal{I}$. A *graph property* is a non-empty isomorphism-closed subclass of $\mathcal{I}$. Notation and terminology of concepts related to graph properties are taken from [1] and of concepts related to products of graphs are taken from [5].

The fact that H is a subgraph of G is denoted by $H \subseteq G$ and $H \leq G$ means that H is an induced subgraph of G. The disjoint union of two graphs G and H is denoted by $G \cup H$. A property $\mathcal{P}$ is called *hereditary* if $G \in \mathcal{P}$ and $H \subseteq G$ implies $H \in \mathcal{P}$; $\mathcal{P}$ is called *induced-hereditary* if $G \in \mathcal{P}$ and $H \leq G$ implies $H \in \mathcal{P}$; $\mathcal{P}$ is called *additive* if $G \cup H \in \mathcal{P}$ whenever $G \in \mathcal{P}$ and $H \in \mathcal{P}$.

Example 1.1. *Some well-known additive hereditary properties are given in the list below.*

$\mathcal{O} = \{G \in \mathcal{I} : E(G) = \emptyset\}$

$\mathcal{S}_k = \{G \in \mathcal{I} : \textit{the maximum degree of } G \textit{ is at most } k\}$

$\mathcal{I}_k = \{G \in \mathcal{I} : G \textit{ does not contain } K_{k+2}\}$

The properties $\mathcal{I}$ and $\mathcal{O}$ are defined to be the *trivial properties* and an edgeless graph is called a *trivial graph*. We use the phrase *G has property $\mathcal{P}$* to denote the fact that $G \in \mathcal{P}$.

2. Decomposability

Let $\mathcal{P}_1, \ldots, \mathcal{P}_n$ be graph properties. A $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-*decomposition* of a graph G is a partition $E_1, \ldots, E_n$ of $E(G)$ such that $G[E_i]$, the subgraph of G induced by E_i, has property $\mathcal{P}_i$, for $i = 1, \ldots, n$. (In this context it is convenient to regard the empty set $\emptyset$ as a set inducing a subgraph with every property $\mathcal{P}$.) We denote by $\mathcal{P}_1 \oplus \cdots \oplus \mathcal{P}_n$ the property $\{G \in \mathcal{I} : G \text{ has a } (\mathcal{P}_1, \ldots, \mathcal{P}_n)\text{-decomposition}\}$. It is easy to see that if $\mathcal{P}_i$ is additive and (induced-)hereditary for every i, then $\mathcal{P}_1 \oplus \cdots \oplus \mathcal{P}_n$ is also additive and (induced-)hereditary.

If $\mathbb{K}$ is a set of properties and $\mathcal{P} \in \mathbb{K}$ then $\mathcal{P}$ is said to be *decomposable in* $\mathbb{K}$ if there exist non-trivial properties $\mathcal{P}_1$ and $\mathcal{P}_2$ in $\mathbb{K}$ such that $\mathcal{P} = \mathcal{P}_1 \oplus \mathcal{P}_2$;

otherwise $\mathcal{P}$ is said to be *indecomposable in* $\mathbb{K}$. We usually use for $\mathbb{K}$ the lattice $\mathbb{L}^a$ of all additive hereditary properties of graphs or the lattice $\mathbb{L}^a_{\leq}$ of all additive induced-hereditary graph properties – see [1] for more details on these lattices.

The property $\mathcal{P} \circ \mathcal{Q}$ is the vertex-analogue of $\mathcal{P} \oplus \mathcal{Q}$. For the sake of completeness we give the necessary definitions: For given properties $\mathcal{P}_1, \ldots, \mathcal{P}_n$, a *vertex* $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$*-partition* of a graph G is a partition $V_1, \ldots, V_n$ of $V(G)$ such that for each $i = 1, \ldots, n$ the induced subgraph $G[V_i]$ has property $\mathcal{P}_i$. The *product* $\mathcal{P}_1 \circ \cdots \circ \mathcal{P}_n$ of the properties $\mathcal{P}_1, \ldots, \mathcal{P}_n$ is now defined as the set of all graphs having a vertex $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partition. Each $\mathcal{P}_i$ is called a *factor* of this product. If $\mathcal{P}_1 = \cdots = \mathcal{P}_n = \mathcal{P}$, then we write $\mathcal{P}^n = \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_n$. As an example we note that $\mathcal{O}^k$ denotes the class of all k-colourable graphs.

