Stieltjes interlacing of zeros of Laguerre polynomials from different sequences

Kathy Driver

Department of Mathematics and Applied Mathematics, University of Cape Town, Private Bag X3, Rondebosch 7701, Cape Town, South Africa

Kerstin Jordaan

Department of Mathematics and Applied Mathematics, University of Pretoria, Pretoria, 0002, South Africa

Abstract

Stieltjes' Theorem (cf. [11]) proves that if $\{p_n\}_{n=0}^{\infty}$ is an orthogonal sequence, then between any two consecutive zeros of p_k there is at least one zero of p_n for all positive integers k, $k < n$; a property called Stieltjes interlacing. We prove that Stieltjes interlacing extends across different sequences of Laguerre polynomials L_n^{α}, $\alpha > -1$. In particular, we show that Stieltjes interlacing holds between the zeros of $L_{n-1}^{\alpha+t}$ and L_{n+1}^{α}, $\alpha > -1$, when $t \in \{1, \ldots, 4\}$ but not in general when $t > 4$ or $t < 0$ and provide numerical examples to illustrate the breakdown of interlacing. We conjecture that Stieltjes interlacing holds between the zeros of $L_{n-1}^{\alpha+t}$ and those of L_{n+1}^{a} for $0 < t < 4$. More generally, we show that Stieltjes interlacing occurs between the zeros of L_{n+1}^{α} and the zeros of the kth derivative of L_n^{α}, as well as with the zeros of $L_{n-k}^{\alpha+k+t}$ for $t \in \{1, 2\}$ and $k \in \{1, 2, \ldots, n-1\}$. In each case, we identify associated polynomials, analogous to the de Boor-Saff polynomials (cf. [3], [7]), that are completely determined by the coefficients in a mixed three term recurrence relation, whose zeros complete the interlacing process.

Key words:
Interlacing of zeros; Stieltjes' Theorem; Laguerre polynomials; de Boor-Saff polynomials.
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Email addresses: kathy.driver@uct.ac.za (Kathy Driver), kjordaan@up.ac.za (Kerstin Jordaan)

1. Introduction

A classical theorem of Stieltjes (cf. [11, Theorem 3.3.3]) proves that if $\{p_n\}_{n=0}^{\infty}$ is any sequence of orthogonal polynomials then the zeros of p_k and p_n, $k < n$, are interlacing in the sense that each open interval of the form $(-\infty, z_1)$, (z_1, z_2),...,(z_{k-1}, z_k), (z_k, ∞), where $z_1 < z_2 < ... < z_k$ are the zeros of p_k, contains at least one zero of p_n. If we assume that p_k and p_n have no common zeros, the same argument used by Stieltjes shows that there exist k open intervals, with endpoints at successive zeros of p_n, each of which contains exactly one zero of p_k.

For polynomials p_k and p_n with $k < n-1$, there are clearly not enough zeros of p_k to provide the required number of points to interlace fully with the zeros of p_n. However, this deficit in the number of points needed to complete the interlacing process between the zeros of polynomials of non-consecutive degree in an orthogonal sequence is well understood. In [2, Theorem 4], Beardon proves that if $\{p_n\}_{n=0}^{\infty}$ is any sequence of orthogonal polynomials with p_m and p_n having no common zeros for $m \neq n$, then there exist real polynomials S_{n-m-1} of degree $n-m-1$ whose real simple zeros, together with the zeros of p_m, interlace with the zeros of p_n for $m < n$. The polynomials S_{n-m-1} were first observed, albeit in a rather different context, by de Boor and Saff in [3] and are also studied by Vinet and Zhedanov in [12]. An important feature of the polynomials S_{n-m-1} is that they are completely determined by the coefficients in the three term recurrence relation satisfied by the orthogonal sequence $\{p_n\}_{n=0}^{\infty}$ (cf. [2]). Independently, it also follows immediately from Segura's result (cf [9], Theorem 1) that the zero of the linear polynomial that completes the interlacing of the zeros of p_{n-1} with those of p_{n+1} is given by one of the coefficients in the three term recurrence relation satisfied by the orthogonal sequence $\{p_n\}_{n=0}^{\infty}$.

