GENERALIZED BESICOVITCH SPACES AND APPLICATION TO DETERMINSTIC HOMOGENIZATION

MAMADOU SANGO, NILS SVANSTEDT, AND JEAN LOUIS WOUKENG

ABSTRACT. The purposes of this paper is to introduce a framework which enables us to study nonlinear homogenization problems. The starting point for this work is the theory of *algebras with mean value*. Very often in physics, from very simple experimental data, one gets sometimes complicated structure phenomena. These phenomena are represented by functions which are permanent in mean, but complicated in detail, which functions are subject to verify a functional equation often nonlinear. The problem is therefore to give an interpretation of these phenomena by functions having the following qualitative properties: they are functions that represent a phenomenon on a large scale, which vary irregularly, undergoing many nonperiodic oscillations. In this work we study the qualitative properties of spaces of such functions, say *generalized Besicovitch spaces*, and we prove general compactness results related to these spaces. We then apply these results to study some new homogenization problems. One important achievement of this work is the resolution of the generalized weakly almost periodic homogenization problem for a nonlinear pseudo monotone parabolic-type operator. We also give the answer to the question raised by Frid and Silva in their paper [26] (Homogenization of nonlinear pde's in the Fourier-Stieltjes algebras, SIAM J. Math. Anal., Vol. 41, No. 4, pp. 1589-1620) to know whether there exist or not ergodic algebras that are not subalgebras of the Fourier-Stieltjes algebra, and further we show that the theory of homogenization algebras by Nguetseng in [36] (Homogenization structures and applications I, Zeit. Anal. Anw., **22** (2003) 73-107) cannot handle weakly almost periodic homogenization problems.

1. INTRODUCTION

The concepts of algebras with mean value [47] (algebras w.m.v., in short) and of homogenization algebras [36] (H-algebras, in short) were introduced to extend to the context of more general classes of oscillatory functions (such as almost periodic functions and others) the theory of periodic homogenization. This gives rise to the theory of deterministic/individual homogenization.

An algebra w.m.v. A in $\mathbb{R}^N$ is defined as a Banach subalgebra of the algebra of bounded uniformly continuous real-valued functions $\mathcal{B}(\mathbb{R}^N)$, which is translation invariant, contains the constants and whose elements possess a mean value. Thus, it generalizes the concept of almost periodic functions. In this sense therefore, from the definition of algebra w.m.v., one easily introduces generalized Besicovitch spaces B_A^p associated to an algebra w.m.v. A. Moreover to each algebra w.m.v. A is

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associated a compact topological space $\Delta(A)$ (called its spectrum) such that every element of A can be seen as an element of $\mathcal{C}(\Delta(A))$, in such a way that the space $L^p(\Delta(A))$ is isometrically isomorphic to some quotient of the space B_A^p, denoted by $\mathcal{B}_A^p$. This last result is of great interest and will be the starting point of our work.

An H-algebra is a separable Banach subalgebra of the algebra of bounded continuous real-valued functions which contains the constants and whose elements possess a mean value. Here we have the same properties as above. However, in order to take full advantage of the above two concepts we will restrict ourselves to the bounded uniformly continuous functions.

In this paper we rely on the concepts of H-algebra and of algebra w.m.v. to introduce a framework for the study of nonlinear homogenization problems. As said above the choice to work with bounded uniformly continuous functions allows us to consider the concept of H-algebra as a particular case of that of algebra w.m.v. As discussed later, this choice is fully justified since some essential properties of H-algebras used in homogenization systematically derive from the invariance by translations' property of the algebra under consideration.

Next, we define according to Ambrosio et al. [2], the notion of vector valued algebras which is a generalization of that of product H-algebras [36]. In fact, being given two H-algebras A_1 and A_2 on $\mathbb{R}^{m_1}$ and $\mathbb{R}^{m_2}$ respectively, if we denote by $A_1 \odot A_2$ the product H-algebra defined as the closure (in the space of bounded uniformly continuous functions on $\mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$) of the tensor product $A_1 \otimes A_2$, we see that $A_1 \odot A_2 = A_1(\mathbb{R}^{m_1}; A_2)$, where the latter space is defined as the set of functions $f \in \mathcal{B}(\mathbb{R}^{m_1}; A_2)$ satisfying the following conditions:

- For each $L \in A_2'$ (topological dual of the Banach space A_2), $\langle L, f\rangle \in A_1$;
- The set $\{\langle L, f\rangle : L \in A_2' \text{ with } \|L\| \leq 1\}$ is relatively compact in A_1.

We rely on the concept of vector valued algebras to prove a general compactness result (Theorem 3.6) which generalizes to all points of view, those already obtained in the papers [36, 40, 41, 42]. In particular in the above-cited papers, this result is available in the only case when $p = 2$ in the context e.g. of the algebra of almost periodic functions. Here we prove it for any real number $p > 1$ by using the *ergodicity* key assumption of the algebra under consideration.

To illustrate the wide scope of application of our result we study homogenization problems associated with some pseudo monotone operators. We solve a few new problems as the generalized weakly almost periodic homogenization problems (see Subsections 4.4.4-4.4.6).

The paper is organized as follows. In Section 2 we define and give some properties of the generalized Besicovitch spaces. We also define and study some spaces connected to the above spaces, which are useful in the rest of the work. Section 3 deals with the concept of *two-scale A-convergence*. We state and prove there our compactness results, and we give a few examples of algebras satisfying the hypotheses of our results, notably the weakly almost periodic algebras. We show there that weakly almost periodic algebras are not H-algebras, which result is a more significant improvement as the deterministic homogenization theory is concerned. Finally in Section 4 we apply the general framework established in the previous sections to the homogenization problem of a nonlinear pseudo monotone parabolic-type operator.

Finally, to simplify the presentation of the results in this work, we assume that all functions are assumed real valued and all function spaces are considered over $\mathbb{R}$.

2. The generalized Besicovitch spaces

In this section we define and give some properties of the generalized Besicovitch spaces. To begin with, we first state some fundamentals of algebras with mean value (algebras w.m.v.), a concept which was first introduced by Zhikov and Krivenko [47] (see also [30]) as a generalization of the algebra of almost periodic functions [8, 7]. It is the main tool suitable to tackle nonperiodic homogenization problems. In the same direction we also mention the works by Frid et al. [2] and Casado and Gayte [12, 13].

2.1. **Algebras w.m.v.: an overview of basic concepts.** Let m be a positive integer. Given $\varepsilon > 0$, let

$$u^{\varepsilon}(x) = u\left(\frac{x}{\varepsilon}\right) \quad (x \in \mathbb{R}^m)$$

for $u \in L^1_{\text{loc}}(\mathbb{R}^m)$. Then $u^{\varepsilon} \in L^1_{\text{loc}}(\mathbb{R}^m)$. More generally if u lies in $L^p_{\text{loc}}(\mathbb{R}^m)$ $(1 \leq p < \infty)$ then so also is u^{ε}.

With this in mind, a function $u \in \mathcal{B}(\mathbb{R}^m_y)$ (the space of bounded uniformly continuous functions on $\mathbb{R}^m_y$) has a mean value if there exists a real number $M(u)$ such that $u^{\varepsilon} \to M(u)$ in $L^{\infty}(\mathbb{R}^m)$-weak $*$ as $\varepsilon \to 0$. $M(u)$ is called the mean value of u, and the mapping $M : u \mapsto M(u)$ is a positive linear form (on the space of those u in $\mathcal{B}(\mathbb{R}^m_y)$ with mean value) invariant by translation, and attaining the value 1 on the constant function 1, and verifying $|M(u)| \leq \|u\|_{\infty} \equiv \sup_{y \in \mathbb{R}^m} |u(y)|$ (thus of norm 1). M is called the mean value on $\mathbb{R}^m$. It is also evident that

$$M(u) = \lim_{r \to +\infty} \frac{1}{|B_r|} \int_{B_r} u(y)dy \tag{2.1}$$

where B_r stands for the bounded open ball in $\mathbb{R}^m$ with radius r, and $|B_r|$ denotes its Lebesgue measure. Expression (2.1) also works for $u \in L^1_{\text{loc}}(\mathbb{R}^m)$ provided that this limit exists.

This being so, by an algebra with mean value is meant any Banach subalgebra A of $\mathcal{B}(\mathbb{R}^m_y)$ with the properties:

(AMV1) A contains the constants
(AMV2) Any $u \in A$ possesses a mean value
(AMV3) A is translation invariant, i.e. for every $u \in A$ and every $a \in \mathbb{R}^m$, $\tau_a u \equiv u(\cdot - a) \in A$.

Let A be an algebra w.m.v. Clearly A (with the norm sup topology) is a commutative $\mathcal{C}^*$-algebra with identity (the involution here being the identity mapping). We denote by $\Delta(A)$ the spectrum of A and by $\mathcal{G}$ the Gelfand transformation on A. We recall that $\Delta(A)$ (a subset of the topological dual A') is the set of all nonzero multiplicative linear forms on A, and $\mathcal{G}$ is the mapping of A into $\mathcal{C}(\Delta(A))$ such that $\mathcal{G}(u)(s) = \langle s, u \rangle$ $(s \in \Delta(A))$, where $\langle\ ,\ \rangle$ denotes the duality pairing between A' and A. We equip $\Delta(A)$ with the relative weak $*$ topology on A'.

In [36] Nguetseng introduced the concept of homogenization algebras (H-algebras) by requiring it to be a *separable* Banach space of the bounded continuous functions in $\mathbb{R}^m$ which contains constants and whose elements possess a mean value. If we glance through the results of the above paper, a careful examination of their proofs tells us that the separability property really is not required in most of the results, so that almost all the results presented in [36] are still valid when replacing an

H-algebra by an algebra .w.m.v. In particular the following result about algebras w.m.v. is worth mentioning; see [36] and [2] for the proof.

Theorem 2.1. *Let A be an algebra w.m.v. on $\mathbb{R}^m_y$. Then*

(i) *The spectrum $\Delta(A)$ is a compact space and the Gelfand transformation $\mathcal{G}$ is an isometric isomorphism identifying A with the algebra $\mathcal{C}(\Delta(A))$ of continuous functions on $\Delta(A)$.*

(ii) *The mean value M considered as defined on A is representable by some Radon probability measure β (called the M-*measure for *A) as follows:*

$$M(u)=\int_{\Delta(A)}\mathcal{G}(u)d\beta \text{ for any } u\in A.$$

For the benefit of the reader we recall some well-known facts: in the case when A is the algebra $\mathcal{C}_{\mathrm{per}}(Y)$ of Y-periodic continuous functions on $\mathbb{R}^m_y$ ($Y=[-1/2,1/2]^m$), then $\Delta(A)$ can be identified with the m-torus $\mathbb{T}^m=(\mathbb{R}/\mathbb{Z})^m$. Let $\mathcal{R}$ be a subgroup of $\mathbb{R}^m_y$. Let $AP_{\mathcal{R}}(\mathbb{R}^m_y)$ denote the algebra of functions on $\mathbb{R}^m_y$ that are uniformly approximated by finite linear combinations of the functions in the set $\{\cos(k\cdot y), \sin(k\cdot y): k\in\mathcal{R}\}$. It is known that $AP_{\mathcal{R}}(\mathbb{R}^m_y)$ is an algebra w.m.v. and its spectrum $\Delta(AP_{\mathcal{R}}(\mathbb{R}^m_y))$ is a compact topological group homeomorphic to the dual group $\widehat{\mathcal{R}}$ of $\mathcal{R}$ consisting of the characters γ_k ($k\in\mathcal{R}$) of $\mathbb{R}^m$, defined by $\gamma_k(y)=\exp(2i\pi k\cdot y)$ ($y\in\mathbb{R}^m$); see [39, Propositions 2.2 and 2.6] for details.

Next, the partial derivative of index i ($1\leq i\leq m$) on $\Delta(A)$ is defined as the mapping $\partial_i=\mathcal{G}\circ\frac{\partial}{\partial y_i}\circ\mathcal{G}^{-1}$ (usual composition) of $\mathcal{D}^1(\Delta(A))=\{\varphi\in\mathcal{C}(\Delta(A)):\mathcal{G}^{-1}(\varphi)\in A^1\}$ into $\mathcal{C}(\Delta(A))$, where $A^1=\{\psi\in\mathcal{C}^1(\mathbb{R}^m_y):\psi,\frac{\partial\psi}{\partial y_i}\in A\ (1\leq i\leq m)\}$. Higher order derivatives can be defined analogously (see [36]). Now, let A^∞ be the space of $\psi\in\mathcal{C}^\infty(\mathbb{R}^m_y)$ such that $D^\alpha_y\psi=\frac{\partial^{|\alpha|}\psi}{\partial y_1^{\alpha_1}\cdots\partial y_m^{\alpha_m}}\in A$ for every multi-index $\alpha=(\alpha_1,\cdots,\alpha_m)\in\mathbb{N}^m$, and let $\mathcal{D}(\Delta(A))=\{\varphi\in\mathcal{C}(\Delta(A)):\mathcal{G}^{-1}(\varphi)\in A^\infty\}$. Endowed with a suitable locally convex topology (see [36]), A^∞ (resp. $\mathcal{D}(\Delta(A))$) is a Fréchet space and further, $\mathcal{G}$ viewed as defined on A^∞ is a topological isomorphism of A^∞ onto $\mathcal{D}(\Delta(A))$.

Analogously to the Schwartz's space $\mathcal{D}'(\mathbb{R}^m)$ of usual distributions, we now define the space of distributions on $\Delta(A)$ as the space of all continuous linear forms on $\mathcal{D}(\Delta(A))$. We denote it by $\mathcal{D}'(\Delta(A))$ and we endow it with the strong dual topology. The following result allows us to view the spaces $L^p(\Delta(A))$ ($1\leq p\leq\infty$) as subspaces of $\mathcal{D}'(\Delta(A))$.

Proposition 2.2. *Let A be an algebra w.m.v. Then the space A^∞ is dense in A.*

Proof. We just give a sketch of the proof. Let $u\in A$, and let $\varphi\in\mathcal{D}(\mathbb{R}^m)$; then $u*\varphi\in\mathcal{C}^\infty(\mathbb{R}^m)$. Since u is uniformly continuous, the function $u*\varphi$ can be uniformly approximated by a finite linear combination of translates of u, so that, as A is translation invariant and closed, $u*\varphi\in A$. Now, let $(\theta_n)_{n\geq1}$ be a mollifier on $\mathbb{R}^m$. Clearly $u*\theta_n\in A^\infty$ and $u*\theta_n\to u$ in A as $n\to\infty$. Whence the result. □

The above result amounts to saying that $\mathcal{D}(\Delta(A))$ is dense in $\mathcal{C}(\Delta(A))$, so that, since $\mathcal{C}(\Delta(A))$ is dense in $L^p(\Delta(A))$, one easily sees that $L^p(\Delta(A))\subset\mathcal{D}'(\Delta(A))$ ($1\leq p\leq\infty$) with continuous embedding. We may therefore define the Sobolev spaces on $\Delta(A)$ as

$$W^{1,p}(\Delta(A))=\{u\in L^p(\Delta(A)):\ \partial_i u\in L^p(\Delta(A))\ (1\leq i\leq m)\}\ (1\leq p<\infty)$$

where the derivative $\partial_i u$ is taken in the distribution sense on $\Delta(A)$ (exactly as the Schwartz derivative in the classical case). This is a Banach space with the norm

$$||u||_{W^{1,p}(\Delta(A))} = \left(||u||^p_{L^p(\Delta(A))} + \sum_{i=1}^{N} ||\partial_i u||^p_{L^p(\Delta(A))} \right)^{\frac{1}{p}} \quad \left(u \in W^{1,p}(\Delta(A))\right).$$

To that space are attached some other spaces as its closed subspace

$$W^{1,p}(\Delta(A))/\mathbb{R} = \left\{ u \in W^{1,p}(\Delta(A)) : \int_{\Delta(A)} u(s)d\beta(s) = 0 \right\}$$

and its separated completion $W^{1,p}_{\#}(\Delta(A))$; see [42] for a documented presentation of these spaces.

We can also define the notion of a product algebra w.m.v. For this purpose, let A_1 (resp. A_2) be an algebra w.m.v. on $\mathbb{R}^{m_1}$ (resp. $\mathbb{R}^{m_2}$); then we define the product algebra w.m.v. $A_1 \odot A_2$ as the closure in $\mathcal{B}(\mathbb{R}^{m_1} \times \mathbb{R}^{m_2})$ of the tensor product $A_1 \otimes A_2 = \{\sum_{\text{finite}} u_{i_1} \otimes u_{i_2} : u_{i_j} \in A_j,\ j = 1, 2\}$.

2.2. **Generalized Besicovitch spaces.** The notations are those of the preceding subsection. Let A be an algebra w.m.v. Let $1 \leq p < \infty$. If $u \in A$, then $|u|^p \in A$ with $\mathcal{G}(|u|^p) = |\mathcal{G}(u)|^p$. Hence the limit $\lim_{r\to+\infty} \frac{1}{|B_r|}\int_{B_r} |u(y)|^p\, dy$ exists and we have

$$\lim_{r\to+\infty} \frac{1}{|B_r|}\int_{B_r} |u(y)|^p\, dy = M(|u|^p) = \int_{\Delta(A)} |\mathcal{G}(u)|^p\, d\beta.$$

Hence, for $u \in A$, put

$$\|u\|_p = (M(|u|^p))^{1/p}.$$

This defines a seminorm on A with which A is not complete. we denote by B^p_A the completion of A with respect to $\|\cdot\|_p$. B^p_A is a Fréchet space, and an argument due to Besicovitch [7] states that B^p_A is a complete subspace of $L^p_{\text{loc}}(\mathbb{R}^m_y)$. We have the following properties that can be achieved using the theory of the completion; see e.g., [9].

Proposition 2.3. *The following hold true:*

(i) *A is dense in B^p_A;*
(ii) *If F is a Banach space then any continuous linear mapping l from A to F extends by continuity to an unique continuous linear mapping L, of B^p_A into F.*

Now, let $1 \leq p \leq q < \infty$. Obviously we have $B^q_A \subset B^p_A$, so that one may naturally define the space B^∞_A as follows:

$$B^\infty_A = \left\{ f \in \bigcap_{1\leq p<\infty} B^p_A : \sup_{1\leq p<\infty} \|f\|_p < \infty \right\}.$$

We endow B^∞_A with the seminorm $[f]_\infty = \sup_{1\leq p<\infty} \|f\|_p$, which makes it a Fréchet space.

Next, thanks to the preceding proposition, the following properties are worth noting:

1) The Gelfand transformation $\mathcal{G}: A \to \mathcal{C}(\Delta(A))$ extends by continuity to an unique continuous linear mapping, still denoted by $\mathcal{G}$, of B_A^p into $L^p(\Delta(A))$. Furthermore if $u \in B_A^p \cap L^\infty(\mathbb{R}_y^m)$ then $\mathcal{G}(u) \in L^\infty(\Delta(A))$ and $\|\mathcal{G}(u)\|_{L^\infty(\Delta(A))} \leq \|u\|_{L^\infty(\mathbb{R}_y^m)}$.

2) The mean value M viewed as defined on A, extends by continuity to a positive continuous linear form (still denoted by M) on B_A^p satisfying $M(u) = \int_{\Delta(A)} \mathcal{G}(u) d\beta$ $(u \in B_A^p)$. Furthermore, $M(\tau_a u) = M(u)$ for each $u \in B_A^p$ and all $a \in \mathbb{R}^m$, where $\tau_a u(y) = u(y-a)$ for almost all $y \in \mathbb{R}^m$.

3) Let $1 \leq p, q, r < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} \leq 1$. The usual multiplication $A \times A \to A$; $(u, v) \mapsto uv$, extends by continuity to a bilinear form $B_A^p \times B_A^q \to B_A^r$ with

$$\|uv\|_r \leq \|u\|_p \|v\|_q \text{ for } (u, v) \in B_A^p \times B_A^q.$$

The following result will be of great interest in the work.

Proposition 2.4. *Let A be an algebra w.m.v. Then A^∞ is dense in B_A^p.*

Proof. Indeed by Proposition 2.2 A^∞ is dense in A. The result follows therefore by [part (i) of] Proposition 2.3. □

Now, let $u \in B_A^p$ $(1 \leq p < \infty)$; then $|u|^p \in B_A^1$ (this is easily seen) and so, by part **2)** above one has $M(|u|^p) = \int_{\Delta(A)} |\mathcal{G}(u)|^p \, d\beta = \|\mathcal{G}(u)\|_{L^p(\Delta(A))}^p$. Thus for $u \in B_A^p$ we have $\|u\|_p = (M(|u|^p))^{1/p}$, and $\|u\|_p = 0$ if and only if $\mathcal{G}(u) = 0$. Unfortunately, the mapping $\mathcal{G}$ (defined on B_A^p) is not in general injective. So let $\mathcal{N} = Ker\mathcal{G}$ (the kernel of $\mathcal{G}$) and let

$$\mathcal{B}_A^p = B_A^p / \mathcal{N}.$$

Endowed with the norm

$$\|u + \mathcal{N}\|_{\mathcal{B}_A^p} = \|u\|_p \quad (u \in B_A^p),$$

$\mathcal{B}_A^p$ is a Banach space with the following property.

Theorem 2.5. *The mapping $\mathcal{G}: B_A^p \to L^p(\Delta(A))$ induces an isometric isomorphism $\mathcal{G}_1$ of $\mathcal{B}_A^p$ onto $L^p(\Delta(A))$.*

Proof. Since $\mathcal{G}$ is an isometry (indeed $\|\mathcal{G}(u)\|_{L^p(\Delta(A))} = \|u\|_p$ for $u \in \mathcal{B}_A^p$) it suffices to show that $\mathcal{G}$ is surjective so that, using the first isomorphism theorem, $\mathcal{G}$ will induce an algebraic isomorphism $\mathcal{G}_1 : \mathcal{B}_A^p \to L^p(\Delta(A))$ which will be moreover a topological isomorphism.

So, first and foremost, $\mathcal{G}(B_A^p)$ is dense in $L^p(\Delta(A))$: this comes on one hand from the density of $\mathcal{C}(\Delta(A)) = \mathcal{G}(A)$ in $L^p(\Delta(A))$ and on the other hand from the inclusion $\mathcal{C}(\Delta(A)) \subset \mathcal{G}(B_A^p) \subset L^p(\Delta(A))$. Now, let $v \in L^p(\Delta(A))$. Then there exists a sequence $(u_n)_n \subset B_A^p$ such that $\mathcal{G}(u_n) \to v$ in $L^p(\Delta(A))$ as $n \to \infty$. But we have $\|u_n - u_{n'}\|_p = \|\mathcal{G}(u_n) - \mathcal{G}(u_{n'})\|_{L^p(\Delta(A))} \to 0$ as $n, n' \to \infty$. Thus $(u_n)_n$ is a Cauchy sequence in B_A^p. As B_A^p is complete there exists a function $u \in B_A^p$ such that $u_n \to u$ in B_A^p as $n \to \infty$, hence $\mathcal{G}(u_n) \to \mathcal{G}(u)$ in $L^p(\Delta(A))$ as $n \to \infty$. Whence $v = \mathcal{G}(u)$. This shows that $\mathcal{G}$ is surjective. Therefore, by the first isomorphism theorem, the mapping $\mathcal{G}_1 : \mathcal{B}_A^p = B_A^p/\mathcal{N} \to L^p(\Delta(A))$ defined by

$$\mathcal{G}_1(u + \mathcal{N}) = \mathcal{G}(u) \text{ for } u \in B_A^p$$

is an algebraic isomorphism. But $\mathcal{G}_1$ is a topological isometric isomorphism since

$$\|\mathcal{G}_1(u+\mathcal{N})\|_{L^p(\Delta(A))} = \|u\|_p = \|u+\mathcal{N}\|_{\mathcal{B}^p_A} \text{ for } u \in B^p_A.$$

This completes the proof. □

As a first consequence of the preceding theorem one can also define the mean value of $u+\mathcal{N}$ (for each $u \in B^p_A$) as follows:

$$M_1(u+\mathcal{N}) = M(u), \text{ so that } M_1(u+\mathcal{N}) = \lim_{r\to+\infty} \frac{1}{|B_r|}\int_{B_r} u(y)dy.$$

One crucial result that can be derived from the preceding theorem is the following

Corollary 2.6. *The following hold true:*

(i) *The spaces $\mathcal{B}^p_A$ are reflexive for $1 < p < \infty$;*

(ii) *The topological dual of the space $\mathcal{B}^p_A$ $(1 \le p < \infty)$ is the space $\mathcal{B}^{p'}_A$ $(p' = p/(p-1))$, the duality being given by*

$\langle u+\mathcal{N}, v+\mathcal{N}\rangle_{\mathcal{B}^{p'}_A, \mathcal{B}^p_A} = M(uv) = \int_{\Delta(A)} \mathcal{G}_1(u+\mathcal{N})\mathcal{G}_1(v+\mathcal{N})d\beta$

for $u \in B^{p'}_A$ and $v \in B^p_A$.

