

# Graded Tensor Products and the Problem of Tensor Grade Computation and Reduction

Andrei Khrennikov

International Center for Mathematical Modelling  
in Physics and Cognitive Sciences

Linnaeus University, Växjö, Sweden

Elemer E. Rosinger and Augustin van Zyl

Department of Mathematics, University of Pretoria  
South-African Republic

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## **Abstract**

We consider a non-negative integer valued grading function on tensor products which aims to measure the extent of entanglement. This grading, unlike most of the other measures of entanglement, is defined exclusively in terms of the tensor product. It gives a possibility to approach the notion of entanglement in a more refined manner, as the non-entangled elements are those of grade zero or one, while the rest of elements with grade at least two are entangled, and the higher its grade, the more entangled an element of the tensor product is. The problem of computing and reducing the grade is studied in products of arbitrary vector spaces over arbitrary fields.

# 1 Introduction

The notion of entanglement plays a fundamental role in modern quantum mechanics and especially quantum information theory, see, e.g., [1]. In this note we study entanglement by using a non-negative integer valued grading function defined on tensor products. This provides a possibility to bring further light upon the notoriously complex structure of entangled states. Instead of the standard dichotomy, entangled/separable, we suggest a more refined scale of degrees of entanglement, with the elements of larger grade having a higher level of entanglement. We further study the problem of the computation of the grade of a tensor, as well as the reduction of that grade. This study is done in the case of tensor products of arbitrary vector spaces over arbitrary fields, although some of the results only hold for more particular cases. This provides a possibility to apply the approach presented here, which is based on the graded tensor products, to generalizations of entanglement not only in the standard quantum mechanics over the field of complex numbers, but even to non-Archimedean quantum models, see [2] for extended review on  $p$ -adic physical models, see [3] on other non-Archimedean physical models, including models over nonstandard numbers, and comparing them with  $p$ -adic theory.

## 2 Grading tensor products

Let  $E, F$  be vector spaces over the field  $\mathbb{K}$ . For  $n \in \mathbf{N} = \{0, 1, 2, 3, \dots\}$ , we denote by

$$E \overset{n}{\otimes} F \tag{1}$$

the null vector subspace of  $E \otimes F$ , if  $n = 0$ , while for  $n \geq 1$ , we denote the set of all elements  $z \in E \otimes F$  which have the form

$$z = x_1 \otimes y_1 + \dots + x_n \otimes y_n \tag{2}$$

for suitable  $x_1, \dots, x_n \in E, y_1, \dots, y_n \in F$ . We call

$$E \overset{n}{\otimes} F$$

the  $n$ -graded subset of the tensor product  $E \otimes F$ . Clearly

$$E \overset{0}{\otimes} F \subseteq E \overset{1}{\otimes} F \subseteq \dots \subseteq E \overset{n}{\otimes} F \subseteq \dots; \quad (3)$$

$$(E \overset{n}{\otimes} F) \cup (E \overset{m}{\otimes} F) \subseteq E \overset{\max\{n,m\}}{\otimes} F \quad (4)$$

$$(E \overset{n}{\otimes} F) + (E \overset{m}{\otimes} F) \subseteq E \overset{n+m}{\otimes} F, \quad n, m \in \mathbf{N}; \quad (5)$$

$$\mathbf{K}(E \overset{n}{\otimes} F) \subseteq E \overset{n}{\otimes} F, \quad n \in \mathbf{N}; \quad (6)$$

$$E \otimes F = \bigcup_{n \in \mathbf{N}} (E \overset{n}{\otimes} F). \quad (7)$$

The difficulty with these  $n$ -graded subsets

$$E \overset{n}{\otimes} F$$

is that, except for  $n = 0$ , they are *not* closed under addition, thus they are not vector subspaces of  $E \otimes F$ .