The question arises as to whether Stieltjes interlacing occurs between the zeros of two polynomials p_n and q_k, $k < n-1$, from different orthogonal sequences $\{p_n\}_{n=0}^{\infty}$ and $\{q_n\}_{n=0}^{\infty}$ and whether polynomials analogous to the de Boor-Saff polynomials exist in this more general situation. Among the classical orthogonal families of Gegenbauer, Laguerre and Jacobi polynomials, natural choices of different orthogonal sequences are those corresponding to different values of the appropriate parameter(s) and also the (orthogonal) sequences of their derivatives. In this context, results in [4] prove that Stieltjes interlacing of zeros occurs across different sequences of Gegenbauer polynomials with $p_n = C_n^{\lambda}$ and $q_k = C_k^{\mu}$, $k \leq n-2$ for a specified range of values of λ and μ, $\lambda \neq \mu$, $\lambda, \mu > -\frac{1}{2}$.

In this paper we prove that Stieltjes interlacing takes place between the zeros of Laguerre polynomials L_{n+1}^{α} and $L_{n-1}^{\alpha+t}$, $n \in \mathbb{N}$, $\alpha > -1$, $t \in \{1, 2, 3, 4\}$ and we identify the polynomials that are analogous to the de Boor-Saff polynomials in these cases. We make two conjectures regarding the location of the (single) extra point that completes the interlacing process for continuous variation of the parameter t in the range $0 < t < 2$ and $2 < t < 4$. Our two conjectures are equivalent when $t = 2$. Numerical examples are provided to illustrate that

Stieltjes interlacing breaks down in general when $t = 5$ or $t = -1$. Our main result proves that Stieltjes interlacing occurs between the zeros of L_{n+1}^{α} and the zeros of the k^{th} derivative of L_n^{α} for all $k \in \{1, \ldots, n-1\}$ and identifies the polynomials analogous to the de Boor-Saff polynomials. Finally, we prove that Stieltjes interlacing holds between the zeros of L_{n+1}^{α} and the zeros of $L_{n-k}^{\alpha+k+t}$ for $t = 1$ and $t = 2$.

We recall that, for $\alpha > -1$, the Laguerre polynomials L_n^{α} are orthogonal with respect to the weight function $w_{\alpha}(x) = x^{\alpha}e^{-x}$ on $(0, \infty)$ and satisfy the three term recurrence relation

$$(n+1)L_{n+1}^{\alpha}(x) = (2n + \alpha + 1 - x)L_n^{\alpha}(x) - (\alpha + n)L_{n-1}^{\alpha}(x). \qquad (1)$$

In Section 2, we discuss the existence of common zeros of Laguerre polynomials. Thereafter, we will assume in all our theorems, lemmas and conjectures that the polynomials under consideration have no common zeros. Our results are stated in Section 3 and the proofs are provided in Section 4.

2. Common zeros of Laguerre polynomials

Bourget's Hypothesis for Bessel functions states that for any integers $n \geq 0$ and $m \geq 1$, the functions J_n and J_{n+m} have no common zeros except for the one at the origin. Siegel proved this famous hypothesis in [10] as a consequence of a more general result. An interesting discussion on this and related results can be found in (cf. [13, p. 484-485]). The analogue of Bourget's Hypothesis for Laguerre polynomials, namely that L_n^{α} and L_{n+m}^{α} have no common zeros for any integers $n \geq 0$ and $m > 1$, does not hold for all $n \in \mathbb{N}$ and $\alpha > -1$. If we compute the zeros of the resultant of L_n^{α} and L_{n+m}^{α} using Mathematica, we see that the two polynomials have a common zero when $n = 2, m = 2$, and $\alpha = 23$. Note that it is evident from (1) that this common zero of L_2^{23} and L_4^{23} is at the point $x = 30 = 2n + \alpha + 1$. It is an open question how many common zeros are possible in general for L_n^{α} and L_{n+m}^{α} , $n \geq 0$ and $m > 1$, although there is a sharp upper bound, namely, $\min\{k, n-k-1\}$, due to Gibson, for the maximum possible number of common zeros of p_k and p_n, $k < n$, in a general orthogonal sequence $\{p_n\}_{n=0}^{\infty}$ (cf. [5]). Across different sequences of Laguerre polynomials, we can compute the zeros of the resultant using Mathematica to show that when $t = 1$, $n = 4$, $m = 2$ and $\alpha = \frac{\sqrt{97}-5}{3}$, the polynomials $L_4^{\alpha+1}$ and L_6^{α} have a common zero at $x = \frac{13+\sqrt{97}}{3}$. Note that this common zero occurs at the point $\alpha + n + 1$ as can be seen from (5).