This result is easily proven by using the properties of L^p-spaces and the above isometric isomorphism.

Remark 2.1. The space $\mathcal{B}^p_A$ is the separated completion of B^p_A and the canonical mapping of B^p_A into $\mathcal{B}^p_A$ is just the canonical surjection of B^p_A onto $\mathcal{B}^p_A$; see once more [9] for the theory of completion.

Another definition which will be of great interest in the ensuing sections is

Definition 2.1. An algebra w.m.v. A on $\mathbb{R}^m$ is *ergodic* if for every $u \in B^1_A$ such that $\|u - u(\cdot + a)\|_1 = 0$ for every $a \in \mathbb{R}^m$ we have $\|u - M(u)\|_1 = 0$.

An equivalent property given by Casado and Gayte [13] is the following proposition.

Proposition 2.7. [13] *An algebra w.m.v. A on $\mathbb{R}^m$ is ergodic if*

$$\lim_{r\to+\infty}\left\| \frac{1}{|B_r|}\int_{B_r} u(\cdot + y)dy - M(u)\right\|_p = 0 \text{ for all } u \in B^p_A,\ 1 \le p < \infty. \tag{2.2}$$

The following result provides us with a few examples of ergodic algebras (see next section for its application).

Lemma 2.8. *Let A be an algebra w.m.v. on $\mathbb{R}^m$ with the following property:*

$$\lim_{r\to+\infty} \frac{1}{|B_r|}\int_{B_r} u(x+y)dx = M(u) \text{ uniformly with respect to } y. \tag{2.3}$$

Then A is ergodic.

Proof. As A is dense in B^p_A, it suffices to check (2.2) for $u \in A$. Let $\eta > 0$ be freely fixed. For such u's, according to (2.3) there exists $r_0 > 0$ such that

$$\left|\frac{1}{|B_r|}\int_{B_r} u(x+y)dx - M(u)\right| \le \eta \text{ for } r > r_0$$

and all $y \in \mathbb{R}^m$. This leads at once at

$$M_y\left(\left|\frac{1}{|B_r|}\int_{B_r} u(x+y)dx - M(u)\right|^p\right) \leq \eta^p \text{ for } r > r_0.$$

The ergodicity of A follows thereby. □

From now on, we will use the same letter u (if there is no danger of confusion) to denote an equivalence class of an element $u \in B_A^p$. The symbol ϱ will denote the canonical surjection of B_A^p onto $\mathcal{B}_A^p = B_A^p/\mathcal{N}$. With this in mind, one can define another space attached to $\mathcal{B}_A^p$. To begin with, however, we need one further definition. Let $u \in \mathcal{B}_A^p$ be such that $\partial_i \mathcal{G}_1(u) \in L^p(\Delta(A))$ for fixed $1 \leq i \leq m$. There exists an unique $u_i \in \mathcal{B}_A^p$ such that $\partial_i \mathcal{G}_1(u) = \mathcal{G}_1(u_i)$. This leads to the following definition.

Definition 2.2. By a *formal derivative of index i of $u \in \mathcal{B}_A^p$* is meant the unique element u_i of $\mathcal{B}_A^p$ (if there exists) denoted by $\overline{\partial} u/\partial y_i$, defined by

$$\mathcal{G}_1\left(\frac{\overline{\partial} u}{\partial y_i}\right) = \partial_i \mathcal{G}_1(u). \tag{2.4}$$

Remark 2.2. For $u \in B_A^{1,p}$ (that is the space of $u \in B_A^p$ such that $D_y u \in (B_A^p)^m$) we have

$$\mathcal{G}_1\left(\varrho\left(\frac{\partial u}{\partial y_i}\right)\right) = \mathcal{G}\left(\frac{\partial u}{\partial y_i}\right) = \partial_i \mathcal{G}(u) = \partial_i \mathcal{G}_1(\varrho(u)) = (\text{by definition})\, \mathcal{G}_1\left(\frac{\overline{\partial}}{\partial y_i}(\varrho(u))\right),$$

hence

$$\varrho\left(\frac{\partial u}{\partial y_i}\right) = \frac{\overline{\partial}}{\partial y_i}(\varrho(u)),$$

or equivalently,

$$\varrho \circ \frac{\partial}{\partial y_i} = \frac{\overline{\partial}}{\partial y_i} \circ \varrho \text{ on } B_A^{1,p}. \tag{2.5}$$

Now, set (for $1 \leq p < \infty$)

$$\mathcal{B}_A^{1,p} = \left\{u \in \mathcal{B}_A^p : \frac{\overline{\partial} u}{\partial y_i} \in \mathcal{B}_A^p,\ 1 \leq i \leq m\right\}.$$

We endow $\mathcal{B}_A^{1,p}$ with the norm

$$\|u\|_{\mathcal{B}_A^{1,p}} = \left[\|u\|_p^p + \sum_{i=1}^m \left\|\frac{\overline{\partial} u}{\partial y_i}\right\|_p^p\right]^{1/p} \quad (u \in \mathcal{B}_A^{1,p}).$$

In this norm, $\mathcal{B}_A^{1,p}$ is a Banach space and further, the restriction of $\mathcal{G}_1$ to $\mathcal{B}_A^{1,p}$ is an isometric isomorphism of $\mathcal{B}_A^{1,p}$ onto $W^{1,p}(\Delta(A))$.

Next, consider the subspace $\mathcal{B}_A^{1,p}/\mathbb{R}$ of $\mathcal{B}_A^{1,p}$ consisting of $u \in \mathcal{B}_A^{1,p}$ with $M_1(u) = 0$. Equipped with the seminorm

$$\|u\|_{\mathcal{B}_A^{1,p}/\mathbb{R}} = \left\|\overline{D}_y u\right\|_p := \left[\sum_{i=1}^m \left\|\frac{\overline{\partial} u}{\partial y_i}\right\|_p^p\right]^{1/p} \quad (u \in \mathcal{B}_A^{1,p}/\mathbb{R})$$

where $\overline{D}_y = (\frac{\overline{\partial}}{\partial y_i})_{1\leq i\leq m}$, $\mathcal{B}_A^{1,p}/\mathbb{R}$ is a locally convex topological space which is in general nonseparate and noncomplete. We denote by $\mathcal{B}_{\#A}^{1,p}$ the separated completion of $\mathcal{B}_A^{1,p}/\mathbb{R}$ with respect to $\|\cdot\|_{\mathcal{B}_A^{1,p}/\mathbb{R}}$, and by J_1 the canonical mapping of $\mathcal{B}_A^{1,p}/\mathbb{R}$ into $\mathcal{B}_{\#A}^{1,p}$. It is a fact (using the theory of completion of the uniform spaces [9]) that the mapping $\frac{\overline{\partial}}{\partial y_i} : \mathcal{B}_A^{1,p}/\mathbb{R} \to \mathcal{B}_A^p$ extends by continuity to an unique continuous linear mapping $\frac{\overline{\partial}}{\partial y_i} : \mathcal{B}_{\#A}^{1,p} \to \mathcal{B}_A^p$ satisfying

$$\frac{\overline{\partial}}{\partial y_i} \circ J_1 = \frac{\overline{\partial}}{\partial y_i} \text{ and } \|u\|_{\mathcal{B}_{\#A}^{1,p}} = \left\|\overline{D}_y u\right\|_p \quad (u \in \mathcal{B}_{\#A}^{1,p}) \tag{2.6}$$

where $\overline{D}_y = (\frac{\overline{\partial}}{\partial y_i})_{1\leq i\leq m}$. Since $\mathcal{G}_1$ is an isometric isomorphism of $\mathcal{B}_A^{1,p}$ onto $W^{1,p}(\Delta(A))$ we have by (2.4) that the restriction of $\mathcal{G}_1$ to $\mathcal{B}_A^{1,p}/\mathbb{R}$ sends isometrically $\mathcal{B}_A^{1,p}/\mathbb{R}$ onto $W^{1,p}(\Delta(A))/\mathbb{R}$. So by [9] there exists an unique isometric isomorphism $\overline{\mathcal{G}}_1 : \mathcal{B}_{\#A}^{1,p} \to W_{\#}^{1,p}(\Delta(A))$ such that

$$\overline{\mathcal{G}}_1 \circ J_1 = J \circ \mathcal{G}_1 \tag{2.7}$$

and

$$\partial_i \circ \overline{\mathcal{G}}_1 = \mathcal{G}_1 \circ \frac{\overline{\partial}}{\partial y_i} \quad (1 \leq i \leq m). \tag{2.8}$$

(We recall that J is the canonical mapping of $W^{1,p}(\Delta(A))/\mathbb{R}$ into its separated completion $W_{\#}^{1,p}(\Delta(A))$ while J_1 is the canonical mapping of $\mathcal{B}_A^{1,p}/\mathbb{R}$ into $\mathcal{B}_{\#A}^{1,p}$.) Furthermore, as $J_1(\mathcal{B}_A^{1,p}/\mathbb{R})$ is dense in $\mathcal{B}_{\#A}^{1,p}$ (this is classical), it follows that $(J_1 \circ \varrho)(A^\infty/\mathbb{R})$ is dense in $\mathcal{B}_{\#A}^{1,p}$, where $A^\infty/\mathbb{R} = \{u \in A^\infty : M(u) = 0\}$.

The properties of $\mathcal{G}_1$ and $\overline{\mathcal{G}}_1$ will come to light in the next section.

3. The two-scale A-convergence

In this section we rewrite the definition of the two-scale convergence. Thanks to the identification $\mathcal{G}_1(\mathcal{B}_A^p) = L^p(\Delta(A))$ we get more accurate convergence results as those obtained in the previous papers by Nguetseng et al. [36, 40, 41, 42]. In all that follow Ω is an open subset of $\mathbb{R}^N$ (integer $N \geq 1$) and A is an algebra w.m.v. on $\mathbb{R}^N$.

Definition 3.1. Let $1 \leq p < \infty$. A sequence $(u_\varepsilon)_{\varepsilon>0} \subset L^p(\Omega)$ is said to *weakly two-scale A-converge* in $L^p(\Omega)$ to some $u_0 \in L^p(\Omega; \mathcal{B}_A^p)$ if as $\varepsilon \to 0$, we have

$$\int_\Omega u_\varepsilon(x) f\left(x, \frac{x}{\varepsilon}\right) dx \to \int_\Omega \int_{\Delta(A)} \widehat{u}_0(x,s) \widehat{f}(x,s) d\beta(s) dx \tag{3.1}$$

for every $f \in L^{p'}(\Omega; A)$ $(1/p' = 1-1/p)$ where $\widehat{u}_0 = \mathcal{G}_1 \circ u_0$ and $\widehat{f} = \mathcal{G} \circ f = (\mathcal{G}_1 \circ \varrho) \circ f$.

Notation. We express this by writing $u_\varepsilon \to u_0$ in $L^p(\Omega)$-weak A.

Remark 3.1. Thanks to the equality $\mathcal{G}_1(\mathcal{B}_A^p) = L^p(\Delta(A))$, the above definition of convergence is exactly the same as the one given by Nguetseng in [36], namely the Σ-*convergence*. We have replaced Σ by A since we do not use the terminology *homogenization structures* [36] here. Thus the uniqueness of the limit u_0 is ensured. It is also a fact that the weak A-convergence in L^p implies the weak convergence in L^p.

In the sequel the letter E will throughout denote a fundamental sequence, that is, any ordinary sequence $E = (\varepsilon_n)_n$ (integers $n \geq 0$) with $0 < \varepsilon_n \leq 1$ and $\varepsilon_n \to 0$ as $n \to \infty$. The following result is the starting point to all the next compactness results we will deal with.

Theorem 3.1. *Any bounded sequence* $(u_\varepsilon)_{\varepsilon\in E}$ *in* $L^p(\Omega)$ $(1<p<\infty)$ *admits a subsequence which is weakly two-scale* A*-convergent in* $L^p(\Omega)$.

Proof. The proof in the general setting of algebras w.m.v. is given in [12, Theorem 4.8]. Here we just give sketch of the proof in the case of H-algebras, that is when A is separable.

Set $L_\varepsilon(f)=\int_\Omega u_\varepsilon f^\varepsilon dx$ $\quad (f\in L^{p'}(\Omega;A))$, where $f^\varepsilon\in L^{p'}(\Omega)$ is defined by $f^\varepsilon(x)=f(x,x/\varepsilon)$, $x\in\Omega$. Then

$$\left|L_\varepsilon(f)\right|\leq c\left\|f\right\|_{L^{p'}(\Omega;A)}\quad(\varepsilon\in E),$$

hence $(L_\varepsilon(f))_{\varepsilon\in E}$ is bounded in $\mathbb{R}$ and therefore, there exist a subsequence $E'(f)$ and a real number $L(f)$ such that, as $E'(f)\ni\varepsilon\to 0$, one has $L_\varepsilon(f)\to L(f)$. Using the separability of $L^{p'}(\Omega;A)$ (note that A is separable) and the diagonal process, one gets the existence of a subsequence E' from E such that, as $E'\ni\varepsilon\to 0$,

$$L_\varepsilon(f)\to L(f)\text{ for any }f\in L^{p'}(\Omega;A).$$

But one also has the inequality $\left|L_\varepsilon(f)\right|\leq c\left\|f^\varepsilon\right\|_{L^{p'}(\Omega)}$ and further, as $E'\ni\varepsilon\to 0$, $\left\|f^\varepsilon\right\|_{L^{p'}(\Omega)}\to\left\|\widehat{f}\right\|_{L^{p'}(\Omega\times\Delta(A))}$ (see e.g. [36]), so that the following holds:

$$\left|L(f)\right|\leq c\left\|\widehat{f}\right\|_{L^{p'}(\Omega\times\Delta(A))}=c\left\|f\right\|_{L^{p'}(\Omega;\mathcal{B}_A^{p'})}.$$

The space $\mathcal{B}_A^{p'}$ being reflexive with dual space $\mathcal{B}_A^p$, there exists a function $u_0\in L^p(\Omega;\mathcal{B}_A^p)$ such that

$$L(f)=\int_\Omega\int_{\Delta(A)}\widehat{u}_0(x,s)\widehat{f}(x,s)d\beta(s)dx.$$

This concludes the proof. □

The next result will need further notion.

Definition 3.2. A sequence $(u_\varepsilon)_{\varepsilon>0}$ in $L^1(\Omega)$ is said to be *uniformly integrable* if $(u_\varepsilon)_{\varepsilon>0}$ is bounded in $L^1(\Omega)$ and further one has $\sup_{\varepsilon>0}\int_X\left|u_\varepsilon\right|dx\to 0$ for any integrable set $X\subset\Omega$ such that $\left|X\right|\to 0$, where $\left|X\right|$ denotes the Lebesgue measure of X.

The analogue of Theorem 3.1 in the case when $p=1$ states as follows.

Theorem 3.2. *Any uniformly integrable sequence* $(u_\varepsilon)_{\varepsilon\in E}$ *in* $L^1(\Omega)$ *admits a subsequence which is weakly two-scale* A*-convergent in* $L^1(\Omega)$.

Proof. See the proof of [12, Theorem 4.10]. □

The next result requires some preliminaries. Beforehand we recall that A is translation invariant. Let $a\in\mathbb{R}^N$; the translation operator $\tau_a:A\to A$ extends by continuity to an unique translation operator still denoted by $\tau_a:B_A^p\to B_A^p$ $(1\leq p<\infty)$. Indeed τ_a is bijective and $\left\|\tau_a u\right\|_p=\left\|u\right\|_p$ since $M(\left|\tau_a u\right|^p)=M(\tau_a\left|u\right|^p)=M(\left|u\right|^p)$ for all $u\in A$. Besides since each element of A is uniformly continuous, the group of unitary operators $\{\tau_a\}_{a\in\mathbb{R}^N}$ thus defined is strongly continuous, i.e., $\tau_a u\to u$ in B_A^p as $\left|a\right|\to 0$ for all $u\in B_A^p$. Moreover

$$M(\tau_a u)=M(u)\text{ for all }u\in B_A^p\text{ and any }a\in\mathbb{R}^N.\tag{3.2}$$

With this in mind, we begin with the following preliminary result.

Lemma 3.3. *Let $(u_\varepsilon)_{\varepsilon\in E}$ be a uniformly integrable sequence in $L^1(\Omega)$ which weakly two-scale A-converges towards $u_0 \in L^1(\Omega;\mathcal{B}^1_A)$. Then*

(i) *For each $a \in \mathbb{R}^N$ the sequence $(v_\varepsilon)_{\varepsilon\in E}$ defined by $v_\varepsilon(x) = u_\varepsilon(x-\varepsilon a)$ $(x \in \Omega)$ weakly two-scale A-converges in $L^1(\Omega)$ towards the function $\tau_a u_0$ defined by $\tau_a u_0(x,y) = u(x,y-a)$ a.e. $(x,y) \in \Omega\times\mathbb{R}^N$;*

(ii) *For each positive real number r the sequence $(w_\varepsilon)_{\varepsilon\in E}$ defined by $w_\varepsilon(x) = \frac{1}{|B_r|}\int_{B_r} u_\varepsilon(x+\varepsilon\rho)d\rho$ $(x \in \Omega)$ weakly two-scale A-converges in $L^1(\Omega)$ towards the function w_0 defined by $w_0(x,y) = \frac{1}{|B_r|}\int_{B_r} u_0(x,y+\rho)d\rho$.*

Proof. Let us beginning by noticing that since $\mathcal{G}_1$ is a bounded linear operator of $\mathcal{B}^1_A$ into $L^1(\Delta(A))$ we have the following property

$$\mathcal{G}_1\left(\int_{B_r} u_0(x,\cdot+\rho)d\rho\right) = \int_{B_r}\mathcal{G}_1(u_0(x,\cdot+\rho))d\rho \tag{3.3}$$

where u_0 is as above.

Now, as part (i) is concerned, let $\varphi \in \mathcal{D}(\Omega)$ and let $f \in A$; we have

$$\begin{aligned}
\int_\Omega v_\varepsilon\varphi f^\varepsilon dx &= \int_\Omega u_\varepsilon(x-\varepsilon a)\varphi(x)f\left(\frac{x}{\varepsilon}\right)dx\\
&= \int_{\Omega-\varepsilon a} u_\varepsilon(x)\varphi(x+\varepsilon a)f\left(\frac{x}{\varepsilon}+a\right)dx\\
&= \int_\Omega u_\varepsilon(x)\varphi(x+\varepsilon a)f\left(\frac{x}{\varepsilon}+a\right)dx - \int_{\Omega\setminus(\Omega-\varepsilon a)} u_\varepsilon(x)\varphi(x+\varepsilon a)f\left(\frac{x}{\varepsilon}+a\right)dx\\
&\quad + \int_{(\Omega-\varepsilon a)\setminus\Omega} u_\varepsilon(x)\varphi(x+\varepsilon a)f\left(\frac{x}{\varepsilon}+a\right)dx\\
&= (I)-(II)+(III).
\end{aligned}$$

As for (I) we have

$$\begin{aligned}
(I) &= \int_\Omega u_\varepsilon(x)\varphi(x)(\tau_{-a}f)\left(\frac{x}{\varepsilon}\right)dx + \int_\Omega u_\varepsilon(x)[\varphi(x+\varepsilon a)-\varphi(x)](\tau_{-a}f)\left(\frac{x}{\varepsilon}\right)dx\\
&= (I_1)+(I_2).
\end{aligned}$$

The algebra A being translation invariant, we have $\tau_{-a}f \in A$, and so

$$(I_1) \to \int_\Omega\int_{\Delta(A)} \widehat{u}_0(x,s)\varphi(x)\widehat{\tau_{-a}f}(s)d\beta dx \text{ as } E \ni \varepsilon \to 0.$$

But

$$\begin{aligned}
\int_{\Delta(A)} \widehat{u}_0(x,s)\widehat{\tau_{-a}f}(s)d\beta &= M(u_0(x,\cdot)\tau_{-a}f)\\
&= M(\tau_{-a}(\tau_a u_0(x,\cdot)f))\\
&= M(\tau_a u_0(x,\cdot)f) \text{ because } M \text{ is translation invariant}\\
&= \int_{\Delta(A)} \widehat{\tau_a u_0}(x,s)\widehat{f}(s)d\beta.
\end{aligned}$$

Note that here we have identified $u_0(x,\cdot) \in \mathcal{B}^1_A$ with its representant $u_0(x,\cdot) \in B^1_A$ so that $M_1(u_0(x,\cdot)) = M(u_0(x,\cdot))$, $u_0(x,\cdot)$ on the left-hand side being an equivalence class whereas $u_0(x,\cdot)$ on the right-hand side is its representant. On the other hand we have

$$(I_2) \leq \varepsilon\,|a|\,(\sup_{x\in\Omega}|D\varphi(x)|)\int_\Omega |u_\varepsilon(x)|\left|f\left(\frac{x}{\varepsilon}+a\right)\right|dx$$

and so $(I_2)\to 0$ as $E\ni\varepsilon\to 0$. Besides, the uniform integrability of $(u_\varepsilon)_\varepsilon$ and the inequality

$$\int_{(\Omega-\varepsilon a)\Delta\Omega}|u_\varepsilon(x)|\,|\varphi(x+\varepsilon a)|\left|f\left(\frac{x}{\varepsilon}+a\right)\right|dx \leq \|\varphi\|_\infty\,\|f\|_\infty\int_{(\Omega-\varepsilon a)\Delta\Omega}|u_\varepsilon(x)|\,dx$$

(where $B\Delta C$ denotes the symmetric difference between the two sets B and C) lead to the fact that (II) and (III) go towards 0 as $E\ni\varepsilon\to 0$. (i) is proved thereby.

As for (ii), let φ and f be as above. We have by the Fubini's theorem,

$$\int_\Omega\left(\frac{1}{|B_r|}\int_{B_r}u_\varepsilon(x+\varepsilon\rho)d\rho\right)\varphi(x)f\left(\frac{x}{\varepsilon}\right)dx=\frac{1}{|B_r|}\int_{B_r}\int_\Omega u_\varepsilon(x+\varepsilon\rho)\varphi(x)f\left(\frac{x}{\varepsilon}\right)dxd\rho.$$

According to (i) we have for each fixed $\rho\in B_r$, as $E\ni\varepsilon\to 0$,

$$\int_\Omega u_\varepsilon(x+\varepsilon\rho)\varphi(x)f\left(\frac{x}{\varepsilon}\right)dx\to\iint_{\Omega\times\Delta(A)}\widehat{\tau_{-\rho}u_0}(x,s)\varphi(x)\widehat{f}(s)dxd\beta.$$

Thus, using the Lebesgue dominated convergence theorem we are led at once at (ii). ☐

The preceding lemma has an important corollary whose usefulness will come to light in the proof of the next compactness result.

Corollary 3.4. *Let $Q=\Omega\times(0,T)$ where $0<T<\infty$ and Ω is an open bounded set in $\mathbb{R}^N$. Let $(u_\varepsilon)_{\varepsilon\in E}$ be a uniformly integrable sequence in $L^1(Q)$ which weakly two-scale A-converges towards $u_0\in L^1(Q;\mathcal{B}_A^1)$ (A being an algebra w.m.v. on $\mathbb{R}_{y,\tau}^{N+1}$). For each positive real number r the sequence $(w_\varepsilon)_{\varepsilon\in E}$ defined by $w_\varepsilon(x,t)=\frac{1}{|B_r|}\int_{B_r}u_\varepsilon(x+\varepsilon\rho,t)d\rho$ $((x,t)\in Q)$ weakly two-scale A-converges in $L^1(Q)$ towards the function $w_0\in L^1(Q;\mathcal{B}_A^1)$ defined by $w_0(x,t,y,\tau)=\frac{1}{|B_r|}\int_{B_r}u_0(x,t,y+\rho,\tau)d\rho$.*

Proof. This is a mere adaptation of the proof of Lemma 3.3 by changing $x\in\Omega$ into $(x,t)\in Q$ and $a\in\mathbb{R}^N$ into $a'=(a,0)\in\mathbb{R}^{N+1}$. ☐

In order to state the next result, we need one further definition. For $1<p<\infty$, set

$$V^p=\{v\in L^p(0,T;W_0^{1,p}(\Omega)):v'=\partial v/\partial t\in L^{p'}(0,T;W^{-1,p'}(\Omega))\}$$

where $0<T<\infty$ and Ω is an open bounded set in $\mathbb{R}^N$. Then V^p is a Banach space under the norm

$$\|v\|_{V^p}=\|v\|_{L^p(0,T;W_0^{1,p}(\Omega))}+\|v'\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))}\quad(v\in V^p).$$

Moreover if $p\geq 2$, then V^p is continuously embedded in the space $\mathcal{C}([0,T];L^2(\Omega))$ and compactly embedded in the space $L^2(Q)$ where, as above,

$$Q=\Omega\times(0,T).$$

We also define according to [2] the notion of vector valued algebra. Let A_τ be an algebra w.m.v. on $\mathbb{R}_\tau$, and let X be a Fréchet space. We denote by $A_\tau(\mathbb{R}_\tau;X)$ the space of functions $f\in\mathcal{B}(\mathbb{R}_\tau;X)$ (the space of bounded uniformly continuous functions of $\mathbb{R}_\tau$ into X) satisfying the following conditions:

(i) For all $L\in X'$ (topological dual of X) we have $\langle L,f\rangle\in A_\tau$;

(ii) The family $\mathcal{F}_f = \{\langle L, f\rangle : L \in X' \text{ with } \|L\| \leq 1\}$ is relatively compact in A_τ.

