Distribution of the product of determinants of noncentral bimatrix beta variates

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Abstract - The product moments of existing and new noncentral bimatrix variate beta distributions with bounded domain are derived. From these, exact expressions for the distributions of statistics are obtained by using the Mellin transform. These distributions add value to multivariate statistical analysis with specific reference to factors of Wilks' statistics and the product of generalized statistics.

Keywords and phrases - Bimatrix variate beta distributions; cumulative distribution function; Fox's H-function; Meijer's G-function; Mellin transforms; product of beta determinants; test statistics; Wilks' statistic.

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1 Introduction

A large variety of hypothesis tests in multivariate analysis make use of the likelihood ratio method to derive appropriate test criteria. Several of the test statistics used are functions of the determinant or product of determinants of matrix or bimatrix beta variates respectively ([1],[30]), the best known of these statistics is the Wilks' statistic [39] defined as $\Lambda \equiv \left|\frac{\boldsymbol{S}}{\boldsymbol{S}+\boldsymbol{B}}\right| = |\boldsymbol{U}|$ with $\boldsymbol{S}$ and $\boldsymbol{B}$ two independent $(p \times p)$ Wishart matrices, i.e. $\boldsymbol{S} \sim W_p(n, \boldsymbol{I}_p)$ and $\boldsymbol{B} \sim W_p(m, \boldsymbol{I}_p; \boldsymbol{\Theta})$, where $\boldsymbol{\Theta}$ is the noncentrality parameter and $n, m \geq p$. Note that $\boldsymbol{U} = (\boldsymbol{S}+\boldsymbol{B})^{-\frac{1}{2}} \boldsymbol{S} (\boldsymbol{S}+\boldsymbol{B})^{-\frac{1}{2}}$ has the noncentral matrix variate beta type I distribution ($\boldsymbol{A}^{\frac{1}{2}}$ is the unique positive definite square root of $\boldsymbol{A}$). The distribution under the nonnull hypothesis is of importance when calculating the power of the test and [3] gave an exact expression for the nonnull distribution of the Wilks' statistic.

In this paper, the main focus is on deriving exact distributions of statistics that developed within the noncentral bimatrix beta variates paradigm. Firstly, Bekker, Roux, Ehlers and Arashi [4] defined the product of two *dependent* Wilks' statistics, i.e.

$$\Lambda_1 \equiv \left|\frac{\boldsymbol{S}_1}{\boldsymbol{S}_1+\boldsymbol{B}}\right| \left|\frac{\boldsymbol{S}_2}{\boldsymbol{S}_2+\boldsymbol{B}}\right| = |\boldsymbol{X}_1 \boldsymbol{X}_2|, \tag{1}$$

where $\boldsymbol{S}_i \sim W_p(n_i, \boldsymbol{I}_p)$, $i = 1, 2$ and $\boldsymbol{B} \sim W_p(m, \boldsymbol{I}_p)$ are independent, and derived an exact expression for the density function of Λ_1. Suppose the columns of a $(p \times m)$ matrix $\boldsymbol{Z}$, a $(p \times n_1)$ matrix $\boldsymbol{Y}_1$, and a $(p \times n_2)$ matrix $\boldsymbol{Y}_2$ are distributed independently in a p-variate normal distribution with a common positive definite covariance matrix $\boldsymbol{\Sigma}$. Also, let $E(\boldsymbol{Z}) = \boldsymbol{M}$, $E(\boldsymbol{Y}_1) = \boldsymbol{0}$ and $E(\boldsymbol{Y}_2) = \boldsymbol{0}$. Then, both Wilks' statistics

$$\Lambda_a = \frac{|\boldsymbol{Y}_1 \boldsymbol{Y}_1'|}{|\boldsymbol{Y}_1 \boldsymbol{Y}_1' + \boldsymbol{Z}\boldsymbol{Z}'|} \text{ and } \Lambda_b = \frac{|\boldsymbol{Y}_2 \boldsymbol{Y}_2'|}{|\boldsymbol{Y}_2 \boldsymbol{Y}_2' + \boldsymbol{Z}\boldsymbol{Z}'|} \tag{2}$$

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can be used to test $H_0 : \boldsymbol{M} = \boldsymbol{0}$ vs $H_A : \boldsymbol{M} \neq \boldsymbol{0}$. Now, in order to use the information available in matrices $\boldsymbol{Y}_1$ and $\boldsymbol{Y}_2$, designate the matrix $\boldsymbol{T} = [\boldsymbol{Y}_1 : \boldsymbol{Y}_2]$. Then one might use the product of two dependent Wilks' statistics, i.e. $\Lambda_1 = \Lambda_a \Lambda_b$ as the likelihood criteria for testing H_0 vs H_A. For example, suppose the original testing problem was based on Λ_a. But now, suppose we are interested in including the information, that were neglected in the beginning phase of the analysis, of the p parameters (that are specific to the characteristics that are examined) from the other source of n_2 experimental units. Also consider also the following case: columns of a $(2p \times m)$ matrix $\boldsymbol{Z}$, a $(2p \times n)$ matrix $\boldsymbol{Y}$ are distributed independently in a $2p$-variate normal distribution with a common positive definite covariance matrix $diag\,(\boldsymbol{\Sigma}, \boldsymbol{\Sigma})$, and let $E(\boldsymbol{Z}) = \boldsymbol{M}$, $E(\boldsymbol{Y}) = \boldsymbol{0}$. Also let the trio matrices $\boldsymbol{Z}$, $\boldsymbol{Y}$ and $\boldsymbol{M}$ be partitioned as:

$$\boldsymbol{Z} = \begin{array}{c} \begin{bmatrix} \boldsymbol{Z}_1 \\ \boldsymbol{Z}_2 \end{bmatrix} \end{array} \begin{array}{c} p \\ p \end{array}, \quad \boldsymbol{Y} = \begin{bmatrix} \boldsymbol{Y}_1 \\ \boldsymbol{Y}_2 \end{bmatrix} \begin{array}{c} p \\ p \end{array}, \quad \boldsymbol{M} = \begin{bmatrix} \boldsymbol{M}_1 \\ \boldsymbol{M}_2 \end{bmatrix} \begin{array}{c} p \\ p \end{array},$$

Consider the following hypotheses

$$\begin{cases} H_o^* : \boldsymbol{M} = \boldsymbol{0} \\ H_A^* : \boldsymbol{M}_1 \neq \boldsymbol{0}, H_1^* : \boldsymbol{M}_2 \neq \boldsymbol{0} \end{cases} \text{,equivalently} \begin{cases} H_o^* : \boldsymbol{M} = \boldsymbol{0} \\ H_A^c : \boldsymbol{M}_1 \neq \boldsymbol{0}, \boldsymbol{M}_2 \neq \boldsymbol{0}. \end{cases}$$

It is important to note that the alternative hypothesis H_A^c in the above is different from that of $\boldsymbol{M} \neq \boldsymbol{0}$. Take $\boldsymbol{M}_1 = (\boldsymbol{M}_{11}, \ldots, \boldsymbol{M}_{1p_1})'$, and $\boldsymbol{M}_2 = (\boldsymbol{M}_{21}, \ldots, \boldsymbol{M}_{2p_2})'$. If $\boldsymbol{M}_1 = \boldsymbol{0}$, then $\boldsymbol{M} = \boldsymbol{0}$, however H_A^c occurs if at least for one $i = 1, \ldots, p_1$ and $j = 1, \ldots, p_2$ $(\boldsymbol{M}_{1i} = \boldsymbol{0}, \boldsymbol{M}_{2j} = \boldsymbol{0})$. Thus the hypotheses above is different from $H_0 : \boldsymbol{M} = \boldsymbol{0}$ vs $H_A : \boldsymbol{M} \neq \boldsymbol{0}$. Then the test Wilks' statistic for the hypotheses H_o^* vs H_A^c may be designated as the product of two dependent Wilks' statistics, similarly as in (2). For $\mathbb{X} = (\boldsymbol{X}_1 : \boldsymbol{X}_2)'$, where $\boldsymbol{X}_i = (\boldsymbol{S}_i + \boldsymbol{B})^{-\frac{1}{2}} \boldsymbol{S}_i (\boldsymbol{S}_i + \boldsymbol{B})^{-\frac{1}{2}}$, $i = 1, 2$, it is said to have the bimatrix variate beta type IV distribution. The latter distribution has been studied independently by [4], [16] and [24]. For $\boldsymbol{B} \sim W_p(m, \boldsymbol{I}_p; \boldsymbol{\Theta})$, $\mathbb{X} = (\boldsymbol{X}_1 : \boldsymbol{X}_2)'$ has the *noncentral bimatrix variate beta type IV distribution*, studied by [15]. In this paper, we derive the density function and the cumulative distribution function (CDF) of $\Lambda_1 \equiv |\boldsymbol{X}_1 \boldsymbol{X}_2|$ in terms of Meijer's G-function for this noncentral case.

Secondly, let $\boldsymbol{S}_i \sim W_p(n_i, \boldsymbol{I}_p)$, $i = 1, 2$, and $\boldsymbol{B} \sim W_p(m, \boldsymbol{I}_p)$ independent, and let $\boldsymbol{U}_i = (\boldsymbol{S}_1 + \boldsymbol{S}_2 + \boldsymbol{B})^{-\frac{1}{2}} \boldsymbol{S}_i (\boldsymbol{S}_1 + \boldsymbol{S}_2 + \boldsymbol{B})^{-\frac{1}{2}}$, $i = 1, 2$, then the distribution of $\mathbb{U} = (\boldsymbol{U}_1 : \boldsymbol{U}_2)'$ is known as the *bimatrix variate beta type I distribution.* The statistic

$$\Lambda_2 \equiv \left| \frac{\boldsymbol{S}_1}{\boldsymbol{S}_1 + \boldsymbol{S}_2 + \boldsymbol{B}} \right|^{\frac{1}{2}n_1} \left| \frac{\boldsymbol{S}_2}{\boldsymbol{S}_1 + \boldsymbol{S}_2 + \boldsymbol{B}} \right|^{\frac{1}{2}n_2} = |\boldsymbol{U}_1|^{\frac{1}{2}n_1} |\boldsymbol{U}_2|^{\frac{1}{2}n_2}, \tag{3}$$

arises when testing whether two normal populations are identical [1]. Testing that two normal distributions are identical has an important place in multivariate analysis (see [36], pp. 1238 and [19]). In this paper, we derive an *exact* expression for the density function and CDF of Λ_2 when $\boldsymbol{B} \sim W_p(m, \boldsymbol{I}_p; \boldsymbol{\Theta})$ (see [17]).

In this paper we focus on the distributional aspect of the generalized statistic Λ_3, where the covariance matrices are *not equal*; we consider the case of proportional covariance matrices. More specifically, $\boldsymbol{S}_i^* \sim W_p(n_i, \alpha_i \boldsymbol{I}_p)$, $i = 1, 2$, and $\boldsymbol{B}^* \sim W_p(m, c\boldsymbol{I}_p; \boldsymbol{\Theta}^*)$ are independent $(\alpha_1, \alpha_2, c > 0)$, with

$$\Lambda_3 \equiv \left| \frac{\boldsymbol{S}_1^*}{\boldsymbol{S}_1^* + \boldsymbol{S}_2^* + \boldsymbol{B}^*} \right|^{\frac{1}{2}n_1} \left| \frac{\boldsymbol{S}_2^*}{\boldsymbol{S}_1^* + \boldsymbol{S}_2^* + \boldsymbol{B}^*} \right|^{\frac{1}{2}n_2}. \tag{4}$$

The study of Λ_3 is a theoretical development of Λ_1 and Λ_2, proposing a more general statistic, with different covariances matrices. The application is still to be explored.

Expression (4) can also be written as

$$\Lambda_3 \equiv \left|\frac{\alpha_1 \boldsymbol{S}_1}{\alpha_1 \boldsymbol{S}_1 + \alpha_2 \boldsymbol{S}_2 + c\boldsymbol{B}}\right|^{\frac{1}{2}n_1} \left|\frac{\alpha_2 \boldsymbol{S}_2}{\alpha_1 \boldsymbol{S}_1 + \alpha_2 \boldsymbol{S}_2 + c\boldsymbol{B}}\right|^{\frac{1}{2}n_2}, \tag{5}$$

with $\boldsymbol{S}_i \sim W_p(n_i, \boldsymbol{I}_p)$, $i = 1, 2$ and $\boldsymbol{B} \sim W_p(m, \boldsymbol{I}_p, \boldsymbol{\Theta})$ independent. This leads to the definition of the *noncentral bimatrix variate beta type V distribution.* In this paper, the density function for this proposed distribution is derived as well as the density function and CDF of Λ_3.

