Izak Broere and Johannes Heidema

Universal $H$-colourable graphs

Department of Mathematics and Applied Mathematics, University of Pretoria
Department of Mathematical Sciences, University of South Africa

Abstract

Rado constructed a (simple) denumerable graph $R$ with the positive integers as vertex set with the following edges: For given $m$ and $n$ with $m < n$, $m$ is adjacent to $n$ if $n$ has a 1 in the $m$’th position of its binary expansion. It is well known that $R$ is a universal graph in the set $I_c$ of all countable graphs (since every graph in $I_c$ is isomorphic to an induced subgraph of $R$) and that it is a homogeneous graph (since every isomorphism between two finite induced subgraphs of $R$ extends to an automorphism of $R$).

In this paper we construct a graph $U(H)$ which is $H$-universal in $→H_c$, the induced-hereditary hom-property of $H$-colourable graphs consisting of all (countable) graphs which have a homomorphism into a given (countable) graph $H$. If $H$ is the (finite) complete graph $K_k$, then $→H_c$ is the property of $k$-colourable graphs. The universal graph $U(H)$ is characterised by showing that it is, up to isomorphism, the unique denumerable, $H$-universal graph in $→H_c$ which is $H$-homogeneous in $→H_c$. The graphs $H$ for which $U(H) \cong R$ are also characterised.

With small changes to the definitions, our results translate effortlessly to hold for digraphs too. Another slight adaptation of our work yields related results for $(k,l)$-split graphs.

AMS Subject Classification (from MSC2010): 05C63

Keywords: Universal graph, hom-property of graphs, extension property of graphs, homogeneous graph, $H$-colourable graph, $k$-colourable graph, $(k,l)$-split graph, Rado graph

Corresponding author: Izak Broere
Email: izak.broere@up.ac.za
Telephone: +27-12-420-2611
Fax: +27-12-420-3893

1 Introduction

All graphs considered here are simple, undirected (except later only in Corollaries 1 and 2), unlabelled and have countable vertex sets. When the vertex set is taken to be the set, or some subset, of the positive integers $\mathbb{N} = \{1, 2, \ldots\}$, number-theoretic properties of the integers may be employed in constructions and proofs. Otherwise, the vertex set of a graph may be indexed by $\mathbb{N}$ or one of its subsets.

Let $\mathcal{P}$ be a class of countable graphs. Following [6], we define a graph $U$ to be a universal graph for $\mathcal{P}$ if every graph in $\mathcal{P}$ is (isomorphic to) an induced subgraph of $U$; it is a universal graph in $\mathcal{P}$ if $U \in \mathcal{P}$ too. We shall often have occasion to refer to two graphs which are isomorphic; in that case we shall refer to (any) one of them as a clone of the other. For any graph property $\mathcal{P}$ (i.e., an isomorphically closed class of graphs) we use the symbols $\mathcal{P}_c$, $\mathcal{P}_d$, $\mathcal{P}_f$ to denote, respectively, the classes of countable, denumerable, and finite graphs of $\mathcal{P}$. 
Rado [11] constructed the denumerable graph $R$ on $\mathbb{N}$ with the following edges: For given $m$ and $n$ with $m < n$, $m$ is adjacent to $n$ in $R$ (i.e., we have an edge $mn$ in $R$) if $n$ has a 1 in the $m$'th position of its binary expansion. It is well known that $R$ is a universal graph in the set $\mathcal{L}_c$ of all countable graphs (since $R \in \mathcal{L}_c$ and every graph in $\mathcal{L}_c$ is isomorphic to an induced subgraph of $R$). Some known constructions of clones of this graph, together with many new constructions, are discussed in [2]. Important properties of $R$ (sometimes called the “random graph”) are discussed in [4].

A homomorphism from a graph $G$ to a graph $H$ is an edge-preserving map from the vertex set $V(G)$ of $G$ into the vertex set $V(H)$ of $H$. If such a map exists, we say that $G$ is homomorphic to $H$ and we write $G \rightarrow H$. Given any (countable) graph $H$, the hom-property $\rightarrow H_c$ is the class of $H$-colourable graphs, i.e., it consists of all (countable) graphs which have a homomorphism into the given graph $H$. In symbols: $\rightarrow H_c = \{ G \mid G \in \mathcal{L}_c, G \rightarrow H \}$. Using the terminology of [1], we note that every hom-property is an additive, induced-hereditary graph property. In [10] the authors describe the construction of a universal graph in the hom-property $\rightarrow H_c$ for any given finite graph $H$. Our aim in this paper is to construct and characterise a graph $U(H)$ which is universal in $\rightarrow H_c$ for any countable graph $H$. If $H$ is the (finite) complete graph $K_k$, then $\rightarrow H_c$ is the property of $k$-colourable graphs denoted (following [1]) by $O^k_\infty$, while $\rightarrow K_{\omega_0}$ is $\mathcal{L}_c$. We also characterise those graphs $H$ for which $U(H) \cong R$.

A slight adaptation of our work yields related results for $(k,l)$-split graphs.

## 2 Constructing universal $H$-colourable graphs

Throughout this section, $H$ is any graph with vertex set $\{w_1, w_2, \ldots\}$ (in the denumerable case) or $\{w_1, w_2, \ldots, w_k\}$ (in the finite case). We introduce the notation $V(H) = \{w_1, w_2, \ldots, w_k\}$ to cover both cases at once. We are going to construct a graph $U(H)$, universal in the property $\rightarrow H_c$, and then investigate its properties in sections 3 and 4.

Let $p_1, p_2, \ldots$ be any enumeration of the denumerable set of prime numbers. For any integer $n \geq 2$ there is a unique sequence $n_1, n_2, \ldots$ for which $n = \prod_{i \geq 1} p_i^{n_i}$. Here each power $n_i$ is a non-negative integer, and at least one (but only finitely many) $n_i \geq 1$.

First we define countably many pairwise disjoint denumerable proper subsets $N_1, N_2, \ldots$ of $\mathbb{N} = \{2, 3, \ldots\}$ by specifying $N_j, 1 \leq j$, to be the set of all integers $n \geq 2$ for which the power of any prime in its prime factorization is either 0 or $j$:

$$N_j := \{ n \in \mathbb{N} \mid \text{every } n_i \in \{0, j\}\}.$$ 

We now define the graph $U(H)$, co-determined by $H$, as follows: The vertex set of $U(H)$ is

$$V(U(H)) := N_1 \cup N_2 \cup \ldots \cup \cup N_k \cup.$$ 

In $U(H)$ there are no edges between vertices from the same $N_j$. If $m \in N_h$ and $n \in N_j$, $h \neq j$, then there is an edge in $U(H)$ between $m$ and $n$ if and only if

(i) there is an edge in $H$ between $w_h$ and $w_j$;

(ii) $m < n$; and

(iii) $n_m = j$.

For a graph $G$ we now want to define the notion “$G$ is $H$-universal in $\rightarrow H_c$” – where this is stronger than “$G$ is universal in $\rightarrow H_c$” – and then prove that the $U(H)$ that we have just defined is $H$-universal in $\rightarrow H_c$. We need some preliminary definitions.

Consider two graphs $F, G \rightarrow H_c$ and two homomorphisms $\lambda : F \rightarrow H$ and $\zeta : G \rightarrow H$, together with a third homomorphism $\nu : F \rightarrow G$. Then we say that $\nu$ is $H$-universal in $H$-

Consider two graphs $F, G \rightarrow H_c$ and two homomorphisms $\lambda : F \rightarrow H$ and $\zeta : G \rightarrow H$, together with a third homomorphism $\nu : F \rightarrow G$. Then we say that $\nu$ is $H$-universal in $H$-universal in $\rightarrow H_c$. We need some preliminary definitions.