Since there is an isometric isomorphism between $A_\tau(\mathbb{R}_\tau; X)$ and $\mathcal{C}(\Delta(A_\tau); X)$ (see [2, Theorem 5.1]), one sees that $A_\tau \otimes X$ is dense in $A_\tau(\mathbb{R}_\tau; X)$. Indeed this follows from the classical fact that $\mathcal{C}(\Delta(A_\tau)) \otimes X$ is dense in $\mathcal{C}(\Delta(A_\tau); X)$ as $\Delta(A_\tau)$ is a compact topological space. In particular we have the following result which shows that the concept of vector valued algebra is a generalization of the notion of product algebras.

Lemma 3.5. *Let A_1 (resp. A_2) be an algebra w.m.v. on $\mathbb{R}^{m_1}$ (resp. $\mathbb{R}^{m_2}$). Then*

$$A_1 \odot A_2 = A_1(\mathbb{R}^{m_1}; A_2).$$

Proof. This follows from the fact that $A_1 \otimes A_2$ is dense in $A_1(\mathbb{R}^{m_1}; A_2)$. □

Now let $u \in A_\tau(\mathbb{R}_\tau; X)$; then due to the compactness of the corresponding family $\mathcal{F}_u$, we have that the function $\tau \mapsto \|u(\tau)\|_X$, denoted by $\|u\|_X$, lies in A_τ (this comes in fact from the equality $\|u(\tau)\|_X = \sup_{L\in X', \|L\|\leq 1} |\langle L, u(\tau)\rangle|$). Hence for any $1 \leq p < \infty$ we have $\|u\|_X^p \in A_\tau$. Therefore we define the vector valued generalized Besicovitch spaces $B^p_{A_\tau}(\mathbb{R}_\tau; X)$ as the completion of the space $A_\tau(\mathbb{R}_\tau; X)$ with respect to the seminorm

$$\|u\|_p = (M(\|u\|_X^p))^{1/p} \quad (u \in A_\tau(\mathbb{R}_\tau; X)).$$

Next, we know that, in the above seminorm, $A_\tau(\mathbb{R}_\tau; X)$ is dense in $B^p_{A_\tau}(\mathbb{R}_\tau; X)$, so that $A_\tau \otimes X$ is dense $B^p_{A_\tau}(\mathbb{R}_\tau; X)$. In particular if A_y is an algebra w.m.v. on $\mathbb{R}^N_y$ then $A_\tau \otimes A_y$ (and hence $A_\tau \odot A_y$) is dense in $B^p_{A_\tau}(\mathbb{R}_\tau; B^p_{A_y})$.

We can also define the corresponding quotient spaces $\mathcal{B}^p_{A_\tau}(\mathbb{R}_\tau; X)$ and it can be proven that the space $\mathcal{B}^p_{A_\tau}(\mathbb{R}_\tau; X)$ is isometrically isomorphic to the space $L^p(\Delta(A_\tau); X)$.

Now, let $A = A_y \odot A_\tau$, A_y and A_τ being as above. The same letter $\mathcal{G}$ will denote the Gelfand transformation on A_y, A_τ and A, as well. Points in $\Delta(A_y)$ (resp. $\Delta(A_\tau)$) are denoted by s (resp. s_0). The compact space $\Delta(A_y)$ (resp. $\Delta(A_\tau)$) is equipped with the M-measure β_y (resp. β_τ), for A_y (resp. A_τ). We have $\Delta(A) = \Delta(A_y) \times \Delta(A_\tau)$ (Cartesian product) and the M-measure for A, with which $\Delta(A)$ is equipped, is precisely the product measure $\beta = \beta_y \otimes \beta_\tau$ (see [36]).

We can now state and prove the compactness theorem we will deal with in the forthcoming sections.

Theorem 3.6. *Let $A = A_y \odot A_\tau$ where A_y (resp. A_τ) is an algebra w.m.v. on $\mathbb{R}^N_y$ (resp. $\mathbb{R}_\tau$). Assume A_y is ergodic. Let $1 < p < \infty$. Finally, let $(u_\varepsilon)_{\varepsilon\in E}$ be a bounded sequence in V^p. There exist a subsequence E' from E and a couple $\mathbf{u} = (u_0, u_1) \in V^p \times L^p(Q; \mathcal{B}^p_{A_\tau}(\mathbb{R}_\tau; \mathcal{B}^{1,p}_{\#A_y}))$ such that, as $E' \ni \varepsilon \to 0$,*

$$u_\varepsilon \to u_0 \text{ in } V^p\text{-weak} \tag{3.4}$$

$$\frac{\partial u_\varepsilon}{\partial x_i} \to \frac{\partial u_0}{\partial x_i} + \frac{\overline{\partial} u_1}{\partial y_i} \text{ in } L^p(Q)\text{-weak } A \ \ (1 \leq i \leq N). \tag{3.5}$$

Proof. Let the hypotheses be those of Theorem 3.6 above. By the reflexivity of the space V^p and also arguing as in Theorem 3.1, there exist a subsequence E' from E and, a function $u_0 \in V^p$ and a vector function $w = (w_i)_{1\leq i\leq N} \in L^p(Q; (\mathcal{B}^p_A)^N)$ such that, as $E' \ni \varepsilon \to 0$, we have (3.4) and $\frac{\partial u_\varepsilon}{\partial x_i} \to w_i$ in $L^p(Q)$-weak A $(1 \leq i \leq N)$. It remains to check that there exists a function $u_1 \in L^p(Q; \mathcal{B}^p_{A_\tau}(\mathbb{R}_\tau; \mathcal{B}^{1,p}_{\# A_y}))$ such that $w_i = \frac{\partial u_0}{\partial x_i} + \frac{\overline{\partial} u_1}{\partial y_i}$ $(1 \leq i \leq N)$. For that purpose, let $r > 0$ be freely fixed. Let $B_{\varepsilon r}$ denote the open ball in $\mathbb{R}^N_y$ with radius εr. By the equalities

$$\begin{aligned} \frac{1}{\varepsilon}\left(u_\varepsilon(x,t) - \frac{1}{|B_{\varepsilon r}|}\int_{B_{\varepsilon r}} u_\varepsilon(x+y,t)dy\right) &= \frac{1}{\varepsilon}\frac{1}{|B_{\varepsilon r}|}\int_{B_{\varepsilon r}} (u_\varepsilon(x,t) - u_\varepsilon(x+y,t))\,dy \\ &= \frac{1}{\varepsilon}\frac{1}{|B_r|}\int_{B_r} (u_\varepsilon(x,t) - u_\varepsilon(x+\varepsilon y,t))\,dy \\ &= -\frac{1}{|B_r|}\int_{B_r} dy \int_0^1 Du_\varepsilon(x+\theta\varepsilon y,t)\cdot y d\theta, \end{aligned}$$

(the dot denoting the usual Euclidean inner product in $\mathbb{R}^N$) we deduce from the boundedness of $(u_\varepsilon)_{\varepsilon\in E'}$ in V^p that the sequence $(v_\varepsilon)_{\varepsilon\in E'}$ defined by

$$v_\varepsilon(x,t) = \frac{1}{\varepsilon}\left(u_\varepsilon(x,t) - \frac{1}{|B_{\varepsilon r}|}\int_{B_{\varepsilon r}} u_\varepsilon(x+\rho,t)d\rho\right)$$

is bounded in $L^p(Q)$. Hence due to Theorem 3.1 there exist a subsequence from E' not relabeled and a function $V_r \in L^p(Q; \mathcal{B}^p_A)$ such that, as $E' \ni \varepsilon \to 0$,

$$v_\varepsilon \to V_r \text{ in } L^p(Q)\text{-weak } A. \tag{3.6}$$

But for $\varphi \in \mathcal{D}(Q)$, $\psi \in A^\infty_y$ and $\chi \in A^\infty_\tau$ we have

$$\begin{aligned} &\int_Q \tfrac{1}{\varepsilon}\left(u_\varepsilon(x,t) - \tfrac{1}{|B_{\varepsilon r}|}\int_{B_{\varepsilon r}} u_\varepsilon(x+\rho,t)d\rho\right)\tfrac{\partial\psi}{\partial y_i}(\tfrac{x}{\varepsilon})\chi(\tfrac{t}{\varepsilon})\varphi(x,t)dxdt \\ &\quad = -\int_Q \left(\tfrac{\partial u_\varepsilon}{\partial x_i}(x,t) - \tfrac{1}{|B_{\varepsilon r}|}\int_{B_{\varepsilon r}} \tfrac{\partial u_\varepsilon}{\partial x_i}(x+\rho,t)d\rho\right)\psi(\tfrac{x}{\varepsilon})\chi(\tfrac{t}{\varepsilon})\varphi(x,t)dxdt \\ &\qquad - \int_Q \left(u_\varepsilon(x,t) - \tfrac{1}{|B_{\varepsilon r}|}\int_{B_{\varepsilon r}} u_\varepsilon(x+\rho,t)d\rho\right)\psi(\tfrac{x}{\varepsilon})\chi(\tfrac{t}{\varepsilon})\tfrac{\partial\varphi}{\partial x_i}(x,t)dxdt. \end{aligned} \tag{3.7}$$

Now since $u_\varepsilon - \frac{1}{|B_{\varepsilon r}|}\int_{B_{\varepsilon r}} u_\varepsilon(\cdot+\rho,\cdot)d\rho = \varepsilon v_\varepsilon \to 0$ in $L^p(Q)$ as $E' \ni \varepsilon \to 0$ (recall that $(v_\varepsilon)_{\varepsilon\in E'}$ is bounded in $L^p(Q)$) we then pass to the limit in (3.7) as $E' \ni \varepsilon \to 0$, and obtain, using Corollary 3.4 and property (3.3),

$$\begin{aligned} &\iint_{Q\times\Delta(A)} \widehat{V}_r \partial_i \widehat{\psi}\widehat{\chi}\varphi dxdtd\beta \\ &\quad = -\iint_{Q\times\Delta(A)} \left(\widehat{w}_i(x,t,s,s_0) - \tfrac{1}{|B_r|}\int_{B_r} \mathcal{G}_1(w_i(x,t,\cdot+\rho,\cdot))d\rho\right)\widehat{\psi}\widehat{\chi}\varphi dxdtd\beta, \end{aligned}$$

and because of the arbitrariness of φ, ψ and χ,

$$\partial_i \widehat{V}_r(x,t,\cdot,\cdot) = \mathcal{G}_1\left(w_i(x,t,\cdot,\cdot) - \frac{1}{|B_r|}\int_{B_r} w_i(x,t,\cdot+\rho,\cdot)d\rho\right) \text{ a.e. } (x,t)\in Q.$$

We recall that here $\widehat{w}_i(x,t,s,s_0) = \mathcal{G}_1(w_i(x,t,\cdot,\cdot))(s,s_0)$ a.e. $(s,s_0) \in \Delta(A_y)\times\Delta(A_\tau)$. But $\partial_i\widehat{V}_r(x,t,\cdot,\cdot) = \partial_i\mathcal{G}_1(V_r(x,t,\cdot,\cdot)) = \mathcal{G}_1\left(\frac{\overline{\partial} V_r}{\partial y_i}(x,t,\cdot,\cdot)\right)$, hence, for $1 \leq i \leq N$,

$$\frac{\overline{\partial} V_r}{\partial y_i}(x,t,\cdot,\cdot) = w_i(x,t,\cdot,\cdot) - \frac{1}{|B_r|}\int_{B_r} w_i(x,t,\cdot+\rho,\cdot)d\rho \text{ a.e. in } (x,t)\in Q$$

since $\mathcal{G}_1$ is an isomorphism of $\mathcal{B}^p_A$ onto $L^p(\Delta(A))$. Set $f_r = V_r - M_1(V_r)$, M_1 being taken with respect to the variable y. Then $M_1(f_r) = 0$ and moreover $\overline{D}_y V_r = \overline{D}_y f_r$. Now since $B^p_A \subset B^p_{A_\tau}(\mathbb{R}_\tau; B^p_{A_y})$ (this comes from the fact that $A_y \otimes A_\tau \subset B^p_{A_\tau}(\mathbb{R}_\tau; B^p_{A_y})$: indeed for every $L \in A'_y$ and any $u \in A_y \otimes A_\tau$ we have $\langle L, u\rangle \in A_\tau$; moreover, the closed unit ball of A'_y being weakly sequentially compact, any sequence from $\mathcal{F}_u$ (for $u \in A_y \otimes A_\tau$) has a compact subsequence in $\mathcal{F}_u$) we have that $w_i \in L^p(Q;\mathcal{B}^p_{A_\tau}(\mathbb{R}_\tau;\mathcal{B}^p_{A_y}))$ and so $\overline{\partial} f_r/\partial y_i \in L^p(Q;\mathcal{B}^p_{A_\tau}(\mathbb{R}_\tau;\mathcal{B}^p_{A_y}))$, that is,

$$f_r \in L^p(Q;\mathcal{B}^p_{A_\tau}(\mathbb{R}_\tau;\mathcal{B}^{1,p}_{A_y}/\mathbb{R})).$$

So let $g_r = J_1 \circ f_r$, where J_1 denotes the canonical mapping of $\mathcal{B}^{1,p}_{A_y}/\mathbb{R}$ into its separated completion $\mathcal{B}^{1,p}_{\#A_y}$. Then $g_r \in L^p(Q;\mathcal{B}^p_{A_\tau}(\mathbb{R}_\tau;\mathcal{B}^{1,p}_{\#A_y}))$ and furthermore,

$$\frac{\overline{\partial} g_r}{\partial y_i}(x,t,\cdot,\cdot) = w_i(x,t,\cdot,\cdot) - \frac{1}{|B_r|}\int_{B_r} w_i(x,t,\cdot+\rho,\cdot)d\rho \ \ (1 \le i \le N)$$

since $\frac{\overline{\partial} g_r}{\partial y_i}(x,t,\cdot,\cdot) = \frac{\overline{\partial} f_r}{\partial y_i}(x,t,\cdot,\cdot) = \frac{\overline{\partial} V_r}{\partial y_i}(x,t,\cdot,\cdot)$. We still view $w_i(x,t,\cdot,\cdot)$ as its representant in B^p_A. With this, we have

$$\begin{aligned}\|g_r(x,t,\cdot,\tau) - g_{r'}(x,t,\cdot,\tau)\|_{\mathcal{B}^{1,p}_{\#A_y}} \le\ & \left\|\overline{D}_y g_r(x,t,\cdot,\tau) - w(x,t,\cdot,\tau) + M(w(x,t,\cdot,\tau))\right\|_p \\ & + \left\|\overline{D}_y g_{r'}(x,t,\cdot,\tau) - w(x,t,\cdot,\tau) + M(w(x,t,\cdot,\tau))\right\|_p .\end{aligned}$$

But

$$\begin{aligned}&\left\|\overline{D}_y g_r(x,t,\cdot,\tau) - w(x,t,\cdot,\tau) + M(w(x,t,\cdot,\tau))\right\|_p \\ &= \left\|\tfrac{1}{|B_r|}\textstyle\int_{B_r} w(x,t,\cdot+\rho,\tau)d\rho - M(w(x,t,\cdot,\tau))\right\|_p .\end{aligned}$$

Therefore, since the algebra A_y is ergodic, we have

$$\|g_r(x,t,\cdot,\tau) - g_{r'}(x,t,\cdot,\tau)\|_{\mathcal{B}^{1,p}_{\#A_y}} \to 0 \text{ as } r, r' \to +\infty.$$

Thus the sequence $(g_r(x,t,\cdot,\tau))_{r>0}$ is a Cauchy sequence in the Banach space $\mathcal{B}^{1,p}_{\#A_y}$, whence the existence of an unique $u_1(x,t,\cdot,\tau) \in \mathcal{B}^{1,p}_{\#A_y}$ such that

$$g_r(x,t,\cdot,\tau) \to u_1(x,t,\cdot,\tau) \text{ in } \mathcal{B}^{1,p}_{\#A_y} \text{ as } r \to +\infty,$$

that is

$$\overline{D}_y g_r(x,t,\cdot,\tau) \to \overline{D}_y u_1(x,t,\cdot,\tau) \text{ in } (\mathcal{B}^p_{A_y})^N \text{ as } r \to +\infty.$$

Once again the ergodicity of A_y and the uniqueness of the limit leads us at once at

$$\overline{D}_y u_1(x,t,\cdot,\tau) = w(x,t,\cdot,\tau) - M(w(x,t,\cdot,\tau)) \text{ a.e. } (x,t,\tau) \in Q \times \mathbb{R}_\tau,$$

hence the existence of $u_1 : Q \to \mathcal{B}^p_{A_\tau}(\mathbb{R}_\tau;\mathcal{B}^{1,p}_{\#A_y})$, $(x,t) \mapsto u_1(x,t,\cdot,\cdot)$, belonging to $L^p(Q;\mathcal{B}^p_{A_\tau}(\mathbb{R}_\tau;\mathcal{B}^{1,p}_{\#A_y}))$ such that

$$w - M(w) = \overline{D}_y u_1.$$

Finally, since $Du_\varepsilon \to w$ in $L^p(Q)^N$-weak A as $E' \ni \varepsilon \to 0$, it follows that $Du_\varepsilon \to M(w)$ in $L^p(Q)^N$-weak A as $E' \ni \varepsilon \to 0$, and so, by (3.4) we get $M(w) = Du_0$, so that

$$w = Du_0 + \overline{D}_y u_1.$$

This concludes the proof. □

Another important result whose proof is quite similar to the preceding one is the following

Theorem 3.7. *Let A be an ergodic algebra w.m.v. on $\mathbb{R}^N_y$. Let $(u_\varepsilon)_{\varepsilon\in E}$ be a bounded sequence in $W^{1,p}(\Omega)$ $(1 < p < \infty)$. There exist a subsequence E' from E and a couple $\mathbf{u} = (u_0, u_1) \in W^{1,p}(\Omega) \times L^p(\Omega; \mathcal{B}^{1,p}_{\#A})$ such that, as $E' \ni \varepsilon \to 0$,*

$$u_\varepsilon \to u_0 \text{ in } W^{1,p}(\Omega)\text{-weak}$$

and

$$\frac{\partial u_\varepsilon}{\partial x_i} \to \frac{\partial u_0}{\partial x_i} + \frac{\overline{\partial} u_1}{\partial y_i} \text{ in } L^p(\Omega)\text{-weak } A \ (1 \leq i \leq N).$$

Naturally the proof of this result being copied on that of the preceding one, is omitted.

Remark 3.2. As it will be seen below, Theorems 3.6 and 3.7 extend all the results of all the previous papers dealing with deterministic homogenization theory [42, 40, 41] since all the H-algebras till encountered so far are ergodic algebras. We will also see that our results apply to particular algebra w.m.v. which are not H-algebras.

We give here below a few examples of algebras which match hypotheses of Theorems 3.6 and 3.7.

3.1. **Example 3.1: Periodic setting.** Let $A_y = \mathcal{C}_{\mathrm{per}}(Y)$ $(Y = (0,1)^N)$ be the algebra of Y-periodic continuous functions on $\mathbb{R}^N_y$. It is classically known that A_y is an ergodic algebra with mean value, so that Theorem 3.7 applies. Also for any algebra w.m.v. A_τ on $\mathbb{R}_\tau$, the algebra $A = \mathcal{C}_{\mathrm{per}}(Y) \odot A_\tau$ satisfies hypotheses of Theorem 3.6.

3.2. **Example 3.2: Almost periodic setting.** Let $A_y = AP(\mathbb{R}^N_y)$ be the algebra of Bohr continuous almost periodic functions $\mathbb{R}^N_y$. We recall that a function $u \in \mathcal{B}(\mathbb{R}^N_y)$ is in $AP(\mathbb{R}^N_y)$ if the set of translates $\{\tau_a u : a \in \mathbb{R}^N\}$ is relatively compact in $\mathcal{B}(\mathbb{R}^N_y)$. An argument due to Bohr [8] specifies that $u \in AP(\mathbb{R}^N_y)$ if and only if u may be uniformly approximated by finite linear combinations of functions in the set $\{\cos(k \cdot y), \sin(k \cdot y) : k \in \mathbb{R}^N\}$.

It is also a classical result that A_y is an ergodic algebra w.m.v.; see e.g. [20, 47]. Hence Theorem 3.7 applies with A_y, and Theorem 3.6 applies with $A = AP(\mathbb{R}^N_y) \odot A_\tau$ for any algebra w.m.v. A_τ on $\mathbb{R}_\tau$.

Now, let $\mathcal{R}$ be a countable subgroup of $\mathbb{R}^N$. We denote by $AP_{\mathcal{R}}(\mathbb{R}^N_y)$ the space of those functions in $AP(\mathbb{R}^N_y)$ that can be uniformly approximated by finite linear combinations in the set $\{\cos(k \cdot y), \sin(k \cdot y) : k \in \mathcal{R}\}$. Then $A_y = AP_{\mathcal{R}}(\mathbb{R}^N_y)$ is an H-algebra [36]. We have the following

Proposition 3.8. *The H-algebra $A_y = AP_{\mathcal{R}}(\mathbb{R}^N_y)$ is an ergodic algebra.*

Proof. Since $AP_{\mathcal{R}}(\mathbb{R}^N_y) \subset AP(\mathbb{R}^N_y)$, the ergodicity of A_y is obtained. Thus our only concern here is to show that A_y is translation invariant. To this end, we know by [31, Theorem 4.1.4] that $\Delta(A_y)$ is a compact Abelian topological group additively written. With this in mind, let the mapping $j : \mathbb{R}^N_y \to \Delta(A_y)$ be given by

$$j(y) = \delta_y \text{ (the Dirac mass at } y \in \mathbb{R}^N).$$

Then thanks to [39] j is a group homomorphism with the following properties:

(a) j is continuous;
(b) $j(\mathbb{R}^N_y)$ is dense in $\Delta(A_y)$.

So let $a \in \mathbb{R}^N$, and let $u \in A_y$. It is a fact that the function $\tau_{j(a)}\mathcal{G}(u)$, defined by $\tau_{j(a)}\mathcal{G}(u)(s) = \mathcal{G}(u)(s - j(a))$ $(s \in \Delta(A_y))$, lies in $\mathcal{C}(\Delta(A_y))$. On the other hand we have, for each $y \in \mathbb{R}^N$,

$$\begin{aligned}\tau_{j(a)}\mathcal{G}(u)(j(y)) &= \mathcal{G}(u)(j(y) - j(a)) = \mathcal{G}(u)(j(y-a)) = \langle \delta_{y-a}, u \rangle \\ &= u(y-a) = (\tau_a u)(y) = \langle \delta_y, \tau_a u \rangle = \mathcal{G}(\tau_a u)(j(y)).\end{aligned}$$

By the density of $j(\mathbb{R}^N_y)$ in $\Delta(A_y)$ and by the continuity of $\mathcal{G}$ it follows that

$$\tau_{j(a)}\mathcal{G}(u) = \mathcal{G}(\tau_a u). \tag{3.8}$$

It turns out that $\tau_a u \in A_y$ because by (3.8), $\mathcal{G}(\tau_a u) \in \mathcal{C}(\Delta(A_y)) = \mathcal{G}(A_y)$. □

In view of Proposition 3.8 the conclusion of:

- Theorem 3.7 holds with $A = AP_{\mathcal{R}}(\mathbb{R}^N_y)$;
- Theorem 3.6 holds with $A = AP_{\mathcal{R}}(\mathbb{R}^N_y) \odot A_\tau$ for any algebra w.m.v. A_τ on $\mathbb{R}_\tau$.