The rest of the paper is organized as follows: In section 2 and 3 the exact expressions for the density functions and CDF's of (1) and (3) if $\boldsymbol{B} \sim W_p(m, \boldsymbol{I}_p; \boldsymbol{\Theta})$ are derived respectively. Subsequently, the noncentral bimatrix variate beta type V distribution is proposed in section 4 and used to derive the density function and CDF for (5). The expressions are given in terms of Meijer's G-function, Fox's H-function, zonal polynomials, hypergeometric functions with matrix argument, or homogeneous invariant polynomials with two or more matrix arguments. The reader is referred to the papers ([6],[7],[8],[9],[10],[11],[26],[27],[28]) on these functions; as well as the reference books ([23],[32],[34]). These density functions of (1), (3) and (5) are complemented with graphical representations in the bivariate as well as the bimatrix case. Note there is no loss of generality in assuming $\boldsymbol{\Sigma} = \boldsymbol{I}_p$ in the derivation of the density function of Λ_i $(i = 1, 2, 3)$.

2 Density function of Λ_1

For $\boldsymbol{S}_i \sim W_p(n_i, \boldsymbol{I}_p)$, $i = 1, 2$ and $\boldsymbol{B} \sim W_p(m, \boldsymbol{I}_p, \boldsymbol{\Theta})$ independent, let

$$\boldsymbol{X}_i = (\boldsymbol{S}_i + \boldsymbol{B})^{-\frac{1}{2}} \boldsymbol{S}_i (\boldsymbol{S}_i + \boldsymbol{B})^{-\frac{1}{2}}, \ i = 1, 2, \tag{6}$$

then $\mathbb{X} = (\boldsymbol{X}_1 : \boldsymbol{X}_2)'$ has the noncentral bimatrix variate beta type IV distribution, denoted as $\mathbb{X} \sim BB_p^{IV}(n_1, n_2, m; \boldsymbol{\Theta})$. The density function is given by

$$\begin{aligned} & f(\boldsymbol{X}_1, \boldsymbol{X}_2) \\ = \ & \left\{\beta_p\left(\tfrac{n_1}{2}, \tfrac{n_2}{2}; \tfrac{m}{2}\right)\right\}^{-1} \prod_{i=1}^{2} |\boldsymbol{X}_i|^{\frac{1}{2}n_i - \frac{1}{2}(p+1)} |\boldsymbol{I}_p - \boldsymbol{X}_1|^{\frac{1}{2}(n_2+m) - \frac{1}{2}(p+1)} |\boldsymbol{I}_p - \boldsymbol{X}_2|^{\frac{1}{2}(n_1+m) - \frac{1}{2}(p+1)} \\ & \cdot |\boldsymbol{I}_p - \boldsymbol{X}_1 \boldsymbol{X}_2|^{-\frac{1}{2}(n_1+n_2+m)} \operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) {}_1F_1\left(\tfrac{n_1+n_2+m}{2}; \tfrac{m}{2}; \tfrac{1}{2}\left[\boldsymbol{I}_p + \sum_{i=1}^{2} \boldsymbol{X}_i (\boldsymbol{I}_p - \boldsymbol{X}_i)^{-1}\right]^{-1} \boldsymbol{\Theta}\right), \end{aligned} \tag{7}$$

$\boldsymbol{0} < \boldsymbol{X}_i < \boldsymbol{I}_p$, $i = 1, 2$, where $n_i > (p-1)$, $i = 1, 2$, $m > (p-1)$, ${}_1F_1(\cdot)$ is the confluent hypergeometric function of matrix argument, $\beta_p(a, b; c) = \frac{\Gamma_p(a)\Gamma_p(b)\Gamma_p(c)}{\Gamma_p(a+b+c)}$ denotes the multivariate beta function, $\Gamma_p(a)$ represents the multivariate gamma function ($\Gamma_p(a) = \int_{\boldsymbol{A} > \boldsymbol{0}} etr(-\boldsymbol{A}) |\boldsymbol{A}|^{a - \frac{1}{2}(p+1)} \mathrm{d}\boldsymbol{A} = \pi^{\frac{1}{4}p(p-1)} \prod_{i=1}^{p} \Gamma\left(a - \frac{1}{2}(i-1)\right)$, $\operatorname{Re}(a) > \frac{1}{2}(p-1)$, see [14], Eq. 2.3). Firstly, we derive the $(h_1, h_2)^{th}$ product moment, $E\left(|\boldsymbol{X}_1|^{h_1} |\boldsymbol{X}_2|^{h_2}\right)$, and use this in an inverse Mellin transform to obtain the density function for Λ_1 (see (1)) in terms of Meijer's G-function. Note that to test equality of the dispersion matrices of two p-variate normal populations ([1], pp.405), the test statistic is based on the product of two dependent Wilks' statistics but it differs from Λ_1. Exact expressions for the density function of two *independent* generalized Wilks' statistics under the null hypothesis was derived by [35].

Lemma 2.1
If $(\boldsymbol{X}_1, \boldsymbol{X}_2) \sim BB_p^{IV}(n_1, n_2, m; \boldsymbol{\Theta})$ *then* $E\left(|\boldsymbol{X}_1|^{h_1}|\boldsymbol{X}_2|^{h_2}\right)$ *is given by*

$$\frac{\left[\Gamma_p\left(\frac{p+1}{2}\right)\right]^2}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)}\operatorname{etr}\left(-\frac{1}{2}\boldsymbol{\Theta}\right)\sum_{\kappa,\lambda,\tau,\rho,J;\phi,\phi^*}\frac{1}{k!t!j!}\binom{\kappa}{\lambda}\binom{\tau}{\rho}g^{\phi}_{\lambda,\rho}\theta^{J,\phi}_{\phi^*}$$
$$\cdot\frac{C_\kappa(\boldsymbol{I}_p)}{C_\lambda(\boldsymbol{I}_p)}\frac{C_\tau(\boldsymbol{I}_p)}{C_\rho(\boldsymbol{I}_p)}\frac{\Gamma_p\left(\frac{n_1}{2}+\frac{p+1}{2},\kappa\right)}{\Gamma_p\left(\frac{n_1}{2}+\frac{p+1}{2},\lambda\right)}\frac{\Gamma_p\left(\frac{n_1}{2}+h_1,\kappa\right)}{\Gamma_p\left(\frac{n_1}{2}+h_1+\frac{p+1}{2},\kappa\right)}\frac{\Gamma_p\left(\frac{n_2}{2}+\frac{p+1}{2},\tau\right)}{\Gamma_p\left(\frac{n_2}{2}+\frac{p+1}{2},\rho\right)}\frac{\Gamma_p\left(\frac{n_2}{2}+h_2,\tau\right)}{\Gamma_p\left(\frac{n_2}{2}+h_2+\frac{p+1}{2},\tau\right)}\frac{\Gamma_p\left(\frac{n_1+n_2+m}{2},\phi^*\right)}{\Gamma_p\left(\frac{m}{2},J\right)}C^{J,\phi}_{\phi^*}\left(\frac{1}{2}\boldsymbol{\Theta},-\boldsymbol{I}_p\right), \tag{8}$$

where $\sum_{\kappa,\lambda,\tau,\rho,J;\phi,\phi^*}=\sum_{k=0}^{\infty}\sum_{\kappa}\sum_{l=0}^{k}\sum_{\lambda}\sum_{t=0}^{\infty}\sum_{\tau}\sum_{r=0}^{t}\sum_{\rho}\sum_{\phi\in\lambda\cdot\rho}\sum_{j=0}^{\infty}\sum_{J}\sum_{\phi^*\in J\cdot\phi}$, $C_\kappa(\cdot)$ *is the zonal polynomial corresponding to* κ *[27],* $C^{J,\phi}_{\phi^*}(\cdot)$ *denotes the invariant polynomial defined by [9],[9],[11] (see also [6]) and* $\theta^{J,\phi}_{\phi^*}$ *and* $g^{\phi}_{\lambda,\rho}$ *as defined in [7].*
Furthermore, the generalized gamma function of weight κ *can be expressed as* $\Gamma_p(a,\kappa)=\pi^{\frac{1}{4}p(p-1)}\prod_{i=1}^{p}\Gamma\left(a+k_i-\frac{1}{2}(i-1)\right)=\Gamma_p(a)(a)_\kappa$, $\left(\operatorname{Re}(a)\geq\frac{p-1}{2}-k_p\right)$ *with the generalized hypergeometric coefficient given by* $(a)_\kappa=\prod_{i=1}^{m}\left(a-\frac{1}{2}(i-1)\right)_{k_i}$ *where* $(a)_k=a(a-1)\ldots(a+k-1)$, $(a)_0=1$.

Proof:
The density of $(\boldsymbol{S}_1,\boldsymbol{S}_2,\boldsymbol{B})$ is given by

$$K\prod_{i=1}^{2}\left[etr\left(-\frac{1}{2}\boldsymbol{S}_i\right)|\boldsymbol{S}_i|^{\frac{1}{2}(n_i-p-1)}\right]\left[etr\left(-\frac{1}{2}\boldsymbol{\Theta}\right)etr\left(-\frac{1}{2}\boldsymbol{B}\right)|\boldsymbol{B}|^{\frac{1}{2}(m-p-1)}\ {}_0F_1\left(\frac{m}{2};\frac{1}{4}\boldsymbol{\Theta}\boldsymbol{B}\right)\right] \tag{9}$$

where $K^{-1}=\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)\Gamma_p\left(\frac{m}{2}\right)2^{\frac{1}{2}(n_1+n_2+m)p}$ ([23], Eq.3.5.1) and ${}_0F_1(\cdot)$ is the hypergeometric function of a matrix argument.

On performing the transformations (6) with Jacobian $J(\boldsymbol{S}_1,\boldsymbol{S}_2\to\boldsymbol{X}_1,\boldsymbol{X}_2)=|\boldsymbol{B}|^{(p+1)}\prod_{i=1}^{2}|\boldsymbol{I}_p-\boldsymbol{X}_i|^{-(p+1)}$, it follows from (9) that

$$E\left(|\boldsymbol{X}_1|^{h_1}|\boldsymbol{X}_2|^{h_2}\right)= Ketr\left(-\frac{1}{2}\boldsymbol{\Theta}\right)\int_{\boldsymbol{B}>\boldsymbol{0}}|\boldsymbol{B}|^{\frac{1}{2}(n_1+n_2+m)-\frac{1}{2}(p+1)}etr\left(-\frac{1}{2}\boldsymbol{B}\right)\ {}_0F_1\left(\frac{m}{2};\frac{1}{4}\boldsymbol{\Theta}\boldsymbol{B}\right)$$
$$\cdot\int_{\boldsymbol{0}<\boldsymbol{X}_1<\boldsymbol{I}_p}u(\boldsymbol{X}_1,\boldsymbol{B})\mathrm{d}\boldsymbol{X}_1\int_{\boldsymbol{0}<\boldsymbol{X}_2<\boldsymbol{I}_p}u(\boldsymbol{X}_2,\boldsymbol{B})\mathrm{d}\boldsymbol{X}_2\mathrm{d}\boldsymbol{B}, \tag{10}$$

where K^{-1} is defined as before and
$u(\boldsymbol{X}_i,\boldsymbol{B})=|\boldsymbol{X}_i|^{\frac{1}{2}n_i+h_i-\frac{1}{2}(p+1)}|\boldsymbol{I}_p-\boldsymbol{X}_i|^{-\frac{1}{2}n_i-\frac{1}{2}(p+1)}etr\left[-\frac{1}{2}\boldsymbol{B}\boldsymbol{X}_i(\boldsymbol{I}_p-\boldsymbol{X}_i)^{-1}\right]$, $i=1,2$.
For any $\boldsymbol{H}\in O(p)=\left\{\boldsymbol{H}\in R^{p\times p}|\boldsymbol{H}'\boldsymbol{H}=\boldsymbol{H}\boldsymbol{H}'=\boldsymbol{I}_p\right\}$ ($\mathrm{d}\boldsymbol{H}$ denotes the normalised Haar measure on $O(p)$ ([34], pp.60), and using ([7], Eq.30), ([27], Eq.29), ([23], Eq.1.6.2) and ([34], pp.283, Eq.7) it follows that