Now, in view of (7), we define the *tensor grading* mapping

$$gr : E \otimes F \longrightarrow \mathbf{N} \quad (8)$$

by

$$gr(z) = \min\{n \in \mathbf{N} \mid z \in E \overset{n}{\otimes} F\}. \quad (9)$$

and  $gr(z)$  is called the *tensor grade* of  $z$ .

As is well known, we have the *injective* mapping

$$E \times F \ni (x, y) \longmapsto x \otimes y \in E \otimes F \quad (10)$$

and obviously, for  $z \in E \otimes F$ , we have

$$gr(z) \leq 1 \iff z = x \otimes y, \text{ for suitable } x \in E, y \in F. \quad (11)$$

Let us now for  $n \in \mathbb{N}$  denote

$$E \underset{n}{\otimes} F = \{z \in E \overset{n}{\otimes} F \mid gr(z) = n\}. \quad (12)$$

Then clearly

$$(E \underset{n}{\otimes} F) \cap (E \underset{m}{\otimes} F) = \phi, \quad n, m \in \mathbb{N}, n \neq m; \quad (13)$$

$$(E \underset{n}{\otimes} F) \cap (E \overset{m}{\otimes} F) = \phi, \quad n, m \in \mathbb{N}, m < n. \quad (14)$$

Further we have

$$E \underset{0}{\otimes} F = E \overset{0}{\otimes} F = \{0\} \quad (15)$$

and the *surjective* mapping

$$E \times F \ni (x, y) \longmapsto x \otimes y \in \{0\} \bigcup_1 (E \underset{1}{\otimes} F). \quad (16)$$

Consequently

$$\mathcal{ENT}(E \otimes F) = \bigcup_{n>1} (E \underset{n}{\otimes} F) \quad (17)$$

is the set of *entangled* elements in the tensor product  $E \otimes F$ .

Obviously

$$E \otimes F = \bigcup_{n \in \mathbb{N}} (E \underset{n}{\otimes} F), \quad (18)$$

thus

$$E \otimes F = \{0\} \bigcup_1 (E \otimes F) \bigcup \mathcal{ENT}(E \otimes F). \quad (19)$$

**Note 1.1.** An interest in the above comes from the quanta. The state space of the composite of quantum systems is given by the tensor product of the state spaces of the component systems. Further, in case  $E \otimes F$ , for instance, is the state space of such a composite system, then the *entangled* elements  $z$  in it are precisely those for which  $gr(z) \geq 2$ , see (1.16), (1.17) above. And as is well known, the phenomenon of quantum entanglement is fundamental with respect

to quanta. By introducing the grading function (8), (9), we obtain a *measure* of entanglement, according to which the higher the grade of a tensor, the more entangled it is. An important feature of the grading function  $\text{gr}$  in (8), (9) is that it is defined *exclusively* in terms of the respective tensor product  $E \otimes F$ , unlike the various measures of entanglement in the literature.

### 3 Tensor Reduction

We are interested in the following general problem of tensor reduction:

$$\forall a_1, \dots, a_n, a_{n+1} \in E, b_1, \dots, b_n, b_{n+1} \in F :$$

$$\left( \begin{array}{l} \exists c_1, \dots, c_n \in E, d_1, \dots, d_n \in F : \\ a_1 \otimes b_1 + \dots + a_n \otimes b_n + a_{n+1} \otimes b_{n+1} = c_1 \otimes d_1 + \dots + c_n \otimes d_n \end{array} \right) \Leftrightarrow ?$$

#### 3.1 Schmidt decomposition

**Theorem 3.1.** *Let  $H_1, H_2$  be Hilbert spaces. Every representation of a tensor  $u \in H_1 \otimes H_2$  can be changed to a Schmidt decomposition, without increasing the number of terms in the representation. In other words, let  $u = \sum_{k=1}^n x_k \otimes y_k$ . Then there is a representation  $u = \sum_{k=1}^n \alpha_k u_k \otimes v_k$ , with the  $\{u_k\}$  and  $\{v_k\}$  each being orthonormal sets, while  $\alpha_k$  are non-negative constants.*