Consider two graphs $F, G \rightarrow H_c$ and two homomorphisms $\lambda : F \rightarrow H$ and $\zeta : G \rightarrow H$, together with a third homomorphism $\nu : F \rightarrow G$. Then we say that $\nu$ is $H$-universal in $H$-universal in $\rightarrow H_c$. We need some preliminary definitions.

Consider two graphs $F, G \rightarrow H_c$ and two homomorphisms $\lambda : F \rightarrow H$ and $\zeta : G \rightarrow H$, together with a third homomorphism $\nu : F \rightarrow G$. Then we say that $\nu$ is $H$-universal in $H$-universal in $\rightarrow H_c$. We need some preliminary definitions.

Consider two graphs $F, G \rightarrow H_c$ and two homomorphisms $\lambda : F \rightarrow H$ and $\zeta : G \rightarrow H$, together with a third homomorphism $\nu : F \rightarrow G$. Then we say that $\nu$ is $H$-universal in $H$-universal in $\rightarrow H_c$. We need some preliminary definitions.

Consider two graphs $F, G \rightarrow H_c$ and two homomorphisms $\lambda : F \rightarrow H$ and $\zeta : G \rightarrow H$, together with a third homomorphism $\nu : F \rightarrow G$. Then we say that $\nu$ is $H$-universal in $H$-universal in $\rightarrow H_c$. We need some preliminary definitions.

Consider two graphs $F, G \rightarrow H_c$ and two homomorphisms $\lambda : F \rightarrow H$ and $\zeta : G \rightarrow H$, together with a third homomorphism $\nu : F \rightarrow G$. Then we say that $\nu$ is $H$-universal in $H$-universal in $\rightarrow H_c$. We need some preliminary definitions.

Consider two graphs $F, G \rightarrow H_c$ and two homomorphisms $\lambda : F \rightarrow H$ and $\zeta : G \rightarrow H$, together with a third homomorphism $\nu : F \rightarrow G$. Then we say that $\nu$ is $H$-universal in $H$-universal in $\rightarrow H_c$. We need some preliminary definitions.

Consider two graphs $F, G \rightarrow H_c$ and two homomorphisms $\lambda : F \rightarrow H$ and $\zeta : G \rightarrow H$, together with a third homomorphism $\nu : F \rightarrow G$. Then we say that $\nu$ is $H$-universal in $H$-universal in $\rightarrow H_c$. We need some preliminary definitions.

Consider two graphs $F, G \rightarrow H_c$ and two homomorphisms $\lambda : F \rightarrow H$ and $\zeta : G \rightarrow H$, together with a third homomorphism $\nu : F \rightarrow G$. Then we say that $\nu$ is $H$-universal in $H$-universal in $\rightarrow H_c$. We need some preliminary definitions.

Consider two graphs $F, G \rightarrow H_c$ and two homomorphisms $\lambda : F \rightarrow H$ and $\zeta : G \rightarrow H$, together with a third homomorphism $\nu : F \rightarrow G$. Then we say that $\nu$ is $H$-universal in $H$-universal in $\rightarrow H_c$. We need some preliminary definitions.

Consider two graphs $F, G \rightarrow H_c$ and two homomorphisms $\lambda : F \rightarrow H$ and $\zeta : G \rightarrow H$, together with a third homomorphism $\nu : F \rightarrow G$. Then we say that $\nu$ is $H$-universal in $H$-universal in $\rightarrow H_c$. We need some preliminary definitions.

Consider two graphs $F, G \rightarrow H_c$ and two homomorphisms $\lambda : F \rightarrow H$ and $\zeta : G \rightarrow H$, together with a third homomorphism $\nu : F \rightarrow G$. Then we say that $\nu$ is $H$-universal in $H$-universal in $\rightarrow H_c$. We need some preliminary definitions.

Consider two graphs $F, G \rightarrow H_c$ and two homomorphisms $\lambda : F \rightarrow H$ and $\zeta : G \rightarrow H$, together with a third homomorphism $\nu : F \rightarrow G$. Then we say that $\nu$ is $H$-universal in $H$-universal in $\rightarrow H_c$. We need some preliminary definitions.
\( \rightarrow H_c \) if there exists a surjective homomorphism \((H\text{-colouring}) \, \zeta : G \to H\) such that for every \( F \in \rightarrow H_c \) and every \( \lambda : F \to H \) there exists a \((\lambda, \zeta)\)-respecting isomorphic embedding \( \nu : F \to G \), i.e., \( \nu : F \cong G[\nu(V(F))] \). This is a stronger property than universality in \( \rightarrow H_c \), which requires only isomorphic embeddings into \( G \). It says that there is some surjective \( H\text{-colouring} \, \zeta \) of \( G \) such that for every \( H\text{-colouring} \, \lambda \) of any \( F \in \rightarrow H_c \) there exists an isomorphic embedding \( \nu \) of \( F \) into \( G \) which preserves the colours (elements of \( V(H) \)) assigned by \( \lambda \) to the vertices of \( F \): for every \( v \in V(F) \), \( \zeta(\nu(v)) = \lambda(v) \). We say then that \( G \) is \( H\)-universal in \( \rightarrow H_c \) with respect to \( \zeta \).

**Theorem 1** For any countable graph \( H \) the graph \( U(H) \) is \( H\)-universal in \( \rightarrow H_c \).

**Proof:**

By the construction of \( U(H) \) it is clear that \( U(H) \in \rightarrow H_c \), since \( n \mapsto w_j \) for every \( n \in \mathbb{N}_j \) induces a homomorphism \( \mu : U(H) \to H \), which is even a surjection. In the sequel we shall call this \( \mu \) the \textit{canonical} homomorphism from \( U(H) \) onto \( H \), or the \textit{canonical} \( H\)-colouring of \( U(H) \). Note that, for every \( j \), \( \mu^{-1}(w_j) = \mathbb{N}_j \). We shall establish \( H\)-universality of \( U(H) \) in \( \rightarrow H_c \) with respect to \( \mu \).

Let \( F \) with \( V(F) = \{v_1, v_2, \ldots, <w_t>\} \) be any countable graph for which \( F \to H \), indeed, let \( \lambda \) be a homomorphism from \( F \) into \( H \). We recursively construct an injection \( \nu : V(F) \to V(U(H)) \) such that \( \nu(V(F)) \) induces a subgraph of \( U(H) \) which, under \( \nu \), is isomorphic to \( F \). By its construction, \( \nu \) will be \((\lambda, \mu)\)-respecting.

We begin by defining subsets \( V_j \) of \( V(F) \) by \( V_j := \lambda^{-1}(w_j) \), \( 1 \leq j \leq k \), and note that \( V_1 \cup V_2 \cup \ldots \cup V_k \) is a partition of \( V(F) \). (For some indices \( j \), \( V_j \) may be empty, since \( \lambda \) need not be surjective onto \( H \).) Furthermore, since \( \lambda \) is a homomorphism from \( F \) into the (simple) graph \( H \), there is no edge in \( F \) between any two vertices from the same \( V_j \). Note also that, for every edge \( uv \) of \( F \), \( \lambda(u) \lambda(v) \) is an edge of \( H \).