The assumption throughout this paper that the pair of polynomials under consideration in each of the lemmas and theorems have no common zeros is necessary in order to ensure that "full" Stieltjes interlacing takes place. However, each of our proofs can be modified to accommodate the situation where there are common zeros and we provide the details of the required modification in the remark after the proof of Theorem 3. Essentially, the common zeros are factored out and Stieltjes interlacing occurs between the remaining (non-common) zeros of the two polynomials under consideration.

3. Main results

Our first theorem proves that the zeros of $L_{n-1}^{\alpha+t}$, together with a point on the real line whose location depends on t, α and n, interlace with the zeros of L_{n+1}^{α} for integer values of t, $t \in \{0, 1, 2, 3, 4\}$. The case $t = 0$ was proved by Segura in [9, Section 3.1]. For the convenience of the reader, we provide an alternative proof of the case $t = 0$.

Theorem 1. *(i) The zeros of L_{n-1}^{α}, together with the point $\alpha + 1 + 2n$, interlace with the zeros of L_{n+1}^{α};*

(ii) The zeros of $L_{n-1}^{\alpha+1}$, together with the point $\alpha + 1 + n$, interlace with the zeros of L_{n+1}^{α};

(iii) The zeros of $L_{n-1}^{\alpha+2}$, together with the point $\alpha + 1$, interlace with the zeros of L_{n+1}^{α};

(iv) The zeros of $L_{n-1}^{\alpha+3}$, together with the point $\dfrac{(\alpha+1)(\alpha+2)}{(\alpha+2+n)}$, interlace with the zeros of L_{n+1}^{α};

(v) The zeros of $L_{n-1}^{\alpha+4}$, together with the point $\dfrac{(\alpha+1)(\alpha+3)}{(\alpha+3+2n)}$, interlace with the zeros of L_{n+1}^{α}.

Remarks

(a) The extra interlacing points in (iv) and (v) respectively are the upper bounds for the smallest zero w_1 of the Laguerre polynomial L_{n+1}^{α} obtained by Gupta and Muldoon in [6, eqns. 2.9 and 2.10], namely $w_1 < \frac{(\alpha+1)(\alpha+2)}{\alpha+2+n}$ and $w_1 < \frac{(\alpha+1)(\alpha+3)}{\alpha+3+2n}$.

(b) We see that Stieltjes interlacing can break down when the restrictions stated in Theorem 1 are not satisfied. For example, Mathematica shows that the zeros of $L_{n+1}^{\alpha}(x)$ and $L_{n-1}^{\alpha+t}(x)$ do not interlace when $n = 5$, $\alpha = -0.9$ and $t = 5$ or $t = -1$. Similarly, the result in Theorem 3 that the zeros of L_{n+1}^{α} and $L_{n-k}^{\alpha+k+t}$ are Stieltjes interlacing for $t \in \{0, 1, 2\}$ does not hold for $t > 2$ or $t < 0$ and this can be illustrated by computing the zeros of $L_{n+1}^{\alpha}(x)$ and $L_{n-k}^{\alpha+k+t}(x)$ when $n = 7$, $\alpha = 1.34$, $k = 3$ with $t = 3$ or $t = -1$ respectively.

Conjectures

(a) The $(n-1)$ zeros of $L_{n-1}^{\alpha+t}$, together with the point $\alpha+1+(2-t)n$, interlace with the zeros of L_{n+1}^{α} for all $0 < t < 2$.

(b) The $(n-1)$ zeros of $L_{n-1}^{\alpha+t}$, together with the point $\dfrac{(\alpha+1)(\alpha+(t-1))}{(\alpha+t-1+(t-2)n)}$, interlace with the zeros of L_{n+1}^{α} for all $2 < t < 4$.

Remark It is evident from Theorem 1 (i), (ii) and (iii) that the interval (w_k, w_{k+1}) with endpoints at successive zeros of L_{n+1}^{α} that does not contain a zero of $L_{n-1}^{\alpha+t}$, $t \in \{0,1,2\}$ changes from the interval containing the point $\alpha+1+2n$ when $t=0$ to the interval containing the point $\alpha+1+n$ when $t=1$ and the interval containing the point $\alpha+1$ when $t=2$. It is clear that since the variation of the extra interlacing point is continuous in t, the point $\alpha+1+(2-t)n$ will coincide with a zero of L_{n+1}^{α} as t varies between 0 and 2. It would be reasonable to expect that when $L_{n+1}^{\alpha}(\alpha+1+(2-t)n)=0$, the point $\alpha+1+(2-t)n$ will also be a zero of $L_{n-1}^{\alpha+t}$ since this is the case when $t=0,1,2$ as is evident from (1), (5) and (8) respectively. Numerical data generated by Mathematica indicates that, modulo the extra interlacing point coinciding with a zero of L_{n+1}^{α}, the above conjectures are true.

