TRIPLY NONCENTRAL BIVARIATE BETA TYPE V DISTRIBUTION

R. Ehlers
Department of Statistics, Faculty of Natural and Agricultural Sciences,
University of Pretoria, South Africa
and
A. Bekker
Department of Statistics, Faculty of Natural and Agricultural Sciences,
University of Pretoria, South Africa
and
J.J.J. Roux
Department of Statistics, Faculty of Natural and Agricultural Sciences,
University of Pretoria, South Africa

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Summary: The *enriched* triply noncentral bivariate beta type V distribution is introduced. This distribution is constructed from independent chi-squared random variables by using the variables-in-common (or trivariate reduction) technique. The marginal density, product moment and the distribution of the product of the correlated components of this distribution are also derived. The effect of the additional parameters on the shape of the density functions and the correlation between the correlated variables is shown. Special cases are highlighted to position this distribution in the bivariate beta distributions context.

1. Introduction

In this paper we introduce the *triply noncentral bivariate beta type V distribution* by letting $S_1 \sim \chi^2(n_1;\delta_1)$, $S_2 \sim \chi^2(n_2;\delta_2)$ and $B \sim$

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$\chi^2(m;\delta_3)$ be independent noncentral chi-square random variables and defining

$$(W_1, W_2) = \left(\frac{\alpha_1 S_1}{\alpha_1 S_1 + \alpha_2 S_2 + cB}, \frac{\alpha_2 S_2}{\alpha_1 S_1 + \alpha_2 S_2 + cB}\right). \tag{1}$$

The transformation in (1) can also be expressed as

$$(W_1, W_2) = \left(\frac{S_1^*}{S_1^* + S_2^* + B^*}, \frac{S_2^*}{S_1^* + S_2^* + B^*}\right)$$

where S_1^*, S_2^* and B^* are independent noncentral gamma variables, i.e. $S_1^* \sim Gam\left(2\alpha_1, \frac{n_1}{2}; \delta_1\right)$, $S_2^* \sim Gam\left(2\alpha_2, \frac{n_2}{2}; \delta_2\right)$ and $B^* \sim Gam\left(2c, \frac{m}{2}; \delta_3\right)$. The variables (W_1, W_2) are said to have the triply noncentral bivariate beta type V distribution and its density function is given by

$$\begin{aligned}
& f_{ncBB^V}\left(w_1, w_2; \tfrac{n_1}{2}, \tfrac{n_2}{2}, \tfrac{m}{2}, \alpha_1, \alpha_2, c; \delta_1, \delta_2, \delta_3\right) \\
\propto \quad & w_1^{\frac{1}{2}n_1 - 1} w_2^{\frac{1}{2}n_2 - 1} (1 - w_1 - w_2)^{\frac{1}{2}m - 1} \\
& \cdot \left(1 + \tfrac{c-\alpha_1}{\alpha_1} w_1 + \tfrac{c-\alpha_2}{\alpha_2} w_2\right)^{-\frac{1}{2}(n_1+n_2+m)} \\
& \cdot \boldsymbol{\Psi}_2^{(3)} \left[\tfrac{n_1+n_2+m}{2}; \tfrac{n_1}{2}, \tfrac{n_2}{2}, \tfrac{m}{2}; \tfrac{\delta_1 c}{2\alpha_1} \tfrac{w_1}{1+\frac{c-\alpha_1}{\alpha_1}w_1 + \frac{c-\alpha_2}{\alpha_2}w_2}, \right. \\
& \left. \tfrac{\delta_2 c}{2\alpha_2} \tfrac{w_2}{1+\frac{c-\alpha_1}{\alpha_1}w_1 + \frac{c-\alpha_2}{\alpha_2}w_2}, \tfrac{\delta_3}{2} \tfrac{1-w_1-w_2}{1+\frac{c-\alpha_1}{\alpha_1}w_1 + \frac{c-\alpha_2}{\alpha_2}w_2}\right],
\end{aligned} \tag{2}$$

where $0 \leq w_1, w_2 \leq 1$, $w_1 + w_2 \leq 1$ and $\boldsymbol{\Psi}_2^{(3)}$ is the confluent hypergeometric function in three variables (see Sánchez et al., 2006). Some statistical properties as well as the distribution of the product of the correlated components of this distribution are also derived in this paper.

The triply noncentral bivariate beta type V distribution with additional parameters α_1, α_2, c and δ_i $(i = 1, 2, 3)$, allows for great flexibility in modeling, thus it responds to the need for a parameter rich family of distributions that have been expressed in literature (Bekker, 1990). The importance of noncentral distributions have been emphasized by several authors (Gupta et al., 2009; Kotz et al., 2000; Sánchez et al., 2006). One of the important applications of the noncentral distribution is to calculate the power of the test of a specific hypothesis.

Several known bivariate beta distributions are special cases of this newly defined distribution with density function given in (2). These bivariate beta distributions mainly arise in the context of a trivariate reduction of three quantities that must sum to 1, that are mutually exclusive and collectively exhaustive (Balakrishnan and Lai, 2009). Examples include probabilities of events, modeling of the proportion of substances in a mixture and brand shares (Chatfield, 1975). Known bivariate beta distributions that are special cases of (2) are divided into two main groups depending depending on whether S_1, S_2 and B have central or noncentral chi-square distributions. Within each of these two groups distributions can be distinguished based on the values of the parameters α_1, α_2 and c in (1). These parameters α_1, α_2 and c can be considered as pathway parameters since it facilitates a transition to the other well known bivariate beta distributions. The following gives a summary of the known bivariate beta distributions that are special cases of (2).

(a) Central distributions ($\delta_1 = \delta_2 = \delta_3 = 0$):

(i) general α_1, α_2, c: bivariate beta type V distribution (Craiu and Craiu, 1969; Rogers and Young, 1973; Nadarajah and Kotz, 2005);

(ii) $\alpha_1 = \alpha_2 = c = 1$: bivariate beta type I distribution (Balakrishnan and Lai, 2009);

(iii) $\alpha_1 = \alpha_2 = 1$: bivariate beta type III distribution (Cardeño et al., 2005 considered $c = 2$).

(b) Noncentral distributions:

(i) $\alpha_1 = \alpha_2 = c = 1$, $\delta_1 = \delta_2 = 0$: noncentral bivariate beta type I distribution (Troskie, 1967);

(ii) $\alpha_1 = \alpha_2 = c = 1$: triply noncentral bivariate beta type I distribution (Sánchez et al., 2006).

The bivariate beta type I distribution is extensively used as a prior in Bayesian statistics (Apostolakis and Moieni, 1987). It serves as the natural conjugate prior for the multinomial distribution where the variables are negatively correlated. In some practical cases random variables may be positively correlated, hence the bivariate beta type I distribution will not be a reasonable choice to be a prior distribution. It will be shown in this paper that the triply noncentral bivariate beta type V distribution accommodates positive correlation for specific choices of the additional parameters and can be used as an alternative in Bayesian analysis. Bodvin et al. (2010) illustrated the use of the bivariate beta type V distribution in the Bayes context. They proposed the use of Shannon entropy when determining the parameters of prior bivariate beta distributions as part of a Bayesian calibration methodology and illustrated the appropriateness of this bivariate beta distribution on Moody's default rate data because of its ability to deal with positive correlation in the underlying data.

Further, we also derive the distribution of $Z = W_1W_2$, the product of the *correlated* variables of the triply noncentral bivariate beta type V distribution. Nagar et al. (2009) studied the importance of these type of distributions where the variables are correlated while Pham-Gia (2000) and Pham-Gia and Turkkan (2002) give applications for the product of *independent* beta variables in the field of reliability.

In Section 2 the transformation in (1) is used to derive the density function of the triply noncentral bivariate beta type V distribution. The effect of the additional and noncentrality parameters on the form of the density of the triply noncentral bivariate beta type V distribution is illustrated. In Section 3 several properties of this distribution, including the marginal distribution and product moments are studied and the effect of the parameters on the correlation between the variables W_1 and W_2 also receives attention. Finally, the distribution of the product of the components of the triply noncentral bivariate beta type V variables is derived in Section 4 and graphs of this density function for several values of the parameters are shown.