Remark 3.3. As was said in Remark 3.2 we see by Proposition 3.8 that our results in this setting extend those in [42] and in [40]. Indeed in the above-mentioned papers, these results were proved only in the case when $p = 2$. Here they are extended to the case when $1 < p < \infty$ is arbitrarily fixed.

3.3. **Example 3.3: The convergence at infinity setting.** Let $\mathcal{B}_\infty(\mathbb{R}^N_y)$ denote the space of all continuous functions on $\mathbb{R}^N$ that converge at infinity, that is of all $u \in \mathcal{B}(\mathbb{R}^N_y)$ such that $\lim_{|y|\to\infty} u(y) \in \mathbb{R}$. The space $\mathcal{B}_\infty(\mathbb{R}^N_y)$ is an H-algebra on $\mathbb{R}^N$ [36]. It is an easy exercise to see that $\mathcal{B}_\infty(\mathbb{R}^N_y)$ is an ergodic algebra w.m.v. Indeed by [20] any $u \in \mathcal{B}_\infty(\mathbb{R}^N_y)$ is uniformly continuous, has a mean value defined by $M(u) = \lim_{|y|\to\infty} u(y)$. Moreover $\mathcal{B}_\infty(\mathbb{R}^N_y)$ is translation invariant, and once more by [20], $\mathcal{B}_\infty(\mathbb{R}^N_y)$ is ergodic (this will be seen in the next subsection). Therefore we have the conclusion of Theorem 3.7 with $A = \mathcal{B}_\infty(\mathbb{R}^N_y)$ and of Theorem 3.6 with $A = \mathcal{B}_\infty(\mathbb{R}^N_y) \odot A_\tau$ for any A_τ as in the preceding subsection.

3.4. **Example 3.4: The weakly almost periodic setting.** We begin with the notion of weak almost periodicity due to Eberlein [20].

Definition 3.3. A continuous function u on $\mathbb{R}^N$ is weakly almost periodic if the set of translates $\{\tau_a u : a \in \mathbb{R}^N\}$ is relatively weakly compact in $\mathcal{B}(\mathbb{R}^N_y)$.

We denote by $WAP(\mathbb{R}^N_y)$ the set of all weakly almost periodic functions on $\mathbb{R}^N_y$; $WAP(\mathbb{R}^N_y)$ is a vector space over $\mathbb{R}$. Endowed with the sup norm topology, $WAP(\mathbb{R}^N_y)$ is a Banach algebra with the usual multiplication.

The space $WAP(\mathbb{R}^N_y)$ is sometimes called the space of Eberlein's functions. As examples of Eberlein's functions we have the continuous Bohr almost periodic functions, the continuous functions vanishing at infinity, the positive definite functions (hence Fourier-Stieltjes transforms); see [20] for more details.

The following properties are worth mentioning (see [20] for details):

(P1) $WAP(\mathbb{R}^N_y)$ is a translation invariant $\mathcal{C}^*$-subalgebra of $\mathcal{B}(\mathbb{R}^N_y)$.
(P2) A weakly almost periodic function is uniformly continuous and bounded.
(P3) A weakly almost periodic function possesses a mean value with

$$M(u) = \lim_{r\to+\infty} \frac{1}{|B_r|} \int_{B_r} u(y+a)dy,$$

the convergence being uniform in $a \in \mathbb{R}^N$.

(P4) If $u, v \in WAP(\mathbb{R}^N_y)$ the convolution defined by the mean value $w(y) = (u \widehat{*} v)(y) = M_z(u(y-z)v(z)) = M_z(u(z)v(y-z))$ is an usual Bohr almost periodic function, where M_z stands for the mean value with respect to z.

(P5) ([21, Theorem 1]) Every $u \in WAP(\mathbb{R}^N_y)$ admits the unique decomposition $u = v + w$, v being a Bohr almost periodic function and w a continuous function of quadratic mean value zero: $M(|w|^2) = 0$.

Property (P5) above, knowing as a *decomposition theorem*, is crucial in the definition of the weak almost periodic algebra. Indeed let $W_0(\mathbb{R}^N_y)$ denote the subset of $WAP(\mathbb{R}^N_y)$ consisting of elements with quadratic mean value zero. One easily observes that the set of bounded continuous functions on $\mathbb{R}^N$ of quadratic mean value zero is a complete vector subspace of the algebra of bounded continuous functions on $\mathbb{R}^N$. Hence $W_0(\mathbb{R}^N_y)$ is a complete vector subspace of $WAP(\mathbb{R}^N_y)$. Property (P5) states as follows: $WAP(\mathbb{R}^N_y)$ is a direct sum of the two spaces $AP(\mathbb{R}^N_y)$ and $W_0(\mathbb{R}^N_y)$:

$$WAP(\mathbb{R}^N_y) = AP(\mathbb{R}^N_y) \oplus W_0(\mathbb{R}^N_y).$$

Another representation of the space $W_0(\mathbb{R}^N_y)$ is given by de Leeuw and Glicksberg [19]:

$$W_0(\mathbb{R}^N_y) = \{u \in WAP(\mathbb{R}^N_y) : M(|u|) = 0\}.$$

With this in mind, let $\mathcal{R}$ be a subgroup of $\mathbb{R}^N$ (not necessarily countable). Set

$$A = AP_{\mathcal{R}}(\mathbb{R}^N_y) \oplus W_0(\mathbb{R}^N_y) \tag{3.9}$$

where $AP_{\mathcal{R}}(\mathbb{R}^N_y)$ is defined in the preceding subsection. Note that in the case when $\mathcal{R} = \mathbb{R}^N$ then we have $A = WAP(\mathbb{R}^N_y)$. The following holds true.

Proposition 3.9. *The vector space $A = AP_{\mathcal{R}}(\mathbb{R}^N_y) \oplus W_0(\mathbb{R}^N_y)$ is an ergodic algebra.*

Definition 3.4. The algebra A defined by (3.9) is called the weak almost periodic algebra with mean value represented by the subgroup $\mathcal{R}$. A is denoted by $WAP_{\mathcal{R}}(\mathbb{R}^N_y)$ when $\mathcal{R} \neq \mathbb{R}^N$.

Proof of Proposition 3.9. **(1)** The ergodicity of A is a consequence of Proposition 2.7 and of property (P3). **(2)** Thanks to the properties (P1)-(P4) our only concern here is to check that A is a Banach subalgebra of $WAP(\mathbb{R}^N_y)$. To this end, because of the decomposition theorem any $u \in WAP(\mathbb{R}^N_y)$ uniquely expresses as $u = f + g$ with $f \in AP(\mathbb{R}^N_y)$ and $g \in W_0(\mathbb{R}^N_y)$. Set

$$\psi(u) = f.$$

This defines a mapping $\psi : WAP(\mathbb{R}^N_y) \to AP(\mathbb{R}^N_y)$. **(a)** Claim: ψ is continuous. In fact let $F = WAP(\mathbb{R}^N)$, $P = AP(\mathbb{R}^N)$ and $G = W_0(\mathbb{R}^N)$. Then $F = P \bigoplus G$, P and G are closed in F, hence $P \times G$ is a Banach space and further, the mapping $i : P \times G \to F$ defined by $i(x, y) = x + y$ is a one-to-one continuous mapping, hence, in view of the open mapping theorem, i is an isomorphism (we endow $P \times G$ with the Hilbertian norm $\|(x, y)\| = (\|x\|^2_\infty + \|y\|^2_\infty)^{1/2}$). But $\psi = p_1 \circ i$, where p_1 is the natural projection of $P \bigoplus G$ onto P (which is naturally continuous), hence ψ is continuous, as claimed. **(b)** Now as $AP_{\mathcal{R}}(\mathbb{R}^N)$ is a closed subspace of $AP(\mathbb{R}^N)$, it turns out that $\psi^{-1}(AP_{\mathcal{R}}(\mathbb{R}^N)) = AP_{\mathcal{R}}(\mathbb{R}^N) \bigoplus W_0(\mathbb{R}^N) = A$ is closed in $WAP(\mathbb{R}^N)$. Moreover A is a Banach algebra. Indeed this will be accomplished if we can check that $\psi(fg) = \psi(f)\psi(g)$ for all $f, g \in WAP(\mathbb{R}^N)$. So let $f = u + v$ and $g = u_1 + v_1$ with $u, u_1 \in AP(\mathbb{R}^N)$ and $v, v_1 \in W_0(\mathbb{R}^N)$; then $fg = uu_1 + u_1 v + uv_1 + vv_1$ with $uu_1 \in AP(\mathbb{R}^N)$. But $u_1 v + uv_1 + vv_1 \in W_0(\mathbb{R}^N)$ since: $|u_1 v| \leq c\,|v|$ ($c = \|u\|_\infty$), whence $M(|u_1 v|) \leq cM(|v|) = 0$, i.e., $M(|v|) = 0$ as M is a positive linear form on $WAP(\mathbb{R}^N)$; $M(|uv_1|) = 0$ and $M(|vv_1|) = 0$ by the same argument. It turns out that $\psi(fg) = uu_1 = \psi(f)\psi(g)$, so that ψ is a homomorphism of algebras. This being so, let $f, g \in A$; then $\psi(f)$ and $\psi(g)$ are in the algebra $AP_{\mathcal{R}}(\mathbb{R}^N)$, hence $\psi(fg) = \psi(f)\psi(g) \in AP_{\mathcal{R}}(\mathbb{R}^N)$, that is $fg \in A$. Therefore A is a closed subalgebra of $WAP(\mathbb{R}^N)$, that is A is an ergodic algebra on $\mathbb{R}^N$.

Returning to the statement of Theorems 3.6 and 3.7, we immediately see that the conclusion of Theorem 3.7 holds for the above $WAP_{\mathcal{R}}(\mathbb{R}^N)$, and that the conclusion of Theorem 3.6 holds for any $A = WAP_{\mathcal{R}}(\mathbb{R}^N) \odot A_\tau$ where A_τ is any algebra w.m.v. on $\mathbb{R}_\tau$.

It is however important to establish the link between the above algebras and some H-algebras defined in [36]. Let $\mathcal{C}_0(\mathbb{R}^N)$ denote the space of continuous functions on $\mathbb{R}^N$ which are vanishing at infinity. Then by [20] we have $AP(\mathbb{R}^N) + \mathcal{C}_0(\mathbb{R}^N) \subset WAP(\mathbb{R}^N)$. Since $\mathbb{R}^N$ is a $[IN]$-*group* in the sense of Grosser and Moskowitz [27] (indeed $V = [-1, 1]^N$ is a compact neighborhood of the origin 0 of $\mathbb{R}^N$ which is invariant under all the inner automorphisms of $\mathbb{R}^N$: the only inner automorphism of $\mathbb{R}^N$ is the identity mapping of $\mathbb{R}^N$), we deduce from [14, Theorem 4.5] that $AP(\mathbb{R}^N) + \mathcal{C}_0(\mathbb{R}^N)$ is a proper subset of $WAP(\mathbb{R}^N)$, i.e.

$$AP(\mathbb{R}^N) + \mathcal{C}_0(\mathbb{R}^N) \subsetneq WAP(\mathbb{R}^N).$$

Now let $\mathcal{R}$ be a countable subgroup of $\mathbb{R}^N$. Set $A_1 = AP_{\mathcal{R}}(\mathbb{R}^N) + \mathcal{C}_0(\mathbb{R}^N)$. Arguing as in part **(2)** of the proof of Proposition 3.9 we see that A_1 is a closed subalgebra of $WAP_{\mathcal{R}}(\mathbb{R}^N)$ which is further strictly contained in $WAP_{\mathcal{R}}(\mathbb{R}^N)$. On the other hand, we have $A_1 = AP_{\mathcal{R}}(\mathbb{R}^N) + \mathcal{B}_\infty(\mathbb{R}^N)$ where $\mathcal{B}_\infty(\mathbb{R}^N) = \mathcal{C}_0(\mathbb{R}^N) \bigoplus \mathbb{R}$ is defined in Subsection 3.3. Thus A_1 is exactly the H-algebra $\mathcal{B}_{\infty,\mathcal{R}}(\mathbb{R}^N)$ defined in [36, Example 2.4] as the *closure* in $\mathcal{B}(\mathbb{R}^N)$ of the space $AP_{\mathcal{R}}(\mathbb{R}^N) + \mathcal{B}_\infty(\mathbb{R}^N)$. It is therefore a fact that speaking in [36, Propositions 3.5 and 3.7] about the "closure" is superfluous since A_1 is already closed as seen above.

Remark 3.4. All the ergodic algebras encountered so far are subalgebras of $WAP(\mathbb{R}^N)$ (see e.g. $\mathcal{C}_{\text{per}}(Y)$, $AP(\mathbb{R}^N)$, $\mathcal{B}_\infty(\mathbb{R}^N)$ and others which can be obtain by combining the previous ones such as $AP(\mathbb{R}^N) + \mathcal{B}_\infty(\mathbb{R}^N)$ etc.). Therefore we are right to ask whether there exist or not ergodic algebras that are not subalgebras of $WAP(\mathbb{R}^N)$; this is an open problem.

The following result will allow us to see that weakly almost periodic algebras are not H-algebras.

Theorem 3.10. *The algebra $W_0(\mathbb{R}^m)$ is nonseparable.*

For the proof of this theorem we need the following lemma.

Lemma 3.11. *Let G and H be two Banach spaces, and let φ be a surjective continuous linear mapping of G onto H. If G is separable then so also is H.*

Proof. Let $(a_n)_{n\geq 1}$ be a countable dense set in G, and let c be a positive constant such that $\|\varphi(g)\|_H \leq c\,\|g\|_G$ for all $g \in G$, where $\|\cdot\|_G$ and $\|\cdot\|_H$ denote respectively the norms in G and in H. Set $B = (\varphi(a_n))_{n\geq 1} \subset H$, and let us show that B is dense in H. For that, let $h \in H$ and let $\varepsilon > 0$ be freely fixed; let finally $g \in G$ be such that $h = \varphi(g)$. There exists some $n_0 \geq 1$ such that $\|g - a_{n_0}\|_G < \varepsilon/c$, hence $\|h - \varphi(a_{n_0})\|_H \leq c\|g - a_{n_0}\|_G < \varepsilon$. Thus B is a countable dense subset of H, and the proof is complete. □

Proof of Theorem 3.10. Set $G = W_0(\mathbb{R}^m)$, $H = W_0(\mathbb{R}^m)/\mathcal{C}_0(\mathbb{R}^m)$ and φ the natural (canonical) homomorphism of G onto H. Assume G is separable (with the sup norm topology), then according to the above lemma, H is separable. But we know by [15, Theorem 4.6] that the quotient space H contains a linear isometric copy of ℓ^∞ and hence is nonseparable. This contradicts our assumption, and hence G is nonseparable.

Now let $\mathcal{R}$ be a subgroup of $\mathbb{R}^m$ (countable or not). Since $W_0(\mathbb{R}^m) \subset WAP(\mathbb{R}^m)$ we conclude by Theorem 3.10 that the ergodic algebras $WAP_{\mathcal{R}}(\mathbb{R}^m)$ are not H-algebras because they are nonseparable with respect to the sup norm topology. So we have in hands an example of algebra w.m.v. which contrarily to the algebra $AP(\mathbb{R}^m)$, induces no H-algebra, say $WAP(\mathbb{R}^m)$. This is therefore sufficient to show that the concept of H-algebras cannot handle weakly almost periodic homogenization problems. This is a true advance as far as the deterministic homogenization theory is concerned.

4. Application: Homogenization of nonlinear pseudo monotone parabolic operators

4.1. **Introduction.** We are concerned here with the homogenization of nonlinear parabolic pseudo monotone operators in a general deterministic setting. Here we present a new approach connected to a concrete physical assumption, contrary to what has been till now done in the deterministic homogenization theory. This allows us firstly to obtain a very general homogenization result and secondly a corrector type result. This also has the advantage of considering concrete homogenization problems from their true perspective, taking into account the discontinuities in general. We finally apply the above result to the resolution of several concrete problems such as the almost periodic one, the weak almost periodic one, and others. We rely on the tools developed in the earlier sections of this paper to achieve this.

To be more precise we are interested in the asymptotic behavior as $0 < \varepsilon \to 0$ of the solutions u_ε of the initial-boundary value problem

$$\begin{aligned} \frac{\partial u_\varepsilon}{\partial t} - \operatorname{div} a\left(x, t, \tfrac{x}{\varepsilon}, \tfrac{t}{\varepsilon}, u_\varepsilon, Du_\varepsilon\right) + a_0\left(x, t, \tfrac{x}{\varepsilon}, \tfrac{t}{\varepsilon}, u_\varepsilon, Du_\varepsilon\right) = f \text{ in } Q \\ u_\varepsilon = 0 \text{ on } \partial\Omega \times (0,T) \\ u_\varepsilon(x,0) = 0 \text{ in } \Omega, \end{aligned} \tag{4.1}$$

where $Q = \Omega \times (0,T)$ and $f \in L^{p'}(0,T;W^{-1,p'}(\Omega))$, Ω being an open bounded set in $\mathbb{R}^N$ with Lipschitz boundary, reals $T > 0$ and $p \geq 2$ with $p' = p/(p-1)$, D and div denoting respectively the gradient and divergence operators in Ω, and where the functions $(x,t,y,\tau,\mu,\lambda) \mapsto a(x,t,y,\tau,\mu,\lambda)$ and $(x,t,y,\tau,\mu,\lambda) \mapsto a_0(x,t,y,\tau,\mu,\lambda)$ from $\overline{Q} \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N$ to $\mathbb{R}^N$ and $\mathbb{R}$, respectively, satisfy the following assumptions:

For each fixed $(x,t) \in \overline{Q}$ and $(\mu,\lambda) \in \mathbb{R} \times \mathbb{R}^N$, the functions $a(x,t,\cdot,\cdot,\mu,\lambda)$ and $a_0(x,t,\cdot,\cdot,\mu,\lambda)$ are measurable (4.2)

$$a(x,t,y,\tau,\mu,0)=0 \text{ almost everywhere (a.e.) in } (y,\tau)\in\mathbb{R}^N\times\mathbb{R} \text{ and, for all } (x,t)\in\overline{Q} \text{ and all } \mu\in\mathbb{R} \tag{4.3}$$

There are three constants c_0, c_1, $c_2>0$ and a continuity modulus ω (i.e., a nondecreasing continuous function on $[0,+\infty)$ such that $\omega(0)=0,\omega(r)>0$ if $r>0$, and $\omega(r)=1$ if $r>1$) such that a.e. in $(y,\tau)\in\mathbb{R}^N\times\mathbb{R}$,

$$\begin{aligned}
&\text{(i) } \left(a(x,t,y,\tau,\mu,\lambda)-a(x,t,y,\tau,\mu,\lambda')\right)\cdot\left(\lambda-\lambda'\right)\geq c_1\left|\lambda-\lambda'\right|^p\\
&\text{(ii) } a_0(x,t,y,\tau,\mu,\lambda)\mu\geq 0\\
&\text{(iii) } |a_0(x,t,y,\tau,\mu,\lambda)|+|a(x,t,y,\tau,\mu,\lambda)|\leq c_2(1+|\mu|^{p-1}+|\lambda|^{p-1})\\
&\text{(iv) } \left|a_0(x,t,y,\tau,\mu,\lambda)-a_0(x',t',y,\tau,\mu',\lambda')\right|+\\
&\qquad+\left|a(x,t,y,\tau,\mu,\lambda)-a(x',t',y,\tau,\mu',\lambda')\right|\\
&\quad\leq\omega(|x-x'|+|t-t'|+|\mu-\mu'|)(1+|\mu|^{p-1}+|\lambda|^{p-1}+|\mu'|^{p-1}\\
&\qquad+\left|\lambda'\right|^{p-1})+c_0\left(1+|\mu|+|\lambda|+|\mu'|+\left|\lambda'\right|\right)^{p-2}\left|\lambda-\lambda'\right|
\end{aligned} \tag{4.4}$$

for all $(x,t),(x',t')\in\overline{Q}$ and all $(\mu,\lambda),(\mu',\lambda')\in\mathbb{R}\times\mathbb{R}^N$, where the dot denotes the usual Euclidean inner product in $\mathbb{R}^N$ and $|\cdot|$ the associated norm.

Remark 4.1. The positivity constraint (ii) in (4.4) is stated in order to establish the a priori estimates. It plays no role in the process of the existence of solutions to (4.1).

Owing to the presence of the term $a_0(x,t,x/\varepsilon,t/\varepsilon,u_\varepsilon,Du_\varepsilon)$, the entire elliptic part of the leading equation in (4.1) is not monotone in general, but pseudo monotone. It is also to be noted that the elliptic part of the leading equation in (4.1) is degenerate so that the problem under consideration is a parabolic non monotone degenerate one. Thus, provided the diffusion term in (4.1) is rigorously defined (this will be accomplished in Section 2), we will see that equation (4.1) admits (at least) a solution u_ε. Thus, in order to have a sequence, we need that the coefficients a and a_0 fulfill some conditions (that we do not specify here) so that, to each ε is associated a unique solution u_ε to (4.1). We will therefore assume that problem (4.1) has a unique solution associated to each fixed $\varepsilon>0$, in the above sense, and our goal will be the investigation of the limiting behavior as $\varepsilon\to 0$, of u_ε.

Using G-convergence method, this class of problems has been studied by Pankov and co-authors in several works. We refer e.g., to [11, 22, 23, 24]. More precisely, in [43], the periodic homogenization of (4.1) is carried out by Pankov. In [11], time averaging in random case is studied by Pankov et al. The general random case of (4.1) is studied in [23], and finally in [24], problem (4.1) is carried out in the almost periodic setting. It is worth noting that in [35], Nandakumaran and Rajesh have studied the particular case where $a_0\equiv 0$.

In all the works cited above, the coefficients a_0 and a of the elliptic part are independent of the macroscopic variables (x,t) as opposed to what is considered here. This is a nonnegligible aspect as the applications are concerned.

In this section, we investigate the asymptotic behavior of u_ε (the solutions of (4.1)) when $\varepsilon\to 0$, under an assumption on $a_i(x,t,y,\tau,\mu,\lambda)$ (with respect to (y,τ)), for fixed x,t,μ,λ and $0\leq i\leq N$ (where a_i, for $1\leq i\leq N$, denotes the i th component of the function a), covering a set of concrete behaviors such as the almost periodicity, the weak almost periodicity, and others. This is the so-called *deterministic* homogenization theory which encompasses the periodic homogenization theory as a particular case. As opposed to what is usually done in the deterministic homogenization theory, we present here a new approach based on the generalized Besicovitch type spaces (see Section 2), which widely opens the scope of application of our main homogenization result, Theorem 4.9, as we will see it therein. In particular we will work out the almost periodic homogenization problem without further assumption on the functions a_i, as was usually the case in all the previous papers dealing with deterministic homogenization theory; see for instance hypothesis (4.34) in Remark 4.6 (which was fundamental in all the previous works [40, 41, 42, 46], and which is purely dropped here).

This section is organized as follows. Subsection 4.2 deals with some trace results. In Subsection 4.3, we state and prove the main homogenization result, Theorem 4.9. Finally in Subsection 4.4, we solve some concrete homogenization problems for (4.1).

4.2. **Preliminaries.**

4.2.1. *Traces.* In this subsection we need to give a meaning to certain functions which will be useful in the forthcoming sections. We begin with one preliminary result.