We now define \( \nu \) by recursion on the indices \( 1, 2, \ldots, <\ell> \) of the vertices of \( F \). Let us suppose that \( v_1 \in V_s \); then we can choose \( \nu(v_1) \) to be any element of \( \mathbb{N}_s \). Next assume that \( \nu(v_1), \nu(v_2), \ldots, \nu(v_{p-1}) \) have already been specified so that for every \( q \) and \( r \) with \( 1 \leq q, r \leq p-1 \)

- if \( q \neq r \), then \( \nu(v_q) \neq \nu(v_r) \);
- if \( v_q \in V_j \), then \( \nu(v_q) \in \mathbb{N}_j \); and
- \( F[\{v_1, v_2, \ldots, v_{p-1}\}] \cong U(H)[\{\nu(v_1), \nu(v_2), \ldots, \nu(v_{p-1})\}] \) under \( \nu[\{v_1, v_2, \ldots, v_{p-1}\}] \).

Note that, for every \( q \) and \( r \) with \( 1 \leq q, r \leq p-1 \), an edge \( v_q v_r \) of \( F \) corresponds to an edge \( \nu(v_q) \nu(v_r) \) of \( U(H) \) and an edge \( \lambda(v_q) \lambda(v_r) \) of \( H \), and that \( \mu(\nu(v_q)) = \lambda(v_q) \).

Now we consider \( v_p \) to decide on \( \nu(v_p) \). Suppose that \( v_p \in V_i \). We shall construct (by specifying its prime factorization) an \( n \in \mathbb{N}_i \) which is a suitable choice for \( \nu(v_p) \). Let \( \{u_1, u_2, \ldots, u_m\} \) be the subset of \( \{v_1, v_2, \ldots, v_{p-1}\} \) of those vertices which are adjacent to \( v_p \) in \( F \), (none of them is with \( v_p \) in \( V_i \), of course). Now, in the prime factorization of \( n \), the primes with indices \( \nu(u_1), \nu(u_2), \ldots, \nu(u_m) \) all occur to the power \( t \), and so does one extra prime with a value so large as to ensure that \( n \) is larger than each of \( \nu(u_1), \nu(u_2), \ldots, \nu(u_m) \); all other primes have power zero in the factorization of \( n \). Then, by specifying \( \nu(v_p) = n \), we have that \( \nu \) establishes the isomorphism \( F[\{v_1, v_2, \ldots, v_p\}] \cong U(H)[\{\nu(v_1), \nu(v_2), \ldots, \nu(v_p)\}] \). Thus a \((\lambda, \mu)\)-respecting isomorphism \( \nu \) from \( F \) onto an induced subgraph of \( U(H) \) (i.e., \( \lambda = \mu \circ \nu \)) is constructed in countably many recursive steps.

Let us here digress for a moment and devote some thought to the case when \( H \) is a directed graph and, correspondingly, consider \( H\)-colourings of directed graphs, i.e., homomorphisms into \( H \) which not only preserve edges, but also the directions of those edges. The obvious way for the directions of the edges of \( H \) to induce directions on the edges of \( U(H) \) is as follows: where we say (in the definition of \( U(H) \)) that “there is an edge in \( U(H) \) between \( m \) and \( n \) if and only if (i) there is an edge in \( H \) between \( w_m \) and \( w_n \); (ii) . . .”, we just add “and the direction of the edge between \( m \) and \( n \) corresponds to the direction of the edge between
any two finite disjoint subsets \( X \subseteq U \). 

In this way \( U(H) \) becomes directed (\( "H\)-directed\), if you like). Similarly, the directions of the edges on \( H \) determine unambiguously the direction of every single edge of \( H \)-colourable graph – irrespective of the choice of a particular \( H \)-colouring of that graph. Colour \( F \) by \( \lambda : F \to H \) and consider an edge in \( F \) between vertices \( v_q \) and \( v_r \). If in \( H \) the edge \((\lambda(v_q), \lambda(v_r))\) is directed from \( \lambda(v_q) \) to \( \lambda(v_r) \), then \((v_q, v_r) \in E(F) \) is directed from \( v_q \) to \( v_r \), since \( \lambda \) preserves directions. And then any other \( H \)-colouring \( \lambda' : F \to H \) would of course yield \((\lambda'(v_q), \lambda'(v_r))\), directed from \( \lambda'(v_q) \) to \( \lambda'(v_r) \) in \( H \). So there is no possibility whatsoever for any edge in \( F \) to be directed in the opposite direction to the one consistent with an arbitrary \( H \)-colouring of \( F \).

The definitions of \( \rightarrow H_c \) and \( H \)-universal in \( \rightarrow H_c \) when \( H \) is directed are also obvious. Thus the proof of Theorem 1 can be rewritten for the directed case with minimal changes. To the crucial \( "an edge \( v_q v_r \) of \( F \) corresponds to an edge \( \nu(v_q) \nu(v_r) \) of \( U(H) \) and an edge \( \lambda(v_q) \lambda(v_r) \) of \( H \)\)”, we can then add \( "with corresponding directions\). To summarise, we obtain

**Corollary 1** For any countable digraph \( H \) the digraph \( U(H) \) is \( H \)-universal in \( \rightarrow H_c \).

## 3 Two characterisations of \( U(H) \)

The Rado graph \( R \), universal in \( \mathcal{I}_c \), has interesting properties, some of which characterise it [4]. It is, up to isomorphism, the unique countable graph with the “extension property”. It is also characterised by being universal in \( \mathcal{I}_c \) and “homogeneous”. (These properties are defined in the sequel.) The relative simplicity of these properties relates to the extreme symmetry of the structure of \( R \). \( H \)-colouring complicates issues somewhat for \( U(H) \). We have, however, already relativised the notion of universality in Section 2 and shall now do so for the extension property and homogeneity to incorporate \( H \)-colourings. This will facilitate characterisations of \( U(H) \) analogous to those of \( R \).

Consider any \( G \in \rightarrow H_c \) and let \( \lambda : G \to H \) be a surjective homomorphism with \( V_1 \cup V_2 \cup \ldots \cup V_k \) of \( V(G) \) the partition associated with it, i.e., for every \( j \), \( V_j = \lambda^{-1}(w_j) \). Then \( V_1 \cup V_2 \cup \ldots \cup V_k \) is called the \( \lambda \)-induced partition of \( V(G) \). Note that every \( G[V_j] \) is edgeless.

We say that a countable graph \( G \) has the \( H \)-extension property if there exists a surjective homomorphism \( \lambda : G \to H \) such that, with respect to the \( \lambda \)-induced partition \( V_1 \cup V_2 \cup \ldots \cup V_k \) of \( V(G) \), for every possible choice of all of the following:

(i) any index \( j \) with \( 1 \leq j \leq k \);

(ii) any finite subset \( Z_j \) of \( V_j \);

(iii) any finite subset \( Q \) of the set of indices \( \{1, 2, \ldots, k\} \) such that, for every \( h \in Q \), \( w_h w_j \in E(H) \);

(iv) any two finite disjoint subsets \( X_h \) and \( Y_h \) of \( V_h \) for each \( h \in Q \),

the following holds: there exists a vertex in \( V_j \) which is not in \( Z_j \) and which is adjacent in \( G \) to every vertex in every \( X_h \) and to no vertex in any \( Y_h \).

If the graph \( G \) has this property, we also say that \( G \) has the \( H \)-extension property with respect to \( \lambda \).