2. Triply noncentral bivariate beta type V distribution

In this section the newly proposed triply noncentral bivariate beta type V distribution will be derived from the transformation in (1) and its position relative to the other bivariate beta distributions is given.

Theorem 1

Let $S_1 \sim \chi^2(n_1;\delta_1)$, $S_2 \sim \chi^2(n_2;\delta_2)$ and $B \sim \chi^2(m;\delta_3)$ be independently distributed. Define

$$(W_1, W_2) = \left(\tfrac{\alpha_1 S_1}{\alpha_1 S_1+\alpha_2 S_2+cB}, \tfrac{\alpha_2 S_2}{\alpha_1 S_1+\alpha_2 S_2+cB}\right), \tag{3}$$

$\alpha_i > 0, c > 0,\ 0 < w_i < 1,\ i = 1, 2$ and $0 < w_1 + w_2 < 1.$

The density function of (W_1, W_2) is given by

$$\begin{aligned}
& f_{ncBB^V}\left(w_1, w_2; \tfrac{n_1}{2}, \tfrac{n_2}{2}, \tfrac{m}{2}, \alpha_1, \alpha_2, c; \delta_1, \delta_2, \delta_3\right) \\
= \quad & \left\{\beta\left(\tfrac{n_1}{2}, \tfrac{n_2}{2}, \tfrac{m}{2}\right)\right\}^{-1} e^{-\frac{1}{2}(\delta_1+\delta_2+\delta_3)} \left(\tfrac{c}{\alpha_1}\right)^{\frac{1}{2}n_1} \left(\tfrac{c}{\alpha_2}\right)^{\frac{1}{2}n_2} \\
& \cdot w_1^{\frac{1}{2}n_1-1} w_2^{\frac{1}{2}n_2-1} (1-w_1-w_2)^{\frac{1}{2}m-1} \\
& \cdot \left(1+\tfrac{c-\alpha_1}{\alpha_1}w_1+\tfrac{c-\alpha_2}{\alpha_2}w_2\right)^{-\frac{1}{2}(n_1+n_2+m)} \\
& \cdot \boldsymbol{\Psi}_2^{(3)}\left[\tfrac{n_1+n_2+m}{2}; \tfrac{n_1}{2}, \tfrac{n_2}{2}, \tfrac{m}{2}; \tfrac{\delta_1 c}{2\alpha_1}\tfrac{w_1}{1+\frac{c-\alpha_1}{\alpha_1}w_1+\frac{c-\alpha_2}{\alpha_2}w_2},\right. \\
& \left.\tfrac{\delta_2 c}{2\alpha_2}\tfrac{w_2}{1+\frac{c-\alpha_1}{\alpha_1}w_1+\frac{c-\alpha_2}{\alpha_2}w_2}, \tfrac{\delta_3}{2}\tfrac{1-w_1-w_2}{1+\frac{c-\alpha_1}{\alpha_1}w_1+\frac{c-\alpha_2}{\alpha_2}w_2}\right].
\end{aligned} \tag{4}$$

The distribution in (4) is the triply noncentral bivariate beta type V distribution and is denoted as $(W_1, W_2) \sim ncBB^V\left(\frac{n_1}{2}, \frac{n_2}{2}, \frac{m}{2}, \alpha_1, \alpha_2, c; \delta_1, \delta_2, \delta_3\right)$.

Proof:

The density function of (S_1, S_2, B) is given by

$$\begin{aligned}
& K e^{-\frac{1}{2}(\delta_1+\delta_2+\delta_3)} s_1^{\frac{1}{2}n_1-1} s_2^{\frac{1}{2}n_2-1} b^{\frac{1}{2}m-1} e^{-\frac{1}{2}(s_1+s_2+b)} \\
& \cdot {}_0F_1\left(\tfrac{n_1}{2}; \tfrac{\delta_1}{4}s_1\right) {}_0F_1\left(\tfrac{n_2}{2}; \tfrac{\delta_2}{4}s_2\right) {}_0F_1\left(\tfrac{m}{2}; \tfrac{\delta_3}{4}b\right)
\end{aligned} \tag{5}$$

where $K^{-1} = \Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{m}{2}\right)2^{\frac{1}{2}(n_1+n_2+m)}$ and ${}_0F_1(a; z) = \sum\limits_{j=0}^{\infty} \frac{\Gamma(a)}{\Gamma(a+j)}\frac{z^j}{j!}$. Considering the transformation in (3) and letting $Z = \alpha_1 S_1 + \alpha_2 S_2 + cB$, the Jacobian is $J((s_1, s_2, b) \to (w_1, w_2, z)) = \frac{z^2}{\alpha_1\alpha_2 c}$. Substituting in (5), we obtain the joint density of W_1, W_2 and Z as

$$\begin{aligned}
& f(w_1, w_2, z) \\
= \quad & K e^{-\frac{1}{2}(\delta_1+\delta_2+\delta_3)} \alpha_1^{-\frac{1}{2}n_1} \alpha_2^{-\frac{1}{2}n_2} c^{-\frac{1}{2}m} z^{\frac{1}{2}(n_1+n_2+m)-1} w_1^{\frac{1}{2}n_1-1} w_2^{\frac{1}{2}n_2-1} \\
& \cdot (1-w_1-w_2)^{\frac{1}{2}m-1} \\
& \cdot \exp\left[-\tfrac{1}{2c}z\left(1+\tfrac{c-\alpha_1}{\alpha_1}w_1+\tfrac{c-\alpha_2}{\alpha_2}w_2\right)\right] {}_0F_1\left(\tfrac{n_1}{2}; \tfrac{\delta_1}{4}\tfrac{zw_1}{\alpha_1}\right) \\
& \cdot {}_0F_1\left(\tfrac{n_2}{2}; \tfrac{\delta_2}{4}\tfrac{zw_2}{\alpha_2}\right) {}_0F_1\left(\tfrac{m}{2}; \tfrac{\delta_3}{4}\tfrac{z(1-w_1-w_2)}{c}\right).
\end{aligned} \tag{6}$$

The joint density of W_1 and W_2 can be obtained by integrating (6) with respect to z. This gives the following integral that must be solved:

$$\begin{aligned} I &= \int_0^\infty z^{\frac{1}{2}(n_1+n_2+m)-1} \exp\left[-\tfrac{1}{2c} z\left(1+\tfrac{c-\alpha_1}{\alpha_1}w_1+\tfrac{c-\alpha_2}{\alpha_2}w_2\right)\right] \\ &\quad \cdot {}_0F_1\left(\tfrac{n_1}{2}; \tfrac{\delta_1}{4}\tfrac{zw_1}{\alpha_1}\right) {}_0F_1\left(\tfrac{n_2}{2}; \tfrac{\delta_2}{4}\tfrac{zw_2}{\alpha_2}\right) \\ &\quad \cdot {}_0F_1\left(\tfrac{m}{2}; \tfrac{\delta_3}{4}\tfrac{z(1-w_1-w_2)}{c}\right) dz. \end{aligned} \tag{7}$$