$$\begin{aligned}
&\int_{\boldsymbol{0}<\boldsymbol{X}_1<\boldsymbol{I}_p}u(\boldsymbol{X}_1,\boldsymbol{B})\,\mathrm{d}\boldsymbol{X}_1\\
=\ &\int_{\boldsymbol{0}<\boldsymbol{X}_1<\boldsymbol{I}_p}\int_{O(p)}u\left(\boldsymbol{X}_1,\boldsymbol{H}\boldsymbol{B}\boldsymbol{H}'\right)\mathrm{d}\boldsymbol{H}\mathrm{d}\boldsymbol{X}_1\\
=\ &\int_{\boldsymbol{0}<\boldsymbol{X}_1<\boldsymbol{I}_p}|\boldsymbol{X}_1|^{\frac{1}{2}n_1+h_1-\frac{1}{2}(p+1)}|\boldsymbol{I}_p-\boldsymbol{X}_1|^{-\frac{1}{2}n_1-\frac{1}{2}(p+1)}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{1}{k!}\frac{C_\kappa\left(-\frac{1}{2}\boldsymbol{B}\right)C_\kappa\left[\boldsymbol{X}_1(\boldsymbol{I}_p-\boldsymbol{X}_1)^{-1}\right]}{C_\kappa(\boldsymbol{I}_p)}\mathrm{d}\boldsymbol{X}_1\\
=\ &\int_{\boldsymbol{0}<\boldsymbol{X}_1<\boldsymbol{I}_p}|\boldsymbol{X}_1|^{\frac{1}{2}n_1+h_1-\frac{1}{2}(p+1)}|\boldsymbol{I}_p-\boldsymbol{X}_1|^{-\frac{1}{2}n_1-\frac{1}{2}(p+1)}\ {}_0F_0^{(p)}\left(-\frac{1}{2}\boldsymbol{B},\boldsymbol{X}_1(\boldsymbol{I}_p-\boldsymbol{X}_1)^{-1}\right)\mathrm{d}\boldsymbol{X}_1\\
=\ &\sum_{k=0}^{\infty}\sum_{\kappa}\frac{1}{k!}L_\kappa^{\frac{1}{2}n_1}\left(\frac{1}{2}\boldsymbol{B}\right)\int_{\boldsymbol{0}<\boldsymbol{X}_1<\boldsymbol{I}_p}|\boldsymbol{X}_1|^{\frac{1}{2}n_1+h_1-\frac{1}{2}(p+1)}\frac{C_\kappa(\boldsymbol{X}_1)}{C_\kappa(\boldsymbol{I}_p)}\mathrm{d}\boldsymbol{X}_1
\end{aligned}$$

where $L_\kappa^\gamma(\cdot)$ is the Laguerre polynomial of a symmetric matrix [34]. Next, using ([9], Eq.3.2) and ([23], Eq.1.7.4) we have

$$\begin{aligned}\int_{\mathbf{0}<\boldsymbol{X}_1<\boldsymbol{I}_p} u(\boldsymbol{X}_1,\boldsymbol{B})\,\mathrm{d}\boldsymbol{X}_1 &= \sum_{k=0}^{\infty}\sum_{\kappa}\frac{1}{k!}L_\kappa^{\frac{1}{2}n_1}\left(\tfrac{1}{2}\boldsymbol{B}\right)\frac{\Gamma_p\left(\frac{n_1}{2}+h_1,\kappa\right)\Gamma_p\left(\frac{p+1}{2}\right)}{\Gamma_p\left(\frac{n_1}{2}+h_1+\frac{p+1}{2},\kappa\right)}\\ &= \sum_{k=0}^{\infty}\sum_{\kappa}\sum_{l=0}^{k}\sum_{\lambda}\frac{1}{k!}\binom{\kappa}{\lambda}\frac{C_\kappa(\boldsymbol{I}_p)}{C_\lambda(\boldsymbol{I}_p)}\frac{\Gamma_p\left(\frac{n_1}{2}+\frac{p+1}{2},\kappa\right)}{\Gamma_p\left(\frac{n_1}{2}+\frac{p+1}{2},\lambda\right)}\frac{\Gamma_p\left(\frac{n_1}{2}+h_1,\kappa\right)\Gamma_p\left(\frac{p+1}{2}\right)}{\Gamma_p\left(\frac{n_1}{2}+h_1+\frac{p+1}{2},\kappa\right)}C_\lambda(-\tfrac{1}{2}\boldsymbol{B}).\end{aligned} \tag{11}$$

Similarly,

$$\int_{\mathbf{0}<\boldsymbol{X}_2<\boldsymbol{I}_p} u(\boldsymbol{X}_2,\boldsymbol{B})\,d\boldsymbol{X}_2 = \sum_{t=0}^{\infty}\sum_{\tau}\sum_{r=0}^{t}\sum_{\rho}\frac{1}{t!}\binom{\tau}{\rho}\frac{C_\tau(\boldsymbol{I}_p)}{C_\rho(\boldsymbol{I}_p)}\frac{\Gamma_p\left(\frac{n_2}{2}+\frac{p+1}{2},\tau\right)}{\Gamma_p\left(\frac{n_2}{2}+\frac{p+1}{2},\rho\right)}\frac{\Gamma_p\left(\frac{n_2}{2}+h_2,\tau\right)\Gamma_p\left(\frac{p+1}{2}\right)}{\Gamma_p\left(\frac{n_2}{2}+h_2+\frac{p+1}{2},\tau\right)}C_\rho(-\tfrac{1}{2}\boldsymbol{B}). \tag{12}$$

Substituting (11) and (12) in (10) yields

$$\begin{aligned}&E\left(|\boldsymbol{X}_1|^{h_1}|\boldsymbol{X}_2|^{h_2}\right)\\ &= K etr\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\left[\Gamma_p\left(\tfrac{p+1}{2}\right)\right]^2\sum_{k=0}^{\infty}\sum_{\kappa}\sum_{l=0}^{k}\sum_{\lambda}\frac{1}{k!}\binom{\kappa}{\lambda}\frac{C_\kappa(\boldsymbol{I}_p)}{C_\lambda(\boldsymbol{I}_p)}\frac{\Gamma_p\left(\frac{n_1}{2}+\frac{p+1}{2},\kappa\right)}{\Gamma_p\left(\frac{n_1}{2}+\frac{p+1}{2},\lambda\right)}\frac{\Gamma_p\left(\frac{n_1}{2}+h_1,\kappa\right)}{\Gamma_p\left(\frac{n_1}{2}+h_1+\frac{p+1}{2},\kappa\right)}\\ &\quad\cdot\sum_{t=0}^{\infty}\sum_{\tau}\sum_{r=0}^{t}\sum_{\rho}\frac{1}{t!}\binom{\tau}{\rho}\frac{C_\tau(\boldsymbol{I}_p)}{C_\rho(\boldsymbol{I}_p)}\frac{\Gamma_p\left(\frac{n_2}{2}+\frac{p+1}{2},\tau\right)}{\Gamma_p\left(\frac{n_2}{2}+\frac{p+1}{2},\rho\right)}\frac{\Gamma_p\left(\frac{n_2}{2}+h_2,\tau\right)}{\Gamma_p\left(\frac{n_2}{2}+h_2+\frac{p+1}{2},\tau\right)}\int_{\boldsymbol{B}>\mathbf{0}}v(\boldsymbol{B})\mathrm{d}\boldsymbol{B}.\end{aligned} \tag{13}$$

Applying ([9], Eq.2.8 and Eq.2.10), ([7], Eq.25) and ([6], Eq.3.21) it follows that

$$\begin{aligned}&\int_{\boldsymbol{B}>\mathbf{0}}v(\boldsymbol{B})\mathrm{d}\boldsymbol{B}\\ &= \int_{\boldsymbol{B}>\mathbf{0}}|\boldsymbol{B}|^{\frac{1}{2}(n_1+n_2+m)-\frac{1}{2}(p+1)}etr\left(-\tfrac{1}{2}\boldsymbol{B}\right)\,{}_0F_1\left(\tfrac{m}{2};\tfrac{1}{4}\boldsymbol{\Theta}\boldsymbol{B}\right)C_\lambda\left(-\tfrac{1}{2}\boldsymbol{B}\right)C_\rho\left(-\tfrac{1}{2}\boldsymbol{B}\right)\mathrm{d}\boldsymbol{B}\\ &= \sum_{\phi\in\lambda\cdot\rho}g_{\lambda,\rho}^{\phi}\int_{\boldsymbol{B}>\mathbf{0}}|\boldsymbol{B}|^{\frac{1}{2}(n_1+n_2+m)-\frac{1}{2}(p+1)}etr\left(-\tfrac{1}{2}\boldsymbol{B}\right)\,{}_0F_1\left(\tfrac{m}{2};\tfrac{1}{4}\boldsymbol{\Theta}\boldsymbol{B}\right)C_\phi\left(-\tfrac{1}{2}\boldsymbol{B}\right)\mathrm{d}\boldsymbol{B}\\ &= \sum_{\phi\in\lambda\cdot\rho}g_{\lambda,\rho}^{\phi}\sum_{j=0}^{\infty}\sum_{J}\frac{1}{j!}\frac{1}{\left(\frac{m}{2}\right)_J}\sum_{\phi^*\in J\cdot\phi}\theta_{\phi^*}^{J,\phi}\int_{\boldsymbol{B}>\mathbf{0}}|\boldsymbol{B}|^{\frac{1}{2}(n_1+n_2+m)-\frac{1}{2}(p+1)}etr\left(-\tfrac{1}{2}\boldsymbol{B}\right)\;C_{\phi^*}^{J,\phi}\left(\tfrac{1}{4}\boldsymbol{\Theta}\boldsymbol{B},-\tfrac{1}{2}\boldsymbol{B}\right)\mathrm{d}\boldsymbol{B}\\ &= 2^{\frac{1}{2}(n_1+n_2+m)p}\sum_{\phi\in\lambda\cdot\rho}g_{\lambda,\rho}^{\phi}\sum_{j=0}^{\infty}\sum_{J}\frac{1}{j!}\frac{1}{\left(\frac{m}{2}\right)_J}\sum_{\phi^*\in J\cdot\phi}\theta_{\phi^*}^{J,\phi}\Gamma_p\left(\tfrac{n_1+n_2+m}{2},\phi^*\right)C_{\phi^*}^{J,\phi}\left(\tfrac{1}{2}\boldsymbol{\Theta},-\boldsymbol{I}_p\right).\end{aligned} \tag{14}$$

Substituting (14) in (13) completes the proof. ∎

Now using this result (8), we are in a position to derive the exact expression for the density function and the CDF of Λ_1.

Theorem 2.1
Let $(\boldsymbol{X}_1,\boldsymbol{X}_2)\sim BB_p^{IV}(n_1,n_2,m;\boldsymbol{\Theta})$ *and* $\Lambda_1=|\boldsymbol{X}_1\boldsymbol{X}_2|$.

Then the density of Λ_1 *is given by*

$$\frac{\left[\Gamma_p\left(\frac{p+1}{2}\right)\right]^2}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)}etr\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\sum_{\kappa,\lambda,\tau,\rho,J;\phi,\phi^*}\frac{1}{k!t!j!}\binom{\kappa}{\lambda}\binom{\tau}{\rho}g_{\lambda,\rho}^{\phi}\theta_{\phi^*}^{J,\phi}\frac{C_\kappa(\boldsymbol{I}_p)}{C_\lambda(\boldsymbol{I}_p)}\frac{C_\tau(\boldsymbol{I}_p)}{C_\rho(\boldsymbol{I}_p)} \tag{15}$$

$$\cdot\frac{\Gamma_p\left(\frac{n_1}{2}+\frac{p+1}{2},\kappa\right)}{\Gamma_p\left(\frac{n_1}{2}+\frac{p+1}{2},\lambda\right)}\frac{\Gamma_p\left(\frac{n_2}{2}+\frac{p+1}{2},\tau\right)}{\Gamma_p\left(\frac{n_2}{2}+\frac{p+1}{2},\rho\right)}\frac{\Gamma_p\left(\frac{n_1+n_2+m}{2},\phi^*\right)}{\Gamma_p\left(\frac{m}{2},J\right)}C_{\phi^*}^{J,\phi}\left(\tfrac{1}{2}\boldsymbol{\Theta},-\boldsymbol{I}_p\right)G_{2p,2p}^{2p,0}\left(\lambda_1\middle|{}_{b_1,\ldots,b_{2p}}^{a_1,\ldots,a_{2p}}\right),$$