A property $\mathcal{R}$ is *reducible* if there are properties $\mathcal{P}$ and $\mathcal{Q}$ such that $\mathcal{R} = \mathcal{P} \circ \mathcal{Q}$; otherwise it is *irreducible*. This paper is motivated by the following unique factorisation theorem [6] (see also [7]).

Theorem 2.1. *Every reducible property $\mathcal{P} \neq \mathcal{I}$ in $\mathbb{L}^a_{\leq}$ is uniquely factorisable into irreducible factors in $\mathbb{L}^a_{\leq}$ (up to the order of the factors).* □

The following result shows that there is no corresponding result for decompositions of properties.

Theorem 2.2. *Let $\mathcal{P}_1 = \{G \in \mathcal{I}$: Every component of G is either a triangle or triangle-free$\}$. Then $\mathcal{P}_1 \oplus \mathcal{S}_1 = \mathcal{I}_1 \oplus \mathcal{S}_1$ from which it follows that $\mathcal{I}_1 \oplus \mathcal{S}_1$ is not uniquely decomposable.*

Proof. For the non-trivial inclusion, let $G \in \mathcal{P}_1 \oplus \mathcal{S}_1$ and let E_1, E_2 be a $(\mathcal{P}_1, \mathcal{S}_1)$-decomposition of $E(G)$. Let E' consist of exactly one edge from each component of $G[E_1]$ isomorphic to K_3 and let $E'' = \{e \in E_2 : e \text{ is adjacent to an edge of } E'\}$. Let $E_1' = (E_1 \setminus E') \cup E''$ and $E_2' = (E_2 \setminus E'') \cup E'$. Clearly $G[E_2'] \in \mathcal{S}_1$. Also, $G[E_1'] \in \mathcal{I}_1$ since it is obtained from the triangle-free graph $F = G[E_1 \setminus E']$ by adding a set of disjoint edges E'' such that every edge in E'' has its vertices in different components of F. □

A similar argument shows that the above example is but a special case of the following: For all positive integers k and m such that $k \leq m$, $\mathcal{S}_k \oplus \mathcal{I}_m = \mathcal{S}_k \oplus \mathcal{P}_m$ where $\mathcal{P}_m = \{G \in \mathcal{I}$: Every component of G is either a K_{m+2} or K_{m+2}-free$\}$.

3. The unique decomposability of $\mathcal{O}^k$

In order to prove that $\mathcal{O}^k$ is uniquely decomposable in $\mathbb{L}^a_{\leq}$ we need a few results on homomorphism properties.

A *homomorphism* of a graph G to a graph H is a function f from $V(G)$ into $V(H)$ such that if $uv \in E(G)$ then $f(u)f(v) \in E(H)$; if such a function exists, we write $G \to H$. For a given graph H we denote by $\to H$ the (additive hereditary) property $\{G \in \mathcal{I} : G \to H\}$. $\to H$ is called a *hom property*.

The *disjunction* of two graphs G and H, denoted by $G \vee H$, is the graph with vertex set $V(G) \times V(H) = \{(g,h) : g \in V(G) \text{ and } h \in V(H)\}$ and edge set $\{(g_1,h_1)(g_2,h_2) : g_1g_2 \in E(G) \text{ or } h_1h_2 \in E(H)\}$.

Using the standard notation $\overline{H}$ for the complement of a graph H we write $G[n]$ for $G \vee \overline{K_n}$ and call $G[n]$ a *multiplication* of G.

Some basic properties of the disjunction, multiplications and homomorphism properties are given below.

Lemma 3.1. *For all graphs G, H and F and positive integers k and n:*

1. $G \vee H = H \vee G$.
2. $(G \vee H) \vee F = G \vee (H \vee F)$.
3. $G \to H$ *iff* $G \subseteq H[k]$ *for some* k.
4. $\to G = \to H$ *iff* $G \to H$ *and* $H \to G$.
5. $\to H = \to H[k]$.
6. $\mathcal{O}^k = \to K_k$. □

Theorem 3.2. *Let G and H be graphs. Then* $\to G \oplus \to H = \to (G \vee H)$.