Setting $t = \frac{1}{2c} z\left(1+\frac{c-\alpha_1}{\alpha_1}w_1+\frac{c-\alpha_2}{\alpha_2}w_2\right)$ and using the following result (Sánchez et al., 2006)

$$\mathbf{\Psi}_2^{(m)}\left[a; c_1, \ldots, c_m; z_1, \ldots, z_m\right] = \frac{1}{\Gamma(a)} \int_0^\infty \exp(-t)\, t^{a-1} \prod_{i=1}^m F_1(c_i; tz_i)\, dt$$

where $\mathbf{\Psi}_2^{(m)}$ is the confluent hypergeometric function in m variables, the integral in (7) equals

$$\begin{aligned} & (2c)^{\frac{1}{2}(n_1+n_2+m)} \int_0^\infty t^{\frac{1}{2}(n_1+n_2+m)-1} e^{-t} \\ & \cdot \left(1+\tfrac{c-\alpha_1}{\alpha_1}w_1+\tfrac{c-\alpha_2}{\alpha_2}w_2\right)^{-\frac{1}{2}(n_1+n_2+m)} \\ & \cdot {}_0F_1\left(\tfrac{n_1}{2}; \tfrac{\delta_1 c}{2\alpha_1}\tfrac{w_1}{1+\frac{c-\alpha_1}{\alpha_1}w_1+\frac{c-\alpha_2}{\alpha_2}w_2} t\right) {}_0F_1\left(\tfrac{n_2}{2}; \tfrac{\delta_2 c}{2\alpha_2}\tfrac{w_2}{1+\frac{c-\alpha_1}{\alpha_1}w_1+\frac{c-\alpha_2}{\alpha_2}w_2} t\right) \\ & \cdot {}_0F_1\left(\tfrac{m}{2}; \tfrac{\delta_3}{2}\tfrac{1-w_1-w_2}{1+\frac{c-\alpha_1}{\alpha_1}w_1+\frac{c-\alpha_2}{\alpha_2}w_2} t\right) dt \\ = \quad & (2c)^{\frac{1}{2}(n_1+n_2+m)} \left(1+\tfrac{c-\alpha_1}{\alpha_1}w_1+\tfrac{c-\alpha_2}{\alpha_2}w_2\right)^{-\frac{1}{2}(n_1+n_2+m)} \Gamma\left(\tfrac{n_1+n_2+m}{2}\right) \\ & \cdot \mathbf{\Psi}_2^{(3)}\left[\tfrac{n_1+n_2+m}{2}; \tfrac{n_1}{2}, \tfrac{n_2}{2}, \tfrac{m}{2}; \tfrac{\delta_1 c}{2\alpha_1}\tfrac{w_1}{1+\frac{c-\alpha_1}{\alpha_1}w_1+\frac{c-\alpha_2}{\alpha_2}w_2}, \right. \\ & \left. \tfrac{\delta_2 c}{2\alpha_2}\tfrac{w_2}{1+\frac{c-\alpha_1}{\alpha_1}w_1+\frac{c-\alpha_2}{\alpha_2}w_2}, \tfrac{\delta_3}{2}\tfrac{1-w_1-w_2}{1+\frac{c-\alpha_1}{\alpha_1}w_1+\frac{c-\alpha_2}{\alpha_2}w_2}\right]. \end{aligned} \tag{8}$$

Thus result (4) follows from (6) and (8). ■

Remarks

1. If $\delta_1 = \delta_2 = \delta_3 = 0$ in (4) the density function of (W_1, W_2) simplifies to

$$\begin{aligned} & f_{BB^V}\left(w_1, w_2; \tfrac{n_1}{2}, \tfrac{n_2}{2}, \tfrac{m}{2}, \alpha_1, \alpha_2, c\right) \\ = \quad & \left\{\beta\left(\tfrac{n_1}{2}, \tfrac{n_2}{2}, \tfrac{m}{2}\right)\right\}^{-1} \left(\tfrac{c}{\alpha_1}\right)^{\frac{1}{2}n_1} \left(\tfrac{c}{\alpha_2}\right)^{\frac{1}{2}n_2} \\ & \cdot w_1^{\frac{1}{2}n_1 - 1} w_2^{\frac{1}{2}n_2 - 1} (1 - w_1 - w_2)^{\frac{1}{2}m - 1} \\ & \cdot \left(1 + \tfrac{c - \alpha_1}{\alpha_1} w_1 + \tfrac{c - \alpha_2}{\alpha_2} w_2\right)^{-\frac{1}{2}(n_1 + n_2 + m)} \end{aligned} \tag{9}$$

   and we refer to this as the bivariate beta type V distribution. It is denoted as

   $(W_1, W_2) \sim BB^V\left(\frac{n_1}{2}, \frac{n_2}{2}, \frac{m}{2}, \alpha_1, \alpha_2, c\right)$.

2. Further, by using the series expansion of $\boldsymbol{\Psi}_2^{(3)}$ (see Sánchez et al., 2006), the triply noncentral bivariate beta type V distribution in (4) can be represented as an infinite mixture of the bivariate beta type V distribution given in (9):

$$\begin{aligned} & f_{ncBB^V}\left(w_1, w_2; \tfrac{n_1}{2}, \tfrac{n_2}{2}, \tfrac{m}{2}, \alpha_1, \alpha_2, c; \delta_1, \delta_2, \delta_3\right) \\ = \quad & \sum_{j_1, j_2, j_3 = 0}^{\infty} \left[\prod_{i=1}^{3} \left(\tfrac{\delta_i}{2}\right)^{j_i} \frac{e^{-\frac{1}{2}\delta_i}}{j_i!}\right] \\ & \cdot f_{BB^V}\left(w_1, w_2; \tfrac{n_1}{2} + j_1, \tfrac{n_2}{2} + j_2, \tfrac{m}{2} + j_3, \alpha_1, \alpha_2, c\right). \end{aligned} \tag{10}$$

3. If $\delta_1 = \delta_2 = 0$ in (4), then (W_1, W_2) has the noncentral bivariate beta type V distribution, denoted as $(W_1, W_2) \sim ncBB^V\left(\frac{n_1}{2}, \frac{n_2}{2}, \frac{m}{2}, \alpha_1, \alpha_2, c; \delta_3\right)$, therefore

$$\begin{aligned} & f_{ncBB^V}\left(w_1, w_2; \tfrac{n_1}{2}, \tfrac{n_2}{2}, \tfrac{m}{2}, \alpha_1, \alpha_2, c; \delta_3\right) \\ = \quad & f_{BB^V}\left(w_1, w_2; \tfrac{n_1}{2}, \tfrac{n_2}{2}, \tfrac{m}{2}, \alpha_1, \alpha_2, c\right) e^{-\frac{1}{2}\delta_3} \\ & \cdot {}_1F_1\left(\tfrac{n_1+n_2+m}{2}; \tfrac{m}{2}; \tfrac{\delta_3}{2} \tfrac{1-w_1-w_2}{1+\frac{c-\alpha_1}{\alpha_1}w_1+\frac{c-\alpha_2}{\alpha_2}w_2}\right), \end{aligned} \tag{11}$$

where $f_{BB^V}\left(w_1, w_2; \frac{n_1}{2}, \frac{n_2}{2}, \frac{m}{2}, \alpha_1, \alpha_2, c\right)$ is given in (9) and ${}_1F_1(\cdot)$ is the confluent hypergeometric function (see Mathai, 1993, Definition 2.2, page 96).

4. Similarly, if $\delta_2 = \delta_3 = 0$ in (4), the density function of (W_1, W_2) is given by

$$\begin{aligned} & f_{ncBB^V}\left(w_1, w_2; \tfrac{n_1}{2}, \tfrac{n_2}{2}, \tfrac{m}{2}, \alpha_1, \alpha_2, c; \delta_1\right) \\ = \quad & f_{BB^V}\left(w_1, w_2; \tfrac{n_1}{2}, \tfrac{n_2}{2}, \tfrac{m}{2}, \alpha_1, \alpha_2, c\right) e^{-\frac{1}{2}\delta_1} \\ & \cdot {}_1F_1\left(\tfrac{n_1+n_2+m}{2}; \tfrac{n_1}{2}; \tfrac{\delta_1 c}{2\alpha_1} \tfrac{w_1}{1+\frac{c-\alpha_1}{\alpha_1}w_1+\frac{c-\alpha_2}{\alpha_2}w_2}\right), \end{aligned} \tag{12}$$

where $f_{BB^V}\left(w_1, w_2; \frac{n_1}{2}, \frac{n_2}{2}, \frac{m}{2}, \alpha_1, \alpha_2, c\right)$ is given in (9). In this case we also refer to the distribution of (W_1, W_2) given in (12) as the noncentral bivariate beta type V distribution and denote it as $(W_1, W_2) \sim ncBB^V\left(\frac{n_1}{2}, \frac{n_2}{2}, \frac{m}{2}, \alpha_1, \alpha_2, c; \delta_1\right)$.