We will need the following lemma in the proof of Theorem 3.

Lemma 2. *For each $k \in 1,\ldots,n-1$, $n \in \mathbb{N}$,*

$$x^k L_{n-k}^{\alpha+k}(x) = G_k(x)L_n^{\alpha}(x) - H_{k-1}(x)L_{n+1}^{\alpha}(x) \tag{2}$$

where G_k and H_k are polynomials of degree k.

Theorem 3. *Let $n \in \mathbb{N}$ and $k \in \{1,\ldots,n-1\}$ and let G_k and H_k be the polynomials defined by (2). Then*

(i) the zeros of the k^{th} derivative of L_n^{α}, together with the k zeros of G_k, interlace with the zeros of L_{n+1}^{α};

(ii) the zeros of $L_{n-k}^{\alpha+k+1}$, together with the k zeros of $G_k - H_{k-1}$, interlace with the zeros of L_{n+1}^{α};

(iii) the zeros of $L_{n-k}^{\alpha+k+2}$, together with the k zeros of $\alpha(G_k - H_{k-1})(x) - xH_{k-1}(x)$, interlace with the zeros of L_{n+1}^{α};

Our final result shows the symmetric structure of the zeros of the de Boor-Saff polynomial associated with L_{n+1}^{α} and L_{n-2}^{α}.

Theorem 4. *The zeros of L_{n-2}^{α}, together with the points $2n+\alpha+\sqrt{1+n(n+\alpha)}$ and $2n+\alpha-\sqrt{1+n(n+\alpha)}$, interlace with the zeros of L_{n+1}^{α}.*

4. Proofs

Proof of Theorem 1.

(i) Since $L_{n+1}^{\alpha}(\alpha+1+2n) \neq 0$ by (1) and our assumption that L_{n+1}^{α} and L_{n-1}^{α} are coprime, evaluating (1) at successive zeros w_k and w_{k+1} of L_{n+1}^{α}, we obtain

$$\frac{L_n^{\alpha}(w_k)L_n^{\alpha}(w_{k+1})}{L_{n-1}^{\alpha}(w_k)L_{n-1}^{\alpha}(w_{k+1})} = \frac{(\alpha+n)^2}{(2n+\alpha+1-w_k)(2n+\alpha+1-w_{k+1})}. \quad (3)$$

The right-hand side of (3) is positive if and only if $2n+\alpha+1 \notin (w_k, w_{k+1})$, while $L_n^{\alpha}(w_k)L_n^{\alpha}(w_{k+1}) < 0$ for each $k \in \{1,\dots,n\}$ because the zeros of L_{n+1}^{α} and L_n^{α} are interlacing. We deduce that, provided $2n+1+\alpha \notin (w_k, w_{k+1})$, L_{n-1}^{α} has a different sign at successive zeros of L_{n+1}^{α} and therefore has an odd number of zeros in each interval (w_j, w_{j+1}), $j \in \{1,\dots,n\}$, that does not contain the point $\alpha+1+2n$. It follows from the Intermediate Value Theorem that the $n-1$ zeros of L_{n-1}^{α}, together with the point $\alpha+1+2n$, interlace with the $n+1$ zeros of L_{n+1}^{α}.

(ii) From [8, p.203]

$$(n+1)L_{n+1}^{\alpha}(x) = (\alpha+n+1)L_n^{\alpha}(x) - xL_n^{\alpha+1}(x). \quad (4)$$

From [1, eqn(22.7.30)] and (4)

$$\begin{aligned}(n+1)L_{n+1}^{\alpha} &= (\alpha+n+1)L_n^{\alpha}(x) - x\big[L_{n-1}^{\alpha+1}(x) + L_n^{\alpha}(x)\big]\\ &= (\alpha+n+1-x)L_n^{\alpha}(x) - xL_{n-1}^{\alpha+1}(x). \quad (5)\end{aligned}$$

Since L_{n+1}^{α} and $L_{n-1}^{\alpha+1}$ are co-prime by assumption, evaluating (5) at successive zeros w_k and w_{k+1} of L_{n+1}^{α} we obtain

$$\frac{L_n^{\alpha}(w_k)L_n^{\alpha}(w_{k+1})}{L_{n-1}^{\alpha+1}(w_k)L_{n-1}^{\alpha+1}(w_{k+1})} = \frac{w_k w_{k+1}}{(\alpha+n+1-w_k)(\alpha+n+1-w_{k+1})}. \quad (6)$$

Since $w_k > 0$ for all $k \in 1,\dots,n+1$, the stated result follows from (6) in the same way as the result in (i) followed from (3).