*Proof.* Let  $J = \text{span}\{x_k\}_{k=1}^n$  and  $K = \text{span}\{y_k\}_{k=1}^n$ . Then  $J$  inherits a Hilbert space structure from  $H_1$ , and has dimension  $m_1 \leq n$ . Similarly  $K$  inherits the Hilbert space structure from  $H_2$ , and has dimension say  $m_2 \leq n$ . Observe that  $u \in J \otimes K$ . By the Schmidt decomposition theorem, there exists orthonormal sets  $\{u_k\}$  in  $J$ , and  $\{v_k\}$  in  $K$  so that  $u = \sum_{k=1}^m \alpha_k u_k \otimes v_k$ , with the scalars  $\{\alpha_k\}$  being non-negative, and  $m \leq \min m_1, m_2 \leq n$ .  $\square$

As a result, every grade-optimal representation can be changed to a Schmidt representation with the same number of terms, namely

**Corollary 3.2.** *Let  $H_1, H_2$  be Hilbert spaces and  $u = \sum_{k=1}^n x_k \otimes y_k \in H_1 \otimes H_2$ , with  $\text{gr}(u) = n$ . Then there exists a Schmidt decomposition*

$u = \sum_{k=1}^n \alpha_k u_k \otimes v_k$ , with orthonormal sets  $\{u_k\}$  and  $\{v_k\}$ , and with the scalars  $\{\alpha_k\}$  being non-negative.

In the above theorem and corollary, the field  $\mathbb{K}$  can be either  $\mathbb{R}$  or  $\mathbb{C}$ , as is customary for Hilbert spaces.

We also note that, as is well known, the Schmidt decomposition can be *effectively computed* by usual methods of linear algebra, since it is closely related to the singular decomposition of matrices.

## 3.2 A criterion for tensor reduction

Let  $X, Y$  be vector spaces over any field  $\mathbb{K}$ , and let the superscript “#” denote linear functionals. The following proposition is essentially in [4, Proposition 1.2]. We include a proof in the Appendix.

**Proposition 3.3.** *The following are equivalent for  $u = \sum_{k=1}^n x_k \otimes y_k \in X \otimes Y$ :*

- (i)  $u = 0$ ;
- (ii)  $\sum_{k=1}^n \phi(x_k)\psi(y_k) = 0$  for every  $\phi \in X^\#, \psi \in Y^\#$ .
- (iii)  $\sum_{k=1}^n \phi(x_k)y_k = 0$  for every  $\phi \in X^\#$ .
- (iv)  $\sum_{k=1}^n x_k\psi(y_k) = 0$  for every  $\psi \in Y^\#$ .

We use this to provide a criterion for tensor reduction.

**Theorem 3.4.** *Let  $X, Y$  be vector spaces over a field  $\mathbb{K}$ , and representation*

$$u = \sum_{k=1}^n x_k \otimes y_k \in X \otimes Y$$

*Then  $gr(u) < n$ , if and only if the following criterion is satisfied: the collection  $\{x_k\}$  or the collection  $\{y_k\}$  are linearly dependent.*

*Proof.* To prove the sufficiency of the criterion, suppose that the collection  $\{x_k\}$  is linearly dependent. Then one can replace  $x_n$  with  $\sum_{k=1}^{n-1} r_k x_k$ , where the  $r_k$ 's are scalars, and use the bilinearity of “ $\otimes$ ” to get a representation of  $u$  in terms of  $n - 1$  elementary tensors. Similar reasoning applies if not  $\{x_k\}$  but  $\{y_k\}$  is linearly dependent.