$0<\lambda_1<1$, *where* $\sum\limits_{\kappa,\lambda,\tau,\rho,J;\phi,\phi^*} = \sum\limits_{k=0}^{\infty}\sum\limits_{\kappa}\sum\limits_{l=0}^{k}\sum\limits_{\lambda}\sum\limits_{t=0}^{\infty}\sum\limits_{\tau}\sum\limits_{r=0}^{t}\sum\limits_{\rho}\sum\limits_{\phi\in\lambda\cdot\rho}\sum\limits_{j=0}^{\infty}\sum\limits_{J}\sum\limits_{\phi^*\in J\cdot\phi}$, $G(\cdot)$ *denotes Meijer's G-function ([32], pp.60) and*

$$a_i = \begin{cases} \frac{n_1}{2}+\frac{p-1}{2}+k_{(i+1)/2}-\frac{1}{4}(i-1) & \textit{for} \quad i=1,3,5,\ldots,2p-1 \\ \frac{n_2}{2}+\frac{p-1}{2}+t_{i/2}-\frac{1}{4}(i-2) & \textit{for} \quad i=2,4,6,\ldots,2p, \end{cases}$$

$$b_i = \begin{cases} \frac{n_1}{2}-1+k_{(i+1)/2}-\frac{1}{4}(i-1) & \textit{for} \quad i=1,3,5,\ldots,2p-1 \\ \frac{n_2}{2}-1+t_{i/2}-\frac{1}{4}(i-2) & \textit{for} \quad i=2,6,10,\ldots,2p. \end{cases}$$

Proof:

Using (8) the Mellin transform (see [32], Eq.1.8.1) of $f(\lambda_1)$ is

$$\begin{aligned} M_f(h) &\equiv E\left(\Lambda_1^{h-1}\right) \\ &= E\left[\left(|\boldsymbol{X}_1\boldsymbol{X}_2|\right)^{h-1}\right] \\ &= \frac{\left[\Gamma_p\left(\frac{p+1}{2}\right)\right]^2}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)} etr\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) \sum_{\kappa,\lambda,\tau,\rho,J;\phi,\phi^*} \frac{1}{k!t!j!}\binom{\kappa}{\lambda}\binom{\tau}{\rho} g_{\lambda,\rho}^{\phi}\theta_{\phi^*}^{J,\phi}\frac{C_\kappa(\boldsymbol{I}_p)}{C_\lambda(\boldsymbol{I}_p)}\frac{C_\tau(\boldsymbol{I}_p)}{C_\rho(\boldsymbol{I}_p)} \\ &\quad \cdot\frac{\Gamma_p\left(\frac{n_1}{2}+\frac{p+1}{2},\kappa\right)}{\Gamma_p\left(\frac{n_1}{2}+\frac{p+1}{2},\lambda\right)}\frac{\Gamma_p\left(\frac{n_2}{2}+\frac{p+1}{2},\tau\right)}{\Gamma_p\left(\frac{n_2}{2}+\frac{p+1}{2},\rho\right)}\frac{\Gamma_p\left(\frac{n_1+n_2+m}{2},\phi^*\right)}{\Gamma_p\left(\frac{m}{2},J\right)}\frac{\Gamma_p\left(\frac{n_1}{2}+h-1,\kappa\right)}{\Gamma_p\left(\frac{n_1}{2}+h+\frac{p-1}{2},\kappa\right)}\frac{\Gamma_p\left(\frac{n_2}{2}+h-1,\tau\right)}{\Gamma_p\left(\frac{n_2}{2}+h+\frac{p-1}{2},\tau\right)}C_{\phi^*}^{J,\phi}\left(\tfrac{1}{2}\boldsymbol{\Theta},-\boldsymbol{I}_p\right). \end{aligned} \tag{16}$$

The generalized gamma functions of weights κ and τ respectively in (16) can be written as

$$\Gamma_p\left(\tfrac{n_1}{2}+h+\tfrac{p-1}{2},\kappa\right)\Gamma_p\left(\tfrac{n_2}{2}+h+\tfrac{p-1}{2},\tau\right) = \pi^{\frac{1}{2}p(p-1)}\prod_{i=1}^{2p}\Gamma(a_i+h), \tag{17}$$

and

$$\Gamma_p\left(\tfrac{n_1}{2}+h-1,\kappa\right)\Gamma_p\left(\tfrac{n_2}{2}+h-1,\tau\right) = \pi^{\frac{1}{2}p(p-1)}\prod_{i=1}^{2p}\Gamma(b_i+h) \tag{18}$$

with a_i and b_i as defined above.

Now, substituting (17) and (18) in (16) gives

$$\begin{aligned} M_f(h) \equiv\ & \frac{\left[\Gamma_p\left(\frac{p+1}{2}\right)\right]^2}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)} etr\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) \sum_{\kappa,\lambda,\tau,\rho,J;\phi,\phi^*} \frac{1}{k!t!j!}\binom{\kappa}{\lambda}\binom{\tau}{\rho} g_{\lambda,\rho}^{\phi}\theta_{\phi^*}^{J,\phi}\frac{C_\kappa(\boldsymbol{I}_p)}{C_\lambda(\boldsymbol{I}_p)}\frac{C_\tau(\boldsymbol{I}_p)}{C_\rho(\boldsymbol{I}_p)} \\ & \cdot\frac{\Gamma_p\left(\frac{n_1}{2}+\frac{p+1}{2},\kappa\right)}{\Gamma_p\left(\frac{n_1}{2}+\frac{p+1}{2},\lambda\right)}\frac{\Gamma_p\left(\frac{n_2}{2}+\frac{p+1}{2},\tau\right)}{\Gamma_p\left(\frac{n_2}{2}+\frac{p+1}{2},\rho\right)}\frac{\Gamma_p\left(\frac{n_1+n_2+m}{2},\phi^*\right)}{\Gamma_p\left(\frac{m}{2},J\right)}C_{\phi^*}^{J,\phi}\left(\tfrac{1}{2}\boldsymbol{\Theta},-\boldsymbol{I}_p\right)\frac{\prod_{i=1}^{2p}\Gamma(b_i+h)}{\prod_{i=1}^{2p}\Gamma(a_i+h)}. \end{aligned} \tag{19}$$

Applying the inverse Mellin transform on (19) (see [32], Eq.1.8.2), the result (15) follows. ■

Theorem 2.2

Let $(\boldsymbol{X}_1,\boldsymbol{X}_2)\sim BB_p^{IV}(n_1,n_2,m;\boldsymbol{\Theta})$ *and* $\Lambda_1=|\boldsymbol{X}_1\boldsymbol{X}_2|$ *with density function given by (15).*

Then the CDF of Λ_1 *is given by*

$$\begin{aligned} F(\lambda_1)=P(\Lambda_1\le\lambda_1) =\ & \frac{\left[\Gamma_p\left(\frac{p+1}{2}\right)\right]^2}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)} etr\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) \sum_{\kappa,\lambda,\tau,\rho,J;\phi,\phi^*} \frac{1}{k!t!j!}\binom{\kappa}{\lambda}\binom{\tau}{\rho} g_{\lambda,\rho}^{\phi}\left(\theta_{\phi^*}^{J,\phi}\right)^2\frac{C_\kappa(\boldsymbol{I}_p)}{C_\lambda(\boldsymbol{I}_p)}\frac{C_\tau(\boldsymbol{I}_p)}{C_\rho(\boldsymbol{I}_p)} \\ & \cdot\frac{\Gamma_p\left(\frac{n_1}{2}+\frac{p+1}{2},\kappa\right)}{\Gamma_p\left(\frac{n_1}{2}+\frac{p+1}{2},\lambda\right)}\frac{\Gamma_p\left(\frac{n_2}{2}+\frac{p+1}{2},\tau\right)}{\Gamma_p\left(\frac{n_2}{2}+\frac{p+1}{2},\rho\right)}\frac{\Gamma_p\left(\frac{n_1+n_2+m}{2},\phi^*\right)}{\Gamma_p\left(\frac{m}{2},J\right)}\frac{C_J\left(\frac{1}{2}\boldsymbol{\Theta}\right)C_{\phi^*}(-\boldsymbol{I}_p)}{C_J(\boldsymbol{I}_p)}G_{2p+1,2p+1}^{2p,1}\left(\lambda_1\Big|_{b_1+1,\ldots,b_{2p}+1,0}^{1,a_1+1,\ldots,a_{2p}+1}\right) \end{aligned}$$

$0 < \lambda_1 < 1$, *where* $\sum\limits_{\kappa,\lambda,\tau,\rho,J;\phi,\phi^*} = \sum\limits_{k=0}^{\infty}\sum\limits_{\kappa}\sum\limits_{l=0}^{k}\sum\limits_{\lambda}\sum\limits_{t=0}^{\infty}\sum\limits_{\tau}\sum\limits_{r=0}^{t}\sum\limits_{\rho}\sum\limits_{\phi\in\lambda\cdot\rho}\sum\limits_{j=0}^{\infty}\sum\limits_{J}\sum\limits_{\phi^*\in J\cdot\phi}$, *and* a_i *and* b_i *as specified in Theorem 2.1.*

Proof:

Applying ([33], Eq 2.53) and ([32], Eq. 2.2.1) completes the proof. ■

For the bivariate case, $p = 1$, Corollary 2.1 gives the density function and CDF of $\Lambda_1 = X_1X_2$ where $(X_1, X_2) \sim BB_1^{IV}(n_1, n_2, m; \theta)$.

Corollary 2.1

If $(X_1, X_2) \sim BB_1^{IV}(n_1, n_2, m; \theta)$ *then the*

(a) *density function of* $\Lambda_1 = X_1X_2$ *is*

$$\begin{aligned}
f(\lambda_1) &= \frac{1}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} e^{-\frac{1}{2}\theta}\lambda_1^{\frac{1}{2}n_2-1}\sum_{k=0}^{\infty}\frac{1}{k!}(1-\lambda_1)^{\frac{1}{2}m+k-1}\frac{\Gamma\left(\frac{n_2+m}{2}+k\right)\Gamma\left(\frac{n_1+m}{2}+k\right)\Gamma\left(\frac{n_1+n_2+m}{2}+k\right)}{\Gamma\left(\frac{m}{2}+k\right)\Gamma\left(\frac{n_1+n_2+2m}{2}+2k\right)}\left(\frac{\theta}{2}\right)^k \\
&\cdot {}_2F_1\left(\tfrac{n_2+m}{2}+k, \tfrac{n_2+m}{2}+k; \tfrac{n_1+n_2+2m}{2}+2k; 1-\lambda_1\right), \quad 0 < \lambda_1 < 1, \\
&= \frac{1}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} e^{-\frac{1}{2}\theta}\sum_{k=0}^{\infty}\sum_{t=0}^{\infty}\frac{1}{k!t!}\left(\frac{\theta}{2}\right)^k\frac{\Gamma\left(\frac{n_2+m}{2}+k\right)\Gamma\left(\frac{n_1+m}{2}+k\right)\Gamma\left(\frac{n_1+n_2+m}{2}+k+t\right)}{\Gamma\left(\frac{m}{2}+k\right)} \\
&\cdot G_{2,2}^{2,0}\left(\lambda_1\Big|_{\frac{n_1}{2}+t-1,\frac{n_2}{2}+t-1}^{\frac{n_1+n_2+m}{2}+k+t-1,\frac{n_1+n_2+m}{2}+k+t-1}\right) \qquad (20)
\end{aligned}$$

and

(b) *CDF of* $\Lambda_1 = X_1X_2$ *is*

$$\begin{aligned}
F(\lambda_1) = P(\Lambda_1 \leq \lambda_1) = \quad & e^{-\frac{1}{2}\theta}\sum_{k=0}^{\infty}\sum_{t=0}^{\infty}\frac{1}{k!t!}\left(\frac{\theta}{2}\right)^k\frac{\Gamma\left(\frac{n_2+m}{2}+k\right)\Gamma\left(\frac{n_1+m}{2}+k\right)\Gamma\left(\frac{n_1+n_2+m}{2}+k+t\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)\Gamma\left(\frac{m}{2}+k\right)} \\
& \cdot G_{3,3}^{2,1}\left(\lambda_1\Big|_{\frac{n_1}{2}+t,\frac{n_2}{2}+t,0}^{1,\frac{n_1+n_2+m}{2}+k+t,\frac{n_1+n_2+m}{2}+k+t}\right), \quad 0 < \lambda_1 < 1,
\end{aligned}$$

where ${}_2F_1(\cdot)$ *is the Gauss hypergeometric function with scalar argument (see [20]).*

Expression (20), in terms of the Gauss hypergeometric function, was studied and used by [25] to calculate percentage points. The effect of the noncentrality parameter on the form of the pdf of Λ_1 will be illustrated. Figure 1 shows the effect of the noncentrality parameter θ on $f(\lambda_1)$, given by (20), where $(X_1, X_2) \sim BB_1^{IV}(8, 8, 8; \theta)$. As θ increases the density $f(\lambda_1)$ shifts towards smaller values of Λ_1.

Secondly, in Figure 2 we consider the bimatrix case, $p = 2$, to illustrate the effect of the noncentrality parameter $\boldsymbol{\Theta}$ on the density function of Λ_1 (see (15)) where $(\boldsymbol{X}_1, \boldsymbol{X}_2) \sim BB_2^{IV}(8, 8, 8; \boldsymbol{\Theta})$, $\boldsymbol{\Theta} = \theta\boldsymbol{I}_2$. We note that as θ increases the density function shifts towards smaller values of Λ_1.