In the following part the effect of the parameters α_1, α_2, c and δ_i $(i = 1, 2, 3)$, on the shape of the density of the triply noncentral bivariate beta type V distribution is illustrated. In each case all the parameters except one is held constant. Without loss of generality we consider the central and noncentral bivariate beta type V distributions given in (9), (11) and (12) respectively. Figures 1a and 1b show graphs of the density of the central bivariate beta type V distribution for different values of α_2 and c respectively. As α_2 increases with all the other parameters constant, the density shifts towards smaller values of W_1 and larger values of W_2. The opposite will be observed for increasing values of α_1. Increasing the value of the parameter c in Figure 1b with all other parameters constant causes $f(w_1, w_2)$ to shift towards smaller values of W_1 and W_2. The effect of the noncentrality parameters δ_1 and δ_3 on the form of the density function of the triply noncentral bivariate beta type V is illustrated in Figures 1c and 1d respectively. As δ_1 increases the density shifts towards larger values of W_1 and smaller values of W_2. With an increase in δ_3 the density shifts towards smaller values of both W_1 and W_2.

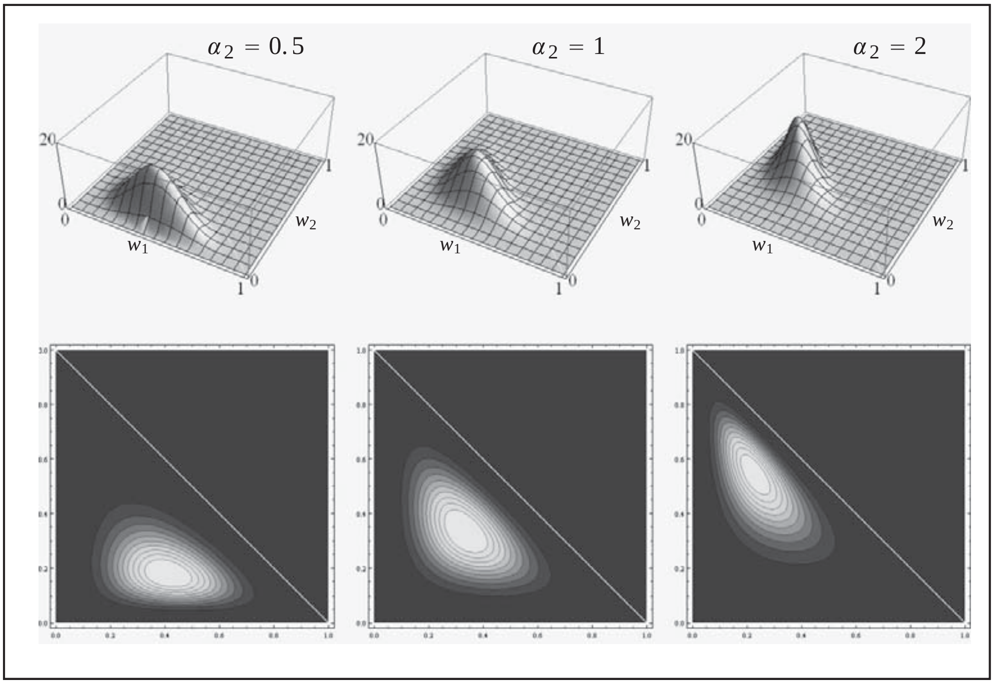

Figure 1a. Effect of α_2 on $f(w_1, w_2), (W_1, W_2) \sim BB^V(5, 5, 5, 1, \alpha_2, 1)$

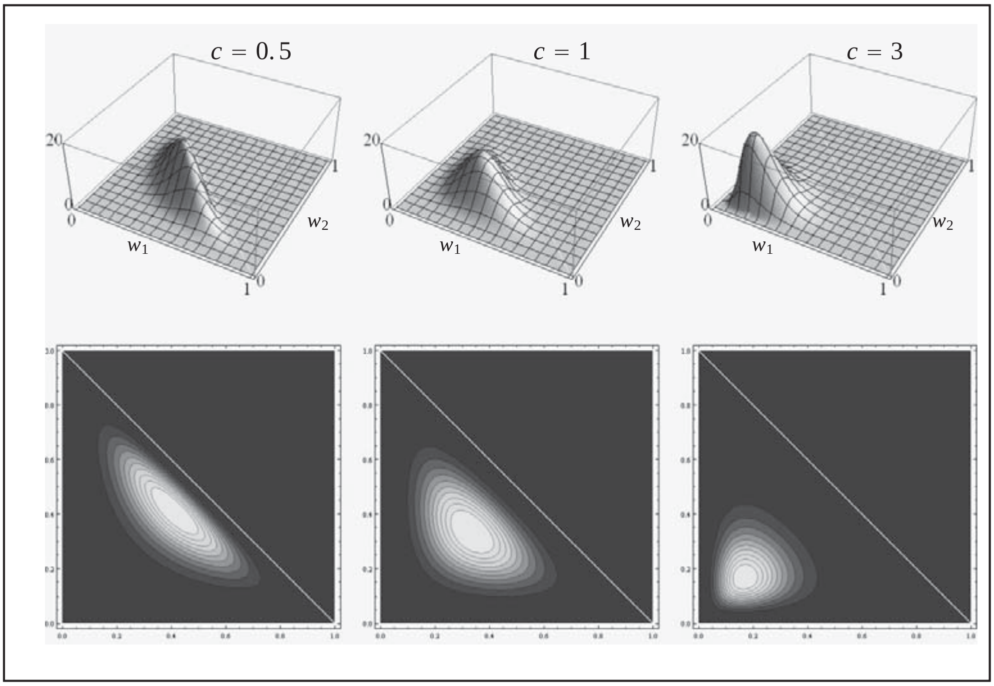

Figure 1b. Effect of c on $f(w_1, w_2), (W_1, W_2) \sim BB^V(5, 5, 5, 1, 1, c)$

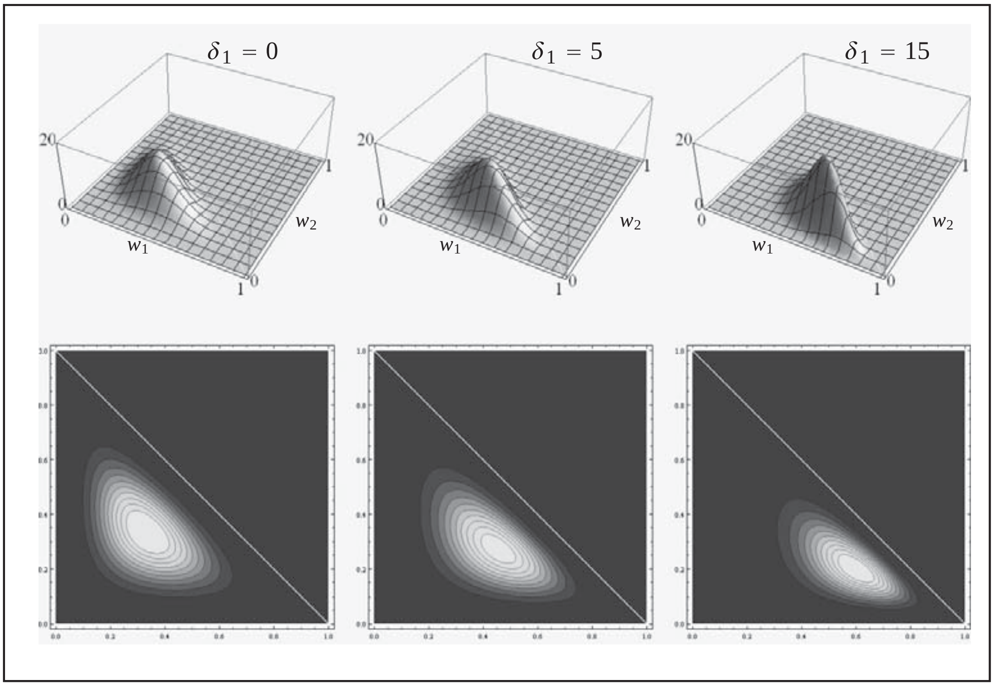

Figure 1c. Effect of δ_1 on $f(w_1, w_2)$,$(W_1, W_2) \sim ncBB^V(5, 5, 5, 1, 1, 1; \delta_1)$