To prove the necessity of the criterion, suppose  $u$  has a representation

$$u = \sum_{k=1}^m x'_k \otimes y'_k$$

for some  $m < n$  and  $x'_k$ 's and  $y'_k$ 's in  $X, Y$  respectively, and that the criterion is not satisfied. Then, using the linear independence of the  $\{x_k\}$ , one can choose, for any integer  $j$ ,  $1 \leq j \leq n$ , a  $\phi \in X^\#$  so that  $\phi(x_j) = 1$  and  $\phi(x_k) = 0$  for  $k \neq j$ . So, by Proposition 3.3 we get

$$y_j = \sum_{k=1}^n \phi(x_k) y_k = \sum_{k=1}^m \phi(x'_k) y'_k.$$

This shows that  $y_j$  is in the span of  $\{y'_k\}_{k=1}^m$ . Since  $j$  is arbitrary, we get that the  $y_j$ 's span a subspace of dimension  $m$  and are therefore linearly dependent, contradicting the assumption that the criterion is not satisfied.  $\square$

Although the results below can possibly be obtained by other means, it is interesting that they are also straightforward consequences of the above theorem.

**Corollary 3.5.** *Given  $u = \sum_{k=1}^n x_k \otimes y_k \in X \otimes Y$ , then  $gr(u)$  is the minimum of the dimensions of  $span\{x_k\}$  and  $span\{y_k\}$ .*

**Corollary 3.6.** *If  $X, Y$  have dimensions  $m, n$  respectively, then for all  $u \in X \otimes Y$  we have  $gr(u) \leq \min\{m, n\}$*

A result of special interest which gives a more detailed information on the structure of the *minimum* number of terms in an entangled element  $u \in X \otimes Y$ , is in

**Corollary 3.7.** *Given  $u = \sum_{i=1}^n x_i \otimes y_i \in X \otimes Y$ , then, after a prior permutation of the terms in this sum, and possibly of the factors  $x_i$  with  $y_i$ , the vector  $u \in X \otimes Y$  can be reduced to one of the following two forms*

$$u = \sum_{j=1}^m x_j \otimes v_j$$



where  $m = \text{gr}(u) \leq n$  and  $v_j$  is in the span of  $y_1, \dots, y_n$ , or alternatively, it can be reduced to

$$u = \sum_{j=1}^m u_j \otimes y_j$$

where  $u_j$  is in the span of  $x_1, \dots, x_n$ .

*Proof.* If  $m = n$ , then the proof is completed by taking  $\lambda_j = 1$ . Let therefore  $m < n$ . Then according to Theorem 3.4., at least one of the sets  $\{x_1, \dots, x_n\}$  or  $\{y_1, \dots, y_n\}$  is linearly dependent. Let us assume that this is the case with  $\{x_1, \dots, x_n\}$ . Then in view of Corollary 3.5., and after a suitable permutation of the terms in the sum  $u = \sum_{i=1}^n x_i \otimes y_i$ , and possibly, of the factors  $x_i$  with  $y_i$ , we can assume that the subset  $\{x_1, \dots, x_m\}$  is linearly independent and has the same span with  $\{x_1, \dots, x_n\}$ . Therefore, each  $x_j$ , with  $m < j \leq n$ , is a linear combination of  $x_1, \dots, x_m$ , namely

$$x_j = \sum_{i=1}^m \mu_{j,i} x_i, \quad m < j \leq n$$

with  $\mu_{j,i} \in \mathbb{K}$ . Consequently, we have

$$\begin{aligned} u &= \sum_{i=1}^m x_i \otimes y_i + \sum_{j=m+1}^n \sum_{i=1}^m \mu_{j,i} x_i \otimes y_j = \\ &= \sum_{i=1}^m x_i \otimes y_i + \sum_{i=1}^m \sum_{j=m+1}^n \mu_{j,i} x_i \otimes y_j = \\ &= \sum_{i=1}^m x_i \otimes (y_i + \sum_{j=m+1}^n \mu_{j,i} y_j) \quad \square \end{aligned}$$

### 3.3 Conclusions

The basic result regarding the computation of the grade of an element in a tensor product is in Corollary 3.5. which is valid for arbitrary vector spaces over arbitrary fields. And that computation reduces to the computation of the dimension of two finite dimensional vector subspaces.