Fig.1. Effect of θ on $f(\lambda_1)$, $\Lambda_1 = X_1X_2$, $(X_1, X_2) \sim BB_1^{IV}(8,8,8;\theta)$

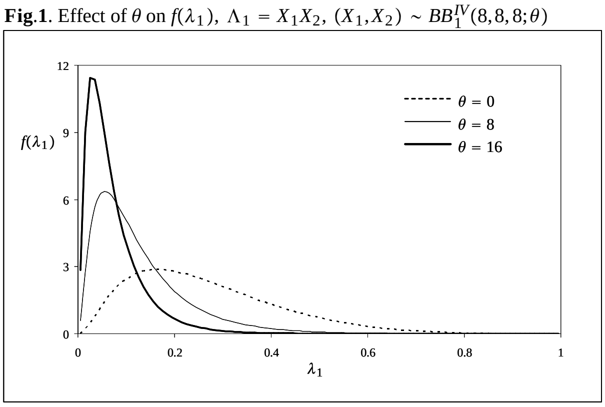
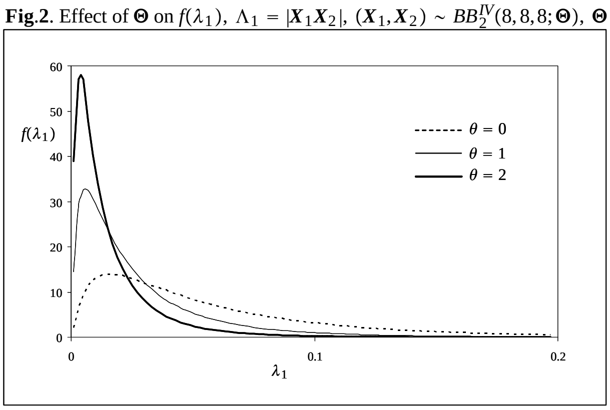


Fig.2. Effect of $\boldsymbol{\Theta}$ on $f(\lambda_1)$, $\Lambda_1 = |\boldsymbol{X}_1\boldsymbol{X}_2|$, $(\boldsymbol{X}_1, \boldsymbol{X}_2) \sim BB_2^{IV}(8,8,8;\boldsymbol{\Theta})$, $\boldsymbol{\Theta} = \theta\boldsymbol{I}_2$



3 Density function of Λ_2

If $\mathbb{U} = (\boldsymbol{U}_1 : \boldsymbol{U}_2)'$ has the noncentral bimatrix variate beta type I distribution, denoted as $\mathbb{U} \sim BB_p^I(n_1, n_2, m; \boldsymbol{\Theta})$, the density function is given by

$$\begin{aligned} f(\boldsymbol{U}_1, \boldsymbol{U}_2) = \quad & \left\{\beta_p\left(\tfrac{n_1}{2}, \tfrac{n_2}{2}; \tfrac{m}{2}\right)\right\}^{-1} \prod_{i=1}^{2} |\boldsymbol{U}_i|^{\frac{1}{2}n_i - \frac{1}{2}(p+1)} \left|\boldsymbol{I}_p - \sum_{i=1}^{2} \boldsymbol{U}_i\right|^{\frac{1}{2}m - \frac{1}{2}(p+1)} \\ & \cdot etr\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) \, {}_1F_1\left(\tfrac{n_1+n_2+m}{2}; \tfrac{m}{2}; \tfrac{1}{2}\left(\boldsymbol{I}_p - \sum_{i=1}^{2} \boldsymbol{U}_i\right)\boldsymbol{\Theta}\right), \end{aligned} \tag{21}$$

$\boldsymbol{0} < \boldsymbol{U}_i < \boldsymbol{I}_p$, $i = 1, 2$, $\boldsymbol{0} < \sum_{i=1}^{2} \boldsymbol{U}_i < \boldsymbol{I}_p$, where $n_i > (p-1)$, $i = 1, 2$, $m > (p-1)$ and with product moment [see [12]]

$$\begin{aligned} E\left(|\boldsymbol{U}_1|^{h_1} |\boldsymbol{U}_2|^{h_2}\right) \quad = \quad & \frac{\Gamma_p\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)} \frac{\Gamma_p\left(\frac{n_1}{2}+h_1\right)\Gamma_p\left(\frac{n_2}{2}+h_2\right)}{\Gamma_p\left(\frac{n_1+n_2+m}{2}+h_1+h_2\right)} \\ & \cdot etr\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) {}_1F_1\left(\tfrac{n_1+n_2+m}{2}; \tfrac{n_1+n_2+m}{2} + h_1 + h_2; \tfrac{1}{2}\boldsymbol{\Theta}\right), \end{aligned} \tag{22}$$

where $\mathrm{Re}\left(\frac{n_i}{2} + h_i\right) > \frac{1}{2}(p-1)$, $i = 1, 2$. Subsequently, we now derive an exact expression for the density function and CDF of Λ_2. For asymptotic distribution of a suitable function of Λ_2 the reader is referred to [12] and [22].

Theorem 3.1
Let $(\boldsymbol{U}_1, \boldsymbol{U}_2) \sim BB_p^I(n_1, n_2, m; \boldsymbol{\Theta})$ *and* $\Lambda_2 = |\boldsymbol{U}_1|^{\frac{1}{2}n_1} |\boldsymbol{U}_2|^{\frac{1}{2}n_2}$.
Then the density function of Λ_2 *is given by*

$$\frac{\pi^{\frac{1}{4}p(p-1)}}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)} etr\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) \sum_{k=0}^{\infty} \sum_{\kappa} \tfrac{1}{k!} \Gamma_p\left(\tfrac{n_1+n_2+m}{2}, \kappa\right) C_\kappa\left(\tfrac{1}{2}\boldsymbol{\Theta}\right) H_{p,2p}^{2p,0}\left(\lambda_2 \middle|_{(b_1,\beta_1),\ldots,(b_{2p},\beta_{2p})}^{(a_1,\alpha_1),\ldots,(a_p,\alpha_p)}\right), \tag{23}$$

$0 < \lambda_2 < 1$, *where* $H(\cdot)$ *denotes Fox's H-function ([30], pp.140 and [33]) and*

$a_j = \frac{m}{2} + k_j - \frac{1}{2}(j-1)$ *for* $j = 1, 2, \ldots, p$, $\qquad \alpha_j = \frac{n_1+n_2}{2}$ *for* $j = 1, 2, \ldots, p$,

$$b_j = \begin{cases} -\frac{1}{4}(j-1) & \text{for } j = 1, 3, 5, \ldots, 2p-1 \\ -\frac{1}{4}(j-2) & \text{for } j = 2, 4, 6, \ldots, 2p, \end{cases} \qquad \beta_j = \begin{cases} \frac{n_1}{2} & \text{for } j = 1, 3, 5, \ldots, 2p-1 \\ \frac{n_2}{2} & \text{for } j = 2, 4, 6, \ldots, 2p. \end{cases}$$

Proof:
Using (22), ([7], Eq.25), the inverse Mellin transform and definition of the H-function, the result (23) follows.■

Theorem 3.2
Let $(\boldsymbol{U}_1, \boldsymbol{U}_2) \sim BB_p^I(n_1, n_2, m; \boldsymbol{\Theta})$ *and* $\Lambda_2 = |\boldsymbol{U}_1|^{\frac{1}{2}n_1} |\boldsymbol{U}_2|^{\frac{1}{2}n_2}$.
Then the CDF of Λ_2 *is given by*

$$\begin{aligned} F(\lambda_2) = P(\Lambda_2 \le \lambda_2) = \quad & \frac{\pi^{\frac{1}{4}p(p-1)}}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)} etr\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) \sum_{k=0}^{\infty} \sum_{\kappa} \tfrac{1}{k!} \Gamma_p\left(\tfrac{n_1+n_2+m}{2}, \kappa\right) C_\kappa\left(\tfrac{1}{2}\boldsymbol{\Theta}\right) \\ & \cdot H_{p+1,2p+1}^{2p,1}\left(\lambda_2 \middle|_{(b_1+\beta_1,\beta_1),\ldots,(b_{2p}+\beta_{2p},\beta_{2p}),(0,1)}^{(1,1),(a_1+\alpha_1,\alpha_1),\ldots,(a_p+\alpha_p,\alpha_p)}\right) \end{aligned} \tag{24}$$

$0 < \lambda_2 < 1$, *with* $a_j, \alpha_j, b_j, \beta_j$ *as specified in Theorem 3.1.*

Proof:

Applying ([33], Eq.2.53) and ([33], Eq.1.60) completes the proof. ■

For the bivariate case, $p = 1$, Corollary 3.1 gives the density function and CDF of $\Lambda_2 = U_1^{\frac{1}{2}n_1} U_2^{\frac{1}{2}n_2}$ where $(U_1, U_2) \sim BB_1^I(n_1, n_2, m; \theta)$.

Corollary 3.1

If $(U_1, U_2) \sim BB_1^I(n_1, n_2, m; \theta)$ *then the*

(a) *density function of* $\Lambda_2 = U_1^{\frac{1}{2}n_1} U_2^{\frac{1}{2}n_2}$ *is*

$$f(\lambda_2) = \frac{1}{\Gamma(\frac{n_1}{2})\Gamma(\frac{n_2}{2})} e^{-\frac{1}{2}\theta} \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{\theta}{2}\right)^k \Gamma\left(\frac{n_1+n_2+m}{2}+k\right) \cdot H_{1,2}^{2,0}\left(\lambda_2 \middle| {}^{\left(\frac{m}{2}+k, \frac{n_1+n_2}{2}\right)}_{\left(0,\frac{n_1}{2}\right),\left(0,\frac{n_2}{2}\right)}\right), \qquad 0 < \lambda_2 < 1, \tag{25}$$

and

(b) *CDF of* $\Lambda_2 = U_1^{\frac{1}{2}n_1} U_2^{\frac{1}{2}n_2}$ *is*

$$F(\lambda_2) = P(\Lambda_2 \le \lambda_2) = \frac{1}{\Gamma(\frac{n_1}{2})\Gamma(\frac{n_2}{2})} e^{-\frac{1}{2}\theta} \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{\theta}{2}\right)^k \Gamma\left(\frac{n_1+n_2+m}{2}+k\right) \cdot H_{2,3}^{2,1}\left(\lambda_2 \middle| {}^{(1,1)\left(\frac{m}{2}+\frac{n_1+n_2}{2}+k, \frac{n_1+n_2}{2}\right)}_{\left(\frac{n_1}{2},\frac{n_1}{2}\right),\left(\frac{n_2}{2},\frac{n_2}{2}\right),(0,1)}\right), \quad 0 < \lambda_2 < 1.$$

The effect of the noncentrality parameter on the density function of Λ_2, $f(\lambda_2)$, is shown in Figures 3 and 4 where $(U_1, U_2) \sim BB_1^I(2, 2, 2; \theta)$ and $(\boldsymbol{U}_1, \boldsymbol{U}_2) \sim BB_2^I(2, 2, 2; \boldsymbol{\Theta})$ $(\boldsymbol{\Theta} = \theta \boldsymbol{I}_2)$ respectively. In Figure 3, at smaller values of Λ_2 the density function increases as θ increases, whilst for $p = 2$, i.e. Figure 4, the density function shifts towards smaller values of Λ_2 for increasing θ.