Proof. First we show that $G \vee G' \in \to G \oplus \to G'$ for all G'. An appropriate $(\to G, \to G')$-decomposition E_1, E_2 of $G \vee G'$ is given by letting $(u_1,v_1)(u_2,v_2) \in E_1$ iff $u_1u_2 \in E(G)$.

In order to prove now that $\to (G \vee H) \subseteq \to G \oplus \to H$ we suppose that $K \in \to (G \vee H)$. Then, by Lemma 3.1(3), $K \subseteq (G \vee H)[k]$ for some k. But, by the definition of $G[k]$ and Lemma 3.1(2), $(G \vee H)[k] = (G \vee H) \vee \overline{K_k} = G \vee (H \vee \overline{K_k}) = G \vee (H[k])$. Therefore, with $G' = H[k]$, it follows that $K \in \to G \oplus \to H[k] = \to G \oplus \to H$, using Lemma 3.1(5).

Now suppose that $F \in \to G \oplus \to H$ and let E_1, E_2 be a $(\to G, \to H)$-decomposition of F. Then there exist homomorphisms $g : (V(F), E_1) \to G$ and $h : (V(F), E_2) \to H$. Now define $f : F \to G \vee H$ by $f(v) = (g(v), h(v))$ for all $v \in V(F)$. In order to show that f is a homomorphism, let $uv \in E(F)$. Then $f(u)f(v) = (g(u), h(u))(g(v), h(v))$. If $uv \in E_1$ then $g(u)g(v) \in E(G)$ hence $f(u)f(v) \in E(G \vee H)$. Similarly, if $uv \in E_2$ then $f(u)f(v) \in E(G \vee H)$. Therefore f is a homomorphism, proving that $F \in \to (G \vee H)$. □

Corollary 3.3. *For all positive integers a and b, $\mathcal{O}^{ab} = \mathcal{O}^a \oplus \mathcal{O}^b$.*

Proof. $\mathcal{O}^{ab} = \to K_{ab} = \to (K_a \vee K_b) = \to K_a \oplus \to K_b = \mathcal{O}^a \oplus \mathcal{O}^b$. $\square$

For graphs G and H we define the *lexicographic product* $H \circ G$ of G and H to be the graph with vertex set $V(H) \times V(G)$ and edge set $\{(u_1, v_1)(u_2, v_2) : u_1 = u_2 \text{ and } v_1v_2 \in E(G) \text{ or } u_1u_2 \in E(H)\}$. We let $H \circ \mathcal{P}$ be the class of all subgraphs of graphs of the form $H \circ G, G \in \mathcal{P}$.

The edges of the lexicographic product $H \circ G$ of two graphs H and G take the following two forms:

- For a given vertex $u_1 \in V(H)$, the edges of the form $(u_1, v_1)(u_1, v_2)$ with $v_1v_2 \in E(G)$; these we call *edges of type* u_1.
- For a given edge $u_1u_2 \in E(H)$, the edges of the form $(u_1, v_1)(u_2, v_2)$ with $v_1, v_2 \in V(G)$; these we call *edges of type* u_1u_2.

A colouring of the edge set $E(F)$ of a subgraph F of $H \circ G$ is called *good* if, for each $u_1 \in V(H)$, all the edges of type u_1 have the same colour and, for each $u_1u_2 \in E(H)$, all the edges of type u_1u_2 have the same colour. (For different vertices (edges) of H, the colours of the edges of the type associated with these vertices (edges respectively) need not be the same.)

Next we consider two graphs $F \subseteq H \circ G$ and $F' \subseteq H \circ G'$. If there is an isomorphism $f : V(F) \to V(F')$ of F onto F' such that, for all $(u, v) \in V(F)$, $f(u, v) \in \{u\} \times V(G')$, then we say that f is *position-sensitive* and we write $F \cong_{ps} F'$.