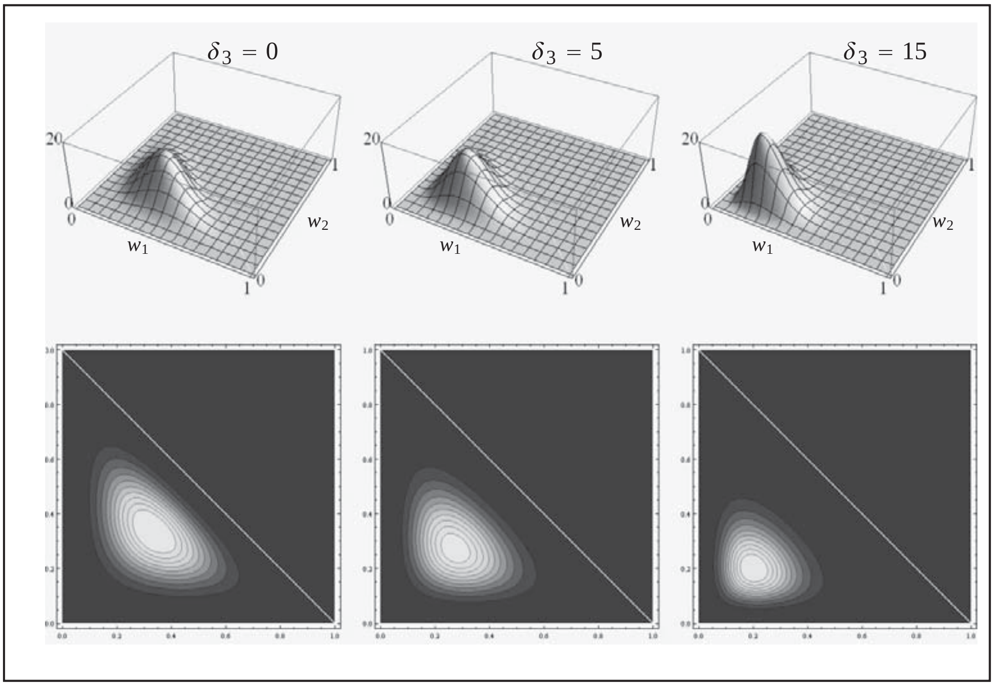

Figure 1d. Effect of δ_3 on $f\,(w_1, w_2)$,$(W_1, W_2) \sim ncBB^V\,(5, 5, 5, 1, 1, 1; \delta_3)$

3. Properties

In this section we will study some properties of this triply noncentral bivariate beta type V distribution.

Theorem 2

If $(W_1, W_2) \sim ncBB^V\left(\frac{n_1}{2}, \frac{n_2}{2}, \frac{m}{2}, \alpha_1, \alpha_2, c; \delta_1, \delta_2, \delta_3\right)$, then the density function of W_1 is given by

$$\sum_{j_1,j_2,j_3=0}^{\infty}\left[\prod_{i=1}^{3}\left(\frac{\delta_i}{2}\right)^{j_i}\frac{e^{-\frac{1}{2}\delta_i}}{j_i!}\right]\left\{\beta\left(\frac{n_1}{2}+j_1, \frac{n_2+m}{2}+j_2+j_3\right)\right\}^{-1}$$
$$\cdot\left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{1}{2}n_1+j_1}\left(\frac{\alpha_2}{c}\right)^{\frac{1}{2}m+j_3} w_1^{\frac{1}{2}n_1+j_1-1}$$
$$\cdot(1-w_1)^{\frac{1}{2}(n_2+m)+j_2+j_3-1}\left(1+\frac{\alpha_2-\alpha_1}{\alpha_1}w_1\right)^{-\left[\frac{1}{2}(n_1+n_2+m)+j_1+j_2+j_3\right]}$$
$$\cdot\,{}_2F_1\left(\frac{m}{2}+j_3, \frac{n_1+n_2+m}{2}+j_1+j_2+j_3; \frac{n_2+m}{2}+j_2; \frac{c-\alpha_2}{\alpha_2}\frac{1-w_1}{1+\frac{\alpha_2-\alpha_1}{\alpha_1}w_1}\right),$$
$$0<w_1<1, \tag{13}$$

where ${}_2F_1(\cdot)$ is the Gauss hypergeometric function (see Mathai, 1993, Definition 2.2, page 96).

Proof:

From (10), $f_{ncBB^V}\left(w_1; \frac{n_1}{2}, \frac{n_2}{2}, \frac{m}{2}, \alpha_1, \alpha_2, c; \delta_1, \delta_2, \delta_3\right)$ is

$$\sum_{j_1,j_2,j_3=0}^{\infty}\left[\prod_{i=1}^{3}\left(\frac{\delta_i}{2}\right)^{j_i}\frac{e^{-\frac{1}{2}\delta_i}}{j_i!}\right]$$
$$\cdot\int_0^{1-w_1} f_{BB^V}\left(w_1, w_2; \frac{n_1}{2}+j_1, \frac{n_2}{2}+j_2, \frac{m}{2}+j_3, \alpha_1, \alpha_2, c\right)dw_2 \tag{14}$$

where $f_{BB^V}(w_1, w_2)$ is given in (9). By change of variable $z=\frac{w_2}{1-w_1}$ the integral in (14) can be written as

$$
\begin{aligned}
&K w_1^{\frac{1}{2}n_1+j_1-1}\left(1-w_1\right)^{\frac{1}{2}(n_2+m)+j_2+j_3-1} \\
&\cdot\left(1+\tfrac{c-\alpha_1}{\alpha_1}w_1\right)^{-\left[\frac{1}{2}(n_1+n_2+m)+j_1+j_2+j_3\right]} \\
&\cdot\int_0^1 z^{\frac{1}{2}n_2+j_2-1}\left(1-z\right)^{\frac{1}{2}m+j_3-1} \\
&\cdot\left(1+\frac{\frac{c-\alpha_2}{\alpha_2}(1-w_1)}{1+\frac{c-\alpha_1}{\alpha_1}w_1}z\right)^{-\left[\frac{1}{2}(n_1+n_2+m)+j_1+j_2+j_3\right]} \qquad dz,
\end{aligned} \tag{15}
$$

where $K=\left\{\beta\left(\frac{n_1}{2}+j_1,\frac{n_2}{2}+j_2,\frac{m}{2}+j_3\right)\right\}^{-1}\left(\frac{c}{\alpha_1}\right)^{\frac{1}{2}n_1+j_1}\left(\frac{c}{\alpha_2}\right)^{\frac{1}{2}n_2+j_2}$.

Using Gradshteyn and Ryzhik (2000, equation 3.197(3), page 314 and equation 9.131(1), page 998) the above integral (15) simplifies to

$$
\begin{aligned}
&\frac{\Gamma\left(\frac{n_2}{2}+j_2\right)\Gamma\left(\frac{m}{2}+j_3\right)}{\Gamma\left(\frac{n_2+m}{2}+j_2+j_3\right)}\left[\left(1+\tfrac{c-\alpha_1}{\alpha_1}w_1\right)^{-1}\right. \\
&\left.\cdot\,\tfrac{c}{\alpha_2}\left(1+\tfrac{\alpha_2-\alpha_1}{\alpha_1}w_1\right)\right]^{-\left[\frac{1}{2}(n_1+n_2+m)+j_1+j_2+j_3\right]} \\
&\cdot{}_2F_1\left(\tfrac{m}{2}+j_3,\tfrac{n_1+n_2+m}{2}+j_1+j_2+j_3;\tfrac{n_2+m}{2}+j_2;\tfrac{c-\alpha_2}{\alpha_2}\frac{1-w_1}{1+\frac{\alpha_2-\alpha_1}{\alpha_1}w_1}\right).
\end{aligned} \tag{16}
$$

Therefore from (14) and (16) follows that the marginal density function of W_1 is given by (13). ■