(iii) Replacing α by $\alpha+1$ in (5) and n by $n+1$ in [1, eqn(22.7.29)], we obtain

$$(n+1)L_{n+1}^{\alpha+1}(x) = (\alpha+n+2-x)L_n^{\alpha+1}(x) - xL_{n-1}^{\alpha+2}(x), \quad (7)$$

and

$$xL_{n+1}^{\alpha+1}(x) = (x-n-1)L_{n+1}^{\alpha}(x) + (\alpha+n+1)L_n^{\alpha}(x). \quad (8)$$

Substituting from (4) and (7) into (8) yields

$$(n+1)(\alpha+1)L_{n+1}^{\alpha}(x) = (\alpha+n+1)(\alpha+1-x)L_n^{\alpha}(x) - x^2L_{n-1}^{\alpha+2}(x). \quad (9)$$

Since L_{n+1}^{α} and $L_{n-1}^{\alpha+2}$ are co-prime by assumption, evaluating (9) at successive zeros w_k and w_{k+1}, $k \in \{1,\dots,n\}$ of $L_{n+1}^{\alpha}(x)$, we obtain

$$\frac{L_n^{\alpha}(w_k)L_n^{\alpha}(w_{k+1})}{L_{n-1}^{\alpha+2}(w_k)L_{n-1}^{\alpha+2}(w_{k+1})} = \frac{w_k^2 w_{k+1}^2}{(\alpha+n+1)^2(\alpha+1-w_k)(\alpha+1-w_{k+1})}. \quad (10)$$

Again, the result follows from (10) using the same argument as in the proof of (i).

(iv) Replacing α by $\alpha+1$ in (9), we have

$$x^2 L_{n-1}^{\alpha+3}(x) = (\alpha+n+2)(\alpha+2-x)L_n^{\alpha+1}(x) - (n+1)(\alpha+2)L_{n+1}^{\alpha+1}(x) \quad (11)$$

and substituting from (4) and (8) into (11) yields

$$\begin{aligned} x^3 L_{n-1}^{\alpha+3}(x) &= (\alpha+n+1)\big[(\alpha+2)(\alpha+1) - (\alpha+2+n)x\big]L_n^{\alpha}(x) \\ &- (n+1)\big[(\alpha+2)(\alpha+1) - nx\big]L_{n+1}^{\alpha}(x). \end{aligned} \quad (12)$$

The result follows as in (i).

(v) Replacing α by $\alpha+1$ in (12) yields

$$\begin{aligned} x^3 L_{n-1}^{\alpha+4}(x) &= (\alpha+n+2)\big[(\alpha+3)(\alpha+2) - (\alpha+3+n)x\big]L_n^{\alpha+1}(x) \\ &\quad -(n+1)\big[(\alpha+3)(\alpha+2) - nx\big]L_{n+1}^{\alpha+1}(x). \end{aligned} \quad (13)$$

Using (4) and (8) together with (13) we obtain

$$\begin{aligned} x^4 L_{n-1}^{\alpha+4}(x) &= (\alpha+2)(\alpha+n+1)\big[(\alpha+1)(\alpha+3) - (\alpha+3+2n)x\big]L_n^{\alpha}(x) \\ &+ (n+1)\big[nx^2 + 2nx(\alpha+2) - (\alpha+1)(\alpha+2)(\alpha+3)\big]L_{n+1}^{\alpha}(x) \end{aligned}$$

and the stated result again follows using the same argument as in (i).