With F and F' as in the previous paragraph (but not necessarily isomorphic), we write $F' \to^g F$ if for every 2-colouring of $E(F')$ there is an induced subgraph $K \leq F'$ such that the inherited colouring of $E(K)$ is a good colouring and $F \cong_{ps} K$. $F' \to^g F$ means that, with respect to any 2-edge colouring of F', there is a well-coloured position-sensitive copy of F in F'.

A property $\mathcal{P} \in \mathbb{L}^a_{\leq}$ is called *H-Ramsey* if for every $F \in H \circ \mathcal{P}$ there is an $F' \in H \circ \mathcal{P}$ such that $F' \to^g F$; if $H = K_2$ it is called a *bipartite Ramsey* property. The well-known Bipartite Ramsey Lemma (see for instance Lemma 9.3.3 of [4]) states that the property $\mathcal{O}$ is bipartite Ramsey.

Lemma 3.4. *Let $\mathcal{P}$ be a bipartite Ramsey property and let H be any graph. Then $\mathcal{P}$ is H-Ramsey.*

Proof. We imitate the partite construction due to Nešetřil and Rödl in [8] where the special case with $\mathcal{P} = \mathcal{O}$ (and $H = K_n$) is proved. We first prove the following statement: For any $e = u_1u_2 \in E(H)$ and $G \in H \circ \mathcal{P}$ there is a $G' \in H \circ \mathcal{P}$ such that $G' \to^e G$, where we mean by this notation that for any 2-colouring of $E(G')$ there is a $K \leq G'$ such that $G \cong_{ps} K$ and all type u_1u_2 edges have the same colour, all type u_1 edges have the same colour, and all type u_2 edges have the same colour in the 2-colouring K inherits from G'.

We construct G' as follows: For $i = 1, 2$, let $V_i = \{(u, v) \in V(G) : u = u_i\}$. Let $B \in K_2 \circ \mathcal{P}$ be the subgraph of G induced by $V_1 \cup V_2$. Since $\mathcal{P}$ is bipartite Ramsey, there exists a $B' \in K_2 \circ \mathcal{P}$ such that $B' \to^g B$. For every induced subgraph B'' of B' such that $B'' \cong_{ps} B$ we add a copy of $G - E(B)$ to B' and we identify the vertices corresponding to vertices of $V_1 \cup V_2$ with the corresponding vertices of B''. It is easy to see that G' has the required properties.

Now let $E(H) = \{e_1, \ldots, e_m\}$. For any $G \in \mathcal{P}$, we repeat the above construction to obtain graphs $G_1, \ldots, G_m$ such that $G_m \to^{e_m} G_{m-1} \to^{e_{m-1}} G_{m-2} \to^{e_{m-2}} \cdots \to^{e_2} G_1 \to^{e_1} G$ from which it follows that $G_m \to^g G$. $\square$

In our next result we use the notation $H = H_1 \uplus H_2$ to denote that $V(H) = V(H_1) = V(H_2)$ and $E(H) = E(H_1) \cup E(H_2)$, with $E(H_1) \cap E(H_2) = \emptyset$.

Theorem 3.5. *Let $\to H \subseteq \mathcal{P} \oplus \mathcal{Q}$, $\mathcal{P}, \mathcal{Q} \in \mathbb{L}^a_{\leq}$. Then there exist graphs H_1 and H_2 such that $\to H \subseteq \to H_1 \oplus \to H_2$ with $\to H_1 \subseteq \mathcal{P}$, $\to H_2 \subseteq \mathcal{Q}$ and $H = H_1 \uplus H_2$.*

Proof. Let G be any graph in $\to H$. Then $G \subseteq H[k] = H \circ \overline{K_k}$ for some k. By Lemma 3.4 (with $\mathcal{P} = \mathcal{O}$), there exists a graph $G' \subseteq H \circ \overline{K_\ell}$, for some ℓ, such that $G' \to^g G$. Then $G' \in \to H$, so that $G' \in \mathcal{P} \oplus \mathcal{Q}$. Consider therefore any $(\mathcal{P}, \mathcal{Q})$-colouring c of $E(G')$. By the Lemma there is a K such that c restricted to $E(K)$ is a good colouring of K and $K \cong_{ps} G$. Therefore every $G \in \to H$ has a good $(\mathcal{P}, \mathcal{Q})$-colouring, if we regard G as a subgraph of $H \circ \overline{K_k}$ for some k.