Consider a graph \( G \in \rightarrow H_c \) and the partition \( V(G) = V_1 \cup V_2 \cup \ldots \cup V_k \) induced by a homomorphism \( \lambda : G \to H \). Let \( \alpha : C \to D \) be any isomorphism between two induced subgraphs \( C \) and \( D \) of \( G \). Then we say that \( \alpha \) is \( \lambda \)-isochromatic if, for every \( j \), \( \alpha(V(C) \cap V_j) \subseteq V_j \). We also say that such a graph \( G \) is \( \lambda \)-homogeneous if every \( \lambda \)-isochromatic isomorphism \( \alpha \) between two finite induced subgraphs of \( G \) extends to a \( \lambda \)-isochromatic automorphism \( \alpha^+ \) of \( G \). Note
that, for every \( j, \alpha^+|V_j \) is then a permutation of \( V_j \). When \( G \rightarrow H \) is \( \lambda \)-homogeneous for some surjective \( \lambda : G \rightarrow H \), then we say that \( G \) is \( H \)-homogeneous in \( \rightarrow H \) with respect to \( \lambda \). This is in general a property different from homogeneity (as defined in [8]), which requires the extendability to an ordinary automorphism of any isomorphism between two finite induced subgraphs. (It is known that every homogeneous graph belongs to one of only three types, one of which is “clone of \( R \)” [9].)

**Theorem 2** Let \( G \) be any denumerable graph. Then the following three conditions on \( G \) are equivalent:

(a) \( G \) is a clone of \( U(H) \)

(b) \( G \) has the \( H \)-extension property

(c) There exists a surjective \( H \)-colouring of \( G \) with respect to which \( G \) is both \( H \)-universal and \( H \)-homogeneous in \( \rightarrow H \).

**Proof:**

(a) implies (b): Since an isomorphism preserves all graph theoretical properties, it suffices to prove that \( U(H) \) has the properties ascribed to \( G \) in (b).

We shall prove that \( U(H) \) has the \( H \)-extension property with respect to the partition \( N_1 \cup N_2 \cup \ldots < \cup \lambda \) of \( V(U(H)) \) which is induced by the canonical surjective homomorphism \( \mu : U(H) \rightarrow H \) defined by \( n \rightarrow w_j \) for every \( n \in N_j \).

Consider for any index \( j \) with \( 1 \leq j \leq k \) any finite subset \( Z_j \) of \( N_j \), a finite \( Q \subseteq \{1, 2, \ldots < k\} \) with \( w_h w_j \in E(H) \) for every \( h \in Q \), and any two finite disjoint subsets \( X_h \) and \( Y_h \) of \( V_h \) for each \( h \in Q \). Construct \( n \in N_j \) by having in its prime factorization a factor \( p_j^i \) for every \( u \) in any \( X_h \); a factor \( p_j^i \) for some prime \( p \) large enough to ensure that \( n \) is larger than all the numbers in \( Z_j \cup \{X_h \cup Y_h \mid h \in Q\} \); and all other factors \( p_j^i = 1 \).

It is easy to see that this vertex \( n \) has the required properties to ensure that \( U(H) \) has the \( H \)-extension property with respect to \( \mu \).

(b) implies (a): Assume that \( G \) has the \( H \)-extension property with respect to the surjective homomorphism \( \lambda : G \rightarrow H \) with concomitant \( \lambda \)-induced partition \( V_1 \cup V_2 \cup \ldots < \cup \lambda \) of \( V(G) = \{v_1, v_2, \ldots \} \). For \( V(U(H)) := N_1 \cup N_2 \cup \ldots < \cup \lambda \) we consider any enumeration \( V(U(H)) = \{u_1, u_2, \ldots \} \). By a construction going back and forth between \( G \) and \( U(H) \), recursive on both the indices of \( v_1, v_2, \ldots \) as well as those of \( u_1, u_2, \ldots \), we shall now build an isomorphism \( \alpha : G \rightarrow U(H) \) with the property that, for every \( j, \alpha \mid V_j \) is a bijection from \( V_j \) to \( N_j \), i.e., \( \alpha \) is \((\lambda, \mu)\)-respecting.

Suppose \( v_1 \in V_i \). Define \( \alpha(v_1) \) to be any element of \( N_i \). Let \( u_r \) be the vertex of \( U(H) \) with smallest index \( r \) such that \( u_r \notin \{\alpha(v_1)\} \), and suppose that \( u_r \in N_j \). Now define the finite subset \( Z_j := V_j \setminus \{v_1\} \) of \( V_j \), \( Q = \{i\} \) if \( w_i w_j \in E(H) \) and \( Q = \emptyset \) otherwise, and, for each \( h \in Q \), two finite disjoint subsets \( X_h \) and \( Y_h \) of \( V_h \) as follows:

\[
X_h := \{x \in V_h \mid \alpha(x) \in \{\alpha(v_1)\} \text{ and } \alpha(x) u_r \in E(U(H))\};
\]

\[
Y_h := \{y \in V_h \mid \alpha(y) \in \{\alpha(v_1)\} \text{ and } \alpha(y) u_r \notin E(U(H))\}.
\]

(An explanatory remark: By writing, say \( \alpha(x) \), we of course mean “\( \alpha \) has already been defined on \( x \) at this stage and . . . ”.)

Employing the \( H \)-extension property of \( G \) with respect to \( \lambda \), we now have a vertex, say \( v_s \), in \( V_j \) which is not in \( Z_j \), i.e., \( v_s \notin \{v_1\} \), and \( v_s \) is adjacent in \( G \) to \( v_1 \) if and only if \( \alpha(v_1) \) is adjacent in \( U(H) \) to \( u_r \). By defining \( \alpha(v_s) = u_r \), we have that

- \( v_1 \in V_i \) and \( \alpha(v_1) \in N_i \); \( v_s \in V_j \) and \( \alpha(v_s) \in N_j \); and
• $v_1v_2 \in E(G)$ if and only if $\alpha(v_1)\alpha(v_2) \in E(U(H))$; so
• $G[\{v_1, v_2\}] \cong U(H)[\{\alpha(v_1), \alpha(v_2)\}]$ under $\alpha|\{v_1, v_2\}$.

The next step in the back and forth construction of $\alpha$ has to start again in $G$ with vertex $v_t$, say, with lowest index $t$ of those outside $\{v_1, v_2\}$. But let us rather describe the recursive step in general, distinguishing the two cases, starting from either vertex $v_t \in V(G)$ with smallest index $t$ of some sort, or starting from vertex $u_r \in V(U(H))$ with smallest index $r$ of some sort. In both cases the starting situation before the recursive step is the same:

$I$ is a finite subset of the index set $\{1, 2, \ldots\}$ of $V(G) = \{v_1, v_2, \ldots\}$, and $V_I = \{v_i \in V(G) \mid i \in I\}$. For each $i \in I$ we have now already defined $\alpha(v_i) \in V(U(H))$, with, for each $j$, elements of $V_j$ mapping into $N_j$ and $G[V_I] \cong U(H)[\alpha(V_I)]$ under $\alpha|V_I$.