Theorem 3

If $(W_1, W_2) \sim ncBB^V\left(\frac{n_1}{2}, \frac{n_2}{2}, \frac{m}{2}, \alpha_1, \alpha_2, c; \delta_1, \delta_2, \delta_3\right)$, the product moment $E\left(W_1^{h_1} W_2^{h_2}\right)$ equals

$$\sum_{j_1,j_2,j_3=0}^{\infty}\left[\prod_{i=1}^{3}\left(\frac{\delta_i}{2}\right)^{j_i}\frac{e^{-\frac{1}{2}\delta_i}}{j_i!}\right]\left(\frac{c}{\alpha_1}\right)^{\frac{1}{2}n_1+j_1}\left(\frac{c}{\alpha_2}\right)^{\frac{1}{2}n_2+j_2}$$
$$\cdot\frac{\Gamma\left(\frac{n_1+n_2+m}{2}+j_1+j_2+j_3\right)}{\Gamma\left(\frac{n_1}{2}+j_1\right)\Gamma\left(\frac{n_2}{2}+j_2\right)}\frac{\Gamma\left(\frac{n_1}{2}+j_1+h_1\right)\Gamma\left(\frac{n_2}{2}+j_2+h_2\right)}{\Gamma\left(\frac{n_1+n_2+m}{2}+j_1+j_2+j_3+h_1+h_2\right)}$$
$$\cdot F_1\left(\frac{n_1+n_2+m}{2}+j_1+j_2+j_3, \frac{n_1}{2}+j_1+h_1, \frac{n_2}{2}+j_2+h_2,\right.$$
$$\left.\frac{n_1+n_2+m}{2}+j_1+j_2+j_3+h_1+h_2; \frac{\alpha_1-c}{\alpha_1}, \frac{\alpha_2-c}{\alpha_2}\right) \tag{17}$$

where $F_1(\cdot)$ is the Appell function of the first kind (see Gradshteyn and Ryzhik, 2000, equation 9.180 (1), page 1008).

Proof:

From (9) and (10) follows that the $(h_1, h_2)^{th}$ moment is given by

$$E\left(W_1^{h_1} W_2^{h_2}\right) = \sum_{j_1,j_2,j_3=0}^{\infty}\left[\prod_{i=1}^{3}\left(\frac{\delta_i}{2}\right)^{j_i}\frac{e^{-\frac{1}{2}\delta_i}}{j_i!}\right]E_{j_1,j_2,j_3}\left(W_1^{h_1} W_2^{h_2}\right) \tag{18}$$

where $E_{j_1,j_2,j_3}\left(W_1^{h_1} W_2^{h_2}\right)$ denotes the $(h_1, h_2)^{th}$ moment of the bivariate beta type V distribution, i.e.
$(W_1, W_2) \sim BB^V\left(\frac{n_1}{2}+j_1, \frac{n_2}{2}+j_2, \frac{m}{2}+j_3, \alpha_1, \alpha_2, c\right)$.

From (9) follows that

$$E_{j_1,j_2,j_3}\left(W_1^{h_1}W_2^{h_2}\right)$$

$$= \int\limits_{\substack{0<w_1+w_2<1\\0<w_i<1,\, i=1,2}}\int \frac{\Gamma\left(\frac{n_1+n_2+m}{2}+j_1+j_2+j_3\right)}{\Gamma\left(\frac{n_1}{2}+j_1\right)\Gamma\left(\frac{n_2}{2}+j_2\right)\Gamma\left(\frac{m}{2}+j_3\right)}\left(\frac{c}{\alpha_1}\right)^{\frac{1}{2}n_1+j_1}$$

$$\cdot\left(\frac{c}{\alpha_2}\right)^{\frac{1}{2}n_2+j_2} w_1^{\frac{1}{2}n_1+j_1+h_1-1} w_2^{\frac{1}{2}n_2+j_2+h_2-1}$$

$$\cdot(1-w_1-w_2)^{\frac{1}{2}m+j_3-1}$$

$$\cdot\left(1+\frac{c-\alpha_1}{\alpha_1}w_1+\frac{c-\alpha_2}{\alpha_2}w_2\right)^{-\left[\frac{1}{2}(n_1+n_2+m)+j_1+j_2+j_3\right]} dw_1dw_2.$$

By using Gradshteyn and Ryzhik (2000, equation 9.184(1), page 1011) the above expression is

$$E_{j_1,j_2,j_3}\left(W_1^{h_1}W_2^{h_2}\right)$$

$$= \left(\frac{c}{\alpha_1}\right)^{\frac{1}{2}n_1+j_1}\left(\frac{c}{\alpha_2}\right)^{\frac{1}{2}n_2+j_2}$$

$$\cdot\frac{\Gamma\left(\frac{n_1+n_2+m}{2}+j_1+j_2+j_3\right)}{\Gamma\left(\frac{n_1}{2}+j_1\right)\Gamma\left(\frac{n_2}{2}+j_2\right)}\frac{\Gamma\left(\frac{n_1}{2}+j_1+h_1\right)\Gamma\left(\frac{n_2}{2}+j_2+h_2\right)}{\Gamma\left(\frac{n_1+n_2+m}{2}+j_1+j_2+j_3+h_1+h_2\right)}$$

$$\cdot F_1\left(\frac{n_1+n_2+m}{2}+j_1+j_2+j_3,\frac{n_1}{2}+j_1+h_1,\right.$$
$$\left.\frac{n_2}{2}+j_2+h_2,\frac{n_1+n_2+m}{2}+j_1+j_2+j_3+h_1+h_2;\frac{\alpha_1-c}{\alpha_1},\frac{\alpha_2-c}{\alpha_2}\right) \tag{19}$$

and the result (17) follows directly from (18) and (19). ■

Using the result (17) in Theorem 3, the correlation between W_1 and W_2 where (W_1, W_2) has the triply noncentral bivariate beta type V distribution is studied. Figure 2a shows how $corr(W_1, W_2)$ changes for increasing values of α_2 or c. The correlation shifts towards +1 if α_2 decreases or if c increases. For certain values of the parameters $corr(W_1, W_2) > 0$. Thus, these additional parameters allow for positive correlation between the variables. Figure 2b illustrates the effect of the noncentrality parameters on the correlation between the variables W_1 and W_2. The cases are considered where S_1 or B in (3) has a noncentral chi-square distribution, that is $S_1 \sim \chi^2 (n_1; \delta_1)$ or $B \sim \chi^2 (m; \delta_3)$. With a decrease in δ_1 or an increase in δ_3 values of $corr\,(W_1, W_2)$ shifts towards +1.