Fig.3. Effect of θ on $f(\lambda_2)$, $\Lambda_2 = U_1 U_2$, $(U_1, U_2) \sim BB_1^I(2,2,2;\theta)$

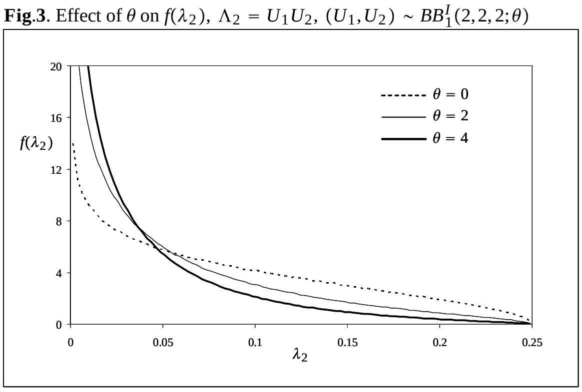
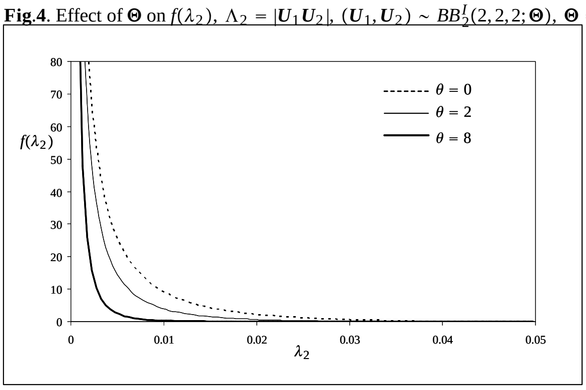


Fig.4. Effect of $\boldsymbol{\Theta}$ on $f(\lambda_2)$, $\Lambda_2 = |\boldsymbol{U}_1 \boldsymbol{U}_2|$, $(\boldsymbol{U}_1, \boldsymbol{U}_2) \sim BB_2^I(2,2,2;\boldsymbol{\Theta})$, $\boldsymbol{\Theta} = \theta \boldsymbol{I}_2$



Remark 3.1

(a) As pointed out by [37], the computation for hypergeometric functions of matrix arguments, or zonal polynomials are in a state of development. Therefore the sequential saddlepoint approximation is used to calculate tail probabilities of $\ln \Lambda_2$. In this method the cumulant generating function $K(s) = \ln M_{\ln \Lambda_2}(s)$ is used, where $M_{\ln \Lambda_2}(s)$ is the moment generating function of $\ln \Lambda_2$. From (22) and ([34], pp.265, Eq.(6)):

$$
\begin{aligned}
M_{\ln \Lambda_2}(s) &= E\left(\Lambda_2^s\right) \\
&= E\left(|\boldsymbol{U}_1|^{\frac{1}{2}n_1 s}\,|\boldsymbol{U}_2|^{\frac{1}{2}n_2 s}\right) \\
&= \frac{\Gamma_p\left(\frac{n_1+n_2+m}{2}\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)}\frac{\Gamma_p\left(\frac{n_1}{2}+\frac{n_1 s}{2}\right)\Gamma_p\left(\frac{n_2}{2}+\frac{n_2 s}{2}\right)}{\Gamma_p\left(\frac{n_1+n_2+m}{2}+\frac{n_1 s+n_2 s}{2}\right)} \\
&\cdot etr\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\ {}_1F_1\left(\tfrac{n_1+n_2+m}{2};\tfrac{n_1+n_2+m}{2}+\tfrac{n_1 s+n_2 s}{2};\tfrac{1}{2}\boldsymbol{\Theta}\right) \\
&= \frac{\prod_{i=1}^{p}\Gamma\left(\frac{n_1+n_2+m}{2}-\frac{1}{2}(i-1)\right)}{\prod_{i=1}^{p}\Gamma\left(\frac{n_1}{2}-\frac{1}{2}(i-1)\right)\prod_{i=1}^{p}\Gamma\left(\frac{n_2}{2}-\frac{1}{2}(i-1)\right)}\frac{\prod_{i=1}^{p}\Gamma\left(\frac{n_1}{2}+\frac{n_1 s}{2}-\frac{1}{2}(i-1)\right)\prod_{i=1}^{p}\Gamma\left(\frac{n_2}{2}+\frac{n_2 s}{2}-\frac{1}{2}(i-1)\right)}{\prod_{i=1}^{p}\Gamma\left(\frac{n_1+n_2+m}{2}+\frac{n_1 s+n_2 s}{2}-\frac{1}{2}(i-1)\right)} \\
&\cdot\ {}_1F_1\left(\tfrac{n_1 s+n_2 s}{2};\tfrac{n_1+n_2+m}{2}+\tfrac{n_1 s+n_2 s}{2};-\tfrac{1}{2}\boldsymbol{\Theta}\right)
\end{aligned}
\tag{26}
$$

The method involves two stages:

(1) replacing the relevant hypergeometric function of matrix argument in (26) by the calibrated Laplace approximation ${}_1\hat{F}_1$;

(2) using the Lugananni and Rice tail probability approximation.

For more detail the reader is referred to [5] and [31]. Table 1 gives values of the tail probabilities, $\hat{P}(\ln \Lambda_2 > y)$, for $p = 2, n_1 = 2, n_3 = 2, m = 2, \boldsymbol{\Theta} = \theta \boldsymbol{I}_2$ for different values of θ.

θ	0	1	8
Approximated values	0.056	0.045	0.001
Simulated empirical values	0.055	0.032	0.001

Table 1: $\hat{P}(\ln \Lambda_2 > -4.5)$.

(b) Consider $\boldsymbol{\Theta} = \theta \boldsymbol{I}_p$, using ([23], Eq.1.5.5) and ([33], Eq.A.69) to write $H(\cdot)$ in a computational form, (24) could be evaluated. However, careful consideration should be given to the gamma function for negative integer values (see also [2]). In commercial software like MATHEMATICA or MAPLE Meijer's G-function is available, but the Fox H-function is still in a developing stage (see also [40]).

4 Density function of Λ_3 - noncentral bimatrix variate beta type V

Firstly, in this section the noncentral bimatrix variate beta type V distribution is introduced, followed by the expression for the product moment. Lastly, we obtain an exact expression for the density function, as well as the CDF of Λ_3 (see (5)). The bimatrix variate beta type V distribution allows for constant factors to be built into the covariance matrices of Wishart matrix variates from which this distribution is generated, and as such may be useful in test statistics requiring this.

Lemma 4.1
Let $\boldsymbol{S}_1 \sim W_p(n_1, \boldsymbol{I}_p)$, $\boldsymbol{S}_2 \sim W_p(n_2, \boldsymbol{I}_p)$ *and* $\boldsymbol{B} \sim W_p(m, \boldsymbol{I}_p; \boldsymbol{\Theta})$ *be independently distributed. Consider the ratios*

$$
\boldsymbol{Q}_i = \left(\alpha_1 \boldsymbol{S}_1 + \alpha_2 \boldsymbol{S}_2 + c\boldsymbol{B}\right)^{-\frac{1}{2}}\left(\alpha_i \boldsymbol{S}_i\right)\left(\alpha_1 \boldsymbol{S}_1 + \alpha_2 \boldsymbol{S}_2 + c\boldsymbol{B}\right)^{-\frac{1}{2}},\ i = 1,2. \tag{27}
$$

Then density function of $\mathbb{Q} = (\boldsymbol{Q}_1 : \boldsymbol{Q}_2)'$, *denoted as* $\mathbb{Q} \sim BB_p^V(n_1, n_2, m, \alpha_1, \alpha_2, c; \boldsymbol{\Theta})$, *is given by*

$$
\begin{aligned}
& f\left(\boldsymbol{Q}_1, \boldsymbol{Q}_2\right) \\
= & \left\{\beta_p\left(\tfrac{n_1}{2}, \tfrac{n_2}{2}; \tfrac{m}{2}\right)\right\}^{-1} \prod_{i=1}^{2}\left|\boldsymbol{Q}_i\right|^{\frac{1}{2} n_i-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\sum_{i=1}^{2} \boldsymbol{Q}_i\right|^{\frac{1}{2} m-\frac{1}{2}(p+1)} \prod_{i=1}^{2}\left(\frac{c}{\alpha_i}\right)^{\frac{1}{2} n_i p}\left|\boldsymbol{I}_p+\sum_{i=1}^{2} \frac{c-\alpha_i}{\alpha_i} \boldsymbol{Q}_i\right|^{-\frac{1}{2}\left(n_1+n_2+m\right)} \\
& \cdot \operatorname{etr}\left(-\tfrac{1}{2} \boldsymbol{\Theta}\right){ }_1 F_1\left(\frac{n_1+n_2+m}{2}; \frac{m}{2}; \frac{1}{2}\left(\boldsymbol{I}_p-\sum_{i=1}^{2} \boldsymbol{Q}_i\right)^{\frac{1}{2}}\left(\boldsymbol{I}_p+\sum_{i=1}^{2} \frac{c-\alpha_i}{\alpha_i} \boldsymbol{Q}_i\right)^{-1}\left(\boldsymbol{I}_p-\sum_{i=1}^{2} \boldsymbol{Q}_i\right)^{\frac{1}{2}} \boldsymbol{\Theta}\right),
\end{aligned} \tag{28}
$$

$\mathbf{0}<\boldsymbol{Q}_i<\boldsymbol{I}_p,\ i=1,2,\ \mathbf{0}<\sum_{i=1}^{2} \boldsymbol{Q}_i<\boldsymbol{I}_p,$ *where* $n_i>(p-1),\ i=1,2$ *and* $m>(p-1)$.

Proof:

On performing the transformations (27) where $\boldsymbol{Q}_i=\boldsymbol{S}^{-\frac{1}{2}}\left(\alpha_i \boldsymbol{S}_i\right) \boldsymbol{S}^{-\frac{1}{2}},\ i=1,2,$ with $\boldsymbol{S}=\alpha_1 \boldsymbol{S}_1+\alpha_2 \boldsymbol{S}_2+c \boldsymbol{B}$, the Jacobian is $J\left(\boldsymbol{S}_1, \boldsymbol{S}_2, \boldsymbol{B} \rightarrow \boldsymbol{Q}_1, \boldsymbol{Q}_2, \boldsymbol{S}\right)=\left(c \prod_{i=1}^{2} \alpha_i\right)^{-\frac{1}{2} p(p+1)}|\boldsymbol{S}|^{(p+1)}$. From (9) follows that

$$
\begin{aligned}
& f\left(\boldsymbol{Q}_1, \boldsymbol{Q}_2, \boldsymbol{S}\right) \\
= & K \operatorname{etr}\left(-\tfrac{1}{2} \boldsymbol{\Theta}\right) \prod_{i=1}^{2} \alpha_i^{-\frac{1}{2} n_i p} c^{-\frac{1}{2} m p} \prod_{i=1}^{2}\left|\boldsymbol{Q}_i\right|^{\frac{1}{2} n_i-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\sum_{i=1}^{2} \boldsymbol{Q}_i\right|^{\frac{1}{2} m-\frac{1}{2}(p+1)} \\
& \cdot|\boldsymbol{S}|^{\frac{1}{2}\left(n_1+n_2+m\right)-\frac{1}{2}(p+1)} \operatorname{etr}\left[-\frac{1}{2 c} \boldsymbol{S}\left(\boldsymbol{I}_p+\sum_{i=1}^{2} \frac{c-\alpha_i}{\alpha_i} \boldsymbol{Q}_i\right)\right]{ }_0 F_1\left(\frac{m}{2}; \frac{1}{4 c} \boldsymbol{\Theta} \boldsymbol{S}^{\frac{1}{2}}\left(\boldsymbol{I}_p-\sum_{i=1}^{2} \boldsymbol{Q}_i\right) \boldsymbol{S}^{\frac{1}{2}}\right),
\end{aligned} \tag{29}
$$

where $K^{-1}=\Gamma_p\left(\frac{n_1}{2}\right) \Gamma_p\left(\frac{n_2}{2}\right) \Gamma_p\left(\frac{m}{2}\right) 2^{\frac{1}{2}\left(n_1+n_2+m\right) p}$.

We consider the symmetrised density function of $\left(\boldsymbol{Q}_1, \boldsymbol{Q}_2\right)$ defined by [21], that is $f_s\left(\boldsymbol{Q}_1, \boldsymbol{Q}_2\right) \equiv \int_{\boldsymbol{S}>\mathbf{0}} \int_{O(p)} f\left(\boldsymbol{H} \boldsymbol{Q}_1 \boldsymbol{H}^{\prime}, \boldsymbol{H} \boldsymbol{Q}_2 \boldsymbol{H}^{\prime}, \boldsymbol{H} \boldsymbol{S} \boldsymbol{H}^{\prime}\right) \mathrm{d} \boldsymbol{H} \mathrm{d} \boldsymbol{S}$ where $\boldsymbol{H} \in O(p)$. Note that $\mathrm{d} \boldsymbol{S}=\mathrm{d} \boldsymbol{H} \boldsymbol{S} \boldsymbol{H}^{\prime}$ [13]. From (29), ([7], Eq.25) and ([17], Eq.2.3.6)

$$
\begin{aligned}
& f_s\left(\boldsymbol{Q}_1, \boldsymbol{Q}_2\right) \\
= & K \prod_{i=1}^{2} \alpha_i^{-\frac{1}{2} n_i p} c^{-\frac{1}{2} m p} \prod_{i=1}^{2}\left|\boldsymbol{Q}_i\right|^{\frac{1}{2} n_i-\frac{1}{2}(p+1)}\left|\boldsymbol{I}_p-\sum_{i=1}^{2} \boldsymbol{Q}_i\right|^{\frac{1}{2} m-\frac{1}{2}(p+1)} \int_{\boldsymbol{S}>\mathbf{0}}|\boldsymbol{S}|^{\frac{1}{2}\left(n_1+n_2+m\right)-\frac{1}{2}(p+1)} \\
& \cdot \operatorname{etr}\left[-\frac{1}{2 c} \boldsymbol{S}\left(\boldsymbol{I}_p+\sum_{i=1}^{2} \frac{c-\alpha_i}{\alpha_i} \boldsymbol{Q}_i\right)\right] \int_{O(p)}{ }_0 F_1\left(\frac{m}{2}; \frac{1}{4 c}\left(\boldsymbol{I}_p-\sum_{i=1}^{2} \boldsymbol{Q}_i\right)^{\frac{1}{2}} \boldsymbol{H}^{\prime} \boldsymbol{\Theta} \boldsymbol{H}\left(\boldsymbol{I}_p-\sum_{i=1}^{2} \boldsymbol{Q}_i\right)^{\frac{1}{2}} \boldsymbol{S}\right) \mathrm{d} \boldsymbol{H} \mathrm{d} \boldsymbol{S}.
\end{aligned} \tag{30}
$$

Integrating (30) with respect to $\boldsymbol{S}$ by using ([23], Eq.1.6.4) and since $f_s\left(\boldsymbol{Q}_1, \boldsymbol{Q}_2\right)=\int_{O(p)} f\left(\boldsymbol{H} \boldsymbol{Q}_1 \boldsymbol{H}^{\prime}, \boldsymbol{H} \boldsymbol{Q}_2 \boldsymbol{H}^{\prime}\right) \mathrm{d} \boldsymbol{H}$ the result (28) follows from applying the result of Greenacre [21] in an inverse way (see [13]). ■

Remark 4.1

Ehlers, Bekker and Roux [18] derived the result in (28) for the bivariate case, that is where $p=1$, and also studied some properties of the noncentral bivariate beta type V distribution.