Any such good colouring induces a colouring of $E(H)$ in a natural way. Since there are finitely many colourings of $E(H)$ there is a colouring $c' = E_1, E_2$ of $E(H)$ such that every graph $G \in \to H$ has a good $(\mathcal{P}, \mathcal{Q})$-colouring that induces c'. (Otherwise we could find a disjoint union of finitely many graphs in $\to H$ with no good $(\mathcal{P}, \mathcal{Q})$-colouring.) Set $H_1 = (V(H), E_1)$ and $H_2 = (V(H), E_2)$. Clearly, $H \in \to H_1 \oplus \to H_2$ and since $\to H_1 \oplus \to H_2$ is a hom-property by Theorem 3.2, it follows that $\to H \subseteq \to H_1 \oplus \to H_2$.

By the choice of c', $\to H_1 \subseteq \mathcal{P}$ and $\to H_2 \subseteq \mathcal{Q}$, and we clearly have $H = H_1 \uplus H_2$. $\square$

Corollary 3.6. *For any graph H, if $\to H$ is decomposable in $\mathbb{L}^a_{\leq}$ then $\to H$ is decomposable in $\mathcal{HOM} = \{\to H : H \in \mathcal{I}\}$.* $\square$

The next result is useful in the proof of our main result. Here we use the following standard notation: $\omega(G)$ is the *clique number* of a graph G, $\chi(G)$ is the *chromatic number* of G and $\alpha(G)$ is the *independence number* of G.

Lemma 3.7. *Let G and H be graphs. Then*

1. $\omega(G \vee H) \leq \omega(G)\chi(H) \leq \chi(G \vee H)$.
2. $\alpha(G \vee H) = \alpha(G)\alpha(H)$.
3. $\rightarrow H = \mathcal{O}^k$ *iff* $\omega(H) = \chi(H) = k$.

Proof.

1. In order to prove the first inequality, let K be a complete subgraph of $G \vee H$ and let F be any edgeless induced subgraph of H. Then $|V(K) \cap (V(G) \times V(F))| \leq \omega(G)$ since $G \vee F = G[d]$ with $d = |V(F)|$, and $\omega(G[d]) = \omega(G)$. Since $V(H)$ can be partitioned into $\chi(H)$ independent sets it follows that $|V(K)| \leq \omega(G)\chi(H)$.
   For the second inequality we take any complete subgraph K of G of order $\omega(G)$. Then $\chi(K \vee H) = \omega(G)\chi(H)$ and $K \vee H \subseteq G \vee H$.
2. If $K = \{(g_1, h_1), \dots, (g_k, h_k)\}$ is an independent subset of $V(G \vee H)$ then $K_G = \{g_1, \dots, g_k\}$ and $K_H = \{h_1, \dots, h_k\}$ are independent subsets of $V(G)$ and $V(H)$, respectively. Then $|K| \leq |K_G \times K_H| = |K_G||K_H| \leq \alpha(G)\alpha(H)$. Also, if K_1 and K_2 are independent subsets of G and H, respectively, then $K_1 \times K_2$ is an independent subset of $G \vee H$, hence $\alpha(G \vee H) = \alpha(G)\alpha(H)$.
3. If $\rightarrow H = \mathcal{O}^k$ then $k \leq \omega(H) \leq \chi(H) \leq k$. If $\omega(H) = \chi(H) = k$ then $H \rightarrow K_k \rightarrow H$ hence $\rightarrow H = \mathcal{O}^k$ by Lemma 3.1. □

Theorem 3.8. *Let $p_1, \dots, p_n$ be prime numbers and let $k = p_1 \cdots p_n$. Then the property $\mathcal{O}^k$ has the unique decomposition $\mathcal{O}^{p_1} \oplus \cdots \oplus \mathcal{O}^{p_n}$ in $\mathbb{L}^a_{\leq}$.*