**Case 1**, starting from $v_t \in V(G)$: Let $v_t \in V(G)$ be the vertex with the lowest index of all those outside $V_I$. We need to define $\alpha(v_t) \in V(U(H))$. Suppose $v_t \in V_I$; we want $\alpha(v_t) \in N_{\ell}$. Define the finite subset $Z_{\ell} := N_{\ell} \cap \alpha(V_I)$ of $N_{\ell}$. Suppose that $\alpha(V_t) \subseteq N_{j_1} \cup N_{j_2} \cup \ldots \cup N_{j_{\lambda t}}$, and define $Q = \{h \in \{j_1, j_2, \ldots, j_{\lambda t}\} \mid w_h w_{\ell} \in E(H)\}$. Now define, for each $h \in Q$, two finite disjoint subsets $X_h$ and $Y_h$ of $N_{h \ell}$ as follows:

\[
X_h := \{x \in N_{h \ell} \mid \text{for some } i \in I, \; x = \alpha(v_i) \text{ and } v_i v_t \in E(G)\};
\]

\[
Y_h := \{y \in N_{h \ell} \mid \text{for some } i \in I, \; y = \alpha(v_i), \; \text{but } v_i v_t \notin E(G)\}.
\]

By the $H$-extension property of $U(H)$ with respect to $\mu$, there exists a vertex, say $u_s \in N_{\ell t}$, which is not in $\alpha(V_I)$, which is adjacent in $U(H)$ to every $\alpha(v_i)$, $i \in I$, for which $v_i v_t \in E(G)$, while $u_s$ is adjacent in $U(H)$ to no $\alpha(v_i)$ for which $v_i v_t \notin E(G)$. By defining $\alpha(v_t) = u_s$, we now have extended $\alpha$ to establish $G[V_I \cup \{v_t\}] \cong U(H)[\alpha(V_t) \cup \{u_s\}]$.

**Case 2**, starting from $u_r \in V(U(H))$: Let $u_r$ be that vertex of $U(H)$ with the least index $r$ among those vertices not already in $\alpha(V_I)$. Suppose that $u_r \in N_{j}$. We need to find a $v_s \in V_J$ suitable for defining $\alpha(v_s) = u_r$. Define the finite subset $Z_j := V_j \cap V_I$ of $V_j$. Suppose that $V_I \subseteq V_{g_1} \cup V_{g_2} \cup \ldots \cup V_{g_{\lambda t}}$, and define $Q = \{h \in \{g_1, g_2, \ldots, g_{\lambda t}\} \mid w_h w_{\ell} \in E(H)\}$. Now define, for each $h \in Q$, the finite disjoint subsets $X_h$ and $Y_h$ of $V_h$ as follows:

\[
X_h := \{x \in V_h \mid x \in V_I \text{ and } \alpha(x)u_r \in E(U(H))\};
\]

\[
Y_h := \{y \in V_h \mid y \in V_I \text{ and } \alpha(y)u_r \notin E(U(H))\}.
\]

Employing the $H$-extension property of $G$ with respect to $\lambda$, we have a vertex, say $v_s \in V_j \setminus V_I$, which is adjacent in $G$ to every $v_i$, $i \in I$, for which $\alpha(v_i)$ is adjacent in $U(H)$ to $u_r$ and to no $v_i$ for which $\alpha(v_i)$ is not adjacent in $U(H)$ to $u_r$. By defining $\alpha(v_s) = u_r$ we extend $\alpha$ to ensure that $G[V_I \cup \{v_t\}] \cong U(H)[\alpha(V_I) \cup \{u_r\}]$.

By alternating the two cases in a back and forth manner through denumerably many recursive steps we construct the isomorphism $\alpha : G \cong U(H)$.

(b) implies (c): Suppose $G$ satisfies (b) with respect to the surjective homomorphism $\zeta : G \to H$. Suppose (again) that $V(G) = \{v_1, v_2, \ldots\}$, with $V_1 \cup V_2 \cup \ldots \cup V_{k_1}$ the partition of $V(G)$ induced by $\zeta$.

To demonstrate $H$-universality of $G$ in $\neg H_c$ with respect to $\zeta$, let $C$ be any graph in $\neg H_c$ and suppose $V(C) = \{u_1, u_2, \ldots, u_{k_1}\}$ with $U_1 \cup U_2 \cup \ldots \cup U_{k_1}$ the induced partition of its vertex set with respect to a homomorphism $\lambda : C \to H$. By recursion on the indices of the vertices of $C$ we construct a $(\lambda, \zeta)$-respecting embedding $\alpha : C \to G$ of $C$ into $G$.

If $u_1 \in U_1$, let $\alpha(u_1)$ be any element of $V_1$. Next suppose that $r$ is the least index of a vertex of $C$ for which $\alpha(u_r)$ has not yet been defined and suppose that $u_r \in U_r$. Let $R = \{\alpha(u_1), \alpha(u_2), \ldots, \alpha(u_{r-1})\}$ and suppose that $R \subseteq V_{g_1} \cup V_{g_2} \cup \ldots \cup V_{g_{\lambda t}}$. Now define the finite
subset $Z_j := V_j \cap R$ of $V_j$, $Q := \{ h \in \{ g_1, g_2, \ldots, g_k \} \ | w_h w_j \in E(H) \}$, and, for each $h \in Q$, two finite disjoint subsets $X_h$ and $Y_h$ of $V_h \cap R$ as follows:

$$X_h := \{ x \in V_h \cap R \ | \ \alpha^{-1}(x) u_r \in E(C) \}$$

and

$$Y_h := \{ y \in V_h \cap R \ | \ \alpha^{-1}(y) u_r \notin E(C) \}.$$  

Employing the $H$-extension property of $G$ with respect to $\zeta$, we have a vertex, say $v_s \in V_j$, which is not in $Z_j$ and which is, for every $h \in Q$, adjacent to $v_s$ in $G$ and not adjacent to $G$ to any $y \in Y_h$. By defining $\alpha(u_r) = v_s$ we extend $\alpha$ and ensure in the process that $C[\{ u_1, u_2, \ldots, u_r \}] \cong G[\{ \alpha(u_1), \alpha(u_2), \ldots, \alpha(u_r) \}]$. Thus a $(\lambda, \zeta)$-respecting isomorphism from $C$ onto an induced subgraph of $G$ is constructed in countably many recursive steps. So $G$ is $H$-universal in $\to H_c$ with respect to $\zeta$.

Next we prove that $G$ is $H$-homogeneous in $\to H_c$ with respect to the same $\zeta$ with respect to which it is $H$-universal in $\to H_c$. Consider any two isomorphic finite induced subgraphs $C$ and $D$ of $G$ with a $\zeta$-isochromatic automorphism $\alpha$ from $C$ onto $D$. We prove that $\alpha$ can be extended to a $\zeta$-isochromatic automorphism $\alpha^+ \cong G$ by a recursive construction on the indices of the vertices in $V(G)$. Hence suppose that $V$ is a finite subset of $V(G)$ containing $V(C)$, and that a $\zeta$-isochromatic isomorphism $\alpha^+$, which extends $\alpha$, has already been defined from $V$ into $V(G)$:

$$\alpha^+ : G[V] \cong G[\alpha^+(V)].$$

Let $r$ be the least index of a vertex of $G$ for which $v_r \notin V$; we want to define $\alpha^+(v_r)$. Suppose that, with respect to the $\zeta$-induced partition of $V(G)$, $v_r \in V_j$. Define $Z_j := \alpha^+(V) \cap V_j$. Suppose $\alpha^+(V) \subseteq V_{g_1} \cup V_{g_2} \cup \ldots \cup V_{g_q}$ and let $Q := \{ h \in \{ g_1, g_2, \ldots, g_q \} \ | w_h w_j \in E(H) \}$. For every $h \in Q$, define

$$X_h := \{ \alpha^+(x) \in V_h \ | \ x \in V \cap V_h \text{ and } xv_r \in E(G) \}$$

and

$$Y_h := \{ \alpha^+(y) \in V_h \ | \ y \in V \cap V_h \text{ and } yv_r \notin E(G) \}.$$  

By the $H$-extension property with respect to the $\zeta$-induced partition of $V(G)$, there exists a vertex $z \in V_j$ which is not in $\alpha^+(V)$ and which is adjacent to every vertex $\alpha^+(x)$ in $\alpha^+(V)$ for which $x$ is adjacent to $v_r$ in $G$ and to no vertex $\alpha^+(y)$ in $\alpha^+(V)$ for which $y$ is not adjacent to $v_r$ in $G$. Hence, by defining $\alpha^+(v_r) = z$, we have extended $\alpha^+$ to a $\zeta$-isochromatic isomorphism on $V \cup \{ v_r \}$ into $G$.

