# Bounds for extreme zeros of some classical orthogonal polynomials

K Driver[1]

*Department of Mathematics and Applied Mathematics, University of Cape Town, Cape Town 7708, South Africa*

K Jordaan[1]

*Department of Mathematics and Applied Mathematics, University of Pretoria, Pretoria, 0002, South Africa*

**Abstract**

We use mixed three term recurrence relations typically satisfied by classical orthogonal polynomials from sequences corresponding to different parameters to derive upper (lower) bounds for the smallest (largest) zeros of Jacobi, Laguerre and Gegenbauer polynomials.

*Keywords:* Bounds for zeros; interlacing of zeros; common zeros of orthogonal polynomials.
*2000 MSC:* 33C45, 42C05

## 1. Introduction

If $\{p_n\}_{n=0}^{\infty}$ is a sequence of orthogonal polynomials, the zeros of $p_n$ are real and simple and each open interval with endpoints at successive zeros of $p_n$ contains exactly one zero of $p_{n-1}$; a property called the interlacing of zeros. Stieltjes (cf. [15], Theorem 3.3.3) extended this interlacing property by proving that if $m < n-1$, provided $p_m$ and $p_n$ have no common zeros, there exist $m$ open intervals, with endpoints at successive zeros of $p_n$, each of which contains exactly one zero of $p_m$. Beardon (cf. [3], Theorem 5) proved that one can say more, namely, for each $m < n-1$, if $p_m$ and $p_n$ are co-prime, there exists a real polynomial $S_{n-m}$ of degree $n-m-1$ whose real simple zeros, together with those of $p_m$, interlace with the zeros of $p_n$. The polynomials $S_{n-m}$ are the dual polynomials introduced by de Boor and Saff in [5] or, equivalently, the associated polynomials analysed by Vinet and Zhedanov in [17]. We prove that constraints on the location of common zeros of two polynomials that satisfy a three term recurrence relation of the type associated with orthogonal polynomials together with a Stieltjes interlacing property lead to lower (upper) bounds for the largest (smallest) zero of Jacobi, Gegenbauer and Laguerre polynomials.

## 2. Laguerre, Jacobi and Gegenbauer polynomials

A special case of the following theorem was proved in [10].

**Theorem 2.1.** *Let $\{p_n\}_{n=0}^{\infty}$ be a sequence of polynomials orthogonal on the (finite or infinite) interval $(c,d)$. Fix $k,n \in \mathbb{N}$ with $k < n-1$ and suppose $g_{n-k}$ is a polynomial of degree $n-k-1$ that satisfies*

$$f(x)g_{n-k}(x) = G_k(x)p_{n-1}(x) + H(x)p_n(x) \tag{1}$$

*where $f(x) \neq 0$ for $x \in (c,d)$ and $H(x)$, $G_k(x)$ are polynomials with $deg(G_k) = k$. Then*

*(i) the $n-1$ real, simple zeros of $G_k g_{n-k}$ interlace with the zeros of $p_n$ if $g_{n-k}$ and $p_n$ are co-prime;*

*(ii) if $g_{n-k}$ and $p_n$ are not co-prime and have $r$ common zeros counting multiplicity, then*

*a) $r \leq min\ \{k, n-k-1\}$;*

*Email addresses:* kathy.driver@uct.ac.za (K Driver), kerstin.jordaan@up.ac.za (K Jordaan )
[1]Research of each author is supported by the National Research Foundation of South Africa

*b) these $r$ common zeros are simple zeros of $G_k$;*

*c) no two successive zeros of $p_n$ , nor its largest or smallest zero, can also be zeros of $g_{n-k}$;*

*d) the $n-2r-1$ zeros of $G_k g_{n-k}$, none of which is also a zero of $p_n$, together with the $r$ common zeros of $g_{n-k}$ and $p_n$, interlace with the $n-r$ remaining (non-common) zeros of $p_n$.*

**Proof of Theorem 2.1.** Let $w_n < \cdots < w_1$ denote the zeros of $p_n$.

(i) From (1), provided $p_n(x) \neq 0$, we have

$$\frac{f(x)g_{n-k}(x)}{p_n(x)} = H(x) + \frac{G_k(x)p_{n-1}(x)}{p_n(x)}. \tag{2}$$

Further,

$$\frac{p_{n-1}(x)}{p_n(x)} = \sum_{j=1}^{n} \frac{A_j}{x-w_j}$$

where $A_j > 0$ for every $j \in \{1, \ldots, n\}$ (cf. [15, Theorem 3.3.5]). Therefore (2) can be written as

$$\frac{f(x)g_{n-k}(x)}{p_n(x)} = H(x) + \sum_{j=1}^{n} \frac{G_k(x)A_j}{x-w_j}, \quad x \neq w_j. \tag{3}$$