Lemma 4.2

If $\left(\boldsymbol{Q}_1, \boldsymbol{Q}_2\right) \sim B B_p^V\left(n_1, n_2, m, \alpha_1, \alpha_2, c; \boldsymbol{\Theta}\right)$ *as given by* (28) *then for* $\alpha_1=\alpha_2=\alpha$, $E\left(\left|\boldsymbol{Q}_1\right|^{h_1}\left|\boldsymbol{Q}_2\right|^{h_2}\right)$ *is given by*

$$
\frac{\Gamma_p\left(\frac{n_1}{2}+h_1\right) \Gamma_p\left(\frac{n_2}{2}+h_2\right)}{\Gamma_p\left(\frac{n_1}{2}\right) \Gamma_p\left(\frac{n_2}{2}\right)}\left(\frac{c}{\alpha}\right)^{-\frac{1}{2} m p} \operatorname{etr}\left(-\tfrac{1}{2} \boldsymbol{\Theta}\right) \sum_{\kappa, \tau ; \phi} \theta_\phi^{\kappa, \tau} \frac{1}{k!t!} \frac{\Gamma_p\left(\frac{m}{2}, \phi\right) \Gamma_p\left(\frac{n_1+n_2+m}{2}, \phi\right)}{\Gamma_p\left(\frac{m}{2}, \kappa\right) \Gamma_p\left(\frac{n_1+n_2+m}{2}+h_1+h_2, \phi\right)} C_\phi^{\kappa, \tau}\left(\frac{\alpha}{2 c} \boldsymbol{\Theta}, \frac{c-\alpha}{c} \boldsymbol{I}_p\right), \tag{31}
$$

where $\sum\limits_{\kappa,\tau;\phi} = \sum\limits_{k=0}^{\infty}\sum\limits_{\kappa}\sum\limits_{t=0}^{\infty}\sum\limits_{\tau}$.

Proof:

From (29) with $\alpha_1 = \alpha_2 = \alpha$ and using ([23], pp.22) yields

$$
\begin{aligned}
& E\left(|\boldsymbol{Q}_1|^{h_1}|\boldsymbol{Q}_2|^{h_2}\right) \\
= \quad & K\alpha^{-\frac{1}{2}(n_1+n_2)p}c^{-\frac{1}{2}mp}etr\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\int_{\boldsymbol{S}>\boldsymbol{0}}|\boldsymbol{S}|^{\frac{1}{2}(n_1+n_2+m)-\frac{1}{2}(p+1)} \\
& \cdot\int\int_{\substack{\boldsymbol{0}<\boldsymbol{Q}_1+\boldsymbol{Q}_2<\boldsymbol{I}_p\\ \boldsymbol{Q}_i>\boldsymbol{0}}}\prod_{i=1}^{2}|\boldsymbol{Q}_i|^{\frac{1}{2}n_i+h_i-\frac{1}{2}(p+1)}f\left(\sum_{i=1}^{2}\boldsymbol{Q}_i\right)\mathrm{d}\boldsymbol{Q}_1\mathrm{d}\boldsymbol{Q}_2\mathrm{d}\boldsymbol{S} \\
= \quad & K\alpha^{-\frac{1}{2}(n_1+n_2)p}c^{-\frac{1}{2}mp}etr\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\beta_p\left(\tfrac{n_1}{2}+h_1,\tfrac{n_2}{2}+h_2\right)\int_{\boldsymbol{S}>\boldsymbol{0}}|\boldsymbol{S}|^{\frac{1}{2}(n_1+n_2+m)-\frac{1}{2}(p+1)}\int_{\boldsymbol{0}<\boldsymbol{Z}<\boldsymbol{I}_p}g(\boldsymbol{Z})\mathrm{d}\boldsymbol{Z}\mathrm{d}\boldsymbol{S},
\end{aligned}
\tag{32}
$$

where

$$g(\boldsymbol{Z}) = |\boldsymbol{Z}|^{\frac{1}{2}(n_1+n_2)+h_1+h_2-\frac{1}{2}(p+1)}|\boldsymbol{I}_p-\boldsymbol{Z}|^{\frac{1}{2}m-\frac{1}{2}(p+1)}etr\left[-\tfrac{1}{2c}\boldsymbol{S}\left(\boldsymbol{I}_p+\tfrac{c-\alpha}{\alpha}\boldsymbol{Z}\right)\right]{}_0F_1\left(\tfrac{m}{2};\tfrac{1}{4c}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{\Theta}\boldsymbol{S}^{\frac{1}{2}}(\boldsymbol{I}_p-\boldsymbol{Z})\right).$$

Let $\boldsymbol{X} = \boldsymbol{I}_p - \boldsymbol{Z}$; using ([7], Eq.25 and Eq.30), ([8], Eq.2.8) and ([6], Eq.3.28) we obtain

$$
\begin{aligned}
& \int_{\boldsymbol{0}<\boldsymbol{Z}<\boldsymbol{I}_p}g(\boldsymbol{Z})\mathrm{d}\boldsymbol{Z} \\
= \quad & etr\left(-\tfrac{1}{2\alpha}\boldsymbol{S}\right)\sum_{k=0}^{\infty}\sum_{\kappa}\sum_{t=0}^{\infty}\sum_{\tau}\sum_{\phi\in\kappa\cdot\tau}\theta_{\phi}^{\kappa,\tau}\frac{1}{k!t!}\frac{1}{\left(\frac{m}{2}\right)_{\kappa}} \\
& \cdot\int_{\boldsymbol{0}<\boldsymbol{X}<\boldsymbol{I}_p}|\boldsymbol{X}|^{\frac{1}{2}m-\frac{1}{2}(p+1)}|\boldsymbol{I}_p-\boldsymbol{X}|^{\frac{1}{2}(n_1+n_2)+h_1+h_2-\frac{1}{2}(p+1)}C_{\phi}^{\kappa,\tau}(\tfrac{1}{4c}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{\Theta}\boldsymbol{S}^{\frac{1}{2}}\boldsymbol{X},\tfrac{c-\alpha}{2c\alpha}\boldsymbol{S}\boldsymbol{X})\mathrm{d}\boldsymbol{X} \\
= \quad & etr\left(-\tfrac{1}{2\alpha}\boldsymbol{S}\right)\sum_{\kappa,\tau;\phi}\theta_{\phi}^{\kappa,\tau}\frac{1}{k!t!}\frac{\Gamma_p\left(\frac{m}{2}\right)}{\Gamma_p\left(\frac{m}{2},\kappa\right)}\frac{\Gamma_p\left(\frac{m}{2},\phi\right)\Gamma_p\left(\frac{n_1+n_2}{2}+h_1+h_2\right)}{\Gamma_p\left(\frac{n_1+n_2+m}{2}+h_1+h_2,\phi\right)}C_{\phi}^{\kappa,\tau}(\tfrac{1}{4c}\boldsymbol{\Theta}\boldsymbol{S},\tfrac{c-\alpha}{2c\alpha}\boldsymbol{S}).
\end{aligned}
\tag{33}
$$

Substituting (33) in (32) and applying ([6], Eq.3.21) completes the proof. ■

Armed with the results in Lemma 4.1 and 4.2, we can derive the key result, namely the density function of Λ_3 (see (5)).

Theorem 4.1

Let $(\boldsymbol{Q}_1,\boldsymbol{Q}_2)\sim BB_p^V(n_1,n_2,m,\alpha,\alpha,c;\boldsymbol{\Theta})$ *with density function given by* (28) *and let* $\Lambda_3 = |\boldsymbol{Q}_1|^{\frac{1}{2}n_1}|\boldsymbol{Q}_2|^{\frac{1}{2}n_2}$. *Then the density function of* Λ_3 *is given by*

$$\frac{\pi^{\frac{1}{4}p(p-1)}}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)}\left(\tfrac{c}{\alpha}\right)^{-\frac{1}{2}mp}etr\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right)\sum_{\kappa,\tau;\phi}\theta_{\phi}^{\kappa,\tau}\frac{1}{k!t!}\frac{\Gamma_p\left(\frac{m}{2},\phi\right)\Gamma_p\left(\frac{n_1+n_2+m}{2},\phi\right)}{\Gamma_p\left(\frac{m}{2},\kappa\right)}C_{\phi}^{\kappa,\tau}(\tfrac{\alpha}{2c}\boldsymbol{\Theta},\tfrac{c-\alpha}{c}\boldsymbol{I}_p)H_{p,2p}^{2p,0}\left(\lambda_3\Big|_{(b_1,\beta_1),\ldots,(b_{2p},\beta_{2p})}^{(a_1,\alpha_1),\ldots,(a_p,\alpha_p)}\right), \tag{34}$$

$0<\lambda_3<1$, *where* $\sum\limits_{\kappa,\tau;\phi} = \sum\limits_{k=0}^{\infty}\sum\limits_{\kappa}\sum\limits_{t=0}^{\infty}\sum\limits_{\tau}\sum\limits_{\phi\in\kappa\cdot\tau}$ *and*

$a_j = \frac{m}{2}+(k_j+t_j)-\frac{1}{2}(j-1)$ *for* $j=1,2,3,\ldots,p$, $\qquad \alpha_j = \frac{n_1+n_2}{2}$ *for* $j=1,2,3,\ldots,p$,

$$b_j = \begin{cases} -\frac{1}{4}(j-1) & \textit{for } j=1,3,5,\ldots,2p-1 \\ -\frac{1}{4}(j-2) & \textit{for } j=2,4,6,\ldots,2p, \end{cases} \qquad \beta_j = \begin{cases} \frac{n_1}{2} & \textit{for } j=1,3,5,\ldots,2p-1 \\ \frac{n_2}{2} & \textit{for } j=2,4,6,\ldots,2p. \end{cases}$$

Proof:

Similar to the proof of Theorem 2.1. ■

Theorem 4.2

Let $(\boldsymbol{Q}_1, \boldsymbol{Q}_2) \sim BB_p^V(n_1, n_2, m, \alpha, \alpha, c; \boldsymbol{\Theta})$ *with density function given by* (28) *and let* $\Lambda_3 = |\boldsymbol{Q}_1|^{\frac{1}{2}n_1} |\boldsymbol{Q}_2|^{\frac{1}{2}n_2}$. *Then the CDF of* Λ_3 *is given by*

$$F(\lambda_3) = P(\Lambda_3 \le \lambda_3) = \frac{\pi^{\frac{1}{4}p(p-1)}}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)} \left(\frac{c}{\alpha}\right)^{-\frac{1}{2}mp} etr\left(-\tfrac{1}{2}\boldsymbol{\Theta}\right) \sum_{\kappa,\tau;\phi} \left(\theta_\phi^{\kappa,\tau}\right)^2 \left(\frac{\alpha}{2c}\right)^k \left(\frac{c-\alpha}{c}\right)^t \frac{\Gamma_p\left(\frac{m}{2},\phi\right)\Gamma_p\left(\frac{n_1+n_2+m}{2},\phi\right)}{\Gamma_p\left(\frac{m}{2},\kappa\right)}$$
$$\cdot \frac{C_\phi(\boldsymbol{I}_p)C_\kappa(\boldsymbol{\Theta})}{k!t!C_\kappa(\boldsymbol{I}_p)} H_{p+1,2p+1}^{2p,1}\left(\lambda_3 \middle| \begin{matrix} (1,1),(a_1+\alpha_1,\alpha_1),\ldots,(a_p+\alpha_p,\alpha_p) \\ (b_1+\beta_1,\beta_1),\ldots,(b_{2p}+\beta_{2p},\beta_{2p}),(0,1) \end{matrix}\right)$$

$0 < \lambda_3 < 1$, *where* $\sum_{\kappa,\tau;\phi} = \sum_{k=0}^{\infty}\sum_{\kappa}\sum_{t=0}^{\infty}\sum_{\tau}\sum_{\phi\in\kappa\cdot\tau}$ *and with* $a_j, \alpha_j, b_j, \beta_j$ *as specified in Theorem 4.1.*

Proof:

Applying ([33], Eq.2.53), ([33], Eq.1.60) and ([9], Eq.2.2, Eq.2.7) completes the proof. ■

For the bivariate case, $p = 1$, that is where $(Q_1, Q_2) \sim BB_1^V(n_1, n_2, m, \alpha_1, \alpha_2, c; \theta)$, the product moment $E\left(Q_1^{h_1} Q_2^{h_2}\right)$ is given for this special case in Corollary 4.1. The density function and the CDF of $\Lambda_3 = Q_1^{\frac{1}{2}n_1} Q_2^{\frac{1}{2}n_2}$ is given in Corollary 4.2 as an immediate result.