Lemma 4.1. *Let $(v_0,\mathbf{v})\in\mathcal{C}(\overline{Q})\times\mathcal{C}(\overline{Q})^N\equiv\mathcal{C}(\overline{Q})^{N+1}$. Let $0\leq i\leq N$. The function $((x,t),(x',t'),(y,\tau))\mapsto a_i(x,t,y,\tau,v_0(x',t'),\mathbf{v}(x',t'))$, of $\overline{Q}\times\overline{Q}\times\mathbb{R}^N_y\times\mathbb{R}_\tau$ into $\mathbb{R}$, lies in $\mathcal{C}(\overline{Q}\times\overline{Q};L^\infty(\mathbb{R}^{N+1}_{y,\tau}))$ so that one can define the trace function $(x,t,y,\tau)\mapsto a_i(x,t,y,\tau,v_0(x,t),\mathbf{v}(x,t))$, from $\overline{Q}\times\mathbb{R}^{N+1}_{y,\tau}$ to $\mathbb{R}$, as element of $\mathcal{C}(\overline{Q};L^\infty(\mathbb{R}^{N+1}_{y,\tau}))$.*

Proof. For any fixed (x,t) and (x',t'), the function $a_i(x,t,\cdot,\cdot,v_0(x',t'),\mathbf{v}(x',t'))$ is measurable (see assumption (4.2)). We conclude by [part (iii) of] (4.4) that this function belongs to $L^\infty(\mathbb{R}^{N+1}_{y,\tau})$. Now, let (x_0,t_0) and (x_0',t_0') be freely fixed in $\overline{Q}$; then

$$\begin{aligned}&\left\|a_i(x,t,\cdot,\cdot,v_0(x',t'),\mathbf{v}(x',t'))-a_i(x_0,t_0,\cdot,\cdot,v_0(x_0',t_0'),\mathbf{v}(x_0',t_0'))\right\|_\infty\\&\leq\omega\left(|x-x_0|+|t-t_0|+|v_0(x',t')-v_0(x_0',t_0')|\right)\left(1+2\left\|v_0\right\|_\infty^{p-1}+2\left\|\mathbf{v}\right\|_\infty^{p-1}\right)\\&\quad+c_0\left(1+2\left\|v_0\right\|_\infty+2\left\|\mathbf{v}\right\|_\infty\right)^{p-2}\left|\mathbf{v}(x',t')-\mathbf{v}(x_0',t_0')\right|.\end{aligned}$$

Thus, as $(x,t)\to(x_0,t_0)$ and $(x',t')\to(x_0',t_0')$, the right-hand side, and hence the left-hand side of the above inequality go to zero. We conclude that the function $((x,t),(x',t'))\mapsto a_i(x,t,\cdot,\cdot,v_0(x',t'),\mathbf{v}(x',t'))$, of $\overline{Q}\times\overline{Q}$ into $L^\infty(\mathbb{R}^{N+1}_{y,\tau})$, lies in $\mathcal{C}(\overline{Q}\times\overline{Q};L^\infty(\mathbb{R}^{N+1}_{y,\tau}))$, and so one defines naturally the trace function $(x,t)\mapsto a_i(x,t,\cdot,\cdot,v_0(x,t),\mathbf{v}(x,t))$ by

$$a_i(x,t,\cdot,\cdot,v_0(x,t),\mathbf{v}(x,t))=a_i(x,t,\cdot,\cdot,v_0(x',t'),\mathbf{v}(x',t'))|_{(x',t')=(x,t)},$$

which sends continuously $\overline{Q}$ into $L^\infty(\mathbb{R}^{N+1}_{y,\tau})$. □

Let $\varepsilon>0$, and let $(v_0,\mathbf{v})\in\mathcal{C}(\overline{Q})^{N+1}$. Thanks to Lemma 4.1 and [39, Proposition 1.5], one can define the trace function $(x,t)\mapsto a_i(x,t,x/\varepsilon,t/\varepsilon,v_0(x,t),\mathbf{v}(x,t))$ on Q, as an element of $L^\infty(Q)$, denoted by $a_i^\varepsilon(-,v_0,\mathbf{v})$. We have the following result whose proof is exactly the same as that of [45, Proposition 3.1] and is therefore omitted.

Proposition 4.2. *Let $2 \leq p < \infty$ and $p' = p/(p-1)$. The transformation $(v_0, \mathbf{v}) \mapsto a_i^\varepsilon(-, v_0, \mathbf{v})$, of $\mathcal{C}(\overline{Q})^{N+1}$ into $L^\infty(Q)$, extends by continuity to a continuous mapping still denoted by $(v_0, \mathbf{v}) \mapsto a_i^\varepsilon(-, v_0, \mathbf{v})$, of $L^p(Q)^{N+1}$ into $L^{p'}(Q)$ verifying*

$$a^\varepsilon(-, v_0, 0) = 0 \text{ a.e. in } Q \tag{4.5}$$

$$(a^\varepsilon(-, v_0, \mathbf{v}) - a^\varepsilon(-, v_0, \mathbf{w})) \cdot (\mathbf{v} - \mathbf{w}) \geq c_1 |\mathbf{v} - \mathbf{w}|^p \text{ a.e. in } Q \tag{4.6}$$

$$a_0^\varepsilon(-, v_0, \mathbf{v}) v_0 \geq 0 \text{ a.e. in } Q \tag{4.7}$$

$$\|a_i^\varepsilon(-, v_0, \mathbf{v})\|_{L^{p'}(Q)} \leq c_2' \left(1 + \|v_0\|_{L^p(Q)}^{p-1} + \|\mathbf{v}\|_{L^p(Q)^N}^{p-1}\right) \tag{4.8}$$

$$\begin{aligned} &\|a_i^\varepsilon(-, v_0, \mathbf{v}) - a_i^\varepsilon(-, v_0, \mathbf{w})\|_{L^{p'}(Q)} \\ &\quad \leq c_0 \|1 + |v_0| + |\mathbf{v}| + |\mathbf{w}|\|_{L^p(Q)}^{p-2} \|\mathbf{v} - \mathbf{w}\|_{L^p(Q)^N} \end{aligned} \tag{4.9}$$

$$\begin{aligned} &|a_i^\varepsilon(-, v_0, \mathbf{v}) - a_i^\varepsilon(-, w_0, \mathbf{v})| \\ &\leq \omega(|v_0 - w_0|) \left(1 + |v_0|^{p-1} + |w_0|^{p-1} + |\mathbf{v}|^{p-1}\right) \text{ a.e. in } Q \end{aligned} \tag{4.10}$$

for all $v_0, w_0 \in L^p(Q)$ and all $\mathbf{v}, \mathbf{w} \in L^p(Q)^N$.

Remark 4.2. Due to Proposition 4.2, the diffusion term in (4.1) is now well defined and rigorously justified. Hence, problem (4.1) is from now on well-posed, and admits (at least) a solution $u_\varepsilon \in L^p(0, T; W_0^{1,p}(\Omega))$ for each fixed $\varepsilon > 0$ (see, e.g., [32, Chap. 2] or [1]). Moreover, u_ε lies in

$$V^p = \{v \in L^p(0, T; W_0^{1,p}(\Omega)) : v' = \partial v/\partial t \in L^{p'}(0, T; W^{-1,p'}(\Omega))\}.$$

We recall that, endowed with the norm

$$\|v\|_{V^p} = \|v\|_{L^p(0,T;W_0^{1,p}(\Omega))} + \|v'\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \quad (v \in V^p)$$

V^p is a Banach space which is continuously embedded in $\mathcal{C}([0, T]; L^2(\Omega))$ (this is a classical result), in such a way that the existence of $u_\varepsilon(0)$ is justified. Therefore u_ε belongs to the space $V_0^p = \{v \in V^p : v(0) = 0\}$, a Banach space under the V^p-norm.

Let $(\psi_0, \Psi) \in \mathcal{C}(\overline{Q}; \mathcal{B}(\mathbb{R}^{N+1})^{N+1})$. Let $(x, t), (x', t') \in \overline{Q}$ and $(z, \theta) \in \mathbb{R}^N \times \mathbb{R}$ be freely fixed, and finally, let $0 \leq i \leq N$. We know by (4.2) that the function $a_i(x, t, \cdot, \cdot, \psi_0(x', t', z, \theta), \Psi(x', t', z, \theta))$ is measurable and thanks to [part (iii) of] (4.4), this function lies in $L^\infty(\mathbb{R}_{y,\tau}^{N+1})$. Moreover, due to inequality (which implies the continuity with respect to (z, θ) for fixed (x, t) and (x', t') in $\overline{Q}$)

$$\begin{aligned} &\|a_i(x, t, \cdot, \cdot, \psi_0(x', t', z, \theta), \Psi(x', t', z, \theta)) - a_i(x, t, \cdot, \cdot, \psi_0(x', t', z_0, \theta_0), \Psi(x', t', z_0, \theta_0))\| \\ &\leq \omega(|\psi_0(x', t', z, \theta) - \psi_0(x', t', z_0, \theta_0)|) \left(1 + 2\|\psi_0\|_\infty^{p-1} + 2\|\Psi\|_\infty^{p-1}\right) + \\ &\quad + c_0 (1 + 2\|\psi_0\|_\infty + 2\|\Psi\|_\infty)^{p-2} |\Psi(x', t', z, \theta) - \Psi(x', t', z_0, \theta_0)|, \end{aligned}$$

we have that the function $(z, \theta) \mapsto a_i(x, t, \cdot, \cdot, \psi_0(x', t', z, \theta), \Psi(x', t', z, \theta))$ sends continuously $\mathbb{R}_{z,\theta}^{N+1}$ into $L^\infty(\mathbb{R}_{y,\tau}^{N+1})$, and so, belongs to $\mathcal{B}(\mathbb{R}_{z,\theta}^{N+1}; L^\infty(\mathbb{R}_{y,\tau}^{N+1}))$ (the space of bounded uniformly continuous functions of $\mathbb{R}_{z,\theta}^{N+1}$ into $L^\infty(\mathbb{R}_{y,\tau}^{N+1})$). Thus, by using [42] one can define the trace function $(y, \tau) \mapsto a_i(x, t, y, \tau, \psi_0(x', t', y, \tau), \Psi(x', t', y, \tau))$ of $\mathbb{R}_{y,\tau}^{N+1}$ into $\mathbb{R}$, as element of $L^\infty(\mathbb{R}_{y,\tau}^{N+1})$.

With this in mind, we have the following result whose proof is similar to that of Lemma 4.1, and is therefore omitted.

Proposition 4.3. *For each* (ψ_0, Ψ) *in* $\mathcal{C}(\overline{Q}; \mathcal{B}(\mathbb{R}^{N+1}_{y,\tau})^{N+1})$ *and each* $0 \leq i \leq N$ *the function* $((x,t),(x',t'),(y,\tau)) \mapsto a_i(x,t,y,\tau,\psi_0(x',t',y,\tau),\Psi(x',t',y,\tau))$*, of* $\overline{Q} \times \overline{Q} \times \mathbb{R}^{N+1}_{y,\tau}$ *into* $\mathbb{R}$*, lies in* $\mathcal{C}(\overline{Q} \times \overline{Q}; L^\infty(\mathbb{R}^{N+1}_{y,\tau}))$*, so that one can define the trace* $(x,t,y,\tau) \mapsto a_i(x,t,y,\tau,\psi_0(x,t,y,\tau),\Psi(x,t,y,\tau))$*, of* $\overline{Q} \times \mathbb{R}^{N+1}_{y,\tau}$ *into* $\mathbb{R}$*, as element of* $\mathcal{C}(\overline{Q}; L^\infty(\mathbb{R}^{N+1}_{y,\tau}))$*.*

Remark 4.3. Let (ψ_0, Ψ) in $\mathcal{B}(\mathbb{R}^{N+1}_{y,\tau})^{N+1}$. For fixed $(x,t) \in \overline{Q}$, the function $a_i(x,t,\cdot,\cdot,\psi_0,\Psi)$ given by $(y,\tau) \mapsto a_i(x,t,y,\tau,\psi_0(y,\tau),\Psi(y,\tau))$, of $\mathbb{R}^{N+1}_{y,\tau}$ into $\mathbb{R}$, is defined in the sense of the preceding proposition and belongs to $L^\infty(\mathbb{R}^{N+1}_{y,\tau})$. This function will be of particular interest in the forthcoming sections.

Corollary 4.4. *Let* $\varepsilon > 0$*. The data being those of Proposition* 4.3*, one can define in the sense of* [39, Proposition 1.5]*, the function* $(x,t) \mapsto a_i(x,t,x/\varepsilon,t/\varepsilon,\psi_0(x,t,x/\varepsilon,t/\varepsilon),\Psi(x,t,x/\varepsilon,t/\varepsilon))$*, from* Q *to* $\mathbb{R}$*, as element of* $L^\infty(Q)$*, denoted by* $a_i^\varepsilon(-,\psi_0^\varepsilon,\Psi^\varepsilon)$*.*

Corollary 4.4 is a direct consequence of Proposition 4.3.

4.3. **The abstract homogenization problem for (4.1).**

4.3.1. ***Setting of the abstract problem and preliminary.*** The notations are those of the preceding sections. Let $A = A_y \odot A_\tau$ be an algebra w.m.v. and let $1 < p < \infty$. It is easy to see that Property (3.1) (in Definition 3.1) still holds for $f \in \mathcal{C}(\overline{Q}; B_A^{p',\infty})$ instead of $f \in L^{p'}(Q;A)$ mutatis mutandis, where $B_A^{p',\infty} = B_A^{p'} \cap L^\infty(\mathbb{R}^{N+1}_{y,\tau})$ and $p' = p/(p-1)$. Furthermore, if we provide the space $B_A^{p',\infty}$ with the $L^\infty(\mathbb{R}^{N+1}_{y,\tau})$-norm, it can be shown that, for $u \in B_A^{p',\infty}$, we have $\mathcal{G}(u) \in L^\infty(\Delta(A))$ and $\|\mathcal{G}(u)\|_{L^\infty(\Delta(A))} \leq \|u\|_{L^\infty(\mathbb{R}^{N+1}_{y,\tau})}$, $\mathcal{G}$ being the canonical mapping of $B_A^{p'}$ into $L^{p'}(\Delta(A))$.

This being so, the main purpose of this section is to investigate the asymptotic analysis, as $\varepsilon \to 0$, of u_ε (the solution of (4.1)) under the hypothesis

$$a_i(x,t,\cdot,\cdot,\mu,\lambda) \in B_A^{p'} \text{ for any } (x,t) \in \overline{Q} \text{ and all } (\mu,\lambda) \in \mathbb{R}^{N+1},\ 0 \leq i \leq N \quad (4.11)$$

where $p' = p/(p-1)$ with $2 \leq p < \infty$.

The following result is the cornerstone of the homogenization process. It allows us to go from a concrete hypothesis to the abstract one which is fundamental in the proof of the main homogenization result in this section.

Proposition 4.5. *Assume* (4.11) *holds true. Then, for every* $(\psi_0, \Psi) \in A \times (A)^N = (A)^{N+1}$ *and every* $(x,t) \in \overline{Q}$*, the function* $(y,\tau) \mapsto a_i(x,t,y,\tau,\psi_0(y,\tau),\Psi(y,\tau))$ *denoted below by* $a_i(x,t,\cdot,\cdot,\psi_0,\Psi)$*, lies in* $B_A^{p'}$*.*

Proof. Let $K \subset \mathbb{R} \times \mathbb{R}^N$ be a compact set such that $(\psi_0(y,\tau),\Psi(y,\tau)) \in K$ for all $(y,\tau) \in \mathbb{R}^N \times \mathbb{R}$. By viewing a_i as a function $(x,t,\mu,\lambda) \mapsto a_i(x,t,\cdot,\cdot,\mu,\lambda)$ of $\overline{Q} \times \mathbb{R} \times \mathbb{R}^N$ into $B_A^{p'}$, we have that a_i belongs to $\mathcal{C}(\overline{Q} \times \mathbb{R} \times \mathbb{R}^N; B_A^{p'})$ (combine (4.11) with [part (iv) of] (4.4)). Still denoting by a_i the restriction of this function to $\overline{Q} \times K$, it immediately follows that $a_i \in \mathcal{C}(\overline{Q} \times K; B_A^{p'})$. Hence using the density of $\mathcal{C}(\overline{Q} \times K) \otimes B_A^{p'}$ in $\mathcal{C}(\overline{Q} \times K; B_A^{p'})$, one may consider a sequence $(q_n)_{n\geq 1}$ in

$\mathcal{C}(\overline{Q}\times K)\otimes B_A^{p'}$ such that

$$\sup_{(x,t)\in\overline{Q}}\sup_{(\mu,\lambda)\in K}\left\|q_n(x,t,\cdot,\cdot,\mu,\lambda)-a_i(x,t,\cdot,\cdot,\mu,\lambda)\right\|_{p'}\to 0$$
as $n\to\infty$.

As
$$\left\|q_n(x,t,\cdot,\cdot,\psi_0,\Psi)-a_i(x,t,\cdot,\cdot,\psi_0,\Psi)\right\|_{p'}\leq \sup_{(x,t)\in\overline{Q}}\sup_{(\mu,\lambda)\in K}\left\|q_n(x,t,\cdot,\cdot,\mu,\lambda)-a_i(x,t,\cdot,\cdot,\mu,\lambda)\right\|_{p'}$$

we have $q_n(x,t,\cdot,\cdot,\psi_0,\Psi)\to a_i(x,t,\cdot,\cdot,\psi_0,\Psi)$ in $B_A^{p'}$ as $n\to\infty$. Thus, the proposition is shown if we can verify that each $q_n(x,t,\cdot,\cdot,\psi_0,\Psi)$ lies in $B_A^{p'}$. However this will follow in an obvious way once we have checked that for any function $q:\overline{Q}\times\mathbb{R}_y^N\times\mathbb{R}_\tau\times\mathbb{R}_\mu\times\mathbb{R}_\lambda^N\to\mathbb{R}$ of the form

$$q(x,t,y,\tau,\mu,\lambda)=\chi(x,t,\mu,\lambda)\Phi(y,\tau)\ (y,\lambda\in\mathbb{R}^N,\ \mu,\tau\in\mathbb{R},\ (x,t)\in\overline{Q})$$
with $\chi\in\mathcal{C}(\overline{Q}\times K)$ and $\Phi\in B_A^{p'}$,

we have $q(x,t,\cdot,\cdot,\psi_0,\Psi)\in B_A^{p'}$. But given q as above, we know by the Stone-Weierstrass theorem that there is a sequence $(f_n)_{n\geq 1}$ of polynomials in $(x,t,\mu,\lambda)\in\overline{Q}\times K$ such that $f_n\to\chi$ in $\mathcal{C}(\overline{Q}\times K)$ as $n\to\infty$, hence $f_n(x,t,\psi_0,\Psi)\to\chi(x,t,\psi_0,\Psi)$ in $\mathcal{B}(\mathbb{R}_y^N\times\mathbb{R}_\tau)$ as $n\to\infty$. Therefore, it follows that $\chi(x,t,\psi_0,\Psi)$ lies in A, since the same is true for each $f_n(x,t,\psi_0,\Psi)$ (recall that A is an algebra). We conclude that

$$q(x,t,\cdot,\cdot,\psi_0,\Psi)=\chi(x,t,\psi_0,\Psi)\Phi\in B_A^{p'}$$

as the product of an element of A by an element of $B_A^{p'}$. This concludes the proof. □

Remark 4.4. Let $\varepsilon>0$ be freely fixed, and let $(\psi_0,\Psi)\in\mathcal{C}(\overline{Q};(A)^{N+1})$. It is an easy exercise to define the function $(x,t)\mapsto a_i^\varepsilon(x,t,x/\varepsilon,t/\varepsilon,\psi_0(x,t,x/\varepsilon,t/\varepsilon),\Psi(x,t,x/\varepsilon,t/\varepsilon))$, from Q to $\mathbb{R}$, as element of $L^\infty(Q)$, denoted by $a_i^\varepsilon(-,\psi_0^\varepsilon,\Psi^\varepsilon)$. Moreover, thanks to the preceding proposition, if we assume that (4.11) holds, then one can also easily define the function $(x,t,y,\tau)\mapsto a_i(x,t,y,\tau,\psi_0(x,t,y,\tau),\Psi(x,t,y,\tau))$ as element of $\mathcal{C}(\overline{Q};B_A^{p',\infty})$, denoted by $a_i(-,\psi_0,\Psi)$ or explicitly by $a_i(x,t,y,\tau,\psi_0,\Psi)$.

The following result will be of great interest in this section. It will allows us to rigorously set the homogenized problem.

Proposition 4.6. *Let $2\leq p<\infty$ and let $0\leq i\leq N$. Suppose* (4.11) *holds. For any $(\psi_0,\Psi)\in\mathcal{C}(\overline{Q};(A)^{N+1})$ we have*

$$a_i^\varepsilon(-,\psi_0^\varepsilon,\Psi^\varepsilon)\to a_i(-,\psi_0,\Psi)\ \text{in}\ L^{p'}(Q)\text{-weak}\ A\ \text{as}\ \varepsilon\to 0. \tag{4.12}$$

Let $a(-,\psi_0,\Psi)=(a_i(-,\psi_0,\Psi))_{1\leq i\leq N}$. The mapping $(\psi_0,\Psi)\mapsto(a_0(-,\psi_0,\Psi),a(-,\psi_0,\Psi))$ of $\mathcal{C}(\overline{Q};(A)^{N+1})$ into $L^{p'}(Q;B_A^{p'})^{N+1}$ extends by continuity to a unique mapping still denoted by (a_0,a), of $L^p(Q;(B_A^p)^{N+1})$ into $L^{p'}(Q;B_A^{p'})^{N+1}$ such that

$$(a(-,u,\mathbf{v})-a(-,u,\mathbf{w}))\cdot(\mathbf{v}-\mathbf{w})\geq c_1\left|\mathbf{v}-\mathbf{w}\right|^p\ \text{a.e. in}\ Q\times\mathbb{R}_y^N\times\mathbb{R}_\tau$$

$$\left\|a_i(-,u,\mathbf{v})\right\|_{L^{p'}(Q;B_A^{p'})}\leq c_2''\left(1+\|u\|_{L^p(Q;B_A^p)}^{p-1}+\|\mathbf{v}\|_{L^p(Q;(B_A^p)^N)}^{p-1}\right)$$

$$\begin{aligned}&\left\|a_i(-,u,\mathbf{v})-a_i(-,u,\mathbf{w})\right\|_{L^{p'}(Q;B_A^{p'})}\\&\quad\leq c_0\left\|1+|u|+|\mathbf{v}|+|\mathbf{w}|\right\|_{L^p(Q;B_A^p)}^{p-2}\left\|\mathbf{v}-\mathbf{w}\right\|_{L^p(Q;(B_A^p)^N)}\end{aligned}\tag{4.13}$$

$$\begin{aligned}&|a_i(x,t,y,\tau,u,\mathbf{w})-a_i(x',t',y,\tau,v,\mathbf{w})|\leq\\&\quad\leq\omega(|x-x'|+|t-t'|+|u-v|)\left(1+|u|^{p-1}+|v|^{p-1}+|\mathbf{w}|^{p-1}\right)\\&a.e.\ in\ Q\times\mathbb{R}_y^N\times\mathbb{R}_\tau\end{aligned}$$

for all $u,v\in L^p(Q;B_A^p)$, $\mathbf{v},\mathbf{w}\in L^p(Q;(B_A^p)^N)$ *and all* $(x,t),(x',t')\in Q$*, where the constant* c_2'' *depends only on* c_2 *and on* Q*.*

Proof. Thanks to Remark 4.4, we see that the function $a_i(-,\psi_0,\Psi)$ lies in $\mathcal{C}(\overline{Q};B_A^{p',\infty})$. Since Property (3.1) (in Definition 3.1) still holds for $f\in\mathcal{C}(\overline{Q};B_A^{p',\infty})$ the convergence result (4.12) follows at once. On the other hand, by the definition of the function $a_i(-,\psi_0,\Psi)$ (for $(\psi_0,\Psi)\in\mathcal{C}(\overline{Q};(A)^{N+1})$) it is immediate that this function verifies properties of the same type as in Proposition 4.2 (see for instance properties (4.5)-(4.10) therein). Therefore arguing as in the proof of [45, Proposition 3.1] we get the remainder of Proposition 4.6. □

The preceding proposition has several corollaries as will be seen below. To see this, for $(\psi_0,\Psi)\in L^p(Q;(B_A^p)^{N+1})$ we set $\widehat{a}_i(-,\widehat{\psi}_0,\widehat{\Psi})=\mathcal{G}(a_i(-,\psi_0,\Psi))$ $(0\leq i\leq N)$, which defines a mapping from $L^p(Q;(B_A^p)^{N+1})$ to $L^{p'}(Q\times\Delta(A))^{N+1}$, where $\widehat{\psi}_0=\mathcal{G}\circ\psi_0$ (a similar definition for $\widehat{\Psi}$), $\mathcal{G}$ being the canonical mapping of $B_A^{p'}$ into $L^{p'}(\Delta(A))$.