By symmetry (between $C$ and $D$), the same proof can now be used to extend it starting with $V(D) \cup \{ z \}$. Hence, in denumerably many such recursive back and forth steps, a $\zeta$-isochromatic automorphism of $G$ which extends $\alpha$ is built, establishing the $H$-homogeneity of $G$ with respect to $\zeta$.

**c) implies b):** Suppose $G$ is $H$-universal in $\to H_c$ and $H$-homogeneous in $\to H_c$, both with respect to the same surjective homomorphism $\zeta : G \to H$ and corresponding partition

$$V(G) = V_1 \cup V_2 \cup \ldots \cup \cup V_k$$

in which each $V_j = \zeta^{-1}(w_j)$. With respect to this $\zeta$-induced partition we show that $G$ has the $H$-extension property. Pick a finite subset $Z_j$ of $V_j$, a finite set $Q$ of indices $h \in \{ 1, 2, \ldots, q \}$ for which $w_h w_j \in E(H)$, and, for each $h \in Q$, two finite disjoint subsets $X_h$ and $Y_h$ of $V_h$.

First we construct a finite graph $A$ (considered as if outside $G$) from the subgraph $F = G[Z_j \cup \cup \{ X_h \cup Y_h \ | \ h \in Q \}]$ of $G$ by adding to $F$ a vertex $m \notin V(G)$ and, for each $q \in \cup \{ X_h \ | \ h \in Q \}$ an edge between $m$ and $q$. From the partitioning

$$V(A) = \cup \{ (X_h \cup Y_h) \ | \ h \in Q \} \cup \{ \{ m \} \cup Z_j \}$$  

it is clear that $A \to H_c$. This is ensured by the homomorphism $\lambda : A \to H$ which, for $h \in Q$, maps each element of $X_h \cup Y_h$ to $w_h$, and each element of $\{ m \} \cup Z_j$ to $w_j$. 

7
By the $H$-universality of $G$ in $\rightarrow H_c$ with respect to $\zeta$, the graph $A$ can $(\lambda, \zeta)$-respectively be isomorphically embedded into $G$ by, say, $\alpha : A \rightarrow B$, where $B = \alpha(A)$ is an induced subgraph of $G$. Suppose $\alpha(m) = n \in V_j$. (That $\alpha(m) \in V_j$ follows from the fact that $\zeta(\alpha(m)) = \lambda(m) = w_j$.) It is clear that the subgraph $F$ of $G$ and the subgraph $B - n$ of $G$ are $\zeta$-isochromatically isomorphic. By the $H$-homogeneity of $G$ in $\rightarrow H_c$ with respect to $\zeta$, this isomorphism can be extended to a $\zeta$-isochromatic automorphism $\beta$ of $G$. Clearly the vertex $\beta^{-1}(n)$, corresponding to $n$, has the properties required to ensure that $G$ has the $H$-extension property with respect to the $\zeta$-induced partition of $V(G)$ with which we started. \hfill $\square$

Properties of $R$, universality, extension, and homogeneity, have now been $H$-"relativised". $R$ is also self-complementary, i.e., isomorphic to its complement. Does $U(H)$ have what we could call its "$H$-complement" to which it is isomorphic? The answer is "yes".

We shall now define a graph $U^*(H)$, which will turn out to be a clone of $U(H)$. Its vertex set is the same as that of $U(H)$:

$$V(U^*(H)) := N_1 \cup N_2 \cup \ldots \cup N_k .$$

In $U^*(H)$ there are no edges between vertices from the same $N_j$. If $m \in N_h$ and $n \in N_j$, $h \neq j$, and $w_h w_j \notin E(H)$, then there is no edge in $U^*(H)$ between $m$ and $n$. If, however, $w_h w_j \in E(H)$, then $mn \in E(U^*(H))$ if and only if $mn \notin E(U(H))$:

$$E(U^*(H)) := \{ mn \mid \mu(m)\mu(n) \in E(H) \text{ but } mn \notin E(U(H)) \} .$$

**Theorem 3** $U^*(H) \cong U(H)$.

**Proof:**
By Theorem 2 it is sufficient to prove that $U^*(H)$ has the $H$-extension property. The same function $\mu : V(U^*(H) \rightarrow V(H))$ that we had before as the canonical $\mu : U(H) \rightarrow H$ is also a surjective $H$-colouring of $U^*(H)$, inducing the same partition of $V(U^*(H))$ into $N_j$'s. We show that $U^*(H)$ has the $H$-extension property with respect to this $\mu$.

Choose a $j$; a finite subset $Z_j$ of $N_j$; a finite set $Q$ of indices $h$ such that $w_h w_j \in E(H)$; and, for each $h \in Q$, two finite disjoint subsets $X^*_h$ and $Y^*_h$ of $N_h$. Now define, for each $h \in Q$, $X_h = Y^*_h$ and $Y_h = X^*_h$. By the $H$-extension property of $U(H)$ there is a vertex $z \in N_j \setminus Z_j$ which is adjacent in $U(H)$ to every vertex in every $X_h$ and to no vertex in any $Y_h$. By the definition of $E(U^*(H))$, $z$ is adjacent in $U^*(H)$ to every vertex in every $X^*_h$ and to no vertex in any $Y^*_h$. Hence $U^*(H)$ has the $H$-extension property with respect to $\mu$, and so is a clone of $U(H)$.

At the end of Section 1 in Corollary 1 we formulated how, with apt small changes to definitions, the $H$-universality of $U(H)$ in $\rightarrow H_c$ (as established in Theorem 1) holds also in the case of digraphs. Similarly now, modulo slight adaptations of the definitions of the "$H$-extension property" and being "$H$-homogeneous in $\rightarrow H_c$", no major conceptual struggle is required to see the truth of

**Corollary 2** Given the obvious definitions of the concepts involved, Theorems 2 and 3 hold for digraphs too.

4 Special cases: $H \cong K_k$, and $U(H) \cong R$

We now investigate the special case of the definitions and results of the previous two sections if we choose $H = K_k$. Note that the existence of a homomorphism $\lambda : G \rightarrow K_k$ is equivalent to the existence of a partition $V_1 \cup V_2 \cup \ldots \cup V_k$ of the vertex set $V(G)$ into subsets inducing edgeless subgraphs of $G$, i.e., the existence of a (classical) $k$-colouring of $G$. Hence the hom-property $\rightarrow (K_k)_c$ is the property of countable $k$-colourable graphs.
Taking these remarks into account, the concepts defined above simplify to:

- $G$ is $K_k$-universal in $\neg(K_k)_c$, if there is a $k$-colouring $\zeta$ of $G$ which uses all $k$ colours, such that for every $k$-colourable graph $F$ (in $\neg(K_k)_c$) and every $K_k$-colouring $\lambda : F \to K_k$ there is a $(\lambda, \zeta)$-respecting isomorphic embedding $\nu : F \to G$; hence for every $k$-colouring $\lambda : V(F) \to \{1, 2, \ldots, k\}$ of $F$ the $k$-colouring $\zeta : V(G) \to \{1, 2, \ldots, k\}$ of $G$ is such that the isomorphism $\nu$ respects the colours assigned to vertices, i.e., if a vertex $x$ of $F$ is assigned the colour $j$ by $\lambda$, then the vertex $\nu(x)$ of $G$ is assigned the same colour $j$ by $\zeta$.