Since $p_{n-1}$ and $p_n$ are always co-prime while $p_n$ and $g_{n-k}$ are co-prime by assumption, it follows from (1) that $G_k(w_j) \neq 0$ for every $j \in \{1, 2, \ldots, n\}$. Suppose that $G_k$ does not change sign in an interval $I_j = (w_{j+1}, w_j)$ where $j \in \{1, 2, \ldots, n-1\}$. Since $A_j > 0$ and the polynomial $H$ is bounded on $I_j$ while the right hand side of (3) takes arbitrarily large positive and negative values on $I_j$, it follows that $g_{n-k}$ must have an odd number of zeros in every interval in which $G_k$ does not change sign. Since $G_k$ is of degree $k$, there are at least $n-k-1$ intervals $(w_{j+1}, w_j)$, $j \in \{1, \ldots, n-1\}$ in which $G_k$ does not change sign and so each of these intervals must contain exactly one of the $n-k-1$ real, simple zeros of $g_{n-k}$. We deduce that the $k$ zeros of $G_k$ are real and simple and, together with the $n-k-1$ zeros of $g_{n-k}$, interlace with the $n$ zeros of $p_n$.

(ii) If $r$ is the total number of common zeros of $p_n$ and $g_{n-k}$ counting multiplicity then each of these $r$ zeros is a simple zero of $p_n$ and it follows from (1) that any common zero of $g_{n-k}$ and $p_n$ is also a zero of $G_k$ since $p_n$ and $p_{n-1}$ are co-prime. Therefore, $r \leq \min\{k, n-k-1\}$ and there must be at least $(n-2r-1)$ open intervals of the form $I_j = (w_{j+1}, w_j)$, $j \in \{1, 2, \ldots, n-1\}$, with endpoints at successive zeros of $p_n$ where neither $w_{j+1}$ nor $w_j$ is a zero of $g_{n-k}$ or $G_k(x)$. If $G_k$ does not change sign in an interval $I_j = (w_{j+1}, w_j)$, it follows from (3), since $A_j > 0$ for every $j \in \{1, 2, \ldots, n\}$, and $H$ is bounded while the right hand side takes arbitrarily large positive and negative values for $x \in I_j$, that $g_{n-k}$ must have an odd number of zeros in that interval. Therefore, in at least $(n-2r-1)$ intervals $I_j$ either $g_{n-k}$ or $G_k$, but not both, must have an odd number of zeros counting multiplicity. On the other hand, $g_{n-k}$ and $G_k$ have at most $(n-k-1-r)$ and $(k-r)$ real zeros respectively that are not zeros of $p_n$. We deduce that there must be at most $(n-2r-1)$ intervals $I_j = (w_{j+1}, w_j)$ with endpoints at successive zeros $w_{j+1}$ and $w_j$ of $p_n$ neither of which is a zero of $g_{n-k}$. It is straightforward to check that if the number of intervals $I_j = (w_{j+1}, w_j)$ with endpoints at successive zeros of $p_n$ neither of which is a zero of $g_{n-k}$ is equal to $n-2r-1$, this is only possible if no pair of consecutive zeros of $p_n$, nor the largest or smallest zero of $p_n$, are also zeros of $g_{n-k}$. This proves a) to c) while d) follows from c).

**Corollary 2.2.** *Suppose (1) holds for $k, n \in \mathbb{N}$ fixed and $k < n-1$. The largest (smallest) zero of $G_k$ is a strict lower (upper) bound for the largest (smallest) zero of $p_n$.*

Mixed three term recurrence relations involving polynomials with the largest possible parameter difference, or alternatively, with no parameter difference but the largest possible degree difference, that satisfy interlacing properties of their zeros, are useful in deriving good bounds for largest (smallest) zeros of $p_n$. We denote the zeros of the polynomial $p_n$ by $w_n < \cdots < w_1$.