Corollary 4.7. *Let* $(u_\varepsilon)_{\varepsilon\in E}$ *be a sequence in* $L^p(Q)$ *such that* $u_\varepsilon\to u_0$ *in* $L^p(Q)$ *(strong) as* $E\ni\varepsilon\to0$*, where* $u_0\in L^p(Q)$*. Let* $\Psi\in\mathcal{C}(\overline{Q};(A)^N)$*, and finally let* $0\leq i\leq N$*. Then, as* $E\ni\varepsilon\to0$*,*

$$a_i^\varepsilon(-,u_\varepsilon,\Psi^\varepsilon)\to a_i(-,u_0,\Psi)\ in\ L^{p'}(Q)\text{-}weak\ A.$$

Proof. Let $f\in L^p(Q;A)$, and let $(\psi_j)_j$ be a sequence in $\mathcal{C}_0^\infty(Q)$ such that $\psi_j\to u_0$ in $L^p(Q)$ as $j\to\infty$. We have

$$\begin{aligned}&\textstyle\int_Q a_i^\varepsilon(-,u_\varepsilon,\Psi^\varepsilon)f^\varepsilon dxdt-\iint_{Q\times\Delta(A)}\widehat{a}_i(-,u_0,\widehat{\Psi})\widehat{f}dxdtd\beta\\&\quad=\textstyle\int_Q\left[a_i^\varepsilon(-,u_\varepsilon,\Psi^\varepsilon)-a_i^\varepsilon(-,u_0,\Psi^\varepsilon)\right]f^\varepsilon dxdt+\\&\quad\quad+\textstyle\int_Q\left[a_i^\varepsilon(-,u_0,\Psi^\varepsilon)-a_i^\varepsilon(-,\psi_j,\Psi^\varepsilon)\right]f^\varepsilon dxdt+\\&\quad\quad\quad+\textstyle\int_Q a_i^\varepsilon(-,\psi_j,\Psi^\varepsilon)f^\varepsilon dxdt-\iint_{Q\times\Delta(A)}\widehat{a}_i(-,u_0,\widehat{\Psi})\widehat{f}dxdtd\beta\\&\quad=A_\varepsilon+B_{\varepsilon,j}+C_{\varepsilon,j}\end{aligned}$$

where:

$$\begin{aligned}A_\varepsilon&=\textstyle\int_Q\left[a_i^\varepsilon(-,u_\varepsilon,\Psi^\varepsilon)-a_i^\varepsilon(-,u_0,\Psi^\varepsilon)\right]f^\varepsilon dxdt,\\B_{\varepsilon,j}&=\textstyle\int_Q\left[a_i^\varepsilon(-,u_0,\Psi^\varepsilon)-a_i^\varepsilon(-,\psi_j,\Psi^\varepsilon)\right]f^\varepsilon dxdt,\\C_{\varepsilon,j}&=\textstyle\int_Q a_i^\varepsilon(-,\psi_j,\Psi^\varepsilon)f^\varepsilon dxdt-\iint_{Q\times\Delta(A)}\widehat{a}_i(-,u_0,\widehat{\Psi})\widehat{f}dxdtd\beta\end{aligned}$$

We proceed in three steps.
Step 1). We first evaluate $\lim_{E\ni\varepsilon\to0}A_\varepsilon$.

We have

$$|A_\varepsilon|\leq\int_Q\omega\left(|u_\varepsilon-u_0|\right)\left(1+|u_\varepsilon|^{p-1}+|u_0|^{p-1}+|\Psi^\varepsilon|^{p-1}\right)|f^\varepsilon|\,dxdt.$$

Let $F_\varepsilon = \left(1 + |u_\varepsilon|^{p-1} + |u_0|^{p-1} + |\Psi^\varepsilon|^{p-1}\right)|f^\varepsilon|$. We have, on one hand, $F_\varepsilon \in L^1(Q)$ and $(F_\varepsilon)_{\varepsilon\in E}$ weakly converges in $L^1(Q)$ as $E \ni \varepsilon \to 0$ (this is easily seen). On the other hand, since $v_\varepsilon \equiv u_\varepsilon - u_0 \to 0$ in $L^p(Q)$ as $E \ni \varepsilon \to 0$, we know by [10, Thm IV-9] that there exist a subsequence E' from E and a function $g \in L^p(Q)$ such that

$$v_\varepsilon \to 0 \text{ (hence } |v_\varepsilon| \to 0) \text{ a.e. in } Q \text{ as } E' \ni \varepsilon \to 0$$
$$|v_\varepsilon| \le g \text{ a.e. in } Q \text{ for all } \varepsilon \in E'.$$

ω being continuous and in particular at 0 with $\omega(0) = 0$, and moreover being increasing, we deduce that $\omega(|v_\varepsilon|)$ is measurable for all ε (each v_ε is measurable and hence $|v_\varepsilon|$ too) and

(i) $\omega(|v_\varepsilon|) \to 0$ a.e. in Q as $\varepsilon \to 0$
(ii) $\omega(|v_\varepsilon|)^{p'} \le \omega(g)^{p'}$ a.e. in Q for all ε.

ω being a continuity modulus, the function $\omega(g) \equiv \omega \circ g$ is measurable and essentially bounded on Q, i.e., $\omega(g) \in L^\infty(Q)$. Thus, the sequence $\omega(|v_\varepsilon|)^{p'}$ is equibounded (see (ii) above) and converges almost pointwise in Q towards 0. Therefore, due to Egorov's theorem, one obtains

$$\int_Q \omega(|v_\varepsilon|)^{p'} F_\varepsilon dxdt \to 0 \text{ as } \varepsilon \to 0.$$

But the above limit being independent of the subsequence v_ε, still holds for the whole sequence v_ε. We deduce from this that $A_\varepsilon \to 0$ as $E \ni \varepsilon \to 0$.

Step 2). As $C_{\varepsilon,j}$ is concerned.

The function $(x,t) \mapsto a_i(x,t,\cdot,\cdot,\psi_j(x,t),\Psi(x,t,\cdot,\cdot))$ belongs to $\mathcal{C}(\overline{Q}; B_A^{p',\infty})$. Thus, using the convergence result (4.12) in Proposition 4.6, we are led at once at

$$C_{\varepsilon,j} \to \iint_{Q\times\Delta(A)} \left(\widehat{a}_i(-,\psi_j,\widehat{\Psi}) - \widehat{a}_i(-,u_0,\widehat{\Psi})\right)\widehat{f}dxdtd\beta \equiv \widehat{C}_j \text{ as } E \ni \varepsilon \to 0.$$

But

$$\left|\widehat{C}_j\right| \le \iint_{Q\times\Delta(A)} \omega\left(\left|\psi_j - u_0\right|\right)\left(1 + \left|\psi_j\right|^{p-1} + |u_0|^{p-1} + \left|\widehat{\Psi}\right|^{p-1}\right)\left|\widehat{f}\right| dxdtd\beta.$$

Therefore, proceeding as we have done it in Step 1) above, we obtain $\widehat{C}_j \to 0$ as $j \to \infty$.

Step 3). For the term $B_{\varepsilon,j}$, the same analysis conducted in Steps 1) and 2) yields

$$\lim_{E\ni\varepsilon\to 0}\lim_{j\to\infty} B_{\varepsilon,j} = 0.$$

Finally, since

$$\begin{aligned}&\lim_{E\ni\varepsilon\to 0}\left(\int_Q a_i^\varepsilon(-,u_\varepsilon,\Psi^\varepsilon)f^\varepsilon dxdt - \iint_{Q\times\Delta(A)} \widehat{a}_i(-,u_0,\widehat{\Psi})\widehat{f}dxdtd\beta\right)\\ &= \lim_{E\ni\varepsilon\to 0} A_\varepsilon + \lim_{E\ni\varepsilon\to 0}\lim_{j\to\infty} B_{\varepsilon,j} + \lim_{E\ni\varepsilon\to 0}\lim_{j\to\infty} C_{\varepsilon,j} = 0,\end{aligned}$$

the result follows. □

Corollary 4.8. *Let $0 \le i \le N$. Let $\psi_0 \in \mathcal{D}(Q)$ and $\psi_1 \in \mathcal{D}(Q) \otimes A^\infty$. For $\varepsilon > 0$, let*

$$\Phi_\varepsilon = \psi_0 + \varepsilon\psi_1^\varepsilon, \tag{4.14}$$

i.e., $\Phi_\varepsilon(x,t) = \psi_0(x,t) + \varepsilon\psi_1(x,t,x/\varepsilon,t/\varepsilon)$ *for* $(x,t) \in Q$. *Let* $(u_\varepsilon)_{\varepsilon\in E}$ *be a sequence in* $L^p(Q)$ *such that* $u_\varepsilon \to u_0$ *in* $L^p(Q)$ *as* $E \ni \varepsilon \to 0$ *where* $u_0 \in L^p(Q)$. *Then, as* $E \ni \varepsilon \to 0$, *one has*

$$(i)\ a_i^\varepsilon(-,u_\varepsilon,D\Phi_\varepsilon) \to a_i(-,u_0,D\psi_0 + D_y\psi_1) \text{ in } L^{p'}(Q)\text{-weak } A.$$

Moreover, if $(v_\varepsilon)_{\varepsilon\in E}$ *is a sequence in* $L^p(Q)$ *such that* $v_\varepsilon \to v_0$ *in* $L^p(Q)$*-weak* A *as* $E \ni \varepsilon \to 0$ *where* $v_0 \in L^p(Q;\mathcal{B}_A^p)$, *then, as* $E \ni \varepsilon \to 0$,

$$(ii)\ \int_Q a_i^\varepsilon(-,u_\varepsilon,D\Phi_\varepsilon)v_\varepsilon dxdt \to \iint_{Q\times\Delta(A)} \widehat{a}_i(-,u_0,D\psi_0 + \partial\widehat{\psi}_1)\widehat{v}_0 dxdtd\beta.$$

Proof. This is a direct consequence of the preceding corollary and of Proposition 4.6. Therefore we just give a rough sketch of the proof.

Let us begin by showing (i). Since $D\Phi_\varepsilon = D\psi_0 + (D_y\psi_1)^\varepsilon + \varepsilon(D\psi_1)^\varepsilon$, it is immediate by [property (4.9) in] Proposition 4.2 (where we have taken there $v_0 = u_\varepsilon$, $\mathbf{v} = D\Phi_\varepsilon$ and $\mathbf{w} = D\psi_0 + (D_y\psi_1)^\varepsilon$) that, as $E \ni \varepsilon \to 0$,

$$a_i^\varepsilon(-,u_\varepsilon,D\Phi_\varepsilon) - a_i^\varepsilon(-,u_\varepsilon,D\psi_0 + (D_y\psi_1)^\varepsilon) \to 0 \text{ in } L^{p'}(Q). \tag{4.15}$$

Therefore using the decomposition (for $f \in L^p(Q;A)$)

$$\begin{aligned}&\textstyle\int_Q a_i^\varepsilon(-,u_\varepsilon,D\Phi_\varepsilon)f^\varepsilon dxdt - \iint_{Q\times\Delta(A)} \widehat{a}_i(-,u_0,D\psi_0 + \partial\widehat{\psi}_1)\widehat{f}dxdtd\beta\\ &= \textstyle\int_Q [a_i^\varepsilon(-,u_\varepsilon,D\Phi_\varepsilon) - a_i^\varepsilon(-,u_\varepsilon,D\psi_0 + (D_y\psi_1)^\varepsilon)]f^\varepsilon dxdt\\ &\quad + \textstyle\int_Q a_i^\varepsilon(-,u_\varepsilon,D\psi_0 + (D_y\psi_1)^\varepsilon)f^\varepsilon dxdt\\ &\qquad - \textstyle\iint_{Q\times\Delta(A)} \widehat{a}_i(-,u_0,D\psi_0 + \partial\widehat{\psi}_1)\widehat{f}dxdtd\beta,\end{aligned}$$

part (i) follows at once by (4.15) and by Corollary 4.7.

As for (ii), by approaching the function u_0 by smooth functions in $\mathcal{C}_0^\infty(Q)$ as in the proof of Corollary 4.7 one can easily show that, as $E \ni \varepsilon \to 0$,

$$\begin{aligned}&\textstyle\int_Q v_\varepsilon a_i^\varepsilon(-,u_\varepsilon,D\psi_0 + (D_y\psi_1)^\varepsilon)dxdt\\ &\qquad \to \textstyle\iint_{Q\times\Delta(A)} \widehat{v}_0\widehat{a}_i(-,u_0,D\psi_0 + \partial\widehat{\psi}_1)dxdtd\beta.\end{aligned} \tag{4.16}$$

Once more using the convergence result (4.15) along with (4.16) we arrive (thanks to a decomposition similar to that in the proof of Corollary 4.7) at part (ii) above. This shows the corollary. □

4.3.2. ***Homogenization results***. The notation and hypotheses are those of the preceding subsections. Let A_y and A_τ be two algebras w.m.v. on $\mathbb{R}_y^N$ and on $\mathbb{R}_\tau$, respectively. We assume from now on that A_y is further ergodic and we set $A = A_y \odot A_\tau$. For $1 \le p < \infty$, we put $\mathcal{H} = \mathcal{B}_{A_\tau}^p(\mathbb{R}_\tau;\mathcal{B}_{\#A_y}^{1,p})$, a Banach space with an obvious norm. The canonical mapping of $\mathcal{B}_{A_y}^{1,p}/\mathbb{R}$ into its separated completion, $\mathcal{B}_{\#A_y}^{1,p}$, is denoted by J_1.

Let $\mathbb{F}_0^{1,p} = V_0^p \times L^p(Q;\mathcal{H})$. We equip $\mathbb{F}_0^{1,p}$ with the norm

$$\|\mathbf{v}\|_{\mathbb{F}_0^{1,p}} = \|v_0\|_{V^p} + \|v_1\|_{L^p(Q;\mathcal{H})} \quad (\mathbf{v} = (v_0,v_1) \in \mathbb{F}_0^{1,p}).$$

In this norm, $\mathbb{F}_0^{1,p}$ is a Banach space which admits $F_0^\infty = \mathcal{D}(Q)\times(\mathcal{D}(Q)\otimes[\varrho_\tau(A_\tau^\infty)\otimes(J_1\circ\varrho_y)(A_y^\infty/\mathbb{R})])$ as a dense subspace; see the end of Section 4.2) where ϱ_τ (resp. ϱ_y) denotes the canonical mapping of $B_{A_\tau}^p$ (resp. $B_{A_y}^p$) onto $\mathcal{B}_{A_\tau}^p$ (resp. $\mathcal{B}_{A_y}^p$).

We are now able to state and prove the main result of this section.

Theorem 4.9. *Let $2 \leq p < \infty$. Assume (4.11) holds with $A = A_y \odot A_\tau$ being as above. For each real $\varepsilon > 0$, let u_ε be a solution of (4.1). There exist a subsequence of $\{\varepsilon\}$, still denoted by $\{\varepsilon\}$, such that, as $\varepsilon \to 0$,*

$$u_\varepsilon \to u_0 \text{ in } L^p(0,T;W_0^{1,p}(\Omega))\text{-weak} \tag{4.17}$$

$$\frac{\partial u_\varepsilon}{\partial t} \to \frac{\partial u_0}{\partial t} \text{ in } L^{p'}(0,T;W^{-1,p'}(\Omega))\text{-weak} \tag{4.18}$$

$$\frac{\partial u_\varepsilon}{\partial x_i} \to \frac{\partial u_0}{\partial x_i} + \frac{\overline{\partial} u_1}{\partial y_i} \text{ in } L^p(Q)\text{-weak } A \ (1 \leq i \leq N) \tag{4.19}$$

where $\mathbf{u} = (u_0, u_1) \in \mathbb{F}_0^{1,p}$ solves the variational equation

$$\begin{array}{l} \int_0^T \langle u_0'(t), v_0(t)\rangle\, dt + \iint_{Q\times\Delta(A)} \widehat{a}(-,u_0,\mathbb{D}\mathbf{u})\cdot \mathbb{D}\mathbf{v} dxdtd\beta + \\ \quad + \iint_{Q\times\Delta(A)} \widehat{a}_0(-,u_0,\mathbb{D}\mathbf{u}) v_0 dxdtd\beta = \int_0^T \langle f(t), v_0(t)\rangle\, dt \\ \text{for all } \mathbf{v} = (v_0,v_1) \in \mathbb{F}_0^{1,p}, \end{array} \tag{4.20}$$

with $\mathbb{D}\mathbf{w} = Dw_0 + \partial\widehat{w}_1$ for $\mathbf{w} = (w_0, w_1) \in \mathbb{F}_0^{1,p}$ where: $\partial\widehat{w}_1 = (\partial_i \widehat{w}_1)_{1\leq i\leq N}$, $\partial_i\widehat{w}_1 = \mathcal{G}_1\left(\frac{\overline{\partial} w_1}{\partial y_i}\right)$. Moreover u_1 is unique and any weak A-limit point in V^p of $(u_\varepsilon)_{\varepsilon>0}$ is a solution to problem (4.20).

Proof. We first show that the sequence $(u_\varepsilon)_{\varepsilon>0}$ is bounded in V_0^p. Let $v \in V_0^p$; then

$$\begin{array}{l} \int_0^T \langle u_\varepsilon'(t), v(t)\rangle\, dt + \int_Q a^\varepsilon(x,t,u_\varepsilon(x,t),Du_\varepsilon(x,t))\cdot Dv(x,t)dxdt + \\ \quad + \int_Q a_0^\varepsilon(x,t,u_\varepsilon(x,t),Du_\varepsilon(x,t))v(x,t)dxdt = \int_0^T \langle f(t), v(t)\rangle\, dt \end{array} \tag{4.21}$$

where $a^\varepsilon(x,t,u_\varepsilon(x,t),Du_\varepsilon(x,t)) = a(x,t,x/\varepsilon,t/\varepsilon,u_\varepsilon(x,t),Du_\varepsilon(x,t))$ and $a_0^\varepsilon(x,t, u_\varepsilon(x,t),Du_\varepsilon(x,t)) = a_0(x,t,x/\varepsilon,t/\varepsilon,u_\varepsilon(x,t),Du_\varepsilon(x,t))$ for $(x,t) \in Q$. Taking in particular $v = u_\varepsilon$ in (4.21) and using the fact that $\int_0^T \langle u_\varepsilon'(t), u_\varepsilon(t)\rangle\, dt = \frac{1}{2}\left\|u_\varepsilon(T)\right\|_{L^2(\Omega)}^2 \geq 0$ and $\int_Q a_0^\varepsilon(x,t,u_\varepsilon(x,t),Du_\varepsilon(x,t))u_\varepsilon(x,t)dxdt \geq 0$ (this is a consequence of (4.7)), we obtain, thanks to property (4.6),

$$\sup_{\varepsilon>0} \|u_\varepsilon\|_{L^p(0,T;W_0^{1,p}(\Omega))} < \infty. \tag{4.22}$$

It therefore comes from (4.22) that

$$\sup_{\varepsilon>0} \|a_0^\varepsilon(-,u_\varepsilon,Du_\varepsilon)\|_{L^{p'}(Q)} + \sup_{\varepsilon>0} \|a^\varepsilon(-,u_\varepsilon,Du_\varepsilon)\|_{L^{p'}(Q)} < \infty. \tag{4.23}$$

It follows that $\sup_{\varepsilon>0} \left\|\frac{\partial u_\varepsilon}{\partial t}\right\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} < \infty$, and hence $(u_\varepsilon)_{\varepsilon>0}$ is bounded in V^p, and so in V_0^p. Thus, given an arbitrary fundamental sequence E, there exist a subsequence E' from E and a couple $\mathbf{u} = (u_0, u_1) \in \mathbb{F}_0^{1,p}$ such that (4.17)-(4.19) hold when $E' \ni \varepsilon \to 0$. The next point is to show that $\mathbf{u}$ verifies the variational equation (4.20). For that, let $\Phi = (\psi_0, (J_1 \circ \varrho)\circ \psi_1) \in F_0^\infty$ with $\psi_0 \in \mathcal{D}(Q)$, $\psi_1 \in \mathcal{D}(Q) \otimes [A_\tau^\infty \otimes (A_y^\infty/\mathbb{R})]$. Define Φ_ε as in Corollary 4.8. Then, $\Phi_\varepsilon \in \mathcal{D}(Q)$ and, due to property (4.6) one has

$$\begin{array}{l} 0 \leq \int_0^T \langle f(t) - u_\varepsilon'(t), u_\varepsilon(t) - \Phi_\varepsilon(\cdot,t)\rangle\, dt - \int_Q a^\varepsilon(-,u_\varepsilon,D\Phi_\varepsilon)\cdot D(u_\varepsilon - \Phi_\varepsilon)dxdt \\ \quad - \int_Q a_0^\varepsilon(-,u_\varepsilon,Du_\varepsilon)(u_\varepsilon - \Phi_\varepsilon)dxdt, \end{array}$$

or using the equality $\int_0^T \langle u_\varepsilon'(t), u_\varepsilon(t)\rangle\, dt = \frac{1}{2}\left\|u_\varepsilon(T)\right\|_{L^2(\Omega)}^2$,

$$\begin{array}{l} \frac{1}{2}\left\|u_\varepsilon(T)\right\|_{L^2(\Omega)}^2 \leq \int_0^T \langle f(t), u_\varepsilon(t) - \Phi_\varepsilon(\cdot,t)\rangle\, dt + \int_0^T \langle u_\varepsilon'(t), \Phi_\varepsilon(\cdot,t)\rangle\, dt - \\ \quad - \int_Q a^\varepsilon(\text{-},u_\varepsilon,D\Phi_\varepsilon)\cdot D(u_\varepsilon - \Phi_\varepsilon)dxdt - \int_Q a_0^\varepsilon(\text{-},u_\varepsilon,Du_\varepsilon)(u_\varepsilon - \Phi_\varepsilon)dxdt. \end{array} \tag{4.24}$$

On one hand, as $\varepsilon \to 0$, one has

$$\frac{\partial \Phi_\varepsilon}{\partial x_j} \to \frac{\partial \psi_0}{\partial x_j} + \frac{\partial \psi_1}{\partial y_j} \text{ in } L^p(Q)\text{-weak } A \ (1 \leq j \leq N).$$

Thus, combining (4.19) with part (ii) of Corollary 4.8 we get

$$\int_Q a^\varepsilon(-, u_\varepsilon, D\Phi_\varepsilon) \cdot D(u_\varepsilon - \Phi_\varepsilon) dxdt \to \iint_{Q \times \Delta(A)} \widehat{a}(-, u_0, \mathbb{D}\Phi) \cdot \mathbb{D}(\mathbf{u} - \Phi) dxdtd\beta.$$

Besides, observing that

$$\int_0^T \langle u_\varepsilon'(t), \Phi_\varepsilon(\cdot, t)\rangle \, dt = -\int_Q u_\varepsilon \frac{\partial \Phi_\varepsilon}{\partial t} dxdt$$

and in view of the compactness of the embedding $V^p \hookrightarrow L^2(Q)$ (from which results the convergence result: $u_\varepsilon \to u_0$ in $L^2(Q)$), we see immediately that

$$\int_0^T \langle u_\varepsilon'(t), \Phi_\varepsilon(\cdot, t)\rangle \, dt \to \int_0^T \langle u_0'(t), \psi_0(\cdot, t)\rangle \, dt \text{ as } E' \ni \varepsilon \to 0.$$

On the other hand, the transformation $v \to \|v(T)\|^2_{L^2(\Omega)}$ being continuous on V_0^p and since $u_\varepsilon \to u_0$ in V_0^p-weak as $E' \ni \varepsilon \to 0$, one has

$$\|u_0(T)\|^2_{L^2(\Omega)} \leq \liminf_{E' \ni \varepsilon \to 0} \|u_\varepsilon(T)\|^2_{L^2(\Omega)} .$$

The sequence $(a_0^\varepsilon(-, u_\varepsilon, Du_\varepsilon))_{\varepsilon > 0}$ is bounded in $L^{p'}(Q)$ (see (4.23)), thus, there exist a subsequence from E', still denoted by E' and a function $\chi \in L^{p'}(Q; \mathcal{B}_A^{p'})$ such that, as $E' \ni \varepsilon \to 0$,

$$a_0^\varepsilon(-, u_\varepsilon, Du_\varepsilon) \to \chi \text{ in } L^{p'}(Q)\text{-weak } A. \tag{4.25}$$

Now, using the compactness of the embedding $V^p \hookrightarrow L^p(Q)$ (this is a classical result), we have, as $E' \ni \varepsilon \to 0$, $u_\varepsilon - \Phi_\varepsilon \to u_0 - \psi_0$ in $L^p(Q)$; hence, as $E' \ni \varepsilon \to 0$,

$$\int_Q a_0^\varepsilon(-, u_\varepsilon, Du_\varepsilon)(u_\varepsilon - \Phi_\varepsilon) dxdt \to \iint_{Q \times \Delta(A)} \widehat{\chi}(u_0 - \psi_0) dxdtd\beta.$$

Therefore, taking the $\liminf_{E' \ni \varepsilon \to 0}$ of both sides of (4.24) and using equality

$$\frac{1}{2} \|u_0(T)\|^2_{L^2(\Omega)} = \int_0^T \langle u_0'(t), u_0(t)\rangle \, dt,$$

we are led at once at

$$\begin{array}{l} 0 \leq \int_0^T \langle f(t) - u_0'(t), u_0(t) - \psi_0(\cdot, t)\rangle \, dt - \iint_{Q \times \Delta(A)} \widehat{a}(-, u_0, \mathbb{D}\Phi) \cdot \mathbb{D}(\mathbf{u} - \Phi) dxdtd\beta \\ \quad - \iint_{Q \times \Delta(A)} \widehat{\chi}(u_0 - \psi_0) dxdtd\beta. \end{array} \tag{4.26}$$