In the case of more particular tensor products, namely, of Hilbert spaces, which are of interest related to quanta, the computation of the grade of an element in such tensor products can also be made with

the help of the well known and understood Schmidt decomposition.

In this way, the study of the extent of entanglement of any given element  $u$  in a tensor product, study done with the help of its grade  $gr(u)$ , is well established. Namely, if  $gr(u) \leq 1$ , then  $u$  is not entangled. On the other hand, if  $gr(u) \geq 2$ , then the larger  $gr(u)$  is, the more entangled is  $u$ .

## Appendix : Proof of Proposition 3.3

*Proof.* We prove  $(i) \Rightarrow (ii) \Leftrightarrow (iii) \Rightarrow (i)$ .

The equivalence  $(ii) \Leftrightarrow (iv)$  is similar to  $(ii) \Leftrightarrow (iii)$ .

Recall that the universal mapping property of tensor products guarantees the existence, for any bilinear functional  $A : X \times Y \rightarrow \mathbb{F}$ , a linear mapping we denote  $T_A : X \otimes Y \rightarrow \mathbb{F}$ , that satisfies  $T_A(x, y) = A(x, y)$  for all  $(x, y) \in X \times Y$ .

$(i) \Rightarrow (ii)$  : Let  $\phi, \psi$  be given. Every linear mapping  $T$  has the property  $u = 0 \Rightarrow T(u) = 0$ . Let  $T = T_A$ , where  $A(x, y) = \phi(x)\psi(y)$ .

$(ii) \Leftrightarrow (iv)$  :

$$\begin{aligned} & \sum_{k=1}^n \phi(x_k)\psi(y_k) = 0 \quad \forall \phi \in X^\#, \psi \in Y^\# \\ \Leftrightarrow & \psi(\sum_{k=1}^n \phi(x_k)y_k) = 0 \quad \forall \phi \in X^\#, \psi \in Y^\# \\ \Leftrightarrow & \sum_k \phi(x_k)y_k = 0 \quad \forall \phi \in X^\# \end{aligned}$$

$(iii) \Rightarrow (i)$  : Recall that, due to the linear structure,  $T(u) = 0$  for all  $T : X \otimes Y \rightarrow \mathbb{F}$  implies  $u = 0$ . And every such  $T$  is the linearization of the bilinear map  $A(x, y) := T(x \otimes y)$ . So it is sufficient to prove that  $T_A(u) = 0$  for all bilinear  $A : X \times Y \rightarrow \mathbb{F}$ .

Let  $A : X \times Y \rightarrow \mathbb{F}$  be given. Let  $E, F$  respectively be the spans of  $\{x_k\}$  and  $\{y_k\}$ , and  $B = A|_{E \times F}$ .

Then because of the finite dimensions of  $E$  and  $F$  we have a representation  $B(x, y) = \sum_{j=1}^m \alpha_j(x)\beta_j(y)$ , where each  $\alpha_j$  belongs to  $E^\#$  and each  $\beta_j$  belongs to  $F^\#$ . Without changing the notation, we can extend each  $\alpha_k$  to belong to  $X^\#$ , and  $\beta_k$  to belong to  $Y^\#$ .

So

$$\begin{aligned} T_A(u) &= \sum_{k=1}^n A(x_k, y_k) \\ &= \sum_{k=1}^n B(x_k, y_k) \\ &= \sum_{k=1}^n \sum_{j=1}^n \alpha_j(x_k) \beta_j(y_k) \\ &= \sum_{j=1}^n \beta_j \left( \sum_k \alpha(x_k) y_k \right) \\ &= 0, \end{aligned}$$

using (iii). Thus  $T_A(u) = 0$  for each  $A \in B(X \times Y)$ .  $\square$

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