■

Proof of Lemma 2. We prove the result by induction on k. For $k=1$, (5) yields (2) with $G_1(x) = \alpha+n+1-x$ and $H_0(x) = -(n+1)$ so the result is true for $k=1$. Now assume that, for $m = 1,2,\dots,k$,

$$x^m L_{n-m}^{\alpha+m}(x) = G_m(x)L_n^{\alpha}(x) - H_{m-1}(x)L_{n+1}^{\alpha}(x) \quad (14)$$

with $deg(G_m) = m$ and $deg(H_m) = m$. Then

$$\begin{aligned} & x^{k+1} L_{n-k-1}^{\alpha+k+1}(x) \\ &= x\left[x^k L_{(n-1)-k}^{(\alpha+1)+k}(x)\right] \\ &= G_k(x)xL_{n-1}^{\alpha+1}(x) - H_{k-1}(x)xL_n^{\alpha+1}(x) \text{ from (14)} \\ &= G_k(x)\left[(\alpha+n+1-x)L_n^{\alpha}(x) - (n+1)L_{n+1}^{\alpha}(x)\right] \\ &\quad - H_{k-1}\left[(\alpha+n+1)L_n^{\alpha}(x) - (n+1)L_{n+1}^{\alpha}(x)\right] \text{ from (5), (4)} \end{aligned}$$

$$= \quad G_{k+1}(x)L_n^{\alpha}(x) - H_k(x)L_{n+1}^{\alpha}(x)$$

where

$$G_{k+1}(x) = (\alpha + n + 1 - x)G_k(x) - (\alpha + n + 1)H_{k-1}(x)$$

and

$$H_k(x) = (n+1)[G_k(x) - H_{k-1}(x)].$$

Therefore (14) holds for $m = k + 1$ and the result follows by induction on k. ■

Proof of Theorem 3.

(i) We note (cf. [1]) that $D^k L_n^{\alpha} = (-1)^k L_{n-k}^{\alpha+k}$, $k \in \{0, 1, \ldots, n-1\}$, where D^k denotes the k-th derivative. From (2), if $L_{n+1}^{\alpha}(x) \neq 0$, we have

$$\frac{x^k L_{n-k}^{\alpha+k}(x)}{L_{n+1}^{\alpha}(x)} = \frac{G_k(x)L_n^{\alpha}(x)}{L_{n+1}^{\alpha}(x)} - H_{k-1}(x). \tag{15}$$

Now (cf. [2])

$$\frac{L_n^{\alpha}(x)}{L_{n+1}^{\alpha}(x)} = \sum_{j=1}^{n+1} \frac{A_j}{x - w_j}$$

where $\{w_j\}_{j=1}^{n}$ are the zeros of L_{n+1}^{α} and $A_j > 0$ for $j = 1, \ldots, n+1$, so that (15) can be written as

$$\frac{x^k L_{n-k}^{\alpha+k}(x)}{L_{n+1}^{\alpha}(x)} = \sum_{j=1}^{n+1} \frac{G_k(x)A_j}{x - w_j} - H_{k-1}(x). \tag{16}$$

Since L_{n+1}^{α} and L_n^{α} are always co-prime while L_{n+1}^{α} and $L_{n-k}^{\alpha+k}$ are co-prime by assumption, it follows from (2) that $G_k(w_j) \neq 0$ for any $j \in \{1, 2, \ldots, n+1\}$. Now, if $G_k(x)$ has no zeros in the interval (w_j, w_{j+1}) for some $j = 1, 2, \ldots, n$, then G_k does not change sign in this interval while H_{k-1} is bounded for all $x \in (w_j, w_{j+1})$ and $A_j > 0$. However, the expression on the right hand side of (16) will take (arbitrarily) large positive and negative values for $x \in (w_j, w_{j+1})$ and we deduce that $L_{n-k}^{\alpha+k}$ must have an odd number of zeros in this interval. Since there are n intervals (w_j, w_{j+1}) while G_k has degree k and $L_{n-k}^{\alpha+k}$ has $n-k$ real simple zeros, it follows that G_k must have k real simple zeros and we deduce that the zeros of the product $G_k L_{n-k}^{\alpha+k}$ interlace with the zeros of L_{n+1}^{α}.

(ii) Replacing α by $\alpha + 1$ in (2) and using (4) and (8) we obtain

$$x^{k+1} L_{n-k}^{\alpha+k+1}(x) = A_k(x)L_n^{\alpha}(x) - B_k(x)L_{n+1}^{\alpha}(x) \tag{17}$$

where

$$A_k(x) = (\alpha + n + 1)[G_k(x) - H_{k-1}(x)]$$

and

$$B_k(x) = (n+1)G_k(x) + (x - n - 1)H_{k-1}(x).$$

The stated result follows from the same argument used in (i).