Since F_0^∞ is dense in $\mathbb{F}_0^{1,p}$, (4.26) still holds for any $\Phi \in \mathbb{F}_0^{1,p}$. Taking in (4.26) the particular functions $\Phi = \mathbf{u} - \lambda \mathbf{v}$ with $\lambda > 0$ and $\mathbf{v} = (v_0, v_1) \in \mathbb{F}_0^{1,p}$, then dividing both sides of the resultant inequality by λ, and letting $\lambda \to 0$, and finally changing $\mathbf{v}$ into $-\mathbf{v}$, leads to

$$\begin{array}{l} \int_0^T \langle u_0'(t), v_0(t)\rangle \, dt + \iint_{Q \times \Delta(A)} \widehat{a}(-, u_0, \mathbb{D}\mathbf{u}) \cdot \mathbb{D}\mathbf{v} dxdtd\beta + \\ \quad + \iint_{Q \times \Delta(A)} \widehat{\chi} v_0 dxdtd\beta = \int_0^T \langle f(t), v_0(t)\rangle \, dt \text{ for all } \mathbf{v} = (v_0, v_1) \in \mathbb{F}_0^{1,p}. \end{array} \tag{4.27}$$

The last point to check is to show that $\chi = a_0(-, u_0, \overline{\mathbb{D}}_y \mathbf{u})$ where $\overline{\mathbb{D}}_y \mathbf{u} = Du_0 + \overline{D}_y u_1$. To this end let $0 < \eta < 1$ be arbitrarily fixed. Let $B_\#(u_1, \eta)$ (resp. $B_0(u_0, \eta)$)

denote the closed ball of $L^p(Q;\mathcal{H})$ (resp. $L^p(0,T;W_0^{1,p}(\Omega))$) centered at u_1 (resp. u_0) and of radius η. Since $L^p(Q;\mathcal{H})$ and $L^p(0,T;W_0^{1,p}(\Omega))$ are reflexive, these balls are weakly compact. Set

$$d=\sup_{v_1\in B_{\#}(u_1,\eta)}\sup_{v_0\in B_0(u_0,\eta)}\left\|1+|u_0|+\left|Du_0+\overline{D}_y v_1\right|+\left|Dv_0+\overline{D}_y v_1\right|\right\|_{L^p(Q;\mathcal{B}_A^p)}^{p-2}<\infty$$

and $k=c_0 d+1$; then $k>1$. By using the density of $\mathcal{D}(Q)$ in $L^p(0,T;W_0^{1,p}(\Omega))$ and that of $\mathcal{D}(Q)\otimes[A_\tau^\infty\otimes(J_1\circ\varrho)(A_y^\infty/\mathbb{R})]$ in $L^p(Q;\mathcal{H})$, we deduce the existence of $\psi_0\in\mathcal{D}(Q)$ and of $\psi_1\in\mathcal{D}(Q)\otimes[A_\tau^\infty\otimes(A_y^\infty/\mathbb{R})]$ such that

$$\left\|u_0-\psi_0\right\|_{L^p(0,T;W_0^{1,p}(\Omega))}<\frac{\eta}{4k}\text{ and }\left\|u_1-(J_1\circ\varrho_y)(\psi_1)\right\|_{L^p(Q;\mathcal{H})}<\frac{\eta}{4k}.\qquad(4.28)$$

Clearly

$$\psi_0\in B_0\left(u_0,\eta\right)\text{ and }(J_1\circ\varrho_y)(\psi_1)\in B_{\#}\left(u_1,\eta\right).$$

Set $\Phi=(\psi_0,(J_1\circ\varrho_y)\circ\psi_1))$ and define Φ_ε as in (4.14). First we have

$$\begin{aligned}&\left\|a_0(-,u_0,\overline{\mathbb{D}}_y\mathbf{u})-\chi\right\|_{L^{p'}(Q;\mathcal{B}_A^{p'})}\leq\\&\leq\left\|a_0(-,u_0,\overline{\mathbb{D}}_y\mathbf{u})-a_0(-,u_0,Du_0+D_y\psi_1)\right\|_{L^{p'}(Q;\mathcal{B}_A^{p'})}+\\&\quad+\left\|a_0(-,u_0,Du_0+D_y\psi_1)-a_0(-,u_0,\overline{\mathbb{D}}_y\Phi)\right\|_{L^{p'}(Q;\mathcal{B}_A^{p'})}+\\&\qquad+\left\|a_0(-,u_0,\overline{\mathbb{D}}_y\Phi)-\chi\right\|_{L^{p'}(Q;\mathcal{B}_A^{p'})}\end{aligned}$$

where, for $\mathbf{v}=(v_0,v_1)\in\mathbb{F}_0^{1,p}$ we have put $\overline{\mathbb{D}}_y\mathbf{v}=Dv_0+\overline{D}_y v_1$, and $D_y\psi_1=\overline{D}_y(J_1\circ\varrho)(\psi_1)$. But, on one hand

$$\begin{aligned}&\left\|a_0(-,u_0,\overline{\mathbb{D}}_y\mathbf{u})-a_0(-,u_0,Du_0+D_y\psi_1)\right\|_{L^{p'}(Q;\mathcal{B}_A^{p'})}\\&\leq c_0\left\|1+|u_0|+\left|\overline{\mathbb{D}}_y\mathbf{u}\right|+|Du_0+D_y\psi_1|\right\|_{L^p(Q;\mathcal{B}_A^p)}^{p-2}\left\|\overline{D}_y(u_1-(J_1\circ\varrho_y)(\psi_1))\right\|_{L^p(Q;\mathcal{B}_A^p)^N}\\&<\tfrac{\eta}{4}\end{aligned}$$

since

$$\left\|\overline{D}_y(u_1-(J_1\circ\varrho_y)(\psi_1))\right\|_{L^p(Q;\mathcal{B}_A^p)^N}=\left\|u_1-(J_1\circ\varrho_y)(\psi_1)\right\|_{L^p(Q;\mathcal{H})};$$

and

$$\begin{aligned}&\left\|a_0(-,u_0,Du_0+D_y\psi_1)-a_0(-,u_0,\overline{\mathbb{D}}_y\Phi)\right\|_{L^{p'}(Q;\mathcal{B}_A^{p'})}\\&\leq c_0\left\|1+|u_0|+|Du_0+D_y\psi_1|+\left|\overline{\mathbb{D}}_y\Phi\right|\right\|_{L^p(Q;\mathcal{B}_A^p)}^{p-2}\left\|D(u_0-\psi_0)\right\|_{L^p(Q;\mathcal{B}_A^p)^N}\\&<\tfrac{\eta}{4}\end{aligned}$$

since

$$\left\|D(u_0-\psi_0)\right\|_{L^p(Q;\mathcal{B}_A^p)^N}=\left\|u_0-\psi_0\right\|_{L^p(0,T;W_0^{1,p}(\Omega))}.$$

On the other hand, combining part (ii) of Corollary 4.8 (notice that $u_\varepsilon\to u_0$ in $L^p(Q)$ as $E'\ni\varepsilon\to 0$) with convergence result (4.25) we get

$$\left\|a_0(-,u_0,\overline{\mathbb{D}}_y\Phi)-\chi\right\|_{L^{p'}(Q;\mathcal{B}_A^{p'})}\leq\liminf_{E'\ni\varepsilon\to 0}\left\|a_0^\varepsilon(-,u_\varepsilon,D\Phi_\varepsilon)-a_0^\varepsilon(-,u_\varepsilon,Du_\varepsilon)\right\|_{L^{p'}(Q)}.$$

But

$$\begin{aligned}&\left\|a_0^\varepsilon(-,u_\varepsilon,D\Phi_\varepsilon)-a_0^\varepsilon(-,u_\varepsilon,Du_\varepsilon)\right\|_{L^{p'}(Q)}\\&\leq c_0\left\|1+|u_\varepsilon|+|Du_\varepsilon|+|D\Phi_\varepsilon|\right\|_{L^p(Q)}^{p-2}\left\|Du_\varepsilon-D\Phi_\varepsilon\right\|_{L^p(Q)^N},\end{aligned}$$

and using the fact that $|D\Phi_\varepsilon|\leq|Du_\varepsilon-D\Phi_\varepsilon|+|Du_\varepsilon|$, which leads to

$$1+|u_\varepsilon|+|Du_\varepsilon|+|D\Phi_\varepsilon|\leq 1+|u_\varepsilon|+2\left|Du_\varepsilon\right|+\left|Du_\varepsilon-D\Phi_\varepsilon\right|,\text{ we get}$$

$$\int_Q (1+|u_\varepsilon|+|Du_\varepsilon|+|D\Phi_\varepsilon|)^p\,dxdt \leq \int_Q (1+|u_\varepsilon|+2\,|Du_\varepsilon|+|Du_\varepsilon-D\Phi_\varepsilon|)^p\,dxdt,$$

whence

$$\begin{aligned}\|1+|u_\varepsilon|+|Du_\varepsilon|+|D\Phi_\varepsilon|\|_{L^p(Q)} &\leq \|1+|u_\varepsilon|+2\,|Du_\varepsilon|\|_{L^p(Q)}+\\ &\quad+\|Du_\varepsilon-D\Phi_\varepsilon\|_{L^p(Q)^N}\\ &\leq c+\|Du_\varepsilon-D\Phi_\varepsilon\|_{L^p(Q)^N}\end{aligned}$$

where $c>0$ is a constant independent of ε, the last inequality above being obtained thanks to (4.22) and to the Poincaré inequality. Thus,

$$\begin{aligned}&\left\|a_0(-,u_0,\overline{\mathbb{D}}_y\Phi)-\chi\right\|_{L^{p'}(Q;\mathcal{B}_A^{p'})}\leq\\ &\leq \liminf_{E'\ni\varepsilon\to 0}\left(c+\|Du_\varepsilon-D\Phi_\varepsilon\|_{L^p(Q)^N}\right)^{p-2}\|Du_\varepsilon-D\Phi_\varepsilon\|_{L^p(Q)^N}.\end{aligned}$$

Besides, in view of (4.6) we have

$$\begin{aligned}&c_1\,\|Du_\varepsilon-D\Phi_\varepsilon\|_{L^p(Q)^N}^p\leq\\ &\quad\textstyle\int_Q (a^\varepsilon(-,u_\varepsilon,Du_\varepsilon)-a^\varepsilon(-,u_\varepsilon,D\Phi_\varepsilon))\cdot(Du_\varepsilon-D\Phi_\varepsilon)dxdt.\end{aligned}$$

Proceeding as we have done it to obtain (4.27), we are led to (as $E'\ni\varepsilon\to 0$)

$$\begin{aligned}B_\varepsilon&\equiv\textstyle\int_Q (a^\varepsilon(-,u_\varepsilon,Du_\varepsilon)-a^\varepsilon(-,u_\varepsilon,D\Phi_\varepsilon))\cdot(Du_\varepsilon-D\Phi_\varepsilon)dxdt\\ &\to\textstyle\iint_{Q\times\Delta(A)}(\widehat{a}(-,u_0,\mathbb{D}\mathbf{u})-\widehat{a}(-,u_0,\mathbb{D}\Phi))\cdot\mathbb{D}(\mathbf{u}-\Phi)dxdtd\beta\equiv B.\end{aligned}$$

Thus there exists $\varepsilon_0>0$ such that $E'\ni\varepsilon\leq\varepsilon_0$ yields $B_\varepsilon\leq B+\frac{\eta}{4}$. But according to Hölder's inequality and to Proposition 4.6 we get that

$$B\leq c_0\,\|1+|u_0|+|\mathbb{D}\mathbf{u}|+|\mathbb{D}\Phi|\|_{L^p(Q\times\Delta(A))}^{p-2}\,\|\mathbb{D}(\mathbf{u}-\Phi)\|_{L^p(Q\times\Delta(A))^N}^2.$$

But

$$\begin{aligned}\|1+|u_0|+|\mathbb{D}\mathbf{u}|+|\mathbb{D}\Phi|\|_{L^p(Q\times\Delta(A))}&=\|1+|u_0|+|\mathbb{D}\mathbf{u}|+|\mathbb{D}\Phi|\|_{L^p(Q;L^p(\Delta(A)))}\\ &=\left\|1+|u_0|+\left|\overline{\mathbb{D}}_y\mathbf{u}\right|+\left|\overline{\mathbb{D}}_y\Phi\right|\right\|_{L^p(Q;\mathcal{B}_A^p)}\\ &<k\end{aligned}$$

and

$$\begin{aligned}\|\mathbb{D}(\mathbf{u}-\Phi)\|_{L^p(Q\times\Delta(A))^N}&=\left\|\overline{\mathbb{D}}_y(\mathbf{u}-\Phi)\right\|_{L^p(Q;\mathcal{B}_A^p)^N}\\ &\leq\|u_0-\psi_0\|_{L^p(0,T;W_0^{1,p}(\Omega))}+\|u_1-(J_1\circ\varrho)(\psi_1)\|_{L^p(Q;\mathcal{H})}\\ &<\frac{\eta}{2k}\quad\text{(see (4.28))}.\end{aligned}$$

Thus $B<k(\eta/2k)^2=\eta^2/4k$. Since $k>1$ and $\eta<1$ we get $B<\eta/4$ and so, $B_\varepsilon\leq\eta/2$ for $E'\ni\varepsilon\leq\varepsilon_0$, hence

$$c_1\,\|Du_\varepsilon-D\Phi_\varepsilon\|_{L^p(Q)^N}^p\leq\frac{\eta}{2}\quad\text{for }E'\ni\varepsilon\leq\varepsilon_0.$$

Therefore

$$\begin{aligned}&\liminf_{E'\ni\varepsilon\to 0}\left(c+\|Du_\varepsilon-D\Phi_\varepsilon\|_{L^p(Q)^N}\right)^{p-2}\|Du_\varepsilon-D\Phi_\varepsilon\|_{L^p(Q)^N}\leq\\ &\quad\leq\left(\tfrac{\eta}{2c_1}\right)^{\frac{1}{p}}\left[c+\left(\tfrac{\eta}{2c_1}\right)^{\frac{1}{p}}\right]^{p-2}.\end{aligned}$$

Finally, we get that

$$\left\|a_0(-,u_0,\overline{\mathbb{D}}_y\mathbf{u})-\chi\right\|_{L^{p'}(Q;\mathcal{B}_A^{p'})}\leq\frac{\eta}{2}+\left(\frac{\eta}{2c_1}\right)^{\frac{1}{p}}\left[c+\left(\frac{\eta}{2c_1}\right)^{\frac{1}{p}}\right]^{p-2}.$$

The above inequality holds true for any positive real $\eta<1$. Hence on letting $\eta\to0$ we arrive at once at $\chi=a_0(-,u_0,\overline{\mathbb{D}}_y\mathbf{u})$.

As the uniqueness of u_1 is concerned, let $(x,t)\in Q$ and let $(r,\xi)\in\mathbb{R}\times\mathbb{R}^N$ be freely fixed, and let $\pi(x,t,r,\xi)$ be defined by the so-called cell problem

$$\begin{array}{l}\pi(x,t,r,\xi)\in\mathcal{H}:\\ \int_{\Delta(A)}\widehat{a}(-,r,\xi+\partial\widehat{\pi}(x,t,r,\xi))\cdot\partial\widehat{w}d\beta=0\text{ for all }w\in\mathcal{H}.\end{array}\tag{4.29}$$

According to Proposition 4.6, we get by [32, Chap. 2] that problem (4.29) has at least a solution. However this solution is unique. In fact, if $\pi_1\equiv\pi_1(x,t,r,\xi)$ and $\pi_2\equiv\pi_2(x,t,r,\xi)$ are two solutions of (4.29), then

$$\int_{\Delta(A)}(\widehat{a}(-,r,\xi+\partial\widehat{\pi}_1)-\widehat{a}(-,r,\xi+\partial\widehat{\pi}_2))\cdot(\partial\widehat{\pi}_1-\partial\widehat{\pi}_2)\,d\beta=0,$$

hence $\partial\widehat{\pi}_1=\partial\widehat{\pi}_2$ $(1\leq i\leq N)$, i.e. $\partial_i\widehat{\pi}_1=\partial_i\widehat{\pi}_2$, or $\mathcal{G}_1\left(\frac{\overline{\partial}\pi_1}{\partial y_i}\right)=\mathcal{G}_1\left(\frac{\overline{\partial}\pi_2}{\partial y_i}\right)$ by (2.8). It therefore comes that $\frac{\overline{\partial}\pi_1}{\partial y_i}=\frac{\overline{\partial}\pi_2}{\partial y_i}$, $1\leq i\leq N$ which amounts to saying that $\pi_1=\pi_2$ since they belong to $\mathcal{H}$ (recall that $\mathcal{H}=\mathcal{B}^p_{A_\tau}(\mathbb{R}_\tau;\mathcal{B}^{1,p}_{\#A_y})$). Now, taking in particular $r=u_0(x,t)$ and $\xi=Du_0(x,t)$ with (x,t) arbitrarily fixed in Q, and then choosing in (4.20) the particular test functions $\mathbf{v}=(0,v_1)$ such that $v_1(x,t)=\varphi(x,t)w$ $((x,t)\in Q)$ with $\varphi\in\mathcal{D}(Q)$ and $w\in\mathcal{H}$, and finally comparing the resultant equation with (4.29), it follows (by the uniqueness argument) that $u_1=\pi(\cdot,\cdot,u_0,Du_0)$, where the right-hand side of this equality stands for the function $(x,t)\mapsto\pi(x,t,u_0(x,t),Du_0(x,t))$ of Q into $\mathcal{H}$. The uniqueness of u_1 is therefore established, and the proof is complete. □

The variational problem (4.20) is called *global* homogenized problem for (4.1) under the assumption (4.11). Our goal here is to derive the macroscopic homogenized problem for (4.1). Let

$$\begin{array}{l}q(x,t,r,\xi)=\int_{\Delta(A)}\widehat{a}(-,r,\xi+\partial\widehat{\pi}(x,t,r,\xi))d\beta\text{ and}\\ q_0(x,t,r,\xi)=\int_{\Delta(A)}\widehat{a}_0(-,r,\xi+\partial\widehat{\pi}(x,t,r,\xi))d\beta\end{array}$$

for fixed $(x,t)\in Q$ and $(r,\xi)\in\mathbb{R}\times\mathbb{R}^N$. Substituting $u_1=\pi(\cdot,\cdot,u_0,Du_0)$ in (4.20) and choosing there the particular test functions $\mathbf{v}=(\varphi,0)$ with $\varphi\in\mathcal{D}(Q)$, we are led to the macroscopic homogenized problem for (4.1):

$$\begin{array}{c}\frac{\partial u_0}{\partial t}-\operatorname{div}q(\cdot,\cdot,u_0,Du_0)+q_0(\cdot,\cdot,u_0,Du_0)=f\text{ in }Q\\ u_0=0\text{ on }\partial\Omega\times(0,T)\\ u_0(x,0)=0\text{ in }\Omega.\end{array}$$

Thanks to (4.20), the above problem has (at least) a solution.

The next result deals with a corrector type result. Before we can state it, we need to fix some basis: from now on, we suppose that E' denotes the subsequence for which (4.17)-(4.19) hold.

Proposition 4.10. *Let the hypotheses and notation be as in Theorem* 4.9*. There exists a continuous increasing function* $\nu:[0,\infty)\to[0,\infty)$ *with* $\nu(0)=0$ *such that* *for all* $\Phi = (\psi_0, (J_1 \circ \varrho_y)(\psi_1))$ *with* $\psi_0 \in L^p(0,T;W_0^{1,p}(\Omega))$ *and* $\psi_1 \in L^p(0,T;W_0^{1,p}(\Omega)) \otimes [A_\tau \otimes (A_y^1/\mathbb{R})]$, *if we define* Φ_ε *as in* (4.14), *then*

$$\limsup_{E' \ni \varepsilon \to 0} \|Du_\varepsilon - D\Phi_\varepsilon\|_{L^p(Q)^N} \leq \nu \left(\left\| \overline{\mathbb{D}}_y \mathbf{u} - \overline{\mathbb{D}}_y \Phi \right\|_{L^p(Q;\mathcal{B}_A^p)^N} \right). \tag{4.30}$$

Proof. Let F_0^1 be the vector space of all Φ as in the statement of Proposition 4.10. Endowed with its natural topology, F_0^1 has F_0^∞ as a dense subspace (this is straightforward). Thus, we begin by show (4.30) for Φ in F_0^∞. But, in view of (4.6), for $\Phi \in F_0^\infty$,

$$c_1 \|Du_\varepsilon - D\Phi_\varepsilon\|_{L^p(Q)^N} \leq$$
$$\int_Q (a^\varepsilon(-, u_\varepsilon, Du_\varepsilon) - a^\varepsilon(-, u_\varepsilon, D\Phi_\varepsilon) \cdot D(u_\varepsilon - \Phi_\varepsilon) dxdt \equiv B_\varepsilon.$$

A quick survey of the proof of Theorem 4.9 reveals that, as $E' \ni \varepsilon \to 0$,

$$B_\varepsilon \to \iint_{Q \times \Delta(A)} (\widehat{a}(-, u_0, \mathbb{D}\mathbf{u}) - \widehat{a}(-, u_0, \mathbb{D}\Phi)) \cdot \mathbb{D}(\mathbf{u} - \Phi) dxdtd\beta \equiv B,$$

where $\mathbf{u} = (u_0, u_1)$ is as in Theorem 4.9. Thus, $\limsup_{E' \ni \varepsilon \to 0} \|Du_\varepsilon - D\Phi_\varepsilon\|_{L^p(Q)^N} \leq \left(\frac{B}{c_1}\right)^{\frac{1}{p}}$. But using Hölder's inequality and [(4.13) in] Proposition 4.6, we get

$$B \leq c_0 \| 1 + |u_0| + |\mathbb{D}\mathbf{u}| + |\mathbb{D}\Phi| \|_{L^p(Q\times\Delta(A))}^{p-2} \|\mathbb{D}(\mathbf{u} - \Phi)\|_{L^p(Q\times\Delta(A))^N}^2 ,$$

and by the obvious inequality $|\mathbb{D}\Phi| \leq |\mathbb{D}\mathbf{u} - \mathbb{D}\Phi| + |\mathbb{D}\mathbf{u}|$,

$$B \leq c_0 \left(\| 1 + |u_0| + 2\,|Du_0| \|_{L^p(Q\times\Delta(A))} + \|\mathbb{D}(\mathbf{u} - \Phi)\|_{L^p(Q\times\Delta(A))^N} \right)^{p-2} \times$$
$$\times \|\mathbb{D}(\mathbf{u} - \Phi)\|_{L^p(Q\times\Delta(A))^N}^2 .$$

Now, set $\alpha = \| 1 + |u_0| + 2\,|Du_0| \|_{L^p(Q\times\Delta(A))}$ and

$$\nu(r) = \frac{c_0}{c_1^{\frac{1}{p}}} r^{\frac{2}{p}} (\alpha + r)^{1-\frac{2}{p}} \text{ for } r \geq 0.$$

Then the function ν is independent of Φ and satisfies hypotheses stated in Proposition 4.10 (this is straightforward by observing that $\|\mathbb{D}(\mathbf{u} - \Phi)\|_{L^p(Q\times\Delta(A))^N} = \left\| \overline{\mathbb{D}}_y \mathbf{u} - \overline{\mathbb{D}}_y \Phi \right\|_{L^p(Q;\mathcal{B}_A^p)^N}$). Whence (4.30) is shown for Φ in F_0^∞.