Proof. Let k be any positive integer. We show that if $\mathcal{O}^k = \mathcal{P} \oplus \mathcal{Q}$, with $\mathcal{P}, \mathcal{Q} \in \mathbb{L}^a_{\leq}$, then there exists an integer a such that $\mathcal{P} = \mathcal{O}^a$. Then, if $\mathcal{O}^k = \mathcal{P}_1 \oplus \cdots \oplus \mathcal{P}_m$ with $\mathcal{P}_i$ indecomposable for every i, it follows that for every i, $\mathcal{P}_i = \mathcal{O}^{q_i}$ for some q_i. Since $\mathcal{P}_i$ is indecomposable q_i must be prime by Corollary 3.3. The result then follows from the unique factorisation of integers and Corollary 3.3.

Suppose therefore that $\mathcal{O}^k = \mathcal{P} \oplus \mathcal{Q}$, $\mathcal{P}, \mathcal{Q} \in \mathbb{L}^a_{\leq}$. Since $\mathcal{O}^k = \rightarrow K_k$ we have, by Theorem 3.5 and Theorem 3.2, that there exist H_1 and H_2 such that $\mathcal{O}^k = \rightarrow (H_1 \vee H_2)$, $\rightarrow H_1 \subseteq \mathcal{P}$, $\rightarrow H_2 \subseteq \mathcal{Q}$ and $H_1 \uplus H_2 = K_k$. First we show that $\rightarrow H_1 = \mathcal{O}^a$ for some a. By Lemma 3.7 we must show that $\omega(H_1) = \chi(H_1)$: By the same lemma we have that $k = \omega(H_1 \vee H_2) \leq \omega(H_2)\chi(H_1) \leq \chi(H_1 \vee H_2) = k$, hence $k = \omega(H_2)\chi(H_1)$. Also, since $H_1 \uplus H_2 = K_k$, we have that $\overline{H_1} = H_2$ so that $\omega(H_1) = \alpha(H_2)$ and $\omega(H_2) = \alpha(H_1)$. Now, $k = \chi(H_1 \vee H_2) \geq \frac{|V(H_1 \vee H_2)|}{\alpha(H_1 \vee H_2)} = \frac{|V(H_1)||V(H_2)|}{\alpha(H_1)\alpha(H_2)} = \frac{k^2}{\omega(H_2)\omega(H_1)}$.

Hence $\omega(H_1) \geq \frac{k}{\omega(H_2)} = \chi(H_1)$, from which it follows that $\omega(H_1) = \chi(H_1)$.

Similarly, $\rightarrow H_2 = \mathcal{O}^b$ for some b. Since $\mathcal{O}^k = \rightarrow H_1 \oplus \rightarrow H_2$ it follows that $k = ab$. Suppose now that $\mathcal{O}^a \subset \mathcal{P}$ and let $G \in \mathcal{P}$ be such that $\chi(G) > a$.

Then the graph $F = G \vee K_b$ has chromatic number greater than $ab = k$ but $F \in \mathcal{P} \oplus \mathcal{Q}$, a contradiction. Therefore $\mathcal{P} = \mathcal{O}^a$. □

4. Conclusion

It would be of interest to characterise those properties which are uniquely decomposable in $\mathbb{L}^a$ (or $\mathbb{L}^a_{\leq}$). In particular, it is easy to see that for every product of properties $\mathcal{P}^k$ we have $\mathcal{P}^k = \mathcal{P} \oplus \mathcal{O}^k$, and hence $\mathcal{P} \oplus \mathcal{O}^{p_1} \oplus \cdots \oplus \mathcal{O}^{p_n}$ if $k = p_1 \cdots p_n$, and the following question arises: For which indecomposable $\mathcal{P}$ is this the unique decomposition of $\mathcal{P}^k$ into indecomposable properties?

We can construct a hom property $\to H$ which does not have a unique decomposition into indecomposable properties, even if we restrict the properties to hom properties. Our proof relies on the fact that the complementary graph $\overline{H}$ is disconnected. We do not know if there is such a graph H with a connected complement.

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