*2.1. Laguerre polynomials*

For $\alpha > -1$, the Laguerre polynomials $L_n^{\alpha}$ satisfy the mixed three term recurrence relation

$$x^5 L_{n-3}^{\alpha+5}(x) \quad = \quad (n+\alpha)\left((\alpha+1)_4 - (\alpha+2)_2(3n+2\alpha+2)x + (n+\alpha+1)_2 x^2\right) L_{n-1}^{\alpha}(x) + H(x)L_n^{\alpha}(x) \tag{4}$$

that follows from [9, eqn. (13)], [9, eqn. (4)] and the three term recurrence relation for Laguerre polynomials (cf. [15]) where $(\alpha)_k = \alpha(\alpha+1)\ldots(\alpha+k-1), \quad k \in \mathbb{N}$, is Pochhammer's symbol. The smallest zero of the polynomial coefficient of $L_{n-1}^{\alpha}$ in (4) is

$$\frac{(\alpha+2)_2(3n+2\alpha+2) - \sqrt{(\alpha+2)_2(-4(\alpha+1)^2(\alpha+2) + 4n(\alpha+1)(\alpha^2+4\alpha+6) + (5\alpha^2+25\alpha+38)n^2)}}{2(n+\alpha+1)_2} \tag{5}$$

which provides a strict upper bound for the smallest zero of $L_n^{\alpha}$. Numerical calculations indicate that (5) compares favourably with the upper bound $\dfrac{(\alpha+1)(\alpha+2)(\alpha+4)(2n+\alpha+1)}{(\alpha+1)^2(\alpha+2)+(5\alpha+11)n(n+\alpha+1)}$ obtained by Gupta and Muldoon in [11, eqn. (2.11)] although the Gupta-Muldoon bound is sharper for $n$ large. Iterating the three term recurrence relation for Laguerre polynomials (cf. [15]) we obtain

$$\begin{aligned}&(\alpha+n-2)_2 L_{n-3}^{\alpha}(x)\\ &\quad = \quad (x^2 - 2(2n+\alpha-2)x + 3n^2 + 3\alpha n + \alpha^2 - 6n - 3\alpha - 1)L_{n-1}^{\alpha}(x) - n(2n+\alpha-3-x) + H(x)L_n^{\alpha}(x)\end{aligned} \tag{6}$$

and the largest zero of the polynomial coefficient of $L_{n-1}^{\alpha}$ in (6) yields the lower bound

$$w_1 > 2n + \alpha - 2 + \sqrt{n^2 + n(\alpha-2) - (\alpha-2)} \tag{7}$$

for the largest zero of $L_n^{\alpha}$ which is sharper than the lower bound $2n+\alpha-1$ found by Szegő (cf. [15, eqn.(6.2.14)]). The lower bound $3n-4$ for the largest zero obtained by Neumann in [14] compares favourably with (7) only when $\alpha$ is close to $-1$ while the lower bound $4n+\alpha-16\sqrt{2n}$ given by Bottema (cf. [4]) is better than (7) for $n$ large.

*2.2. Jacobi polynomials*

For Jacobi polynomials $P_n^{\alpha,\beta}$, $\alpha, \beta > -1$, it was proved in [10, Thm 2.1(i)(c)] that (1) holds for $k = 1$ with

$$g_{n-1} = P_{n-2}^{\alpha+4,\beta}, \; G_1(x) = x - A_n, \; A_n = \frac{2(n-1)(n+\alpha+\beta+2) + (\alpha+3)(\beta-\alpha)}{2(n-1)(n+\alpha+\beta+2) + (\alpha+3)(\alpha+\beta+2)} \quad \text{and} \quad p_n = P_n^{\alpha,\beta}$$

for $n > 1$, $n \in \mathbb{N}$. It follows from Corollary 2.2 that for all $\alpha, \beta > -1$, $n \in \mathbb{N}$,

$$w_1 > 1 - \frac{2(\alpha+1)(\alpha+3)}{2(n-1)(n+\alpha+\beta+2) + (\alpha+3)(\alpha+\beta+2)} = 1 - O(\frac{1}{n^2}) \tag{8}$$

which is sharper than the lower bound $1 - \dfrac{2(\alpha+1)}{2n+\alpha+\beta} = 1 - O(\dfrac{1}{n})$ given by Szegö in [15, eqns. (6.2.11)].

Since $P_n^{\alpha,\beta}(x) = (-1)^n P_n^{\beta,\alpha}(-x)$, we deduce from (8) that

$$w_n < -1 + \frac{2(\beta+1)(\beta+3)}{2(n-1)(n+\alpha+\beta+2) + (\beta+3)(\alpha+\beta+2)}.$$

*2.3. Gegenbauer polynomials*

For the Gegenbauer polynomials $C_n^{\lambda}$, $\lambda > -\frac{1}{2}$, Szegő gives a lower bound for the largest zero $w_1$, namely,

$$w_1^2 \geq 1 - \frac{2\lambda+1}{n+2\lambda} = 1 - O(\frac{1}{n})$$

(cf. [15, eqn. (6.2.13)]). From [8, Theorem 2 (ii)d] and Corollary 2.2, we obtain a sharper bound

$$w_1^2 > 1 - \frac{(2\lambda+1)(2\lambda+3)}{(n-1)(n+2\lambda+1) + (2\lambda+1)(2\lambda+3)} = 1 - O(\frac{1}{n^2}).$$

Another lower bound for the largest zero, namely,

$$w_1 > 1 - \frac{(2\lambda+1)(2\lambda+5)}{4(n-1)(n+2\lambda+1)+(2\lambda+1)(2\lambda+5)} = 1 - O(\frac{1}{n^2})$$

follows from (8) with $\alpha = \beta = \lambda - \frac{1}{2}$.