Now, let $\Phi \in F_0^1$. Let $(\Psi_j)_j$ be a sequence in F_0^∞ such that $\Psi_j \to \Phi$ in F_0^1 as $j \to \infty$. Set

$$\Psi_j = (\varphi_{0j}, (J_1 \circ \varrho_y)(\varphi_{1j})) \text{ and } \Phi = (\psi_0, (J_1 \circ \varrho_y)(\psi_1)),$$

and define

$$\Psi_{j,\varepsilon} = \varphi_{0j} + \varepsilon \varphi_{1j}^\varepsilon \text{ and } \Phi_\varepsilon = \psi_0 + \varepsilon \psi_1^\varepsilon \text{ as in (4.14).}$$

We have

$$\limsup_{E' \ni \varepsilon \to 0} \|Du_\varepsilon - D\Phi_\varepsilon\|_{L^p(Q)^N} \leq \limsup_{E' \ni \varepsilon \to 0} \|Du_\varepsilon - D\Psi_{j,\varepsilon}\|_{L^p(Q)^N} +$$
$$\limsup_{E' \ni \varepsilon \to 0} \|D\Psi_{j,\varepsilon} - D\Phi_\varepsilon\|_{L^p(Q)^N}$$
$$\leq \nu \left(\left\| \overline{\mathbb{D}}_y \mathbf{u} - \overline{\mathbb{D}}_y \Psi_j \right\|_{L^p(Q;\mathcal{B}_A^p)^N} \right) + \limsup_{E' \ni \varepsilon \to 0} \|D\Psi_{j,\varepsilon} - D\Phi_\varepsilon\|_{L^p(Q)^N} .$$

Now, since $\Psi_j \to \Phi$ in F_0^1, we get $\mathbb{D}\Psi_j \to \mathbb{D}\Phi$ in $L^p(Q \times \Delta(A))^N$ as $j \to \infty$. On the other hand, it can be easily shown that $\lim_{j\to\infty} \lim_{E' \ni \varepsilon \to 0} \|D\Psi_{j,\varepsilon} - D\Phi_\varepsilon\|_{L^p(Q)^N} = 0$.

Hence, taking the limit (as $j\to\infty$) of both sides of the last inequality above, we are led to (4.30). □

As a consequence of the preceding result, we have the following corrector-type result.

Corollary 4.11. *Let the hypotheses and notation be as in Theorem* 4.9. *Assume further that*

$$u_1\in L^p(0,T;W_0^{1,p}(\Omega))\otimes[A_\tau\otimes(J_1\circ\varrho_y)(A_y^1/\mathbb{R})].$$

Then, as $E'\ni\varepsilon\to 0$,

$$u_\varepsilon-u_0-\varepsilon u_1^\varepsilon\to 0 \text{ in } L^p(0,T;W^{1,p}(\Omega)).$$

Proof. It is clear that, on one hand, $\varepsilon u_1^\varepsilon\to 0$ in $L^p(Q)$ as $E'\ni\varepsilon\to 0$; and on the other hand, due to (4.17)-(4.18) and to the compactness of the embedding $V^p\hookrightarrow L^p(Q)$, we have $u_\varepsilon-u_0$ in $L^p(Q)$. Thus $u_\varepsilon-u_0-\varepsilon u_1^\varepsilon\to 0$ in $L^p(Q)$ as $E'\ni\varepsilon\to 0$. It remains to show that $D(u_\varepsilon-u_0-\varepsilon u_1^\varepsilon)\to 0$ in $L^p(Q)^N$ as $E'\ni\varepsilon\to 0$. But, if we set $\Phi_\varepsilon=u_0+\varepsilon u_1^\varepsilon$, then applying (4.30), we get

$$\limsup_{E'\ni\varepsilon\to 0}\|Du_\varepsilon-D\Phi_\varepsilon\|_{L^p(Q)^N}=0$$

since $\nu\left(\left\|\overline{\mathbb{D}}_y\mathbf{u}-\overline{\mathbb{D}}_y\Phi\right\|_{L^p(Q;\mathcal{B}_A^p)^N}\right)=\nu(0)=0$, and so $\lim\limits_{E'\ni\varepsilon\to 0}\|Du_\varepsilon-D\Phi_\varepsilon\|_{L^p(Q)^N}=0$. The result follows thereby. □

Remark 4.5. Assuming a_0 to be the null function, and replacing hypotheses (4.3) and [part (i) of] (4.4) by

$$(a(x,t,y,\tau,\mu,\lambda)-a(x,t,y,\tau,\mu,\lambda'))\cdot(\lambda-\lambda')>0 \text{ for } \lambda\neq\lambda',$$
$$a(x,t,y,\tau,\mu,\lambda)\cdot\lambda\geq c_1\left|\lambda\right|^p,$$

we also reach the conclusion of Theorem 4.9.

4.4. **Some concrete applications of Theorem 4.9.**

4.4.1. *Problem I (Periodic homogenization).* Here we mean to study the problem of homogenizing (4.1) under the periodicity hypothesis on the functions $(y,\tau)\mapsto a_i(x,t,y,\tau,\mu,\lambda)$ (for each fixed $(x,t,\mu,\lambda)\in\overline{Q}\times\mathbb{R}\times\mathbb{R}^N$ and $0\leq i\leq N$), i.e.

$$\begin{array}{l}\text{For each } k\in\mathbb{Z}^N \text{ and any } l\in\mathbb{Z}, \text{ we have}\\ a_i(x,t,y+k,\tau+l,\mu,\lambda)=a_i(x,t,y,\tau,\mu,\lambda)\\ \text{a.e. in } (y,\tau)\in\mathbb{R}^N\times\mathbb{R}.\end{array}\tag{4.31}$$

(4.31) is very often expressed by saying that the function $(y,\tau)\mapsto a_i(x,t,y,\tau,\mu,\lambda)$ is Y-periodic in $y\in\mathbb{R}^N$ and $\mathcal{T}$-periodic in $\tau\in\mathbb{R}$ where

$$Y=(0,1)^N \text{ and } \mathcal{T}=(0,1).$$

Surprisingly, this problem has never been studied before. Our aim here is to provide a better understanding and accurate results in this setting by means of deterministic homogenization theory.

Let $L_{\mathrm{per}}^{p'}(Y)$ denote the space of Y-periodic functions in $L_{\mathrm{loc}}^{p'}(\mathbb{R}_y^N)$. It is classically known that $L_{\mathrm{per}}^{p'}(Y)$ is the closure of $\mathcal{C}_{\mathrm{per}}(Y)$ in $L_{\mathrm{loc}}^{p'}(\mathbb{R}_y^N)$ with respect to the norm $\|\cdot\|_{p'}$ (which is here defined by $\|u\|_{p'}=\left(M(|u|^{p'})\right)^{1/p'}=\left(\int_Y|u(y)|^{p'}dy\right)^{1/p'}$). One also has $L^{p'}_{\mathrm{per}}(Y) = B^{p'}_{\mathcal{C}_{\mathrm{per}}(Y)} = \mathcal{B}^{p'}_{\mathcal{C}_{\mathrm{per}}(Y)}$ (this is easily seen). By hypothesis (4.31) we deduce that

$$a_i(x,t,\cdot,\cdot,\mu,\lambda) \in L^{p'}_{\mathrm{per}}(Y \times \mathcal{T}) \text{ for all } (x,t) \in \overline{Q},\ (\mu,\lambda) \in \mathbb{R} \times \mathbb{R}^N,\ 0 \leq i \leq N.$$

This suggests us that the appropriate algebra for this study is the periodic algebra $A = \mathcal{C}_{\mathrm{per}}(Y \times \mathcal{T}) = \mathcal{C}_{\mathrm{per}}(Y) \odot \mathcal{C}_{\mathrm{per}}(\mathcal{T})$, the algebra of $Y \times \mathcal{T}$-periodic continuous functions on $\mathbb{R}^N_y \times \mathbb{R}_\tau$. Since A satisfies hypotheses of Theorem 3.6, the conclusion of Theorem 4.9 follows under hypothesis (4.31).

4.4.2. *Problem II.* Let F be a Banach subalgebra of $\mathcal{B}(\mathbb{R}^m)$. Let $\mathcal{B}_\infty(\mathbb{R}^d;F)$ denote the space of all continuous functions $\psi \in \mathcal{C}(\mathbb{R}^d;F)$ such that $\psi(\zeta)$ has a limit in F as $|\zeta| \to \infty$. In particular, it is known that $\mathcal{B}_\infty(\mathbb{R}^d;\mathbb{R}) \equiv \mathcal{B}_\infty(\mathbb{R}^d)$.

With this in mind, our goal here is to homogenize problem (4.1) under the hypothesis

$$\begin{array}{l} a_i(x,t,\cdot,\cdot,\mu,\lambda) \in \mathcal{B}_\infty(\mathbb{R}_\tau; L^{p'}_{\mathrm{per}}(Y)) \text{ for any } (\mu,\lambda) \in \mathbb{R} \times \mathbb{R}^N \\ \text{and for all } (x,t) \in \overline{Q}\ (0 \leq i \leq N) \end{array} \tag{4.32}$$

where $Y = (0,1)^N$.

With this in mind it is an easy task to see that the appropriated algebra here is the product algebra $A = \mathcal{C}_{\mathrm{per}}(Y) \odot \mathcal{B}_\infty(\mathbb{R}_\tau)$, which satisfies hypothesis of Theorem 3.6. The conclusion of Theorem 4.9 follows under the hypothesis (4.32).

4.4.3. *Problem III (Almost periodic homogenization).* Our objective here is to homogenize problem (4.1) under the assumption

$$\begin{array}{l} a_i(x,t,\cdot,\cdot,\mu,\lambda) \in B^{p'}_{AP}(\mathbb{R}^{N+1}_{y,\tau}) \text{ for any } (\mu,\lambda) \in \mathbb{R}^{N+1} \\ \text{and any } (x,t) \in \overline{Q}\ (0 \leq i \leq N), \end{array} \tag{4.33}$$

where $B^{p'}_{AP}(\mathbb{R}^{N+1}_{y,\tau})$ denotes the space of functions in $L^{p'}_{\mathrm{loc}}(\mathbb{R}^{N+1}_{y,\tau})$ that are almost periodic in the Besicovitch sense [6, 7].

Appeal to [40, Corollary 4.1] yields the existence of three countable subgroups $\mathcal{R}_y$ and $\mathcal{R}_\tau$ of $\mathbb{R}^N$ and $\mathbb{R}$ respectively, such that

$$\begin{array}{l} a_i(x,t,\cdot,\cdot,\mu,\lambda) \in B^{p'}_{AP,\mathcal{R}_y \times \mathcal{R}_\tau}(\mathbb{R}^{N+1}_{y,\tau}) \text{ for all } (x,t) \in \overline{Q}, (\mu,\lambda) \in \mathbb{R}^{N+1} \\ (0 \leq i \leq N) \end{array}$$

where $B^{p'}_{AP,\mathcal{R}_y \times \mathcal{R}_\tau}(\mathbb{R}^{N+1}_{y,\tau})$ is the generalized Besicovitch space associated to the algebra w.m.v. $A = AP_{\mathcal{R}_y \times \mathcal{R}_\tau}(\mathbb{R}^{N+1}_{y,\tau})$. This suggests us to take $A = AP_{\mathcal{R}_y}(\mathbb{R}^N_y) \odot AP_{\mathcal{R}_\tau}(\mathbb{R}_\tau)$, an H-algebra satisfying hypotheses of Theorem 3.6. Whence the homogenization of (4.1) under hypothesis (4.33).

Remark 4.6. In all the previous papers dealing with deterministic homogenization theory (see for instance [40, 41, 42, 46]) the almost periodic homogenization problem were stated by combining hypothesis (4.33) above with the following one

$$\begin{array}{l} \text{For each } (\psi_0,\Psi) \in AP(\mathbb{R}^N_y \times \mathbb{R}_\tau)^{N+1} \text{ and each } (x,t) \in \overline{Q} \text{ we have} \\ \sup_{k\in\mathbb{Z}^{N+1}} \int_{k+Y\times\mathcal{T}} (\mid a_i(x,t,y-\eta,\tau-\delta,\psi_0(y,\tau),\Psi(y,\tau)) - \\ \qquad -a_i(x,t,y,\tau,\psi_0(y,\tau),\Psi(y,\tau)) \mid^{p'} dyd\tau) \to 0 \\ \text{as } |\eta| + |\delta| \to 0. \end{array} \tag{4.34}$$

We observe that here we have one significant improvement: hypothesis (4.34) above on the uniform equicontinuity of the a_i is purely dropped. This is a true advance as far as the applications in the almost periodic setting are concerned, and our contribution in this framework is clearly highlighted.

4.4.4. *Problem IV (Weakly almost periodic homogenization I).* In order to rigorously and judiciously set the homogenization problem for (4.1) in that case, however, we need some preliminaries.
Preliminaries. We begin with the

Proposition 4.12. [15] *Let N and m denote two positive integers. Then we have*

$$\begin{aligned} WAP(\mathbb{R}^N)\odot WAP(\mathbb{R}^m) &= \{f\in WAP(\mathbb{R}^N\times\mathbb{R}^m):\{f_y:y\in\mathbb{R}^N\} \\ &\qquad \textit{is relatively compact in } WAP(\mathbb{R}^m)\}\end{aligned}$$

where $f_y\in WAP(\mathbb{R}^m)$ is defined by $f_y(z)=f(y,z),\ z\in\mathbb{R}^m$.

One of the most significant consequence of the preceding result is summarized in the following corollary.

Corollary 4.13. *One has $WAP(\mathbb{R}^N)\odot WAP(\mathbb{R}^m)\subsetneq WAP(\mathbb{R}^N\times\mathbb{R}^m)$.*

Proof. It will be sufficient to exhibit a function $f\in WAP(\mathbb{R}^N\times\mathbb{R}^m)$ such that the set $\mathcal{F}_f=\{f_y:y\in\mathbb{R}^N\}$ is not relatively compact in $WAP(\mathbb{R}^m)$. To this end, in order to construct such an f, we follow closely the proof of [15, Theorem 4.6]. Set $Y=[-1/2,1/2]^N$ and $Z=[-1/2,1/2]^m$. Pick $a_1\in\mathbb{Z}^N$, and for $n\geq 2$, choose inductively $a_n\in\mathbb{Z}^N$ such that

$$a_n\notin\left\{\sum_{i=1}^{7}y_i:y_i\in Y\cup\{\pm a_1,\dots,\pm a_{n-1}\}\right\}.$$

Construct also a sequence $(b_n)_{n\geq1}\subset\mathbb{Z}^m$ such that

$$b_n\notin\left\{\sum_{i=1}^{7}z_i:z_i\in Z\cup\{\pm b_1,\dots,\pm b_{n-1}\}\right\}.$$

Then we have

$$(a_k+Y)\cap(a_n+Y)=\varnothing \text{ and } (b_k+Z)\cap(b_n+Z)=\varnothing \text{ if } n\neq k.$$

Now, choose $g\in\mathcal{C}_0^\infty(\mathbb{R}^N)$ and $h\in\mathcal{C}_0^\infty(\mathbb{R}^m)$ such that $g(0)=h(0)=1$, $0\leq g\leq 1$, $0\leq h\leq 1$, and supp$g\subset Y$, supp$h\subset Z$ (where supp stands for the support). Set

$$f(y,z)=\sum_{n=1}^{\infty}g(y-a_n)h(z-b_n),\ (y,z)\in\mathbb{R}^N\times\mathbb{R}^m.$$

Then by [15, theorem 4.6] we have that $f\in WAP(\mathbb{R}^N\times\mathbb{R}^m)$. But for $z\in\mathbb{R}^m$ we have

$$\begin{aligned} f_{a_k}(z) &= f(a_k,z)=\sum_{n=1}^{\infty}g(a_k-a_n)h(z-b_n) \\ &= h(z-b_k) \text{ since } g(0)=1 \text{ and } (a_k+Y)\cap(a_n+Y)=\varnothing \text{ if } n\neq k. \\ &= (\tau_{b_k}h)(z),\end{aligned}$$

hence $f_{a_k}=\tau_{b_k}h$. Since supp$(\tau_{b_k}h)\subset b_k+Z$ and $(b_k+Z)\cap(b_n+Z)=\varnothing$ if $n\neq k$, we see that $\{f_{a_k}:k\geq 1\}=\{\tau_{b_k}h:k\geq1\}$ is not relatively compact in $\mathcal{B}(\mathbb{R}^m)$. Therefore $f\notin WAP(\mathbb{R}^N)\odot WAP(\mathbb{R}^m)$. Whence the proof. □

Statement of the problem. Let B^p_{WAP} ($B^p_{WAP}(\mathbb{R}^N)$ if there is a danger of confusion) denote the completion of the algebra $WAP(\mathbb{R}^N)$ with respect to the seminorm $\|\cdot\|_p$ (see Section 2.2 for the definition of $\|\cdot\|_p$). It is known that if B^p_{AP} (resp. B^p_0) denotes the space of Besicovitch almost periodic functions (resp. the space of $u \in B^p_{WAP}$ such that $M(|u|) = 0$, which is obtained as the completion with respect to $\|\cdot\|_p$ of the algebra $W_0(\mathbb{R}^N)$) then $B^p_{AP} + B^p_0 \subset B^p_{WAP}$ since $AP(\mathbb{R}^N) \subset WAP(\mathbb{R}^N)$ and $W_0(\mathbb{R}^N) \subset WAP(\mathbb{R}^N)$. It can be shown that $B^p_{WAP} = B^p_{AP} \oplus B^p_0$ (direct sum); see [6, Chap. VIII, Theorem V]. Therefore we have the following characterization of elements in B^p_{WAP}:

- $u \in B^p_{WAP}$ if and only if $u \in L^p_{\mathrm{loc}}(\mathbb{R}^N)$ and the set of translates $\{\tau_a u : a \in \mathbb{R}^N\}$ is weakly relatively compact with respect to the Besicovitch topology defined by the seminorm $\|\|u\|\|_p = \left(\limsup_{r\to+\infty} \frac{1}{|B_r|}\int_{B_r} |u(y)|^p\, dy\right)^{1/p}$.

Such an u's is called *generalized weakly almost periodic function*.

With this in mind, our goal here is to homogenize problem (4.1) under the assumption

$$a_i(x,t,\cdot,\cdot,\mu,\lambda) \in B^{p'}_{WAP}(\mathbb{R}_\tau; B^{p'}_{WAP}(\mathbb{R}^N_y)) \text{ for any } (\mu,\lambda) \in \mathbb{R}\times\mathbb{R}^N \text{ and for all } (x,t) \in \overline{Q}\ (0 \le i \le N). \tag{4.35}$$

Before we can solve the above problem, however we need to clarify the choice of assumption (4.35).

It is known that $AP(\mathbb{R}^N_y \times \mathbb{R}_\tau) = AP(\mathbb{R}^N_y) \odot AP(\mathbb{R}_\tau)$; see Berglund and Milnes [5, Theorem 2.3] for a simple proof. Thus the almost periodic homogenization problem can as was earlier seen in Subsection 4.4.3, be set in terms of the general, say in terms of $B^{p'}_{AP}(\mathbb{R}^{N+1}_{y,\tau})$. However, in the weakly almost periodic setting, this does not hold, that is, one cannot set the above-mentioned problem in terms of $B^{p'}_{WAP}(\mathbb{R}^{N+1}_{y,\tau})$. Indeed by Corollary 4.13 above we have $WAP(\mathbb{R}^N_y) \odot WAP(\mathbb{R}_\tau) \neq WAP(\mathbb{R}^N_y \times \mathbb{R}_\tau)$. That is why assumption (4.35) is the best one suitable to set the weakly almost periodic homogenization problem in our situation.

Now returning to the statement of the problem we see immediately that the suitable algebra for our study is the product algebra $A = WAP(\mathbb{R}^N_y) \odot WAP(\mathbb{R}_\tau)$, an algebra satisfying conditions of Theorem 3.6 in Section 3; see Subsection 3.4. Therefore the homogenization problem for (4.1) is achieved under hypothesis (4.35).

4.4.5. *Problem V (Weakly almost periodic setting II).* The problem to study states here as follows: solve the homogenization problem for (4.1) under the assumption

$$a_i(x,t,\cdot,\cdot,\mu,\lambda) \in B^{p'}_{AP}(\mathbb{R}_\tau; B^{p'}_{WAP}(\mathbb{R}^N_y)) \text{ for any } (\mu,\lambda) \in \mathbb{R}\times\mathbb{R}^N \text{ and for all } (x,t) \in \overline{Q}\ (0 \le i \le N). \tag{4.36}$$

One also reaches the conclusion of Theorem 4.9 under hypothesis (4.36) with $A = WAP(\mathbb{R}^N_y) \odot AP(\mathbb{R}_\tau)$, an algebra w.m.v. satisfying conditions of Theorem 3.6; see Section 3.2.

Remark 4.7. Hypothesis (4.36) contains the particular case (4.33) since $AP(\mathbb{R}^{N+1}_{y,\tau}) = AP(\mathbb{R}^N_y; AP(\mathbb{R}_\tau)) = AP(\mathbb{R}^N_y) \odot AP(\mathbb{R}_\tau)$ and $AP(\mathbb{R}_\tau) \subset B^{p'}_{WAP}(\mathbb{R}_\tau)$.

4.4.6. *Problem VI (Weakly almost periodic setting III: Homogenization in the Fourier-Stieltjes algebra).* We begin by defining the Fourier-Stieltjes algebra on $\mathbb{R}^N$.

Definition 4.1. The Fourier-Stieltjes algebra on $\mathbb{R}^N$ is defined as the closure in $\mathcal{B}(\mathbb{R}^N)$ of the space

$$FS_*(\mathbb{R}^N) = \left\{ f : \mathbb{R}^N \to \mathbb{R},\ f(x) = \int_{\mathbb{R}^N} \exp(ix \cdot y) d\nu(y) \text{ for some } \nu \in \mathcal{M}_*(\mathbb{R}^N) \right\}$$

where $\mathcal{M}_*(\mathbb{R}^N)$ denotes the space of complex valued measures ν with finite total variation: $|\nu|(\mathbb{R}^N) < \infty$. We denote it by $FS(\mathbb{R}^N)$.

Since by [20] any function in $FS_*(\mathbb{R}^N)$ is a weakly almost periodic continuous function, we have that $FS(\mathbb{R}^N) \subset WAP(\mathbb{R}^N)$. Moreover thanks to [14, Theorem 4.5] $FS(\mathbb{R}^N)$ is a proper subalgebra of $WAP(\mathbb{R}^N)$, such that the question raised by Frid and Silva [26] to know whether there exist ergodic algebras that are not subalgebras of $FS(\mathbb{R}^N)$ find its answer here: The ergodic algebra $WAP(\mathbb{R}^N)$ fulfills the above required conditions since it contains $FS(\mathbb{R}^N)$ as proper ergodic subalgebra.

As $FS(\mathbb{R}^N)$ is an ergodic algebra which is translation invariant (this is easily seen: indeed $FS_*(\mathbb{R}^N)$ is translation invariant) we see that hypotheses of Theorem 3.6 are satisfied with any algebra $A = FS(\mathbb{R}^N) \odot A_\tau$, A_τ being any algebra w.m.v. on $\mathbb{R}_\tau$.

With all this in mind, our concern here is to solve the homogenization problem for (4.1) under the assumption

$$\begin{array}{l} a_i(x,t,\cdot,\cdot,\mu,\lambda) \in B^{p'}_{A_\tau}(\mathbb{R}_\tau; B^{p'}_{FS}(\mathbb{R}^N_y)) \text{ for any } (\mu,\lambda) \in \mathbb{R} \times \mathbb{R}^N \\ \text{and for all } (x,t) \in \overline{Q}\ (0 \le i \le N). \end{array} \tag{4.37}$$

where: $B^{p'}_{FS}(\mathbb{R}^N_y)$ denotes the completion of the algebra $FS(\mathbb{R}^N_y)$ with respect to the seminorm $\|\cdot\|_{p'}$, and A_τ is any arbitrary algebra w.m.v. on $\mathbb{R}_\tau$. Then arguing as in the preceding Subsections we reach the conclusion of Theorem 4.9 under hypothesis (4.37) with $A = FS(\mathbb{R}^N) \odot A_\tau$.

4.4.7. *Concluding remarks.* In this paper we have proved a new compactness theorem (Theorem 3.6) which enables us to solve new homogenization problems associated to a parabolic degenerate-type operator. We also state its stationary analogue (Theorem 3.7) which can permits us to handle homogenization problems for elliptic operators. We have applied Theorem 3.6 to solve several new problems (to the best of our knowledge this is the first time that such problems are considered); see particularly Problems IV-VI. These problems involve the algebra $W_0(\mathbb{R}^m)$ which is nonseparable. Therefore, as pointed out in Section 3, the above-mentioned problems cannot be handled by the theory of H-algebras. This is a true advance as far as the applications are concerned.

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Mamadou Sango, Department of Mathematics and Applied Mathematics, University of Pretoria/Mamelodi Campus, Pretoria 0002, South Africa
E-mail address: mamadou.sango@up.ac.za

N. Svanstedt, Department of Mathematical Sciences, Göteborg University, SE-412 96 Göteborg, Sweden
E-mail address: nilss@math.chalmers.se

Jean Louis Woukeng, Department of Mathematics and Computer Science, University of Dschang, P.O. Box 67, Dschang, Cameroon
Current address: Jean Louis Woukeng, Department of Mathematics and Applied Mathematics, University of Pretoria/Mamelodi Campus, Pretoria 0002, South Africa
E-mail address: jwoukeng@yahoo.fr