- In this strong sense of universality, $U(K_k)$ is $K_k$-universal in $O_c^k$ by Theorem 1.

We can also simplify the characterisation of $U(H)$ when taking $H = K_k$. Again we start with the simplified definitions:

- Consider (again) a $k$-colourable graph $G$ and consider any $k$-colouring $\lambda : G \to K_k$ of it with $V_1 \cup V_2 \cup \ldots V_k$ its colour classes. Let $\alpha : C \to D$ be any isomorphism between induced subgraphs $C$ and $D$ of $G$. Then $\alpha$ is $\lambda$-isochromatic if it preserves the colours assigned to the vertices of $C$. Furthermore, $G$ is $\lambda$-homogeneous if every $\lambda$-isochromatic isomorphism between finite induced subgraphs $C$ and $D$ of $G$ extends to a $\lambda$-isochromatic automorphism of $G$. Furthermore, a $k$-colourable graph $G \in O_c^k$ is $K_k$-homogeneous in $O_c^k$ if it is $\lambda$-homogeneous for some surjective $k$-colouring $\lambda : G \to K_k$.

- These concepts can now be used to characterise the clones of $U(K_k)$ by simply formulating a special case of Theorem 2.

The concept of a universal graph in $O_c^k$ is revisited in the next section where a second construction of such a graph is given.

We now turn our attention to the special case $H = K_{n_0}$. In this case, it is immediate to see that $U(H)$ is universal in $\mathcal{T}_c$ since $\neg(K_{n_0})_c = \mathcal{T}_c$. Indeed, we shall see that $U(H)$ is a clone of the Rado graph $R$, the classical universal graph in $\mathcal{T}_c$. Hence our construction of $U(H)$, using the prime factorizations of positive integers, supplements the many constructions of $R$ discussed in [2].

Our result uses the fact that the Rado graph is characterised by the (classical) extension property (for which see [4]): A graph $G$ is said to have the extension property if for every two finite disjoint sets of vertices $X$ and $Y$ of $G$ there is a vertex of $G$ outside $X \cup Y$ which is adjacent in $G$ to every vertex of $X$ and to no vertex of $Y$. Furthermore, a graph $G$ is here said to have the weak extension property if for every finite set of vertices $X$ of $G$ there is a vertex of $G$ outside $X$ which is adjacent in $G$ to every vertex of $X$.

**Theorem 4** The graph $U(H)$ has the extension property (i.e., $U(H) \cong R$) if and only if $H$ has the weak extension property.

**Proof:**

$\Rightarrow$: Suppose $U(H)$ has the extension property. Let $X = \{w_{i_1}, w_{i_2}, \ldots, w_{i_p}\}$ be any finite subset of $V(H)$. Pick a finite set $X' = \{u_1, u_2, \ldots, u_p\}$ of vertices in $V(U(H))$ with $u_j \in N_{i_j}$ for $1 \leq j \leq p$ and $Y' = \emptyset$ if you insist. By the extension property of $U(H)$ there is a vertex $u \in V(U(H))$, say $u \in N_{i_\ell}$, with $u_j u \in E(U(H))$ for every $j$. By the definition of adjacency in $U(H)$, $N_{i_\ell} \neq N_{i_j}$ for every $j$. Since the canonical map $U(H) \to H$ preserves adjacency, it follows that $w_{i_j} w_{i_\ell} \in E(H)$ for every $j$, establishing the weak extension property of $H$.

$\Leftarrow$: Suppose $H$ has the weak extension property. Let $X = \{x_1, x_2, \ldots, x_p\}$ and $Y = \{y_1, y_2, \ldots, y_q\}$ be any two finite disjoint subsets of $V(U(H))$. Assume that $X \subset N_{i_1} \cup N_{i_2} \cup \ldots \cup N_{i_p}$ where the sets in this union need not all be different. Consider the finite set $\{w_{i_1}, w_{i_2}, \ldots, w_{i_p}\} \subseteq V(H)$ (not necessarily all different vertices). By the weak extension property of $H$, there is a vertex $w_{i_\ell} \in V(H)$ with $w_{i_j} w_{i_\ell} \in E(H)$ for every $j$. By specifying its prime factorization, we define a vertex $n \in N_{i_\ell}$ of $U(H)$ which is adjacent in $U(H)$ to every element of $X$ and to no element of $Y$.

- $n_m = \ell$ if $m \in X$;
Clarify, K extension property it also has the different property of being homogeneous, like its clone R of a non-complete (denumerable) graph G. A U \leq (1 \kappa (together with an extra element, if needed); it is the vertex with the required properties. The weak extension property easily follows by considering the union of finitely many finite sets from only \{n\}, i.e., a (finite) power series reassured by Theorem 1 since O[5]. The existence of a universal graph in an induced-hereditary graph property (of finite character, by the Compactness Theorem – see 5.) Another universal graph in \Gamma is isomorphic to \Gamma (Theorem 2), when \Gamma has the extension property it also has the different property of being homogeneous, like its clone R.

Since K_{\aleph_0} obviously has the weak extension property, we immediately have that \Gamma(K_{\aleph_0}) \cong R. Clearly, K_{\aleph_0} does not have the extension property. We conclude this section with an example of a non-complete (denumerable) graph G which also has the weak extension property but not the extension property: Let V(G) be the set of all finite subsets of N and let AB \in E(G) if and only if A and B are different comparable subsets of N, i.e., subsets satisfying A \subset B or B \subset A. The weak extension property easily follows by considering the union of finitely many finite sets (together with an extra element, if needed); it is the vertex with the required properties. The fact that G does not have the extension property follows by considering, for example, for X any singleton \{F\} consisting of any finite set F with at least two elements and for Y the finite set of its proper subsets \{H \mid H \subset F\}. Clearly, every (finite) subset of N comparable to F is then comparable to some element of Y.

5 Another universal graph in k-colourable countable graphs

A graph G is called k-colourable (k \geq 1) if its vertex set can be partitioned into k subsets such that every edge has its endpoints in two of these (different) sets. The graph property of countable k-colourable graphs is \mathcal{O}_c^k = \{G \mid G is a countable k-colourable graph\}; it is also an induced-hereditary graph property (of finite character, by the Compactness Theorem – see [5]). The existence of a universal graph in \mathcal{O}_c^k is already guaranteed by the results of [10] (and reassured by Theorem 1 since \mathcal{O}_c^k = (\rightarrow K_k)_c).

Every positive integer n has a unique (k + 1)-ary expansion with entries from \{0, 1, \ldots, k\}, i.e., a (finite) power series n = \sum_{i=0}^{\infty} n_i(k+1)^i with k \geq 1 and 0 \leq n_i \leq k. We shall refer to n_{i-1} (i \geq 1) as the entry in the i^{\text{th}} position of the expansion. We use this fact to define k denumerable, pairwise disjoint, proper subsets M_1, M_2, \ldots, M_k of N = \{1, 2, \ldots\}: M_i (1 \leq i \leq k) is the set of all those positive integers whose (k + 1)-ary expansion has all entries from only \{0, i\}. We now define a graph U_k as follows: The vertex set of U_k is

\[ V(U_k) := M_1 \cup M_2 \cup \ldots \cup M_k. \]

In U_k there are no edges between vertices from the same M_i. If m \in M_i and n \in M_j (i \neq j), then there is an edge in U_k between m and n if and only if m < n and n has the entry j in position number m of its (k + 1)-ary expansion: n_{m-1} = j. For this graph we shall now prove universality in the set \mathcal{O}_c^k.