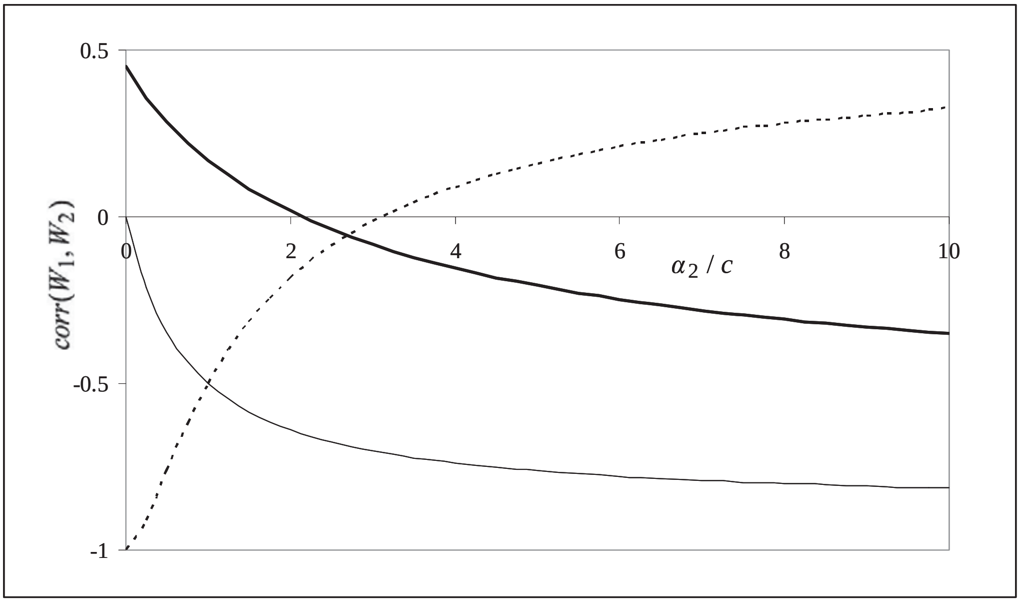

Figure 2a. Effect of α_2 and c on $corr\,(W_1, W_2)$

- - - - - (i) $(W_1, W_2) \sim BB^V (5, 5, 5, 1, 1, c)$

______ (ii) $(W_1, W_2) \sim BB^V (5, 5, 5, 1, \alpha_2, 1)$

______ (iii) $(W_1, W_2) \sim BB^V (5, 5, 5, 0.5, \alpha_2, 6)$

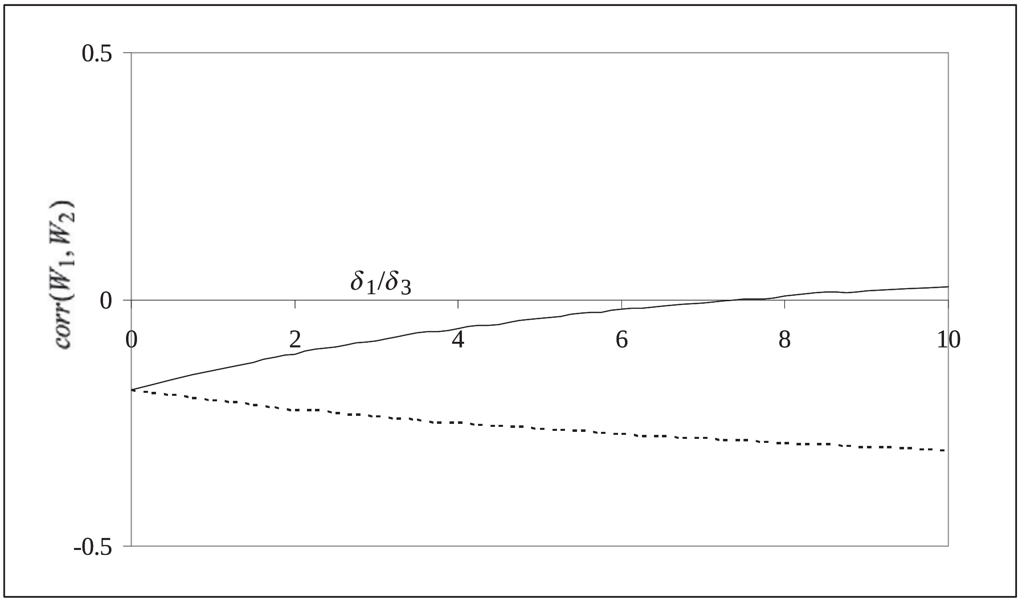

Figure 2b. Effect of δ_1 and δ_3 on $corr\,(W_1, W_2)$
- - - - - (i) $(W_1, W_2) \sim ncBB^V\,(5, 5, 5, 1, 1, 2; \delta_1)$
______ (ii) $(W_1, W_2) \sim ncBB^V\,(5, 5, 5, 1, 1, 2; \delta_3)$

4. Distribution of product of dependent components W_1 and W_2

The importance of the distribution of the product of correlated variables is highlighted by several authors e.g. Nagar et al. (2009) and Gupta et al. (2009). Thus, in this section the distribution of the product of the components of the triply noncentral bivariate beta type V distribution is derived.

Theorem 5

Let $(W_1, W_2) \sim ncBB^V\left(\frac{n_1}{2}, \frac{n_2}{2}, \frac{m}{2}, \alpha_1, \alpha_2, c; \delta_1, \delta_2, \delta_3\right)$. Then the density function of $Z = W_1 W_2$ is given by

$$\sum_{j_1,j_2,j_3=0}^{\infty}\left[\prod_{i=1}^{3}\left(\frac{\delta_i}{2}\right)^{j_i}\frac{e^{-\frac{1}{2}\delta_i}}{j_i!}\right]\left(\frac{c}{\alpha_1}\right)^{\frac{1}{2}n_1+j_1}\left(\frac{c}{\alpha_2}\right)^{\frac{1}{2}n_2+j_2} \tag{20}$$

$$\cdot\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\left(\frac{\alpha_1-c}{\alpha_1}\right)^k\left(\frac{\alpha_2-c}{\alpha_2}\right)^l}{k!l!}\frac{\Gamma\left(\frac{n_1+n_2+m}{2}+j_1+j_2+j_3+k+l\right)}{\Gamma\left(\frac{n_1}{2}+j_1\right)\Gamma\left(\frac{n_2}{2}+j_2\right)}$$

$$\cdot H_{1,2}^{2,0}\left(z\middle|_{(b_1,\beta_1),(b_2,\beta_2)}^{(a_1,\gamma_1)}\right).$$

where $a_1 = \frac{n_1+n_2+m}{2} + j_1 + j_2 + j_3 - 2 + k + l$ and $\gamma_1 = 2$; $b_1 = \frac{n_1}{2} + j_1 + k - 1$, $b_2 = \frac{n_2}{2} + j_2 + l - 1$ and $\beta_1 = \beta_2 = 1$ and where $H(\cdot)$ is the H-function (see Mathai, 1993, Definition 3.1, page 140).

Proof:

Using (17) the Mellin transform (see Mathai, 1993, Definition 1.8, page 23) of $f(z)$ is given by

$$M_f(h)$$

$$= \sum_{j_1,j_2,j_3=0}^{\infty}\left[\prod_{i=1}^{3}\left(\frac{\delta_i}{2}\right)^{j_i}\frac{e^{-\frac{1}{2}\delta_i}}{j_i!}\right]\left(\frac{c}{\alpha_1}\right)^{\frac{1}{2}n_1+j_1}\left(\frac{c}{\alpha_2}\right)^{\frac{1}{2}n_2+j_2}$$

$$\cdot\frac{\Gamma\left(\frac{n_1+n_2+m}{2}+j_1+j_2+j_3\right)}{\Gamma\left(\frac{n_1}{2}+j_1\right)\Gamma\left(\frac{n_2}{2}+j_2\right)}\frac{\Gamma\left(\frac{n_1}{2}+j_1+h-1\right)\Gamma\left(\frac{n_2}{2}+j_2+h-1\right)}{\Gamma\left(\frac{n_1+n_2+m}{2}+j_1+j_2+j_3+2h-2\right)}$$

$$\cdot F_1\left(\frac{n_1+n_2+m}{2}+j_1+j_2+j_3, \frac{n_1}{2}+j_1+h-1, \frac{n_2}{2}+j_2+h-1, \frac{n_1+n_2+m}{2}+j_1+j_2+j_3+2h-2; \frac{\alpha_1-c}{\alpha_1}, \frac{\alpha_2-c}{\alpha_2}\right). \tag{21}$$

From Gradshteyn and Ryzhik (2000, equation 9.180 (1), page 1008) and (21) follows that

$$
\begin{aligned}
&M_f(h) \\
&= \sum_{j_1,j_2,j_3=0}^{\infty}\left[\prod_{i=1}^{3}\left(\frac{\delta_i}{2}\right)^{j_i}\frac{e^{-\frac{1}{2}\delta_i}}{j_i!}\right]\left(\frac{c}{\alpha_1}\right)^{\frac{1}{2}n_1+j_1}\left(\frac{c}{\alpha_2}\right)^{\frac{1}{2}n_2+j_2} \\
&\quad\cdot\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\left(\frac{\alpha_1-c}{\alpha_1}\right)^{k}\left(\frac{\alpha_2-c}{\alpha_2}\right)^{l}}{k!l!}\frac{\Gamma\left(\frac{n_1+n_2+m}{2}+j_1+j_2+j_3+k+l\right)}{\Gamma\left(\frac{n_1}{2}+j_1\right)\Gamma\left(\frac{n_2}{2}+j_2\right)}\frac{\prod_{j=1}^{2}\Gamma\left(b_j+\beta_j h\right)}{\prod_{j=1}^{1}\Gamma\left(a_j+\gamma_j h\right)}
\end{aligned}
\tag{22}
$$

where a_j, γ_j, b_j and β_j are given in the Theorem above. The desired result (20) follows from (22), the inverse Mellin transform (see Mathai, 1993, Definition 1.8, page 23) and the definition of the H-function (see Mathai, 1993, Definition 3.1, page 140). ■

The effect of the additional parameters on the shape of the density of the triply noncentral bivariate beta type V distribution is illustrated in Figures 3a and 3b. In each case the values of all the parameters except one is held constant. Without loss of generality we consider the central and noncentral bivariate beta type V distributions given in (9), (11) and (12) respectively.