Corollary 4.1

If $(Q_1, Q_2) \sim BB_1^V(n_1, n_2, m, \alpha_1, \alpha_2, c; \theta)$ *then from [18],*

$$E\left(Q_1^{h_1} Q_2^{h_2}\right) = \frac{\Gamma\left(\frac{n_1}{2}+h_1\right)\Gamma\left(\frac{n_2}{2}+h_2\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} \left(\frac{c}{\alpha_1}\right)^{\frac{1}{2}n_1} \left(\frac{c}{\alpha_2}\right)^{\frac{1}{2}n_2} e^{-\frac{1}{2}\theta} \sum_{l=0}^{\infty} \frac{1}{l!} \frac{\Gamma\left(\frac{n_1+n_2+m}{2}+l\right)}{\Gamma\left(\frac{n_1+n_2+m}{2}+l+h_1+h_2\right)} \left(\frac{\theta}{2}\right)^l$$
$$\cdot F_1\left(\frac{n_1+n_2+m}{2}+l, \frac{n_1}{2}+h_1, \frac{n_2}{2}+h_2, \frac{n_1+n_2+m}{2}+l+h_1+h_2; \frac{\alpha_1-c}{\alpha_1}, \frac{\alpha_2-c}{\alpha_2}\right). \tag{35}$$

where $F_1(\cdot)$ is the Appell function of the first kind.

Corollary 4.2

If $(Q_1, Q_2) \sim BB_1^V(n_1, n_2, m, \alpha_1, \alpha_2, c; \theta)$ *then the*

(a) *density function of* $\lambda_3 = Q_1^{\frac{1}{2}n_1} Q_2^{\frac{1}{2}n_2}$ *(see [18])*

$$f(\lambda_3) = \frac{1}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} \left(\frac{c}{\alpha_1}\right)^{\frac{1}{2}n_1} \left(\frac{c}{\alpha_2}\right)^{\frac{1}{2}n_2} e^{-\frac{1}{2}\theta} \sum_{k=0}^{\infty}\sum_{t=0}^{\infty}\sum_{l=0}^{\infty} \frac{1}{k!t!l!}\Gamma\left(\frac{n_1+n_2+m}{2}+k+t+l\right)$$
$$\cdot \left(\frac{\theta}{2}\right)^l \left(\frac{\alpha_1-c}{\alpha_1}\right)^k \left(\frac{\alpha_2-c}{\alpha_2}\right)^t H_{1,2}^{2,0}\left(\lambda_3 \middle| \begin{matrix} \left(\frac{m}{2}+k+t+l, \frac{n_1+n_2}{2}\right) \\ \left(k, \frac{n_1}{2}\right), \left(t, \frac{n_2}{2}\right) \end{matrix}\right), \quad 0 < \lambda_3 < 1. \tag{36}$$

and

(b) *CDF of* λ_3 *is*

$$F(\lambda_3) = P(\Lambda_3 \le \lambda_3) = \frac{1}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} \left(\frac{c}{\alpha_1}\right)^{\frac{1}{2}n_1} \left(\frac{c}{\alpha_2}\right)^{\frac{1}{2}n_2} e^{-\frac{1}{2}\theta} \sum_{k=0}^{\infty}\sum_{t=0}^{\infty}\sum_{l=0}^{\infty} \frac{1}{k!t!l!}\Gamma\left(\frac{n_1+n_2+m}{2}+k+t+l\right)$$
$$\cdot \left(\frac{\theta}{2}\right)^l \left(\frac{\alpha_1-c}{\alpha_1}\right)^k \left(\frac{\alpha_2-c}{\alpha_2}\right)^t H_{2,3}^{2,1}\left(\lambda_3 \middle| \begin{matrix} (1,1), \left(\frac{m}{2}+\frac{n_1+n_2}{2}+k+t+l, \frac{n_1+n_2}{2}\right) \\ \left(k+\frac{n_1}{2}, \frac{n_1}{2}\right), \left(t+\frac{n_2}{2}, \frac{n_2}{2}\right), (0,1) \end{matrix}\right), \quad 0 < \lambda_3 < 1.$$

Subsequently, graphical representations will show the effect of the parameters α_1, α_2 and c on the form of the density function of Λ_3. Figure 5 shows the effect of α_1 on $f(\lambda_3)$ (see (36)) where $(Q_1, Q_2) \sim BB_1^V(2,2,2,\alpha_1,1,1)$. At smaller values of Λ_3 the density function, $f(\lambda_3)$, increases as α_1 decreases. Figure 6 illustrates the shape of $f(\lambda_3)$ (see (34)) for increasing values of α where $(\boldsymbol{Q}_1, \boldsymbol{Q}_2) \sim BB_2^V(2,2,2,\alpha,\alpha,1)$. We note that as α increases the density function shifts towards larger values of Λ_3.

Fig.5. Effect of α_1 on $f(\lambda_3)$, $\Lambda_3 = Q_1Q_2$, $(Q_1,Q_2) \sim BB_1^V(2,2,2,\alpha_1,1,1)$

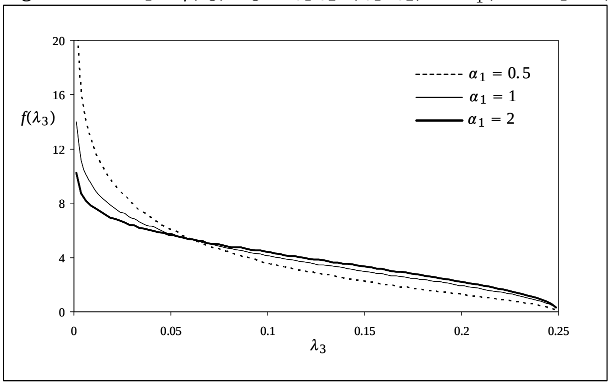
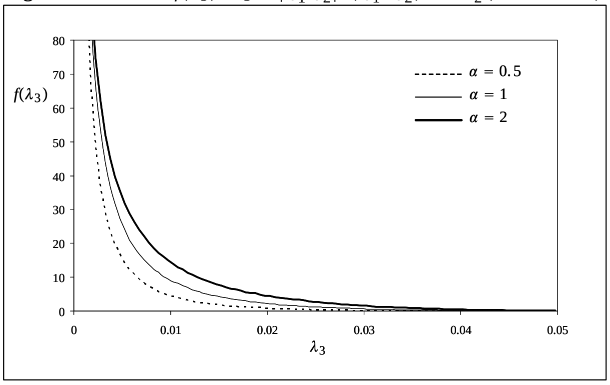


Fig.6. Effect of α on $f(\lambda_3)$, $\Lambda_3 = |\boldsymbol{Q}_1\boldsymbol{Q}_2|$, $(\boldsymbol{Q}_1,\boldsymbol{Q}_2) \sim BB_2^V(2,2,2,\alpha,\alpha,1)$



In Figure 7 the effect of the additional parameter c on $f(\lambda_3)$ (see (36)) was studied where $(Q_1, Q_2) \sim BB_1^V(2,2,2,1,1,c)$. At smaller values of Λ_3 the density function, $f(\lambda_3)$, increases as c increases. Figure 8 illustrates the shape of $f(\lambda_3)$ (see (34)) for increasing values of c where $(\boldsymbol{Q}_1, \boldsymbol{Q}_2) \sim BB_2^V(2,2,2,1,1,c)$.

Fig.7. Effect of c on $f(\lambda_3)$, $\Lambda_3 = Q_1Q_2$, $(Q_1,Q_2) \sim BB_1^V(2,2,2,1,1,c)$

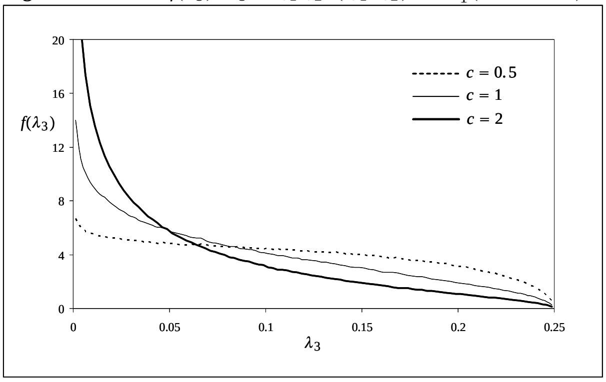
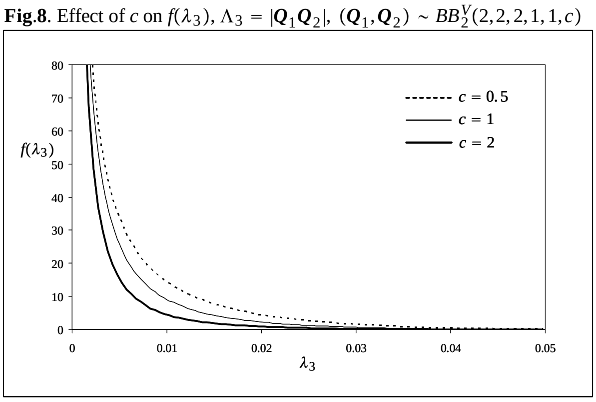


Fig.8. Effect of c on $f(\lambda_3)$, $\Lambda_3 = |\boldsymbol{Q}_1\boldsymbol{Q}_2|$, $(\boldsymbol{Q}_1,\boldsymbol{Q}_2) \sim BB_2^V(2,2,2,1,1,c)$



5 Conclusions

Bekker, Roux, Ehlers and Arashi [4] defined $\Lambda_1 \equiv \left|\frac{\boldsymbol{S}_1}{\boldsymbol{S}_1+\boldsymbol{B}}\right| \left|\frac{\boldsymbol{S}_2}{\boldsymbol{S}_2+\boldsymbol{B}}\right|$, the product of two dependent Wilks'statistics, and in this paper we focussed on the case if the common "denominator" of the "ratios" has the *noncentral* Wishart distribution. An exact expression for the density function, as well as the CDF of $\Lambda_2 \equiv \left|\frac{\boldsymbol{S}_1}{\boldsymbol{S}_1+\boldsymbol{S}_2+\boldsymbol{B}}\right|^{\frac{1}{2}n_1} \left|\frac{\boldsymbol{S}_2}{\boldsymbol{S}_1+\boldsymbol{S}_2+\boldsymbol{B}}\right|^{\frac{1}{2}n_2} = |\boldsymbol{U}_1|^{\frac{1}{2}n_1} |\boldsymbol{U}_2|^{\frac{1}{2}n_2}$ was given for $\boldsymbol{B}$ having a noncentral Wishart distribution. The noncentral bimatrix variate beta type V distribution, that allows for different covariance structure, was introduced with the corresponding generalized statistic Λ_3 and its density function expression. The effect of specific parameters on the density functions of Λ_i, $i = 1,2,3$ were shown. This paper makes a substantial contribution to the field of multivariate statistical analysis with the potential to be applied to hypothesis testing where two samples are present. A reason for the lack of exact expressions for the distributions of the test statistics under the nonnull hypothesis in the past is because of the limitation of software packages to handle the final expressions which are quite complicated. These functions are becoming more computable due to the availability of packages and algorithms, see [29]. Since exact expressions for the density functions of these statistics are now available exact confidence intervals can also be determined.

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