**Theorem 5** For any positive integer k the graph U_k is universal in \mathcal{O}_c^k.

**Proof:**

By the construction of U_k, it is clear that U_k \in \mathcal{O}_c^k, in fact U_k \in \mathcal{O}_d^k. Let G with V(G) = \{v_1, v_2, \ldots\} be any countable k-colourable graph. We recursively construct an injection \alpha : V(G) \to V(U_k) such that \alpha(V(G)) induces a subgraph of U_k which, under \alpha, is isomorphic to G. Since G is k-colourable, there is a partition V(G) = V_1 \cup V_2 \cup \ldots \cup V_k such that G has no edge between any two vertices from the same V_i. Suppose v_1 \in V_i and let \alpha(v_1) be any element of M_i. Next assume that \alpha(v_1), \alpha(v_2), \ldots, \alpha(v_{p-1}) have already been specified so that G[\{v_1, v_2, \ldots, v_{p-1}\}] \cong U_k[\{\alpha(v_1), \alpha(v_2), \ldots, \alpha(v_{p-1})\}]. Suppose that v_p \in V_j.
We shall construct an \( n \in M_j \) which is a suitable choice for \( \alpha(v_p) \) by prescribing its \((k+1)\)-ary expansion. Let \( \{u_1, u_2, \ldots, u_\ell\} \) be the subset of \( \{v_1, v_2, \ldots, v_{p-1}\} \) of those vertices which are adjacent to \( v_p \) in \( G \), (none of them is with \( v_p \) in \( V_j \), of course). Now the \((k+1)\)-ary expansion of \( n \) has entry \( j \) in the positions with numbers \( \{\alpha(u_1), \alpha(u_2), \ldots, \alpha(u_\ell)\} \), as well as in one position with a number so large as to ensure that \( n \) is larger than each of \( \alpha(u_1), \alpha(u_2), \ldots, \alpha(u_\ell) \) – and entry 0 in all other positions. It is clear that by specifying \( \alpha(v_p) = n \) we have that \( G[\{v_1, v_2, \ldots, v_p\}] \cong U_k[\alpha(v_1), \alpha(v_2), \ldots, \alpha(v_p)] \). Thus an isomorphism from \( G \) onto an induced subgraph of \( U_k \) is constructed in countably many steps.

If you now wonder whether \( U_k \) is even \( K_k \)-universal in \( O_k^c \), the answer is “yes”, as will be seen in Theorem 6. Note that both constructions we provide of universal graphs in \( O_k^c \), namely \( U(K_k) \) in Section 2 and \( U_k \) above, have the desirable properties of what is called an \( A \)-type universal graph in [3].

We now proceed to show that the graph \( U_k \) also has, like \( U(K_k) \), the \( K_k \)-extension property with respect to the surjective homomorphism \( \zeta : U_k \rightarrow K_k \) which maps every vertex in \( M_j \) to the vertex \( j \) of \( K_k \), \( 1 \leq j \leq k \).

**Lemma 1** The graph \( U_k \) has the \( K_k \)-extension property with respect to \( \zeta \).

**Proof:**
Let \( M_1 \cup M_2 \cup \ldots \cup M_k \) be the partition of \( V(U_k) \) induced by \( \zeta \). Pick a \( j \), a finite subset \( Z_j \) of \( M_j \), any subset \( Q \) of \( \{1, 2, \ldots, k\} \setminus \{j\} \), and, for every \( h \in Q \), two finite disjoint subsets \( X_h \) and \( Y_h \) of \( M_h \). We define \( v \in M_j \) through its \((k+1)\)-ary expansion by putting an entry \( j \) in every position \( u \) for which \( u \in \bigcup \{X_h \mid h \in Q\} \) and in some position which is larger than all of the numbers in \( Z_j \cup \bigcup \{X_h \cup Y_h \mid h \in Q\} \) and a 0 in every position \( u \) for which \( u \in \bigcup \{Y_h \mid h \in Q\} \); the remaining positions can be filled with 0’s and (finitely many) \( j \)’s at will. It is easy to see that this vertex \( v \) has the required properties.

From Lemma 1, together with Theorem 2, it now follows that \( U_k \) and \( U(K_k) \), although constructed differently, are clones of each other, and hence share the same graph theoretical properties. From Theorem 2 we then have as a special case

**Theorem 6** Let \( G \) be any denumerable graph. Then the following four conditions on \( G \) are equivalent:

(a) \( G \) is a clone of \( U_k \)

(b) \( G \) is a clone of \( U(K_k) \)

(c) \( G \) has the \( K_k \)-extension property

(d) There exists a surjective \( K_k \)-colouring of \( G \) with respect to which \( G \) is both \( K_k \)-universal and \( K_k \)-homogeneous in \( -(K_k)_c \).

6 Split graphs

In logic, a contradiction and a tautology harbour no employable information: no possibility, or every possibility, is allowed. Somewhat analogously, in graph theory, an edgeless graph and a complete graph have similarly inane internal structure. This has the effect that if in \( U_{k+l} \) we replace \( l \) of the edgeless induced subgraphs by complete subgraphs, we easily obtain another universal graph. We now describe this briefly.

Following [7], a graph \( G \) is called a \((k,l)\)-split graph \((k \geq 0, l \geq 0 \text{ and } k+l \geq 2)\) if its vertex set can be partitioned into \( k+l \) (possibly empty) subsets

\[
V_1, V_2, \ldots, V_k, V_{k+1}, V_{k+2}, \ldots, V_{k+l}
\]
such that each induced subgraph $G[V_i], i = 1, 2, \ldots, k$ is edgeless, while each induced subgraph $G[V], j = k + 1, k + 2, \ldots, k + l$ is a complete subgraph of $G$. The graph property of countable $(k, l)$-split graphs is denoted by $S_{k,l}^{(k)} = \{G \mid G$ is a countable $(k, l)$-split graph$\}$; it is also an induced-hereditary graph property. Note that the $(k, 0)$-split graphs are exactly the $k$-colourable graphs, i.e., $S_{k,0}^{(k)} = \mathcal{O}_{k}^{(k)}$.

We now define the graph $U_{k,l}$ with the following adaptation of our previous graph $U_{k+l}$. It has vertex set

$$V(U_{k,l}) = V(U_{k+l}) = M_1 \cup \ldots \cup M_k \cup M_{k+1} \cup \ldots \cup M_{k+l}$$

but now with no edges between vertices from the same $M_i$ when $1 \leq i \leq k$; and all possible edges between vertices from the same $M_j$ when $k + 1 \leq j \leq k + l$. Edges between vertices from $M_g$ and $M_h$ ($g \neq h$) are exactly as in $U_{k+l}$. Then the proof of Theorem 5 can be rewritten with slight adaptations to yield

**Theorem 7** For any integers $k$ and $l$, where $k \geq 0, l \geq 0$ and $k + l \geq 2$, the graph $U_{k,l}$ is universal in $S_{k,l}^{(k)}$.

We have not (yet) been able to prove or disprove that $U_{k,k}$ is self-complementary, but formulate the open

**Conjecture 1** $U_{k,k} \cong \overline{U_{k,k}}$. 

**References**