(iii) Replacing α by $\alpha+1$ in (17) and using (4) and (8) yields

$$x^{k+2}L_{n-k}^{\alpha+k+2}(x) = C_k(x)L_n^{\alpha}(x) - D_{k+1}(x)L_{n+1}^{\alpha}(x)$$

where

$$C_k(x) = (\alpha+n+1)[A_k(x) - B_k(x)]$$

and

$$D_{k+1}(x) = (x-n-1)B_k(x) + (n+1)A_k(x).$$

The result now follows as in (i).

■

Remark The proof of Theorem 3 (i) in the case when $L_{n-k}^{\alpha+k}$ and L_{n+1}^{α} have common zeros can be modified as follows. We observe from (2) that if L_{n+1}^{α} and $L_{n-k}^{\alpha+k}$ have any common zeros, these must also be zeros of G_k since L_n^{α} and $L_{n+1}^{\alpha}$are always co-prime. Let $r(x)$ be the polynomial of degree $r \leq \max\{n-k, (n+1)-(n-k)-1\} = \max\{k, n-k\}$, having zeros at the r common (simple) zeros of $L_{n-k}^{\alpha+k}$ and L_{n+1}^{α}. Then in (16), canceling out all common zeros of $L_{n-k}^{\alpha+k}$ and L_{n+1}^{α}, we obtain

$$\frac{x^k L_{n-k}^{\alpha+k}(x)}{L_{n+1}^{\alpha}(x)} = \sum_{j=1}^{n+1-r} \frac{G_{k-r}(x)A_j}{x-w_j} - H_{k-1}(x)$$

where $G_{k-r}(x) = \frac{G_k(x)}{r(x)}$ is a polynomial of degree $k-r$. The argument proceeds exactly as before and we obtain Stieltjes interlacing of the (non-common) zeros of $L_{n-k}^{\alpha+k}$ and L_{n+1}^{α}.

Proof of Theorem 4. Replacing n by $n-1$ in (1) and eliminating L_{n-1}^{α} from the resulting equation and (1), we obtain

$$\begin{aligned}
&(n+1)(2n+\alpha-1-x)L_{n+1}^{\alpha}(x) \\
&= ((2n+\alpha+1-x)(2n+\alpha-1-x) - n(\alpha+n))L_n^{\alpha}(x) - \\
&\quad (\alpha+n)(\alpha+n-1)L_{n-2}^{\alpha}(x).
\end{aligned}$$

A straightforward calculation yields the result.

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References

[1] M Abramowitz and IA Stegun (Eds.), *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, (Nat. Bur. Standards Appl. Series, vol. 55, US Government Printing Office, Washington, DC, 1964).

[2] AF Beardon, The theorems of Stieltjes and Favard. *CMFT*, **11** (1) (2011), 247-262.

[3] C de Boor &, EB Saff, Finite sequences of orthogonal polynomials connected by a Jacobi matrix, *Linear Algebra Appl.* **75** (1986), 4355.

[4] K Driver, Interlacing of zeros of Gegenbauer polynomials of non-adjacent degree from different sequences, Submitted for publication.

[5] PC Gibson, Common zeros of two polynomials in an orthogonal sequence, *Journal of Approximation Theory*, **105** (2000), 129-132.

[6] DP Gupta and ME Muldoon, Inequalities for the smallest zeros of Laguerre polynomials and their q-analogues, *Journal of inequalities in pure and applied mathematics*, **8** 1 (2007), article 24.

[7] MED Ismail, *Classical and Quantum Orthogonal Polynomials in One Variable.* (Cambridge University Press, Cambridge, 2005)

[8] ED Rainville, *Special Functions*, (The Macmillan Company, New York, 1960).

[9] J Segura, Interlacing of the zeros of contiguous hypergeometric functions, *Numer. Algor.*, **49** (2008), 387-407.

[10] CL Siegel, Über einige Anwendungen diophantischer Approximationen, *Abh. Preuss. Akad. Wiss.* 1, 1929, 1-70.

[11] G Szegő, *Orthogonal Polynomials*, (American Mathematical Society, New York, 1959).

[12] L Vinet & A Zhedanov. A characterization of classical and semiclassical orthogonal polynomials from their dual polynomials. *J. Comput. Appl. Math.* **172** (2004), 41-48.

[13] GN Watson, *A Treatise on the Theory of Bessel Functions 2nd ed.*, (Cambridge University Press, Cambridge 1966).