In Figure 3a the density of $Z = W_1W_2$ shifts towards smaller values as c increases. The same but opposite effect is observed if α_1 or α_2 increases. In Figure 3b, the spread of the density $f(z)$ increases as δ_1 increases. As δ_3 increases the density shifts towards smaller values of Z.

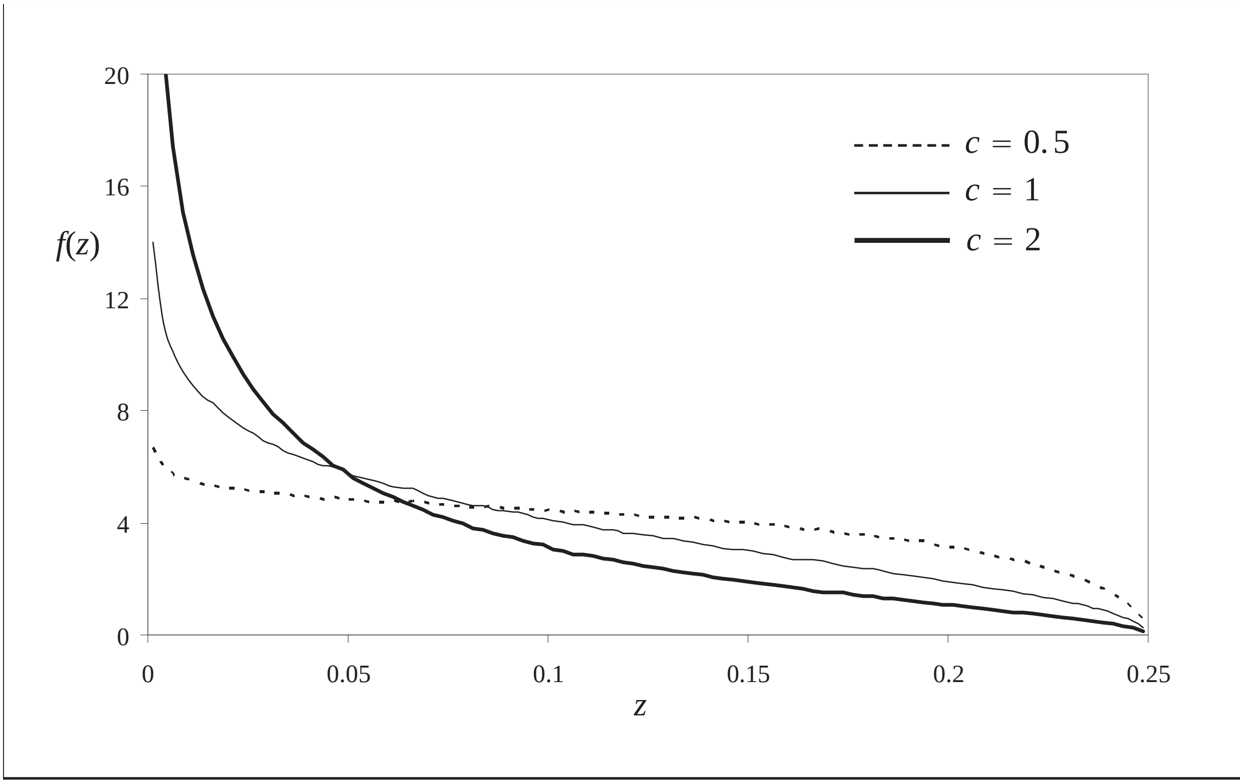

Figure 3a. Effect of c on $f(z), Z = W_1, W_2, (W_1, W_2) \sim BB^V(1, 1, 1, 1, c)$

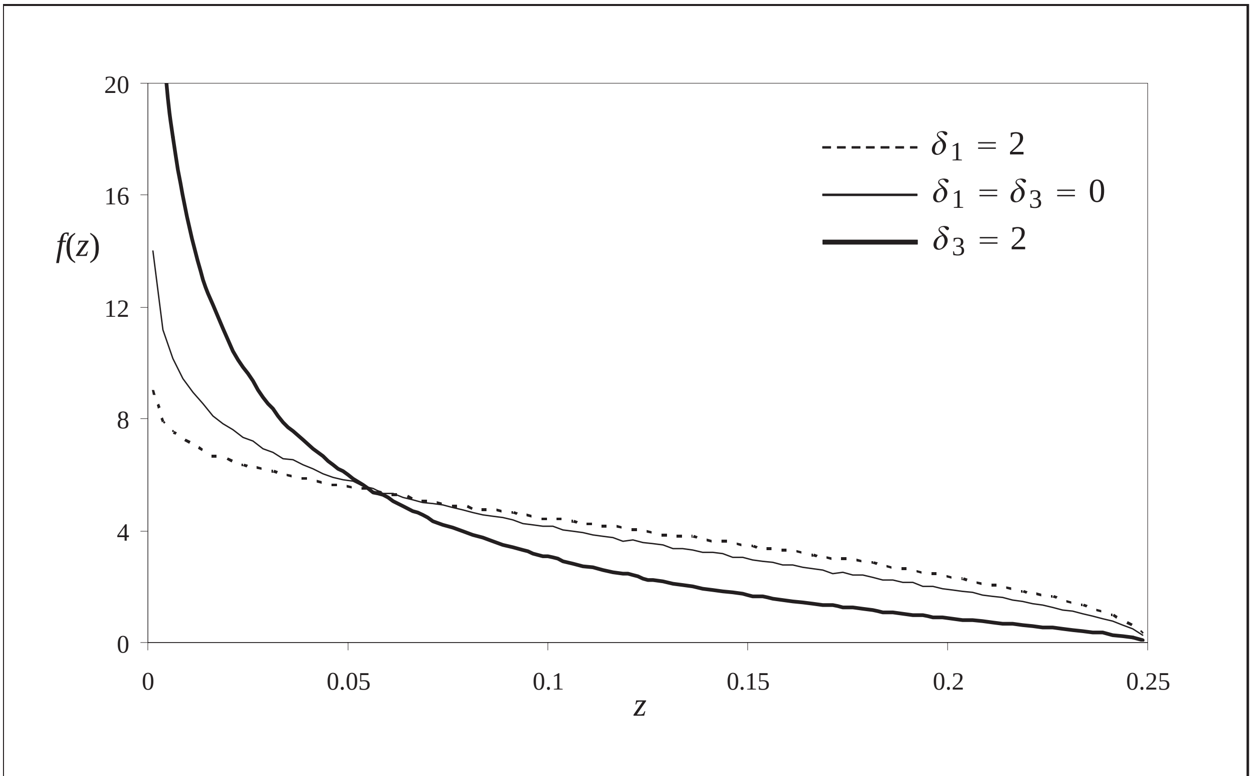

Figure 3b. Effect of δ_i on $f(z)$, $Z = W_1, W_2, (W_1, W_2) \sim BB^V(1,1,1,1,1,1;\delta_i)$

5. Conclusion

In this paper we proposed the triply noncentral bivariate beta type V distribution. Some properties are studied which enhance the possibility of application in different areas. The variety of shapes of the triply noncentral bivariate beta type V density and the density of the product of its correlated components also received attention. Furthermore the positive correlation due to the enriched parameter structure is an added value that justifies the development of this model.

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