**Remarks:**

1. Our results may be viewed as complementary to upper (lower) bounds for the largest (smallest) zeros of classical orthogonal polynomials that have been established by several authors using a wide range of approaches. For Laguerre polynomials, good upper (lower) bounds for the largest (smallest) zero can be found in Ismail and Li [12]; Krasikov [13] and Dimitrov and Rafaeli [6] while a comprehensive summary of the Laguerre case is given in [2]. Sharp limits for the zeros of Gegenbauer and Hermite polynomials are proved in [1] while van Doorn in [16], Dimitrov in [2] and Dimitrov and Nikolov in [7] provide bounds for zeros of Jacobi polynomials.

2. Sharper upper (lower) bounds for the smallest (largest) zeros of Laguerre, Jacobi and Gegenbauer polynomials can be obtained from (1) by putting $k = 3, 4, \ldots$ The calculations become more complicated as the degree of the coefficient polynomial $G_k$ increases.


[1] I. Area, D.K. Dimitrov, E. Godoy, A. Ronveaux, Zeros of Gegenbauer and Hermite polynomials and connection coefficients, Math. Comp. 73 (2004) 1937-1951.

[2] I. Area, D.K. Dimitrov, E. Godoy, F.R. Rafaeli, Inequalities for zeros of Jacobi polynomials via Obrechkoff's theorem, Math. Comp. 81 (2012) 991-1012.

[3] A.F. Beardon, The theorems of Stieltjes and Favard, Lect. Notes Math. 11(1) (2011) 247-262.

[4] O. Bottema, Die Nulstellen gewisser durch Rekursionsformeln definierter Polynome, Proc. Amsterdam 34 (5)(1931) 681-691.

[5] C. de Boor, E.B. Saff, Finite sequences of orthogonal polynomials connected by a Jacobi matrix, Linear Algebra Appl. 75 (1986) 43-55.

[6] D.K. Dimitrov, F.R. Rafaeli, Monotonicity of zeros of Laguerre polynomials, J. Comput. Appl. Math. 223 (2009) 699-702.

[7] D.K. Dimitrov, G.P. Nikolov, Sharp bounds for the extreme zeros of classical orthogonal polynomials, J. Approx. Th. 162 (2010) 1793-1804.

[8] K. Driver, Interlacing of zeros of Gegenbauer polynomials of non-adjacent degree from different sequences, Numer. Math. 120 (2012) 35-44.

[9] K. Driver, K. Jordaan, Stieltjes interlacing of zeros of Laguerre polynomials from different sequences, Indagat. Math. New Ser. 21 (2011) 204-211.

[10] K. Driver, A. Jooste, K. Jordaan, Stieltjes interlacing of zeros of Jacobi polynomials from different sequences, Electronic Transactions of Numerical Analysis 38 (2011) 317-326.

[11] D. E. Gupta, M. E. Muldoon, Inequalities for the smallest zeros of Laguerre polynomials and their $q$-analogues, Journal of Inequalities in Pure and Applied Mathematics 8 (2007).

[12] M. E. H. Ismail, X. Li, Bounds on the extreme zeros of orthogonal polynomials, Proc. Amer. Math. Soc. 115 (1992) 131-140.

[13] I. Krasikov, Bounds for zeros of the Laguerre polynomials, J. Approx. Theory 121 (2003) 287-291.

[14] E.R. Neumann, Beiträge zur Kenntnis der Laguerreschen Polynome, Jahresber. d. DMV 30 (1921) 15-35.

[15] G. Szegő, Orthogonal Polynomials, American Mathematical Society Colloquium Publications, Volume XXIII, Providence, RI, fourth edition, 1975.

[16] E. van Doorn, Representations and bounds for zeros of orthogonal polynomials and eigenvalues of sign-symmetric tri-diagonal matrices, J. Approx. Theory 51 (1987) 254-266.

[17] L. Vinet, A. Zhedanov, A characterization of classical and semiclassical orthogonal polynomials from their dual polynomials, J. Comput. Appl. Math. 172 (2004) 41-48.